MECHANICS AND DYNAMICS OF ICE FLOW

By

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Abstract

The Arctic and Antarctic regions of the globe are experiencing radical changes to their physical structure due to climate change. Naturally, ice flows from land to sea, however the increasing temperatures on Earth are accelerating this process. In this research project, we will understand the mechanics and dynamics of the flow of ice sheets and their role leading up to the calving processes. Variables such as velocity, time, and topography encountered during flow are inputted into a mechanically grounded analytical model. The analytical model was then merged into a computational model created in MATLAB and provides a visual tool to understand relationships between variables. Scientists can then build more advanced climate models to better understand the alteration of Earth's cryosphere due to global warming. Next steps mentioned in the research include developing a physical scale-model to test and verify the models. This is because there is no simple way to directly test the quantitative predictions of ice models in the Antarctic because of the slow velocities of ice flow, the uncertainties in the ice properties, and the overall complex topography.

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INTRODUCTION:

Ice flow is a natural phenomenon that occurs in the Antarctic region. Creating analytical models centered around ice sheet flow is a particularly helpful method in being able to understand the mechanics and dynamics of ice flow over Antarctic topography, before the mass enters the sea. An analytical model was derived in order to relate a pressure gradient to the velocity of ice flow in the Antarctic region, wherein the velocity is a consequence of the conservation of momentum of a thin layer of ice relative to a thick layer of land. Velocity, *U*, is a function of the height of the ice (H_i (x,t)), height of the land (H_l (x,t)), change in the height of ice, change in the height of land, constant in the Glen-Nye Flow Law (n), and a temperature-dependent flow constant (K). By applying a force balance in the vertical and horizontal directions, the velocity of the ice is predicted to be:

$$U(x,Y,t) = -\left(-\rho g \frac{\partial H}{\partial x}\right)^n \frac{KH_i(x)^{n+1}}{n+1}((Y-1)^{n+1}-1) \tag{1}$$

In the above formula, *Y* is simply a numerical input between 0 and 1, that characterizes the vertical position in the flowing ice current. The formula is useful and helps form an underlying understanding of a velocity profile for the ice sheet. Although this model can create a velocity output based on any topographical and environmental input, it is particularly rigid in the sense that it requires the user of the model to know the ice thickness data ahead of time and also is characterized by a constant ice thickness, H_i (x), which is not ideal, because the height of ice changes with respect to position and time.

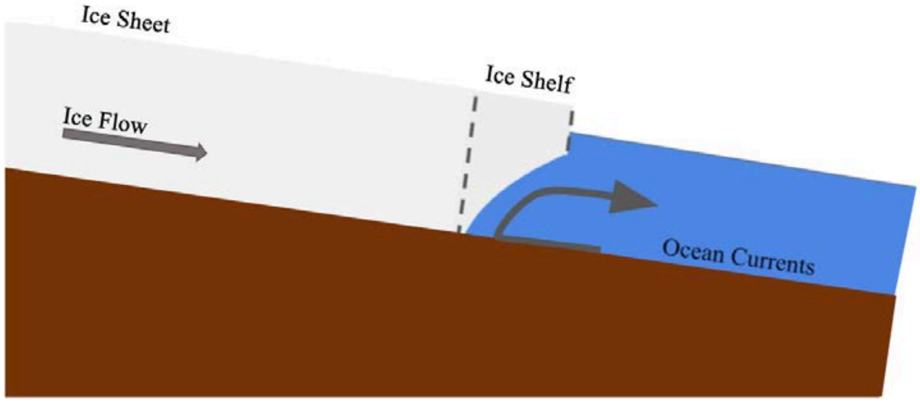

Figure 1: Ice sheet flow with constant ice thickness, H_i [1]

Overall, developing an analytical model that instead uses known quantities such as land topography and mass flux conditions, will allow one to better model ice sheet flow behavior. Figure 2 below shows a realistic visual representation of how ice should flow, assuming no snow or ice is added to the system.

