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Publication Date

1986-09-01

ECONOMIC EVALUATION OF REMUNERATION FROM PATENTS $\qquad \qquad \text{AND TECHNOLOGY TRANSFER}$

September 1986

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First Draft

September 1986

1. <u>Introduction</u>

One of the most critical and complex issues to be resolved between a would-be supplier (licensor) and recipient (licensee) of technological know-how is the monetary value, or price, of the "merchandise" transferred. This price is a function of a number of factors, including the nature of the item in question and the risk and return associated with its commercialization.

The direct monetary compensation may take the form of lump sum payment or "fixed royalties" - a pre-determined amount to be paid in one or a few installments; or, alternatively, of "variable royalties" - recurring payments, determined as a function of the economic usefulness or the achievements of the technology (measured, for example, by the volume or amount of the sales, or by the profit). There are qualitative reasons for choosing one form of payment over the other. The advantages of a lump-sum payment are:

- (a) The cost of the technology is known in advance.
- (b) The licensor can dispense with examination of the licensee's account.
- (c) Competitive licensing offers are readily compared.
 While its disadvantages are:
- (a) The licensor incurs no income risk and thus is less motivated to maintain interest in the licensee's enterprise. Expansion of the latter's market (for example, through process improvements) may be hindered if it brings no additional income to the licensor.
- (b) The licensor has no share in the achievements of the technology if it proves successful.

These disadvantages are obviated in a variable-royalties scheme, so that a combination of both forms can often be beneficial to either or both parties. $^{\footnotesize 1}$

The empirical evidence on royalty arrangements is quite scarce, and usually in aggregated form. In a study by Contractor (1980) on 102 technology transfer licenses from the U.S.A., it is shown that many forms of royalty payments exist. In his sample, variable royalties appeared in 80 percent of the contracts, with rates ranging from 3 to 10 percent, with 5 percent being the most typical figure. The duration of most agreements is 5 to 10 years (Kopits, 1976). Similar findings were obtained by Hersovic (1976) who studied the import of technological know-how through licensing agreements to Israel. He states that in 36 percent of the contracts involved, lump-sum payments and variable royalties appeared in 83 percent, and combinations of various forms of payments in 25 percent. Variable royalties combined with a minimum payment accounted for 15 percent. A mere 11 percent were in pure lump-sum form, while in 60 percent lump-sum payments were absent altogether. There was significant correlation between the type of payment and the industry affiliation.

Herskovic's study indicates also that the royalty rate of know-how agreements is usually lower when royalties are combined with other kinds of payments, than when they appear alone. It was found that the average rate from net sales for all industries is 4.0 percent in the former case and 5.0 percent in the latter.

¹In certain developing countries minimum royalty schemes in importing technologies are prohibited on the grounds that the licensor should share the licensee's market risk. However, it is difficult to distinguish between a lump-sum payment and a minimum-royalty schedule.

A royalty arrangement is a rule of risk-return sharing between the parties, and its economic value should be calculated and compared to that of an alternative arrangement. In this study we compare different royalty schedules and determine the monetary trade-offs among them. A valuation model is used for expressing the alternatives in present-value terms. The effect of the risk associated with the business stemming from the technology, on the value of each royalty arrangement, is specifically analyzed and discussed.

In Section 2 four royalty schemes are described in a one-period framework. Section 3 details the assumptions made in order to derive closed-form analytical solutions. In section 4 the models to determine the present-value of each royalty schedule are derived. The models, based on valuation models for contingent claims, are used for assessing and comparing the four alternative royalty schemes. In Section 5 simulation results are presented and discussed and the trade-offs between the components of the royalty schemes are quantified. The implications of the models and concluding remarks are presented in Section 6.

2. <u>The Royalties Schemes</u>

Based on the empirical evidence, four common combinations of percentage royalties and lump sums are evaluated and compared to a basic payment of a fixed lump sum. The benchmark payment is denoted by $R_{_{\scriptsize O}}=A_{_{\scriptsize O}}$. The four combinations are described as follows:

(a) R_a is the total royalties from a payment of a fixed lump sum of A_a and percentage royalties, r_a , of sales, S = PQ:

$$(1) \quad R_a = A_a + r_a PQ$$

where P is the price of a unit sold and Q is the quantity. The benchmark payment R_0 is a special case of (a), when $A_a = A_0$ and $r_a = 0$.

