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Zeros of Riemann zeta-type functions

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by

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ABSTRACT OF THE DISSERTATION

Zeros of Riemann zeta-type functions

by

Alexander James Dobner

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2021

Professor Terence Chi-Shen Tao, Chair

In this thesis we study two topics concerning the zeros of the zeta function and the zeros of related functions.

The first topic is a proof of a generalization of Newman's conjecture. Newman's original conjecture, which was proved by Rodgers and Tao in 2018, states that certain deformations of the Riemann xi function have zeros off the critical line. We will show that it is possible to formulate an analogue of Newman's conjecture for any function in the extended Selberg class, and we then prove that these analogous conjectures are true in every case. Our proof is necessarily quite different from Rogers and Tao's proof because their work requires information about the zeros of the zeta function which is not known in the general case. Our proof also has the benefit of being simpler and more direct.

The second topic is a result on the distribution of the argument of the Riemann zeta function on the critical line. In particular we will prove an unconditional lower

bound on the tails of this distribution. This can equivalently be stated as a quantitative estimate for how often the number of nontrivial zeta zeros up to a given height differs greatly from the expected number of such zeros.

The dissertation of Alexander James Dobner is approved.

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Rowan Brett Killip

Terence Chi-Shen Tao, Committee Chair

University of California, Los Angeles

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PUBLICATIONS

A New Proof of Newman's Conjecture and a Generalization, arXiv preprint (2020), arXiv:2005.05142. To appear (pending minor revisions) in *Acta Arith.*

Extreme values of the argument of the Riemann zeta function, arXiv preprint (2021), arXiv:2101.01747.

CHAPTER 1

Introduction

1.1 Background

This thesis is a compilation of two papers which are concerned with the zeros of the Riemann zeta function and the zeros of related functions. These papers have been edited to fit together as Chapters 2 and 3 of the thesis, but the reader should have no trouble reading either chapter on its own. In this introduction we will give some brief background on the zeta function and its zeros, and then we will summarize the contents of the subsequent chapters. Further background information can be found in standard textbooks on analytic number theory (e.g. see [31] and [12]).

To start we recall the usual Dirichlet series representation of the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for all complex s in the half-plane $\operatorname{Re} s > 1$. The zeta function also possesses an *Euler product* representation in this half-plane,

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

where the variable p ranges over all primes. By taking logarithms we that

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s},$$

where $\Lambda(n)$ is the von Mangoldt function, which is defined to equal $\log p$ whenever $n = p^r$ for some prime p and some integer $r \geq 1$, and zero otherwise.

The three equations written above are only valid in the half-plane $\operatorname{Re} s > 1$, but it is also possible to meromorphically continue $\zeta(s)$ to the entire complex plane as was shown by Riemann in his famous paper [23]. To prove this fact, Riemann derived a fundamental property of the zeta function known as the *functional equation*. This equation can be nicely expressed as $\xi(1-s) = \xi(s)$, where the function $\xi(s)$ is defined as

$$\xi(s) \coloneqq \frac{1}{2} s(s-1)\Gamma(\tfrac{s}{2})\pi^{-\frac{s}{2}}\zeta(s).$$

This function is referred to as the Riemann xi function.

Using the Euler product and the functional equation it is possible to deduce several facts about the zeros and poles of the zeta function. We list some of these now:

- The only pole of $\zeta(s)$ is a simple pole at $s = 1$.
- The only zeros of $\zeta(s)$ outside of the strip $0 \leq \operatorname{Re} s \leq 1$ are the so-called *trivial zeros* lying at the negative even integers.
- Inside of the strip $0 \leq \operatorname{Re} s \leq 1$ (also known as the *critical strip*) there are infinitely many *nontrivial zeros*. These are the same as the zeros of the Riemann xi function.

- The nontrivial zeros are symmetric about the real axis and about the line $\operatorname{Re} s = \frac{1}{2}$ (also known as the *critical line*).

The behavior of the nontrivial zeros is of much interest to number theorists, primarily because there is a certain Fourier duality between these zeros and the prime numbers. This duality is often expressed via *explicit formulas* which can be used to write sums over primes in terms of sums over zeta zeros and vice versa. The following is a basic example of such an explicit formula.

Theorem 1.1. *Suppose* $w\colon [0,\infty) \to \mathbb{R}$ *is a smooth compactly supported weight function, and suppose the support is contained in* $(1,\infty)$*. Let* ϕ *be the Mellin transform of this weight,*

$$\phi(s) := \int_0^\infty w(u) u^s \, \frac{du}{u}.$$

Then for any $X > 1$*, we have*

$$\sum_n \Lambda(n) w(\tfrac{n}{X}) = X\,\phi(1) - \sum_\rho X^\rho \phi(\rho) - \sum_{k=1}^\infty X^{-2k}\phi(-2k),$$

where the summation variable ρ *ranges over all the nontrivial zeros of the zeta function.*

Explicit formulas like this can be used to prove interesting connections between the primes and the zeta zeros. For example, one can show that the size of the error term in the prime number theorem is controlled by the horizontal locations of the zeta zeros, and the strongest bound on the error term is equivalent in some sense to the strongest statement one can make about the horizontal distribution of the zeros, i.e. the Riemann hypothesis.

Conjecture 1.2 (Riemann hypothesis)**.** *All nontrivial zeros of the zeta function lie on the critical line.*

In Chapter 2 we will study a question which is related to the Riemann hypothesis (and and its generalizations to other L-functions).

While the Riemann hypothesis is a conjecture about the horizontal distribution of zeta zeros, one can also study questions related to the vertical distribution. The most basic result in this direction in the Riemann-von Mangoldt formula, which gives an estimate for the number of nontrivial zeta zeros up to a given height.

Theorem 1.3 (Riemann-von Mangoldt formula)**.** *Let $N(T)$ denote the number of nontrivial zeta zeros whose imaginary part lies between 0 and T (and by convention we also count zeros whose imaginary part equals T with weight $\frac{1}{2}$). For any $T \geq 3$,*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

One may think of this formula as the zeta zero analogue of the prime number theorem since the prime number theorem gives an estimate for the number of primes up to a given "height". It is natural then to want to understand the behavior of the error term in the Riemann-von Mangoldt formula, just as one does with the error term in prime number theorem. This will be the subject of our investigations in Chapter 3.

1.2 Summary of Chapter 2

In Chapter 2 we examine a generalization of a conjecture known as Newman's conjecture. This chapter is an edited version of a paper that is to appear in *Acta Arithmetica*.

Newman's conjecture concerns a certain special family $\{\xi_t\}_{t\in\mathbb{R}}$ of deformations of the Riemann xi function (where $\xi_0 := \xi$). The interesting feature of this family is that there exists an associated real number Λ (called the *de Bruijn-Newman constant*) with the defining property that all of the zeros of $\xi_t(s)$ lie on the critical line if and only if $t \geq \Lambda$. From this property it is immediate that the Riemann hypothesis is equivalent to the statement that $\Lambda \leq 0$, and Newman's conjecture is the complementary inequality $\Lambda \geq 0$ (in other words, ξ_t has a zero off the critical line for every $t < 0$). This conjecture has been studied by many authors, and was proved by Rodgers and Tao in 2018.

We will prove two main results related to Newman's conjecture. The first result states that an analogue of the de Bruijn-Newman constant can be defined for any function in a class of functions known as the *extended Selberg class*. Consequently, one is able to state an analogue of Newman's conjecture for all such functions. This result supersedes some previous generalizations of Newman's conjecture in the literature which were to more restricted classes of functions. The second result we prove in the chapter is that our generalization of Newman's conjecture is true in every case. The proof is necessarily quite different from Rodgers and Tao's proof because they require information about the spacing between the zeros of the zeta function that is not known in the general case. Our proof is also simpler and more direct. Stated in the Riemann xi function case, our argument proceeds by showing that for every $t < 0$ the function ξ_t can be approximated in terms of a Dirichlet series $\zeta_t(s) = \sum_{n=1}^{\infty} \exp(\frac{t}{4} \log^2 n) n^{-s}$ whose zeros then provide infinitely many zeros of ξ_t off the critical line.

1.3 Summary of Chapter 3

In Chapter 3 we will study the argument of the Riemann zeta function on the critical line (i.e. $\operatorname{Im}\log\zeta(\frac{1}{2}+it)$). This quantity is of particular interest because it is the main quantity appearing in the error term of Riemann-von Mangoldt formula (see equation (3.1) for a precise statement of this). In other words, the argument of the Riemann zeta function at the point $\frac{1}{2}+it$ determines the size of the discrepancy between the actual and "expected" number of zeta zeros up to height t. Our main result in this chapter concerns the distribution of this quantity as t varies between T and $2T$.

For simplicity we will just state a corollary of our main result here. The corollary is as follows: for any small $\varepsilon > 0$, any large real number T, and any V in the range $\sqrt{\log\log T} \le V \le (\log T)^{\frac{1}{3}-\varepsilon}$ we have the lower bound

$$\operatorname{meas}\{t \in [T, 2T]\colon \operatorname{Im}\log\zeta(\tfrac{1}{2}+it) \ge V\} \ge T\exp(-D_\varepsilon V^2/\log\log T) \tag{1.1}$$

for some positive constant D_ε depending only on ε. This bound is interesting because it is consistent with the heuristic that the distribution of $\operatorname{Im}\log\zeta(\frac{1}{2}+it)$ is approximately Gaussian with a variance proportional to $\log\log T$ even in a large deviations regime. Previously the only results of this kind in the literature were for much smaller ranges of V.

1.4 Notation

Throughout this thesis we will use the notation $X \ll Y$, $Y \gg X$ and $X = O(Y)$ to indicate that there exists a positive absolute constant C such that $|X| \le CY$, as is standard in analytic number theory. If the constant C depends on other parameters then this will be indicated in the text or by a subscript in the notation (i.e. if C depends on Z then we write $X \ll_Z Y$ or $X = O_Z(Y)$). Furthermore, the notation $X \asymp Y$ indicates $X \ll Y \ll X$, and the notation $X = o_{T\to\infty}(Y)$ indicates that there exists a bound $|X| \leq c(T)Y$ where $c(T) \to 0$ as $T \to \infty$. Further notation will be introduced when needed.

CHAPTER 2

A proof of Newman's conjecture for the extended Selberg class

2.1 Introduction

We start by recalling the definition of the Riemann xi function,

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The xi function is entire and satisfies the functional equation $\xi(1-s) = \xi(s)$. Furthermore, the Riemann hypothesis is equivalent to the statement that all the zeros of the xi function lie on the critical line.

In order to study the behavior of ξ on the critical line, one may consider the function $z \mapsto \xi\left(\frac{1+iz}{2}\right)$ which has a Fourier representation (see [31, p. 255]),

$$\xi\left(\frac{1+iz}{2}\right) = \int_{-\infty}^{\infty} \Phi(u)e^{izu}\, du, \tag{2.1}$$

where

$$\Phi(u) := 4\sum_{n=1}^{\infty}\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right)e^{-\pi n^2 e^{4u}} \tag{2.2}$$

is a super-exponentially decaying even function. (The evenness of Φ follows from the fact that it is the Fourier transform of an even, real-valued function.)

In the course of his investigations on the Riemann hypothesis, Pólya [19] proved the rather remarkable result that if one replaces Φ in (2.1) with a certain approximation coming from the first term of (2.2), then the resulting "deformed" xi function has all of its zeros on the critical line. Moreover, he showed that the average spacing of these zeros is the same as the average spacing of the zeros of the true xi function. This work initiated the study of various other deformations of ξ which come about by modifying Φ in some way.

The deformations we will be studying in this chapter come from the work of de Bruijn [11] and later Newman [16], who considered the family $\{\xi_t\}_{t\in\mathbb{R}}$ of entire functions defined such that

$$\xi_t\left(\frac{1+iz}{2}\right) := \int_{-\infty}^{\infty} e^{tu^2}\Phi(u)e^{izu}\,du.$$

One may verify that each of these functions satisfies the functional equation $\xi_t(1-s) = \xi_t(s)$ as a consequence of the fact that Φ is even. Furthermore, a result of Pólya [20] implies that if ξ_{t_0} has all of its zeros on the critical line for some $t_0 \in \mathbb{R}$, then the same is true of ξ_t for all $t > t_0$. De Bruijn proved that for all $t \geq 1/2$ the zeros of ξ_t all lie on the critical line, and Newman subsequently proved that there is some $t < 1/2$ for which this is not the case. Newman concluded that there exists a real number $\Lambda \leq 1/2$ (now called the *de Bruijn-Newman constant*) with the defining property that ξ_t has all of its zeros on the critical line if and only if $t \geq \Lambda$.

Because $\xi = \xi_0$, it is clear that the Riemann hypothesis is equivalent to the statement that $\Lambda \leq 0$. Newman made the complementary conjecture that $\Lambda \geq 0$, and in doing so he pointed out that this is a quantitative form of the assertion that "the Riemann hypothesis, if true, is only barely so." A progression of lower bounds on Λ (see [7], [30], [9], [17], [8], [10], [18], [25]) culminated in a proof of Newman's conjecture by Rodgers and Tao [24] in 2018. Their proof is inspired by previous work of Csordas, Smith, and Varga [10] who noted that zeros of ξ_t exhibit a repulsion effect as t varies. Rodgers and Tao were able to use this repulsion to show via a rather involved argument that if $\Lambda < 0$, then as $t \to 0^-$ the zeros of ξ_t spread out in such a way as to contradict known results about the gaps between zeta zeros.

In this chapter we will give a simpler proof of Newman's conjecture which does not rely on any information about the zeros of the zeta function. In fact, because we use such limited information about ζ, we can state and prove our results in a generalized L-function setting. We will set up our generalization in Section 2.2 by giving the definition of a large class of Dirichlet series $\mathcal{S}^\sharp$ (known as the *extended Selberg class* in the literature) for which it is reasonable to define a de Bruijn-Newman constant. This class contains the Riemann zeta function as well as all Dirichlet L-functions associated to primitive characters. The main restriction on membership in $\mathcal{S}^\sharp$ is that every Dirichlet series $F \in \mathcal{S}^\sharp$ must have a corresponding "completed" version ξ^F satisfying a certain functional equation. Given such an F, we will show that one can always define a set of deformations $\{\xi_t^F\}_{t\in\mathbb{R}}$ analogous to the ξ_t functions we defined above, and we prove the following theorem:

Theorem 2.1. *For every $F \in \mathcal{S}^\sharp$, there is a real number Λ_F such that all the zeros of ξ_t^F lie on the critical line if and only if $t \geq \Lambda_F$.*

Previous authors have defined generalized de Bruijn-Newman constants but only for certain restricted classes of L-functions (see [29] for the case of quadratic Dirichlet L-functions and [1, Section 2.4] for the case of certain automorphic L-functions). Our definition of Λ_F for $F \in \mathcal{S}^\sharp$ subsumes these definitions, and we are also able to prove the analogue of Newman's conjecture in this general case,

Theorem 2.2. $\Lambda_F \geq 0$ *for every* $F \in \mathcal{S}^\sharp$.

We now give the idea behind our proof of Theorem 2.2 (restricting to the zeta function case for ease of exposition). The main tool will be an approximation for ξ_t that we establish for every $t < 0$ and which we use to locate zeros of ξ_t off the critical line. This approximation is of the form

$$\xi_t(J_t(s)) \approx [\text{gamma-like factor}] \cdot \zeta_t(s) \tag{2.3}$$

where

$$\zeta_t(s) := \sum_{n=1}^{\infty} \exp\left(\frac{t}{4}\log^2 n\right) n^{-s}$$

is an everywhere absolutely convergent Dirichlet series, and

$$J_t(s) := s + \frac{|t|}{4}\operatorname{Log}\left(\frac{s}{2\pi}\right)$$

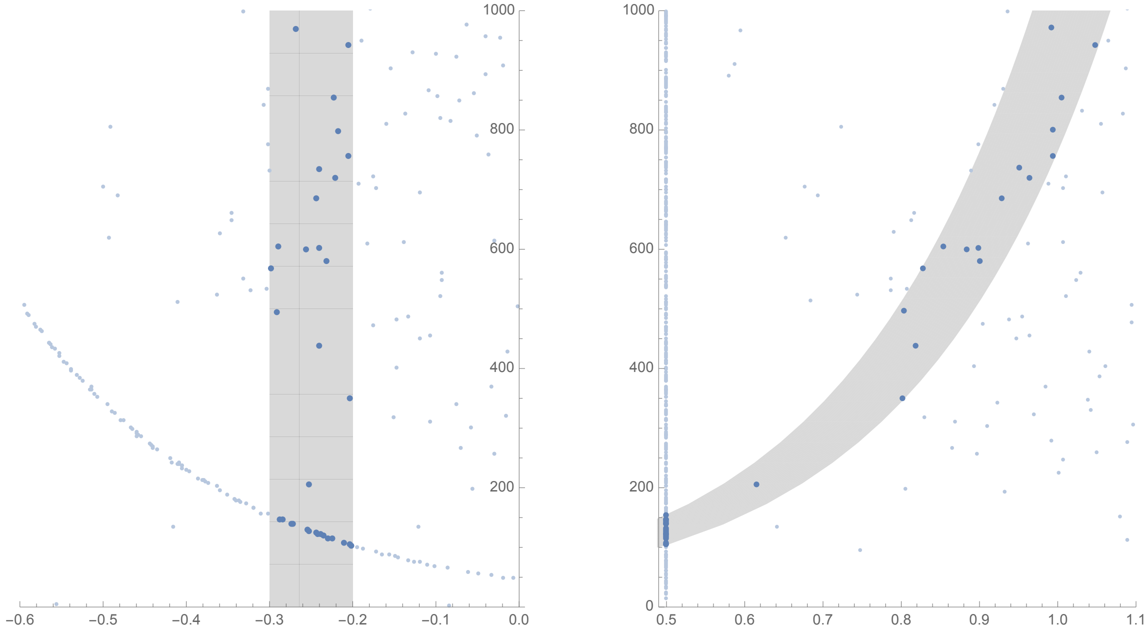
is a nonlinear change of coordinates. If s is restricted to lie in a vertical strip, a crucial feature of our approximation is that it becomes increasingly accurate as the imaginary part of s increases.

Such an approximation hints that for any $t < 0$ and any vertical strip V, there should be a close correspondence between the zeros of ζ_t in V and the zeros of ξ_t in the curved region $J_t(V)$. We will prove that this is the case and also that ζ_t must always have some vertical strip with zeros at arbitrarily height heights. Since the region $J_t(V)$ eventually lies to the right of the the critical line at high enough heights, this argument produces zeros of ξ_t which are off the critical line, thus implying Newman's conjecture.

An example of the correspondence between the zeros of ζ_t and ξ_t in the case $t = -1$ and $V = \{-.3 \leq \operatorname{Re} s \leq -.2\}$ is depicted in Figure 2.1. The figure also indicates that this correspondence seems to extend quite a bit beyond the range we have just described. Indeed, one can clearly see that the zeros of ξ_t on the critical line show up as zeros of ζ_t near the inverse image of the critical line under the J_t map. We note that the results we will be proving are insufficient to explain this broader correspondence because the rigorous form of (2.3) that we prove does not remain accurate if one chooses $J_t(s)$ to lie in a vertical strip rather than s.

For large negative t values, a different phenomenon occurs that is readily apparent in Figure 2.2 (which depicts zeros for $t = -30$). In this case one sees that the zeros of ξ_t begin to congregate near deterministic curves, and as t becomes increasingly negative more of these curves appear. This phenomenon was originally discovered by Rudolph Dwars (see the comments at `terrytao.wordpress.com/2018/12/28`) while doing extensive numerical work on the zeros of ξ_t. Our results can be used to give a rigorous explanation for these curves, but since the case of large negative t values is not relevant to Newman's conjecture we do not pursue this direction here.

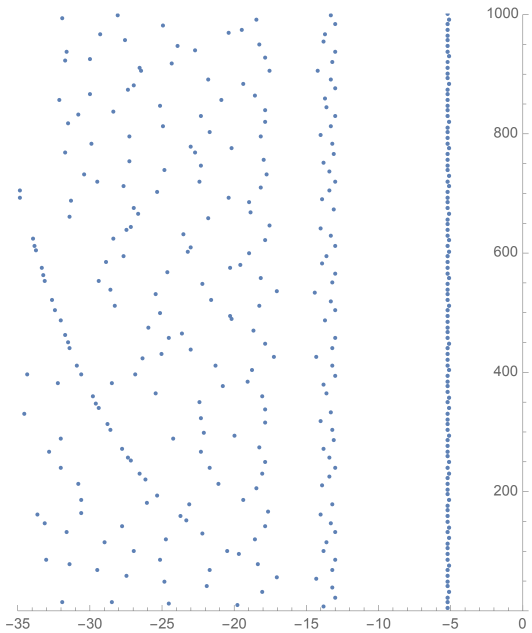
To prove the approximation (2.3) the starting point will be to rewrite ξ_t, for $t < 0$, as a certain contour integral of the xi function times a complex Gaussian. This integral is taken over a vertical line in the complex plane, and we are able to estimate it by shifting the contour to the right so that $\zeta(s)$ can be written in terms of its absolutely convergent Dirichlet series. Interestingly, in the $t > 0$ case (which we do not consider) there is an analogous contour integral representation of ξ_t, but the integral is taken over a horizontal line instead, so it is not possible to proceed in the same manner. It turns out that is possible to derive an estimate for ξ_t in terms of partial sums of ζ_t in this case. This is done in [21, Thm. 1.3], where the estimate is is used to prove an upper bound on the de Bruijn-Newman constant.