Figure 2: Ice sheet flow with changing ice thickness over a sloped landform

With a concrete understanding of why an analytical model should include changing ice thickness, such a model can be derived using topographical, environmental, velocity, time, and distance input dependencies.

CALCULATIONS:

The slope of the land directly impacts the flow of the ice in either a downwards or upwards direction. A positive slope means the ice will flow backwards and a negative slope means that the ice will flow downwards. Therefore, attention to sign convention in this derivation is imperative in order preserve the meaning of the ice flow model. In order to prevent sign errors, absolute values can be introduced to the equation, where the sign (+ or -) can be added at the end of the derivation based on the slope of the land.

$$U = \operatorname{sgn}\left(\frac{\partial H}{\partial x}\right) \left|\rho g \frac{\partial H}{\partial x}\right|^{n} \frac{KH_{i}(x)^{n+1}}{n+1} \left(|Y-1|^{n+1}-1\right) \tag{2}$$

Next, we can now consider how velocity plays a role in reaching an equation that measures the change in ice thickness. In general, an analysis of a control volume of ice creates a relationship wherein the net mass of ice into the control volume must equal the rate of change of mass in the control volume. In order to form this relationship, the average velocity must be used to calculate the mass flux. As mentioned before, (2) is the velocity at a specific height. This height is based on a normalized scale using a global set of axes. In order to reach this point, the relationship:

$$q = \rho \int_{H_l(x)}^{H_i(x)+h_l(x)} U(y)dy \tag{3}$$

can be used in order to relate mass flux, q, to the velocity. However, one will see that there is an issue where the velocity used to calculate mass flux is dependent on a y-value, or numerical height value, however, (2) uses a normalized height value. Therefore, a scaling adjustment must be made in order to integrate (3). Differentiating the relation $Y = \frac{y-H_l}{H_i}$ yields d$y =$ $H_i(x)\ dY$ which, when substituted into (3) shows that:

$$q = \rho \int_0^1 U H_i(x) dY \tag{4}$$

Plugging in the original limits of integration with respect to Y, yields limits of integration from 0 to 1. Now, substituting (2) into (4) is the following:

$$q = \rho \int_0^1 \operatorname{sgn}\left(\frac{\partial H}{\partial x}\right) \left|\rho g \frac{\partial H}{\partial x}\right|^n \frac{K H_i(x)^{n+1}}{n+1} (|Y-1|^{n+1} - 1) H_i(x) dY \tag{5}$$

Integrating shows that the mass flux reduces to:

$$q = -\operatorname{sgn}\left(\frac{\partial H}{\partial x}\right) \rho^{n+1} g^n K \left|\frac{\partial H}{\partial x}\right|^n \frac{H_i(x)^{n+2}}{n+2} \tag{6}$$

For convenience we define a new constant $\alpha_n = \frac{\rho^n g^n K}{n+2}$ that combines the relavant physical parameters, so the mass flux equation is simply:

$$q = -\operatorname{sgn}\left(\frac{\partial H}{\partial x}\right) \rho \alpha_n \left|\frac{\partial H}{\partial x}\right|^n H_i(x)^{n+2} \tag{7}$$

The rate of change of mass in the control volume is the key component of introducing time as a parameter into the model. Conservation of the mass of ice requires that the change in ice thickness with respect to time is proportional to the change in mass flux with respect to distance, or:

$$\frac{\partial q}{\partial x} + \rho \frac{\partial H_i}{\partial t} = 0 \tag{8}$$

Substituting (7) into (8) yields:

$$\frac{\partial}{\partial x}\left[\rho \alpha_n \left|\frac{\partial H}{\partial x}\right|^n H_i(x)^{n+2}\left\{-sgn\left(\frac{\partial H}{\partial x}\right)\right\}\right] + \rho \frac{\partial H_i}{\partial t} = 0 \tag{9}$$

which is a partial differential equation that governs the change in ice thickness with respect to time, *t*, and position, *x*. It depends on the properties of the ice, characterized by α_n, and the topography of the land.

