Lawrence Berkeley National Laboratory

Recent Work

Title

ONE-PARTICLE OPERATORS AND LOCAL INTERNAL SYMMETRIES

Permalink

<https://escholarship.org/uc/item/5sp62916>

Author

Herbst, Ira W.

Publication Date

1971-03-01

Submitted to
Journal of Mathematical Physics;
also filed as a Ph.D. thesis

I. Herbst

UCRL-20648
Preprint

C.2

TWO-WEEK LOAN COPY

*This is a Library Circulating Copy
which may be borrowed for two weeks.
For a personal retention copy, call
Tech. Info. Division, Ext. 5545*

RECEIVED
LIBRARY
ADMINISTRATIVE

DOCUMENTS SECTION

**ONE-PARTICLE OPERATORS AND
LOCAL INTERNAL SYMMETRIES**

Ira W. Herbst

March 31, 1971

AEC Contract No. W-7405-eng-48

34d

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

ONE-PARTICLE OPERATORS AND LOCAL INTERNAL SYMMETRIES^{*†}

Ira W. Herbst

Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

March 31, 1971

ABSTRACT

The structure of local operators linear in the creation and destruction operators of a finite number of particles is investigated. It is shown that these operators determine a set of local and relatively local free fields relative to which the original operators are local. This result is used to study local internal symmetries in a theory of interactions and to show that these symmetries commute with the Lorentz group. The assumptions of Haag and Ruelle which lead to a complete particle interpretation are also discussed. Asymptotic many-particle states are constructed under less restrictive assumptions.

I. INTRODUCTION

In this article we will first investigate the structure of a particularly simple local theory. In a Fock space of N free particles of mass $m > 0$ and arbitrary spins, we consider local Hermitian operators linear in the creation and destruction operators of these particles. These "one-particle operators" are associated with bounded regions of space-time in a way familiar from the local algebra theories of Haag and Araki,^{1,2} with one important difference: The set of space-time regions to which the one-particle operators are localized need not include regions of arbitrarily small size. (For a precise description of this set of regions, see theorem 1 of Sec. III.) We show that such a set of one-particle operators determines a standard set of relatively local free fields relative to which the original one-particle operators are local. This is the subject of Sec. III.

In Sec. IV we use this result to analyze local internal symmetries of a type considered by L. J. Landau and E. Wichmann.^{3,4} These authors have demonstrated the translational invariance of such symmetries, and Landau has further shown that in a field theory with an interpolating field⁵ for each particle, these symmetries must commute with the Lorentz group. In this section we extend this result to the apparently more general local algebra theories and field theories for which a complete set of interpolating fields is not assumed. We also discuss the relationship of the Haag-Ruelle theory of asymptotic states to selection rules. Asymptotic many-particle states are constructed under less restrictive assumptions.

To establish notation we first review the Poincare transformation properties of one-particle states.

II. ONE-PARTICLE REPRESENTATIONS OF THE POINCARÉ GROUP

The representation $\Gamma_{m,s}$ of the Poincaré group (denoted $\bar{g}$) appropriate to a particle of mass $m > 0$ and spin s can be realized by unitary operators $U(x,g) = U(x) U(g)$ [x a four-vector, g in $g \equiv SL(2,C)$] on a Hilbert space of $2s + 1$ component functions $f(p)$ in the following way:

$$(U(x,g)f)(p) = e^{ix \cdot p} D^s(g) f(\Lambda^{-1}(g)p); \quad x \cdot p = t\omega - \mathbf{x} \cdot \mathbf{p},$$

$$\omega(p) = (\mathbf{p}^2 + m^2)^{\frac{1}{2}}. \quad (1)$$

Here $D^s(g)$ is the standard $2s + 1$ dimensional representation $(0,s)$ of g and the notation $\Lambda(g)p$ means the spatial part of $\Lambda(g)p$, where $\Lambda(g)$ is the 4 by 4 Lorentz matrix. The invariant scalar product is

$$\langle f|h \rangle = \int \frac{d^3p}{2\omega} B(f,h; p),$$

where $B(f,h; p) = f^\dagger(p) D^s(\tilde{p}) h(p)$, $\tilde{p} = [\omega(p) - \mathbf{p} \cdot \boldsymbol{\sigma}] / m$, and the σ_i are Pauli matrices. Note that since $D^s(\tilde{p})$ is a positive definite Hermitian matrix, for fixed p $B(f,h; p)$ is a scalar product of the $2s + 1$ component vectors $f(p)$ and $h(p)$. We also mention here that $D^s(g)$ is a homogeneous polynomial of degree $2s$ in the matrix elements of g and that therefore $D^s(\tilde{p})$ is a polynomial in the components of $p = (p, \omega)$.

We will, in addition, make use of the infinitesimal operators of the representation $U(x,g)$. A simple calculation gives the following results:

(a) The generators of rotations and velocity transformations are

given by $\tilde{J} = \tilde{S} + \tilde{L}$, $\tilde{K} = i(\tilde{S} - \omega \tilde{V})$ respectively. Here $\tilde{S}$ is the standard $2s + 1$ dimensional angular momentum matrix and $\tilde{L} = -ip \times \nabla$. Note that $\tilde{S}$ is not Hermitian relative to the scalar product $\langle f|g \rangle$ and hence is not the spin observable.

(b) The operators $\tilde{S}$ and $\tilde{L}$ are expressible in terms of the generators of Poincaré transformations. The formula is

$$m^2 \tilde{L} = p \times [ip \times (\tilde{J} + i\tilde{K})] - i\omega p \times (\tilde{J} + i\tilde{K}). \quad (2)$$

Finally we will make use of the creation and destruction operators appropriate to a Fock space description of this particle. Hence we introduce the operators $a_\mu^\dagger(p)$ such that if $f(p)$ is the $2s + 1$ component function transforming according to Eq. (1), then

$$|f\rangle = \sum_{\mu=-s}^s \int \frac{d^3p}{2\omega} f_\mu(p) a_\mu^\dagger(p) |0\rangle$$

is the one-particle Fock space state it represents. Here $|0\rangle$ is the vacuum vector. It is then easy to verify the transformation law

$$U(x,g) a_\mu^\dagger(p) U^{-1}(x,g) = \sum_{\sigma=-s}^s e^{ix \cdot \Lambda(g)p} a_\sigma^\dagger(\Lambda(g)p) D_{\sigma\mu}^s(g). \quad (3)$$

If the particle satisfies either Fermi or Bose statistics, the creation and destruction operators must satisfy the usual anticommutation or commutation relations

$$[a_\mu(p), a_\nu^\dagger(q)]_\pm = 2\omega \delta(p - q) D_{\mu\nu}^s(\tilde{p})$$

respectively.

We introduce the free field $\phi_\mu(x)$ transforming according to $D^s(g)$ as follows:

$$\phi_\mu(x) = \int \frac{d^3p}{2\omega} [a_\mu^\dagger(p) e^{ix \cdot p} + \lambda \theta a_\mu(p) \theta^{-1} e^{-ix \cdot p}] \quad (4)$$

Here θ is the TCP operator given by

$$\theta a_\mu(p) \theta^{-1} = \sum_\sigma a_\sigma(p) [C_s D^s(\tilde{p})]_{\sigma\mu} \lambda^*,$$

where $C_s = \exp(-i\pi S_2)$ and λ is a complex number of modulus unity.

If θ is to act locally on $\phi_\mu(x)$ we must choose $\lambda^2 = (-1)^{2s}$. Note that θ has the following action on the one-particle wavefunctions:

$$(\theta f)(p) = [C_s D^s(\tilde{p}) f(p)]^*$$

and that

$$B(\theta f, \theta h; p) = B(h, f; p), \quad (5a)$$

$$[U(g), \theta] = 0. \quad (5b)$$

Since $C_s D^s(g) C_s^{-1} = D^s(g^{-1})^\dagger$ and $D^{s+}(g) D^s(\tilde{p}) D^s(g) = D^s(\tilde{p}')$, with $p' = \Lambda(g^{-1})p$, we find that

$$U(g) \phi_\mu(x) U^{-1}(g) = \sum_\lambda \phi_\lambda(\Lambda(g)x) D_{\lambda\mu}^s(g)$$

and

$$[\phi_\mu(x), \phi_\lambda(y)]_\pm = i(C_s^{-1})_{\mu\lambda} \Delta_0(x - y; m).$$

III. RELATIVELY LOCAL SETS OF ONE-PARTICLE OPERATORS

We consider a Fock space of N particles with a common mass $m > 0$ and arbitrary spins. Each particle is assumed to be either a fermion or a boson, and normal commutation relations are assumed for the creation and destruction operators of different particles.