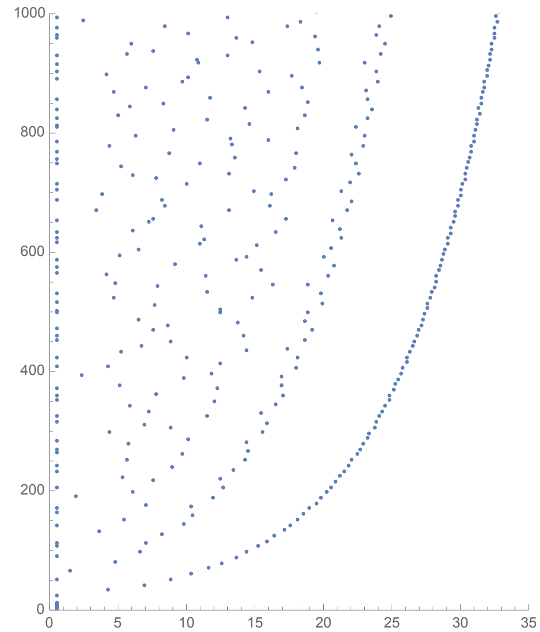
(a) Zeros of ζ_{-1}.

(b) Zeros of ξ_{-1}

Figure 2.1: The zeros of ζ_t in any vertical strip are mapped quite precisely under J_t to zeros of ξ_t in the corresponding curved region. This correspondence provides zeros of ξ_t off the critical line. For ξ_t the zeros are symmetric about the critical line, but in (b) we only depict the zeros on and to the right of the line.



(a) Zeros of ζ_{-30}.



(b) Zeros of ξ_{-30}

Figure 2.2: For large negative t values, the zeros of ξ_t congregate near curves which are the images of certain vertical lines under the J_t map. On these lines the Dirichlet series ζ_t is dominated by two consecutive terms of equal magnitude, which leads to the regular pattern of zeros.

One final note we make about our proof is that it reveals that Newman's conjecture holds for completely analytic rather than arithmetic reasons. To highlight this, we mention that our methods can be used to show that if one selects essentially *any* Dirichlet series F and multiplies F by arbitrary Γ factors to produce a "mock" ξ^F function, then there are corresponding ξ_t^F functions for all $t < 0$ and these functions always have zeros off the critical line (though in this completely general case there is no associated de Bruijn-Newman constant because of the lack of a functional equation). Thus, our proof of Newman's conjecture is quite different from the proof of Rodgers and Tao which depends fundamentally on knowledge about the gaps between zeta zeros and hence on the arithmetic structure of the zeta function.

2.1.1 Notation

In this chapter we will use the notation $\operatorname{Arg} z$ to denote the argument of a complex number, where this value is chosen to lie in the interval $(-\pi, \pi]$. Correspondingly, we use $\operatorname{Log} z$ to denote the standard branch of the complex logarithm (i.e. $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$), and we define $\sqrt{z} := \exp(\frac{1}{2} \operatorname{Log} z)$.

In Section 2.2, we will give the definition of a certain family $\mathcal{S}^\sharp$ of Dirichlet series, and we will subsequently choose an arbitrary $F \in \mathcal{S}^\sharp$ which will remain fixed for the rest of the chapter. We will also choose a certain meromorphic function γ (depending on F) which will be fixed for the rest of the chapter. *All* implicit constants in the estimates that we prove will then be allowed to depend on F and γ. Similarly, if a statement is said to hold for "sufficiently large y" then the meaning of sufficiently large may depend on F and γ. Readers who are primarily interested in the Riemann zeta function case of Theorem 2.2 (i.e. Newman's conjecture) may want to skim the definition of $\mathcal{S}^\sharp$ in Section 2.2 in order to understand the notation and then skip to Section 2.3 and assume that $F \coloneqq \zeta$.

2.2 The generalized de Bruijn-Newman constant Λ_F

In 1989, Selberg [27] introduced a definition for a class of Dirichlet series $\mathcal{S}$ now known as the Selberg class. Among other things, Selberg conjectured that all the functions in $\mathcal{S}$ satisfy a corresponding Riemann hypothesis, and the conditions he imposed on $\mathcal{S}$ were chosen with this in mind.

For the purposes of defining generalized de Bruijn-Newman constants, one of Selberg's conditions (the existence of a functional equation) is completely essential , whereas some of the other conditions (e.g. the existence of an Euler product) are not necessary. Because of this, the Dirichlet series that we consider will be members of a class known as the extended Selberg class $\mathcal{S}^\sharp$ which was defined by Kaczorowski and Perelli [14]. This class may be thought of as capturing only the analytic rather than arithmetic properties among those given by Selberg.

A function $F(s)$ is said to be a member of $\mathcal{S}^\sharp$ if it is not identically zero and it satisfies the following conditions:

(i) (Dirichlet series) F has a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which converges absolutely for all s with $\operatorname{Re} s > 1$.

(ii) (Meromorphic continuation) $(s-1)^m F(s)$ is an entire function of finite order for some nonnegative integer m.

(iii) (Functional Equation) Let m be the order of the pole of F at $s = 1$ (or let $m = 0$ in the case where F has no pole). There exists a function $\gamma(s)$ of the form

$$\gamma(s) \coloneqq \alpha s^m (s-1)^m Q^s \prod_{i=1}^{k} \Gamma(\omega_i s + \mu_i)$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $Q > 0$, $\omega_i > 0$, and $\mu_i \in \mathbb{C}$ with $\operatorname{Re} \mu_i \geq 0$ such that

$$\xi^F(s) \coloneqq \gamma(s) F(s)$$

satisfies

$$\xi^F(s) = \overline{\xi^F(1-\overline{s})}. \tag{2.4}$$

Our notation in (iii) differs slightly from the usual notation given in definitions for $\mathcal{S}$ and $\mathcal{S}^\sharp$ (e.g. [13, p. 160]) because we include the polynomial factor $s^m(s-1)^m$ in the functional equation. This notation will be convenient for us because we require that the "completed" L-function ξ^F be an entire function. In our notation it is clear that if F is the Riemann zeta function, then one may choose $\gamma(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ to make ξ^F the usual Riemann xi function.

There are a few straightforward consequences of conditions (i),(ii), and (iii) which we will need later on, so we state these now. Firstly, from (i) it is clear that $|a_n|$ cannot grow too fast. For instance, note that (i) implies, say, $a_n = O(n^2)$. Secondly, since the Γ function has no poles in the right half-plane, (ii) and (iii) imply that ξ^F is an entire function. Thirdly, more than one of the coefficients a_n must be nonzero because otherwise the functional equation (iii) could not hold. One way to verify this is by noting that if the Dirichlet series F had only a single term then F would have no zeros which is impossible because F satisfies a Riemann-von Mangoldt formula as a consequence of the functional equation. Lastly, for any vertical strip V in the complex plane, if $s \in V$ and s is bounded away from 1, then a bound $F(s) = O\big((1 + |\operatorname{Im} s|)^A\big)$ holds for some $A \geq 0$ depending on V. This follows by applying the Phragmén-Lindelöf principle in the same way one does to get convexity bounds for $\zeta(s)$ in vertical strips.

We will now state a lemma about the magnitude of $\gamma(s)$ in certain regions, which we will need later. The proof, which is an application of Stirling's formula, will be given in Section 2.5.

Lemma 2.3. *Let $\gamma(s)$ be one of the functions described in condition (iii) of the extended Selberg class definition. Let $D > 0$ and $0 \leq \theta < 1$ and let s be a complex number which is at least unit distance away from the poles and zeros of γ, and which satisfies $|\operatorname{Re} s| \leq D|\operatorname{Im} s|^\theta$. There exist $K, K' > 0$ (depending on γ, D, and θ) such that*

$$\exp(-K'|\operatorname{Im} s|) \leq |\gamma(s)| \leq \exp(-K|\operatorname{Im} s|).$$

One immediate consequence of this lemma (and the Phragmén-Lindelöf bound on F) is that $\xi^F(s)$ decays exponentially as $\operatorname{Im}(s) \to \pm\infty$ in any fixed vertical strip. This means that for any $F \in \mathcal{S}^\sharp$, we can perform the same Fourier analytic setup that we did for the Riemann xi function in the introduction. We define

$$\Phi_F(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^F\left(\frac{1 + ix}{2}\right) e^{-ixu}\, dx \tag{2.5}$$

to be the Fourier transform of ξ^F on the critical line, and likewise we define the family of entire functions $\{\xi_t^F\}_{t \in \mathbb{R}}$ so that

$$\xi_t^F\left(\frac{1 + iz}{2}\right) := \int_{-\infty}^{\infty} e^{tu^2} \Phi_F(u) e^{izu}\, du. \tag{2.6}$$

In the proof of Theorem 2.1, we will show that Φ_F has rapid enough decay for this definition to make sense for all $t \in \mathbb{R}$, and that there exists a real number Λ_F for which ξ_t^F has all of its zeros on the critical line if and only if $t \geq \Lambda_F$.

It is important to note that, strictly speaking, for a given Dirichlet series $F \in \mathcal{S}^\sharp$ the choice of γ is not unique. For example, α can be scaled by any nonzero real number and the functional equation (2.4) will still hold. However, it turns out that other than the choice of scaling, γ is unique (see [6, Thm. 2.1] which is stated for $\mathcal{S}$ but whose proof only requires the properties we gave for $\mathcal{S}^\sharp$). Since scaling does not affect the locations of the zeros of ξ_t^F, there will be no ambiguity in our definition of Λ_F.

From now on we shall assume that F has been fixed, and that the parameters α, Q, ω_i, and μ_i have all been chosen as well. All implied constants in error terms will be allowed to depend on these parameters.

In order to prove Theorem 2.1, we will rely on the following theorem of de Bruijn extending a previous theorem of Pólya:

Theorem 2.4 (De Bruijn [11, Thm. 13], cf. Pólya [20, Thm. 1])**.** *Let $\phi\colon \mathbb{R} \to \mathbb{C}$ be an integrable function satisfying $\phi(u) = \overline{\phi(-u)}$ and $\phi(u) = O\left(e^{-|u|^b}\right)$ for some $b > 2$, and let $G(z) := \int_{-\infty}^{\infty} \phi(u) e^{izu}\, du$. For any $t > 0$, let $G_t(z) := \int_{-\infty}^{\infty} e^{tu^2} \phi(u) e^{izu}\, du$. If all the roots of G lie in the strip $|\operatorname{Im} z| \leq \Delta$, then all the roots of G_t lie in the strip*

$$|\operatorname{Im} z| \leq \left(\max(\Delta^2 - 2t, 0)\right)^{\frac{1}{2}}.$$

Proof of Theorem 2.1. In order to apply Theorem 2.4, it is convenient to define a rotated version of ξ^F. Let

$$H(z) := \xi^F\left(\frac{1+iz}{2}\right),$$

so that values of ξ^F on the critical line correspond to values of H on the real line. In this rotated frame of reference, we would like to apply Theorem 2.4 with H corresponding to the function G in the statement of the theorem.

The functional equation (2.4) implies that $H(x) = \overline{H(x)}$ for any real number x, so H is real valued as a function on the real line. Also, $H(x)$ decays exponentially as $x \to \pm\infty$ because of the presence of the Γ factors in γ, so H is a Schwartz function and its Fourier transform is Φ_F by (2.5). Fourier inversion gives,

$$H(x) = \int_{-\infty}^{\infty} \Phi_F(u) e^{ixu}\, du.$$

Since $H(x)$ is real valued for all $x \in \mathbb{R}$, its Fourier transform Φ_F has the conjugate symmetry property $\Phi_F(u) = \overline{\Phi_F(-u)}$. Hence, in order to apply Theorem 2.4 with $\phi := \Phi_F$ it will suffice to verify that Φ_F has the necessary decay as $u \to +\infty$. The decay as $u \to -\infty$ then follows immediately by the symmetry of Φ_F.

To bound Φ_F we begin by performing the substitution $w := \frac{1+ix}{2}$ to rewrite (2.5) as a complex contour integral,

$$\Phi_F(u) = \frac{e^u}{\pi i} \int_{\frac{1}{2}-\infty i}^{\frac{1}{2}+\infty i} \xi^F(w) e^{-2uw}\, dw.$$

After expanding out the definition of ξ^F in the integrand, we wish to interchange the infinite sum coming from the Dirichlet series F with the integral. To justify this, one can first shift the vertical line contour to the right into the half-plane of absolute convergence of F (which can be done because ξ^F decays uniformly exponentially in any vertical strip), and on this new contour the interchange of sum and integral is valid. Hence,

$$\Phi_F(u) = \frac{\alpha e^u}{\pi i} \sum_{n=1}^{\infty} a_n \int_{2-\infty i}^{2+\infty i} w^m (w-1)^m \prod_{j=1}^{k} \Gamma(\omega_j w + \mu_j) \left(\frac{n e^{2u}}{Q}\right)^{-w} dw. \tag{2.7}$$

It is now helpful to view the above integral as an inverse Mellin transform. Letting $\Psi(w) := w^m(w-1)^m \prod_{j=1}^k \Gamma(\omega_j w+\mu_j)$, define $\psi\colon (0,\infty) \to \mathbb{C}$ to be the inverse Mellin transform,

$$\psi(v) := \frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \Psi(w) v^{-w}\, dw.$$

Then (2.7) can be rewritten as

$$\Phi_F(u) = 2\alpha e^u \sum_{n=1}^{\infty} a_n \psi\left(\frac{ne^{2u}}{Q}\right). \tag{2.8}$$

We can now bound ψ using the following lemma whose proof can be found in Section 2.5.

Lemma 2.5. *Let*

$$h(x) := \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} \prod_{j=1}^{k} \Gamma(a_j w + b_j) x^{-w}\, dw$$

for some $a_j > 0$ and $b_j \in \mathbb{C}$ with $\operatorname{Re} b_j \geq 0$. Then there exists a $\delta > 0$ (depending on the a_j and b_j values) such that $h(x) \ll e^{-x^\delta}$ for all $x \geq 1$.

The function ψ is not quite of the same form as h in the lemma, but by expanding out the polynomial factors in the definition of Ψ and then repeatedly applying the relation

$$w\Gamma(aw+b) = \frac{1}{a}\Gamma(aw+b+1) - \frac{b}{a}\Gamma(aw+b).$$

one can write ψ as a linear combination functions to which the lemma does apply. Hence, we can conclude that there exists a $\delta > 0$ be such that $\psi(v) \ll e^{-v^\delta}$ for all $v \geq 1$. Applying this bound in (2.8) we see that for all sufficiently large u,

$$\begin{aligned}\Phi_F(u) &\ll e^u \sum_{n=1}^{\infty} a_n e^{-\left(n^\delta e^{2\delta u}/Q^\delta\right)} \\ &\ll e^u \sum_{n=1}^{\infty} e^{-n^\delta e^{2\delta u}/\left(2Q^\delta\right)}\end{aligned}$$

where the second inequality follows from the fact that $a_n = O(n^2) = e^{n^{o(1)}}$.

Note that for all $c \in [0, 1/2]$, say, we have a bound $\sum_{n=1}^{\infty} c^{n^\delta} \ll_\delta c$ which we can apply to the sum above to get

$$\Phi_F(u) \ll e^u e^{-e^{2\delta u}/(2Q^\delta)} \ll e^{-e^{2\delta u}/(3Q^\delta)}$$

for all u large enough. Hence, Φ_F has rapid enough decay that Theorem 2.4 applies.

We are now in a position to define the de Bruijn-Newman constant Λ_F associated to F. For any $t \in \mathbb{R}$, let

$$H_t(z) := \int_{-\infty}^{\infty} e^{tu^2} \Phi_F(u) e^{ixu}\, du$$

be the functions which are relevant to the conclusions of Theorem 2.4, and let

$$\mathcal{Z} := \{t \in \mathbb{R} \colon \text{ all the roots of } H_t \text{ are real}\}.$$

The functions ξ_t^F defined in (2.6) are simply the un-rotated versions of H_t and so $\mathcal{Z}$ could just as well be defined as the set of t for which all the zeros of ξ_t^F lie on the critical line.

Note that $\mathcal{Z}$ is closed because the roots of H_t vary continuously in t by Hurwitz's theorem. Furthermore, if $t_0 \in \mathcal{Z}$, then by applying Theorem 2.4 with $\phi(u) := e^{t_0 u^2} \Phi_F(u)$, we may conclude $t \in \mathcal{Z}$ for all $t \geq t_0$. Hence, the only possible choices for $\mathcal{Z}$ are the empty set, all of $\mathbb{R}$, or a half line $\{t \geq \Lambda_F\}$ where Λ_F is some real number.

To see that $\mathcal{Z}$ is nonempty, Theorem 2.4 shows (by choosing $\phi := \Phi_F$) that it suffices to check that H has all of its zeros lying in some horizontal strip, or equivalently that ξ^F has all of its zeros in some vertical strip. It is a general fact that any convergent Dirichlet series F has a zero-free half plane because if s has large enough real part, then the first term of $F(s)$ will strictly dominate the sum of all the other terms. This implies ξ^F also has a zero-free half plane, and so by the functional equation for ξ^F all the zeros of ξ^F must lie in a vertical strip.

The fact that $\mathcal{Z} \neq \mathbb{R}$ will be a consequence of the generalized Newman's conjecture which we prove in the next section. Once this is proved it follows that there is a generalized de Bruijn-Newman constant Λ_F associated to F for which $\mathcal{Z} = \{t \geq \Lambda_F\}$. The constant Λ_F is the unique real number for which ξ_t^F has all of its zeros on the critical line if and only if $t \geq \Lambda_F$. $\square$

2.3 Proof of the generalized Newman's Conjecture

To prove $\Lambda_F \geq 0$, we will take a direct approach by showing that for every $t < 0$ the function ξ_t^F has zeros off the critical line. As a starting point, we first rewrite our defintion of ξ_t^F for all $t < 0$ in terms of a new integral which is essentially a convolution of ξ^F with a Gaussian whose variance is proportional to $|t|$.

Inserting the definition (2.5) of Φ_F into the definition (2.6) of ξ_t^F and then applying Fubini's theorem (which can be justified for any $t < 0$) gives

$$\begin{aligned}\xi_t^F\left(\frac{1+iz}{2}\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi^F\left(\frac{1+ix}{2}\right)\int_{-\infty}^{\infty} e^{tu^2} e^{i(z-x)u}\,du\,dx \\ &= \frac{1}{2\sqrt{\pi|t|}}\int_{-\infty}^{\infty} \xi^F\left(\frac{1+ix}{2}\right) e^{\frac{1}{4t}(z-x)^2}\,dx.\end{aligned}$$

Substituting $s \coloneqq \frac{1+iz}{2}$ and $w \coloneqq \frac{1+ix}{2}$ this becomes

$$\xi_t^F(s) = \frac{1}{i\sqrt{\pi|t|}}\int_{\frac{1}{2}-\infty i}^{\frac{1}{2}+\infty i} \xi^F(w) e^{\frac{1}{|t|}(s-w)^2}\,dw.$$

Because it will be convenient later on, we will shift the contour in the above integral to the vertical line $\operatorname{Re} w = 2$ where the Dirichlet series for F converges absolutely. Since $\xi^F(s)$ has no poles and decays exponentially in any vertical strip, this shift will not affect the value of the integral. Hence,

$$\xi_t^F(s) = \frac{1}{i\sqrt{\pi|t|}} \int_{2-\infty i}^{2+\infty i} \xi^F(w) e^{\frac{1}{|t|}(s-w)^2}\, dw. \tag{2.9}$$

We will make use of this new expression for ξ_t^F in the proof of the next theorem, which will be our main tool for proving the generalized Newman's conjecture.

Theorem 2.6. *Let $s = x + iy$ with $x \in \mathbb{R}$, $y > 0$, and let $t < 0$. Suppose $|t| \leq C$ and $|x| \leq Cy^{1/4}$ for some positive constant C. Then for all y sufficiently large (depending on C) the following estimate holds,*

$$\xi_t^F(J_t(s)) = \gamma_t(s)\left(F_t(s) + O\left(y^{-1/5}\exp\left(\frac{10}{|t|}\min(x,-2)^2\right)\right)\right) \tag{2.10}$$

where

$$F_t(s) := \sum_{n=1}^{\infty} \exp\left(-\frac{|t|}{4}\log^2 n\right)\frac{a_n}{n^s}, \tag{2.11}$$

$$J_t(s) := s + \frac{|t|}{2}\log Q + \frac{|t|}{2}\sum_{i=1}^{k} \omega_i \operatorname{Log}(\omega_i s), \tag{2.12}$$

$$\gamma_t(s) := \gamma(s)\exp\left(\frac{1}{|t|}(s - J_t(s))^2\right) \tag{2.13}$$

and where the implicit constant in the error may depend on C.

