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Interim Technical Report
Stanford Research Institute
Menlo Park, California
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STRUCTURAL ENGINEERING LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY CALIFORNIA

Structures and Materials Research
Department of Civil Engineering
Division of Structural Engineering
and Structural Mechanics

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ABSTRACT

A theoretical investigation of the compressive response of a wire embedded in a matrix is reported. In the first two sections consideration is given to the beam-column behavior of an initially crooked wire embedded in a matrix which is subjected to a state of uniaxial stress. A plot of the effective stiffness of the wire as a function of the wire's initial crookedness is given. Secondly, consideration is given to the possibility of buckling, within the matrix, of a compressively loaded straight wire. A plot of the buckling load of the wire as a function of the ratio of the wire's and matrix's shear moduli is given. Two procedures are developed for the prediction of the lateral support offered to the wire by the matrix. Both procedures are developed by treating the matrix as a three-dimensional elastic solid, however, only the first of the two methods completely satisfies the displacement continuity requirements between the wire and the matrix.

NOTATION

A	Cross-sectional area of wire
α	Rotation of wire's cross-section about z axis
α_s	Beam shear deformation coefficient
$\beta = \frac{2\pi}{\ell}$	
$\beta_{cr} = \frac{2\pi}{\ell_{cr}}$	
c_1 and c_2	Constants defined by Equations (30) and (31)
χ_0	Amplitude of the initial shape of the wire
χ	Amplitude of the deflection of the wire
$\Delta\ell$	Change in the x component of the distance between the ends of an element of wire due to deformation
ΔS_0	Length of an undeformed element of wire
ΔS	Length of a deformed element of wire
$\bar{E}$	Young's modulus of the wire

ε_p	Axial strain in wire due to force P
ε_x	The x component of strain in the composite material
F and Q	Components (x and y) of the internal force in the wire
$\bar{G}$	Shear modulus of the wire
I	Moment of inertia of the wire's cross-section
k_1 and k_2	Normal force and moment foundation moduli
$K_i(\xi)$	Modified Bessel function of the second kind
ℓ	Wave length of the deformed wire
ℓ_{cr}	Wave length of the buckled wire
M	Internal moment in the wire
μ	Shear modulus of the matrix
ν	Poisson's ratio of the matrix
$\bar{\nu}$	Poisson's ratio of the wire

P	Axial force in the wire
P_{cr}	Buckling load of the wire
q and m	Normal force and moment reaction of the matrix on the wire
R	Radius of the wire
ρ	Radius of curvature of the deformed wire
S_c	Constant defined by Equation (79)
S_c^*	Constant defined by Equation (85)
θ_0	Initial slope of wire's center line
θ	Final slope of wire's center line
Y_0	Position of the undeformed wire's center line
Y	Position of the deformed wire's center line
Y^*	Deflection of the wire's center line

INTRODUCTION

The structural usage of low modulus materials reinforced by high modulus wires or filaments [1]* has necessitated the determination of the mechanical properties of the resulting composite materials. From results of mechanical property tests it has been observed that such materials often exhibit a different compressive behavior than tensile behavior. It has been suggested that perhaps this difference in behavior is due to the buckling of the wires within the matrix (in contrast to the overall buckling of the composite material [2]). It is the purpose of this paper to attempt to shed some light on this phenomenon by theoretically predicting the compressive (and tensile) response of a wire contained in a matrix. The investigation will be restricted to those configurations where the spacing between the reinforcing wires is relatively large and, thus, the interaction of the compressive behaviors of neighboring wires need not be considered.

MEDIUM DEFORMATION BEAM-COLUMN BEHAVIOR

One plausible explanation for the difference between the compressive response and the tensile response is that it is caused by the lack of initial straightness of the wires, i.e., by a beam-column type behavior of the wires. Because it is desirable to be able to apply this analysis to reinforcement configurations in which the wires are deliberately not straight (such as screen reinforced material) and are, in fact, initially quite crooked, it will be necessary to consider moderately large deflection behavior of the wires. The solution will, however, for the sake of simplicity, be limited to the inclusion of those terms necessary for the assessment of the importance

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*Numbers in brackets denote references listed at the end of the paper.

of the medium deformation effects and to indicate the trends of the solution. In characterizing the supporting strength of the matrix it will be assumed that the deflections are small and that the cavity occupied by the wire is initially straight.

Three-dimensional elasticity theory will be used to characterize the behavior of the surrounding matrix, and beam theory will be used to describe the response of the wire. The ability of simple beam theory to accurately predict the behavior of a wire contained in a continuous matrix, for the range of relative stiffnesses of interest in this study, is verified by illustrating that the inclusion of shear deformation effects has a negligible effect on the final results (this comparison will be presented in a following section).

Consider an initially non-straight wire; the average position of the wire's center line coincides with the x axis and the initial crookedness is in the $x - y$ plane. The wire's initial shape (Y_0) is approximated by the first term in a cosine series (see Figure 1).

$$Y_0(x) = \chi_0 \cos \beta x \tag{1}$$

The dashed line indicates the position of the wire after a uniaxial stress (in the x direction) has been applied to the composite material; the internal forces in the wire (F, M and Q) and the reactions of the matrix on the wire (q and m) are also shown in the sketch (for the problem considered herein the component of the matrix reaction parallel to the fiber is zero and, therefore, is not shown on the sketch). It was because of the inclusion in the analysis of the interaction moment "m" that it was felt that it might be necessary to include beam shear deformation effects, however, this was

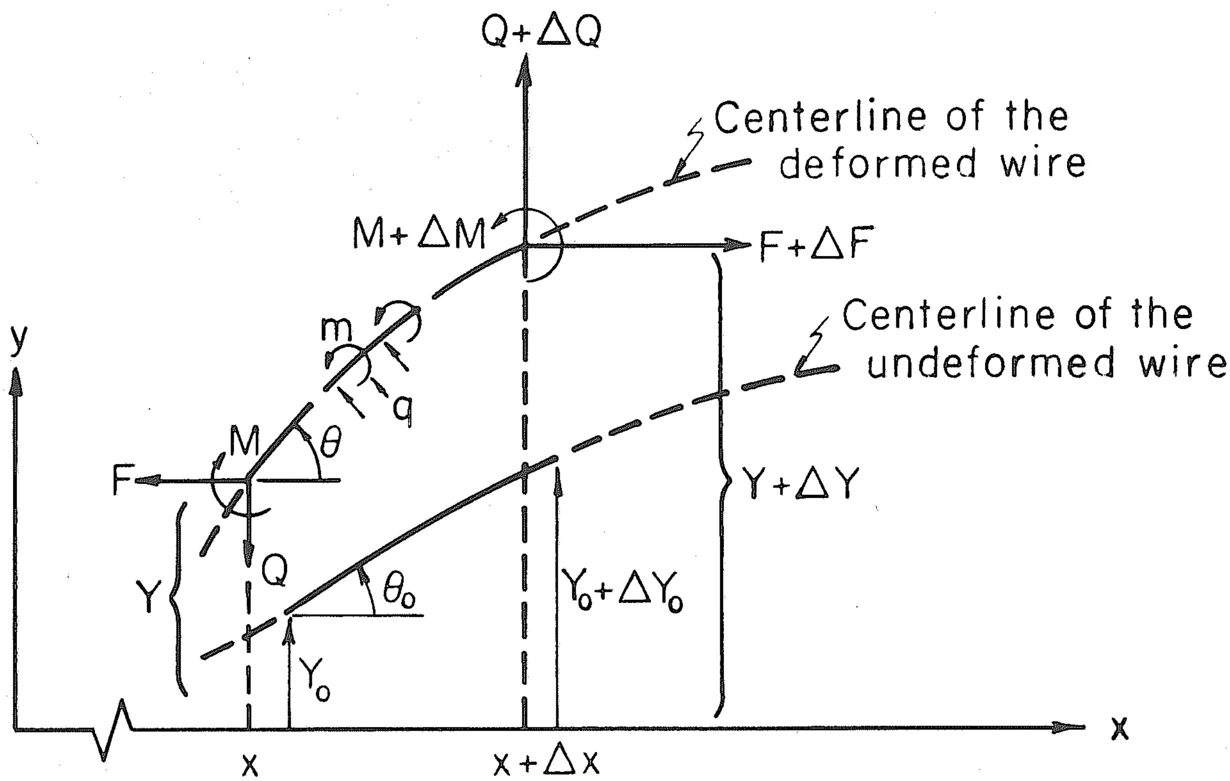

Figure 1 - Wire Element

found not to be the case for the class of problems of interest. As only moderately large deformations are considered it is not necessary to distinguish between the x coordinates of a point in the deformed and undeformed wire. In the following derivations, although it is not explicitly noted, the final equations are obtained by considering the limit as Δx passes to zero. Equilibrium of the wire requires:

$$\frac{dF}{dx} = q \text{ Tan } \theta \tag{2}$$

$$\frac{dQ}{dx} = - q \tag{3}$$

$$\frac{dM}{dx} + Q - F \text{ Tan } \theta = - \frac{m}{\text{Cos } \theta} \tag{4}$$

The shape of the wire's center line in its deformed position is expressed in the form (the first term is the deformation and the second the initial crookedness):

$$Y(x) = (\chi + \chi_0) \text{ Cos } \beta x \tag{5}$$

Hence, the deflection of the beam $Y^*(x)$ is expressed by the equation:

$$Y^*(x) = \chi \text{ Cos } \beta x \tag{6}$$

The reaction of the foundation (i.e., the matrix) on the wire due to the deflection χ is expressed in the form (the derivation of expressions for the foundation moduli k_1 and k_2 is discussed in the appendix):

$$q(x) = k_1 \; Y^* = k_1 \; \chi \; \text{Cos} \; \beta x \tag{7}$$

$$\frac{m(x)}{\text{Cos} \; \theta} = - \frac{k_2}{\beta} \frac{dY^*}{dx} = k_2 \; \chi \; \text{Sin} \; \beta x \tag{8}$$

Introducing Equation (7) into Equation (3) yields:

$$\frac{dQ}{dx} = - \, k_1 \; \chi \; \text{Cos} \; \beta x$$

Hence (note: $Q(0) = 0$):

$$Q(x) = - \frac{k_1 \; \chi}{\beta} \; \text{Sin} \; \beta x \tag{9}$$

Introducing Equation (7) into Equation (2) and noting that $\text{Tan} \; \theta = \frac{dY}{dx}$ yields:

$$\frac{dF}{dx} = - \; \beta \; (\chi + \chi_0) \; \text{Sin} \; \beta x \; k_1 \; \chi \; \text{Cos} \; \beta x \tag{10}$$

Integrating the above expression and denoting the value of $F(x)$ at $x = 0$ by F_0 the following equation is obtained:

$$F(x) = F_0 - \frac{k_1 \; \chi \; (\chi + \chi_0)}{2} \; (\text{Sin} \; \beta x)^2 \tag{11}$$

Substituting Equations (8), (9) and (11) (noting that $\text{Tan}\ \theta = \frac{dY}{dx}$) into Equation (4) yields:

$$\frac{dM}{dx} = \left[\frac{k_1 \chi}{\beta} - k_2 \chi - F_0 \beta (\chi + \chi_0)\right] \text{Sin}\ \beta x$$

$$+ \frac{\beta k_1 \chi (\chi + \chi_0)^2}{2} (\text{Sin}\ \beta x)^3 \tag{12}$$