Suppose $\mathcal{D}$ is a Poincare invariant class of bounded open space-time regions. That is, if $D \in \mathcal{D}$ then $[\Lambda(g)D + x_0] \in \mathcal{D}$, where $\Lambda(g)D + x_0$ is the region $\{x | x = \Lambda(g)x' + x_0, x' \in D\}$. With each $D \in \mathcal{D}$ we associate a set $S(D)$ of Hermitian one-particle operators each of which is linear in a subset of the N creation and destruction operators. Each operator in $S(D)$ is assumed to involve either fermions alone or bosons alone defining the sets $S_+(D)$ and $S_-(D)$ respectively; $S(D) = S_+(D) \cup S_-(D)$. To insure the completeness of the description of the theory in terms of the $S(D)$ we assume that the linear span of $\{\bigcup_{D \in \mathcal{D}} S(D)|0\rangle\}$ is dense in the one-particle Hilbert space. The operators of $S(D)$ are "localized" to the region D in the following sense:

1. $U(x, g) S(D) U^{-1}(x, g) = S(\Lambda(g)D + x)$;
2. If $D_1 \subseteq D_2'$ then $[S_\pm(D_1), S_\pm(D_2)]_\pm = 0$.

(Here D' is the region space-like to the closure of D .) To simplify the geometrical aspects of our study we will assume in what follows that all regions $D \in \mathcal{D}$ are of a particular kind called "diamonds." A diamond with vertices x_1 and x_2 (where $x_2 - x_1$ is forward time-like) is the intersection of the open backward light cone from x_2 with the open forward cone from x_1 . Note that $\mathcal{D}$ may consist of the Poincare transforms of only a single diamond.

The assumptions stated above will allow us to prove the following:

Theorem 1: There exists a set of N free fields $\phi^k(x)$, $k = 1, \dots, N$ of the type discussed in Sec. II [ϕ^k transforms according to $D^s(g)$, where s depends on k] which satisfy

$$[\phi_\mu^k(x), \phi_\nu^\ell(y)]_\pm = i \delta_{k\ell} (C_s^{-1})_{\mu\nu} \Delta_0(x-y; m) \text{ and which are local}$$

relative to the $S(D)$. That is, if $Q \in S_\pm(D)$ and $x \in D'$ then

$$[Q, \phi_\mu^k(x)]_- = 0 \text{ if at least one of } Q \text{ and } \phi^k \text{ are Bose operators}$$

and $[Q, \phi_\mu^k(x)]_+ = 0$ if both are Fermi operators.

As we shall see later the above set of fields is essentially unique. Furthermore each operator of $S(D)$ is essentially a sum of fields "smeared" with functions with support in D . Hence the structure of the operators of $S(D)$ is determined by that of a free field theory.

In proving theorem 1 we will want to integrate and differentiate quantities such as $U(x, g) Q U^{-1}(x, g)$ with $Q \in S(D)$. While only the commutation properties of these operators which follow from assumptions 1 and 2 above will be used, for notational convenience we will say that the integrated or differentiated operators are members of the appropriate $S(D)$. A series of three lemmas will now be given which result in theorem 1.

We first show that only one spin at a time need be considered.

Lemma 1. Suppose $Q \in S(D)$ with $U(x, g) Q |0\rangle \in C^\infty$ in (x, g) . Then each Hermitian operator Q_k which occurs in the decomposition of Q into a sum of operators involving particles of spin $j(k)$ is also in $S(D)$.

Proof: We introduce the polarization four-vector operator

$W^k = (1/2m) \sum \epsilon^{klpq} P_\ell J_{pq}$ (Here P_ℓ and J_{pq} are the usual generators of translations and Lorentz transformations.) The operator

$W^2 = \sum W_k W^k$ has the property that for a one-particle state ψ of spin j and mass m , $[W^2 + j(j+1)]\psi = 0$. The action of W^2 on Q is defined by the appropriate repeated differentiations of $U(x, g) Q U^{-1}(x, g)$. For example

$$(P_\ell J_{pq})(Q) = [i \frac{d}{d\lambda} i \frac{d}{d\lambda'} U(x, g) Q U^{-1}(x, g)]_{\lambda=\lambda'=0}, \text{ where}$$

$$U(x, g) = e^{-i\lambda P_\ell} e^{-i\lambda' J_{pq}}.$$

Since this operation is local, i.e., for $R \in S(D_1)$ and $D_1 \subseteq D'$, $[W^2(Q), R]_\pm = 0$, we have $W^2(Q) \in S(D)$. With $Q = \sum_{k=1}^M Q_k$ and $\alpha_k = j(k)[j(k) + 1]$, we note that

$$Q_i = \left(\prod_{k=1}^M \left[\frac{W^2 + \alpha_k}{\alpha_k - \alpha_i} \right] \right) (Q),$$

where the prime sign indicates that the factor containing the index i is to be omitted. Hence $Q_i \in S(D)$.

We now consider one of the Hermitian operators Q_k in $S(D)$ (hereafter called Q) involving only particles of spin s . Poincare invariance allows us to transform D to the special form $D_r = \{x | ||x|| + |t| < r\}$ for some r . The operator Q can be written

$$Q = \sum_{k=1}^n \sum_{\mu=-s}^s \int \frac{d^3 p}{2\omega} [a_\mu^\dagger(k, p) f_\mu(k, p) + a_\mu(k, p) f_\mu^*(k, p)].$$

Here $a_\mu^\dagger(k, p)$ is a creation operator for particle k as discussed in Sec. II and $[a_\mu(k, p), a_\nu^\dagger(\ell, q)]_\pm = 0$ for $k \neq \ell$. We use the notation $f_1(p) = \{f_{-s}(i, p), \dots, f_s(i, p)\}$, $f(p) = \{f_1(p), \dots, f_n(p)\}$, and $D(g)$ for the $(2s+1) \times n$ dimensional matrix whose elements consist of n matrices $D^s(g)$ on the diagonal. Thus $f_1(p)$ and $f(p)$ are respectively $2s+1$ and $(2s+1) \times n$ component column vectors. Without loss of generality we will assume that the $f_1(p)$ are linearly independent vector valued functions of p . [If there are $m < n$

linearly independent $f_i(p)$ we "rename" the particles by choosing

$$b_\mu^\dagger(k, p) = \sum_{\ell=1}^n a_\mu^\dagger(\ell, p) \beta_{\ell k} \quad \text{with unitary } \beta. \quad \text{For a suitable choice}$$

of β the new wavefunctions $f'_i(p) = \sum_{\ell=1}^n \beta_{i\ell}^{-1} f_\ell(p)$ will satisfy

$f'_i(p) \equiv 0$ for $i > m$, the remainder being linearly independent.] Note

that by virtue of the differentiability assumption in lemma 1, the

$f_\mu(k, p)$ are in the function class $\mathcal{S}$.

We want to establish the result that $f(p)$ is the restriction of an entire function $\hat{f}(p)$ of the four-vector p to $p_0 = \omega(p)$ such that the x -space Fourier transforms of $\hat{f}(p, \omega) + \hat{f}(p, -\omega)$ and of $[\hat{f}(p, \omega) - \hat{f}(p, -\omega)]/\omega$ have support in $||x|| \leq r$. These facts and certain other results of an algebraic nature will be needed to establish theorem 1.

Hence consider the function

$$\mathcal{F}(x; g_1, g_2) = [U(x, g_1) Q U^{-1}(x, g_1), U(g_2) Q U^{-1}(g_2)]_+,$$

which by assumption has compact support in x at $t = 0$ along with all its derivatives with respect to x and g_1 . With F_1 and F_2 respectively the Fourier transforms of $[i\partial_t \mathcal{F}(x; g_1, g_2)]_{t=0}$ and $[\mathcal{F}(x; g_1, g_2)]_{t=0}$, we calculate

$$B(U(g_1)f, U(g_2)f; p) = F_1(p; g_1, g_2) + \omega F_2(p; g_1, g_2) = A(p; g_1, g_2), \quad (6)$$

where $B(U(g_1)f, U(g_2)f; p) = \sum_{k=1}^n B(U(g_1)f_k, U(g_2)f_k; p)$. Note that

$A(p; g_1, g_2)$ has the following two properties:

1. It is the restriction to $p_0 = \omega(p)$ of an entire function $\hat{A}(p; g_1, g_2)$ of the four-vector p ; for example,

$\hat{A}(p; g_1, g_2) = F_1(p; g_1, g_2) + p_0 F_2(p; g_1, g_2)$. This is also true of all derivatives of $A(p; g_1, g_2)$ with respect to g_i . [For brevity, functions of the three-vector p which are restrictions to $p_0 = \omega(p)$ of entire functions of the four-vector p will sometimes be called entire functions of p .]

2. For real p and $p_0 = \omega$, $\hat{A}$ satisfies the reality condition $\hat{A}^*(p; g_1, g_2) = \hat{A}(-p; g_1, g_2)$. These properties of A can be used to prove the following:

Lemma 2. (a) The function $f(p) = \hat{f}(p, \omega)$, where $\hat{f}(p)$ is an entire function of p .