One should think of Theorem 2.6 as showing that ξ_t^F is analogous to ξ^F in that it can be expressed (approximately) as the product of a Dirichlet series and some special additional factors. We stress that the error term in (2.10) is not optimal, but for our application to Newman's conjecture, we only require the following qualitative version of the theorem: if $t < 0$, V is some vertical strip, and $s = x + iy \in V$ with y sufficiently large, then

$$\xi_t^F(J_t(s)) = \gamma_t(s)(F_t(s) + o_{y\to\infty}(1)) \tag{2.14}$$

where the decay of the error term is uniform in x (but not necessarily in t).

The appearance of the Dirichlet series F_t in the theorem can be explained heuristically from (2.9) as follows. Suppose one could pull the γ factor of $\xi^F(w) = \gamma(w)F(w)$ outside of the integral in (2.9), leaving an integral of the form

$$\frac{1}{i\sqrt{\pi|t|}} \int_{2-\infty i}^{2+\infty i} F(w) e^{\frac{1}{|t|}(s-w)^2}\, dw.$$

Interchanging the sum and the integral and then computing the result gives,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \frac{1}{i\sqrt{\pi|t|}} \int_{2-\infty i}^{2+\infty i} n^{-w} e^{\frac{1}{|t|}(s-w)^2}\, dw &= \sum_{n=1}^{\infty} a_n \exp\left(-\frac{|t|}{4}\log^2 n\right) n^{-s} \\ &= F_t(s).\end{aligned}$$

Further explanation of Theorem 2.6 including an explanation for the presence of the J_t function and a full proof of theorem will be left for Section 2.4. We will now show how this theorem implies the generalized Newman's conjecture without much additional work.

2.3.1 Deducing Theorem 2.2 from Theorem 2.6

Assume that $t < 0$ is fixed. Theorem 2.6 suggests that if F_t has a zero at s_0, then ξ_t^F should have a zero near $J_t(s_0)$. In order to make this rigorous, we must somehow deal with the fact that there are error terms present in the correspondence between F_t and ξ_t^F given in the theorem.

A crucial feature of F_t that we will use is that it is everywhere absolutely convergent. To see this, recall that

$$F_t(s) \coloneqq \sum_{n=1}^{\infty} \exp\left(-\frac{|t|}{4}\log^2 n\right)\frac{a_n}{n^s},$$

and the factor $\exp\left(-\frac{|t|}{4}\log^2 n\right)$ decays faster than $O(n^{-d})$ for any d whereas $a_n = O(n^2)$.

Because F_t is everywhere absolutely convergent it is entire, and it has a property known as *almost periodicity* which was introduced and studied extensively by Bohr [3]. Roughly speaking, almost periodicity for F_t means that for any vertical strip there is an ample supply of shifts $\tau \in \mathbb{R}$ for which $F_t(s)$ and $F(s+i\tau)$ are uniformly close.

A precise version of one of Bohr's results can be stated as follows,

Theorem 2.7 ([3, Thm. 1])**.** *Let $G(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be a Dirichlet series which is absolutely convergent for all $\operatorname{Re} s > \sigma$. For any $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{R}$ such that $\sigma < \alpha < \beta$, there exists a sequence of shifts*

$$0 < \tau_1 < \tau_2 < \cdots$$

satisfying

$$\liminf_{m\to\infty}(\tau_{m+1} - \tau_m) > 0 \quad \textit{and} \quad \limsup_{m\to\infty}\frac{\tau_m}{m} < \infty$$

such that

$$|G(s) - G(s + i\tau_m)| < \varepsilon$$

for all $\alpha \le \operatorname{Re} s \le \beta$.

Because F_t is everywhere absolutely convergent, the above theorem holds for F_t for any choice of $\alpha < \beta$.

In order to proceed with the proof, we suppose for the moment that one is able to locate a single zero s_0 of F_t. Let C be a circle centered at s_0 of radius r chosen so that C does not pass through any other zeros of F_t, and let

$$\delta := \min_{s \in C} |F_t(s)| > 0.$$

Let V denote the vertical strip $\{|\operatorname{Re} s - \operatorname{Re} s_0| \leq r\}$ and recall that the qualitative form of Theorem 2.6 states

$$\xi_t^F(J_t(s)) = \gamma_t(s)(F_t(s) + o_{y\to\infty}(1)) \tag{2.15}$$

for $s = x + iy \in V$, y sufficiently large, where the decay of the error term is uniform in x.

Let

$$h(s) := \xi_t^F(J_t(s))/\gamma_t(s),$$

and note that h is analytic in the upper half plane, and

$$h(s) = F_t(s) + o_{y\to\infty}(1) \tag{2.16}$$

for $s = x + iy \in V$, $y > 0$ by (2.15).

Applying Theorem 2.7 to F_t on the strip V and taking $\varepsilon := \delta/3$, let $0 < \tau_1 < \tau_2 < \ldots$ be the resulting sequence of shifts.

By picking τ_m which is sufficiently large, one can ensure that for any s on the circle C the shift $s + i\tau_m$ will have a large enough imaginary part that the error term in (2.16) when evaluating $h(s + i\tau_m)$ is uniformly less than $\delta/3$. Consequently, for all $s \in C$ we have

$$|h(s + i\tau_m) - F_t(s)| \leq |F_t(s + i\tau_m) - F_t(s)| + \frac{\delta}{3} \leq \frac{2\delta}{3} < |F_t(s)|,$$

so by Rouché's theorem h has a zero inside the shifted circle $C + i\tau_m$.

By taking larger and larger shifts τ_m, the argument above yields an infinite collection of zeros of h which are arbitrarily high up the strip V. From the definition of h and the fact that poles of γ_t only exist up to some bounded height, we can deduce that $\xi_t^F(J_t(s))$ also has an infinite collection of zeros and these zeros exist arbitrarily high up the strip V. Hence, ξ_t^F has infinitely many zeros within the curved region $J_t(V)$ at arbitrarily high heights. Past a certain height $J_t(V)$ lies completely to the right of the critical line so in particular we have shown that ξ_t^F has zeros off the critical line.

This completes the deduction of the generalized Newman's conjecture from Theorem 2.6 and the assumption that F_t has at least one zero. We will assert this latter fact as a lemma for now, and the proof will be given in Section 2.5.

Lemma 2.8. *For any $t < 0$, F_t has a zero.*

2.4 Proof of Theorem 2.6

We are interested in estimating $\xi_t^F(J_t(s))$, so by (2.9) we know

$$\xi_t^F(J_t(s)) = \frac{1}{i\sqrt{\pi|t|}} \int_{2-\infty i}^{2+\infty i} \xi^F(z) e^{\frac{1}{|t|}(J_t(s)-z)^2} \, dz$$

for all $t < 0$. By inserting the definition $\xi^F(z) = \gamma(z) \sum_{n=1}^{\infty} a_n n^{-z}$ and then interchanging the sum and integral (which can be justified since the contour is in the half-plane of absolute convergence of the Dirichlet series F) we get

$$\xi_t^F(J_t(s)) = \sum_{n=1}^{\infty} a_n B_{t,n}(s)$$

where

$$B_{t,n}(s) := \frac{1}{i\sqrt{\pi|t|}} \int_{2-\infty i}^{2+\infty i} \gamma(z) e^{\frac{1}{|t|}(J_t(s)-z)^2} n^{-z}\, dz. \tag{2.17}$$

Our goal is to estimate $B_{t,n}(s)$ in the regime of s and t values we are interested in. We will prove the following lemma:

Lemma 2.9. *Let t, $s = x + iy$, and C satisfy the same restrictions as in Theorem 2.6. Then for all y sufficiently large (depending on C), the following estimates for $B_{t,n}(s)$ hold for small, medium, and large n respectively (where the implicit constants may depend on C):*

(i) For $1 \le n \le \exp(y^{1/3}/|t|)$,

$$B_{t,n}(s) = \gamma_t(s) \exp\left(-\frac{|t|}{4} \log^2 n\right) n^{-s} (1 + O(y^{-1/5})).$$

(ii) For $1 \le n \le \exp(y^{3/5}/|t|)$,

$$B_{t,n}(s) = O\left(|\gamma_t(s)| \exp\left(-\frac{|t|}{8} \log^2 n\right) n^{-x}\right).$$

(iii) For $n > \exp(y^{3/5}/|t|)$,

$$B_{t,n}(s) = O\left(\exp\left(-\frac{|t|}{10} \log^2 n\right)\right).$$

Before giving the proof we will summarize our method. Looking at the integral in (2.17), one may note that the integrand exhibits very rapid decay along the vertical line contour. Hence, a reasonable approach to estimate $B_{t,n}(s)$ is to split the contour into a finite segment where the mass of the integrand is concentrated, and a tail whose size can be crudely bounded. It then remains to estimate the integral over the finite segment. Unfortunately, this problem is still nontrivial because the integrand can be quite oscillatory on this segment. One way to proceed is to use the method of stationary phase to estimate the resulting oscillatory integral (e.g. see [31, Ch. 4] for useful lemmas in this direction). We will instead use the complex analytic analogue of this method known as the method of steepest descent. The basic principle is to start by shifting the contour so that the mass of the integrand is concentrated somewhere where the integrand is not so oscillatory. Upon doing so, one may then estimate the resulting integral and get good error terms without needing to be too sophisticated about handling cancellation.

We are now in a position to describe the significance of the $J_t(s)$ function. This function has been defined such that when performing the contour shift to estimate the value of $\xi_t^F(J_t(s))$, the region where the integrand oscillates negligibly is near the point s (although the precise shift will depend on the value of n as well).

We now give a heuristic argument for Lemma 2.9 which will also serve as a sketch to be made rigorous. We will skip over any details about contour shifting for now.

The first step is to locally approximate the integrand of $B_{t,n}(s)$ near s because it turns out that this is approximately where the dominant contribution is. We will see in an upcoming lemma that for all z suitably close to s one has

$$\gamma(z) \approx \gamma(s) \exp\left(\left(\log Q + \sum_{i=1}^{k} \omega_i \operatorname{Log}(\omega_i s) \right)(z-s) + \frac{\sum_{i=1}^{k} \omega_i}{2s}(z-s)^2 \right) \tag{2.18}$$

where this is essentially coming from a Taylor expansion of $\operatorname{Log}\gamma(z)$ around s. A similar Taylor expansion of the Gaussian factor in the integrand of $B_{t,n}(s)$ yields

$$e^{\frac{1}{|t|}(J_t(s)-z)^2} = e^{\frac{1}{|t|}(J_t(s)-s)^2} \exp\left(\frac{2}{|t|}(s-J_t(s))(z-s) + \frac{1}{|t|}(z-s)^2 \right) \tag{2.19}$$

where in this case the Taylor expansion is exact because the phase is simply a quadratic polynomial.

Taking the product of (2.18) and (2.19), the constant term in the resulting Taylor expansion is

$$\gamma(s)e^{\frac{1}{|t|}(J_t(s)-s)^2} = \gamma_t(s),$$

and the coefficient of the $(z-s)$ term is

$$\log Q + \sum_{i=1}^{k} \omega_i \operatorname{Log}(\omega_i s) + \frac{2}{|t|}(s - J_t(s)) = 0$$

where this equality holds because we have chosen the definition of $J_t(s)$ to make it so.

Putting this information together, we get the heuristic approximation

$$B_{t,n}(s) \approx \gamma_t(s)\frac{1}{i\sqrt{\pi|t|}}\int_{2-\infty i}^{2+\infty i} \exp\bigl(A(z-s)^2\bigr)n^{-z}\,dz$$

where $A := \frac{1}{|t|} + \frac{\sum_{i=1}^{k}\omega_i}{2s}$.

The remaining integral can now be computed directly. By completing the square,

$$\begin{aligned}\exp\bigl(A(z-s)^2\bigr)n^{-z} = \exp&\left(-\frac{1}{4A}\log^2 n\right)n^{-s}\\ &\times \exp\left(A\left(z-\left(s+\frac{1}{2A}\log n\right)\right)^2\right)\end{aligned}$$

which means

$$\begin{aligned}B_{t,n}(s) \approx \gamma_t(s)&\exp\left(-\frac{1}{4A}\log^2 n\right)n^{-s}\\ &\times \frac{1}{i\sqrt{\pi|t|}}\int_{2-\infty i}^{2+\infty i}\exp\left(A(z-(s+\frac{1}{2A}\log n))^2\right)dz. \quad (2.20)\end{aligned}$$

The value of this last integral is just $i\sqrt{\pi/A}$. Inserting this and then removing all instances of A in the resulting expression by using the approximate equality $A \approx \frac{1}{|t|}$ gives the desired main term in (i).

Before giving a rigorous version of the heuristic calculations above, we first list several lemmas that we will need. The proofs of these lemmas will be given in Section 2.5. The first is a rigorous version of (2.18):

Lemma 2.10. *For any $\varepsilon > 0$, define the region $S_\varepsilon := \{|\operatorname{Arg} w| < \pi - \varepsilon, |w| > \varepsilon\}$. Let $z, z_0 \in S_\varepsilon$ and $|z - z_0| \leq D|z_0|^{2/3}$ for some $D > 0$. Then for all z, z_0 that are at least unit distance away from the poles and zeros of γ, we have the estimate*

$$\gamma(z) = \gamma(z_0) \exp\Bigg(\Bigg(\log Q + \sum_{j=1}^{k} \omega_j \operatorname{Log}(\omega_j z_0)\Bigg)(z - z_0) + \frac{\sum_{j=1}^{k} \omega_j}{2z_0}(z - z_0)^2\Bigg) \times \left(1 + O\left(\frac{1 + |z - z_0|}{|z_0|} + \frac{|z - z_0|^3}{|z_0|^2}\right)\right)$$

where the implicit constant may depend on ε and D.

The above lemma can be used to approximate the integrand of $B_{t,n}(s)$ when z is relatively close to s. It will also be useful to have an upper bound on the integrand when z is far away from s.

Lemma 2.11. *Let t, $s = x + iy$, and C satisfy the same hypotheses as in Theorem 2.6. Let $n \leq \exp(y^{3/5}/|t|)$ (i.e. the small/medium case of Lemma 2.9) and let*

$$\mathcal{I}(z) := \gamma(z) e^{\frac{1}{|t|}(J_t(s) - z)^2} n^{-z}$$

be the integrand of (2.17). Then for all y sufficiently large (depending on C) the following estimates hold (where the implicit constants may also depend on C):

(i) If $|x - \operatorname{Re} z| \leq y^{3/5}$ and $\frac{1}{2}y^{2/3} \leq |y - \operatorname{Im} z| \leq 2y^{2/3}$ then

$$\mathcal{I}(z) \ll \exp\left(-\frac{y^{4/3}}{10|t|}\right).$$

(ii) If $\operatorname{Re} z = 2$ *and* $|y - \operatorname{Im} z| \geq y^{2/3}$ *then*

$$\mathcal{I}(z) \ll \exp\left(-\frac{1}{2|t|}(y - \operatorname{Im} z)^2\right).$$

We will also need a weak bound on γ at some point,

Lemma 2.12. *There is some* $K > 0$ *such that* $|\gamma(z)| \leq \exp(K(\operatorname{Re} z)^{1.1})$ *uniformly for any* z *with* $\operatorname{Re} z \geq 1$.

Proof of Lemma 2.9. Recall that $s = x + iy$, and we assume that $y > 0$, $|x| \leq Cy^{1/4}$ and $-C < t < 0$. The lemma is only asserted to hold when y is sufficiently large, and throughout the proof we will frequently assume that y is large enough (depending on C) to make various statements hold. We also allow all implicit constants to depend on C.

We start by proving (i) and (ii) using the heuristic calculations given above as framework. These are the cases of small and medium n where $|t| \log n \leq y^{1/3}$ and $|t| \log n \leq y^{3/5}$ respectively. We will not need to distinguish between these two cases until the very end of the proof, so for now we will just assume that the latter bound holds.

Recall that in the heuristic calculations, the final integral in (2.20) was a complex Gaussian centered at the point $s + \frac{1}{2A} \log n$ where $A = \frac{1}{|t|} + \frac{1}{2s} \sum_{j=1}^{k} \omega_j$. We will shift our contour to pass through this point. The shifted contour we select is the piecewise linear contour passing though $2 - \infty i$, $2 + (y - y^{2/3})i$, $s + \frac{1}{2A} \log n - y^{2/3} i$, $s + \frac{1}{2A} \log n + y^{2/3} i$, $2 + (y + y^{2/3})i$, and $2 + \infty i$. Let V_1, H_1, M, H_2, and V_2 denote these linear pieces respectively.

We will make frequent use of the estimate

$$A = \frac{1}{|t|}\left(1 + O\left(\frac{|t|}{y}\right)\right) \asymp \frac{1}{|t|}, \tag{2.21}$$

starting with the fact that this implies $\frac{1}{2A}\log n \ll y^{3/5}$. From this and our assumption $x \ll y^{1/4}$, it follows that the contour shift we are performing occurs completely within a $O(y^{2/3})$ radius ball around s. One consequence of this is that for all large y the contour shift does not pass over any poles of the integrand (which come from the Γ factors) because the imaginary parts of these poles are uniformly bounded above. Hence, by Cauchy's theorem

$$\int_{2-\infty i}^{2+\infty i} = \int_{V_1} + \int_{H_1} + \int_{M} + \int_{H_2} + \int_{V_2}$$

where the M integral will contribute the main term of our estimate, and the other integrals are negligible in comparison. We will apply Lemma 2.10 to estimate the M integral and Lemma 2.11 to bound the other four.

M integral

As noted above, the segment M lies completely within a circle of radius $O(y^{2/3})$ around s, so Lemma 2.10 gives an approximation for $\gamma(z)$ in terms of $\gamma(s)$ for every $z \in M$. Applying the lemma and then manipulating the integrand in the same way that we did in our heuristic calculations (but now carrying along the error terms coming from Lemma 2.10), the integral over M becomes

$$\begin{aligned}&\gamma_t(s)\exp\left(-\frac{1}{4A}\log^2 n\right)n^{-s}\frac{1}{i\sqrt{\pi|t|}}\\ &\times\int_M \exp\left(A(z-(s+\frac{1}{2A}\log n))^2\right)\left(1+O\left(\frac{1+|z-s|}{|s|}+\frac{|z-s|^3}{|s|^2}\right)\right)dz. \quad (2.22)\end{aligned}$$

Parametrizing M as $z(u) = s + \frac{1}{2A}\log n + ui$ for $u = -y^{2/3}$ to $u = y^{2/3}$, and then using the fact that $|z(u) - s| \le |u| + \left|\frac{1}{2A}\log n\right| \ll |u| + y^{3/5}$ and $|s| \asymp y$ to simplify the error terms we deduce that (2.22) equals

$$\gamma_t(s)\exp\left(-\frac{1}{4A}\log^2 n\right)n^{-s}\frac{1}{\sqrt{\pi|t|}}\int_{-y^{2/3}}^{y^{2/3}}\exp\big(-Au^2\big)\left(1+O\left(\frac{1+|u|^3}{y^{1/5}}\right)\right)du. \quad (2.23)$$

The integral of the error term can be bounded via the triangle inequality,

$$\begin{aligned}\int_{-y^{2/3}}^{y^{2/3}}\exp\big(-Au^2\big)O\left(\frac{1+|u|^3}{y^{1/5}}\right)du &\ll y^{-1/5}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2|t|}u^2\right)\big(1+|u|^3\big)\,du\\ &\ll \sqrt{|t|}y^{-1/5},\end{aligned}$$

where we have used the fact that $\operatorname{Re}A \geq \frac{1}{2|t|}$ which comes from (2.21) and taking y sufficiently large.

Estimating the main term now we see that

$$\begin{aligned}\int_{-y^{2/3}}^{y^{2/3}}\exp\big(-Au^2\big)\,du &= \int_{-\infty}^{\infty}\exp\big(-Au^2\big)\,du + O\left(\int_{y^{2/3}}^{\infty}\exp\big(-(\operatorname{Re}A)u^2\big)\,du\right)\\ &= \sqrt{\frac{\pi}{A}} + O\left(|t|\exp\left(-\frac{y^{4/3}}{2|t|}\right)\right),\end{aligned}$$

where the second line follows from the first by using $\operatorname{Re}A \geq \frac{1}{2|t|}$ again.

Inserting these estimates into (2.23) and using $\frac{1}{\sqrt{A}} = \sqrt{|t|}(1+O(y^{-1}))$, which follows from (2.21), we see that the value of the M contour integral is

$$= \gamma_t(s)\exp\left(-\frac{1}{4A}\log^2 n\right)n^{-s}(1+O(y^{-1/5}))$$

which we will see later is the main term of our estimate.

H_1 and H_2 integrals

The H_1 integral is

$$\frac{1}{i\sqrt{\pi|t|}}\int_{H_1}\gamma(z)e^{\frac{1}{|t|}(J_t(s)-z)^2}n^{-z}\,dz,$$

so applying the ML-inequality gives the bound

$$\ll \frac{1}{\sqrt{|t|}}|H_1| \max_{z\in H_1}\left|\gamma(z)e^{\frac{1}{|t|}(J_t(s)-z)^2}n^{-z}\right| \tag{2.24}$$

where $|H_1|$ denotes the length of the contour.