Or (note: $M\left(\frac{\pi}{2\beta}\right) = 0$):

$$M(x) = \left[F_0 (\chi + \chi_0) + \frac{k_2 \chi}{\beta} - \frac{k_1 \chi}{\beta^2}\right] \text{Cos}\ \beta x$$

$$+ \frac{\chi k_1 (\chi + \chi_0)^2}{2} \left[\frac{(\text{Cos}\ \beta x)^3}{3} - \text{Cos}\ \beta x\right] \tag{13}$$

The right hand side of Equation (13) is approximated by retaining only the first term in a $\text{Cos}\ n\beta x$ series expansion.

$$M(x) = \left[F_0 (\chi + \chi_0) + \frac{k_2 \chi}{\beta} - \frac{k_1 \chi}{\beta^2} - \frac{3 \chi k_1 (\chi + \chi_0)^2}{8}\right] \text{Cos}\ \beta x \tag{14}$$

For moderately large deflections the change in the curvature of the wire may be expressed by the following equation:

$$\Delta\left(\frac{1}{\rho}\right) = \left[\frac{\frac{d^2Y}{dx^2}}{\left[1 + \left(\frac{dY}{dx}\right)^2\right]^{3/2}} - \frac{\frac{d^2Y_0}{dx^2}}{\left[1 + \left(\frac{dY_0}{dx}\right)^2\right]^{3/2}}\right] \tag{15}$$

The relationship between the internal moment M and the change of curvature $\Delta\left(\frac{1}{\rho}\right)$ for a beam (i.e., the wire) whose initial radius of curvature is much larger than the radius of the wire is ($\bar{E}$ and I denote respectively the elastic modulus and the moment of inertia of the cross-section of the wire):

$$M = \bar{E}\ I\ \Delta\left(\frac{1}{\rho}\right) \tag{16}$$

Combining Equations (15) and (16) yields:

$$M = \bar{E}\ I \left[\frac{\frac{d^2Y}{dx^2}}{\left[1 + \left(\frac{dY}{dx}\right)^2\right]^{3/2}} - \frac{\frac{d^2Y_0}{dx^2}}{\left[1 + \left(\frac{dY_0}{dx}\right)^2\right]^{3/2}} \right] \tag{17}$$

For moderately large deflections the above equation may be written in the following form:

$$M = \bar{E}\ I \left\{ \frac{d^2Y}{dx^2}\left[1 - \frac{3}{2}\left(\frac{dY}{dx}\right)^2\right] - \frac{d^2Y_0}{dx^2}\left[1 - \frac{3}{2}\left(\frac{dY_0}{dx}\right)^2\right] \right\} \tag{18}$$

Introducing Equations (1) and (5) into Equation (18) yields:

$$M = \bar{E}\ I \left\{ - \beta^2 \text{ Cos } \beta x \left[\chi - \frac{3\ \beta^2}{2}\left((\chi + \chi_0)^3 - \chi_0^3\right)(\text{Sin } \beta x)^2 \right] \right\} \tag{19}$$

Expanding the right hand side of the above equation in a Cos $n\beta x$ series and retaining the first term yields:

$$M(x) = - \bar{E}\ I\ \beta^2 \left\{ \chi - \frac{3\ \beta^2}{8}\left[(\chi + \chi_0)^3 - \chi_0^3\right] \right\} \text{Cos } \beta x \tag{20}$$

Equating Equations (14) and (20) and solving for F_0, yields:

$$F_0 = \frac{1}{\chi + \chi_0} \left\langle \frac{k_1 \chi}{\beta^2} + \frac{3 \chi k_1 (\chi + \chi_0)^2}{8} - \frac{k_2 \chi}{\beta} - \bar{E} I \beta^2 \left\{ \chi \right.\right.$$

$$\left.\left. - \frac{3 \beta^2}{8} [(\chi + \chi_0)^3 - \chi_0^3] \right\} \right\rangle \qquad (21)$$

Denoting the axial force in the wire by P the following expression is obtained (see Figure 1):

$$P = F \cos \theta + Q \sin \theta \qquad (22)$$

Introducing Equations (9) and (11) into the above expression yields:

$$P = \left[F_0 - \frac{k_1 \chi (\chi + \chi_0)}{2} (\sin \beta x)^2 \right] \cos \theta - \frac{k_1 \chi}{\beta} \sin \beta x \sin \theta \qquad (23)$$

Utilizing Equation (5) the following expressions are found:

$$\cos \theta = \frac{1}{\sqrt{1 + \left(\frac{dY}{dx}\right)^2}} = \frac{1}{\sqrt{1 + (\chi + \chi_0)^2 \beta^2 (\sin \beta x)^2}} \qquad (24)$$

$$\sin \theta = \frac{dY}{dx} \cos \theta = - \frac{(\chi + \chi_0) \beta \sin \beta x}{\sqrt{1 + (\chi + \chi_0)^2 \beta^2 (\sin \beta x)^2}} \qquad (25)$$

Hence Equation (23) may be written in the form:

$$P = \frac{F_0 + \frac{k_1 \chi (\chi + \chi_0)}{2} (\text{Sin } \beta x)^2}{\sqrt{1 + (\chi + \chi_0)^2 \beta^2 (\text{Sin } \beta x)^2}} \tag{26}$$

The above equation is approximated as follows:

$$P = F_0 + \frac{(\chi + \chi_0)}{2} [k_1 \chi - F_0 \beta^2 (\chi + \chi_0)] (\text{Sin } \beta x)^2 \tag{27}$$

The axial strain ε_p in the wire due to the axial force P is (A denotes the cross-sectional area of the wire):

$$\varepsilon_p = \frac{P}{A \bar{E}} \tag{28}$$

Or (using Equation (27))

$$\varepsilon_p = c_1 + c_2 (\text{Sin } \beta x)^2 \tag{29}$$

Where

$$c_1 = \frac{F_0}{\bar{E} A} \tag{30}$$

$$c_2 = \frac{(\chi + \chi_0)}{2 \bar{E} A} [k_1 \chi - F_0 \beta^2 (\chi + \chi_0)] \tag{31}$$

The squared length of the deformed wire element is expressed by the equation:

$$(\Delta S)^2 = (\Delta S_0)^2 \; (1 + \varepsilon_p)^2 \tag{32}$$

The lengths of the undeformed and deformed wire element may be expressed in the following form (where $\Delta \ell$ is the increase in the x distance between the ends of the wire element due to deformation):

$$\Delta S_0 = \sqrt{(\Delta x - \Delta \ell)^2 + (\Delta Y_0)^2} \tag{33}$$

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta Y)^2} \tag{34}$$

Hence Equation (32) may be expressed in the form:

$$(\Delta x - \Delta \ell)^2 + (\Delta Y_0)^2 = \frac{(\Delta x)^2 + (\Delta Y)^2}{(1 + \varepsilon_p)^2} \tag{35}$$

Noting that $\varepsilon_p << 1$ the above equation yields:

$$(\Delta x)^2 - 2\Delta x \; \Delta \ell + (\Delta \ell)^2 + (\Delta Y_0)^2 = [(\Delta x)^2 + (\Delta Y)^2] \; (1 - 2\varepsilon_p) \tag{36}$$

Solving for $\Delta \ell$ and passing to the limit the following equation is obtained:

$$\frac{d\ell}{dx} = \left\{1 - \sqrt{1 - \left[\left(\frac{dY_0}{dx}\right)^2 - \left(\frac{dY}{dx}\right)^2 + 2\varepsilon_p + 2\varepsilon_p \left(\frac{dY}{dx}\right)^2\right]}\right\} \tag{37}$$

Noting that the second term under the square root sign is small:

$$\frac{d\ell}{dx} = \frac{1}{2}\left\{\left(\frac{dY_0}{dx}\right)^2 - \left(\frac{dY}{dx}\right)^2 + \frac{1}{4}\left[\left(\frac{dY_0}{dx}\right)^2 - \left(\frac{dY}{dx}\right)^2\right]^2\right.$$

$$\left. + \varepsilon_p\left[2 + \varepsilon_p + \left(\frac{dY_0}{dx}\right)^2 + \left(\frac{dY}{dx}\right)^2\right]\right\} \tag{38}$$

Assuming that the wire remains bended to the matrix the x component of strain (ε_x) of the composite material is approximately expressed by the following equation:

$$\varepsilon_x = \frac{\beta}{\pi}\int_0^{\frac{\pi}{\beta}} \frac{d\ell}{dx}\, dx \tag{39}$$

Introducing Equation (38) into Equation (39) (using Equations (1), (5) and (29)) and integrating yields:

$$\varepsilon_x = c_1 + \frac{c_1^2}{2} + \frac{1}{4}\left\{- \beta^2 \chi (\chi + 2\chi_0) + 2c_2 (1 + c_1) + c_1 \beta^2 \left[\chi_0^2 + (\chi_0 + \chi)^2\right]\right\}$$

$$+ \frac{3}{16}\left\{c_2^2 + c_2 \beta^2 \left[\chi_0^2 + (\chi + \chi_0)^2\right] + \frac{\beta^4 \chi^2}{4} (\chi + 2\chi_0)^2\right\} \tag{40}$$

Equations (21) and (40) relate the force (F_0) carried by the initially nonstraight wire with the strain ε_x in the composite material. These two nonlinear parametric equations (in terms of the parameter χ) are easily solved by assigning values to χ and calculating the resulting values of F_0 and ε_x. The effective stiffness of the reinforcing wire is given by the ratio F_0/ε_x.

SMALL DEFLECTION BEAM-COLUMN BEHAVIOR

If the initial crookedness of the wire is small and if the analysis is restricted to additional small deflections the foregoing equations are considerably simplified (the many approximations that were made in the previous section are "exact" within the framework of small deflection theory); Equations (21) and (40) become:

$$F_0 = \frac{\chi}{\chi + \chi_0} \frac{k_1}{\beta^2} - \frac{k_2}{\beta} - \bar{E}\, I\, \beta^2 \tag{41}$$

$$\varepsilon_x = \frac{F_0}{\bar{E}\, A} - \frac{\beta^2\, \chi\, (\chi + 2\chi_0)}{4} \tag{42}$$

BUCKLING ANALYSIS

In this section the possibility of buckling (within the matrix) of an initially straight wire (i.e., $Y_0 = 0$) is investigated. For completeness the effects of beam shear deformations are included in the analysis (their insignificance for the class of problems considered in this report will be demonstrated in a following section). The curvature of the beam is given by the following equation [3] ($\bar{G}$ denotes the shear modulus of the wire):

$$\frac{d^2 Y^*}{dx^2} = \frac{M}{\bar{E}\, I} + \frac{\alpha_s}{A\, \bar{G}} \frac{dQ}{dx} \tag{43}$$

The coefficient α_s characterizes the effect of beam shear deformation (for beams with circular cross-sections the value of α_s has been reported to lie between 1.13 and 1.25). Substituting Equation (3) into the above expression and solving for M yields the following expression:

$$M = \bar{E}\, I \frac{d^2Y^*}{dx^2} + \frac{\bar{E}\, I\, \alpha_s}{A\, \bar{G}}\, q \tag{44}$$

With the addition of the shear deformation term Equation (41) becomes (for buckling considerations the initial crookedness is not considered, hence $\chi_0 = 0$; let $P = - F_0$):

$$P = \left[\bar{E}\, I\, \beta^2 + \frac{k_2}{\beta} - k_1 \left(\frac{1}{\beta^2} + \frac{\bar{E}\, I\, \alpha_s}{A\, \bar{G}}\right)\right] \tag{45}$$

The above expression represents the compressive load necessary to hold the wire in the deformed shape:

$$Y^* = \chi \operatorname{Cos} \beta x \tag{46}$$

The buckling load P_{cr} and the critical wave length $\ell_{cr} = \frac{2\pi}{\beta_{cr}}$ are determined by selecting β such that Equation (45) yields a minimum value for P. This selection is complicated by the fact that for a realistic foundation model (i.e., a realistic representation of the matrix behavior) the values of the foundation moduli k_1 and k_2 (see the Appendix) are very involved functions of β. The solution is obtained by numerically minimizing Equation (45).