(b) A suitable renaming of the n particles can be carried out to give for the n new wavefunctions [again denoted $f_i(p)$] and for real p : $f^\theta(p) = e^{i\pi s} \hat{f}(-p, -\omega)$. Here the abbreviation $f^\theta(p) = e^{-i\pi s} [CD(\tilde{p}) f(p)]^*$ has been introduced.

The proof of this lemma has essentially been carried out by Epstein.⁷ We present a somewhat different proof:

Let $\mathcal{A}$ be the algebra of all polynomials in the representatives of the generators of the Poincare group. For each p let $\mathcal{V}(p)$ be that subspace of $(2s + 1) \times n$ dimensional complex space equal to $\{Af(p)\}$ [i.e., $\mathcal{V}(p) = \{w | w = (Rf)(p), R \in \mathcal{A}\}$]. Suppose $M = \max_p$ [dimension of $\mathcal{V}(p)$]. We note the following important facts:

1. If $R_1, R_2 \in \mathcal{A}$ then $B(R_1 f, R_2 f; p)$ is an entire function of p .
2. We can choose $R_1, \dots, R_M \in \mathcal{A}$ with $R_1 = 1$ such that the Gram determinant $G(p) = \det[B(R_i f, R_j f; p)]$ does not vanish identically. [Note also that $G(p) \geq 0$ for real p .]

3. Equation (2) implies that $\omega \in A$.

Statements 1, 2, and 3 then imply that for p real and $G(p) \neq 0$,

$$\frac{\partial(R_1 f)(p)}{\partial p_k} = \sum_{j=1}^M (R_j f)(p) \alpha_{ji}^k(p). \quad (7)$$

Here $\alpha_{ji}^k(p)$ can be calculated by Cramer's rule in terms of the inner products $B(\omega \nabla_k R_1 f, R_j f; p)$ and $B(R_1 f, R_j f; p)$ and is seen to equal $(\omega G(p))^{-1} \chi$ (an entire function of p).

Substituting $\frac{\partial^2(R_1 f)(p)}{\partial p_\ell \partial p_k} = \frac{\partial^2(R_1 f)(p)}{\partial p_k \partial p_\ell}$, and noting the linear independence of the column vectors $(R_1 f)(p)$ at real points p for which $G(p) > 0$, we derive the integrability conditions

$$\frac{\partial \alpha^k(p)}{\partial p_\ell} - \frac{\partial \alpha^\ell(p)}{\partial p_k} = [\alpha^k(p), \alpha^\ell(p)] \quad (8)$$

for all points $p \in \mathbb{C}^3$ for which $\omega(p) G(p) \neq 0$. The matrices $\alpha^k(p)$ are analytic at least at these points.

The analyticity properties of the solutions of differential equations such as Eq. (7) (total differential equations) with coefficients satisfying (8) are well known:^{8,9} In any simply connected neighborhood (real or complex) of a point z_0 where the α^k are analytic, the equation has a unique solution (with given initial conditions). This solution is analytic in this neighborhood and has an analytic continuation along any path of analyticity of the coefficients.

To determine additional analyticity properties of $f(p)$ we will make use of finite Lorentz transformations. Choose a polycylinder C_0 , centered at $p_0 = p_0^*$, in which $\omega(z) G(z) \neq 0$. We

solve the D.E. there to get a function $F(z)$ analytic in C_0 and equal to $f(p)$ in R_0 , the intersection of C_0 with the real space. Let us assume that the particle wavefunctions $f_1(p), \dots, f_n(p)$ are linearly independent (as vector valued functions of p) in R_0 . [If this is not so we arrange a renaming of the particles so that there are n' ($n' < n$) linearly independent $f_i(p)$ in R_0 , the remainder being zero in R_0 .] Since $F(p) = f(p)$ for $p \in R_0$, the equation

$$B(U(g)F, F; p) = A(p; g, 1) \equiv A(p; g) \quad (6a)$$

holds for all g in some real neighborhood $N(1)$ of the identity and all p in some real neighborhood R_1 of R_0 .

We will now use Eq. (6a) to show that in fact $F(z)$ can be analytically continued along any path $z(t)$ which begins at p_0 and does not pass through the surface $\omega(z) = 0$. Hence suppose $F(z)$ is analytic at all points $z(t)$, $0 \leq t < t_1$. We will show that $F(z)$ is also analytic at $z(t_1)$. We choose $g_0 \in N(1)$ and a neighborhood $N(g_0)$ of g_0 with $N(g_0) \subseteq N(1)$ such that

- (a) $|\omega(\Lambda^{-1}(g)z(t))| > 0$ for $0 \leq t \leq t_1$, $g \in N(g_0)$,
- (b) $|G(\Lambda^{-1}(g)z(t))| > 0$ for $t_1 - \epsilon \leq t \leq t_1$, $g \in N(g_0)$ with an $\epsilon > 0$ which depends on g_0 and $N(g_0)$.

Such a choice is possible because the condition $G(\Lambda^{-1}(g)z(t_1)) = 0$ for all g in a real neighborhood of the identity implies that G vanishes in a complex neighborhood of the identity and therefore that $G(z)$ vanishes in a complex neighborhood of $z(t_1)$. [Note that $\Lambda^{-1}z$ is the spatial component of $\Lambda^{-1}(z, \omega(z))$, where the value of $\omega(z)$ depends on the path taken to the point z .] Let $J(z; g_1, \dots, g_\beta)$, $\beta = (2s + 1) \times n$, be the matrix whose rows are the vectors

$v_i = [D(g_i) F(\Lambda^{-1}(g_i)z^*)]^*$. Let us now choose a set of β elements $g_i \in N(g_0)$ such that $\det J(z; g_1, \dots, g_\beta) \neq 0$ for z in a neighborhood of $z(t_1)$. This choice is possible for the following reason: The vanishing of $\det J(z(t_1); g_1, \dots, g_\beta)$ for all $g_i \in N(g_0)$ is equivalent to the statement that the v_i do not span the β dimensional space no matter how the g_i are chosen in $N(g_0)$; i.e., there is a nonzero vector v orthogonal to $v(g) \equiv [D(g) F(\Lambda^{-1}(g)z(t_1))]^*$ for all $g \in N(g_0)$. But $v^\dagger v(g)$ is therefore zero in a complex neighborhood N_0 of g_0 . We choose $u \in SU(2)$ and g_1 such that $\Lambda^{-1}(g_1)z(t_1) = 0$. Then if $g^{-1} = h^{-1} g_1 u g_1^{-1}$, $h \in N_0$ we have $v^\dagger v(g) = 0$ for all $u \in SU(2)$. A short calculation then shows the $F_i^*(z^*)$ to be linearly dependent in a neighborhood of $\Lambda^{-1}(g_0)z(t_1)$. Continuing back to R_0 gives the linear dependence of the $f_i(p)$ in R_0 , contradicting our assumption that they are linearly independent in R_0 .

We will now use Eq. (6a) to show that $F(z)$ is analytic in a neighborhood of $z(t_1)$. We choose a path $w(t)$ such that (a) $w(t)$ coincides with p and $z(t_1 - \epsilon)$ at its end points; (b) $w(t)$ is close enough to $z(t)$ for $0 \leq t \leq t_1 - \epsilon$ so that it can be continuously distorted to $z(t)$ without crossing any singularities of $F(z)$; and (c) for $0 \leq t \leq t_1 - \epsilon$ $|G(\Lambda^{-1}(g_i)w(t))| > 0$ $i = 1, \dots, \beta$. With these three conditions met we can continue the equations

$$[D(g_i) F(\Lambda^{-1}(g_i)z)]^\dagger D(\tilde{z}) F(\tilde{z}) = A(\tilde{z}; g_i) \quad (6b)$$

along $w(t)$ to $z(t_1 - \epsilon)$, resulting in the same $F(z)$ as would have been obtained by continuation along $z(t)$. [Note that conditions (b) and (c) can easily be met, for example in the following way: First choose a polygonal path which approximates $z(t)$ well enough to satisfy

(a) and (b) and whose component line segments neither begin nor end at zeroes of $G(\Lambda^{-1}(g_i)z)$ for any i . If for each straight path segment $w_k(t)$ we choose a linear parametrization, $G(\Lambda^{-1}(g_i)w_k(t))$ is analytic in t and hence $w_k(t)$ can be distorted infinitesimally in the t complex plane to miss the zeroes of $G(\Lambda^{-1}(g_i)z)$, $i = 1, \dots, \beta$. The new path will satisfy (a), (b), and (c).] Then the Eqs. (6b) can be continued further from $z(t_1 - \epsilon)$ along $z(t)$. If $A(z)$ is the column vector whose components are $A(z; g_i)$ we thus obtain the representation

$$F(z) = [J(z; g_1, \dots, g_\beta) D(\tilde{z})]^{-1} A(\tilde{z}) \quad (9)$$

in a neighborhood of $z(t_1)$. This shows explicitly that $F(z)$ is analytic in this neighborhood. By repeated application of this argument we see that no singularities can appear on the path $z(t)$. The monodromy theorem then shows $F(z)$ to be analytic in any simply connected domain not containing points of the surface $\omega(z) = 0$.