Recall that H_1 is the line segment from $2+(y-y^{2/3})i$ to $s+\frac{1}{2A}\log n - y^{2/3}i$, and since $\left|\frac{1}{2A}\log n\right| \leq y^{3/5} \leq y^{2/3}/2$ for all sufficiently large y, the hypotheses of Lemma 2.11(i) are satisfied, so

$$\max_{z\in H_1}\left|\gamma(z)e^{\frac{1}{|t|}(J_t(s)-z)^2}n^{-z}\right| \ll \exp\left(-\frac{y^{4/3}}{10|t|}\right).$$

Also,

$$|H_1| = \left|x+\frac{1}{2A}\log n - 2\right| \ll y^{1/4}+y^{3/5}+2 \ll y,$$

so, we conclude that the integral over H_1 is bounded by

$$\ll \frac{y}{\sqrt{|t|}}\exp\left(-\frac{y^{4/3}}{10|t|}\right) \ll \exp\left(-\frac{y^{4/3}}{20|t|}\right)$$

since $|t| \leq C$ and y is large.

For the H_2 integral, an identical bound holds and the derivation is the same.

V_1 and V_2 integrals

First consider the integral over V_1. Applying the triangle inequality and Lemma 2.11(ii) we get

$$\begin{aligned}\frac{1}{i\sqrt{\pi|t|}}\int_{V_1}\gamma(z)\exp\biggl(\frac{1}{|t|}(J_t(s)-z)^2\biggr)n^{-z}\,dz&\\ &\ll \frac{1}{\sqrt{|t|}}\int_{V_1}\exp\biggl(\frac{1}{2|t|}(y-\operatorname{Im} z)^2\biggr)\,|dz|\\ &=\frac{1}{\sqrt{|t|}}\int_{-\infty}^{-y^{2/3}}\exp\biggl(-\frac{u^2}{2|t|}\biggr)\,du\\ &\ll \exp\biggl(-\frac{y^{4/3}}{20|t|}\biggr).\end{aligned}$$

A matching argument gives the same bound for the V_2 integral.

Conclusion of (i) and (ii)

Adding together our estimates for all five integrals we get

$$B_{t,n}(s)=\gamma_t(s)\exp\biggl(-\frac{1}{4A}\log^2 n\biggr)n^{-s}(1+O(y^{-1/5}))+O\biggl(\exp\biggl(-\frac{y^{4/3}}{20|t|}\biggr)\biggr). \tag{2.25}$$

In order to absorb the additive error term at the end into the multiplicative error term it suffices to show the following lower bound on the size of the main term:

$$\biggl|\gamma_t(s)\exp\biggl(-\frac{1}{4A}\log^2 n\biggr)n^{-s}\biggr|\gg\exp\biggl(-\frac{2y^{6/5}}{|t|}\biggr). \tag{2.26}$$

(This is sufficient because the exponent 6/5 is less than 4/3.)

To derive this lower bound, we will bound each of the factors individually. For the first factor, recall that $\gamma_t(s) := \gamma(s)\exp\Bigl(\frac{1}{|t|}(s-J_t(s))^2\Bigr)$. Since $x \ll y^{1/4}$, we can apply Lemma 2.3 to get

$$|\gamma(s)|\geq\exp(-K'y)$$

for some $K' > 0$, and furthermore

$$\begin{aligned}\left|\exp\left(\frac{1}{|t|}(s - J_t(s))^2\right)\right| &= \left|\exp\left(\frac{1}{|t|}\operatorname{Re}\big((s - J_t(s))^2\big)\right)\right| \\ &\geq \left|\exp\left(-\frac{1}{|t|}(\operatorname{Im}(s - J_t(s)))^2\right)\right| \\ &\geq \exp(-O(1))\end{aligned}$$

where the last inequality comes from the fact that $\operatorname{Im}(s - J_t(s)) = O(|t|)$ by definition (2.12) of $J_t(s)$. Combining these lower bounds gives

$$|\gamma_t(s)| \gg \exp(-K'y) \tag{2.27}$$

for all sufficiently large y.

To lower bound the other two factors of (2.26), note $\operatorname{Re}\big(\frac{1}{A}\big) \leq 2|t|$ for all large y by (2.21), and $\log n \leq y^{3/5}/|t|$ by assumption, so

$$\left|\exp\left(-\frac{1}{4A}\log^2 n\right)\right| \geq \exp\left(-\frac{|t|}{2}\log^2 n\right) \geq \exp\left(-\frac{y^{6/5}}{2|t|}\right)$$

and also

$$|n^{-s}| = \exp(-x\log n) \geq \exp\left(-\frac{y^{6/5}}{2|t|}\right)$$

since $x \ll y^{1/4}$.

Hence, (2.26) is true, so we deduce that

$$B_{t,n}(s) = \gamma_t(s)\exp\left(-\frac{1}{4A}\log^2 n\right)n^{-s}(1 + O(y^{-1/5})). \tag{2.28}$$

From this estimate we shall now deduce (i) and (ii) of the lemma. By taking absolute values in (2.28) and applying $\operatorname{Re}\big(\frac{1}{A}\big) \geq \frac{|t|}{2}$, which follows from (2.21), we deduce (ii) immediately. To get (i), note that since $\log n \leq y^{1/3}/|t|$ in this case

$$\begin{aligned}-\frac{1}{4A}\log^2 n &= -\frac{|t|}{4}\left(1 + O\left(\frac{|t|}{y}\right)\right)\log^2 n \\ &= -\frac{|t|}{4}\log^2 n + O(y^{-1/3}),\end{aligned}$$

where we have used (2.21) once again in the first line. Inserting this estimate into (2.28) and using $\exp(O(y^{-1/3})) = 1 + O(y^{-1/3})$ gives (i).

Proof of (iii)

Recall that the integral defining $B_{t,n}(s)$ is

$$B_{t,n}(s) = \frac{1}{i\sqrt{\pi|t|}} \int_{2-\infty i}^{2+\infty i} \gamma(z) e^{\frac{1}{|t|}(J_t(s)-z)^2} n^{-z}\, dz.$$

We now work under the assumption that n is large enough that $|t|\log n > y^{3/5}$.

We can rewrite the exponentials in the integrand by completing the square

$$e^{\frac{1}{|t|}(J_t(s)-z)^2} n^{-z} = \exp\left(-\left(J_t(s) + \frac{|t|}{4}\log n\right)\log n\right) e^{\frac{1}{|t|}\left(z-\left(J_t(s)+\frac{|t|}{2}\log n\right)\right)^2}, \tag{2.29}$$

so if we choose to integrate over the vertical line $\operatorname{Re} z = \operatorname{Re}\left(J_t(s) + \frac{|t|}{2}\log n\right)$, we see that there would be no oscillation coming these exponentials. Let L denote this vertical line, and we now claim that L lies in the right half plane. Indeed, because $\operatorname{Re} J_t(s) = x + O(\log y) = O(y^{1/4})$, and $|t|\log n > y^{3/5}$, we see that

$$|\operatorname{Re} J_t(s)| \leq \frac{|t|}{10}\log n \tag{2.30}$$

for y large enough, and so in particular $\operatorname{Re}\left(J_t(s) + \frac{|t|}{2}\log n\right) > 0$.

In the right half plane the integrand has no poles and decays rapidly in any vertical strip, so by Cauchy's theorem

$$B_{t,n}(s) = \frac{1}{i\sqrt{\pi|t|}} \int_L \gamma(z) e^{\frac{1}{|t|}(J_t(s)-z)^2} n^{-z}\, dz. \tag{2.31}$$

For all $z \in L$, note that (2.30) implies $\operatorname{Re} z \asymp |t|\log n$ so we can apply Lemma 2.12 to bound $\gamma(z)$, which gives $\gamma(z) \ll \exp(K''|t|\log^{1.1} n)$ for some $K'' > 0$. Applying this bound and (2.29) to (2.31), we can bound $B_{t,n}(s)$ by,

$$\begin{aligned}
&\ll \frac{\exp(K''|t|\log^{1.1} n)}{\sqrt{|t|}} \int_L \left| e^{\frac{1}{|t|}(J_t(s)-z)^2} n^{-z} \right| |dz| \\
&= \exp\bigg(-\operatorname{Re}\bigg(J_t(s) + \frac{|t|}{4}\log n\bigg)\log n\bigg) \frac{\exp(K''|t|\log^{1.1} n)}{\sqrt{|t|}} \int_{-\infty}^{\infty} e^{-u^2/|t|}\, du \\
&\ll \exp\bigg(-\operatorname{Re}\bigg(J_t(s) + \frac{|t|}{4}\log n\bigg)\log n\bigg) \exp(K''|t|\log^{1.1} n).
\end{aligned}$$

From (2.30) it follows that $\operatorname{Re}\Big(J_t(s) + \frac{|t|}{4}\log n\Big) \geq \frac{|t|}{8}\log n$, so

$$B_{t,n}(s) \ll \exp\bigg(-\frac{|t|}{8}\log^2 n\bigg)\exp(K'|t|\log^{1.1} n) \ll \exp\bigg(-\frac{|t|}{10}\log^2 n\bigg)$$

for all y sufficiently large (because this implies $\log n$ is large). $\square$

2.4.1 Deducing Theorem 2.6 from Lemma 2.9

Recall that $\xi_t^F(J_t(s)) = \sum_{n=1}^{\infty} a_n B_{t,n}(s)$. We will split this sum into small, medium, and large n as designated by Lemma 2.9, and then we will approximate each of the terms using the lemma.

It will be notationally convenient to define,

$$\widetilde{F}_t(x) := \sum_{n=1}^{\infty} \exp\bigg(-\frac{|t|}{4}\log^2 n\bigg)\frac{|a_n|}{n^x}$$

which is the sum one gets when applying the triangle inequality to $|F_t(s)|$.

For the small n terms of the sum (i.e. those for which $|t|\log n \leq y^{1/3}$) the lemma gives

$$\begin{aligned}
\sum_{\text{small } n} a_n B_{t,n}(s) &= \gamma_t(s) \sum_{\text{small } n} \exp\bigg(-\frac{|t|}{4}\log^2 n\bigg)\frac{a_n}{n^s}(1 + O(y^{-1/5})) \\
&= \gamma_t(s)\Bigg(\sum_{\text{small } n} \exp\bigg(-\frac{|t|}{4}\log^2 n\bigg)\frac{a_n}{n^s} + O(y^{-1/5}\widetilde{F}_t(x))\Bigg).
\end{aligned}$$

Rewriting the sum in the main term as $F_t(s)$ minus a tail sum, we get

$$= \gamma_t(s)\left(F_t(s) + O\left(y^{-1/5}\widetilde{F}_t(x) + \sum_{\log n > y^{1/3}/|t|} \exp\left(-\frac{|t|}{4}\log^2 n\right)\frac{|a_n|}{n^x}\right)\right). \qquad (2.32)$$

For the medium n terms (which satisfy $y^{1/3} < |t|\log n \leq y^{3/5}$) the lemma gives

$$\sum_{\text{medium } n} a_n B_{t,n}(s) = \gamma_t(s) \times O\left(\sum_{\log n > y^{1/3}/|t|} \exp\left(-\frac{|t|}{8}\log^2 n\right)\frac{|a_n|}{n^x}\right). \qquad (2.33)$$

Adding (2.32) and (2.33), we get an estimate for the sum over all small and medium n,

$$= \gamma_t(s)\left(F_t(s) + O\left(y^{-1/5}\widetilde{F}_t(x) + \sum_{\log n > y^{1/3}/|t|} \exp\left(-\frac{|t|}{8}\log^2 n\right)\frac{|a_n|}{n^x}\right)\right).$$

To bound the tail sum in the error term note that since $|t|\log n > y^{1/3}$ and $a_n = O(n^2)$,

$$\begin{aligned}\exp\left(-\frac{|t|}{8}\log^2 n\right)\frac{|a_n|}{n^x} &\ll \left(n^{-y^{1/3}/8}\right)\left(n^{2-x}\right)\\ &\leq n^{-\frac{y^{1/3}}{10}}\end{aligned}$$

where the second inequality holds for all sufficiently large y because $x = O(y^{1/4})$. Summing this bound over $n > \exp(y^{1/3}/|t|)$ and applying the bound $\sum_{n>R} n^{-M} \leq 2R^{1-M}$ (which holds for all $R, M \geq 2$, say) we conclude that

$$\sum_{\log n < \frac{y^{3/5}}{|t|}} a_n B_{t,n}(s) = \gamma_t(s)\left(F_t(s) + O\left(y^{-1/5}\widetilde{F}_t(x) + \exp\left(-\frac{y^{2/3}}{15|t|}\right)\right)\right). \qquad (2.34)$$

for all sufficiently large y.

Lastly, for large n (which satisfy $|t|\log n > y^{3/5}$) the lemma tells us

$$\sum_{\text{large } n} a_n B_{t,n}(s) = O\left(\sum_{\log n > \frac{y^{3/5}}{|t|}} \exp\left(-\frac{|t|}{10}\log^2 n\right)|a_n|\right),$$

and by using the same method that was just used to bound the other tail sum, we deduce the bound

$$\sum_{\text{large } n} a_n B_{t,n}(s) = O\left(\exp\left(-\frac{y^{6/5}}{15|t|}\right)\right).$$

In order to add this into the small and medium n estimate (2.34) and absorb this error term into the existing error terms we first need to divide through by the $\gamma_t(s)$ factor. Since we have seen previously in (2.27) that $\gamma_t(s)$ only decays exponentially in y, we can write the large n sum bound as

$$\sum_{\text{large } n} a_n B_{t,n}(s) = \gamma_t(s) \times O\left(\exp\left(-\frac{y^{6/5}}{20|t|}\right)\right).$$

Adding together all of the small, medium, and large n then gives

$$\xi_t^F(J_t(s)) = \gamma_t(s)\left(F_t(s) + O\left(y^{-1/5}\widetilde{F}_t(x) + \exp\left(-\frac{y^{2/3}}{20|t|}\right)\right)\right). \tag{2.35}$$

We conclude the proof by bounding $\widetilde{F}_t(x)$ to simplify the error term. First we consider the case when $x \leq -2$. By applying $a_n = O(n^2)$, we see that

$$\widetilde{F}_t(x) \ll \sum_{n=1}^{\infty} \exp\left(-\frac{|t|}{4}\log^2 n + 2|x|\log n\right).$$

Truncating this sum to only those values of n satisfying $\frac{|t|}{4}\log n \leq 3|x|$ will incur only an $O(1)$ error because this condition implies that all the terms in the tail are $O(n^{-2})$, so we have

$$\widetilde{F}_t(x) \ll \sum_{n \leq \exp(12|x|/|t|)} \exp\left(-\frac{|t|}{4}\log^2 n + 2|x|\log n\right) + O(1).$$

It is easy to verify that the maximum possible value of a term in the sum is $\exp\left(\frac{4|x|^2}{|t|}\right)$, and so bounding each term by this quantity gives

$$\widetilde{F}_t(x) \ll \exp\left(\frac{4|x|^2}{|t|}\right)\exp\left(\frac{12|x|}{|t|}\right) + O(1) \ll \exp\left(\frac{10|x|^2}{|t|}\right),$$

where the second inequality holds because $|t| \leq C$ and because we are in the case $x \leq -2$.

For $x > -2$, one can prove a bound which decays exponentially in x by similar methods to what we used above, but in the interest of keeping the error term simple we instead choose to use the trivial bound coming from the monotonicity of $\widetilde{F}_t(x)$. Since $\widetilde{F}_t(x)$ is decreasing in x, we naturally have

$$\widetilde{F}_t(x) \ll \exp\left(\frac{10}{|t|}\min(x,-2)^2\right). \tag{2.36}$$

Inserting this into (2.35) gives the theorem.

2.5 Proof of lemmas

Lemma 2.3. *Let $\gamma(s)$ be one of the functions described in condition (iii) of the extended Selberg class definition. Let $D > 0$ and $0 \leq \theta < 1$ and let s be a complex number which is at least unit distance away from the poles and zeros of γ, and which satisfies $|\operatorname{Re} s| \leq D|\operatorname{Im} s|^\theta$. There exist $K, K' > 0$ (depending on γ, D, and θ) such that*

$$\exp(-K'|\operatorname{Im} s|) \leq |\gamma(s)| \leq \exp(-K|\operatorname{Im} s|).$$

Proof. Recall that

$$\gamma(s) \coloneqq \alpha s^m (s-1)^m Q^s \prod_{i=1}^{k} \Gamma(\omega_i s + \mu_i).$$

We may assume $|\operatorname{Im} s|$ is large, because the small $|\operatorname{Im} s|$ case follows by compactness (since $|\operatorname{Re} s| \le D|\operatorname{Im} s|^\theta$ and since s is bounded away from the poles and zeros of γ by our hypothesis).

Note that the polynomial and exponential factors in $\gamma(s)$ are insignificant because if $|\operatorname{Im} s|$ is large and $|\operatorname{Re} s| \le D|\operatorname{Im} s|^\theta$, then

$$\exp\Big(-|\operatorname{Im} s|^{\theta'}\Big) \le |s^m(s-1)^m Q^s| \le \exp\Big(|\operatorname{Im} s|^{\theta'}\Big)$$

for some $\theta < \theta' < 1$. Hence, it is enough to show that each Γ factor obeys

$$\exp(-K_i'|\operatorname{Im} s|) \le |\Gamma(\omega_i s + \mu_i)| \le \exp(-K_i|\operatorname{Im} s|) \tag{2.37}$$

for $K_i', K_i > 0$.

Fix some i and let $z \coloneqq \omega_i s + \mu_i$. Since $\omega_i \in \mathbb{R}$ and $|\operatorname{Im} s|$ is large, we have $|\operatorname{Im} z| \asymp |\operatorname{Im} s|$ and $|\operatorname{Re} z| \ll |\operatorname{Im} z|^\theta$, and we may assume $|\operatorname{Im} z|$ is large as well.

Stirling's approximation gives

$$\begin{aligned} |\Gamma(z)| &= \exp\bigg(\operatorname{Re}\bigg(z \operatorname{Log} z - z + \frac{1}{2}\operatorname{Log}\frac{2\pi}{z}\bigg) + O(1/|z|)\bigg) \\ &= \exp\Big(-(\operatorname{Im} z)\operatorname{Arg} z + O\Big(|\operatorname{Im} z|^{\theta'}\Big)\Big). \end{aligned}$$

Since $|\operatorname{Re} z| \ll |\operatorname{Im} z|^\theta$ and $|\operatorname{Im} z|$ is large, we deduce that $\pi/4 \le \operatorname{Arg} z \le 3\pi/4$ if $\operatorname{Im} z > 0$ and $-3\pi/4 \le \operatorname{Arg} z \le -\pi/4$ if $\operatorname{Im} z < 0$. Hence, we can conclude

$$\exp(-C'|\operatorname{Im} z|) \le |\Gamma(z)| \le \exp(-C|\operatorname{Im} z|),$$

and then (2.37) follows since $|\operatorname{Im} z| \asymp |\operatorname{Im} s|$. $\square$

Lemma 2.5. *Let*

$$h(x) \coloneqq \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} \prod_{j=1}^{k} \Gamma(a_j w + b_j) x^{-w}\, dw$$

for some $a_j > 0$ and $b_j \in \mathbb{C}$ with $\operatorname{Re} b_j \geq 0$. Then there exists a $\delta > 0$ (depending on the a_j and b_j values) such that $h(x) \ll e^{-x^\delta}$ for all $x \geq 1$.

Proof. Let $\mathcal{R}$ denote the set of functions $f \in C((0,\infty))$ for which,

(i) there exists a $\delta > 0$ such that $f(x) \ll e^{-x^\delta}$ for all $x \geq 1$, and

(ii) for every $\kappa > 0$, the bound $f(x) \ll_\kappa x^{-\kappa}$ holds for all $x > 0$.

To prove the statement it clearly suffices to show that $h \in \mathcal{R}$. In order to do this we first make several observations about $\mathcal{R}$ and about Mellin transforms.

Note that for any function $f \in \mathcal{R}$, the bound (ii) implies that the Mellin transform

$$F(w) := \int_0^\infty f(x) x^w \, \frac{dx}{x}$$

of f is defined for all $\operatorname{Re} w > 0$. Moreover, if F is integrable over some vertical line $\operatorname{Re} w = c > 0$, then f can be recovered from F via the inverse Mellin transform,

$$f(x) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} F(w) x^{-w} \, dw.$$

This follows directly from the Fourier inversion formula for L^1 functions if one performs the necessary changes of variables.

If f and g are functions in $\mathcal{R}$, we define their multiplicative convolution to be

$$f \star g(x) := \int_0^\infty f\left(\frac{x}{y}\right) g(y) \, \frac{dy}{y}, \tag{2.38}$$

and we claim that $\mathcal{R}$ is closed under this operation. Note that if $f, g \in \mathcal{R}$, then $f \star g$ is continuous by a standard application of the dominated convergence theorem, so it suffices to show that this function satisfies the bounds (i) and (ii).

