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A COVARIANT APPROXIMATION SCHEME FOR GREEN'S FUNCTIONS OF COUPLED FIELDS

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A COVARIANT APPROXIMATION SCHEME FOR GREEN'S FUNCTIONS OF COUPLED FIELDS

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ABSTRACT

The scattering and production Green's functions for one nucleon and an arbitrary number of mesons are related by an infinite set of coupled linear integral equations. The first N of these equations contain Green's f functions involving 0, 1, 2, ... N external meson lines. The set of equations may be cut off at any point by making an assumption as to the structure of the Green's function with the highest number of external meson lines. In particular this function is approximated by decomposing it into products of lower order Green's functions, the physical assumption being that one of the mesons interacts weakly with the remaining meson-nucleon system. This leads to a closed set of equations which are linear if vacuum polarization is neglected. Examples of successive approximations are derived. The formalism is also applied to the two-nucleon case and to the three-fields problem, the latter being treated in a manifestly gauge covariant manner.

I. INTRODUCTION

Recent investigations of the problem of the interaction between the meson and nucleon fields have been aimed at dispensing with methods only suited to weakly coupled systems. In particular, attention has been centered upon covariant two-body equations[1] (in which the approximation of a perturbation expansion of the interaction operator is made), and on the non-adiabatic three-dimensional Tamm-Dancoff method[2] (where the approximation consists of setting the amplitudes for systems containing more than a given number of particles to zero). While the Bethe-Salpeter approach is a covariant one, a serious drawback lies in its perturbation treatment of the interaction term. Not only is the convergence of this in grave doubt, but one has no criteria for deciding how many (and which) irreducible graphs are to be included for the description of a given process. In the Tamm-Dancoff scheme no perturbation approximation is used. Here again however, there is no guide enabling one to determine at what number of particles the infinite set of equations should be cut off. Furthermore, as has recently been pointed out[3], the lack of

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1 J. Schwinger, Proc. Nat. Acad. Sci. U. S., **37**, 452 (1951) and E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

2 Dyson, Schweber, and Visscher, Phys. Rev. **90**, 372 (1953).

3 M. Gell-Mann and M. L. Goldberger, private communication from M. L. Goldberger.

symmetry between the "crossed" and "uncrossed" diagrams at any stage of the approximation in the meson-nucleon case, will yield an incorrect zero energy limit for the exchange scattering.

The purpose of this paper is to utilize a new approximation procedure for dealing with the Green's functions appropriate to a given problem. Taking cognizance of the relationship between the vertex operator and kernels involving external meson lines, a system of coupled integral equations for these Green's functions may be written down. Following a suggestion of M. Neuman[4], we treat one of the mesons as interacting weakly with the rest of the system, by decomposing the highest Green's function considered, into products of lower Green's functions, thus cutting off the infinite set of equations. This scheme has the advantages of being Lorentz covariant (and for photon processes, gauge covariant), of containing no perturbation features, and of providing some physical insight into the meaning of the approximation. The basic symmetry of the Green's functions in the meson variables avoids the difficulty with the exchange scattering mentioned above. If certain aspects of vacuum polarization are neglected, the resulting equations are linear.

In the following section the formalism is developed in connection with the meson-nucleon system. Later sections deal with its applicability to the two-nucleon and the three-fields problems.

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[4] M. Neuman, Phys. Rev. **92**, 1021 (1953). The formalism developed below is closely related to that of Dr. Neuman's paper.

II. ONE-NUCLEON SYSTEMS

(A) Definitions and Notation.

The equations of motion for the pseudoscalar meson operator ϕ_i and the nucleon field operator ψ interacting via symmetric pseudoscalar coupling in the absence of external fields are

$$(\gamma p + m + g\,\gamma_5 \tau_i \phi_i)\,\psi = 0$$

$$(p^2 + \mu^2)\,\phi_i + \tfrac{1}{2} g\,[\bar\psi, \gamma_5 \tau_i \psi] = 0$$

where $p_\mu = -i\partial_\mu$ [5].

A generalized Green's function governing processes involving n nucleons and m external meson lines may be defined as follows

$$G(x_1 \ldots x_n,\ x_1' \ldots x_n';\ \xi_1 \ldots \xi_m)$$

$$= i^n\, i^{[m/2]} \left\langle \left(\psi(x_1) \ldots \psi(x_n)\,\bar\psi(x_1') \ldots \bar\psi(x_n')\,\phi(\xi_1) \ldots \phi(\xi_n)\right)_+ \right\rangle$$

$$x \in (x_1, \ldots x_n')$$

where for any Heisenberg operator F(x), the symbol $\langle F(x) \rangle$ stands for

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5 Throughout the paper the notation is that of J. Schwinger, (reference 1). Thus, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$; $\gamma_5^2 = -1$

$\langle vac., \sigma_1 | F(x) | vac., \sigma_2 \rangle / \langle vac., \sigma_1 | vac., \sigma_2 \rangle$; σ_1 and σ_2 representing two spacelike surfaces at $\pm \infty$ respectively; $\left[\frac{m}{2}\right]$ is the integral part of $\frac{m}{2}$; $\in (x_1 \ldots x_n')$ is $+1$ for an even permutation of the order of the times $x_{o_1} \ldots x'_{o_n}$ and -1 for an odd permutation. The spinor and isotopic indices have been suppressed. In this section we shall concern ourselves with problems involving only one nucleon. For simplicity of notation the nucleon co-ordinates will also be suppressed and matrix multiplication over these variables will be assumed. Thus we write

$$G(\xi_1 \ldots \xi_m) = i^{1+[m/2]} \langle (\psi(x)\bar{\psi}(x')\phi(\xi_1)\ldots\phi(\xi_m))_+ \rangle \in (x-x') \tag{2.4}$$

It will also be found useful to define the quantity $\gamma(\xi)$:

$$(x | \gamma_{5i}(\xi) | x') = \gamma_5 \tau_i \, \delta(x-x') \delta(\xi - x') \tag{2.5}$$

(B) Derivation of the Green's Functions Equations.(5a)

We begin by obtaining the equation for the one-nucleon, no-meson Green's function. Operating on the quantity G with $(\gamma p + m)$ and invoking the equations of motion (2.1) one finds

$$(\gamma p + m) G(xx') = \delta(x-x') - ig\gamma_5 \tau_i \langle (\psi(x)\bar{\psi}(x')\phi_i(x))_+ \rangle \in (x-x') \tag{2.6}$$