We now go on to examine $F(z)$ in a neighborhood of a point z_0 on this surface. We can easily choose $g_i \in N(1)$, $i = 1, \dots, \beta$ so that $\omega(\Lambda^{-1}(g_i)z_0) \neq 0$, $\det J(z_0; g_1, \dots, g_\beta) \neq 0$, and so that the $\Lambda^{-1}(g_i)z_0$ lie in a neighborhood of a single point z_1 containing no points of the surface $\omega(z) = 0$. Note that $J(z; g_1, \dots, g_\beta)$ can be written as $J_1(z; g_1, \dots, g_\beta) + \omega(z) J_2(z; g_1, \dots, g_\beta)$, where the $J_i(z)$ are analytic in a neighborhood of z_0 . Again Eq. (9) gives an explicit representation for $F(z)$ in a neighborhood of z_0 . This representation shows two things. Firstly, since all curves $z(t)$ which encircle $-m^2$ twice in the z^2 plane can be continuously distorted to lie arbitrarily close to z_0 without crossing $\omega(z) = 0$,

we have $F(z) = F_1(z) + \omega(z) F_2(z)$, where $F_1(z)$ is single valued and analytic for $\omega(z) \neq 0$. Secondly, Eq. (9) gives an explicit representation for the $F_i(z)$ in a neighborhood of z_0 , showing that they are analytic there. Since z_0 is arbitrary the $F_i(z)$ are in fact entire functions.

To complete part (a) of the lemma, it remains to show that the branch of $F(z)$ resulting from a continuation along a real path from p_0 is in fact equal to $f(p)$ for all real p . To accomplish this we will demonstrate the equality on all real straight lines emanating from p_0 . Suppose $p(t) = p_0 + p_1 t$ is such a line. Then along $p(t)$ Eq. (7) reduces to an ordinary linear differential equation in t . The coefficients are analytic functions of t in a neighborhood of the real axis except at the zeroes of $G(p(t))$, and hence $F(p(t)) = f(p(t))$ for $t \geq 0$ up to the nearest zero of $G(p(t))$. We now consider the solution to the original D.E. [Eq. (7)] in a neighborhood of a point on the line between this and the next zero of $G(p(t))$. This solution, $H(z)$, has the same analyticity properties as $F(z)$ and $H(p(t)) = f(p(t))$ between the first and second zeroes of $G(p(t))$. But since $H(p(t))$ and $F(p(t))$ are analytic at the first zero and since $f(p(t))$ is infinitely differentiable, we have equality of all derivatives with respect to t of H and F at the first zero. Hence $H(p(t)) = F(p(t)) = f(p(t))$ for $t \geq 0$ up to the second zero of $G(p(t))$. Repeating this argument we reach any point on the line in a finite number of steps. This completes the proof of part (a) of the lemma.

We now derive the connection between the two branches of the function $f(p) = \hat{f}(p, \omega)$ which is implied by

$\hat{A}^*(p; g_1, g_2) = \mp \hat{A}(-p; g_1, g_2)$ for real p and $p_0 = \omega$. Let $\bar{f}(p) = \hat{f}(-p, -\omega)$ and $h(p) = [CD(\tilde{p})\bar{f}(p)]^*$ for p real. Note that since $D(\tilde{p})$ is a homogeneous polynomial of degree $2s$ in the matrix elements of $\tilde{p} = [\omega(p) - p \cdot \underline{g}]/m$, we have $D(-\tilde{p}) = (-1)^{2s} D(\tilde{p})$. We now consider Eq. (6). We continue both sides around the branch point $z^2 = -m^2$ in the z^2 plane and back to the real domain, and then we replace the argument p by $-p$. (These operations correspond to the variable transformation $p \rightarrow -p$.) We thus obtain

$$B(U(g_1)\bar{f}, U(g_2)\bar{f}; p) = (-1)^{2s} \hat{A}(-p, -\omega; g_1, g_2).$$

If we now take the complex conjugate of this equation and make use of the above property of A and the relations (5) we obtain the result

$$B(U(g_1)h, U(g_2)h; p) = \mp (-1)^{2s} B(U(g_1)f, U(g_2)f; p). \quad (10)$$

Setting $g_1 = 1$ we find that for f to be nonzero we must have the usual connection between spin and statistics.¹⁰ Setting $g_2^{-1} = g^{-1} v(p) u v^{-1}(p)$ [$u \in SU(2)$ and $v(p)$ the boost to momentum p] and noting the linear independence of the functions $D_{\mu\nu}^s(u)$ we conclude

$$\sum_{i=1}^n h_i(p) h_i^\dagger(q) = \sum_{i=1}^n f_i(p) f_i^\dagger(q). \quad (10a)$$

Since the $f_i(p)$ are linearly independent (10a) implies that in fact

$$f_i(p) = \sum_{j=1}^n r_{ij} h_j(p) \text{ with } r \text{ a unitary matrix.}$$

From this and from the definition of $h(p)$ we find that $r^* r = 1$ and hence that r has the representation $r = \beta \beta^T$ with β unitary. A renaming of the

particles with the unitary transformation β leads to part (b) of the lemma for the new wavefunctions $f'(p) = \beta^{-1} f(p)$.

In terms of these new wavefunctions [which we again call $f_i(p)$] and new creation operators [again called $a_\mu^\dagger(k, p)$], Q can be written

$$Q = \sum_{k=1}^n \sum_{\mu=-s}^s \int \frac{d^3 p}{2\omega} (a_\mu^\dagger(k, p) \hat{f}_\mu(k, p) + a_\mu(k, p) [c_s D^s(\tilde{p}) \hat{f}_k(-p)]_\mu)_{p_0=\omega} \\ = \sum_{k=1}^n \sum_{\mu=-s}^s \int d^3 x [\phi_\mu^k(x) \tilde{f}_\mu^1(k, x) - \partial_t \phi_\mu^k(x) \tilde{f}_\mu^2(k, x)]_{t=0} \quad (11)$$

where here $f(p)$ has been expressed as $f^1(p) - i\omega(p) f^2(p)$, with the $f^i(p)$ entire functions of p , and where $\tilde{f}^i$ is the Fourier transform of f^i . The field $\phi^k(x)$ is given by Eq. (4) with $a_\mu^\dagger(p)$ replaced by $a_\mu^\dagger(k, p)$. Note that by virtue of part (b) of lemma 2 the $f^i(p)$ are in the function class $\mathcal{J}$ when considered as functions of the real variable p . We shall next prove the following:

Lemma 3.¹¹ The functions $\tilde{f}_\mu^i(k, x)$ have support in $||x|| \leq r$ where r is the radius of the base of the diamond D_r .

This means that the operator Q commutes (or anticommutes, depending on s) with the fields $\phi^k(x)$, $k = 1, \dots, n$ for x space-like relative to the region D_r . Equation (11) shows that Q depends only on the field variables inside D_r . Note that because the fields satisfy the Klein-Gordon equation, Q is expressible in terms of the field variables at $t = 0$.

To prove lemma 3 we first estimate

$c_k(g_1, g_2) = \sup_x |D^k \mathcal{J}(x; g_1, g_2)|$ for certain (g_1, g_2) . (Here D^k is a k th derivative with respect to x .) Let v be a pure velocity transformation in $\mathcal{J}$. Suppose $\epsilon > 0$ and let $g \in \mathcal{J}$ be such that $||\Lambda(g) - I|| < \epsilon/\sqrt{2}$. (Here $||M||^2 = \text{trace } M^\dagger M$.) We set $g_1 = v g$, $g_2 = v^{-1}$, and $r = \Lambda_{00}(v)$. From the fact that $U(x) Q |0\rangle$ is C^∞ in x we easily derive the inequality $c_k(g_1, g_2) < c_{Nk}(\epsilon) r^{-N}$ for every N and every k . Using locality, a simple calculation shows that $[D^k \mathcal{J}(x; g_1, g_2)]_{t=0}$ vanishes for $||x|| > 2r r_\epsilon$, with $r_\epsilon = r(1 + \epsilon)$, leading to estimates on the Fourier transforms $F_i(p; g_1, g_2)$:

$$||z||^k |F_i(z; g_1, g_2)| < c'_{Nk}(\epsilon) r^{-N} \exp(2r r_\epsilon ||\text{Im } z||) \quad (12)$$

uniformly in g whenever $||\Lambda(g) - I|| < \epsilon/\sqrt{2}$.