RESULTS AND DISCUSSION:

The expression (9) is a complicated equation and cannot be solved fully by hand calculations.[4] In order to understand (9) and its meaning, scaling arguments can be utilized to simplify it to basic relationships between variables. To do this, the first step is to simplify (9).

$$\frac{\partial H_i}{\partial t} = \frac{\partial}{\partial x}\left[\alpha_n {H_i}^{n+2}\left|\frac{\partial H}{\partial x}\right|^n\right] \tag{10}$$

Then, using a scaling argument, removes the delta from all variables. The equation then reduces down to:

$$\frac{H_i}{t} \sim \frac{1}{x}\left[\alpha_n {H_i}^{n+2}\left|\frac{\partial H}{\partial x}\right|^n\right] \tag{11}$$

However, before the delta is removed from $\left|\frac{\partial H}{\partial x}\right|^n$, another scaling argument can be used. $\left|\frac{\partial H}{\partial x}\right|^n$ is really $\left|\frac{\partial H_i}{\partial x} + \frac{\partial H_l}{\partial x}\right|^n$. Using length scale arguments, at a specific length of land, either the slope of land is greater than the change in ice thickness, or the change in ice thickness is greater than the slope of the land. Figure 3 and 4 describes this concept visually.

Case 1: $\frac{\partial H_l}{\partial x} >> \frac{\partial H_i}{\partial x}$

Figure 3 – Comparison of slope of land versus change in ice thickness

In Figure 3 we see that the height of the ice, changes relatively little compared to the slope of the land. This in turn provides the ability to say that the change in the height of the ice is negligible, thus creating the first case of parameters.

Figure 4 – Comparison of change in ice thickness versus slope of the land

In Figure 4 we notice that as the ice flows down the slope, its thickness decreases with distance and time at a certain rate that is greater than the slope of the land. Many factors, such as temperature of the ice during certain seasons, can affect how quickly the height of the ice changes over distance in time. [2] Ice that is warmer during the summer months when compared to temperatures of winter months, can cause greater changes in the height of the ice. This is out of scope for this report, but the overall notion is that the length scale argument essentially provides the ability to neglect either the $\frac{\partial H_i}{\partial x}$ or $\frac{\partial H_l}{\partial x}$ terms in the $\left|\frac{\partial H}{\partial x}\right|^n$ term. In addition, the slope of the land is readily available information that can be found by measurements already made by scientists, or new data found through various instruments. This is also out of scope of this current research problem, but based on the assumption that $\frac{\partial H_l}{\partial x}$ is a readily available constant, one can replace the term by constant *S.* Therefore, we can now move forward in developing further simplification of (11) based on the two cases shown in Figure 4. The first case, where $\frac{\partial H_l}{\partial x} >> \frac{\partial H_i}{\partial x}$, is shown as:

$$\frac{H_i}{t} \sim \frac{1}{x}[\alpha_n H_i^{n+2}|S|^n] \tag{12}$$

Let $x = l$, where l is the length that the ice has flowed over time t. Therefore (12) is now:

$$\frac{H_i}{t} \sim \frac{1}{l}[\alpha_n H_i^{n+2}|S|^n] \tag{13}$$

The final goal is to develop two equations, H_i as a function for t and l as a function of t. Let us first look at $H_i(t)$. In order to reparametrize (13), a secondary equation is needed, which spawns from the considering the beginning state of the ice as a rectangular prism. The volume of the prism is: $V = H_i l w$. Rearranging the volume yields: $l = \frac{V}{H_i w}$. Substituting l into (13) reveals:

$$H_i \sim \frac{V^{1/(n+2)}}{w^{1/(n+2)} t^{1/(n+2)} \alpha_n{}^{1/(n+2)} |S|^{n/(n+2)}} \tag{14}$$