(5a) Similar equations have been derived by J. Schwinger, unpublished lecture notes, Harvard University where $\delta\,(x - x')$ arises from the time derivative of the discontinuity of G. Referring to (2.3) we notice that the last term on the r.h.s. is equal to $-g\gamma_5 \tau\, G(xx';x)$, so that (2.6) may be rewritten as

$$(\gamma p + m)\, G = 1 - g \int \gamma(\xi)\, G(\xi)\, d\xi \tag{2.7}$$

where $(x|1|x') = \delta\,(x - x')$. Proceeding in a similar manner the equation for $G(\xi)$ may be obtained:

$$(\gamma p + m)\, G(\xi) = ig \int \gamma(\xi')\, G(\xi'\xi)\, d\xi' \tag{2.8}$$

No term corresponding to $\delta\,(x - x')$ appears here since $\langle \phi(\xi) \rangle = 0$ in the absence of external meson fields. The application of $(\gamma p + m)$ to $G(\xi'\xi)$ leads to the appearance of the quantity $i\langle(\phi_i(\xi)\phi_j(\xi'))_+\rangle$, to be denoted by $\mathcal{G}_{ij}(\xi\xi')$ (the one-meson (no-nucleon) Green's function), as well as $G(\xi''\xi'\xi)$:

$$(\gamma p + m)\, G(\xi'\xi) = \mathcal{G}(\xi'\xi) - g \int \gamma(\xi'')\, G(\xi''\xi'\xi)\, d\xi'' \tag{2.9}$$

The next equation in the scheme is:

$$(\gamma p + m)\, G(\xi''\xi'\xi) = ig \int \gamma(\xi''')\, G(\xi'''\xi''\xi'\xi)\, d\xi''' \tag{2.10}$$

The equation for $\mathcal{G}$ may easily be obtained by operating with $(k^2 + \mu^2)$ (where $k_\mu = -i\,\partial/\partial\xi_\mu$) upon $\mathcal{G}$ and using (2.2). Thus,

$$(k^2+\mu^2)\,\mathcal{G}(\xi\xi') = \delta(\xi-\xi') + g\,\mathrm{Tr}\,\gamma(\xi)\,G(\xi') \tag{2.11}$$

We may mention at this point that the substitution of (2.8) into (2.7) and (2.11) yields equations (1"a) and (1"b) of Neuman[4].

$$(\gamma p+m)\,G = 1 - ig^2\int\gamma(\xi)\,G_0\,\gamma(\xi')\,G(\xi'\xi)\,d\xi\,d\xi' \tag{2.12a}$$

$$(k^2+\mu^2)\,\mathcal{G}(\xi\xi') = \delta(\xi-\xi') + ig^2\,\mathrm{Tr}\int\gamma(\xi)\,G_0\,\gamma(\xi'')\,G(\xi''\xi')\,d\xi'' \tag{2.12b}$$

We shall denote by $\mathcal{G}_0(\xi\xi')$ the solution of (2.11) with the second term on the right omitted.

(C) The Approximation Scheme.

The functions G, $G(\xi)$, $G(\xi,\xi')$, etc. of the preceding section govern rigorously the motion of one nucleon and various numbers of mesons, the functions with an odd number of meson indices referring to the production of at least one meson (rather than a scattering). Thus G is the propagation function for a nucleon with all possible self-energy processes occurring. As is customarily done, we shall represent such "clothed" particles by thick lines. In Fig. 1, we give the diagrammatic representations of the first few Green's functions with the times arranged so that a minimum number of "thick lines" occur at a given time. We also notice that any Green's function may be symbolically represented by a set of three indices (x, y, z) where x = the number of nucleons, $y = [m/2]$ where m is the number of meson variables, and the number $z = 1$ or 0 depending on whether m is odd or even. The sum of these three indices yields the maximum number of thick lines in the corresponding diagrams time-ordered as in Fig. 1.

The equations (2.7), (2.8), (2.9) etc. coupling the various Green's functions may be schematically respresented by an array of linking "boxes" (labeled by (x,y,z)), each box describing a particular Green's function (Fig. 2). The scheme proposed consists of limiting oneself in a given calculation, to a finite number of thick lines which corresponds to a finite portion of the array of "boxes". If this is done, one in general obtains N equations involving $N + 1$ unknowns, the last Green's function $G(\xi_1 \ldots \xi_N)$ having one more line than the number allowed by the scheme. Following the suggestion made by M. Neuman[4], the last function is now treated approximately. The scheme utilized is to factor this function into products of Green's functions already appearing in the equations, some of these being replaced by their zeroth order forms so that the number of thick lines does not exceed the maximum allowed. Such a factorization may of course be carried out in a number of ways. A procedure that appears most reasonable at low energies is the factoring out, in a symmetric fashion, of a single meson propagation function: $G(\xi_1 \ldots \xi_N) \cong G_0(\xi_1 \xi_2) G(\xi_3 \ldots \xi_N) + \ldots$. The physical interpretation of this procedure is that one of the mesons does not interact strongly with the remaining meson-nucleon complex.

(D) Examples of Approximations.

(i) Zero'th Approximation.

To this order one allows only one thick line. Since in (2.7), $G(\xi)$ has two lines and its factorization in the absence of an external meson field gives zero $\left(G(\xi) \simeq G\langle\phi(\xi)\rangle = 0 \right)$, Eq. (2.7) reduces to

$$(\gamma p + m)\, G_0 = 1 \tag{2.13}$$

the equation for the non-interacting nucleon kernel.

(ii) First Approximation.

In this approximation we still restrict ourselves to one line. Here we break up

$$G(\xi\xi') \simeq G\, \mathcal{G}_0(\xi\xi') \tag{2.14}$$

which corresponds to the explicit assumption of no meson-nucleon scattering. Inserting (2.14) in (2.12a) one obtains:

$$(\gamma p + m)\, G = 1 - ig^2 \left(\int \gamma(\xi)\, \mathcal{G}_0(\xi\xi')\, G_0\, \gamma(\xi')\, d\xi\, d\xi' \right) G \tag{2.15}$$

The G defined by the above equation has previously been studied[6] in order to ascertain its effect on the suppression of pairs in meson-nucleon scattering. It is to be noted that in the present scheme, its use in any discussion of the meson-nucleon equation would be inconsistent due to the approximation made in Eq. (2.14).