Let $\delta > 0$ be small enough so that both $f(y) \ll e^{-y^\delta}$ and $g(y) \ll e^{-y^\delta}$ for all $y \geq 1$. For $x \geq 1$, we bound the $f \star g(x)$ integral by splitting it into two parts,

$$f \star g(x) = \left(\int_0^{x^{1/2}} + \int_{x^{1/2}}^{\infty} \right) f\left(\frac{x}{y}\right) g(y) \, \frac{dy}{y},$$

which we denote by I and II respectively.

To bound II, apply $f\left(\frac{x}{y}\right) \ll \left(\frac{x}{y}\right)^{-\delta}$ and $g(y) \ll e^{-y^\delta}$ to get

$$\mathrm{II} \ll \int_{x^{1/2}}^{\infty} \left(\frac{x}{y}\right)^{-\delta} e^{-y^\delta} \, \frac{dy}{y} = x^{-\delta} \int_{x^{1/2}}^{\infty} y^{\delta-1} e^{-y^\delta} \, dy \ll e^{-x^{\delta/2}}.$$

To bound I, note that the substitution $y' := \frac{x}{y}$ turns this integral into an integral which is identical to II but with the roles of f and g reversed. Hence the same bound holds, and we can conclude that

$$f \star g(x) \ll e^{-x^{\delta/2}}$$

for all $x \geq 1$.

It remains to show that $f \star g(x) \ll_\kappa x^{-\kappa}$ for all $x > 0$ and $\kappa > 0$. Fixing some $\kappa > 0$, it suffices to prove this bound for $0 < x < 1$ because we have already proved a superior bound when $x \geq 1$. To obtain the bound when $x < 1$, we split the $f \star g(x)$ integral into three parts

$$f \star g(x) = \left(\int_0^{x} + \int_x^{1} + \int_1^{\infty} \right) f\left(\frac{x}{y}\right) g(y) \, \frac{dy}{y}$$

which we denote by I, II, and III.

For III, the bounds $f\left(\frac{x}{y}\right) \ll \left(\frac{x}{y}\right)^{-\kappa}$ and $g(y) \ll e^{-y^\delta}$ hold, so

$$\mathrm{III} \ll \int_1^{\infty} \left(\frac{x}{y}\right)^{-\kappa} e^{-y^\delta} \, \frac{dy}{y} \ll x^{-\kappa}.$$

For II, the bounds $f\left(\frac{x}{y}\right) \ll \left(\frac{x}{y}\right)^{-\kappa/2}$ and $g(y) \ll y^{-\kappa/2}$ hold, so

$$\mathrm{II} \ll \int_x^1 \left(\frac{x}{y}\right)^{-\kappa/2} y^{-\kappa/2} \frac{dy}{y} = x^{-\kappa/2} \log \frac{1}{x} \ll x^{-\kappa}.$$

For I, we can again apply the substitution $y' := \frac{x}{y}$ to turn this integral into an identical integral to III but with the roles of f and g reversed. Hence,

$$f \star g(x) \ll x^{-\kappa},$$

so we can conclude that $f \star g \in \mathcal{R}$.

Now suppose that $f, g \in \mathcal{R}$ with F, G their respective Mellin transforms. Using Fubini's theorem and a change of variables, it is straightforward to show that the Mellin transform of $f \star g$ is the product FG. Furthermore, if $F(w)G(w)$ is integrable over a line $\operatorname{Re} w = c > 0$ then our previous remark about Mellin inversion implies that $f \star g$ is the inverse Mellin transform of FG. We may now apply this result inductively to prove the lemma.

Using the fact that the inverse Mellin transform of $\Gamma(w)$ is e^{-x}, one can verify that the inverse Mellin transform of $\Gamma(a_j w + b_j)$ is $\frac{1}{a_j} x^{b_j/a_j} e^{-x^{1/a_j}}$, which is a member of $\mathcal{R}$. Any product of these gamma functions will be integrable over the line $\operatorname{Re} w = 2$ because of the exponential decay of the gamma function on vertical lines. Hence, the inverse Mellin transform

$$h(x) = \frac{1}{2\pi i} \int_{2-\infty i}^{2+\infty i} \prod_{j=1}^{k} \Gamma(a_j w + b_j) x^{-w} \, dw$$

is a multiplicative convolution of k functions each of which are in $\mathcal{R}$, so h is in $\mathcal{R}$ as well.

$\square$

Lemma 2.8. *For any $t < 0$, F_t has a zero.*

Proof. Fix $t < 0$, and recall that $F_t(s) \coloneqq \sum_{n=1}^{\infty} \exp\Big(-\frac{|t|}{4}\log^2 n\Big)\frac{a_n}{n^s}$ is an entire function. Furthermore, the bound (2.36) on the function $\widetilde{F}_t(x) \coloneqq \sum_{n=1}^{\infty} \exp\Big(-\frac{|t|}{4}\log^2 n\Big)\frac{|a_n|}{n^x}$ implies

$$|F_t(x+iy)| \le \widetilde{F}_t(x) \ll \exp\bigg(\frac{10}{|t|}\min(x,-2)^2\bigg),$$

so F_t is of order at most two.

Now suppose for the sake of contradiction that F_t has no zeros. Then by the Hadamard factorization theorem, $F_t(s) = \exp(P(s))$ where P is a polynomial of degree at most two. Using the fact that $F_t(s)$ is uniformly bounded in all half-planes $\{\operatorname{Re} s > c\}$, one may verify that the only possible choices for $P(s)$ are polynomials of the form $P(s) = -\lambda s + \rho$ where $\lambda \ge 0$, $\rho \in \mathbb{C}$. Hence,

$$\sum_{n=1}^{\infty} \exp\bigg(-\frac{|t|}{4}\log^2 n\bigg)\frac{a_n}{n^s} = e^{\rho}\exp(-\lambda s)$$

for all $s \in \mathbb{C}$. Now note that the right-hand side of this equality is a (generalized) Dirichlet series with only one term whereas the left-hand side has multiple terms because as we have noted previously a_n is nonzero for more than one n. By the uniqueness of coefficients of generalized Dirichlet series such an equality is impossible, so we get a contradiction. $\square$

Lemma 2.10. *For any $\varepsilon > 0$, define the region $S_\varepsilon \coloneqq \{|\operatorname{Arg} w| < \pi - \varepsilon, |w| > \varepsilon\}$. Let $z, z_0 \in S_\varepsilon$ and $|z - z_0| \le D|z_0|^{2/3}$ for some $D > 0$. Then for all z, z_0 that are at least unit distance away from the poles and zeros of γ, we have the estimate*

$$\gamma(z) = \gamma(z_0)\exp\Bigg(\bigg(\log Q + \sum_{j=1}^{k}\omega_j \operatorname{Log}(\omega_j z_0)\bigg)(z - z_0) + \frac{\sum_{j=1}^{k}\omega_j}{2z_0}(z-z_0)^2\Bigg)$$
$$\times\bigg(1 + O\bigg(\frac{1+|z-z_0|}{|z_0|} + \frac{|z-z_0|^3}{|z_0|^2}\bigg)\bigg)$$

where the implicit constant may depend on ε and D.

Proof. We may assume that $|z_0|$ is large, as the small $|z_0|$ case holds by compactness. We allow all implicit constants to depend on ε and D.

Recall that $\gamma(z) := \alpha z^m(z-1)^m Q^z \prod_{j=1}^{k} \Gamma(\omega_i z + \mu_i)$. We will consider the factors of γ separately and show how each one differs when evaluating at z versus z_0.

To handle the polynomial factor $\alpha z^m(z-1)^m$, note that for all $|z_0|$ large we have

$$z = z_0\left(1 + O\left(\frac{|z - z_0|}{|z_0|}\right)\right) \quad \text{and} \quad z - 1 = (z_0 - 1)\left(1 + O\left(\frac{|z - z_0|}{|z_0|}\right)\right),$$

hence

$$\alpha z^m(z-1)^m = \alpha z_0^m(z_0 - 1)^m\left(1 + O\left(\frac{|z - z_0|}{|z_0|}\right)\right). \tag{2.39}$$

For the exponential term in γ, there is a trivial equality

$$Q^z = Q^{z_0} \exp((\log Q)(z - z_0)). \tag{2.40}$$

To get estimates for the Γ factors in γ we recall Stirling's formula which states that for any $w \in S_\varepsilon$ with $|w| \geq 1$,

$$\Gamma(w) = \sqrt{2\pi} \exp\left(w \operatorname{Log} w - w - \frac{1}{2} \operatorname{Log} w\right)\left(1 + O\left(\frac{1}{|w|}\right)\right). \tag{2.41}$$

Suppose $w, w_0 \in S_\varepsilon$ and $|w - w_0| \ll |w_0|^{2/3}$. By assuming $|w_0|$ is sufficiently large we have $\left|\operatorname{Arg}\left(1 + \frac{w - w_0}{w_0}\right)\right| < \varepsilon$, and we then deduce

$$\operatorname{Log} w = \operatorname{Log} w_0 + \operatorname{Log}\left(1 + \frac{w - w_0}{w_0}\right). \tag{2.42}$$

By applying Stirling's formula at w and w_0 (and assuming $|w_0|$ is large), then using (2.42) and the fact that $|w| \asymp |w_0|$ gives

$$\begin{aligned}\frac{\Gamma(w)}{\Gamma(w_0)} &= \exp\biggl(w \operatorname{Log} w - w - w_0 \operatorname{Log} w_0 + w_0 - \frac{1}{2}\operatorname{Log}\biggl(1 + \frac{w - w_0}{w_0}\biggr)\biggr) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times \biggl(1 + O\biggl(\frac{1}{|w_0|}\biggr)\biggr) \\ &= \exp(w \operatorname{Log} w - w - w_0 \operatorname{Log} w_0 + w_0)\biggl(1 + O\biggl(\frac{1 + |w - w_0|}{|w_0|}\biggr)\biggr)\end{aligned}$$

where the second line follows from the first because $\operatorname{Log}(1 + s) = O(|s|)$ for $|s| \le \frac{1}{2}$, and $\exp(s) = 1 + O(|s|)$ for s bounded.

If we consider just the expression inside the exponential, one can show that

$$\begin{aligned}w \operatorname{Log} w - w - w_0 \operatorname{Log} w_0 + w_0 = \operatorname{Log}(w_0)(w - w_0) + \frac{1}{2w_0}(w - w_0)^2 \\ + O\biggl(\frac{|w - w_0|^3}{|w_0|^2}\biggr)\end{aligned}$$

by applying (2.42) to the logarithm, then applying the Taylor approximation $\operatorname{Log}(1+s) = s - \frac{s^2}{2} + O(s^3)$ which holds for all $|s| \le \frac{1}{2}$, and gathering together error terms.

Plugging this back into our expression for $\frac{\Gamma(w)}{\Gamma(w_0)}$ and then applying $\exp(1 + s) = 1 + O(|s|)$ for s bounded again,

$$\begin{aligned}\frac{\Gamma(w)}{\Gamma(w_0)} = \exp\biggl(\operatorname{Log}(w_0)(w - w_0) + \frac{1}{2w_0}(w - w_0)^2\biggr) \\ \times \biggl(1 + O\biggl(\frac{1 + |w - w_0|}{|w_0|} + \frac{|w - w_0|^3}{|w_0|^2}\biggr)\biggr). \end{aligned} \tag{2.43}$$

Taking $w \coloneqq \omega_i z + \mu_i$ and $w_0 \coloneqq \omega_i z_0 + \mu_i$ note that from the assumptions that $z, z_0 \in S_\varepsilon$, $|z - z_0| \le D|z_0|^{2/3}$, and $|z_0|$ is large, we get the corresponding facts that $w, w_0 \in S_{\varepsilon'}$, $|w - w_0| \ll |w_0|^{2/3}$, and $|w_0|$ is large. Hence, by (2.43)

$$\begin{aligned}\frac{\Gamma(\omega_i z + \mu_i)}{\Gamma(\omega_i z_0 + \mu_i)} = \exp\biggl(\omega_i \operatorname{Log}(\omega_i z_0 + \mu_i)(z - z_0) + \frac{\omega_i^2}{2(\omega_i z_0 + \mu_i)}(z - z_0)^2\biggr) \\ \times \biggl(1 + O\biggl(\frac{1 + |z - z_0|}{|z_0|} + \frac{|z - z_0|^3}{|z_0|^2}\biggr)\biggr). \end{aligned} \tag{2.44}$$

When $|z_0|$ is large,

$$\operatorname{Log}(\omega_i z_0 + \mu_i) = \operatorname{Log}(\omega_i z_0) + O\left(\frac{1}{|z_0|}\right)$$

and

$$\frac{1}{\omega_i z_0 + \mu_i} = \frac{1}{\omega_i z_0} + O\left(\frac{1}{|z_0|^2}\right).$$

Inserting these into (2.44) gives

$$\frac{\Gamma(\omega_i z + \mu_i)}{\Gamma(\omega_i z_0 + \mu_i)} = \exp\left(\omega_i \operatorname{Log}(\omega_i z_0)(z - z_0) + \frac{\omega_i}{2z_0}(z - z_0)^2\right) \\ \times \left(1 + O\left(\frac{1 + |z - z_0|}{|z_0|} + \frac{|z - z_0|^3}{|z_0|^2}\right)\right). \quad (2.45)$$

Applying (2.45) to each Γ factor in γ and combining this with (2.39) and (2.40) gives the lemma.

□

Lemma 2.11. *Let t, $s = x + iy$, and C satisfy the same hypotheses as in Theorem 2.6. Let $n \leq \exp(y^{3/5}/|t|)$ (i.e. the small/medium case of Lemma 2.9) and let*

$$\mathcal{I}(z) := \gamma(z) e^{\frac{1}{|t|}(J_t(s) - z)^2} n^{-z}$$

be the integrand of (2.17). *Then for all y sufficiently large (depending on C) the following estimates hold (where the implicit constants may also depend on C):*

(i) If $|x - \operatorname{Re} z| \leq y^{3/5}$ and $\frac{1}{2}y^{2/3} \leq |y - \operatorname{Im} z| \leq 2y^{2/3}$ then

$$\mathcal{I}(z) \ll \exp\left(-\frac{y^{4/3}}{10|t|}\right).$$

(ii) If $\operatorname{Re} z = 2$ and $|y - \operatorname{Im} z| \geq y^{2/3}$ then

$$\mathcal{I}(z) \ll \exp\left(-\frac{1}{2|t|}(y - \operatorname{Im} z)^2\right).$$

Proof. We shall bound the three factors of $\mathcal{I}(z)$ individually and prove (i) and (ii) in parallel. We allow all implicit constants to depend on C, and we freely assume that y large enough to make various assumptions hold.

From the definition (2.12) of $J_t(s)$, and the assumptions that $x = O(y^{1/4})$ and $t = O(1)$ it is clear that

$$\operatorname{Im} J_t(s) = y + O(1) \quad \text{and} \quad \operatorname{Re} J_t(s) = x + O(\log y),$$

so if z satisfies the hypotheses in either (i) or (ii), then

$$\begin{aligned}\operatorname{Re}\big((J_t(s) - z)^2\big) &= (x + O(\log y) - \operatorname{Re} z)^2 - (y + O(1) - \operatorname{Im} z)^2 \\ &\leq -\frac{1}{2}(y - \operatorname{Im} z)^2.\end{aligned}$$

for all y sufficiently large (where we have used the assumption that $x \ll y^{1/4}$ again in the (ii) case). Hence we can bound the first exponential factor in $\mathcal{I}(z)$ by

$$\left| e^{\frac{1}{|t|}(J_t(s)-z)^2} \right| \leq \exp\left(-\frac{1}{2|t|}(y - \operatorname{Im} z)^2 \right). \tag{2.46}$$

To handle the $\gamma(z)$ factor we can apply Lemma 2.3. Note that the hypotheses in (i) imply

$$|\operatorname{Re} z| \leq |x| + y^{3/5} \ll y^{3/5} \quad \text{and} \quad \operatorname{Im} z \asymp y$$

and the in case (ii) we have $\operatorname{Re} z = 2$, so in both cases Lemma 2.3 certainly implies that $\gamma(z) = O(1)$.

In (ii) it is clear that the $n^{-z} = O(1)$, and in (i) note that $\operatorname{Re} z \ll y^{3/5}$ and $\log n \leq y^{3/5}/|t|$, so

$$n^{-z} = \exp(-(\operatorname{Re} z) \log n) \leq \exp\left(\frac{K y^{6/5}}{|t|} \right).$$

Hence, to get the conclusion in (i), we put together the above bounds and the fact that $|y - \operatorname{Im} z| \geq \frac{1}{2} y^{2/3}$ which gives

$$\begin{aligned} \mathcal{I}(z) &\ll \exp\left(-\frac{1}{2|t|}(y - \operatorname{Im} z)^2\right) \exp\left(\frac{Ky^{6/5}}{|t|}\right) \\ &\leq \exp\left(-\frac{y^{4/3}}{8|t|}\right) \exp\left(\frac{Ky^{6/5}}{|t|}\right) \\ &\leq \exp\left(-\frac{y^{4/3}}{10|t|}\right) \end{aligned}$$

for all y sufficiently large.

Similarly, the conclusion of (ii) is immediate from (2.46) and the $O(1)$ bounds on the other two factors.

$\square$

Lemma 2.12. *There is some $K > 0$ such that $|\gamma(z)| \leq \exp(K(\operatorname{Re} z)^{1.1})$ uniformly for any z with $\operatorname{Re} z \geq 1$.*

Proof. Recall that

$$\gamma(z) := \alpha z^m (z-1)^m Q^z \prod_{i=1}^{k} \Gamma(\omega_i z + \mu_i).$$

Consider two cases where either $|\operatorname{Im} z| > (\operatorname{Re} z)^2$ or $|\operatorname{Im} z| \leq (\operatorname{Re} z)^2$. In the first case, we see $\gamma(z) = O(1)$ by Lemma 2.3. In the second case, one can bound each of the Γ factors in γ with the bound

$$|\Gamma(\omega_i z + \mu_i)| \leq \Gamma(\operatorname{Re}(\omega_i z + \mu_i)) \leq \exp(K_0(\operatorname{Re} z)^{1.1})$$

for some $K_0 > 0$. Since, $|z| \ll (\operatorname{Re} z)^2$ and $|Q^z| = \exp((\log Q)(\operatorname{Re} z))$ the bound follows. $\square$

CHAPTER 3

Extreme values of the argument of the Riemann zeta function

3.1 Introduction

Let

$$S(t) := \frac{1}{\pi} \operatorname{Im} \log \zeta\left(\frac{1}{2} + it\right),$$

such that the logarithm of $\zeta(s)$ is defined with branch cuts extending horizontally to the left of each zero and pole, and the branch is chosen so that $\log \zeta(2)$ is real. When $\frac{1}{2} + it$ lies on a branch cut, $S(t)$ has a jump discontinuity, and by convention we set $S(t) := \lim_{\varepsilon \to 0}(S(t+\varepsilon) + S(t-\varepsilon))/2$ in this case.

The behavior of $S(t)$ is closely connected to the locations of the zeros of the zeta function as one may see from the Riemann-von Mangoldt formula,

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\big(t^{-1}\big), \quad t \geq 1. \tag{3.1}$$

Here $N(t)$ denotes the number of zeta zeros $\rho = \beta + i\gamma$ for which $0 < \gamma < t$, and any zeros with imaginary part equal to t are counted with weight $1/2$ in order to be consistent with our definition of $S(t)$. The classical bound $S(t) = O(\log t)$ which is due to Backlund [2] shows that the macroscopic behavior of $N(t)$ is dictated by the first two terms in the Riemann-von Mangoldt formula, but at smaller scales the irregularities in the vertical distribution of zeta zeros are governed by the oscillations of the $S(t)$ term. In this chapter we are interested in studying the distribution of $S(t)$ on an interval $[T, 2T]$ where T is large. In particular, we would like to understand the tails of this distribution.

The statistical properties of $S(t)$ were first studied by Selberg [26] who showed that on the interval $[T, 2T]$ the moments of $S(t)$ are asymptotic to the moments of a normal distribution with mean zero and variance $\frac{1}{2\pi^2} \log \log T$. Using standard results in probability theory, these moment estimates allow one to deduce the *Selberg central limit theorem*, which states that for fixed $\Delta > 0$,

$$\frac{1}{T} \operatorname{meas}\left\{ t \in [T, 2T] \colon \frac{\pi S(t)}{\sqrt{\frac{1}{2} \log \log T}} \geq \Delta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\Delta}^{\infty} e^{-u^2/2}\, du + o(1) \tag{3.2}$$

as T tends to infinity. Note that this result as we have stated it says nothing about the large deviations behavior of $S(t)$ where Δ is allowed to grow with T. However, if we were to extend the heuristic that $S(t)$ behaves like a Gaussian random variable with variance proportional to $\log \log T$, then we would predict that the measure of the set $\{t \in [T, 2T] \colon S(t) \geq V\}$ is uniformly bounded below by $T \exp(-DV^2/\log \log T)$ for some positive absolute constant D even if V chosen to be much larger than $\sqrt{\log \log T}$. A lower bound of this shape was proved by Soundararajan [28] for the analogous problem of $\operatorname{Re} \log \zeta(\frac{1}{2} + it)$ rather than $S(t)$ for the range $\log \log T \leq V \leq (\log T)^{\frac{1}{2} - \varepsilon}$. Unfortunately, it is not possible to directly adapt Soundararajan's method to get the same result for $S(t)$ because the possibility of zeros of the zeta function off the critical line leads to complications which do not appear for $\operatorname{Re} \log \zeta(\frac{1}{2} + it)$. We are able to prove the following result instead.