Letting $\underline{e}$ be a real unit vector and p_0 a real three-vector perpendicular to $\underline{e}$, we define $f(z) \equiv f(z\underline{e} + p_0)$. Denoting $\lambda = (p_0^2 + m^2)^{\frac{1}{2}}$, we see that $f(z)$ is analytic on a two-sheeted Riemann surface with branch points at $\omega(z) \equiv (z^2 + \lambda^2)^{\frac{1}{2}} = 0$. We cut the z plane along the imaginary axis from $i\lambda$ to $i\infty$ and from $-i\lambda$ to $-i\infty$. We first want to show that for any non-negative integer k , $||z^k f(z)|| < c_k \exp(r_\epsilon ||\text{Im } z||)$. Hence we choose $\beta = (2s + 1) \times n$ elements $g_i \in \mathcal{J}$ with $||\Lambda(g_i) - I|| < \epsilon/\sqrt{2}$ so that $J(z) \equiv J(z\underline{e} + p_0; g_1, \dots, g_\beta)$ has a determinant which does not vanish identically as a function of z . We next choose two circles (one on each sheet) centered at $z = 0$ with equal radii greater than λ and such that on these curves $\det J(z)$ has no zeroes. Let $\mathcal{C}$ denote the union of the two circles. We now consider velocity transformations v in the $\underline{e}$ direction, and since these do not change p_0 , we write

$z(v)\underline{e} + p_0 = \Lambda(v)(z\underline{e} + p_0)$. Since $A(\Lambda(g_0)p; g_0 g_1, g_0 g_2) = A(p; g_1, g_2)$, Eq. (6) implies that

$$B(U(g_1)f, U^{-1}(v^2)f; p) = A(\Lambda(v)p; v g_1, v^{-1}),$$

or, when written in terms of the matrix J ,

$$f(z(v^2)) = D(v^2)[J(z) D(z)]^{-1} A(z(v)), \quad (13)$$

where $D(z) \equiv D(\tilde{q})$, $q = (z\underline{e} + p_0, \omega(z))$, and $A(z(v))$ is the column vector with components $A(z(v); v g_1, v^{-1})$. With $w = z(v^2)$ we easily calculate $2\gamma z(v) = w + z$ and therefore the estimate (12) results in

$$\left\| \left(\frac{w+z}{2\gamma} \right)^k f(w) \right\| < c_{Nk}''(\epsilon) r^{-N} \exp[r_\epsilon |\operatorname{Im}(w+z)|] \quad (14)$$

for z on the "double circle" $\mathcal{C}$. Here we have used the fact that $\|D(v^2)\| r^{-2s}$ is bounded. This gives the result

$$\|w^k f(w)\| < c_k e^{r_\epsilon |\operatorname{Im} w|} \quad (15)$$

for all w which can be reached from some z on the double circle $\mathcal{C}$ with a real velocity transformation. That these w in fact make up the whole two-sheeted Riemann surface can be seen by direct calculation or by the following argument: We consider the real two-vectors $p = (\operatorname{Re} z, \operatorname{Re}(z^2 + \lambda^2)^{\frac{1}{2}})$ and $q = (\operatorname{Im} z, \operatorname{Im}(z^2 + \lambda^2)^{\frac{1}{2}})$. Since the invariant $I(z) = p \cdot p + q \cdot q = |z^2 + \lambda^2| - |z|^2$ varies in the interval $[-\lambda^2, \lambda^2]$ and since $p \cdot p = (I + \lambda^2)/2$ and $q \cdot q = (I - \lambda^2)/2$, p and q are respectively time-like or null and space-like or null. Hence the real velocity transformations preserve $\operatorname{sign}(\operatorname{Re}(z^2 + \lambda^2)^{\frac{1}{2}})$ (and therefore the sheet structure) and $\operatorname{sign}(\operatorname{Im} z)$. It is not difficult to see that all points z with the same value of the invariants are related by a real velocity transformation. The

demonstration is then completed by noting that each curve described by a fixed value of the invariants has an intersection with one of the circles. (These curves are the hyperbolas $(1+t)^{\frac{1}{2}} \operatorname{Im} z = \pm [(1-t)(\operatorname{Re} z)^2 + (1-t^2)(\lambda^2/2)]^{\frac{1}{2}}$ for $t \equiv I/\lambda^2$ in the interval $(-1, 1]$ and the lines $\pm \operatorname{Im} z \geq \lambda$, $\operatorname{Re} z = 0$.) We then have Eq. (15) for all w and hence the $f^i(w + p_0)$ also satisfy Eq. (15). This is just the condition which insures that

$$\int_{-\infty}^{\infty} dp f^i(p\underline{e} + p_0) e^{ipx} = 0 \quad \text{for } |x| > r_\epsilon.$$

Since this holds for arbitrary $\underline{e}$ and p_0 , integration over all p_0 perpendicular to $\underline{e}$ gives $\tilde{f}^i(x\underline{e}) = 0$ for $|x| > r_\epsilon$ and all $\underline{e}$. Since $\epsilon > 0$ is arbitrary, $\tilde{f}^i(x) = 0$ for $||x|| > r$, which establishes lemma 3.

Thus the operator Q of Eq. (10) determines a set of fields relative to which Q is local. We would now like to show that two different operators, Q_1 and Q_2 , of the same type as Q , determine relatively local free fields. Thus suppose

$$Q_1 = \sum_{k=1}^{n_1} \sum_{\mu=-s}^s \int \frac{d^3 p}{2\omega} [a_\mu^\dagger(k, p) f_\mu(k, p) + a_\mu(k, p) f_\mu^*(k, p)],$$

$$Q_2 = \sum_{k=1}^{n_2} \sum_{\mu=-s}^s \int \frac{d^3 p}{2\omega} [b_\mu^\dagger(k, p) h_\mu(k, p) + b_\mu(k, p) h_\mu^*(k, p)],$$

with $[a_\mu(k, p), b_\nu^\dagger(l, q)]_+ = \epsilon_{k\ell} D_{\mu\nu}^s(\tilde{p}) 2\omega \delta(p - q)$. Here $\epsilon_{k\ell}$ is an n_1 by n_2 matrix. As was shown in lemma 2 the particle creation operators can be selected so that the wavefunctions $f_k(p)$ are

linearly independent and satisfy $f^\theta(p) = e^{i\tau s} \hat{f}(-p, -\omega)$,
 $f(p) = \hat{f}(p, \omega)$. The same applies to the $h_k(p)$. The analog of Eq. (6)
for the function

$$\mathcal{F}_{12}(x; g_1, g_2) = [U(x, g_1) Q_1 U^{-1}(x, g_1), U(g_2) Q_2 U^{-1}(g_2)]_+$$

is

$$B(U(g_1)f, U(g_2)eh; p) = A_{12}(p; g_1, g_2), \quad (6')$$

where $(eh)_k(p) = \sum_{\ell=1}^{n_2} \epsilon_{k\ell} h_\ell(p)$, and where A_{12} has the same
analyticity and reality properties as the function $A(p; g_1, g_2)$. The
analog of Eq. (10) is therefore

$$B(U(g_1)f, (\epsilon - \epsilon^*) U(g_2)h; p) = 0, \quad (10')$$

leading to

$$\sum_{k=1}^{n_1} ((\epsilon - \epsilon^*)h)_k(p) f_k^\dagger(q) = 0, \quad (10a')$$

which implies $\epsilon = \epsilon^*$. This is just the condition that the fields
constructed from the "a" creation operators are local relative to
those constructed from the "b" operators. It also guarantees that
the "b" fields anticommute or commute with Q_1 when these fields
are at a point x space-like separated from the region of localization
of Q_1 . We have thus demonstrated that the set of all $Q \in \{\bigcup_{D \in \mathcal{D}} S(D)\}$
which also have the property that $U(x, g) Q|0\rangle$ is C^∞ in (x, g)
determines a set $\mathcal{L}$ of relatively local fields which are also local
relative to those Q . It is now a simple matter to construct the
standard set of fields referred to in the theorem.¹² Choosing N_s

linearly independent creation operators $b_\mu^\dagger(k, p)$, $k = 1, \dots, N_s$ of
spin s from those associated with fields in $\mathcal{L}$, we note that
 $\langle 0 | b_\mu(k, p) b_\nu^\dagger(\ell, q) | 0 \rangle = 2\omega \delta(p - q) D_{\mu\nu}^s(\tilde{p}) M_{k\ell}$ where M is real,
symmetric, and positive definite. If we choose
 $c_\mu^\dagger(k, p) = \sum_{\ell} b_\mu^\dagger(\ell, p) \alpha_{\ell k}$ and construct fields $\phi^k(x)$ from these new
creation operators, we obtain fields that are local relative to this set of
 Q 's if and only if α is real. Since under this change of creation
operators $M \rightarrow M' = \alpha^\dagger M \alpha$, we can achieve $M' = I$ with a suitable real
 α . This gives the standard set of fields referred to in the theorem.
We finally show that these fields are also local relative to $Q \in S(D)$
for which $U(x, g)Q|0\rangle$ is not C^∞ in (x, g) . The proof goes via a
limiting process: Let $Q_n = \int d^4x dg f_n(x, g) U(x, g) Q U^{-1}(x, g)$, where
 $f_n(x, g)$ is a sequence of real, non-negative C^∞ functions with
compact support Δ_n converging to the point $(0, 1)$. With
 $\int d^4x dg f_n(x, g) = 1$ it is easy to see that $Q_n|0\rangle$ converges strongly
to $Q|0\rangle$. If $h(x) \in \mathcal{D}$ has support in D' , then for large enough n ,
 $\langle 0 | [Q_n, \phi_\mu^k(h)]_+ | 0 \rangle = 0$. [Here $\phi_\mu^k(h) = \int d^4x \phi_\mu^k(x) h(x)$.]
Therefore $[Q, \phi_\mu^k(h)]_+ = 0$ and the proof of theorem 1 is complete.