NUMERICAL RESULTS

Buckling:

In Figure 2 a plot is given of Equation (45); the minimum point

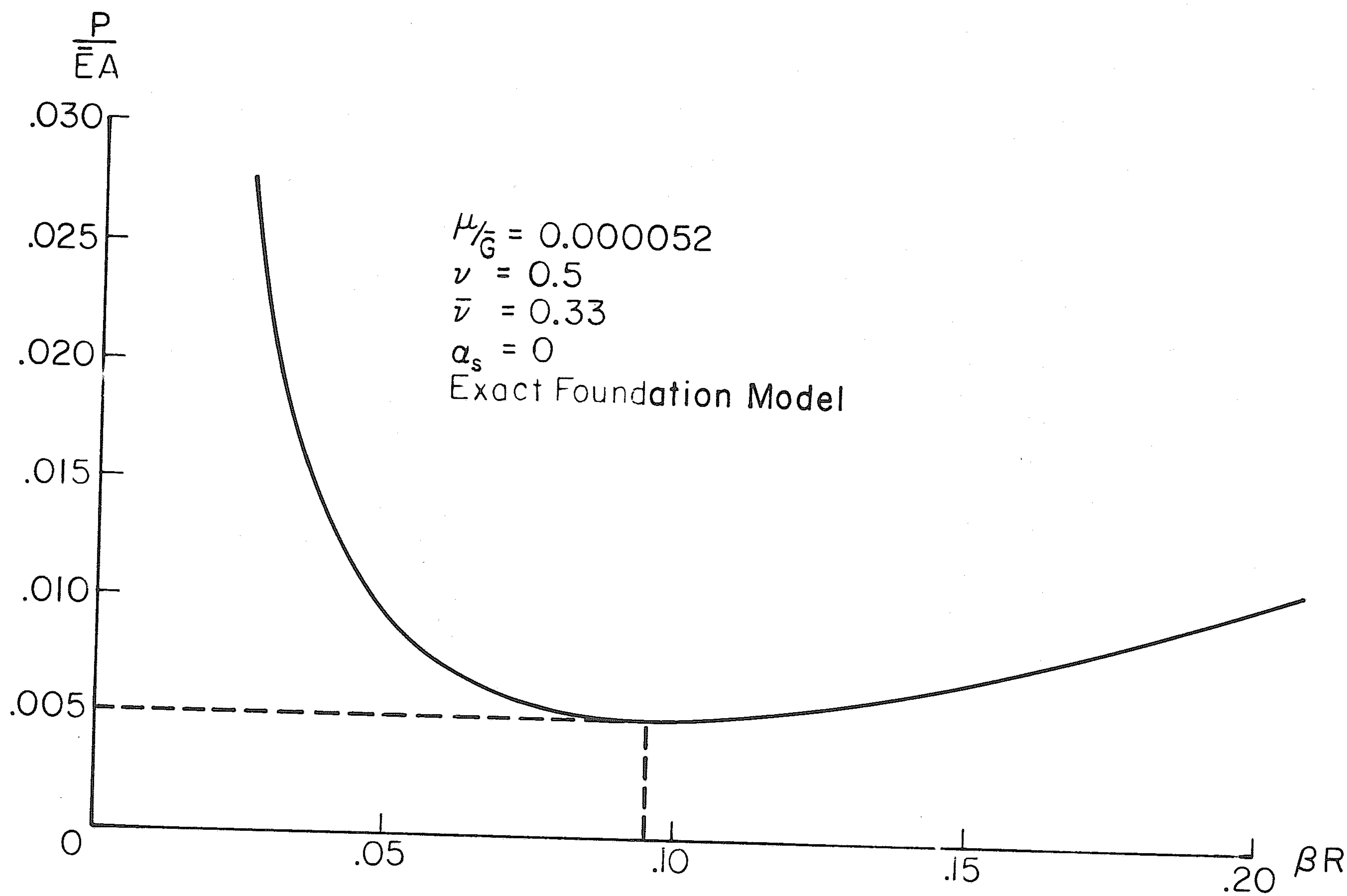

Figure 2 - Plot of Equation (45) (Exact Foundation Model)

represents the buckling load. Figure 3 is a plot of the buckling load as a function of the ratio of the shear modulus of the matrix to that of the wire. Figure 4 is the corresponding plot of β_{cr}. The results of interest for the current study are for $\mu/\bar{G} << .01$; Figures 5-8 give enlarged views of these results. From Figures 3 and 4 it is observed that the effects of beam shear deformation are negligible over the range of interest ($\mu/\bar{G} << .01$) and thus will not be included in subsequent results (for the case of a stiff beam resting on a soft foundation a similar conclusion was reported in reference [4]). It may also be observed that the differences in results obtained from the "exact" and the "approximate" foundation models are negligible for $\mu/\bar{G} << .01$ and, hence, only the results obtained using the approximate model are reported on subsequent figures.

Beam-column behavior:

In the numerical evaluation of the beam-column solutions (Equations (21) and (40) or Equations (41) and (42)) the most critical state of initial imperfection was considered, i.e., the imperfection was considered to have the shape of the buckling mode (the critical value of β was taken from Figure 7). Figure 9 illustrates the compressive stress-strain curves for the wire that are predicted by the small deflection beam-column analysis for varying degrees of initial imperfection; Figure 10 gives the corresponding results predicted by the medium deflection analysis. Figure 11 is a plot of the effective stiffness (i.e., the values that would be observed in uniaxial tests of the composite material) of the reinforcing wire as a function of the uniaxial composite strain and the initial imperfection of the wire (use was made of the medium deflection analysis). Figure 12 is a corresponding plot of the small deflection predictions. Thus, for example, if the initial