Theorem 3.1. *There exist constants $\kappa, D > 0$ such that for all T sufficiently large* *and all V in the range $\sqrt{\log\log T} \le V \le \kappa\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}$ we have*

$$\operatorname{meas}\{t \in [T, 2T] \colon S(t) \ge V\} \ge T \exp\left(-D\frac{V^2}{\log \frac{\log T}{V^3 \log V}}\right).$$

The same is true if one replaces $S(t)$ with $-S(t)$.

The lower endpoint on the range of V in the theorem could be adjusted to be any constant multiple of $\sqrt{\log\log T}$ but this is not so important since the Selberg central limit theorem already applies in this regime. Notably, for $V \le (\log T)^{\frac{1}{3}-\varepsilon}$ the lower bound in Theorem 3.1 matches the prediction described in the previous paragraph. For values of V near the endpoint $\kappa\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}$ the lower bound degrades but is still nontrivial (cf. Theorem 1 of [28] where the behavior of the bound near the endpoint is quite similar).

To compare Theorem 1 with what is already known about $S(t)$, we note that the best Ω results[1] for $S(t)$ are

$$S(t) = \begin{cases} \Omega_{\pm}\left(\left(\frac{\log t}{\log\log t}\right)^{\frac{1}{3}}\right) \\ \Omega_{\pm}\left(\sqrt{\frac{\log t \log\log\log t}{\log\log t}}\right) \text{ on RH.} \end{cases}$$

The unconditional result is due to Tsang [32] which is itself an improvement in the power of the $\log\log t$ factor over a result of Selberg [26]. In the conditional Ω result, the order of magnitude comes from a recent breakthrough of Bondarenko and Seip [4] which improves a longstanding theorem of Montgomery [15] by a $\sqrt{\log\log\log t}$ factor. The fact that Bondarenko and Seip's method yields large values of either sign was noted by Chirre and Mahatab [5].

[1]An Ω result is any statement of the form $f(x) = \Omega(g(x))$ meaning that $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$. The notation $f(x) = \Omega_{\pm}(g(x))$ indicates that $\limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0$ and $\liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0$.

We now see that Theorem 3.1 implies a quantitative strengthening of Tsang's Ω result by giving a lower bound on the measure of the set where $S(t)$ is large. By comparison, Tsang's proof does not give such a bound. To our knowledge, the only other measure result of this kind in the literature is a theorem of Radziwiłł [22, Rmk. (1)] which gives an asymptotic for the measure of $\{t \in [T, 2T] \colon S(t) \geq V\}$ rather than just a lower bound but for the range $V \ll (\log\log T)^{\frac{1}{2}+\frac{1}{10}-\varepsilon}$ instead.

Broadly speaking, the proof of Theorem 3.1 consists of three parts. The first part is an application of a convolution formula of Selberg [26] which expresses a smoothed version of $S(t)$ in terms a contribution of zeta zeros and a contribution of a Dirichlet series. The second part is an upper bound on how often the contribution of the zeta zeros in the convolution formula is large. This requires an application of a zero density estimate and theorem of Tsang on the moments of $S(t+h) - S(t)$ to get some control over the zeta zeros in the horizontal and vertical directions respectively. The third part of the proof is a lower bound on how often the Dirichlet series contribution in the convolution formula is large. This will be an application of the *resonance method* which was developed by Soundararajan in [28] to prove the result about $\operatorname{Re}\log\zeta(\frac{1}{2}+it)$ mentioned previously.

Before giving the full proof of Theorem 3.1, in the next section we will give a heuristic proof which describes the main ideas in more detail and will illustrate why the upper endpoint $\kappa\Big(\frac{\log T}{\log\log T}\Big)^{\frac{1}{3}}$ in our theorem appears to be optimal barring some new idea for bounding the contribution of zeta zeros off the critical line.

As a final remark we note that if we were to assume the Riemann hypothesis, then our arguments would simplify since there is no longer a possible contribution of zeta zeros in Selberg's convolution formula. In this case it is possible to get an improvement in Theorem 1 which is essentially of the same strength as Soundararajan's measure result for $\operatorname{Re}\log(\frac{1}{2}+it)$.

3.2 Proof Outline

In order to study $S(t)$ we would like to express it in terms of a sum over primes. Recall that for any s in the half plane $\operatorname{Re}(s) \geq 1+\varepsilon$ we have the classical formula $\log\zeta(s) = \sum_p p^{-s} + O_\varepsilon(1)$. Naively extending this to the case where $s = \frac{1}{2}+it$ we obtain the heuristic

$$S(t) \approx \frac{1}{\pi}\operatorname{Im}\sum_p \frac{1}{\sqrt{p}}p^{-it}.$$

Of course this is not a rigorous approximation because the summation on the right-hand side does not converge. To remedy this, we can damp out the contribution of the higher frequencies (i.e. the larger primes) by convolving $S(t)$ with an approximation to the identity. This idea was first used by Selberg in his work on large values of $S(t)$ (see [26, Section 7]). Selberg proved what we will refer to as the *convolution formula*, which expresses a smoothing of $S(t)$ in terms of a sum over primes and a sum over zeta zeros.

Roughly speaking, the convolution formula can be stated as follows: if one convolves $S(t)$ with an approximation to the identity of width λ^{-1} (where λ is some parameter that we may choose) then the resulting smoothed version of $S(t)$ is approximately

$$\frac{1}{\pi}\operatorname{Im}\sum_{\log p \ll \lambda}\frac{1}{\sqrt{p}}p^{-it} + \sum_{\substack{\beta>1/2 \\ |\gamma-t|\ll\lambda^{-1}}} G_{t,\lambda}(\rho) \tag{3.3}$$

where $G_{t,\lambda}$ is a function whose precise definition is not so important for now. The first summation is over all primes up to a certain cutoff determined by the parameter

λ, and the second summation is over all zeta zeros (denoted $\rho = \beta + i\gamma$) which lie to the right of the critical line and which are close to height t. Instead of trying to understand the distribution of $S(t)$ directly, it will suffice for us to try to understand the distribution of the quantity (3.3). Indeed, we should expect that large values of $S(t)$ will occur at least as often as large values of a smoothing of $S(t)$.

Note that if we were to assume the Riemann hypothesis then (3.3) would have no contribution coming from zeta zeros. In this case the problem would be reduced to understanding large values of the imaginary part of a certain Dirichlet polynomial. This can be done in several ways: by computing moments of the Dirichlet polynomial (see Tsang [32]), by using Dirchlet's approximation theorem (see Montgomery [15]), or by using the resonance method (see Bondarenko and Seip [4]). However, since our goal is to prove an unconditional result, we must account for the possibility of a contribution from zeta zeros off the critical line. Tsang's proof of his unconditional Ω theorem for $S(t)$ involves comparing high moments of the Dirichlet polynomial contribution in (3.3) with high moments of the zeros contribution and then applying a clever lemma to deduce the existence of large values of $S(t)$. Our approach will be more direct. Given any V we first prove that the magnitude of the contribution of zeta zeros in the convolution formula is less than V for all $t \in [T, 2T]$ outside of some exceptional set of small measure. Next we show that the Dirichlet polynomial contribution is larger than $2V$ on a set whose measure much larger than the exceptional set. From this we can deduce a bound on how often the sum of these two contributions is greater than V.

In order to carry out the argument we have just described, we need to understand how often the Dirichlet polynomial contribution and the zeta zeros contribution in (3.3) are large. We start by giving a heuristic analysis of the Dirichlet polynomial.

Given any V (which we think of as growing slowly with T) we want to know how often the Dirichlet polynomial is greater than $2V$. First observe that the terms in the sum coming from primes smaller than V (say) are inconsequential because they can contribute at most $\sum_{p \le V} \frac{1}{\sqrt{p}} = o(V)$ to the total. This immediately implies that we must have $\lambda \gg \log V$ if the sum is ever to be larger than V. For the remaining terms, we can model the sum by a random sum since we expect that the phases p^{-it} behave roughly like i.i.d. random variables with mean zero. The variance of this random sum is proportional to

$$\sum_{\log V \le \log p \ll \lambda} \frac{1}{p} = \log \lambda - \log\log V + O(1)$$

by Mertens' theorem. Hence, if we assume that the distribution of the sum is approximately Gaussian then we expect it to be greater than $2V$ with probability at least $\exp(-CV^2/(\log \lambda - \log\log V))$ for some positive constant C. If this probabilistic model is accurate, then we expect that our deterministic sum is greater than $2V$ on a subset of $[T, 2T]$ of measure

$$T \exp\left(-C \frac{V^2}{\log \frac{\lambda}{\log V}}\right). \tag{3.4}$$

We shall see in Section 3.5 that it is possible to prove a bound of precisely this form (assuming that λ and V lie in some appropriate ranges) using Soundararajan's resonance method.

Next we must analyze the sum over zeta zeros. Recall that the average spacing of zeta zeros near height T is $\frac{2\pi}{\log T}$, so we expect that there are at most $O(\lambda^{-1} \log T)$ terms in this sum. Moreover, we will see later on that the magnitude of each summand $G_{t,\lambda}(\rho)$ is (roughly speaking) $O(\lambda^2(\beta - \frac{1}{2})^2)$. Hence, if we let $\theta_t := \max(\beta - \frac{1}{2})$

where the maximum is taken over all the zeros in the sum, then we expect that

$$\sum_{\substack{\beta>1/2 \\ |\gamma-t|\ll\lambda^{-1}}} G_{t,\lambda}(\rho) \ll \lambda^2 \sum_{\substack{\beta>1/2 \\ |\gamma-t|\ll\lambda^{-1}}} (\beta - \frac{1}{2})^2 \ll \lambda\theta_t^2 \log T. \tag{3.5}$$

If this bound holds for a given $t \in [T, 2T]$, then the contribution of the zeros is guaranteed to be less than V whenever

$$\theta_t \ll \left(\frac{V}{\lambda \log T}\right)^{\frac{1}{2}} \tag{3.6}$$

for some sufficiently small implicit constant. Hence, we would like to bound the measure of the exceptional set of t values where (3.6) fails to hold. To do this we need a bound on the number of zeta zeros lying a specified distance off the critical line. Such bounds are commonly known in the literature as *zero density estimates*. We will see in Lemma 3.7 that by using a classical zero density estimate we can bound the measure of our exceptional set by

$$T \exp(10\lambda) \exp\left(-c\left(\frac{V \log T}{\lambda}\right)^{\frac{1}{2}}\right) \tag{3.7}$$

where c is some positive absolute constant.

We can now complete the proof sketch. Given a particular V, we would like to select a value of the parameter λ so that the lower bound (3.4) is larger than the upper bound (3.7). It turns out that this is only possible if V is not too large. To find the endpoint, we start by noting that if V is as large as a fixed power of $\log T$, then we saw in our analysis of the Dirichlet series that we must have $\lambda \gg \log V \gg \log\log T$. A short calculation shows that V can be taken as large as $\kappa\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}$ for some small constant $\kappa > 0$ if we let $\lambda = \log\log T$, and we cannot do any better than this. For values of V less than this endpoint, one needs to choose a larger value of λ in order to get the right measure bound in the conclusion of Theorem 3.1. The complete details for this may be found in Section 3.6, but as an example we can consider the case where $V = (\log T)^{\frac{1}{3}-\varepsilon}$ for some $0 < \varepsilon < \frac{1}{3}$. Letting $\lambda = (\log T)^{\mu}$ for some small μ depending on ε, one sees that the lower bound (3.4) is at least

$$T \exp\biggl(-D_\varepsilon \frac{V^2}{\log\log T}\biggr) \tag{3.8}$$

for some constant $D_\varepsilon > 0$ depending on ε. One can also check that the upper bound (3.7) on the measure of the exceptional set is much smaller than this as long as μ is chosen to be small enough. Hence, we conclude that the sum of the Dirichlet polynomial contribution and zeros contribution is at least V on a set of measure $\gg T \exp\Bigl(-D_\varepsilon \frac{V^2}{\log\log T}\Bigr)$ which matches the prediction we have described previously.

3.3 Convolution formula

We now begin the full proof Theorem 1. For the rest of the chapter we will assume $t \in [T, 2T]$ where T is large. We start by quoting a lemma given in [32] which is a general form of the convolution formula for $\log\zeta$ discovered by Selberg.

Lemma 3.2. *Suppose $1/2 \le \sigma < 1$ and let $K(x+iy)$ be an analytic function in the horizontal strip $\sigma - 2 \le y \le 0$ satisfying the growth condition*

$$W(x) := \sup_{\sigma-2\le y\le 0} |K(x+iy)| = O\bigl((|x|\log^2|x|)^{-1}\bigr)$$

as $|x| \to \infty$. For any $t \neq 0$, we have

$$\int_{-\infty}^{\infty} \log\zeta(\sigma + i(t+u))K(u)\,du = \sum_{n=2}^{\infty} \hat{K}(\log n)\frac{\Lambda(n)}{\log n} n^{-\sigma-it} + 2\pi \sum_{\substack{\rho=\beta+i\gamma \\ \beta>\sigma}} \int_0^{\beta-\sigma} K(\gamma - t - i\alpha)\,d\alpha + O(W(t)). \tag{3.9}$$

The second summation above runs over all zeta zeros lying strictly to the right of the line $\mathrm{Re}(s) = \sigma$. The convention used here for the Fourier transform is $\hat{K}(x) := \int K(u)e^{-ixu}\,du$.

In order to prove Theorem 1, we will be applying the lemma with a kernel K_λ where $\lambda \in [1, \log T]$ is a certain parameter depending on V and T to be chosen later. To define K_λ, recall the following Fourier pair,

$$\omega(u) := \frac{1}{2\pi}\left(\frac{\sin(u/2)}{u/2}\right)^2 \iff \hat{\omega}(x) := \max(0, 1 - |x|).$$

Let

$$K_\lambda(z) := \lambda\omega(\lambda z).$$

One may verify that K_λ has the following properties:

(i) $K_\lambda(u) \geq 0$ for $u \in \mathbb{R}$,

(ii) $\int K_\lambda(u)\,dx = 1$,

(iii) $\mathrm{Im}\, K_\lambda(u + iv) \ll \frac{\lambda^2 |v| e^{\lambda|v|}}{1+(\lambda u)^2+(\lambda v)^2}$,

(iv) $\widehat{K_\lambda}(x) = \hat{\omega}\big(\frac{x}{\lambda}\big)$.

(For a proof of property (iii) see [26, pg. 52].)

Applying Selberg's convolution formula with K_λ, we can prove that a short average of $S(t)$ can be expressed in terms of a Dirichlet polynomial and some contribution from zeta zeros.

Lemma 3.3. *Suppose* $1 \leq \lambda \leq \log T$ *and* T *is large. For any* $t \in [T, 2T]$,

$$\int_{-\log T}^{\log T} S(t+u)K_\lambda(u)\,du = \frac{1}{\pi}\,\mathrm{Im}\,D_\lambda(t) + Z_\lambda(t) + O(1)$$

where

$$D_\lambda(t) = \sum_{k=2}^{\infty} \hat{\omega}\left(\frac{\log k}{\lambda}\right) \frac{\Lambda(k)}{(\log k)\sqrt{k}}\, k^{-it},$$

$$Z_\lambda(t) = 2\,\mathrm{Im} \sum_{\beta > \frac{1}{2}} \int_0^{\beta - \frac{1}{2}} K_\lambda(\gamma - t - i\alpha)\, d\alpha.$$

Proof. Applying Lemma 3.2 with the kernel K_λ and $\sigma = 1/2$, we see that

$$\int_{-\infty}^{\infty} \log \zeta\left(\frac{1}{2} + i(t+u)\right) K_\lambda(u)\, du = \sum_{n=2}^{\infty} \hat{\omega}\left(\frac{\log n}{\lambda}\right) \frac{\Lambda(n)}{\log n} n^{-\frac{1}{2} - it}$$

$$+ 2\pi \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \frac{1}{2}}} \int_0^{\beta - \frac{1}{2}} K_\lambda(\gamma - t - i\alpha)\, d\alpha + O(W(t)).$$

where

$$W(t) = \sup_{-3/2 \le y \le 0} |K_\lambda(t + iy)|.$$

Taking imaginary parts and dividing through by π we get

$$\int_{-\infty}^{\infty} S(t+u) K_\lambda(u)\, du = \frac{1}{\pi} \mathrm{Im}\, D_\lambda(t) + Z_\lambda(t) + O(W(t)). \tag{3.10}$$

Using the assumptions $t \in [T, 2T]$ and $\lambda \le \log T$ and the inequality $|\sin(a+ib)| \ll e^{|b|}$, we see that

$$W(t) \ll \log T \frac{e^{\frac{3}{2}\log T}}{T^2} \ll 1.$$

All that remains is to show that the tails of the integral in (3.10) are small. Using the classical bound $S(t) = O(\log(|t| + 2))$ and the bound $K_\lambda(u) \ll \lambda^{-1} u^{-2}$, we see

$$\int_{|u| > \log T} S(t+u) K_\lambda(u)\, du \ll \int_{\log T}^{\infty} (\log T + \log u) \lambda^{-1} u^{-2}\, du \ll 1.$$

Hence, the tails may be absorbed into the error term. □

3.4 Contribution of the zeros

Our goal in this section will be to prove that the zeros contribution $Z_\lambda(t)$ is small outside of some exceptional set whose measure we can bound.

Recall that the average gap between consecutive zeta zeros with imaginary parts in $[T, 2T]$ is asymptotic to $\frac{2\pi}{\log T}$, so we expect that in most short subintervals $[t, t+h]$ there should be approximately $\frac{h}{2\pi}\log T$ zeros. We now prove a quantitative estimate on how often a short interval contains "too many" zeros.

Lemma 3.4. *There exists an absolute constant $d > 0$ such that for all $T \geq 3$ and $1/\log T \leq h \leq 2$ the bound*

$$\operatorname{meas}\{t \in [T, 2T] \colon N(t+h) - N(t) \geq h \log T\} \ll T \exp(-dh \log T) \tag{3.11}$$

holds uniformly.

The proof will be an application of the following moment estimate due to Tsang.

Lemma 3.5 (Tsang [32, Thm. 4])**.** *For any positive integer k and any $0 < h < 1$,*

$$\begin{aligned}\int_T^{2T} &(S(t+h) - S(t))^{2k}\, dt \\ &= A_k T (\log(2 + h\log T))^k + O\Big(T(ck)^k\Big(k^k + (\log(2 + h\log T))^{k-1/2}\Big)\Big)\end{aligned} \tag{3.12}$$

where c is a positive constant and

$$A_k := \frac{(2k)!}{2^k \pi^{2k} k!}.$$

Proof of Lemma 3.4. We may assume $h \log T$ is larger than some fixed constant to be chosen later (and hence T is large as well) because for small $h \log T$ the conclusion of the lemma is trivial.

From the Riemann-von Mangoldt formula one may verify that

$$N(t+h) - N(t) = \frac{h}{2\pi} \log \frac{t}{2\pi} + S(t+h) - S(t) + O(T^{-1})$$

for any $t \in [T, 2T]$. Hence we see that

$$\{t \in [T,2T] \colon N(t+h) - N(t) \geq h \log T\} \subseteq \left\{ t \in [T,2T] \colon |S(t+h) - S(t)| \geq \frac{1}{2} h \log T \right\}$$

for all sufficiently large T. Let G denote the set on the right-hand side. It suffices to show that

$$\operatorname{meas} G \ll T \exp(-dh \log T). \tag{3.13}$$

To do this we use Markov's inequality and the moment estimate from Lemma 3.5.

Note that since $h \geq 1/\log T$ we have

$$\log(2 + h \log T) \leq 1 + h \log T \leq 2h \log T.$$

Inserting this into Lemma 3.5 and applying Stirling's approximation to estimate A_k we get an upper bound

$$\int_T^{2T} (S(t+h) - S(t))^{2k} \, dt \leq T(c'k)^k ((h \log T)^k + k^k)$$

where $c' > 0$ is some absolute constant and k can be taken to be any positive integer. From Markov's inequality we have

$$\operatorname{meas} G \leq \frac{T(c'k)^k((h \log T)^k + k^k)}{\left(\frac{1}{2} h \log T\right)^{2k}}.$$

If $h \log T \geq 8c'$, then applying this bound with

$$k = \left\lfloor \frac{1}{8c'} h \log T \right\rfloor \geq 1$$

gives (3.13). Hence we have proved the lemma for $h \log T$ larger than some absolute constant. $\square$

We will also require a *zero density estimate* on the number of zeros off the critical line. Let $N(\sigma, T)$ denote the number of zeta zeros $\rho = \beta + i\gamma$ for which $\beta > \sigma$ and $0 \le \gamma \le T$. For our purposes, the following classical result will suffice.