(b) The second combination is based on the principle of "Minimum Royalties", the payment thus being:

(2)
$$R_b = MAX (A_b, r_bPQ)$$

Namely, the licensee is obligated to pay at least $A_{\hat{b}}$ dollars; if the percentage royalties $r_{\hat{b}}^{PQ}$ exceed $A_{\hat{b}}^{}$, the company will pay the larger sum. Hence

$$R_b = MAX (r_b S_b^*, r_b S) = r_b MAX (S_b^*, S)$$

where $S_b^* = A_b/r_b$ is the critical sales volume above which variable royalties will be paid, and S = PQ

(c) In the third combination ("Minimum Sales") royalties are only paid after a minimum amount of sales S_c^* is achieved. Hence the payment R_c is given by:

(3)
$$R_c = MAX (0, -A_c + r_c PQ)$$

or

$$R_{c} = \text{MAX } [0, r_{c}(S - S_{c}^{*})] = r_{c} \text{ MAX } [0, (S - S_{c}^{*})]$$
 where $S_{c}^{*} = A_{c}/r_{c}$

(d) In the fourth combination ("Maximum Royalties"), a rate r_d is paid on sales up to a ceiling denoted by A_d . The payment R_d is therefore given by

(4)
$$R_d = MIN (r_d PQ, A_d)$$

= $r_d MIN (S, S_d^*)$

where $S_d^* = A_d/r_d$ is the critical sales volume above which no additional royalties will be paid. This type of payment is common in contingent loans guaranteed by government agencies with a view to encouraging investments in R & D. The loan will be paid out of future sales until the principal and the interest are fully paid.

The four combinations and the benchmark are described in Figure 1, for two alternative critical sale values (\$500 and \$1000). The revenues for each combination are depicted as a function of end-of period sales. Since sales are uncertain, each schedule means a completely different distribution of

Insert Figure 1

payments. Figure 2 shows the cumulative distribution of future royalty payments (FRP) under the assumption of log-normal distribution of the sales.² It should be noted that the parameters assumed in the royalty schedules are

Insert Figure 2

based on the models developed below, in such a way that the present values are the same for all. As can be seen, each royalty schedule has its unique distribution with its specific volatility characteristics.

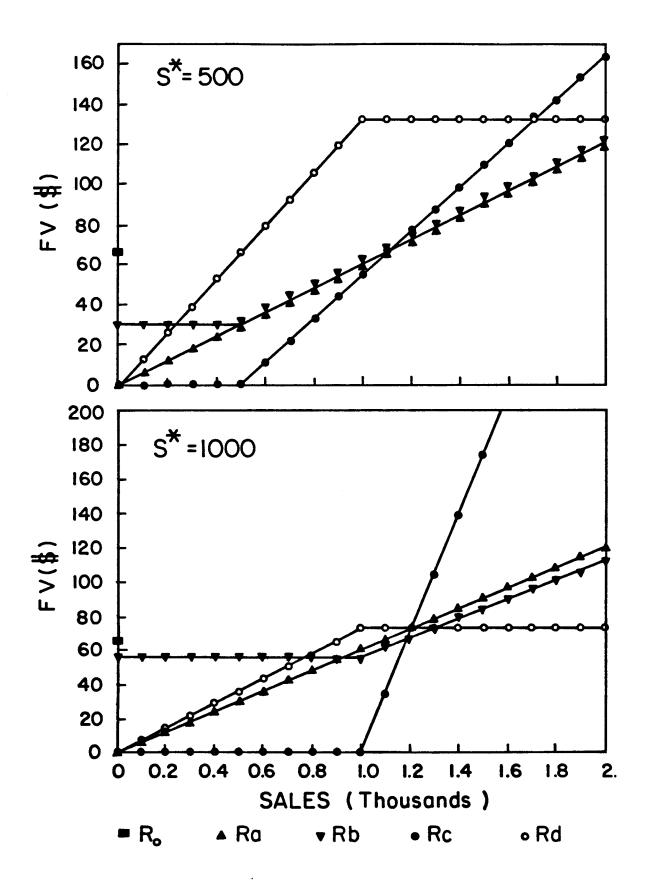
The actual payoff for each schedule is dependent on the realization of sales for the firm. The decision making problem for the licensee and licensor is to determine, ex-ante, the present value of each schedule so that alternatives can be compared.