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[6] Brueckner, Gell-Mann and Goldberger, Phys. Rev. 90, 476 (1953); and Karplus, Kivelson and Martin, Phys. Rev. 90, 1072 (1953).

(iii) Second Approximation.

Here we consider the case of two thick lines, i.e. the second set of boxes (Fig. 2), described by Eqs. (2.7), (2.8), (2.9) and (2.11). We approximate the three-line quantity appearing here by

$$G(\xi''\xi'\xi) \simeq G(\xi'')\,\mathcal{G}_0(\xi'\xi) + G(\xi')\,\mathcal{G}_0(\xi\xi'') + G(\xi)\,\mathcal{G}_0(\xi''\xi') \tag{2.16}$$

Eliminating $G(\xi)$ in terms of $G(\xi\xi')$ via (2.8) and substituting into (2.9) one obtains an equation for $G(\xi\xi')$:

$$\begin{aligned} G(\xi'\xi) = {} & G_0\,\mathcal{G}_0(\xi'\xi) - ig^2 \int G_0\gamma(\xi'')G_0\gamma(\xi''')G(\xi'''\xi'')\,\mathcal{G}_0(\xi'\xi) \\ & + ig^2 G_0\,\mathrm{Tr} \int \mathcal{G}_0(\xi\xi'')\gamma(\xi'')G_0\gamma(\xi''')\,G(\xi'''\xi') \\ & - ig^2 \Big[\int G_0\gamma(\xi'')\,G_0\gamma(\xi''')\Big\{ G(\xi'''\xi')\,\mathcal{G}_0(\xi''\xi) \\ & + G(\xi'''\xi)\,\mathcal{G}_0(\xi''\xi')\Big\}\Big] \end{aligned} \tag{2.17}$$

The second and third terms on the r.h.s. of (2.17) are only self-energy structures on external lines and may therefore be absorbed into the first term. In obtaining (2.17), use was made of Eqs. (2.12a,b).

The quantity $G(\xi''\xi'\xi)$ describes one-meson production in a meson-nucleon scattering process. The approximation (2.16) does not allow a real production of this type to occur since $G(\xi)$ has vanishing matrix elements between states conserving energy-momentum. Experimentally the meson-production cross section in meson-nucleon collisions appears to be very small up to energies $\sim$ 1 Bev. Thus this approximation may be adequate at low energies, since one may hope that any expected rise in the matrix element for virtual production at energies $>>$ 1 Bev (in intermediate states) will be damped by the energy denominators.

An integral equation for $G(\xi)$, which is closely related to the vertex operator, may be derived in this approximation. Substituting (2.16) into (2.9) and eliminating $G(\xi\xi')$ in (2.8) via (2.9) one obtains

$$\begin{aligned} G(\xi) = {} & ig \int G_0 \gamma(\xi') G_0 \left[g_0(\xi'\xi) \right. \\ & \left. + ig^2 g_0(\xi\xi'') \, \mathrm{Tr}\, \gamma(\xi'') G(\xi) \right] \\ & - ig^2 \int G_0 \gamma(\xi') G_0 \gamma(\xi'') \left[G(\xi'') g_0(\xi\xi') \right. \\ & \left. + G(\xi) g_0(\xi''\xi') + G(\xi') g_0(\xi''\xi) \right] \end{aligned} \tag{2.18}$$

(iv) Third Approximation.

In this case, we eliminate $G(\xi''\xi'\xi)$ in (2.9) via (2.10) and decompose the Green's function $G(\xi'''...\xi)$ so as to allow no more than two thick lines:

$$
\begin{aligned}
G(\xi'''\xi''\xi'\xi) \cong\ & [G(\xi'''\xi'') - G\mathcal{G}(\xi'''\xi'')]\mathcal{G}(\xi'\xi) \qquad (2.19) \\
& + [G(\xi'''\xi') - G\mathcal{G}(\xi'''\xi')]\mathcal{G}(\xi''\xi) + [G(\xi'''\xi) - G\mathcal{G}(\xi'''\xi)]\mathcal{G}(\xi''\xi') \\
& + [G(\xi''\xi') - G\mathcal{G}(\xi''\xi')]\mathcal{G}(\xi'''\xi) + [G(\xi''\xi) - G\mathcal{G}(\xi''\xi)]\mathcal{G}(\xi'''\xi') \\
& + [G(\xi'\xi) - G\mathcal{G}(\xi'\xi)]\mathcal{G}(\xi'''\xi'') \\
& + G\Big(\mathcal{G}(\xi'''\xi'')\mathcal{G}(\xi'\xi) + \mathcal{G}(\xi'''\xi')\mathcal{G}(\xi''\xi) \\
& \qquad + \mathcal{G}(\xi'''\xi)\mathcal{G}(\xi''\xi')\Big)
\end{aligned}
$$

The first terms in (2.19) represent the diagrams in which only one meson can propagate independently. Since $G(\xi'''\xi''\xi'\xi)$ must also include the possibility of two disconnected meson lines, the second set of terms must be added in. Eq. (2.19) may be rewritten as follows:

$$
\begin{aligned}
G(\xi'''\xi''\xi'\xi) \simeq \Big[& G(\xi'''\xi'')\mathcal{G}(\xi'\xi) + G(\xi'''\xi')\mathcal{G}(\xi''\xi) \\
& + G(\xi'''\xi)\mathcal{G}(\xi''\xi')\Big] + \Big[\{G(\xi''\xi') - G\mathcal{G}(\xi''\xi')\}\mathcal{G}(\xi'''\xi) \\
& + \{G(\xi''\xi) - G\mathcal{G}(\xi''\xi)\}\mathcal{G}(\xi'''\xi') \\
& + \{G(\xi'\xi) - G\mathcal{G}(\xi'\xi)\}\mathcal{G}(\xi'''\xi'')\Big]
\end{aligned} \tag{2.20}
$$

The first three terms are the ones appearing in the second approximation. Substituting (2.20) into (2.10), one sees that in this case the three remaining terms allow single meson production in a meson-nucleon collision. The approximation made in (2.20) assumes that two real mesons and a nucleon do not form a closely bound system, or, for a different time ordering, one is neglecting two-meson production in meson-nucleon scattering. An equation for the meson-nucleon Green's function may easily be derived and is linear when vacuum polarization effects (closed loops) are neglected. Some of the lowest order diagrams included in this approximation are shown in Fig. 3.