Lemma 3.6 (Zero density estimate)**.**

$$N(\sigma, T) \ll T^{\frac{3}{2}-\sigma} \log^5 T$$

Proof. See Titchmarsh [31, Theorem 9.19(A)]. $\square$

We can now bound the contribution of $Z_\lambda(t)$ outside of an exceptional set.

Lemma 3.7. *Suppose* $\log\log T \le \lambda \le \log T$ *and* $2 \le V \le \log T/\lambda$. *Let*

$$E_V := \{t \in [T, 2T] \colon |Z_\lambda(t)| > V\}.$$

For all T sufficiently large, we have the uniform bound

$$\operatorname{meas} E_V \ll T \exp(10\lambda) \exp\left(-d'\left(\frac{V \log T}{\lambda}\right)^{1/2}\right) \tag{3.14}$$

for some absolute constant $d' > 0$.

Proof. First we note that

$$\begin{aligned}
Z_\lambda(t) &= \operatorname{Im} \sum_{\beta > \frac{1}{2}} \int_0^{\beta - \frac{1}{2}} K_\lambda(\gamma - t - i\alpha)\, d\alpha \\
&\ll \sum_{\beta > \frac{1}{2}} \int_0^{\beta - \frac{1}{2}} \frac{\lambda^2 \alpha e^{\lambda\alpha}}{1 + (\lambda(\gamma - t))^2}\, d\alpha \\
&\ll \sum_{\beta > \frac{1}{2}} \frac{\lambda^2 \left(\beta - \frac{1}{2}\right)^2 e^{\lambda\left(\beta - \frac{1}{2}\right)}}{1 + (\lambda(\gamma - t))^2},
\end{aligned}$$

where the second inequality can be deduced from the fact that the integrand is increasing in α. In the final sum, each zero $\rho = \beta + i\gamma$ contributes a certain amount depending on the horizontal distance $\beta - 1/2$ and the vertical distance $|\gamma - t|$. We will split up the sum depending on whether $|\gamma - t|$ is small or large.

We will only have good control over $Z_\lambda(t)$ if there are not "too many" zeros in short intervals near height t. To make this precise, we define an exceptional set

$$E' := \left\{ t \in [T, 2T] \ : \ \begin{array}{r} N(t + \lambda^{-1}(k+1)) - N(t + \lambda^{-1}k) \geq \lambda^{-1} \log T \\ \text{for some } k = -\lfloor e^{2\lambda} \rfloor, \ldots, \lfloor e^{2\lambda} \rfloor - 1 \end{array} \right\}.$$

The measure of E' can be bounded using Lemma 3.4. Indeed, note that E' may covered by $2\lfloor e^{2\lambda} \rfloor$ translates of the set

$$\left\{ t \in [T, 2T] \colon N(t + \lambda^{-1}) - N(t) \geq \lambda^{-1} \log T \right\}$$

in addition to some small sets of length $\ll \lambda^{-1} e^{2\lambda}$ near the endpoints T and $2T$. So by the union bound and Lemma 3.4 we see that

$$\begin{aligned} \operatorname{meas} E' &\ll \lambda^{-1} e^{2\lambda} + T \exp(2\lambda) \exp(-d\lambda^{-1} \log T) \\ &\ll T \exp(2\lambda) \exp(-d\lambda^{-1} \log T). \end{aligned}$$

Hence, the measure of E' is less than the right-hand side of (3.14) (as long as d' is chosen to be small enough later) by our assumption that $V \leq \log T/\lambda$. This means that it will suffice for us to bound $Z_\lambda(t)$ for $t \in [T, 2T] \setminus E'$ from now on.

Fix a particular $t \in [T, 2T] \setminus E'$ and define intervals $I_k := (t + \lambda^{-1} k, t + \lambda^{-1}(k+1)]$ for $k = -\lfloor e^{2\lambda} \rfloor, \ldots, \lfloor e^{2\lambda} \rfloor - 1$. Also define

$$\theta_t := \max_{\substack{\rho = \beta + i\gamma \\ |\gamma - t| \leq e^{2\lambda}}} (\beta - 1/2).$$

Any zeta zero whose imaginary part lies in one of the intervals I_k will have real part at most $\frac{1}{2}+\theta_t$. Therefore,

$$\sum_{\beta>\frac{1}{2}} \frac{\lambda^2\left(\beta-\frac{1}{2}\right)^2 e^{\lambda\left(\beta-\frac{1}{2}\right)}}{1+(\lambda(\gamma-t))^2} \le \sum_k \sum_{\gamma\in I_k} \frac{\lambda^2\theta_t^2 e^{\lambda\theta_t}}{1+(\lambda(\gamma-t))^2} + \sum_{|\gamma-t|\ge\lambda^{-1}\lfloor e^{2\lambda}\rfloor} \frac{\lambda^2(\frac{1}{2})^2 e^{\lambda/2}}{1+(\lambda(\gamma-t))^2}$$

where we have trivially bounded the real part of the zeta zeros in last sum by 1.

Since t is not in the exceptional set E', each interval I_k contains the imaginary parts of at most $\lambda^{-1}\log T$ zeta zeros, and so

$$\sum_k \sum_{\gamma\in I_k} \frac{\lambda^2\theta_t^2 e^{\lambda\theta_t}}{1+(\lambda(\gamma-t))^2} \ll \sum_k \frac{(\lambda^2\theta_t^2 e^{\lambda\theta_t})(\lambda^{-1}\log T)}{1+(\min(|k|-1,0))^2} \ll \lambda\theta_t^2 e^{\lambda\theta_t}\log T.$$

For the remaining sum where $|\gamma-t|$ is large, we may apply the classical bound $N(t'+1)-N(t')=O(\log t')$, and we get

$$\begin{aligned}
\sum_{|\gamma-t|\ge\lambda^{-1}\lfloor e^{2\lambda}\rfloor} \frac{\lambda^2(\frac{1}{2})^2 e^{\lambda/2}}{1+(\lambda(\gamma-t))^2} &\ll e^{\lambda/2} \sum_{|\gamma-t|\ge\lambda^{-1}\lfloor e^{2\lambda}\rfloor} \frac{1}{(\gamma-t)^2} \\
&\ll e^{\lambda/2} \sum_{k=\lfloor\frac{1}{2}\lambda^{-1}e^{2\lambda}\rfloor}^{\infty} \frac{1}{k^2}\log(T+k) \\
&\ll e^{\lambda/2}\frac{\lambda(\log T+\lambda)}{e^{2\lambda}}.
\end{aligned}$$

The quantity in the last line tends to zero as T goes to infinity because of the assumption that $\lambda \ge \log\log T$. Hence if T is sufficiently large we may conclude that the magnitude of this whole sum is less than 1 say.

Putting together the bounds above we conclude for all $t \in [T,2T]\setminus E$,

$$|Z_\lambda(t)| \le C\lambda\theta_t^2 e^{\lambda\theta_t}\log T + 1$$

for some absolute constant $C>0$.

Suppose now that $\theta_t \leq d'\left(\frac{V}{\lambda \log T}\right)^{1/2}$ for a small absolute constant $d' > 0$. Then

$$Z_\lambda(t) \leq C\lambda\left(d'^2 \frac{V}{\lambda \log T}\right) e^{d'\left(\frac{V\lambda}{\log T}\right)^{1/2}} \log T + 1 \leq Cd'^2 e^{d'} V + 1.$$

This means that if d' is a small enough constant (depending on C), then we may conclude that $|Z_\lambda(t)| \leq V$.

We may now use our zero density estimate to show that θ_t is not larger than $d'\left(\frac{V}{\lambda \log T}\right)^{1/2}$ very often. Indeed, by Lemma 3.6 we see that

$$N\left(\frac{1}{2} + d'\left(\frac{V}{\lambda \log T}\right)^{1/2}, 3T\right) \ll T \log^5 T \exp\left(-d'\left(\frac{V \log T}{\lambda}\right)^{1/2}\right).$$

This implies that $\theta_t > d'\left(\frac{V}{\lambda \log T}\right)^{1/2}$ on a set of measure

$$\ll T \log^5 T \exp(2\lambda) \exp\left(-d'\left(\frac{V \log T}{\lambda}\right)^{1/2}\right).$$

The result follows. $\square$

3.5 Contribution of the Dirichlet polynomial

Recall that

$$D_\lambda(t) = \sum_k \hat{\omega}\left(\frac{\log k}{\lambda}\right) \frac{\Lambda(k)}{(\log k)\sqrt{k}} k^{-it}.$$

We would like to prove a rigorous version of the probabilistic heuristics discussed in Section 3.2.

We do not strive for the most general lemma possible but rather one which is sufficient for our purposes.

Lemma 3.8. *Let $\theta \in [0, 2\pi)$ be an angle. Suppose $40 \log\log T \le \lambda \le \frac{1}{40} \log T$ and $\sqrt{\log \lambda} \le V \le \frac{1}{20}\left(\frac{\log T}{\lambda}\right)^{\frac{1}{2}}$. Let*

$$F_V := \left\{ t \in [T, 2T] \colon \operatorname{Re}\left(e^{i\theta} D_\lambda(t)\right) \ge V \right\}.$$

Then we have

$$\operatorname{meas} F_V \ge T \exp\left(-500 \frac{V^2}{\log \frac{\lambda}{\log V}} \right).$$

To prove the lemma we will be using Soundararajan's resonance method which we describe now. We start by defining a Dirichlet polynomial

$$R(t) := \sum_{n \le N} f(n) n^{-it}$$

where f is an arithmetic function and N is a real parameter both of which we leave unspecified for now. This Dirichlet polynomial is called the *resonator*.

Let Φ be a smooth function supported in $[1, 2]$, such that $0 \le \Phi \le 1$ and $\Phi(t) = 1$ for $t \in (5/4, 7/4)$, and define the following quantities,

$$\begin{aligned}
M_1(R, T) &:= \int |R(t)|^2 \Phi(\tfrac{t}{T})\, dt, \\
M_2(R, T) &:= \int \operatorname{Re}\left(e^{i\theta} D_\lambda(t)\right) |R(t)|^2 \Phi(\tfrac{t}{T})\, dt, \\
M_3(R, T) &:= \int |R(t)|^4 \Phi(\tfrac{t}{T})\, dt.
\end{aligned}$$

For any choice of resonator, it is clear that we have the inequality

$$\max_{t \in [T, 2T]} \operatorname{Re}\left(e^{i\theta} D_\lambda(t)\right) \ge \frac{M_2(R, T)}{M_1(R, T)}.$$

Hence, in order to prove that $\operatorname{Re}(e^{i\theta} D_\lambda(t))$ attains large values, one can try to find a resonator that makes the ratio $M_2(R, T)/M_1(R, T)$ as large as possible.

It is illuminating to look at the inequality above from a probabilistic perspective. In particular, if one thinks of $|R(t)|^2\Phi(\frac{t}{T})$ as a (non-normalized) probability density, then the inequality states that the maximum value of $\mathrm{Re}(e^{i\theta}D_\lambda(t))$ is at least as large as the the expected value of this quantity. Using a variant of the Paley-Zygmund inequality from probability theory, we can also find a lower bound on the measure of F_V. Indeed, observe that

$$\begin{aligned} M_2(R,T) &\leq V\,M_1(R,T) + \int_{F_V} \mathrm{Re}\big(e^{i\theta}D_\lambda(t)\big)|R(t)|^2\Phi\big(\tfrac{t}{T}\big)\,dt \\ &\leq V\,M_1(R,T) + \big(\mathrm{meas}\,F_V\big)^{\frac{1}{4}}\left(\int_{-\infty}^{\infty}|D_\lambda(t)|^4\Phi\big(\tfrac{t}{T}\big)\,dt\right)^{\frac{1}{4}} M_3(R,T)^{\frac{1}{2}} \end{aligned}$$

where the second line follows from the first by two applications of Cauchy's inequality. If we can choose the resonator so that $M_2(R,T) \geq 2V\,M_1(R,T)$, then rearranging the inequality above gives

$$\mathrm{meas}\,F_V \gg \frac{M_2(R,T)^4}{M_3(R,T)^2}\left(\int_{-\infty}^{\infty}|D_\lambda(t)|^4\Phi\big(\tfrac{t}{T}\big)\,dt\right)^{-1}. \tag{3.15}$$

This is the bound which we will use to prove the lemma. As we shall see, in order to make this bound as strong as possible the most important factor to control is $M_3(R,T)$ (i.e. we want $M_3(R,T)$ to be as small as possible). Hence, our objective is as follows: given some V (and λ and T) we need to pick a resonator so that $M_2(R,T) \geq 2V\,M_1(R,T)$ and $M_3(R,T)$ is as small as possible.

In order to proceed with the proof of the lemma, we will first show how to estimate the integrals M_1, M_2, etc. using standard techniques for estimating mean values of Dirichlet polynomials. Once this is done we will show how to make a suitable choice of f and N.

3.5.1 Integral estimates

If $N \le T^{1-\varepsilon}$, we are able to estimate M_1 and M_2 as follows. Note that

$$M_1(R,T) = \int_{-\infty}^{\infty} |R(t)|^2 \Phi(\tfrac{t}{T})\, dt = T \sum_{m,n\le N} f(m)\overline{f(n)}\, \hat{\Phi}(T\log(m/n)).$$

For the terms where $m \ne n$, we have that $|\log(m/n)| \gg T^{\varepsilon-1}$. Hence $\hat{\Phi}(T\log(m/n)) \ll_\varepsilon T^{-2}$ (say) since $\hat{\Phi}$ is rapidly decaying. Therefore

$$\begin{aligned} M_1(R,T) &= T\hat{\Phi}(0)\sum_{n\le N}|f(n)|^2 + O\left(T^{-1}\sum_{m,n\le N}|f(m)||f(n)|\right) \\ &= T\hat{\Phi}(0)(1+O(T^{-1}))\sum_{n\le N}|f(n)|^2. \end{aligned}$$

Similarly, for $M_2(R,T)$ we have

$$M_2(R,T) = T\,\mathrm{Re}\, e^{i\theta} \sum_{\substack{m,n\le N\\ k}} \hat{\omega}\left(\frac{\log k}{\lambda}\right)\frac{\Lambda(k)}{(\log k)\sqrt{k}} f(m)\overline{f(n)}\, \hat{\Phi}(T\log(mk/n)).$$

If $mk \ne n$ then $|\log(mk/n)| \gg T^{\varepsilon-1}$, so $\hat{\Phi}(T\log(mk/n)) \ll_\varepsilon T^{-2}$ in this case. Hence, the total contribution of these off-diagonal terms (i.e. the terms where $mk \ne n$) is bounded by

$$T^{-1}\sum_k \hat{\omega}\left(\frac{\log k}{\lambda}\right)\frac{\Lambda(k)}{(\log k)\sqrt{k}} \sum_{m,n\le N}|f(m)||f(n)|.$$

By the hypothesis of the lemma $\lambda \le \frac{1}{40}\log T$, the above sum over k can be trivially bounded above by $e^{\lambda} \le T^{\frac{1}{40}}$. Hence we may conclude that

$$M_2(R,T) = T\hat{\Phi}(0)\,\mathrm{Re}\, e^{i\theta} \sum_{\substack{mk=n\\ m,n\le N}} \hat{\omega}\left(\frac{\log k}{\lambda}\right)\frac{\Lambda(k)}{(\log k)\sqrt{k}} f(m)\overline{f(n)} + O\left(T^{\frac{1}{40}}\sum_{n\le N}|f(n)|^2\right).$$

Note that the precise power of T in the error term is not significant here. We could get any power saving we want since $\hat{\Phi}$ is rapidly decaying.

To estimate $M_3(R,T)$, the method is essentially the same as above. In this case we must assume $N \leq T^{\frac{1}{2}-\varepsilon}$ to bound the contribution of the off-diagonal terms, and we will also make the assumption that the resonator coefficients satisfy $|f(n)| \leq 1$ for all n. We then have the estimate

$$\begin{aligned} M_3(R,T) &= T \sum_{m,n,m',n' \leq N} f(m)f(n)\overline{f(m')f(n)}\,\hat{\Phi}(T \log \tfrac{mn}{m'n'}) \\ &= T\hat{\Phi}(0) \sum_{\substack{m,n,m',n' \leq N \\ mn=m'n'}} f(m)f(n)\overline{f(m')f(n')} + O\Bigg(T^{-3} \sum_{m,n,m',n' \leq N} 1\Bigg) \\ &= T\hat{\Phi}(0) \sum_{\substack{m,n,m',n' \leq N \\ mn=m'n'}} f(m)f(n)\overline{f(m')f(n')} + O(T^{-1}). \end{aligned}$$

In the second equality we used the bound $\hat{\Phi}(T \log \frac{mn}{m'n'}) \ll T^{-4}$ when $mn \neq m'n'$, and the bound $|f(n)| \leq 1$. Once again we could have gotten any power saving of T that we wanted in the error term.

Lastly, using the same methods as above and the assumption that $\lambda \leq \frac{1}{40} \log T$ to bound off-diagonal terms one may check that

$$\int_{-\infty}^{\infty} |D_\lambda(t)|^4 \Phi\big(\tfrac{t}{T}\big)\, dt = T\hat{\Phi}(0)(1 + O(T^{-1})) \sum_n |c_n|^2$$

where c_n denotes the nth coefficient of the Dirichlet polynomial $D_\lambda(t)^2$. We only need an upper bound on this integral, so we can proceed by upper bounding c_n. Note that if c_n is nonzero then n must be a prime power or a product of two prime powers. It is not hard to check that

$$c_n \ll \begin{cases} \frac{\log(r+2)}{r} \frac{1}{\sqrt{p^r}} & \text{if } n = p^r \leq e^{2\lambda}, \\ \frac{1}{rs} \frac{1}{\sqrt{p^r q^s}} & \text{if } n = p^r q^s \text{ such that } p \neq q \text{ and } p^r, q^s \leq e^\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\begin{aligned}
\sum_n |c_n|^2 &\ll \sum_{p^r, q^s \le e^\lambda} \frac{1}{r^2 s^2} \frac{1}{p^r q^s} + \sum_{p^r \le e^{2\lambda}} \frac{\log^2(r+2)}{r^2} \frac{1}{p^r} \\
&\ll \left(\sum_{p^r \le e^\lambda} \frac{1}{r^2} \frac{1}{p^r} \right)^2 + \sum_{p \le e^{2\lambda}} \frac{1}{p} \\
&\ll \left(\sum_{p \le e^\lambda} \frac{1}{p} \right)^2 + \sum_{p \le e^{2\lambda}} \frac{1}{p} \\
&\ll (\log \lambda)^2,
\end{aligned}$$

where in the last line we have used the classical bound $\sum_{p \le x} \frac{1}{p} \ll \log\log x$. We conclude that

$$\int_{-\infty}^{\infty} |D_\lambda(t)|^4 \Phi\big(\tfrac{t}{T}\big)\, dt \ll T(\log \lambda)^2.$$

3.5.2 Resonator coefficients

From now on we set $N = T^{\frac{1}{4}}$ since this is sufficient for the integral estimates discussed above to hold. Given V, λ, and T satisfying the hypotheses of the lemma, we would like to optimize our choice of the coefficient function $f(n)$. As we have stated previously, this means choosing an f such that $M_3(R,T)$ is as small as possible while also ensuring that $M_2(R,T)/M_1(R,T)$ is at least $2V$.

Rather than asserting a complete definition of f right away, for now we just impose some restrictions on f and then we will continue with our analysis assuming these restrictions hold. The restrictions are

(a) $|f(n)| \le 1$ (we already used this in our estimate of $M_3(R,T)$ above),

(b) f is a multiplicative function supported on squarefree numbers composed of primes in the set $\mathcal{P} \coloneqq \{P \le p \le Q\}$ for some parameters $P, Q \ge 3$,

(c) $f(p) = e^{i\theta}|f(p)|$,

(d) $-\frac{\log N}{\log Q} + \frac{2}{\log Q}\sum_p (\log p)|f(p)|^2 \le -5$.

The reasons for each of these restrictions will become clear shortly when we use them to deduce a lower bound for $M_2(R,T)/M_1(R,T)$ and an upper bound for $M_3(R,T)$ each of which takes a relatively simple form.