3. The Basic Assumptions

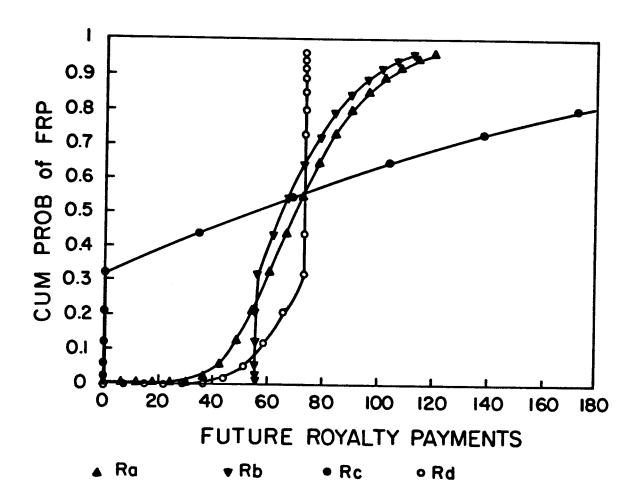
In order to derive closed-form, analytical solutions for the present value calculations, the following assumptions are made:

(A.1) The price per unit is constant and exogenous.

²The assumption will be further discussed below.



<u>Figure 1</u>: The Future Value (FV) of Royalty Payments as a Function of Sales for the Benchmark Royalty (R_o) and Four Alternative Royalty Schedules (R_a , R_b , R_c and R_d)



<u>Figure 2</u>: The Cumulative Distribution of Future Royalty Payments for the Four Royalty Schedules ($S^* = 1000).

(A.2) The quantity of sales is stochastic and its distribution is given $\label{eq:by3} \text{by}^3$

1g Q ~ N
$$(\mu, \sigma^2)$$
.

Given the two assumptions, the distribution of sales is also log normal lg S ~ N (μ + lgP, σ^2).

- (A.3) The risk-free rate i is known and constant.
- (A.4) The firm's beta-risk parameter is known and constant, hence the expected rate of return, k, is also known and constant.
- (A.5) All sales and royalties are to be realized at the end of the period.

If the firm's sales, S, are perfectly correlated with its future market value, V, (which is a reasonable assumption in the one-period framework), then the discount rate k can be applied to future sales. Denote by V_{o} the present value of the expected revenues at time 1:

(5)
$$V_Q = PV (PQ \mid k) = \overline{S}e^{-k}$$

where \overline{S} denotes expected sales revenues, and e^{-k} is the discount factor.

4. The Valuation Models

Combination (a) can then be given by the present value of $\mathbf{R}_{\mathbf{a}}^{}$, denoted by $\mathbf{R}_{\mathbf{a},\,\mathbf{0}}^{}$:

(6)
$$R_{a,o} = A_{a,o} + r_{a,o}$$

where $A_{a,o}$ is the present value of the future (constant and known) lump-sum $A_{a,o}$ and $A_{a,o} = A_a e^{-i}$

³This assumption is intended to facilitate the computation and permit derivation of analytical solutions to the models. Alternative distributions can be handled by using numerical methods, but the major implications remain unchanged.

Valuation of such combinations as (b) or (c) is more complicated, as the discount rate k is incapable of direct application: The risk associated with contingent royalties being related but different from that of S and no longer stationary. Hence the distribution of the relevant R varies stochastically with time. This aspect will be further explored below.

Under the above set of assumptions, the option pricing model developed by Black and Scholes (B-S) (1973) and extended by Merton (1973) can be applied for valuation of the more complicated contingent royalty combinations.

A call option is the right to buy a certain quantity of the underlying asset during a given time period for a predetermined price (denoted by K). If the value of the asset at maturity is $\mathbf{V}_{\mathbf{T}}$, the corresponding value of the call, $\mathbf{C}_{\mathbf{T}}$, is

$$C_{T} = MAX (0, V_{T} - K)$$

The isomorphic relationship between a call and combination (c) can be seen by comparing the above with expression (3),

(3)
$$R_c = r_c MAX[0, (S-S_c^*)]$$

Hence, if S behaves like V, $R_{\rm c}$ can be derived in a similar way in valuing call options. The option-pricing model is described in Appendix 1.