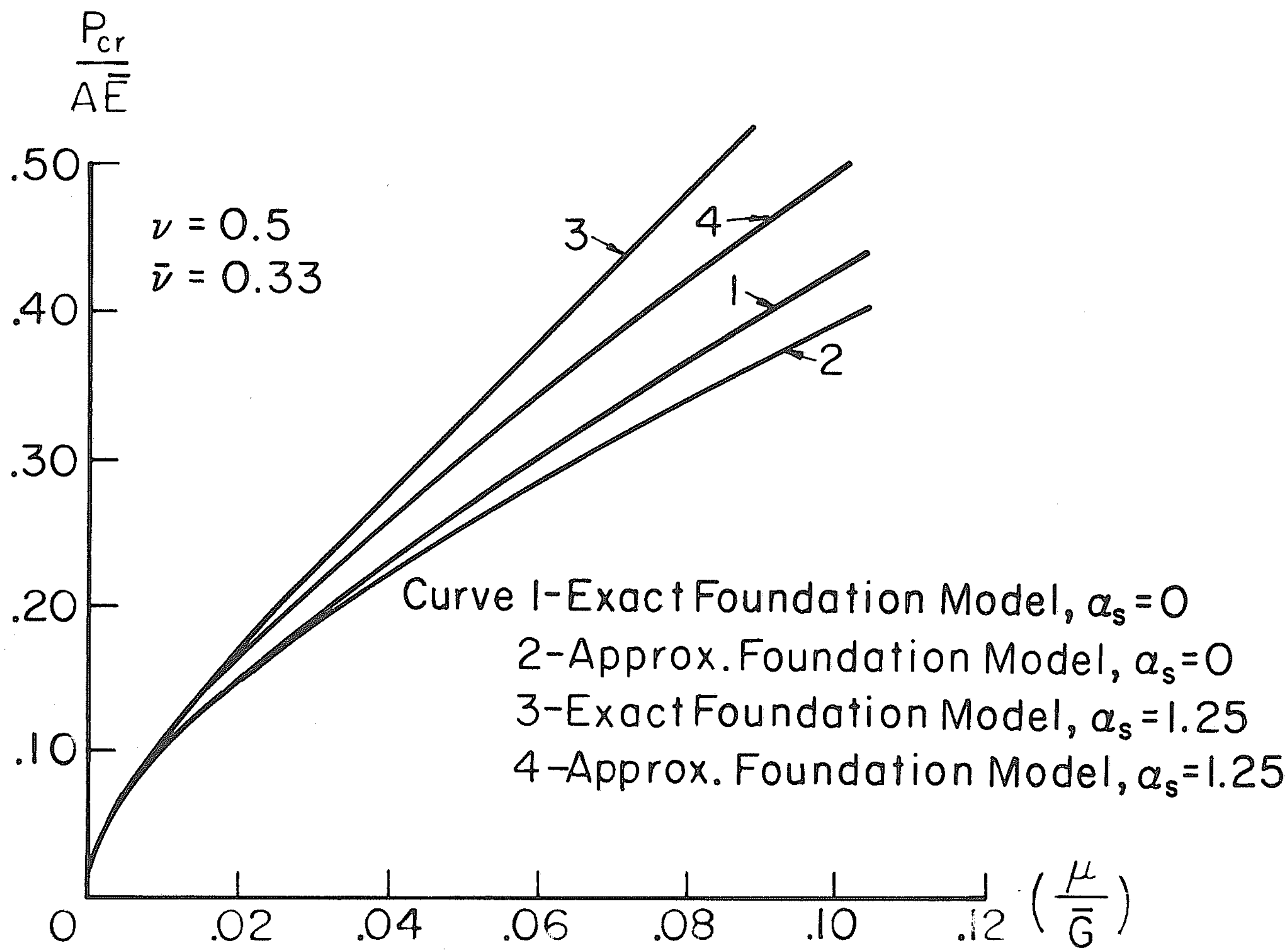

Figure 3 - Wire Buckling Load

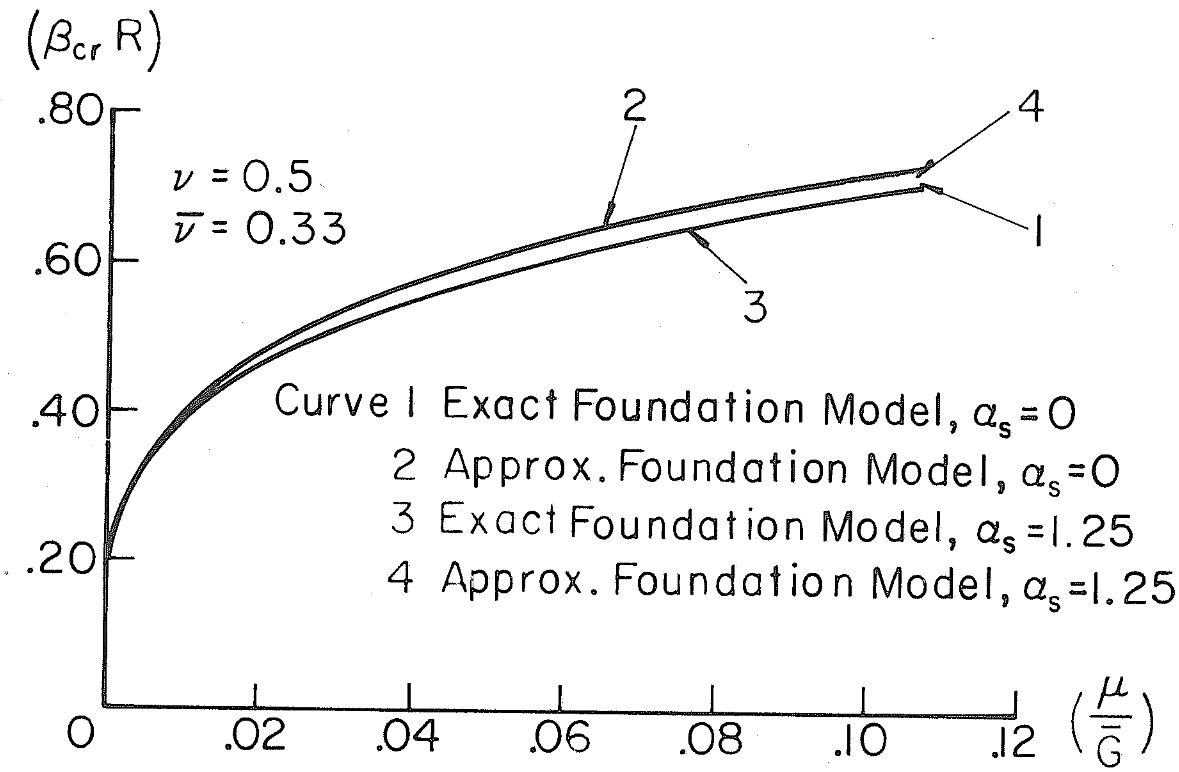

Figure 4 - Wave Number of Buckled Shape

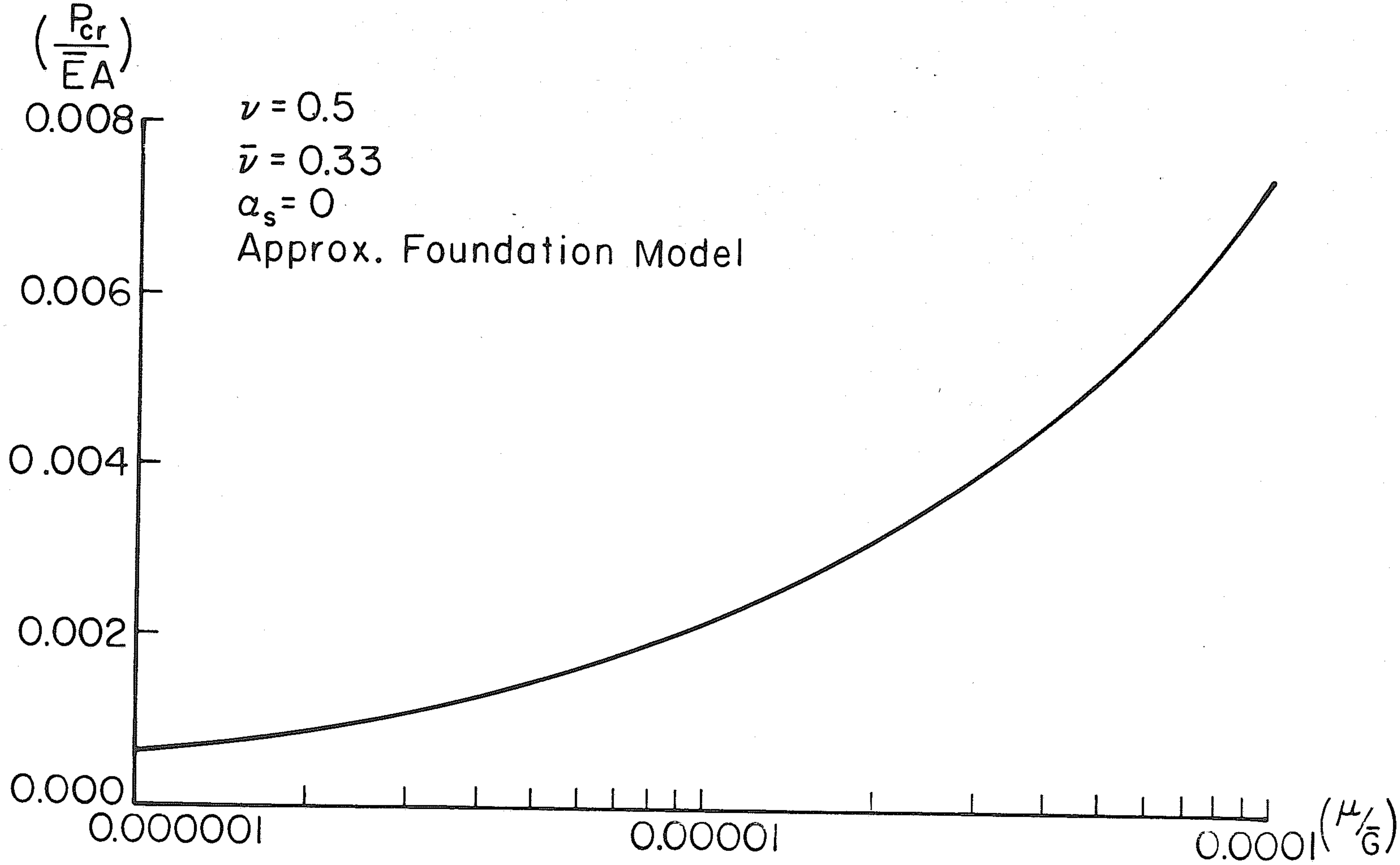

Figure 5 - Wire Buckling Load

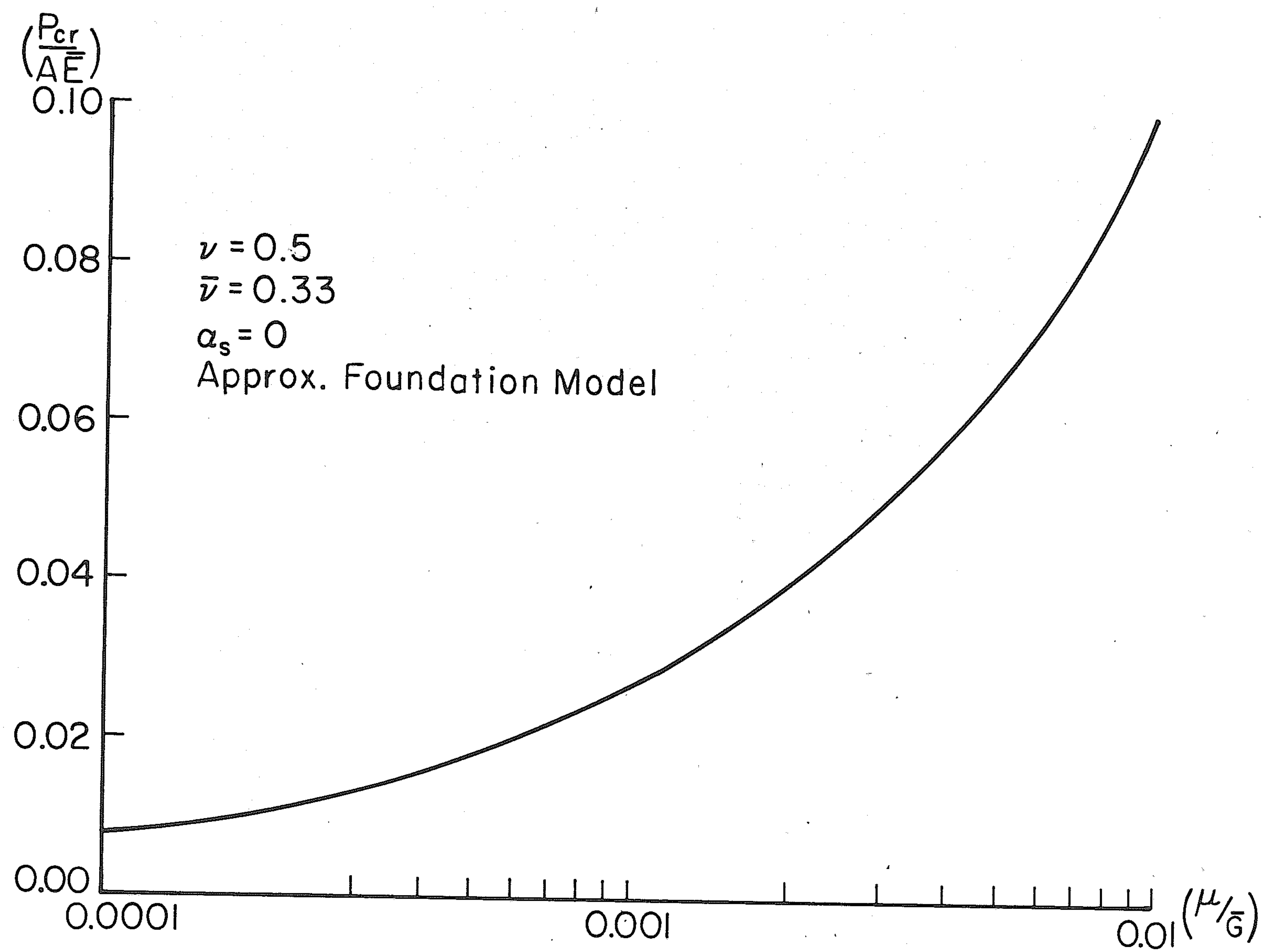

Figure 6 - Wire Buckling Load

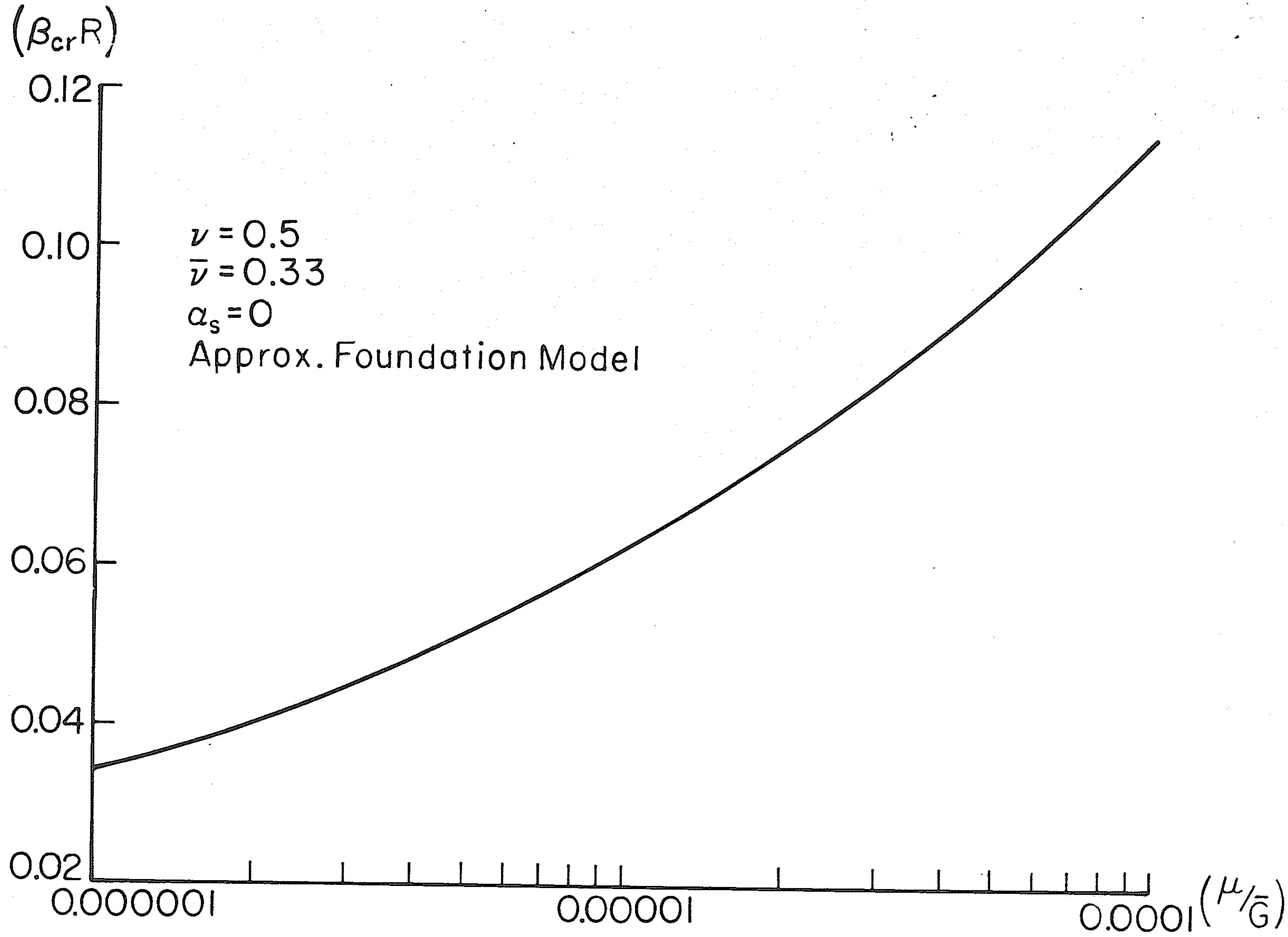