III. TWO-NUCLEON SYSTEMS

The formalism developed in Section II can easily be applied to the two-nucleon problem. The quantity of central interest is the two-nucleon, no-meson Green's function, G_{12}, defined by

$$G_{12} = - \langle \left(\psi(x_1) \psi(x_2) \bar{\psi}(x_1') \bar{\psi}(x_2') \right)_+ \rangle \in \tag{3.1}$$

$$\in \; = \in(x_1 - x_2) \in (x_1 - x_1') \in (x_1 - x_2') \in (x_1' - x_2) \in (x_1' x_2') \in (x_2 - x_2')$$

As before we will denote the two-nucleon, m-meson kernel by $G_{12}(\xi_1, \ldots, \xi_m)$.

We begin by operating with $(\gamma_1 p_1 + m)$ on G_{12} to obtain

$$(\gamma_1 p_1 + m) G_{12} = (1_1 G_2)_A - g \int \gamma_1(\xi) G_{12}(\xi) \, d\xi \tag{3.2}$$

where in general

$$(x_1 x_2 | A_A | x_1' x_2') = (x_1 x_2 | A | x_1' x_2') - (x_1 x_2 | A | x_2' x_1') \tag{3.3}$$

In (3.2) the inhomogeneous term comes from the two non-vanishing discontinuities in the time derivative of (3.1). Continuing, one further finds that

$$(\gamma_2 p_2 + m)(\gamma_1 p_1 + m) G_{12} = (1_1 1_2)_A \tag{3.4}$$

$$- g \left\{ 1_1 \int \gamma_2(\xi) G_2(\xi) \, d\xi \right\}_A - g \left\{ 1_2 \int \gamma_1(\xi) G_1(\xi) \, d\xi \right\}_A$$

$$- i g^2 \int \gamma_1(\xi) \gamma_2(\xi') G_{12}(\xi \xi') \, d\xi \, d\xi'$$

The quantity $G_{12}(\xi\,\xi')$ is the kernel describing the scattering of a meson by the two-nucleon system, or, for a different time-ordering, double meson production in a nucleon-nucleon collision. The breakup

$$G_{12}(\xi\xi') \simeq G_{12}\, \mathcal{G}_0(\xi\xi') \tag{3.5}$$

which corresponds to neglecting meson-deuteron scattering, yields, aside from the self-energy terms, the familiar equation[7]

$$(\gamma_1 p_1 + m)(\gamma_2 p_2 + m)\, G_{12} = \{1_1\, 1_2\}_A - i g^2 \left(\int \gamma_1(\xi)\, \mathcal{G}_0(\xi\xi')\, \gamma_2(\xi')\, d\xi\, d\xi' \right) G_{12} \tag{3.6}$$

In our scheme, however, the breakup (3.5) is incomplete in that it contains but one of the possible "two lines" structures. The correct decomposition is

$$G_{12}(\xi\xi') \cong G_{12}\, \mathcal{G}_0(\xi\xi') + \Big[G_1 \big(G_2(\xi\xi') - G_2\, \mathcal{G}_0(\xi\xi') \big) + G_2 \big(G_1(\xi\xi') - G_1\, \mathcal{G}_0(\xi\xi') \big) \Big] \tag{3.7}$$

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7 In its wave function form, the equation analogous to (3.6) in quantum electrodynamics has been investigated by J. Goldstein, Phys. Rev. 91, 1516 (1953), for the case of vanishing total energy. See also R. Arnowitt and S. Gasiorowicz, Phys. Rev., May 1, 1954.

The additional terms may be interpreted as a kind of "impulse approximation" to the meson-deuteron scattering kernel, in that the meson interacts with only one nucleon at a time. The substitution of (3.7) into (3.4) yields (neglecting self-energy terms):

$$(\gamma_1 p_1 + m)(\gamma_2 p_2 + m)\, G_{12} = \{ 1_1\, 1_2 \}_\Lambda \qquad (3.8)$$

$$- ig^2 \left(\int \gamma_1(\xi)\, \mathcal{G}_0(\xi\xi')\, \gamma_2(\xi')\, d\xi\, d\xi' \right) G_{12}$$

$$- ig^2 \int \gamma_1(\xi)\gamma_2(\xi') \left[G_1 \{ G_2(\xi\xi') - G_2\, \mathcal{G}_0(\xi\xi') \} + 1 \leftrightarrow 2 \right]_\Lambda$$

Continuing with the scheme, one may obtain the rigorous equation for $G_{12}(\xi\, \xi')$:

$$(\gamma_1 p_1 + m)(\gamma_2 p_2 + m)\, G_{12}(\xi\xi') = \{ 1_1\, 1_2\, \mathcal{G}(\xi\xi') \}_\Lambda \qquad (3.9)$$

$$- g \left\{ 1_1 \int \gamma_2(\xi'')\, G_2(\xi''\, \xi'\, \xi)\, d\xi'' \quad + \quad 1 \leftrightarrow 2 \right\}_\Lambda$$

$$- ig^2 \int \gamma_2(\xi''')\, \gamma_1(\xi'')\, G_{12}(\xi'''\, \xi''\, \xi\, \xi')\, d\xi''\, d\xi'''$$

In an elimination of $G_{12}(\xi\, \xi')$ in (3.4) via (3.9), the first three terms on the r.h.s. of (3.9) give "impulse" contributions, while the last term includes the effects due to the nucleon-nucleon binding. Making the breakup of $G_{12}(\xi''' \xi'' \xi' \xi)$ analogous to (3.7) in (3.9 yields an integral equation for $G_{12}(\xi \xi')$[8].

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8 In view of the experimental results favoring two-meson production in nucleon-nucleon collisions at energies $\sim$ 1 Bev, it would seem that a decomposition of $G_{12}(\xi \xi')$ as in (3.7) is not a good approximation, and the effects included in (3.9) will have to be taken into account. It is interesting to note that in terms of a perturbation expansion, (3.9) contains the two-pair fourth-order potentials, which have previously been shown to be large.