Following the contingent claim approach the present value of $R_b = r_b \text{MAX}(S_b^*, S)$ is given by

(7)
$$R_{b,0} = PV(r_b S_b^*) + r_b [V_o N(h_1) - S_b^* e^{-i\tau} N(h_2)]$$

=
$$r_b[S_b^*e^{-i\tau}N(-h_2) + V_0N(h_1)]$$

where

$$\begin{aligned} \mathbf{h}_1 &= \{ \lg \ [\mathbf{V_o/S_b^*}] \ + \ (\mathbf{i} \ + \ \sigma^2/2) \ \tau \} / \ \sigma \sqrt{\tau} \\ \mathbf{h}_2 &= \mathbf{h}_1 \ - \ \sigma \sqrt{\tau} \\ \end{aligned}$$
 and
$$\mathbf{PV(r_bS_b^*)} \ = \ \mathbf{A_be^{-i\tau}}$$

N(') is the cumulative standard normal distribution, σ the instantaneous standard deviation of 1g S, τ the time left until the end of the period, 4 i the risk-free interest rate, S_b^* the critical sales above which the variable royalty rate r_b is paid.

The present value of R_b consists of that of minimum royalty A_b , plus the discounted expected value of the contingent variable r_b , which applies only if sales are above $A_b/r_b \equiv S_b^*$. It is also a function of the present value of the firm's sales, V_o , weighted by a probability factor $N(h_1)$, plus the present value of the critical sales S_b^* weighted by the probability factor $N(-h_2)$. The cumulative probability $N(h_1)$ is approximately that of sales exceeding the minimum S_b^* .

It can be shown that the present value $R_{\rm b,o}$ is a positive function of the uncertainty associated with sales. Hence, everything else being the same, the higher the volatility of future sales, the higher is the present value of the minimum-royalty combinations. This stems from the increased probability of

 $^{^4}$ It is important to note that the model allows for valuing the future royalties at any moment during the discrete time period, under the assumption of continuous trading in financial markets. By substituting $\tau=1$ the value of the future payment at the base time zero is calculated. The possibility of valuing the contract during its life can be found useful when the licensor wants to sell its contract to a third party, or when the licensee may be acquired by another firm during the life of the contract.

 $^{^{5}}$ See Jarrow and Rudd (1983) Chapter 7.

realizing sales above S_b^{\star} as σ increases. Additional results, based on simulations, are presented in the next section.

It is interesting to note that in the continuous time framework, within the discrete period for the royalty payment, the volatility of the royalty distribution is nonstationary. Based on equation (7) it can be shown that the standard deviation of the distribution of royalties at any time t (0< t< 1), is given by

$$\sigma(R_{b,t}) = N(h_1)\sigma V_t/R_{b,t}$$

where all the parameters are valued at time t. Since the value of sales and the value of the royalties, as well as the time left to the end of the period, change constantly, therefore, even if σ is constant, the volatility of the royalties is expected to change.

By using the same valuation technique, the present value for the "minimum sales" combination $\boldsymbol{R}_{_{\mathbf{C}}},$ can be derived:

(8)
$$R_{c,0} = r_c [V_o N(g_1) - S_c^* e^{-i\tau} N(g_2)]$$

where

$$\mathbf{g_1} = [\lg (\mathbf{V_o/S_c^*}) + (\mathbf{i} + \sigma^2/_2)\tau]/\sigma\sqrt{\tau}$$
 and $\mathbf{g_2} = \mathbf{g_1} - \sigma\sqrt{\tau}$

This expression is similar in structure to the second term in expression (7), and both represent the present value of contingent royalties which depend on realization of critical sales values. In spite of this similarity of structure, however, the functions yield different quantitative results, as will be seen in the next section.