The Glen-Nye Flow Law constant, n, ranges from 1 to 4. A value of n = 2 represents ice at low stresses, which a value of n = 4 represents ice at high stresses. Of course, the terms "low" and "high" are relative, but the argument is that as the n-value increases so does the ice stress through a power relationship. For the sake of this model, and average of n = 3 will be used as a relative average between low and high stress points of ice.[3] Substituting n = 3 into (14) shows the relationship that:

$$H_i \sim \frac{V^{1/5}}{w^{1/5} t^{1/5} \alpha_3{}^{1/5} |S|^{3/5}} \tag{15}$$

We can say that $H_i \sim A\frac{1}{t^{1/5}}$, or is a power law function with a negative exponent. The meaning of a power law function is that the relationship of $H_i(t)$ reveals that after a short amount of time, the value of the height of the ice decays as time increases. A good way to imagine this deduction is through the usage of flood gates. Flood gates contain a certain volume of fluid, but if the gates were to open the water would flow and height of the water would change most during the first few seconds. But after a short snippet of time, the water will continue flowing, but its height will change little from one point in time to the next.

An analysis for *l(t)* can now be conducted, however, volume will be rearranged so that: $H_i = \frac{V}{lw}$. Substituting this for H_i in (13), as well as substituting n = 3 equates to:

$$l \sim {\alpha_3}^{1/5} t^{1/5} \left[\frac{1}{w^4}\right]^{1/5} [V]^{4/5} |S|^{3/5} \tag{16}$$

Essentially, $l \sim A\, t^{\frac{1}{5}}$ is a power law function. Tracking a control volume in an experimental setting would reveal that as time increases to infinity, the length traveled by the ice has a certain rate of decay. In other words, the length scale of the control volume of the ice sheet can be considered to decay after a certain point in time.

Now, the second case, where $\frac{\partial H_i}{\partial x} \gg \frac{\partial H_l}{\partial x}$, can be evaluated using the same method used for the first case where $\frac{\partial H_l}{\partial x} \gg \frac{\partial H_i}{\partial x}$. Because the algebraic method was shown for the first case, we will go straight to the solutions for the second case. Applying the following parameters: *x = l*, *n = 3* and $l = \frac{V}{H_i\, w}$ to (11) yields:

$$H_i \sim \frac{1}{{\alpha_3}^{1/9}} \frac{1}{t^{1/9}} \left[\frac{V}{w}\right]^{4/9} \tag{17}$$

Reducing (17) shows that $H_i \sim B\, \frac{1}{t^{1/9}}$. The meaning of this scaled estimate contains the same argument as the first case's result for $H_i(t)$.

Now, let us solve for *l(t)*. Just as we did in the first case, substitute: $H_i = \frac{V}{lw}$, *x = l, and n = 3* to (11) yields:

$$l \sim {\alpha_3}^{1/9} t^{1/9} \left[\frac{V}{w}\right]^{5/9} \tag{18}$$

Therefore, $l \sim Bt^{\frac{1}{9}}$.

However, an interesting aspect to understand about both cases is that if, hypothetically, the analysis begins with the assumption of the second case, where $\frac{\partial H_i}{\partial x} >> \frac{\partial H_l}{\partial x}$, over a certain amount of time, the cases will switch. In greater detail, as the ice flows over a sloped land, H_i will decrease and *l* will increase, and at a certain time *t* the case will switch to where $\frac{\partial H_l}{\partial x} >> \frac{\partial H_i}{\partial x}$. The time at which this occurs is when the solutions for $H_i(t)$ and *l(t)* for both cases intersect. Numerically, for $H_i(t)$ this occurs at: $\frac{A}{B} = t^{-4/45}$ and for *l(t)* this occurs at: $\frac{A}{B} = t^{4/45}$. These two solutions make sense because $H_i(t)$ is an inverse power law function of *l(t)*. Plotting both power law functions on the same graph reveals this conclusion graphically. Overall, it is important to understand that as time proceeds, one case does not hold for an infinite amount of time. In fact, at a certain point in time, the cases switch.