IV. THREE-FIELD SYSTEMS

The introduction of the electromagnetic field permits the extension of the technique outlined above to such problems as photo-production, nuclear Compton effect and the nucleon magnetic moments. In order to exhibit the gauge invariance of the scheme, we shall in this section deal only with manifestly gauge-covariant quantities. The three-field equations of motion are:

$$(\gamma p + m - e\gamma_\mu T A_\mu + g\gamma_5 \tau_i \phi_i)\psi = 0 \tag{4.1}$$

$$\left[(p - eSA)^2 + \mu^2\right]\phi_i = -g/2\left[\bar{\psi}, \gamma_5 \tau_i \psi\right] \tag{4.2}$$

$$p^2 A_\mu - p_\mu p_\lambda A_\lambda = e/2\left[\bar{\psi}, \gamma_\mu \psi\right] \tag{4.3}$$

where $T = (1/2)(1 + \tau_3)$ and $S = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In the succeeding development ξ will be used to denote the meson coordinate, η the photon coordinate, and

$$(x|\gamma(\eta)|x') = \gamma_\mu \dot{T}\, \delta(x - x')\, \delta(\eta - x') \tag{4.4}$$

The equation obeyed by the one-nucleon Green's function is

$$(\gamma p + m)G = 1 - g\int \gamma(\xi)G(\xi)d\xi + e\int \gamma(\eta)G(\eta)\,d\eta \tag{4.5}$$

where

$$G(\eta) = i\langle(\psi(x)\bar{\psi}(x')A(\eta))_+\rangle \epsilon(x-x') \tag{4.6}$$

and is not a gauge covariant quantity. The corresponding gauge covariant kernel is[9]

$$\begin{aligned} G^c(\eta) &= i\langle(\psi(x)\bar{\psi}(x')A(\eta))_+\rangle\epsilon(x-x') \\ &\quad - i\langle A(\eta)\rangle\langle(\psi(x)\bar{\psi}(x'))_+\rangle\epsilon(x-x') \\ &= G(\eta) - G A(\eta) \end{aligned} \tag{4.7}$$

Introducting $G^c(\eta)$ into (4.5) one obtains

$$\begin{aligned}(\gamma p+m)G = 1 - g\int\gamma(\xi)G(\xi)d\xi + e\int\gamma(\eta)G^c(\eta)d\eta \\ + e\left(\int\gamma(\eta)A(\eta)\,d\eta\right)G\end{aligned} \tag{4.8}$$

i.e.,

$$[\gamma(p - eTA) + m]G = 1 - g\int\gamma(\xi)G(\xi)d\xi + e\int\gamma(\eta)G^c(\eta)\,d\eta \tag{4.9}$$

in which form the gauge invariance of the equation is clearly exhibited.

---

9 The structure of $G^c(\eta)$ arises naturally using the variational definition of Schwinger. We shall henceforth denote $\langle A(\eta)\rangle$ simply by $A(\eta)$.

Proceeding as before, the kernels $G(\xi)$ and $G^c(\eta)$ obey the equations:

$$(\gamma\pi+m)G(\xi) = iq\int\gamma(\xi')G(\xi'\xi)d\xi' - ie\int\gamma(\eta')G^c(\eta'\xi)\,d\eta' \tag{4.10}$$

$$(\gamma\pi+m)G^c(\eta) = iq\int\gamma(\xi')G^c(\xi'\eta)d\xi' - ie\int\gamma(\eta')G^c(\eta'\eta)\,d\eta \tag{4.11}$$

where $\pi_\mu = p_\mu - e\,\tau A_\mu$ and the covariant photoproduction and Compton kernels are

$$G^c(\xi'\eta) = G(\xi'\eta) - i\,A(\eta)\,G(\xi') \tag{4.12}$$

$$G^c(\eta'\eta) = G(\eta'\eta) - i\,A(\eta)\,G^c(\eta') - i\,A(\eta')\,G^c(\eta) - i\,A(\eta)\,A(\eta')\,G \tag{4.13}$$

respectively. The meson-nucleon scattering, photoproduction and Compton scattering functions further obey the equations[10]

$$(\gamma\pi+m)G(\xi'\xi) = \mathcal{G}(\xi'\xi) - q\int\gamma(\xi'')G(\xi''\xi'\xi)d\xi'' + e\int\gamma(\eta'')G^c(\eta''\xi'\xi)d\eta'' \tag{4.14}$$

$$(\gamma\pi+m)G^c(\eta'\xi) = -q\int\gamma(\xi'')G^c(\xi''\eta'\xi)d\xi'' + e\int\gamma(\eta'')G^c(\eta''\eta'\xi)\,d\eta'' \tag{4.15}$$

$$(\gamma\pi+m)G^c(\eta'\eta) = \mathcal{G}^c(\eta'\eta) - q\int\gamma(\xi')G^c(\xi'\eta'\eta)d\xi' + e\int\gamma(\eta'')G^c(\eta''\eta'\eta)\,d\eta'' \tag{4.16}$$

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10 These equations have also been obtained by M. Neuman by using the variational derivative techniques. (Private communication.)

with definitions for $\mathcal{G}^c(\eta'\eta)$, $G^c(\eta''\xi'\xi)$, $G^c(\eta''\eta'\xi)$ and $G^c(\eta''\eta'\eta)$ analogous to (4.12) and (4.13).

In cutting off the infinite set of equations, it is necessary to decompose the higher meson and photon Green's functions in a gauge-invariant manner, that is, break them up into products of gauge-invariant quantities. Thus in the first approximation we set

$$G(\xi'\xi) \cong G\,\mathcal{G}(\xi'\xi)\ ;\ G^c(\eta'\xi) = 0\ ;\ G^c(\eta'\eta) \cong G\,\mathcal{G}^c(\eta'\eta) \tag{4.17}$$

Eliminating $G(\xi)$ and $G^c(\eta')$ in (4.9) via (4.10) and (4.11) one obtains an equation for G, involving $\mathcal{G}(\xi'\xi)$, $\mathcal{G}^c(\eta'\eta)$ and $G_0^c \equiv (\gamma\pi + m)^{-1}$. Neglecting vacuum polarization effects, the equation for $\mathcal{G}^c(\eta'\eta) \equiv \mathcal{G}_0^c(\eta'\eta)$

$$q^2 \mathcal{G}^c_{\mu\nu}(\eta'\eta) - q_\mu q_\lambda \mathcal{G}^c_{\lambda\nu}(\eta'\eta) = \delta_{\mu\nu}\,\delta(\eta'-\eta) \tag{4.18}$$

where $q_\mu = -i\,\partial/\partial\eta_\mu$. The equation for $\mathcal{G}(\xi'\xi)$ will be considered in connection with the next approximation.