The value of the payment for the "Maximum Royalties" combination (d) is isomorphic to a put^6 option and is given by

$$R_{d,o} = r_{d} [V_{o} N(-f_{1}) + S_{d}^{*} e^{-i\tau} N(f_{2})]$$

where $f_1 = [\ln(V_0/S_d^*) + (i + \sigma^2/2)\tau] / \sigma\sqrt{\tau}$ and

$$f_2 = f_1 - \sigma \sqrt{\tau} .$$

Here, the value of the contract is an increasing function of v_o , A_d , σ and τ and a decreasing function of r_d ; the effect of the risk-free interest rate, i, is ambiguous.

5. Simulation Results

Each combination described in the preceding section has a different pattern, especially as a function of the uncertain sales. These patterns are considered by simulating the models for reasonable sets of parameters.

Our basic case involves the following numerical values:

- Expected end-of-period revenues (\overline{S}) are \$1,200
- The cost of capital of the project (k) is 20 percent
- The variance of the rate of return on the firm (σ^2) is 10 percent
- The risk-free interest rate (i) is 10 percent
- The time until sales are realized (au) is one period.

 $^{^{6}\}text{A}$ put option is the right to <u>sell</u> the specified asset during a given time period for a prespecified price.

- The present value of the benchmark payment $R_{\rm o}$ is \$60. It is 6 percent of the present value of the expected future sales.

Table 1 lists the values of r_b and $r_c,$ for given levels of $\textbf{S}^{\star},$ such that the

Insert Table 1

corresponding present values, $R_{\rm b}$ and $R_{\rm c}$ equal 60. It also contains the values of the constants $A_{\rm b}$ and $A_{\rm c}$ as a function of the given critical value of sales S^* and the calculated royalty rates. For example, for the "Minimum Royalties" combination, if the critical sales are \$1000, the licensee must guarantee the licensor a minimum royalty of \$55.7 or 5.6% of the critical sales. If realized sales fall below \$1000, the licensor receives the guaranteed minimum \$55.7, and if they exceed \$1000 he receives 5.6% of all sales. If the fixed guaranteed amount is raised to \$61.1, the variable royalty rate may decline to 5.1%. Figure 3 shows the royalty rate for the guaranteed minimum royalties case, $r_{\rm b}$, for alternatives of \$40, \$60, \$80 and \$100.

<u>Table 1:</u> The Required Royalty Rates on the "Minimum-Royalty", "Minimum Sales", and "Maximum-Royalty" Combinations with Present Value of Royalties Equal to $R_{0,0}=60$. ($\sigma^2=0.10$, k=0.20, i=0.10)

Critical Sales Value (S*)	"Minimum Royalty" Combination		"Minimum Sales" Combination		"Maximum Royalty" Combination
	r _b	A _b	r _c	_A_c	r _d
0	.060	0.0	.060	0.0	0.663
100	.060	6.0	.066	6.6	0.221
300	.060	18.0	.082	24.7	0.133
500	.060	30.0	.109	54.8	0.096
700	.060	41.7	.160	112.0	0.078
900	.057	51.7	.261	235.1	0.073
1000	.056	55.7	. 347	346.6	0.066
1200	.051	61.1	. 649	778.4	0.062
1500	.043	64.8	NR	NR	NR

INSERT FIGURE 3

In a similar way, for the "Minimum Sales" combination, if the royalty rate is 10.9%, the fixed amount, A_c , should be \$54.8 (or $S_c^* = 500$) in order to maintain the fixed present value of cost of \$60. For sales below 500 dollars, the licensee pays no royalties, but for sales above 500, about 10.9 percent is paid. Figure 4 shows the royalty rate, r_c , for alternative values of A_c (40, 60, 80 and 100 dollars).