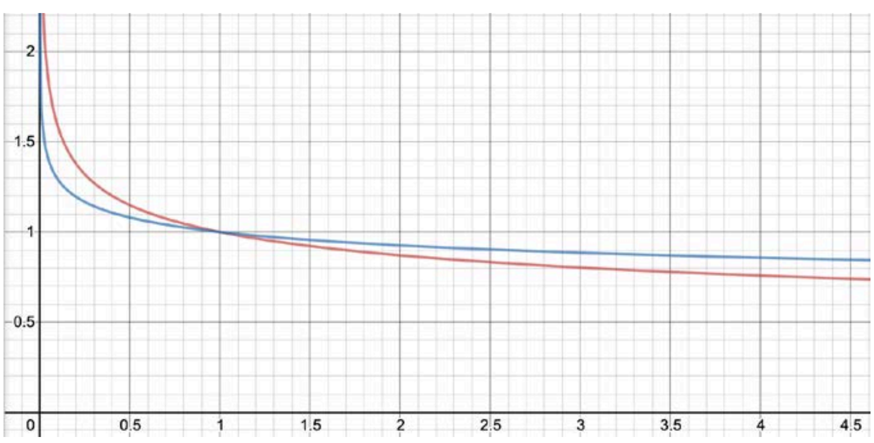

Figure 5 – Comparison of $H_i \sim \frac{1}{t^{1/5}}$ (red) and $H_i \sim \frac{1}{t^{1/9}}$ (blue) where *t* is on the x-axis and H_i is on the y-axis

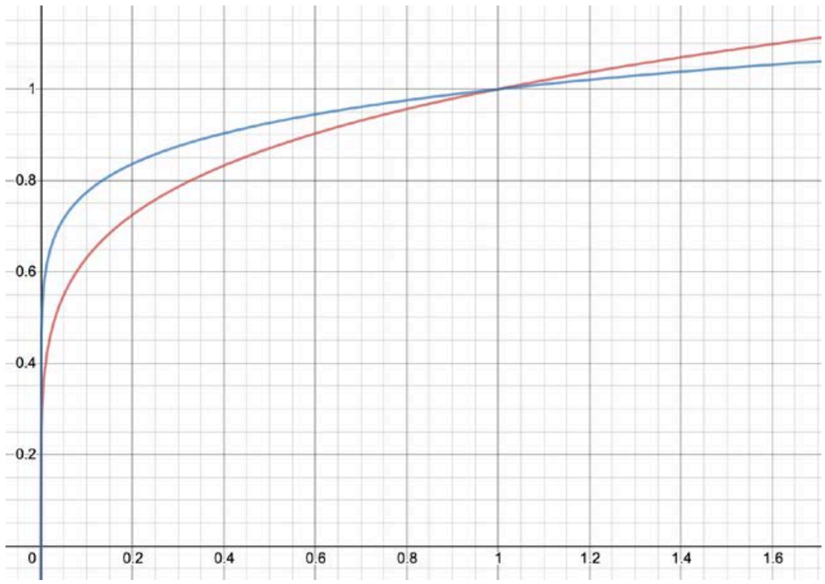

Figure 6 – Comparison of $l \sim t^{1/5}$ (red) and $l \sim t^{1/9}$ (blue) where *t* is on the x-axis and *l* is on the y-axis

The expression (9) provided a possible way to compute the change of the ice thickness with respect to length and time. However, the complexity of the equation is merely impossible to solve by hand, a computational model is needed in order to calculate the mathematical expression. Scaling provided a method in being able to simplify (9) and move into an area where a scaled expression could show certain results and create discussions about the meaning of (9). It is important to note that a scaled expression is by no way a direct solution for (9). However, scaling in general reveals an approximate error within a factor of one or two. Though this may seem like a high error, it is relative, meaning that it is an adequate conclusion on the basis that no computational model or extensive hand calculations were completed. In addition, scaling provided the capability in reaching multiple avenues of discovery from two equations. By relating (11) to the volume of the control volume, we were able to reach four conclusions for $H_i(t)$ and *l(t).* Noting that the two cases eventually change after a certain point in time helped understand the influence of volume, topography, distance, and time upon the thickness of the ice.