In the second approximation, the production kernels of equations (4.14) to (4.16) are decomposed:

$$G(\xi''\xi'\xi) \cong G(\xi'')\,\mathcal{G}(\xi'\xi) + \text{symm. terms} \tag{4.19}$$

$$G^c(\eta''\xi'\xi) \cong G^c(\eta'')\,G(\xi'\xi) + G\,\mathcal{G}^c(\eta''\xi'\xi) \tag{4.20}$$

$$G^c(\eta''\eta'\xi) \cong G(\xi)\mathcal{G}_0^c(\eta''\eta') \tag{4.21}$$ [11]

$$G^c(\eta''\eta'\eta) \cong G^c(\eta'')\mathcal{G}_0^c(\eta'\eta) + \text{symm. terms} \tag{4.22}$$

By the use of Eqs. (4.10) and (4.11), the production kernels (4.19) to (4.22) may be expressed in terms of scattering kernels, which when substituted into (4.14) to (4.16) yield three equations coupling the meson-nucleon scattering, photoproduction and Compton scattering Green's functions:

$$\begin{aligned} G(\xi'\xi) = G\mathcal{G}(\xi'\xi) &- ig^2\int G_0^c\gamma(\xi'')G_0^c\gamma(\xi''')\left[\mathcal{G}(\xi''\xi)G(\xi'''\xi') + \mathcal{G}(\xi''\xi')G(\xi'''\xi)\right]d\xi''d\xi''' \\ &+ ieg\int G_0^c\gamma(\xi'')G_0^c\gamma(\eta''')\left[\mathcal{G}(\xi''\xi)G^c(\eta'''\xi') + \mathcal{G}(\xi''\xi')G^c(\eta'''\xi)\right]d\xi''d\eta''' \\ &+ e\int G_0^c\gamma(\eta'')\mathcal{G}^c(\eta''\xi'\xi)G\,d\eta'' \end{aligned} \tag{4.23}$$

$$\begin{aligned} G^c(\eta'\xi) = &- ig^2\int G_0^c\gamma(\xi''')G_0^c\gamma(\xi'')\mathcal{G}(\xi'''\xi)G^c(\eta'\xi'')d\xi''d\xi''' \\ &+ ieg\int G_0^c\gamma(\xi'')G_0^c\gamma(\eta'')\mathcal{G}(\xi''\xi)G_0^c\mathcal{G}_0^c(\eta''\eta')d\xi''d\eta'' \\ &+ ieg\int G_0^c\gamma(\eta'')G_0^c\gamma(\xi')\mathcal{G}_0^c(\eta''\eta')G(\xi'\xi)d\xi'd\eta'' \\ &- g\int G_0^c\gamma(\xi'')\mathcal{G}^c(\eta'\xi''\xi)G - ie^2\int G_0^c\gamma(\eta'')G_0^c\gamma(\eta''')\mathcal{G}_0^c(\eta''\eta')G^c(\eta'''\xi) \end{aligned} \tag{4.24}$$

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11 The term involving $\mathcal{G}^c(\eta''\eta'\xi)$ is absent, since we have explicitly neglected vacuum polarization. This type of kernel describes the two-photon decay of the neutral meson.

$$G^c(\eta'\eta) = G\, \mathcal{G}_0^c(\eta'\eta) + ieg \int G_0^c \gamma(\eta'') G_0^c \gamma(\xi') G^c(\xi'\eta') \mathcal{G}_0^c(\eta''\eta) \tag{4.25}$$

$$+ ieg \int G_0^c \gamma(\eta'') G_0^c \gamma(\xi') G^c(\xi'\eta) \mathcal{G}_0^c(\eta''\eta')\, d\eta''\, d\xi'$$

$$- ie^2 \int G_0^c \gamma(\eta'') G_0^c \gamma(\eta''') G^c(\eta'''\eta') \mathcal{G}_0^c(\eta''\eta)\, d\eta''\, d\eta'''$$

$$- ie^2 \int G_0^c \gamma(\eta'') G_0^c \gamma(\eta''') G^c(\eta'''\eta) \mathcal{G}_0^c(\eta''\eta')\, d\eta''\, d\eta'''$$

It remains to evaluate $\mathcal{G}(\xi\xi')$ and $\mathcal{G}^c(\eta\xi\xi')$. Starting with equation (4.2) we obtain

$$(k^2 + \mu^2)\mathcal{G}(\xi'\xi) = \delta(\xi'-\xi) + ieS \langle (\{k, A(\xi')\}\phi(\xi')\phi(\xi))_+ \rangle \tag{4.26}$$

$$- ie^2 S^2 \langle (A(\xi')A(\xi')\phi(\xi')\phi(\xi))_+ \rangle$$

Introducing the notation $S(\xi,\eta) = S\ \delta(\xi-\eta)$ and the covariant derivative $\bar{k}_\mu = k_\mu - eSA_\mu$, (4.26) may be written in the form

$$(\bar{k}^2 + \mu^2)\mathcal{G}(\xi'\xi) = \delta(\xi'-\xi) + e\int \{S(\eta\xi'), \bar{k}\}\mathcal{G}^c(\xi'\xi\eta)\, d\eta \tag{4.27}$$

$$+ ie^2 \int S(\eta'\xi') S(\eta\xi') \mathcal{G}^c(\xi'\xi\eta'\eta)\, d\eta\, d\eta'$$

Similarly

$$(\bar{k}^2+\mu^2)\,\mathcal{G}^c(\xi'\xi\eta) = -ie\int\{S(\eta'\xi'),\bar{k}\}\,\mathcal{G}^c(\xi'\xi\eta'\eta)\,d\eta' + ie^2\int S(\eta''\xi')S(\eta'\xi')\,\mathcal{G}^c(\xi'\xi\eta''\eta'\eta) \tag{4.28}$$

Keeping terms of order e only, the above quantities satisfy the equations

$$(\bar{k}^2+\mu^2)\,\mathcal{G}(\xi'\xi) = \delta(\xi'-\xi) \tag{4.29}$$

$$(\bar{k}^2+\mu^2)\,\mathcal{G}^c(\xi'\xi\eta) = -ie\int\{S(\eta'\xi'),\bar{k}\}\,\mathcal{G}(\xi'\xi)\,\mathcal{G}_0^c(\eta'\eta) \tag{4.30}$$