We conclude this section with a few remarks. We first note
that the creation operators used to construct the N fields are unique
up to a real orthogonal transformation. That is, if the equation
 $V c_\mu^\dagger(k, p) V^{-1} = \sum_{\ell} c_\mu^\dagger(\ell, p) V_{\ell k}$ defines a unitary operator V which
commutes with the Poincare group and which satisfies $V|0\rangle = |0\rangle$, then
the fields $V\phi^k(x)V^{-1}$ are local relative to the $S(D)$ if and only if
the matrix $V_{\ell k}$ is real. Another way of stating this is also of
interest: The abbreviation $f^\theta(p)$ used in theorem 1 actually defines
an antiunitary TCP operator when the $f_i(p)$ refer to the standard
creation operators $c_\mu^\dagger(k, p)$. We see that the set of operators

$\Theta S(D)\Theta^{-1}$ is local relative to the fields in the region $[-D]'$. With this definition of Θ , $\psi^k(x)\psi^{-1}$ is local relative to the $S(D)$ if and only if $[\Theta, V] = 0$.^{13,14}

Secondly we remark on the simple representation (11) for the operators $Q \in S(D)$. This representation, with $\tilde{f}^i(\tilde{x}) = 0$ for $||\tilde{x}|| > r$, has been shown to hold for those $Q \in S(D_r)$, $D_r = \{x ||\tilde{x}|| + |t| < r\}$, for which $U(x, g) Q|0\rangle$ is C^∞ in (x, g) . For a general operator $Q \in S(D_r)$ the differentiability condition need not hold; we only require that $Q|0\rangle$ exist. In this case the vanishing of the tempered distributions $[Q, \phi_\mu^k(x)]_\pm$ for $x \in D'_r$ still implies lemma 2 for wavefunctions $f_1(p)$ which may now grow like a power of p in the real domain; hence the representation (11) no longer applies. Lemma 3 can be replaced by the statement that the Fourier transforms of $f^1(p)$ and $f^2(p)$ are tempered distributions with support in $||\tilde{x}|| \leq r$.

IV. APPLICATION TO FIELD THEORIES AND LOCAL ALGEBRA THEORIES

A. Asymptotic Fields

We consider algebras of local operators $\mathcal{F}(D)$ associated with diamonds D . These might be the polynomial algebras of a field theory associated with bounded regions of space-time or Haag-Araki algebras of bounded operators.^{1,2} Considered as a vector space, $\mathcal{F}(D)$ is spanned by the two subspaces $\mathcal{F}_\pm(D)$ containing Fermi and Bose elements respectively. The observables of the theory are, of course, contained in the $\mathcal{F}_-(D)$. Locality is introduced by assuming that for $D_1 \subseteq D'_2$,

$$[\mathcal{F}_+(D_1), \mathcal{F}_+(D_2)]_\pm = 0, \quad (16)$$

$$[\mathcal{F}_+(D_1), \mathcal{F}_+(D_2)]_- = 0.$$

Note that in field theory Eq. (16) contains the statement of normal statistics which in this case involves no loss of generality.¹⁵ We now assume that the theory is asymptotically complete so that there are Fock spaces of "incoming" and "outgoing" particles each of which is in fact equal to the entire Hilbert space. The creation operators are denoted $a_{ex}^\dagger(i, p)$, where i collectively denotes spin and particle type and "ex" means "in" or "out." If the $a_{ex}^\dagger(i, p)$ are defined so that $\langle 0 | a_{ex}(i, p) a_{ex}^\dagger(j, q) | 0 \rangle = \delta_{ij} \delta(p - q)$, then for each operator Q in $\mathcal{F}_\pm(D)$ and each particle mass the formula²

$$Q_{ex} = \sum_i \int d^3p [\langle 0 | a_{ex}(i, p) Q | 0 \rangle a_{ex}^\dagger(i, p) + a_{ex}(i, p) \langle 0 | Q a_{ex}^\dagger(i, p) | 0 \rangle] \quad (17)$$

defines a certain one-particle operator. For a fixed D , the set of all such Q_{ex} is denoted $\mathcal{F}_\pm^{ex}(D)$. The sum is over all particles of a given mass m and hence $\mathcal{F}_\pm^{ex}(D)$ depends implicitly on the mass of

the particles involved. Landau has shown⁴ that the $\mathcal{F}_{\pm}^{\text{ex}}(D)$ satisfy the locality properties assumed for the $S_{\pm}(D)$ in Sec. III. Hence restricting our further attention to theories with a minimum mass > 0 and at most a finite number of particles at each mass, theorem 1 implies the existence of a relatively local set of asymptotic fields $\phi_{\text{ex}}^k(x)$ which are local relative to the $\mathcal{F}_{\pm}^{\text{ex}}(D)$.

B. Local Internal Symmetries

Following Landau and Wichmann^{3,4} we call a unitary operator V a local internal symmetry if $V|0\rangle = |0\rangle$ and if for $D_1 \subseteq D_2'$ we have in analogy to (16),

$$\begin{aligned} [V\mathcal{F}_{\pm}(D_1)V^{-1}, \mathcal{F}_{\pm}(D_2)]_{\pm} &= 0, \\ [V\mathcal{F}_{\pm}(D_1)V^{-1}, \mathcal{F}_{\pm}(D_2)]_{\pm} &= 0. \end{aligned} \quad (18)$$

Under these assumptions $[U(x), V] = 0$, as was shown in Ref. 3. Furthermore, Landau has shown,⁴ using the Haag-Ruelle construction,^{16,17} that V commutes with the S matrix and that

$$V a_{\text{ex}}^{\dagger}(i, p) V^{-1} = \sum_j a_{\text{ex}}^{\dagger}(j, p) \mathcal{V}_{ji}(p). \quad (19)$$

In a field theory, with the additional assumption of the existence of an interpolating field for each particle, Landau was able to show that

$$[U(x, g), V] = 0 \quad (20)$$

and

$$V \psi_{\text{ex}}^k(x) V^{-1} = \sum_{\ell} \psi_{\text{ex}}^{\ell}(x) \mathcal{K}_{\ell k}. \quad (21)$$

Here the $\psi_{\text{ex}}^k(x)$ are suitably chosen asymptotic fields associated with the interpolating fields and $\mathcal{K}$ is a real orthogonal matrix. We will now derive (20) and (21) without assuming the existence of interpolating fields. Equation (21) will then be true for the asymptotic fields $\phi_{\text{ex}}^k(x)$ discussed in IV.A above.

We first note that the assumption (18) coupled with the result of Landau mentioned in IV.A shows that for $D_1 \subseteq D_2'$ the vacuum expectation values of the expressions

$$[V\mathcal{F}_{\pm}^{\text{ex}}(D_1)V^{-1}, \mathcal{F}_{\pm}^{\text{ex}}(D_2)]_{\pm} \text{ and } [V\mathcal{F}_{\pm}^{\text{ex}}(D_1)V^{-1}, \mathcal{F}_{\pm}^{\text{ex}}(D_2)]_{\pm}$$

both vanish. The problem is thus reduced to one involving one-particle operators alone. Let us choose a set of creation operators $c_{\mu}^{\text{tex}}(k, p)$ transforming according to Eq. (3) and such that $\phi_{\mu\text{ex}}^k(x)$ is linear in $c_{\mu}^{\text{tex}}(k, p)$ and its Hermitian conjugate. If Eq. (19) is rewritten in terms of these creation operators, then the matrix $\mathcal{V}(p)$ is replaced by a matrix which we denote $K(p)$. In order to examine $K(p)$ we choose $Q_1, Q_2 \in \mathcal{F}_{\pm}^{\text{ex}}(D)$ such that for each spin the particle wave-functions $f_i(p)$ [relative to the $c_{\mu}^{\text{tex}}(k, p)$] associated with $Q_1|0\rangle$ are linearly independent, and similarly for the $h_i(p)$ associated with $Q_2|0\rangle$. Furthermore we require that $Q_1|0\rangle$ contain all the different types of particles of mass m and half-integer (or integer) spin, and that both $U(x) Q_1|0\rangle$ and $U(x) Q_1^{\dagger}|0\rangle$ are C^{∞} in the variable x . We then consider the function

$$\mathcal{F}(x; g_1, g_2, g) \equiv \langle 0 | [U(x, g_1) Q_1^{\dagger} U^{-1}(x, g_1), V(g) U(g_2) Q_2 U^{-1}(g_2) V^{-1}(g)]_{\pm} | 0 \rangle, \quad (22)$$

where $V(g) \equiv U(g) V U^{-1}(g)$ is also a local internal symmetry. If we fix $\epsilon > 0$ and require $\|A(g_i) - I\| < \epsilon$ for $i = 1, 2$, then