As stated before, the scaling law is accurate to a factor or two. Now that the scaling analysis reveals a reasonable understanding of (9), the computational model can be developed in MATLAB rather than solving the complicated numerical expression by hand. One may ask why MATLAB was not used in the first place, which is a valid question. The answer is that numerical expressions in the engineering field are best calculated using a scaling analysis. This helps provide an intermediate step to state whether such outputs are even remotely valid. If not, then it can be assumed that there is something wrong with the preliminary assumptions and/or math computations.

The expression (9) is a partial differential expression. The structure of the overall expression is that of a heat equation. The exact name of solver in MATLAB is “pdepe.” It is notable that the shape for Figure 5 and Figure 6 relate closely to the shape of Figure 7 and Figure 8, respectively. The MATLAB script developed is useful to now create an input-output scenario that provides precise numerical analysis of (9) rather than scaling law outputs which again, are useful but not numerically precise. Additionally, what is useful about the MATLAB script is that it allows the user to input a specified equation of the height of the ice at the beginning of time, along with specified Arrhenius constants for the properties of ice depending on the temperature. Ultimately, such customizations in a computational model provide a relative surface plot, which is where Figure 7 and Figure 8 were taken from. The surface plot is in fact a three-dimensional plot where the user can rotate the axes and view visual relationships of two-dimensional variables. Further steps to help draw more conclusive points would be to output \ 2D plots relative to Figure 5 and Figure 6, rather than using the rotation tool for the 3D plot. Then, by extracting the appropriate data from the 2D plot, error approximations can be compiled to compare scaling law results with MATLAB results. The conclusions drawn from Figures 5-8 are done merely through visual comparisons of the graphs. This is still useful to check whether the abstract and computational modeling is accurate, however the next goal would be for MATLAB to confirm preciseness of the abstract and computational models. Further steps for the MATLAB model are considered in the conclusion section.

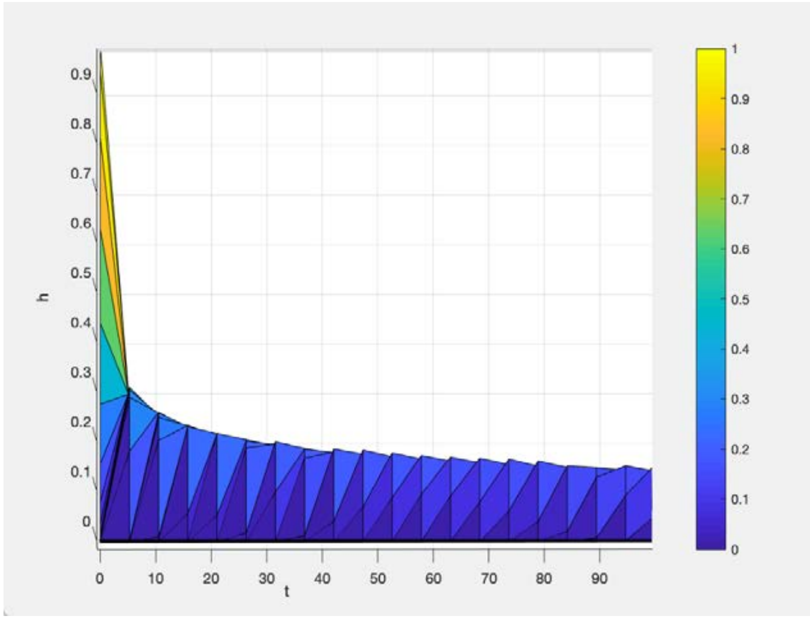

Figure 7 – Comparison of H_i and t where *t* is on the x-axis and H_i is on the y-axis