Figure 7 - Wave Number of Buckled Shape

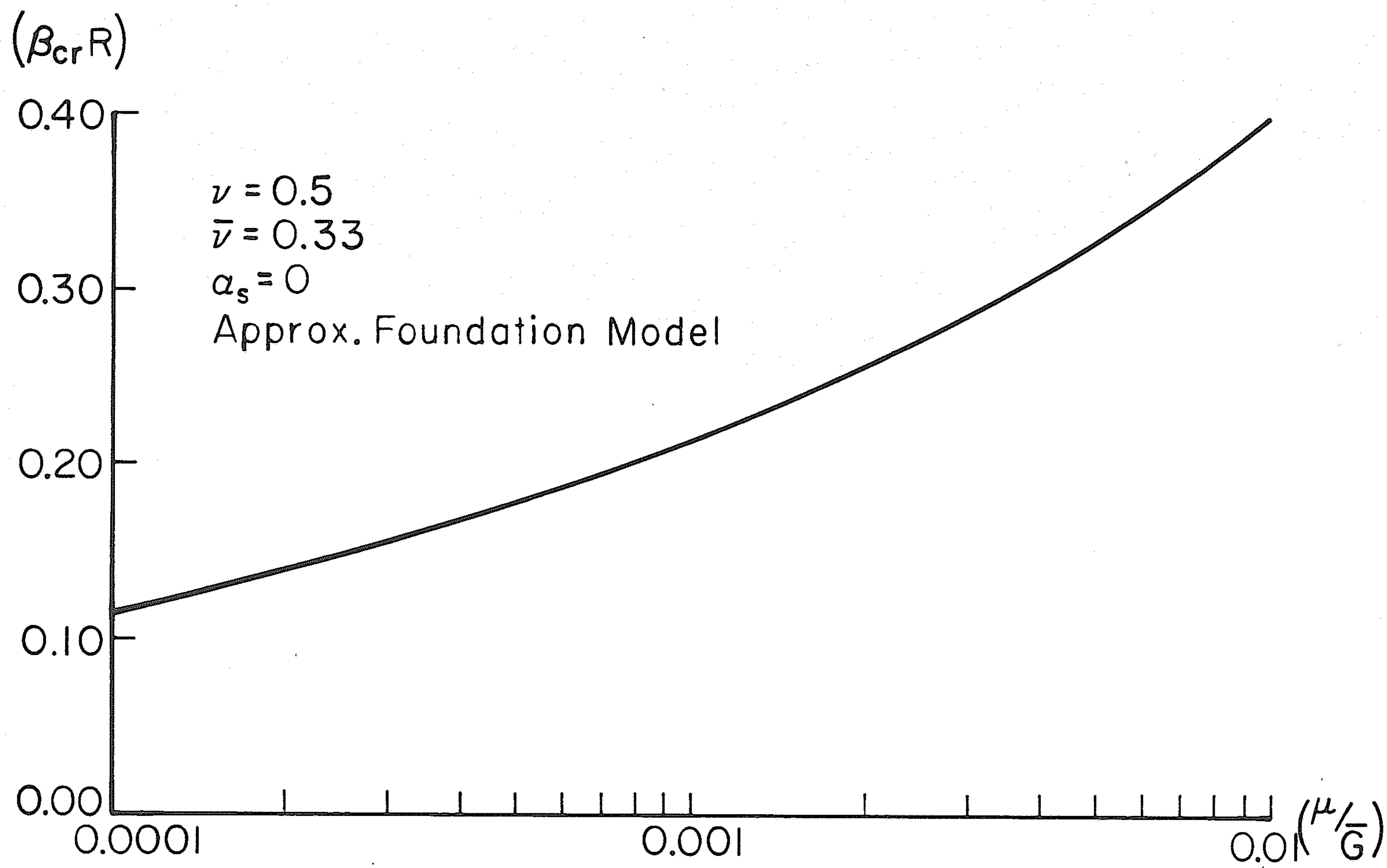

Figure 8 - Wave Number of Buckled Shape

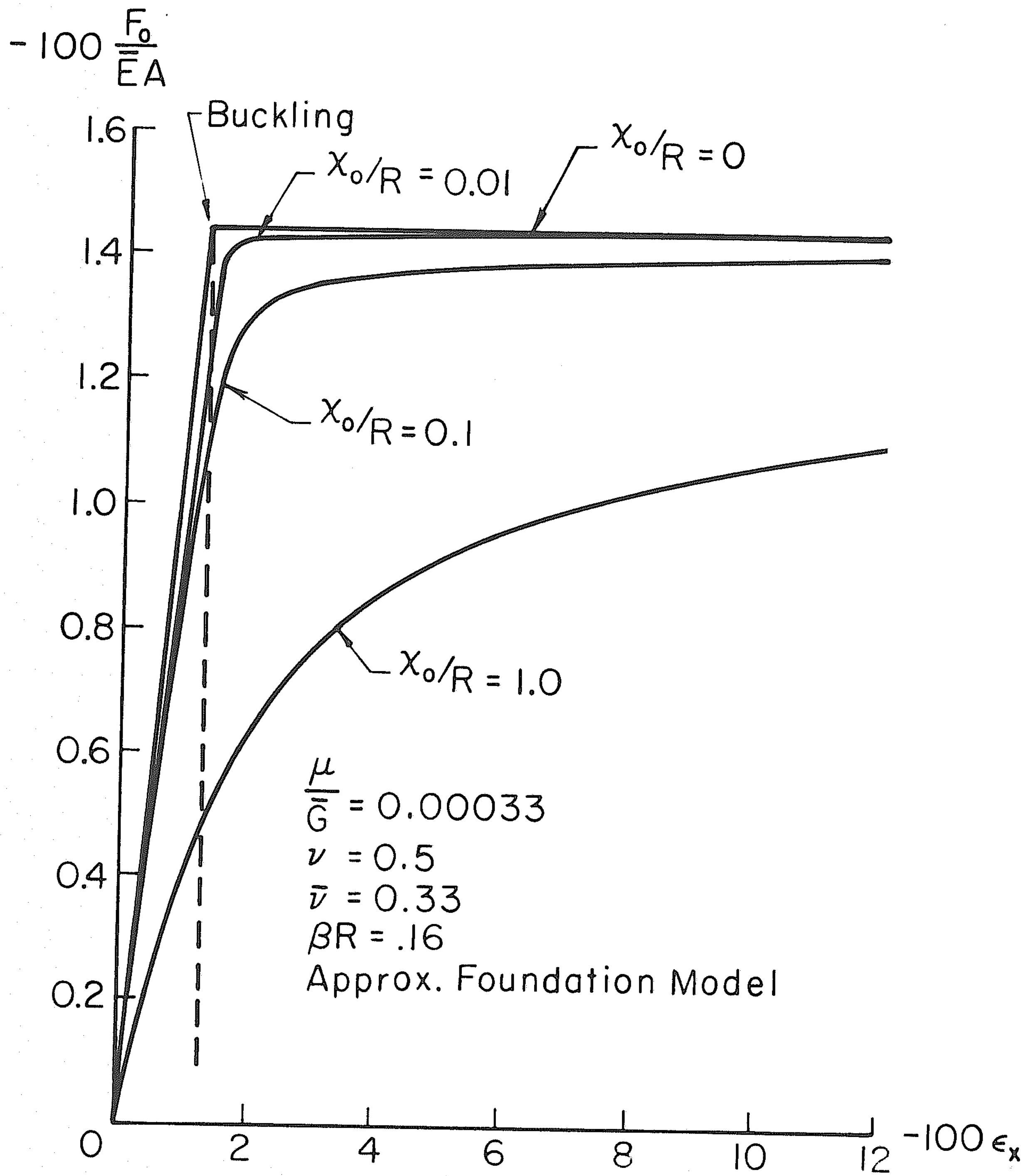

Figure 9 - Uniaxial Compressive Stress-Strain Curve of an Initially Crooked Wire as Predicted by the Small Deformation Beam-Column Analysis