To lowest order in e (4.23) to (4.25) become

$$G(\xi'\xi) = G\,\mathcal{G}_0(\xi'\xi) - ig^2\int G_0^c\,\gamma(\xi'')\,G_0^c\,\gamma(\xi''')\left[\mathcal{G}_0(\xi''\xi)\,G(\xi'''\xi') + \mathcal{G}_0(\xi''\xi')\,G(\xi'''\xi)\right]d\xi''\,d\xi''' \tag{4.31}$$

$$\begin{aligned}G^c(\eta'\xi) = {}& -ig^2\int G_0^c\,\gamma(\xi''')\,G_0^c\,\gamma(\xi'')\,\mathcal{G}_0(\xi'''\xi)\,G^c(\xi''\eta')\,d\xi''\,d\xi''' \\ & + ieg\int G_0^c\,\gamma(\xi'')\,G_0^c\,\gamma(\eta'')\,\mathcal{G}_0(\xi''\xi)\,G^c(\eta''\eta) + ieg\int G_0^c\,\gamma(\eta'')\,G_0^c\,\gamma(\xi'')\,\mathcal{G}_0^c(\eta''\eta')\,G(\xi''\xi) \\ & + ieg\int G_0^c\,\gamma(\xi'')\,\mathcal{G}_0(\xi''\xi''')\,\{S(\eta''\xi'''),\bar{k}\}\,\mathcal{G}(\xi'''\xi)\,\mathcal{G}_0^c(\eta''\eta)\,G\end{aligned} \tag{4.32}$$

$$\begin{aligned}G^c(\eta'\eta) = {}& G\,\mathcal{G}_0^c(\eta'\eta) + ieg\int G_0^c\,\gamma(\eta'')\,G_0^c\,\gamma(\xi')\left[G^c(\xi'\eta')\,\mathcal{G}_0^c(\eta''\eta) + G^c(\xi'\eta)\,\mathcal{G}_0^c(\eta''\eta')\right] \\ & - ie^2\int G_0^c\,\gamma(\eta'')\,G_0^c\,\gamma(\eta''')\left[G_0^c\,\mathcal{G}_0^c(\eta'''\eta')\,\mathcal{G}_0^c(\eta''\eta) + G_0^c\,\mathcal{G}_0^c(\eta'''\eta)\,\mathcal{G}_0^c(\eta''\eta')\right]\end{aligned} \tag{4.33}$$

It should be noted that a knowledge of the meson-nucleon scattering Green's function is required for the solution of the photo-production equation, and the latter for the nuclear Compton effect. Since the equations are gauge covariant one may of course choose any gauge, in particular the usual Lorentz gauge.

V. CONCLUSION

In the preceding sections an attempt has been made to outline a procedure for obtaining a closed set of covariant equations describing meson processes in an approximate fashion. The approximation does not depend for its validity on the weakness of the coupling between the meson and the nucleon fields, but rather on the correctness of the conjectured form of the highest order Green's function appearing in a subset of the rigorous equations. More explicitly, since any subset of the rigorous equations correctly describes the whole theory, a knowledge of the Green's function containing the highest number of variables in such a set allows one in principle to calculate any physical process. The approximation made is to assume that one of the mesons in the highest Green's function does not interact with the remaining meson-nucleon structure. The successive approximations allow more and more mesons to interact closely with the nucleon. Although it seems mathematically impossible at this stage to prove convergence, the scheme appears reasonable in view of the fact that highly-multiple meson production is not observed experimentally.

A remark should be made on the consistency of the decomposition scheme. The rules for making the breakup apply only to the first Green's function that is not treated rigorously. The question arises whether the equations containing the higher Green's functions are then consistent. It can be shown that to make them so, one must make the same physical assumptions in these equations as are implied by the original breakup. For example, if the decomposition (2.14) is substituted into (2.9), consistency is obtained provided one assumes $G(\xi''\xi'\xi) \cong G(\xi'')\mathcal{G}_0(\xi'\xi)$. Comparison with (2.16) shows that this term appears there along with two other terms which give rise to meson-nucleon scattering. The latter, however, must be neglected in accordance with the assumptions leading to (2.14), namely that no meson-nucleon scattering be allowed.

The neglect of vacuum polarization effects in the low energy region seems reasonable. For high energies, however, one might expect the incident meson to interact mainly with the meson cloud around the nucleon, and hence polarization effects would predominate. It is possible to rearrange the equations so that these effects are more explicitly exhibited, while the direct meson-nucleon interaction is deemphasized.

Unlike the approximation in the Tamm-Dancoff method, the number of mesons and pairs present is not limited to a fixed number. This is due mainly to the covariance of the approach, which allows the "bending back" of nucleon lines in time[12]. The covariance allows as a consequence a preservation of the symmetry in meson indices of the Green's functions, with the resulting inclusion of equal numbers of crossed and uncrossed graphs in the same approximation. There is, of course, the serious practical drawback of having to solve inhomogeneous four-dimensional equations for the Green's functions rather than three-dimensional wave equations; on the other hand the boundary conditions are clearly defined when dealing with the kernels.

In order to get a qualitative feeling for the equations, we have examined the problem of $\pi^+ - p$ scattering in the second approximation in a very crude fashion. Eq. (2.18) was simplified by neglecting vacuum

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12 The scheme resembles more closely a Tamm-Dancoff procedure in "proper time".

polarization and dropping terms which yield only self-energy contributions to the external lines[13]. Rewriting the integral equation in momentum space, an ansatz was made that the solution be expressible in terms of a product of the lowest order perturbation expression times a slowly varying function, which could be taken out of the integral[14]. The resulting algebraic equation was solved for the slowly varying function which was found to be expressible in terms of the second order mass and vertex operators. Substitution of this result into Eq. (2.17) yielded an expression for the meson-nucleon Green's function. In this particular case, a perturbation theory renormalization of the second order mass and vertex operators gave the same result as a term-by-term renormalization of the perturbation solution of (2.18) followed by the "slowly varying approximation" applied to the resulting finite residue which was then resummed. With a coupling constant $g^2/4\pi \cong 1 - 3$ the low energy data may be fitted roughly. However no particular emphasis should be placed on this result since it is by no means clear that the first step of such an iteration procedure approximates the true solution, or even that the iteration converges.