INSERT FIGURE 4

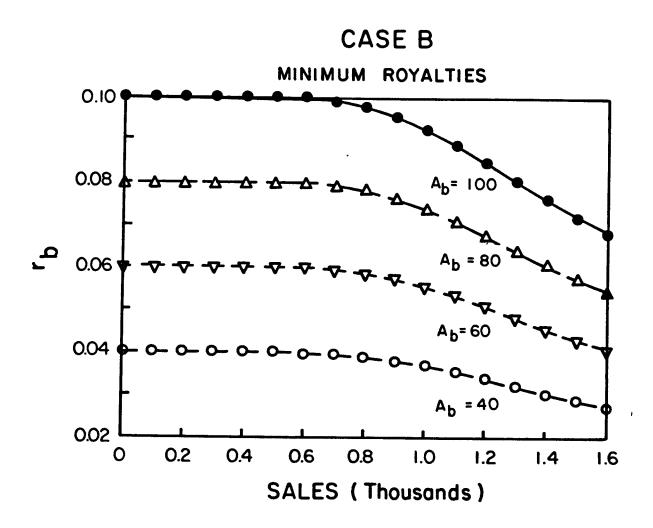
The "Maximum Royalties" combination can also reach a present value of \$60 by increasing the percentage rate paid on initial sales. For example, if royalties are paid on all sales up to \$700, the licensee should pay 9.6 percent. If it is agreed that royalties are paid on sales up to \$1,000, the percentage royalty rate should be reduced to 7.3 to maintain the same present value of \$60.

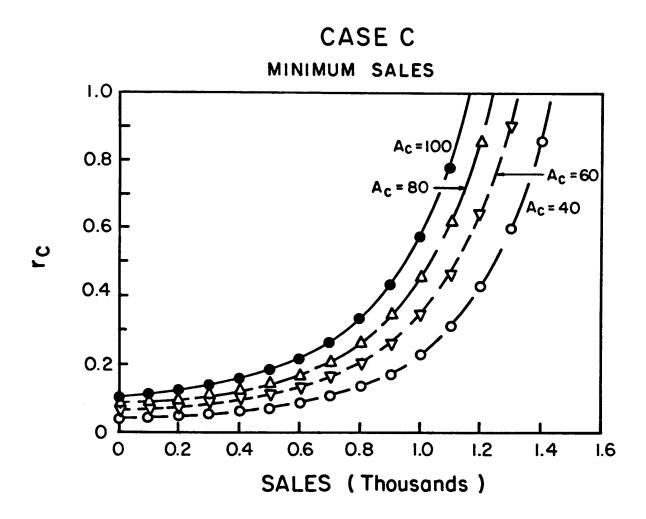
Table 2 presents the sensitivity of the model to changes in the basic

Insert Table 2

parameters. An increase in the volatility of sales leads to reduction of the required royalty rate r_b (to achieve the same present value). The may seem surprising that a higher degree of uncertainty leads to a lower required rate. The reason is that higher volatility means also greater chances of achieving higher future sales and thus, for a given rate, a higher potential income to the licensor. It also should be noted that the royalty rate is very sensitive to changes in σ^2 .

 $^{^{7}}$ It should be noted that increasing the volatility of sales does not necessarily change the present value of sales. According to the CAPM, only changes in the systematic risk will affect the required date of return, k, and hence the value of the firm.





 $\underline{Figure~4}\colon$ The Royalty Rate r_c for Alternative Values of A_c for the "Minimum Sales Schedule."

<u>Table 2:</u> Sensitivity Analysis: Variable Royalties in Different Combinations for Alternative Parameters*

<u>Parameters</u>	r _b	r _c	r _d
$\sigma^2 = 0.1$ (Base case)	.056	. 347	.073
$\sigma^2 = 0.05$.057	. 427	.070
$\sigma^2 = 0.20$.053	. 272	.077
$\sigma^2 = 0.5$.049	.192	.087
T = 5	.056	.128	.0113
T = 0.2	.057	. 906	.064
i = 0.05	.055	.404	.070
i = 0.15	.057	.301	.075

 $^{^{\}star}$ All royalty combinations in the table maintain a present value of \$60. The critical sales value, S^{\star} is assumed to be \$1000 for all cases.

6. Implications of Models for Decision Making

The valuation models can be used by the decision makers in order to determine the cost of alternative royalty combinations. Moreover, each combination may entail a different degree of "moral hazard" or motivation with regard to a certain action. For example, combination b is riskier for the licensee than c, since under the former the licensee is obligated to pay at least A_b dollars even if realized sales are very low. Combination c, on the other hand, requires sales to exceed the minimum value, S_c^* , which is offset by a relatively high value for variable royalties r_c (see Table 1). If a licensee doubts the validity of the information given by the licensor he may prefer c to b. The licensor, on the other hand, may prefer b. The models supply a means of determining the parameters of (c) that would be economically equivalent to those of (b); in present-value terms, the licensor can thus achieve a state of indifference to the choice of combinations.