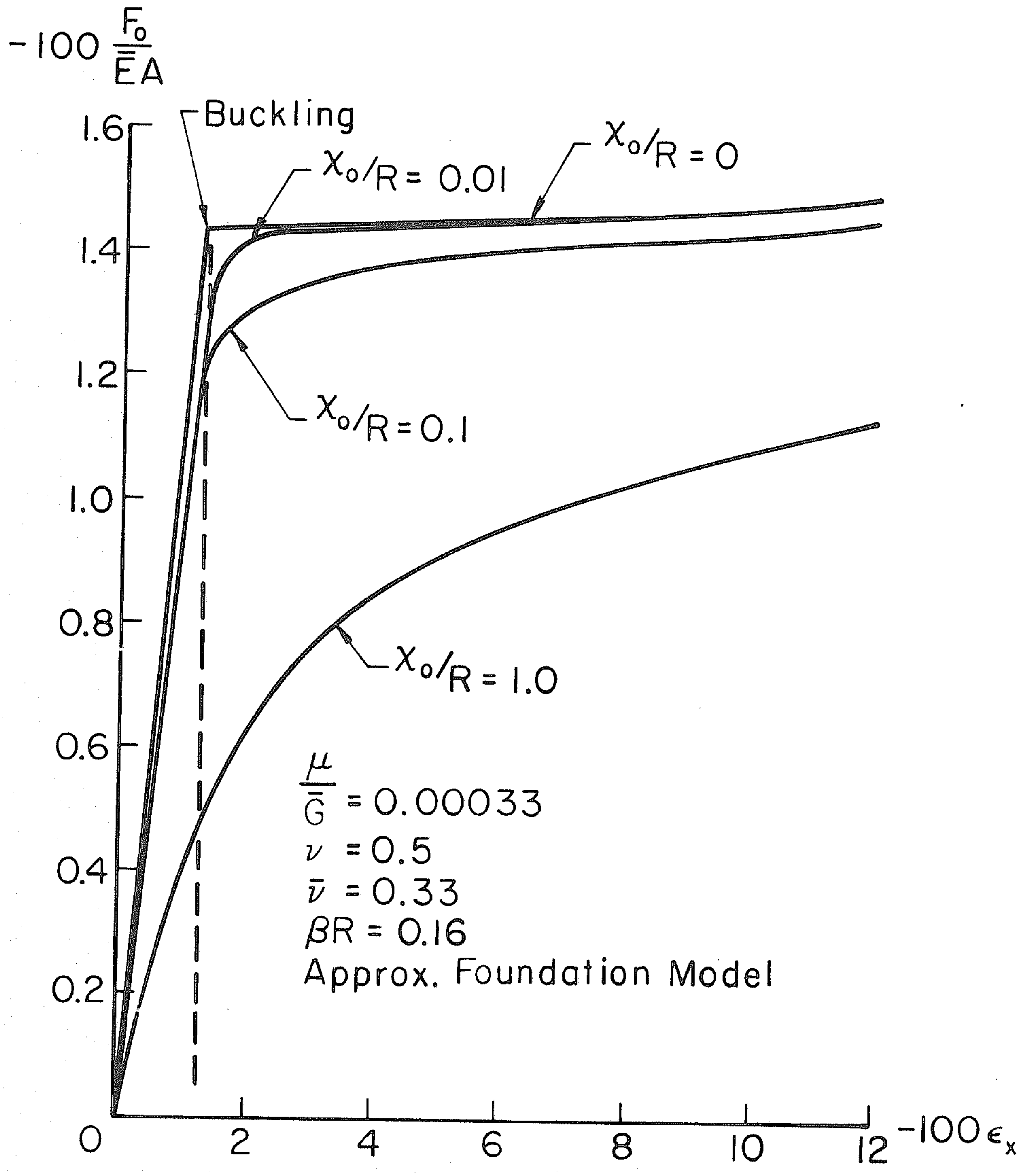

Figure 10 - Uniaxial Compressive Stress-Strain Curve of an Initially Crooked Wire as Predicted by the Medium Deformation Beam-Column Analysis

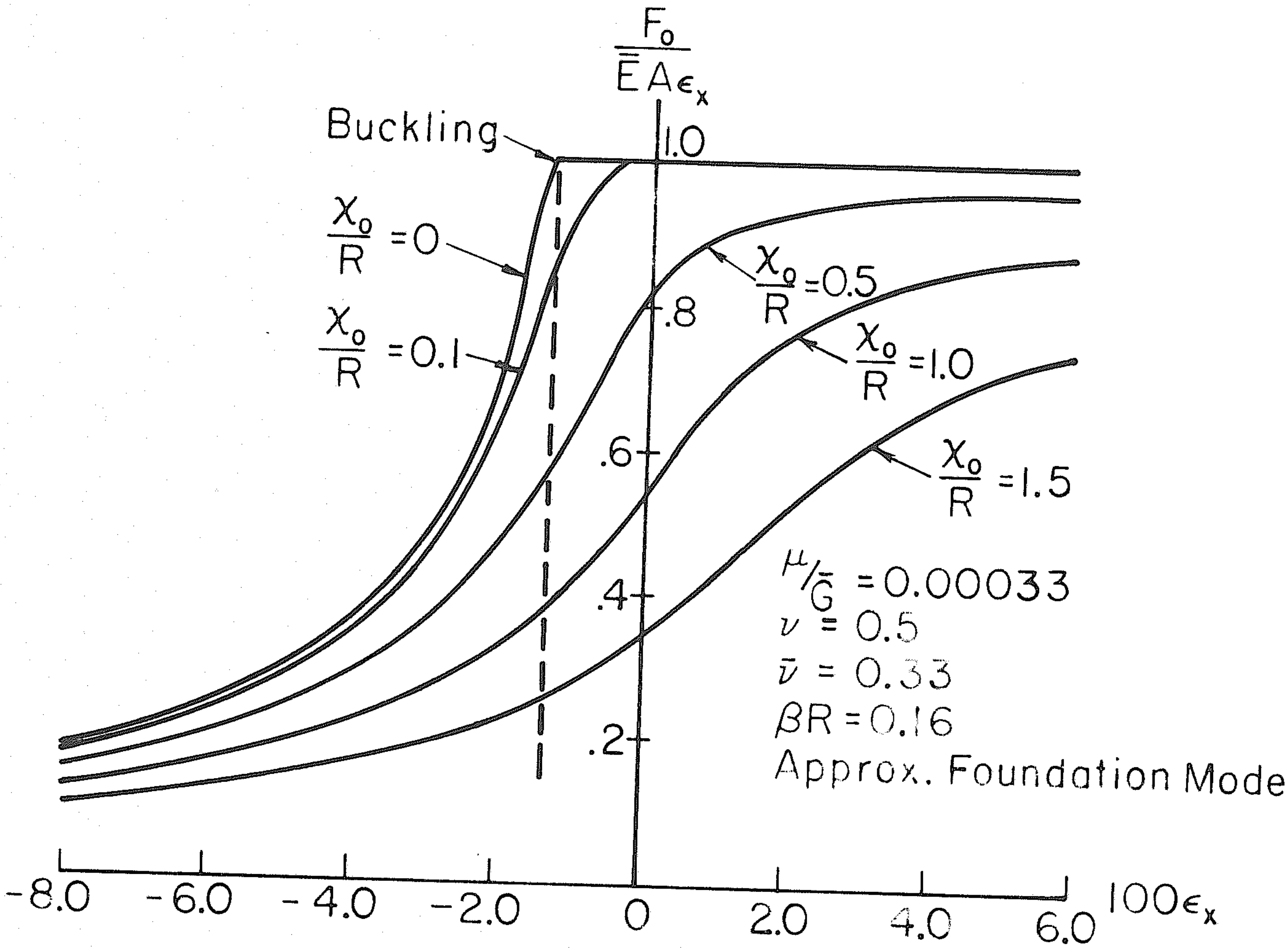

Figure 11 - Effective Stiffness of Reinforcing Wire as Predicted by the Medium Deformation Beam-Column Analysis

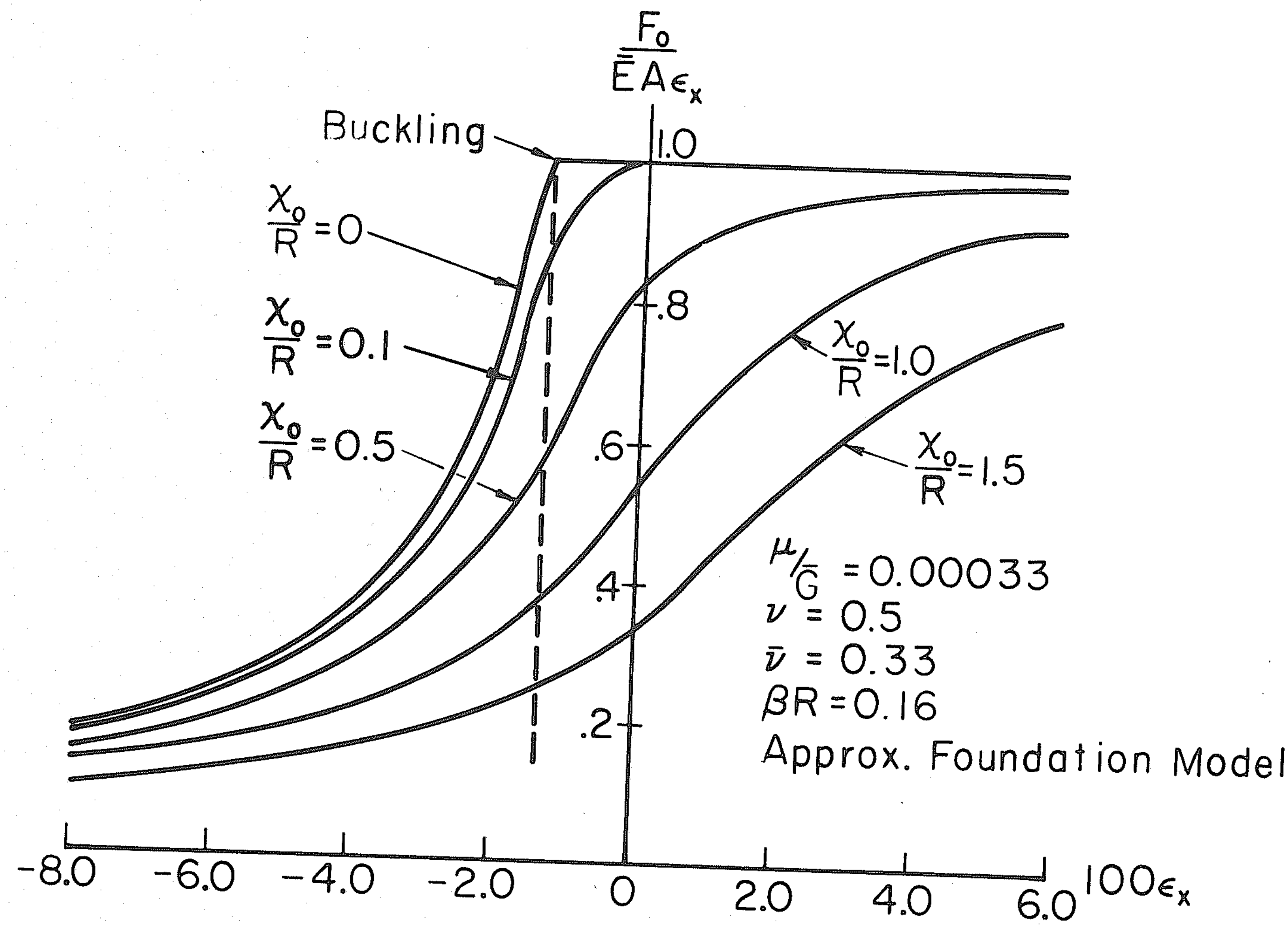

Figure 12 - Effective Stiffness of Reinforcing Wire as Predicted by the Small Deformation Beam-Column Analysis

imperfection is $\chi_0/R = 0.5$ and observations are made at ±1% uniaxial strain, from Figure 11 it is seen that the effective compressive modulus of the wire would be (note that buckling has not taken place):

$$E_c = 0.65\ \bar{E}$$

The effective tensile modulus would be:

$$E_t = 0.9\ \bar{E}$$

Hence, for this situation the effective compressive modulus at -1% strain would be only 72% of the effective tensile modulus at +1% strain.