$\mathcal{F}(x; g_1, g_2, g) = 0$ for x space-like relative to a fixed diamond D^ϵ , uniformly in g . In p -space this leads to the statement that

$$B(U(g_1)f, V(g)U(g_2)h; p) = A(p; g_1, g_2, g), \quad (23)$$

where $A(p; g_1, g_2, g)$ is an entire function of p satisfying

$$||p||^k |A(p; g_1, g_2, g)| < c_k(\epsilon) \exp(\alpha_\epsilon ||\text{Im } p||) \quad (24)$$

with α_ϵ and $c_k(\epsilon)$ independent of g . The Hermitian form $B(f, h; p)$ has the same meaning as in Eq. (6) except that a sum over spins is also understood. Considering along with the expressions (22) the corresponding expressions obtained when $q_1 \in \mathcal{F}_+^{\text{ex}}$ and $q_2 \in \mathcal{F}_+^{\text{ex}}$, we see that both integer and half-integer spins can be assumed to be represented in f and h . Note also that $V(g)$ in (23) corresponds to the matrix $K(p; g) = D(g) K(\Lambda^{-1}(g)p) D^{-1}(g)$. The construction of matrices of the type $J(p)$ as in Eq. (9) from Lorentz-transformed f 's and h 's allows us to invert Eq. (23) to get

$$K(p; g) = [J_1(p) D(\tilde{p})]^{-1} A(p; g) [J_2^T(p)]^{-1} \quad (25)$$

where $J_1^*(p)$ and $J_2(p)$ are constructed respectively from $(U(g_1)f)(p)$ and $(U(g_1')h)(p)$ and where $A_{ij}(p; g) = A(p; g_i, g_j, g)$. Since $f(p)$ and $h(p)$ are entire functions of p , Eq. (25) shows explicitly that $K(p)$ is a meromorphic function of the four-vector p . A suitable choice of the g_i and g_i' can remove any of the zeroes of the denominator, which means that in fact $K(p)$ is an entire function of p . We now consider the growth of $K(p)$ for complex p . As in lemma 3 we examine $K(\Lambda^{-1}p)$ for $p = z\tilde{e} + p_0$. We choose a double circle $\mathcal{C}$ in the z complex space of radius greater than $(p_0^2 + m^2)^{\frac{1}{2}}$ on

which $\det J_1(z)$ and $\det J_2(z)$ have no zeroes. For $z \in \mathcal{C}$, the inequality (24) gives

$$||K(z\tilde{e} + p_0; g)|| < c. \quad (26)$$

We choose g to be a velocity transformation v of velocity v along $\tilde{e}$ and define $r = \Lambda_{00}(v)$. Since $||D(v)||r^{-s}$ is bounded we have

$$||K(w\tilde{e} + p_0)|| < c' r^{2s}, \quad (27)$$

where $w = z(v^{-1}) = r[z - v w(z)]$. Since for large w , v must be near 1 and $z - v w(z)$ cannot be near zero, we have

$$||K(w\tilde{e} + p_0)|| < c''(1 + |w|^{2s}). \quad (28)$$

This means that $K(w\tilde{e} + p_0)$ is in fact a polynomial in w and $\omega(w)$.¹⁸ An argument given by Landau⁴ then suffices to show that V commutes with the Lorentz group. A variant of this argument will be outlined here for the reader's convenience: We first note that the unitarity of V is reflected in the equation

$$K^\dagger(p) D(\tilde{p}) K(p) = D(\tilde{p}) \quad (29)$$

and hence since $D(\tilde{p}) = D^2(v(-p))$, where $v(p) \in \mathfrak{g}$ is the boost to momentum p , that $\Gamma(p) \equiv D^{-1}(v(p)) K(p) D(v(p))$ is a unitary matrix. Let $p = \Lambda(g)(0, m)$, where g is the velocity transformation $\cosh x + \sinh x \tilde{e} \cdot \tilde{g}$. Since $\Lambda(g)$ and $D(g)$ are polynomials in the matrix elements of g and g^* , for fixed $\tilde{e}$ the matrix Γ is a polynomial in $\lambda = e^x$ and $\lambda^{-1} = e^{-x}$. But since Γ must remain bounded as λ varies in $0 < \lambda < \infty$, it is in fact independent of λ . Setting $\lambda = 1$ gives $\Gamma(p) = K(0)$. We now apply the same argument

to $K(p; g)$ with the result that $D^{-1}(v(p))K(p; g)D(v(p))$ is also independent of p . If $g = v(q)$ we can choose $p = q$ in which case the latter expression becomes $K(0)$. This shows that for g a velocity transformation $K(p; g) = K(p)$. But since the velocity transformations generate $\mathcal{G}$, we have $K(p; g) = K(p)$ for all $g \in \mathcal{G}$; i.e., $[U(g), V] = 0$.

Finally, we take $u \in SU(2)$ and note that $K(0; u) = D(u)K(0)D^{-1}(u) = K(0)$ implies that $K(p) = D(v(p))\Gamma(p)D^{-1}(v(p)) = K(0) = \mathcal{K}$. Hence $K(p) = \mathcal{K}$ is a unitary matrix which acts only on the particle indices and does not mix particles of different spin. The closing remarks of Sec. III then suffice to show that $\mathcal{K}$ is a real matrix and thus to complete the demonstration of the Eqs. (20) and (21).

C. Haag-Ruelle Theory

In the discussion of local internal symmetries the Haag-Ruelle construction of asymptotic states was needed to relate the action of V on the one-particle states to its action on the many-particle states. [See the derivation of Eq. (19) in Ref. 4.] We first want to point out that the method of Haag and Ruelle^{16,17} rests on an assumption which is not entirely appropriate in a study of the possible symmetries of a local theory: It is assumed^{16,17} that pure one-particle states can be constructed by letting quasi-local operators¹⁹ Q act on the vacuum. This means that quasi-local operators Q can be found such that the states $Q|0\rangle$ are eigenstates of the mass-operator. In certain situations this supposition follows directly from the assumed mass spectrum: Suppose that the particle in question has mass $m > 0$ and that E_m and $E(\mu^2)$ project onto the subspaces with $p^2 = m^2$ and $p^2 \leq \mu^2$ respectively. If m^2 is an isolated

point in the spectrum of P^2 then it is easy to find the required quasi-local Q .²⁰ For $R \in \mathcal{F}_+(D)$ satisfying $E_m R|0\rangle \neq 0$, the operator $R(x) = U(x)RU(-x)$ can be suitably smeared with a function $f(x) \in \mathcal{S}$ to give a quasi-local $Q = \int d^4x R(x)f(x)$ satisfying $Q|0\rangle = E_m Q|0\rangle = R|0\rangle$. However, it may be the case that m is in the mass-continuum of some many-particle states. In this case one might try to justify the Haag-Ruelle assumption in the following manner: Any continuum contribution to the support of $dE(\mu^2)$ in a neighborhood of m^2 should correspond to states with "quantum numbers" different from those of the particle in question, otherwise the particle would not be stable. At this point one might argue that in a "reasonable theory" there ought to exist local operators carrying the appropriate quantum numbers. For example if these quantum numbers were associated with a compact local internal symmetry group, then local operators carrying definite values of these quantum numbers could be constructed from arbitrary local operators (by means of the Haar integral for example). We could then choose a local operator Q which carried the quantum numbers of the particle in question. This operator would satisfy $dE(\mu^2)(1 - E_m)Q|0\rangle = 0$ in an interval around m^2 , while $E_m Q|0\rangle \neq 0$. By smearing $Q(x)$ as before, one would arrive at a quasi-local operator satisfying the assumption of Haag and Ruelle.

In view of this close connection of the Haag-Ruelle assumption with the symmetries of the theory, we feel that a better understanding of these symmetries could result if asymptotic states could be constructed without making this assumption.

In a theory of strong interactions we are no doubt justified in assuming that $E(\mu^2)(1 - E_m)$ is continuous in a neighborhood of m^2 (for all particle masses, m). This is just the assumption that particle masses have no point of accumulation. If one makes a slightly stronger assumption concerning the smoothness of $E(\mu^2)(1 - E_m)Q|0\rangle$, then as we shall show, it is possible to carry out the Haag-Ruelle construction of asymptotic states. Specifically we assume that for each particle mass, m , operators $Q_i \in \mathcal{F}_+(D_i)$ can be found such that

$$(a) \quad E_m Q_i |0\rangle \neq 0,$$

(b) $E(\mu^2)(1 - E_m)Q_i |0\rangle$ is Hölder continuous at m^2 (i.e., there exist ϵ_i , δ_i , and c_i all greater than zero such that for $|\mu^2 - m^2| < \delta_i$ $\psi_i = (1 - E_m)Q_i |0\rangle$ satisfies $||[E(\mu^2) - E(m^2)]\psi_i|| < c_i |\mu^2 - m^2|^{\epsilon_i}$).