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[13] These contributions were assumed to renormalize the masses and Green's functions on the outgoing lines.

[14] This may be viewed as the first step of an iteration procedure. Thus if the solution obtained below is written as $G_1(\xi) = G_0(\xi) f_1$ where f_1 is the slowly varying function, then the next ansatz would be $G_2(\xi) = G_1(\xi) f_2$ where now f_2 is to be taken out of the integral.

The renormalization used in the above example is basically unsatisfactory due to its limited applicability. In addition, it is questionable whether the renormalizations of the external lines[13] are consistent with those carried out on the approximate solution of the integral equation. The general problem of renormalizing the entire approximation scheme is being investigated by the authors in collaboration with Dr. S. Bludman.

It is a pleasure to acknowledge stimulating discussions with Drs. J. V. Lepore and M. Neuman. This work was performed under the auspices of the Atomic Energy Commission.

FIGURE CAPTIONS

Figure 1: The first three Green's functions with the time ordering arranged so that a minimum number of lines occur at a given time.

Figure 2: Schematic representation of Green's function couplings. The dashed boxes show the kernels included for one, two, and three "thick lines" respectively.

Figure 3: Some of the lower order diagrams included in the third approximation. The corresponding "crossed" diagrams also appear as the Green's function is symmetric in the meson variables.

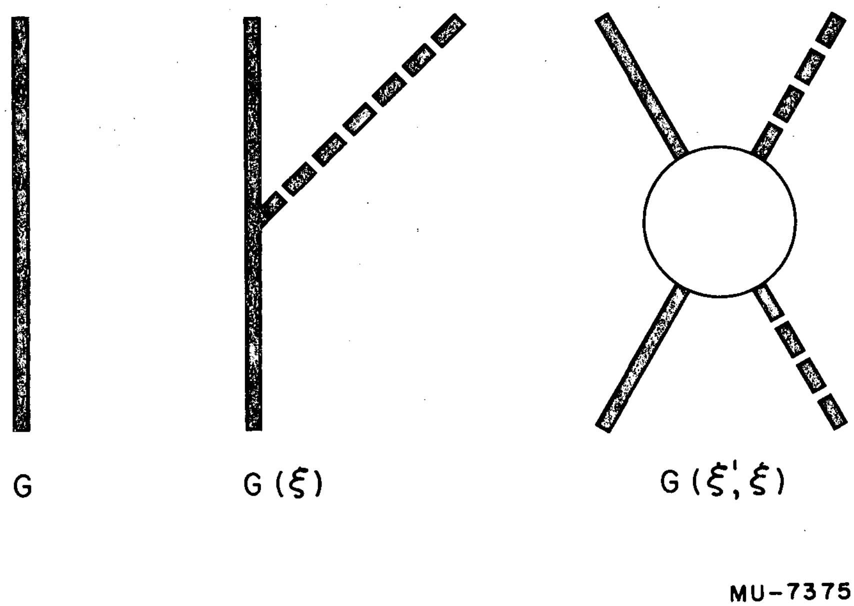

Figure 1

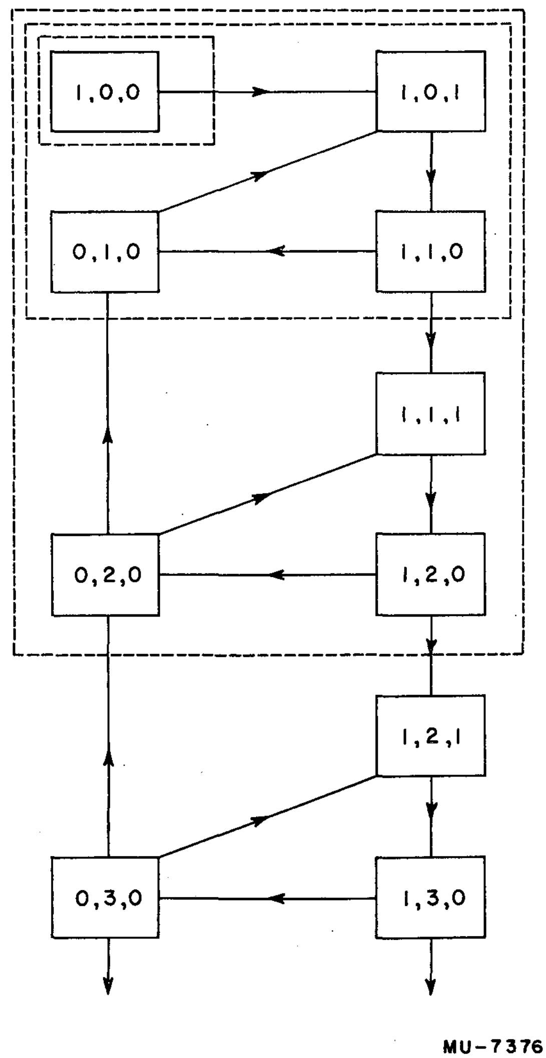

Figure 2

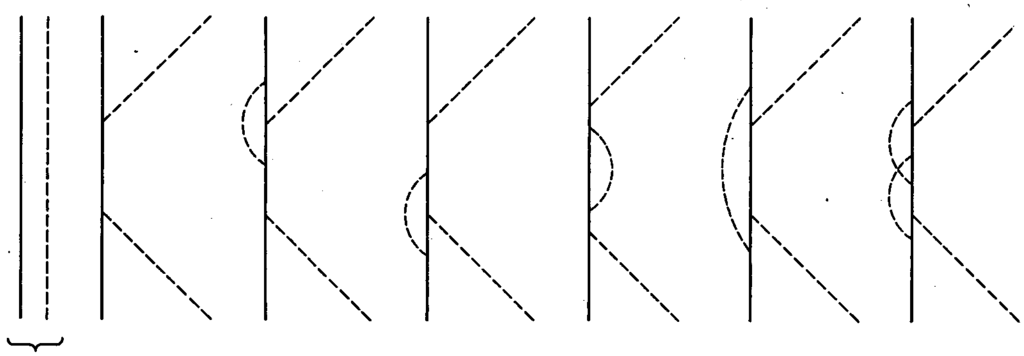

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Figure 3