CONCLUSIONS

Solutions were obtained for the problem of 1) the buckling of a wire embedded in an elastic matrix and 2) the beam-column behavior of an initially non-straight wire embedded in a matrix. The lateral support of the wire by the matrix was characterized by utilizing solutions to the three-dimensional equations of elasticity.

The solutions were evaluated numerically and plots of some of the results of interest were presented. The numerical results suggest that the observed differences in compressive and tensile behavior of certain wire reinforced materials may be explained by the beam-column behavior of initially crooked reinforcing wires.

APPENDIX - FOUNDATION MODELS

Classical foundation models [5,6,7] are relatively empirical in nature, are poor approximations of continuous foundations (see reference [8]) and depend on one or two empirical parameters that are difficult to relate to the measured properties of a continuous foundation. The alternative, that will be applied below, is to treat the foundation as a three-dimensional continuous body and consider equilibrium and displacement continuity between the matrix and the wire at every point along the wire.

Two foundation models for the representation of the support offered the wire by the surrounding matrix will be considered. The first (referred to as the "exact foundation model") satisfies a) all of the equations of elasticity for the matrix and b) all of the displacement and force continuity requirements between the matrix and the beam. The second model (referred to as the "approximate foundation model") will not completely satisfy condition b.

Exact foundation model:

The analysis of an infinite elastic medium subjected to loads applied at the surface of a cylindrical cavity (Figure 13) may be found in reference [9]. The applied loads are arbitrary in the ϕ direction and periodic in the x direction. The solution is expressed in terms of a double infinite series (it is to be noted that the solution given in reference [9] is not restricted to infinite media). For the problem under consideration, the x variation of the loading (see Equations (7) and (8)) involves only the first term of the x series and, in addition, in order to satisfy the conditions of continuity of displacements and forces between the beam and the matrix only one term of the ϕ series is needed; the solution for this problem is given below (the symbol μ denotes the shear modulus of the matrix):

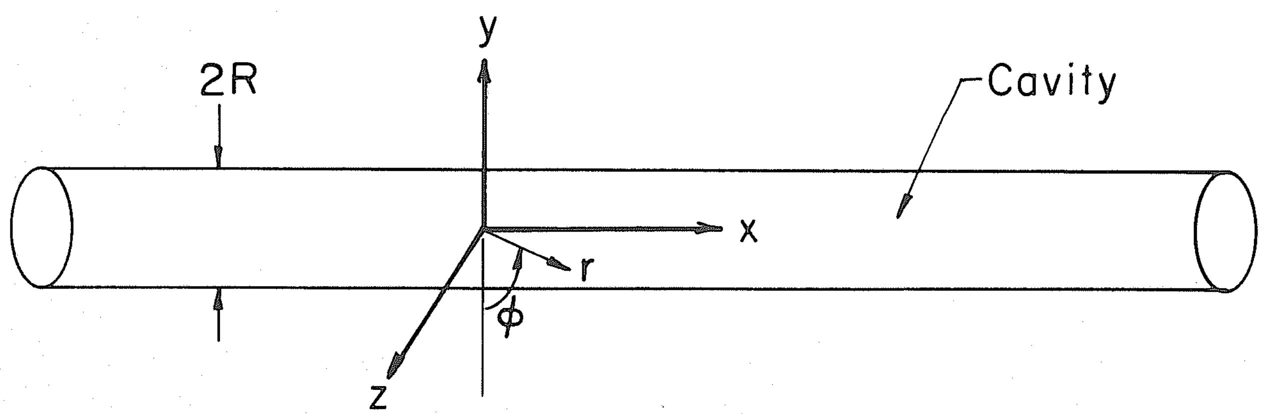

Figure 13 - Infinite Elastic Medium with Cylindrical Cavity

$$u_r = v_r(r) \text{ Cos } \phi \text{ Cos } \beta x \tag{47}$$

$$u_\phi = v_\phi(r) \text{ Sin } \phi \text{ Cos } \beta x \tag{48}$$

$$u_x = v_x(r) \text{ Cos } \phi \text{ Sin } \beta x \tag{49}$$

$$\tau_{rr} = 2\mu \; \sigma_r(r) \text{ Cos } \phi \text{ Cos } \beta x \tag{50}$$

$$\tau_{\phi\phi} = 2\mu \; \sigma_\phi(r) \text{ Cos } \phi \text{ Cos } \beta x \tag{51}$$

$$\tau_{xx} = 2\mu \; \sigma_x(r) \text{ Cos } \phi \text{ Cos } \beta x \tag{52}$$

$$\tau_{r\phi} = 2\mu \; \sigma_{r\phi}(r) \text{ Sin } \phi \text{ Cos } \beta x \tag{53}$$

$$\tau_{rx} = 2\mu \; \sigma_{rx}(r) \text{ Cos } \phi \text{ Sin } \beta x \tag{54}$$

$$\tau_{x\phi} = 2\mu \; \sigma_{x\phi}(r) \text{ Sin } \phi \text{ Sin } \beta x \tag{55}$$

Where

$$v_x(r) = \chi \left[C_1 \; K_1 + C_2 \; r \; K_2 \right] \tag{56}$$

$$v_r(r) = \chi \left\{ C_1 \; K_2 + C_2 \left[r \; K_1 + \frac{5 - 4\nu}{\beta} K_2 \right] + C_3 \frac{1}{r} K_1 \right\} \tag{57}$$

$$v_\phi(r) = \chi\left\{C_1\ K_2 + C_2\ \frac{5 - 4\nu}{\beta}\ K_2 + C_3\left[\beta\ K_2 - \frac{1}{r}\ K_1\right]\right\} \tag{58}$$

$$\sigma_r(r) = \chi\left\{-\ C_1\left[\frac{2}{r}\ K_2 + \beta\ K_1\right] + C_2\left[(2\nu - 3)\ K_1 - \left((5 - 4\nu)\ \frac{2}{\beta\ r}\right.\right.\right.$$

$$\left.\left.\left. + \beta\ r\right)K_2\right] - C_3\ \frac{\beta}{r}\ K_2\right\} \tag{59}$$

$$\sigma_\phi(r) = \chi\left\{C_1\ \frac{2}{r}\ K_2 + C_2\left[(1 - 2\nu)\ K_1 + (5 - 4\nu)\ \frac{2}{\beta\ r}\ K_2\right] + C_3\ \frac{\beta}{r}\ K_2\right\} \tag{60}$$

$$\sigma_x(r) = \chi\left\{C_1\ \beta\ K_1 + C_2\left[\beta\ r\ K_2 - 2\ \nu\ K_1\right]\right\} \tag{61}$$

$$\sigma_{r\phi}(r) = \chi\left\{-\ C_1\left[\frac{2}{r}\ K_2 + \frac{\beta}{2}\ K_1\right] - C_2\left[(5 - 4\nu)\ \frac{2}{\beta\ r}\ K_2 + (3 - 2\nu)\ K_1\right]\right.$$

$$\left. -\ C_3\left[\frac{\beta^2}{2}\ K_1 + \frac{\beta}{r}\ K_2\right]\right\} \tag{62}$$

$$\sigma_{rx}(r) = \chi\left\{C_1\left[\frac{1}{2\ r}\ K_1 - \beta\ K_2\right] - C_2\left[\beta\ r\ K_1 + (3 - 2\nu)\ K_2\right] - C_3\ \frac{\beta}{2\ r}\ K_1\right\} \tag{63}$$

$$\sigma_{x\phi}(r) = \chi\left\{-\ \frac{C_1}{2}\left[\beta\ K_2 + \frac{1}{r}\ K_1\right] - C_2\ (3 - 2\nu)\ K_2 - \frac{C_3}{2}\left[\beta^2\ K_2 - \frac{\beta}{r}\ K_1\right]\right\} \tag{64}$$

$$K_1 = K_1(\beta\ r) \tag{65}$$

$$K_2 = K_2(\beta\ r) \tag{66}$$

The modified Bessel functions of the second kind are denoted by $K_i(\xi)$. The arbitrary constants C_1, C_2 and C_3 are to be selected by applying the continuity and equilibrium conditions at the wire-matrix interface, i.e., at $r=R$. For the beam-column analysis it is assumed that the initial crookedness of the cavity (occupied by the wire) will have only a negligible effect on the stiffness of the matrix and, hence, in calculating the reaction of the matrix upon the wire it is assumed that the cavity is initially straight.

The reaction of the matrix on the wire normal to the wire's surface is:

$$q(x) = -\int_0^{2\pi} (\tau_{rr} \operatorname{Cos} \phi - \tau_{r\phi} \operatorname{Sin} \phi)\Big|_{r=R} R \, d\phi \tag{67}$$

Substituting Equations (50) and (53) into the above expression and integrating yields:

$$q(x) = 2 \pi R \mu [\sigma_{r\phi}(R) - \sigma_r(R)] \operatorname{Cos} \beta x \tag{68}$$

The reaction tangential to the axis of the wire is $\int_0^{2\pi} \tau_{rx} \Big|_{r=R} R \, d\phi$; upon substitution and integration this expression yields a value of zero. The reaction moment "m" (Figure 1) between the wire and the matrix is given by the expression:

$$m(x) = \int_0^{2\pi} \tau_{rx}\Big|_{r=R} R^2 \operatorname{Cos} \phi \, d\phi \tag{69}$$

Substituting Equation (54) into the above expression and integrating yields:

$$m(x) = 2 \pi R^2 \mu \sigma_{rx}(R) \text{ Sin } \beta x \tag{70}$$

The necessary relationships to insure the compatible deformation of the matrix and of the wire will now be established. The average y deflection of the cavity's surface (equivalent to the transverse deflection of the beam, i.e., to $Y^*(x)$) for a particular value of x is:

$$Y^*(x) = \frac{1}{2 \pi R} \int_0^{2\pi} (u_\phi \text{ Sin } \phi - u_r \text{ Cos } \phi)\Big|_{r=R} R \, d\phi \tag{71}$$

Substituting Equations (47) and (48) into the above expression and performing the indicated integration yields:

$$Y^*(x) = \frac{1}{2} [v_\phi(R) - v_r(R)] \text{ Cos } \beta x \tag{72}$$

Equating the above expression to the deflection of the beam (Equation (6)) and employing Equations (57) and (58) yields:

$$- R K_1(\beta R) C_2 + \left[\beta K_2(\beta R) - \frac{2}{R} K_1(\beta R)\right] C_3 = 2 \tag{73}$$

The average rotation α of the surface of the cavity about the z axis is given by the expression:

$$\alpha(x) = \frac{1}{2\ \pi\ R} \int_0^{2\pi} \frac{u_x\big|_{r=R}}{R\ \text{Cos}\ \phi} R\ d\phi \tag{74}$$

Substituting Equation (49) into the above expression and integrating with respect to ϕ yields:

$$\alpha = \frac{v_x(R)}{R} \text{Sin}\ \beta x \tag{75}$$

Equating the above expression to the rotation of the wire (i.e., for the wire $\alpha = \frac{dY^*}{dx} - \frac{\alpha_s\ Q}{A\ \bar{G}}$; the second term on the right is the contribution of the shear deformation) yields:

$$\frac{dY^*}{dx} - \frac{\alpha_s\ Q}{A\ \bar{G}} = \frac{v_x(R)}{R} \text{Sin}\ \beta x \tag{76}$$