Combination (d) does not offer the licensor any special incentives to promote the product, while the licensee benefits the more it sells; this combination is convenient for government agencies which seek to promote an industry while dispensing with close monitoring beyond a certain level of sales. This is achieved with conditional loans that are repaid out of future sales. The monitoring cost for the agency is avoided when sales reach a certain minimal level.

The models bring out the main parameters that determine the economic value of each royalty combination. As was illustrated by the simulations, the risk factor is very important, hence heterogeneous expectations concerning the commercial riskiness of the project affect the negotiations on the terms of the contract.

In some cases the licensor has an incentive to lower the perceived volatility of the project. By doing so the licensee may be induced to pay higher variable royalties. Undoubtedly, the risk assessment is a very important factor in determining the terms of any royalty contract. It should be noted that royalty schedule (a) is not directly sensitive to the risk estimation⁸ and, therefore, when there is a disagreement between the parties they may prefer this type of contract to those implied by either (b) or (c).

On the other hand, new ventures that require high initial investment may opt for schedule (c), which delays royalty payments until a minimum level is achieved.

The model presented in the paper is valid for a discrete one-period, say, a year. The model can be extended to a multi-period case. Since the qualitative results remain the same, we decided to focus on the one period model. The model can also be extended to the case of the licensee going bankrupt. Another interesting extension is to allow prices to be stochastic. If prices and quantities are related by the constant elasticity of demand function the analytical formulation can still be preserved. The incentives to select a specific royalty scheme will be a function of the elasticity of demand. We may expect to find different payment schedules for different products based on their elasticity of demand.

 $^{^{\}rm 8} Indirectly$ it may be affected through the valuation of $\rm V_{\rm O},$ which is assumed to be given in the model.

⁹The interested reader can obtain from the authors the extension of the model to the two-period case, for the "minimum royalty" schedule. The model is derived for two alternative assumptions: first, the payments in the two periods are assumed to be independent based on the sales in each period separately. In the second alternative the second period remuneration is dependent on the realization of sales in the first period.

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Appendix 1

Option Valuation Model

A call option gives its buyer the right to purchase the underlying stock for a prespecified price (the "exercise price") during a given time period. A put option is the right to <u>sell</u> the underlying asset for a prespecified price. Both put and call options are contingent claims and their future value depends on the price of the underlying asset (relative to the exercise price) on the options' expiration day.

The Black-Scholes (1973) pricing model gives the price of a European call option 1 that would be obtained in a perfect capital market when no dividends are expected to be paid on the underlying stock during the life of the option. The inputs to the model consist of four observable variables: the price of the underlying stock (V); the exercise price (K); the time to maturity (τ) ; and the riskless interest (τ) , and one variable that is not observable, the variance of the stock's distribution of rates of return (σ^2) . The model price C is

$$C = VN (d_1) - K e^{-rt} N (d_2)$$
 (2)

where

$$d_1 = \underline{\ln (V/K) + (r + \sigma^2/2) \tau}$$

$$\sigma \sqrt{\tau}$$

 $^{^{1}}$ A European call option can be only exercised on the option's maturity date.

and

$$d_2 = d_1 - \sigma \sqrt{\tau}$$

and N(.) is the cumulative standard normal distribution.

The assumptions needed to derive the model² are as follows: (1) There is a perfect capital market characterized by an absence of taxes and commissions, free access to all available information, and divisibility of all financial assets. In addition, borrowing and short selling as well as free use of all proceeds are permitted to all investors. (2) The short-term interest rate and the variance rate of return are known and constant. (3) The stock price is lognormally distributed at the end of any finite interval.

It can be shown that for European options the following relationship holds:

$$C - P = V - Ke^{-r\tau}$$
 (3)

and by substituting (2) for C and rearranging the model for the put option is: $P = Ke^{-r\tau}N(-d_2) - VN(-d_1).$

 $^{^2}$ These are the basic assumption appearing in the original paper. Other authors showed the model to hold under different sets of assumptions. For example, Rubinstein (1976) derived the model for discrete time trading at the cost of adding assumptions concerning the shape of the utility function of investors.