3.5.2.1 Lower bound on $M_2(R,T)/M_1(R,T)$

From our integral estimates, the main term of the quotient $M_2(R,T)/M_1(R,T)$ is

$$\operatorname{Re}\left(e^{i\theta} \sum_{\substack{mk=n \\ m,n\le N}} \hat{\omega}\left(\frac{\log k}{\lambda}\right) \frac{\Lambda(k)}{(\log k)\sqrt{k}} f(m)\overline{f(n)} \right) \Bigg/ \left(\sum_{n \le N} |f(n)|^2 \right). \tag{3.16}$$

We start by considering the numerator of this ratio. Note that in order for a term in the sum to be nonzero, by restriction (b) it is necessary that k be a prime $p \in \mathcal{P}$ and p be distinct from the primes which divide m. In this case we have $\overline{f(n)} = \overline{f(p)f(m)} = e^{-i\theta}|f(p)|\overline{f(m)}$ by restriction (c). So we see that the numerator equals

$$\sum_p \hat{\omega}\left(\frac{\log p}{\lambda}\right) \frac{|f(p)|}{\sqrt{p}} \sum_{\substack{m \le N/p \\ p \nmid m}} |f(m)|^2.$$

We will now approximate the inner sum by removing the cutoff at N/p. To bound the error (i.e. the sum of tail terms $m > N/p$) we can use Rankin's trick, which is the observation that for any sequence $\{a_m\}_{m \in \mathbb{N}}$ of nonnegative reals and $\alpha > 0$,

$$\sum_{m > X} a_m \le X^{-\alpha} \sum_{m=1}^{\infty} m^\alpha a_m.$$

Applying this trick we see that our numerator is at least

$$\sum_p \hat{\omega}\left(\frac{\log p}{\lambda}\right)\frac{|f(p)|}{\sqrt{p}}\sum_{\substack{m\\ p\nmid m}}|f(m)|^2 - \sum_p \hat{\omega}\left(\frac{\log p}{\lambda}\right)\frac{|f(p)|}{\sqrt{p}}\left(\frac{p}{N}\right)^{\alpha}\sum_{\substack{m\\ p\nmid m}} m^{\alpha}|f(m)|^2 \quad (3.17)$$

for any choice of $\alpha > 0$. We will now show that the error term (i.e. the second double sum) is negligible relative to the main term (the first double sum). For notational convenience, define

$$\Psi := \sum_p \hat{\omega}\left(\frac{\log p}{\lambda}\right)\frac{|f(p)|}{\sqrt{p}}.$$

Note that the inner sum in the main term of (3.17) satisfies

$$\sum_{\substack{m\\ p\nmid m}}|f(m)|^2 = \prod_{\substack{q \text{ prime}\\ q\neq p}}(1+|f(q)|^2) \geq \frac{1}{2}\prod_{q \text{ prime}}(1+|f(q)|^2)$$

since $|f(q)| \leq 1$ by restriction (a). So we see that the main term of (3.17) is

$$\geq \frac{1}{2}\Psi \prod_{q \text{ prime}}(1+|f(q)|^2).$$

To upper bound the error term we will choose $\alpha = 1/\log Q$. This implies $p^{\alpha} \leq e$ for $p \in \mathcal{P}$, so the magnitude of the error term is

$$\begin{aligned}
&\leq \Psi\, eN^{-\alpha}\sum_m m^{\alpha}|f(m)|^2\\
&= \Psi\, eN^{-\alpha}\prod_{q \text{ prime}}(1+q^{\alpha}|f(q)|^2).
\end{aligned}$$

Hence the magnitude of the ratio of the error term to the main term is

$$\begin{aligned}
&\leq 2eN^{-\alpha}\prod_{q \text{ prime}}\frac{1+q^{\alpha}|f(q)|^2}{1+|f(q)|^2}\\
&\leq 2e\exp\left(-\alpha\log N + \sum_{q\in\mathcal{P}}(q^{\alpha}-1)|f(q)|^2\right).
\end{aligned}$$

Since $q^{\alpha} - 1 \leq 2\alpha \log q$ for $q \in \mathcal{P}$, this is

$$\leq 2e \exp\left(-\frac{\log N}{\log Q} + \frac{2}{\log Q} \sum_{q \in \mathcal{P}} (\log q) |f(q)|^2 \right) \leq 2e \exp(-5) \leq \frac{1}{10}$$

by restriction (d). We conclude that the error term in (3.17) is small relative to the main term, so the numerator of (3.16) is at least

$$\left(1 - \frac{1}{10}\right) \frac{1}{2} \Psi \prod_{q \text{ prime}} (1 + |f(q)|^2) = \frac{9}{20} \Psi \prod_{q \text{ prime}} (1 + |f(q)|^2)$$

The denominator of (3.16) is clearly bounded above by $\prod_{q \text{ prime}} (1 + |f(q)|^2)$, so the quotient is at least $\frac{9}{20}\Psi$. Using this along with our integral estimates for $M_1(R,T)$ and $M_2(R,T)$ we conclude that

$$\frac{M_2(R,T)}{M_1(R,T)} \geq (1 + O(T^{-1})) \frac{9}{20} \Psi + O(T^{-\frac{39}{40}}). \tag{3.18}$$

3.5.2.2 Upper bound on $M_3(R,T)$

Note that

$$\begin{aligned} \left| \sum_{\substack{m,n,m',n' \leq N \\ mn = m'n'}} f(m) f(n) \overline{f(m') f(n')} \right| &\leq \prod_p (1 + 4|f(p)|^2 + |f(p)|^4) \\ &\leq \exp\left(5 \sum_p |f(p)|^2 \right) \end{aligned}$$

where we have again used that $|f(p)| \leq 1$ by restriction (b). So by our integral estimate for $M_3(R,T)$ we conclude

$$M_3(R,T) \ll T \exp\left(5 \sum_p |f(p)|^2 \right). \tag{3.19}$$

3.5.2.3 Optimizing parameters

Recall that

$$\Psi = \sum_p \hat{\omega}\left(\frac{\log p}{\lambda}\right)\frac{|f(p)|}{\sqrt{p}}.$$

If f is chosen so that $\Psi \geq 5V$, then the lower bound (3.18) implies that $M_2(R,T)/M_1(R,T) \geq 2V$ for all T greater than some absolute constant. Subject to this constraint $\Psi \geq 5V$, we would like to minimize the sum $\sum_p |f(p)|^2$ since this is the quantity that appears in our upper bound (3.19) on $M_3(R,T)$.

Applying Cauchy's inequality to the definition of Ψ, note that

$$\Psi \leq \left(\sum_{p\in\mathcal{P}} |f(p)|^2\right)^{\frac{1}{2}} \left(\sum_{p\in\mathcal{P}} \hat{\omega}\left(\tfrac{\log p}{\lambda}\right)^2 \frac{1}{p}\right)^{\frac{1}{2}}.$$

Hence, the constraint $\Psi \geq 5V$ implies that

$$\sum_{p\in\mathcal{P}} |f(p)|^2 \geq \frac{25V^2}{\sum_{p\in\mathcal{P}} \hat{\omega}\left(\frac{\log p}{\lambda}\right)^2 \frac{1}{p}}. \tag{3.20}$$

By the equality case of Cauchy's inequality, this lower bound can be attained by choosing $|f(p)|$ to be a certain fixed scalar multiple of $\hat{\omega}\left(\frac{\log p}{\lambda}\right)\frac{1}{\sqrt{p}}$ for each $p \in \mathcal{P}$. Indeed, we see that taking

$$|f(p)| := \frac{5V}{\sum_{q\in\mathcal{P}} \hat{\omega}\left(\frac{\log q}{\lambda}\right)^2 \frac{1}{q}} \hat{\omega}\left(\frac{\log p}{\lambda}\right)\frac{1}{\sqrt{p}}. \tag{3.21}$$

makes (3.20) into an equality and also makes $\Psi = 5V$.

Combining everything that we have seen so far, if all the restrictions (a),(b),(c), and (d) are satisfied when we define $|f(p)|$ according to (3.21) then

$$\frac{M_2(R,T)}{M_1(R,T)} \geq 2V$$

and

$$M_3(R,T) \ll T \exp\left(\frac{125V^2}{\sum_{p\in\mathcal{P}} \hat{\omega}\left(\frac{\log p}{\lambda}\right)^2 \frac{1}{p}}\right).$$

Hence it is advantageous to select P as small as our restrictions allow and Q as large as our restrictions allow so that the upper bound on $M_3(R,T)$ is as strong as possible. We will make the following choice of parameters,

$$P := 25V^2,$$
$$Q := e^{\lambda}.$$

We must now check that the restrictions are satisfied. For restrictions (b) and (c) there is nothing to check. For restriction (a), we first note that a routine calculation using the prime number theorem implies that

$$\begin{aligned}\sum_{p\in\mathcal{P}} \hat{\omega}\left(\tfrac{\log p}{\lambda}\right)^2 \frac{1}{p} &= \log \frac{\lambda}{\log 25V^2} - \frac{3}{2} + 2\frac{\log 25V^2}{\lambda} - \frac{1}{2}\left(\frac{\log 25V^2}{\lambda}\right)^2 + o_{T\to\infty}(1)\\ &= \log\frac{\lambda}{\log V} - \log 2 - \frac{3}{2} + 4\frac{\log V}{\lambda} - 2\left(\frac{\log V}{\lambda}\right)^2 + o_{T\to\infty}(1).\end{aligned}$$

We can conclude that if the ratio $\frac{\lambda}{\log V}$ is large enough then this implies

$$\sum_{p\in\mathcal{P}} \hat{\omega}\left(\tfrac{\log p}{\lambda}\right)^2 \frac{1}{p} \geq 1 \tag{3.22}$$

and also

$$\sum_{p\in\mathcal{P}} \hat{\omega}\left(\tfrac{\log p}{\lambda}\right)^2 \frac{1}{p} \geq \frac{1}{2}\log\frac{\lambda}{\log V} \tag{3.23}$$

for all T larger than some absolute constant. By the hypotheses of the lemma, $\lambda \geq 40\log\log T$ and $\log V \leq \frac{1}{2}\log\log T$. Hence $\frac{\lambda}{\log V} \geq 80$ which one can verify is sufficiently large.

Applying (3.22) and the fact that $\hat{\omega} \leq 1$ to our choice of $|f(p)|$ we see that

$$|f(p)| \leq \frac{5V}{\sqrt{p}} \leq \frac{5V}{\sqrt{P}} \leq 1$$

for any $p \in \mathcal{P}$. Hence, restriction (a) holds.

Lastly we must check restriction (d). Note that (3.22) implies $\sum_p |f(p)|^2 \leq 25V^2$. Using this we see that the left-hand side of restriction (d) is

$$\begin{aligned}
-\frac{\log N}{\log Q} + \frac{2}{\log Q} \sum_{p \in \mathcal{P}} (\log p)|f(p)|^2 &= -\frac{1}{4}\frac{\log T}{\lambda} + \frac{2}{\lambda} \sum_{p \in \mathcal{P}} (\log p)|f(p)|^2 \\
&\leq -\frac{1}{4}\frac{\log T}{\lambda} + 2 \sum_{p \in \mathcal{P}} |f(p)|^2 \\
&\leq -\frac{1}{4}\frac{\log T}{\lambda} + 50V^2 \\
&\leq -\frac{1}{4}\frac{\log T}{\lambda} + \frac{1}{8}\frac{\log T}{\lambda} \\
&\leq -5.
\end{aligned}$$

The second to last inequality follows from the hypothesis that $V \leq \frac{1}{20}\left(\frac{\log T}{\lambda}\right)^{\frac{1}{2}}$, and the last inequality follows from the hypothesis that $\lambda \leq \frac{1}{40}\log T$. Hence restriction (d) is satisfied.

So we have now seen that restrictions (a),(b),(c),(d) all hold. As we have seen, this means $M_2(R,T)/M_1(R,T) \geq 2V$ and

$$M_3(R,T) \ll T \exp\left(\frac{125V^2}{\sum_{p \in \mathcal{P}} \hat{\omega}\left(\frac{\log p}{\lambda}\right)^2 \frac{1}{p}} \right) \leq T \exp\left(\frac{250V^2}{\log \frac{\lambda}{\log V}} \right)$$

where the second inequality follows from (3.23).

We may then conclude from (3.15) that

$$\begin{aligned}
\operatorname{meas} F_V &\gg \frac{M_2(R,T)^4}{M_3(R,T)^2}\left(\int_{-\infty}^{\infty} |D_\lambda(t)|^4 \Phi(\tfrac{t}{T})\,dt\right)^{-1} \\
&\gg T\exp\left(-500\frac{V^2}{\log\frac{\lambda}{\log V}}\right) V^4(\log\lambda)^{-2} \\
&\gg T\exp\left(-500\frac{V^2}{\log\frac{\lambda}{\log V}}\right).
\end{aligned}$$

In the second inequality we have used our integral estimates and that $M_2(R,T) \gg VM_1(R,T) \gg VT$. In the last inequality we have used the hypothesis that $V \geq \sqrt{\log\lambda}$. This completes the proof of the lemma.

3.6 Proof of Theorem 3.1

We will now prove the theorem by combining the convolution formula with the measure bounds proved in the previous two sections. Suppose that V is given to us and satisfies the bounds $\sqrt{\log\log T} \leq V \leq \kappa\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}$ where κ is a small constant to be chosen later. We assume T is large, and hence V is large as well.

To apply the lemmas we will be selecting a value of the parameter λ depending on V. Before we do this, let us reestablish some of the notation from the lemmas. Recall that the convolution formula (i.e. Lemma 3.3) states that

$$\int_{-\log T}^{\log T} S(t+u)K_\lambda(u)\,du = \frac{1}{\pi}\operatorname{Im} D_\lambda(t) + Z_\lambda(t) + O(1).$$

The contribution of the $Z_\lambda(t)$ term was discussed in Lemma 3.7. In particular, as long as λ and V satisfy the necessary hypotheses, the lemma gives a bound on the measure of the set

$$E_V = \{t \in [T,2T] \colon |Z_\lambda(t)| > V\}.$$

Similarly, the contribution of the $\frac{1}{\pi}\operatorname{Im} D_\lambda(t)$ term can be handled by Lemma 3.8. Taking $\theta = \frac{3\pi}{2}$, the lemma can be applied to get a lower bound on the measure of the set

$$F_{4\pi V} = \{t \in [T, 2T] \colon \operatorname{Re}\Big(e^{\frac{3\pi}{2}i} D_\lambda(t)\Big) > 4\pi V\} = \{t \in [T, 2T] \colon \tfrac{1}{\pi}\operatorname{Im} D_\lambda(t) > 4V\}.$$

Let us now define another set G_{2V} as

$$G_{2V} \coloneqq \left\{t \in [T, 2T] \colon \int_{-\log T}^{\log T} S(t+u) K_\lambda(u)\, du > 2V\right\}.$$

The convolution formula implies that $G_{2V} \supseteq F_{4\pi V} \setminus E_V$, which we can use to get a lower bound on the measure of G_{2V}.

We now select the value of the λ parameter. We will consider two cases depending on whether $V > (\log T)^{\frac{2}{9}}$ or $V \leq (\log T)^{\frac{2}{9}}$. In the first case, let $\lambda = (\log T)/V^3$. This implies $\kappa^{-3} \log\log T \leq \lambda < (\log T)^{\frac{1}{3}}$ and one can easily verify that the hypotheses of our lemmas are satisfied as long as κ is somewhat small. Applying Lemma 3.7 and Lemma 3.8 gives

$$\operatorname{meas} E_V \ll T \exp\Big(10(\log T)^{\frac{1}{3}}\Big) \exp(-d' V^2) \ll T \exp\left(-\frac{d'}{2} V^2\right), \text{ and}$$
$$\operatorname{meas} F_{4\pi V} \gg T \exp\left(-500 \frac{(4\pi V)^2}{\log \frac{\log T}{V^3 \log 4\pi V}}\right) \gg T \exp\left(-D' \frac{V^2}{\log \frac{\log T}{V^3 \log V}}\right)$$

where $D' > 0$ is a large absolute constant. By taking κ very small, we can ensure that the quantity $\log \frac{\log T}{V^3 \log V}$ is uniformly very large. In particular, depending on the values of the absolute constants D' and d' we can choose κ small enough so that the measure of $F_{4\pi V}$ is at least say twice the measure of E_V.

Now consider the second case where $\sqrt{\log\log T} \leq V \leq (\log T)^{\frac{2}{9}}$. Here we choose $\lambda = (\log T)^{\frac{1}{10}}$. It is again easy to verify that the hypotheses of Lemmas 3.7 and 3.8

are satisfied, and they imply that

$$\operatorname{meas} E_V \ll T\exp\Big(10(\log T)^{\frac{1}{10}}\Big)\exp(-d'V^{\frac{1}{2}}(\log T)^{\frac{9}{20}}) \ll T\exp\big(-(\log T)^{0.45}\big), \text{ and}$$

$$\operatorname{meas} F_{4\pi V} \gg T\exp\left(-500\frac{(4\pi V)^2}{\log\frac{\log T}{10\log 4\pi V}}\right) \gg T\exp\left(-D'\frac{V^2}{\log\frac{\log T}{V^3\log V}}\right).$$

Once again we have that the measure of $F_{4\pi V}$ is at least twice that of E_V since $\operatorname{meas} F_{4\pi V} \gg T\exp(-(\log T)^{\frac{4}{9}})$ in this case.

Putting together the two cases, we have now seen that for any V satisfying the hypotheses of the theorem it is possible to select λ such that

$$\operatorname{meas} G_{2V} \geq \operatorname{meas} F_{4\pi V} - \operatorname{meas} E_V \gg F_{4\pi V} \gg T\exp\left(-D'\frac{V^2}{\log\frac{\log T}{V^3\log V}}\right). \tag{3.24}$$

This is nearly the bound stated in the conclusion of the theorem, but G_{2V} consists of points in $[T, 2T]$ where a *smoothing* of $S(t)$ is greater than $2V$. It remains to convert this into a lower bound on the measure of the set where $S(t)$ is greater than V.

To pass from 3.24 to the conclusion of the theorem, first note that

$$\int_{G_{2V}}\int_{-\log T}^{\log T} S(t+u)K_\lambda(u)\,du\,dt \geq 2V \ \operatorname{meas} G_{2V}$$

by the definition of G_{2V}. The double integral on the left-hand side can be rewritten (by a change of variables and Fubini's theorem) as an integral of $S(u)$ times a smoothing of the indicator function of G_{2V}. Indeed, letting $1_A(u)$ denote the indicator function of a set A and letting $\widetilde{K_\lambda}(u) := K_\lambda(u)1_{[-\log T,\log T]}(u)$, one quickly verifies that

$$\int_{G_{2V}}\int_{-\log T}^{\log T} S(t+u)K_\lambda(u)\,dudt = \int S(u)(1_{G_{2V}} * \widetilde{K_\lambda})(u)\,du.$$

Note that the support of the convolution $1_{G_{2V}} * \widetilde{K_\lambda}$ is contained in the interval $[T-\log T, 2T+\log T]$ so we can assume the last integral is taken over this bounded interval.

Letting $H_V := \{t \in [T-\log T, 2T+\log T]\colon S(t) > V\}$, and using the same Paley-Zygmund inequality idea as in the previous section we have

$$\begin{aligned}\int S(u)(1_{G_{2V}} * \widetilde{K_\lambda})(u)\,du &\leq V \int (1_{G_{2V}} * \widetilde{K_\lambda})(u)\,du + \int_{H_V} S(u)(1_{G_{2V}} * \widetilde{K_\lambda})(u)\,du \\ &\leq V\,||1_{G_{2V}} * \widetilde{K_\lambda}||_1 + ||1_{G_{2V}} * \widetilde{K_\lambda}||_\infty \int_{H_V} S(u)\,du \\ &\leq V\ \operatorname{meas} G_{2V} + (\operatorname{meas} H_V)^{\frac{1}{2}} \left(\int_{T-\log T}^{2T+\log T} |S(u)|^2\,du \right)^{\frac{1}{2}}.\end{aligned}$$

The last inequality follows from the fact that $||1_{G_{2V}} * \widetilde{K_\lambda}||_1 \leq ||1_{G_{2V}}||_1 ||\widetilde{K_\lambda}||_1 \leq \operatorname{meas} G_{2V}$ and $||1_{G_{2V}} * \widetilde{K_\lambda}||_\infty \leq ||1_{G_{2V}}||_\infty ||\widetilde{K_\lambda}||_1 \leq 1$, and from applying Cauchy's inequality to the integral. Since the left-hand side of this string of inequalities is at least $2V\ \operatorname{meas} G_{2V}$ by our observations in the previous paragraph, rearranging we see that

$$\operatorname{meas} H_V \geq V^2 (\operatorname{meas} G_{2V})^2 \left(\int_{T-\log T}^{2T+\log T} |S(u)|^2\,du \right)^{-1}. \tag{3.25}$$

The moments of $S(t)$ were estimated unconditionally by Selberg in [26, Thm. 6], and from the second moment estimate we can conclude

$$\int_{T-\log T}^{2T+\log T} |S(t)|^2\,dt \ll T \log\log T.$$

Inserting this and the lower bound (3.24) into (3.25) we conclude that

$$\begin{aligned}\operatorname{meas} H_V &\gg V^2 T^2 \exp\left(-2D' \frac{V^2}{\log \frac{\log T}{V^3 \log V}} \right) (T \log\log T)^{-1} \\ &\gg T \exp\left(-D \frac{V^2}{\log \frac{\log T}{V^3 \log V}} \right)\end{aligned}$$

for an absolute constant $D > 0$. In the last inequality we used the assumption that $V \geq \sqrt{\log\log T}$. The intervals $[T - \log T, T)$ and $(2T, 2T + \log T]$ have very small measure compared to this lower bound, so they can safely be thrown away. This gives the conclusion stated in the theorem for positive values of $S(t)$ (after possibly adjusting the constant D to turn the $\gg$ bound we have just proved into a $\geq$ bound). To get the same conclusion for $-S(t)$ the proof is identical except that we select $\theta = \frac{\pi}{2}$ in the application of Lemma 3.8.

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