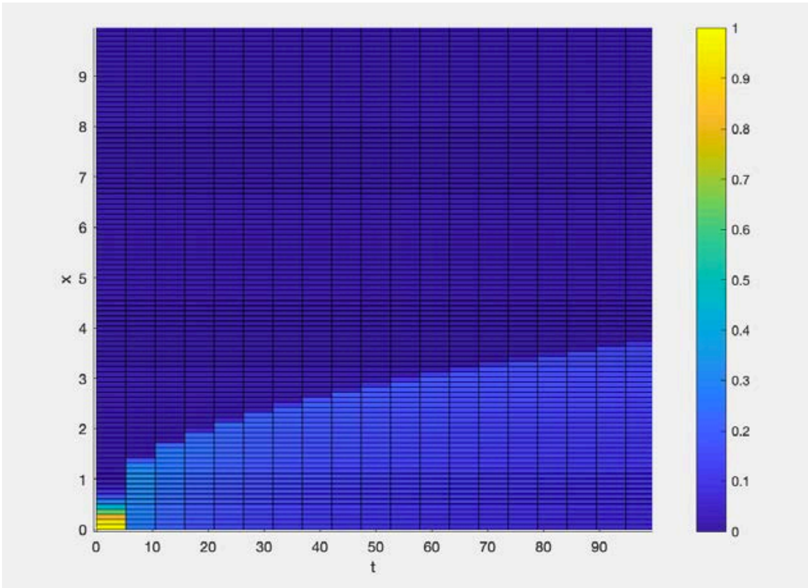

Figure 8 – Comparison of l and t where *t* is on the x-axis and l is on the y-axis

CONCLUSION:

The work completed in Spring 2019, Fall 2019, Winter 2019-2020, and Spring 2020 involved analytical models that revolved around developing equations to describe the velocity and more realistic adaptations of the height of the ice. With a foundation in these two areas, it is worth proceeding in two different ways: 1. Further develop the MATLAB script to solve for other environmental and mechanical/dynamics factors and to contain more features so that the plots and associated equations can be analyzed more closely, and 2. Create experimental models to fix and validate the analytical models with measured data from the tests.

With regards to 1., the control volume analysis, must be adapted in order to account for snow fall, ice formation, snowmelt, and ice melt, which is a very realistic proposition. A new parameter of snowfall and ice development can be added to the volume term. The volume used is the initial volume of the ice at time $t = 0$. In a realistic scenario, the volume changes, especially during different seasons. In the summertime, warmer temperatures relative to Antarctic weather can create ice and snow melt, which can result in water drainage into the ice sheet from slush formation. In the wintertime, snowfall and freezing of water can supply additional volume to an ice sheet. Both of these seasonal principles require a more comprehensive analysis in understanding the process of how volume is added or taken away.

In relation to 2., the reason experimental models must be used is due to limitations of measurements in the field. The analytical model's outputted data would take large amounts of time to collect in the field and cannot be readily measured with instruments in Antarctica. Therefore, creating a lab-scale tool is an appropriate approach in testing the scaling laws developed in this research report. Being able to connect analytical models with field measurements or lab-scale testing is an important aspect in research. The main materials that would need to be used for creating an experimental model are: modeling clay to represent the topography of Antarctica, silicone fluid as the ice sheets, and a high definition camera to track the fluid flow down the topography. In this research report, a value of $n = 3$ was assumed. Silicone oil is versatile in the sense that various viscosities exist on the market. Creating a variable viscosity during tests will help broaden the understanding of n in the results of the scaling laws, and in general the fluid flow. Additionally, the derived scaling laws can be used for any fluid, thus not limiting the analytical model to only ice flow in the Antarctic. Ice flow is a very complex process, and the mathematical expressions created and analyzed in this report only encompass ice flow over land.

All in all, by buttressing the analytical models through computational modeling and experimental techniques, stresses and crack propagation of ice sheets can then be researched.

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