Differentiating the above expression with respect to x and utilizing Equation (3) leads to the following equation:

$$\frac{d^2Y^*}{dx^2} + \frac{\alpha_s\ q}{A\ \bar{G}} = \frac{\beta\ v_x(R)}{R} \text{Cos}\ \beta x \tag{77}$$

Substituting Equations (6), (68), (62), (59) and (56) into the above expression yields:

$$\frac{K_1(\beta R)}{2\ R} (S_c - 2)\ C_1 + K_2(\beta R)\ (S_c - 1)\ C_2 - \frac{S_c\ \beta\ K_1(\beta R)}{2\ R} C_3 = \beta \tag{78}$$

Where

$$S_c = \frac{2\ \pi\ \alpha_s\ R^2\ \mu}{A\ \bar{G}} \tag{79}$$

The third displacement continuity condition that must be considered is the equating of the z displacement components (u_z) of the beam (due to bending) and of the matrix. The z displacement component of a point on the surface of the beam is shown in Figure 14 and is given by the following expression (the symbol $\bar{\nu}$ denotes the value of Poisson's ratio of the beam):

$$u_z = -\frac{\bar{\nu}\ M\ R^2}{\bar{E}\ I}\ \mathrm{Sin}\ \phi\ \mathrm{Cos}\ \phi \tag{80}$$

Utilizing Equation (43) to eliminate M from the above expression yields:

$$u_z = -\ \bar{\nu}\ R^2 \left(\frac{d^2Y^*}{dx^2} + \frac{\alpha_s}{A\ \bar{G}}\ q\right)\ \mathrm{Sin}\ \phi\ \mathrm{Cos}\ \phi \tag{81}$$

The z displacement of a point on the surface of the cavity is given by the following expression:

$$u_z = \left(u_\phi\ \mathrm{Cos}\ \phi + u_r\ \mathrm{Sin}\ \phi\right)\Big|_{r=R} \tag{82}$$

Substituting Equations (47) and (48) into the above expression yields:

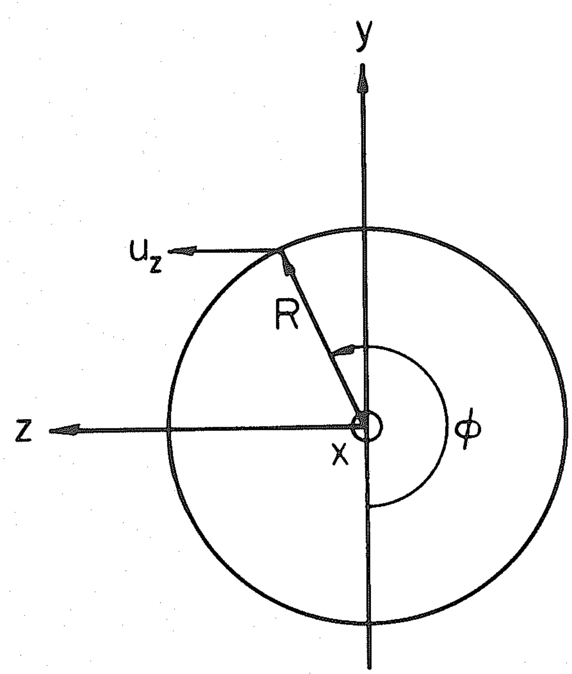

Figure 14 - Cross-Section of the Wire

$$u_z = [v_\phi(R) + v_r(R)] \text{ Cos } \beta x \quad \text{Sin } \phi \quad \text{Cos } \phi \tag{83}$$

Equating Equations (81) and (83) and utilizing Equations (6), (68), (57) and (58) yields:

$$\left[2 \ K_2(\beta R) + S_c^* \ K_1(\beta R)\right] C_1 + \left[R \ K_1(\beta R) + \frac{10 - 8\nu}{\beta} K_2(\beta R)\right.$$

$$\left. + \ 2 \ R \ S_c^* \ K_2(\beta R)\right] C_2 + \beta \left[K_2(\beta R) - S_c^* \ K_1(\beta R)\right] C_3 = \bar{\nu} \ R^2 \ \beta^2 \tag{84}$$

Where

$$S_c^* = \frac{\bar{\nu} \ \pi \ R^3 \ \alpha_s \ \beta \ \mu}{A \ \bar{G}} \tag{85}$$

Equations (73), (78) and (84) may now be solved simultaneously to yield the constants C_1, C_2 and C_3 as functions of the parameter β .

The foundation moduli k_1 and k_2 (see Equations (7) and (8)), may now be found for a given value of β . Comparing Equations (7) and (68) and employing Equations (59) and (62) yields:

$$k_1 = 2 \ \pi \ R \ \mu \left[\frac{\beta \ C_1}{2} K_1(\beta R) + C_2 \ \beta \ R \ K_2(\beta R) - \frac{C_3 \ \beta^2}{2} K_1(\beta R)\right] \tag{86}$$

For the small-deflection beam-column analysis and for the buckling analysis $\text{Cos } \theta \approx 1$, hence, combining Equations (8), (70) and (63) yields

$$k_2 = k_2^* \tag{87}$$

Where

$$k^* = 2\ \pi\ R^2\ \mu\ \left\{C_1 \left[\frac{1}{2\ R}\ K_1(\beta R) - \beta\ K_2(\beta R)\right]\right.$$

$$\left. - C_2 \left[\beta\ R\ K_1(\beta R) + (3 - 2\nu)\ K_2(\beta R)\right] - C_3\ \frac{\beta}{2\ R}\ K_1(\beta R)\right\} \tag{88}$$

For the medium-deflection beam-column analysis Equation (8) gives (utilizing Equations (70), (88) and (24)):

$$k_2 \text{ Sin } \beta x = k_2^* \text{ Sin } \beta x \sqrt{1 + (\chi + \chi_0)^2\ \beta^2\ (\text{Sin } \beta x)^2} \tag{89}$$

The above equation is approximated as follows:

$$k_2 \text{ Sin } \beta x = k_2^* \text{ Sin } \beta x \left[1 + \frac{(\chi + \chi_0)^2\ \beta^2\ (\text{Sin } \beta x)^2}{2}\right] \tag{90}$$

Expanding the right hand side of the above equation in a Sin $n\beta x$ series and neglecting all but the first term yields:

$$k_2 = k_2^* \left[1 + \frac{3\ (\chi + \chi_0)^2\ \beta^2}{8}\right] \tag{91}$$

Thus, for a given value of β, the "exact foundation" moduli k_1 and k_2^* (Equations (86) and (88)) may be calculated.

Approximate foundation model:

Intuitively it appears that the normal foundation reaction is of primary importance (i.e., q(x), Equation (7)). In the development of an approximate foundation model it will be assumed that $m(x) = 0$, i.e., $k_2 = 0$ (Equation (8)) and that the only displacement continuity requirement (between the wire and the matrix) of importance is the equating of the transverse displacement of the wire to the average y displacement of the matrix cavity. The response of the matrix will be approximated by considering an infinite medium subjected to a varying line load body force applied along the x axis; the resulting average displacement of the matrix at a distance R (radius of the wire) from the x axis will be equated to the transverse deflection of the beam. The expression for the y component of displacement (u_y) at a point on the cylindrical surface $r = R$ at the axial station x due to a unit concentrated force applied at ξ was developed by Kelvin [10], see Figure 15; the expression can be written in the form:

$$u_y\,(x, \xi, \phi) = \frac{1}{16\,\pi\,\mu\,(1 - \nu)\,\sqrt{(x - \xi)^2 + R^2}} \left[(3 - 4\nu) + \frac{R^2\,\mathrm{Cos}^2\,\phi}{(x - \xi)^2 + R^2}\right] \tag{92}$$

The average y component of displacement of the surface $r = R$ at position x is given by the expression:

$$\bar{u}_y(x, \xi) = \frac{1}{2\,\pi\,R} \int_0^{2\pi} u_y(x, \xi, \phi)\; R\; d\phi \tag{93}$$

Or

$$\bar{u}_y(x, \xi) = \frac{1}{16\,\pi\,\mu\,(1 - \nu)\,\sqrt{R^2 + (x - \xi)^2}} \left\{(3 - 4\nu) + \frac{R^2}{2\,[R^2 + (x - \xi)^2]}\right\} \tag{94}$$

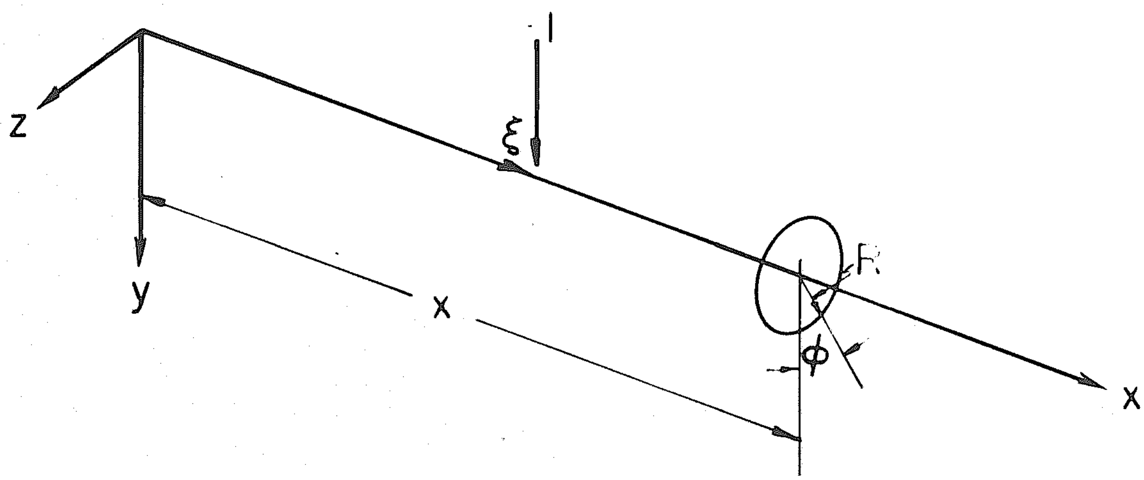

Figure 15 - Sketch of Unit Concentrated Load at Point ξ in an Infinite Medium

The actual applied load (the wire reaction, Equation (7)) is distributed along the x axis according to the equation:

$$q(\xi) = K_1 \chi \operatorname{Cos} \beta\xi \tag{95}$$

Hence the total displacement at x is given by the equation:

$$Y^*(x) = \int_{-\infty}^{\infty} q(\xi)\, \bar{u}_y(x, \xi)\, d\xi \tag{96}$$

Or (substituting Equations (94) and (95), and integrating)

$$Y^*(x) = \frac{\chi k_1 \operatorname{Cos} \beta x}{16 \pi \mu (1 - \nu)} \left\{2 (3 - 4\nu) K_0(\beta R) + R \beta K_1(\beta R)\right\} \tag{97}$$

Equating the above expression to Equation (6) yields:

$$k_1 = \frac{16 \pi \mu (1 - \nu)}{[2 (3 - 4\nu) K_0(\beta R) + R \beta K_1(\beta R)]} \tag{98}$$

Equation (98) coupled with the assumption that $k_2 = 0$ yields the "approximate foundation" moduli.

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