If, furthermore, for each mass, m , the Poincare transforms of $\{E_m Q_i |0\rangle\}$ span the one-particle sub-Hilbert space, asymptotic Fock spaces can be constructed. We note that although the Hölder continuity condition is a considerable relaxation of the Haag-Ruelle assumption, it may not be physically motivated and hence our result should be considered provisional.

The idea of the construction is to use quasi-local "creation operators" $Q_i(f_i, t)$ at finite times t , which as $t \rightarrow \pm\infty$ have a mass spectrum "converging" to the point m . This enables us to construct a dense set of collision states. Details are given in the Appendix.

ACKNOWLEDGMENTS

I would like to thank Professor E. H. Wichmann for suggesting that a result such as the theorem of Sec. III should be true and for many valuable discussions. It is also a pleasure to thank Mr. Peter Mohr for helpful discussions.

I am very grateful to my wife Judith for her encouragement during the course of this work.

APPENDIX

Suppose that for $i = 1, \dots, n$ $Q_i \in \mathcal{F}_+(D_i)$ satisfies the conditions (a) and (b) of Sec. IV.C with m replaced by m_i . Let $h(s)$ be a real C^∞ function of the real variable s with compact support and with $h(0) = 1$. We choose n C^∞ functions $f_i(p)$ of the four-vector p with compact support inside the forward light cone and define

$$Q_i(f_i, t) = \int d^4p \tilde{Q}_i(p) f_i(p) h(s(p^2 - m_i^2)^2) e^{i[p_0 - \omega_i(p)]t},$$

where $\tilde{Q}_i(p)$ is the Fourier transform of $Q_i(x)$, $\omega_i(p) = (p^2 + m_i^2)^{1/2}$, and $s = s(t)$ is a function (to be specified later) which increases with t .

The main tool used in the construction of asymptotic states is a generalization of a space-like cluster property proved by Ruelle.¹⁷ Let $\langle 0 | R_1(x_1) \dots R_n(x_n) | 0 \rangle_T$ be the truncated vacuum expectation value of the product of operators $R_i(x_i)$, with $R_i(0) \in \mathcal{F}_+(D_i)$. Let, furthermore, $f(s; y_1, \dots, y_n)$ be the Fourier transform of a function in $\mathcal{S}$ multiplied by $\prod_{i=1}^n h(s(p^2 - m_i^2)^2)$. Then for some $N_0 \geq 0$ the function

$$\mathcal{F}(s; x_1, \dots, x_n) = \int \prod_{i=1}^n d^4y_i f(s; y_1, \dots, y_n)$$

$$\times \langle 0 | R_1(y_1 + x_1) \dots R_n(y_n + x_n) | 0 \rangle_T$$

satisfies the inequality

$$|D^k \mathcal{F}(s; x_1, \dots, x_n)| < c_{Nk} d^{-N} (|s|^{\frac{N}{2} + N_0} + 1) \quad (A.1)$$

with $d = \max_{i,j} ||x_i - x_j||$. Here D^k is any monomial in the derivatives with respect to the x_i and N is an arbitrary non-negative integer. The proof will not be given since it involves only a straightforward generalization of Ruelle's estimates. Note that the inequality (A.1) might be expected since the function $h(s(p^2 - m^2)^2)$ contributes a $(s)^{\frac{1}{2}}$ "spreading" in coordinate space.

We now investigate the convergence of the vector

$\psi(t) = Q_1(f_1, t) \dots Q_n(f_n, t) | 0 \rangle$ where the $f_i(p)$ are chosen "non-overlapping in velocity space";²¹ that is, for all $i \neq j$, if

$p_i \in \text{supp} f_i$ and $p_j \in \text{supp} f_j$ then $p_i/\omega_i(p_i) \neq p_j/\omega_j(p_j)$. Expanding

$||d\psi(t)/dt||^2$ in truncated vacuum expectation values gives a sum of

products of N -point functions. We first consider an N -point function where $N \geq 3$ and which involves no operator Q_1 which has been differentiated with respect to t . After a renumbering of the operators, the N -point function can be written

$$\int \prod_{i=2}^N d^3p_i \tilde{F}(s; p_2, \dots, p_N) e^{i[\omega_1(p_1) + \dots + \omega_m(p_m) - \omega_{m+1}(p_{m+1}) - \dots - \omega_N(p_N)]t} \quad (A.2)$$

where $F(s; x_2, \dots, x_N) \equiv \langle 0 | Q_1^\dagger(s; 0) Q_2^\dagger(s; -x_2) \dots Q_m^\dagger(s; -x_m) Q_{m+1}(s; x_{m+1}) \dots Q_N(s; x_N) | 0 \rangle_T$ and $Q_i(s; x) \equiv \int d^4p \tilde{Q}_i(p) f_i(p) h(s(p^2 - m_i^2)^2) e^{-ip \cdot x}$.

Note that in (A.2) $p_1 + \dots + p_m = p_{m+1} + \dots + p_N$ and that

$1 \leq m < N$. The estimate (A.1) then shows that if D^ℓ is an ℓ th derivative with respect to $p = (p_2, \dots, p_N)$, then for some N'_0

$$||p||^k |D^\ell \tilde{F}(s; p_2, \dots, p_N)| < c_{k\ell} (|s|^{\frac{\ell}{2} + N'_0} + 1). \quad (A.3)$$

If we now take $s = |t|^{2-\epsilon}$ with $2 \geq \epsilon > 0$, repeated integration by parts in the expression (A.2) shows (see Ref. 21) that it is bounded by $c_k |t|^{-k}$ for any $k \geq 0$. Note that this bound remains valid when one or more $Q_i(f_i, t)$ are replaced by their time derivatives; the differentiation only results in the replacement of $h(s(p^2 - m_i^2)^2)$ by $i(p_0 - \omega_i) h(s(p^2 - m_i^2)^2) + \frac{d}{dt} h(s(p^2 - m_i^2)^2)$.

We now consider the two-point function. The assumption of Hölder continuity is easily seen to imply that if $\epsilon < 2\epsilon_i/(1 + \epsilon_i)$ ($s = |t|^{2-\epsilon}$), then for some $\beta_i > 0$

$$\left\| \frac{dQ_i}{dt}(f_i, t) |0\rangle \right\| < \alpha_i / (1 + |t|)^{1+\beta_i}.$$

Hence for appropriate ϵ each two-point function is bounded, and the terms in the expansion of $\|d\psi(t)/dt\|^2$ containing products of two-point functions only is bounded by $c|t|^{-2-\delta}$ with $\delta > 0$. Thus $\psi(t)$ converges strongly. An argument similar to that given by Hepp²⁰ establishes the Lorentz frame independence of the construction. The existence of asymptotic Fock spaces is thus demonstrated. Note that it is sufficient to consider only sets of functions $f_i(p)$ with nonoverlapping supports in velocity space because the linear span of the vectors constructed using such functions is already dense in Fock space.^{2,21}

We remark finally that once the convergence of $\psi(t)$ to ψ has been demonstrated as above, it is no longer necessary to choose $s(t)$ as we did. In fact suppose $\mu(t)$ is any function of t satisfying

- (a) $\mu(t) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$;
- (b) for some $\sigma > 0$ $|t|^\sigma (\mu(t)/t^2) \rightarrow 0$ as $t \rightarrow \pm\infty$.

Then it is easy to see that if $\phi(t)$ is constructed exactly as $\psi(t)$ above except $s(t)$ is replaced by $\mu(t)$, we have $\|\phi(t) - \psi(t)\| \rightarrow 0$ as $t \rightarrow \pm\infty$ and hence $\phi(t)$ converges to the same vector as $\psi(t)$. Although we will not demonstrate it, this fact can be used to prove strong convergence without the restriction to $f_i(p)$ which are nonoverlapping in velocity space.

FOOTNOTES AND REFERENCES

- * This work was supported in part by the U.S. Atomic Energy Commission.
- † Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant No. AF-AFOSR-68-1471.
1. R. Haag and B. Schroer, J. Math. Phys. 3, 248 (1962).
2. H. Araki and R. Haag, Commun. Math. Phys. 4, 77 (1967).
3. L. J. Landau and E. Wichmann, J. Math. Phys. 11, 306 (1970).
4. L. J. Landau, Commun. Math. Phys. 17, 156 (1970).
5. The field $\phi_\mu(x)$ is an interpolating field for particle i if the projection E_i onto the one-particle states of this particle satisfies $E_i \phi_\mu(x)|0\rangle \neq 0$.
6. S. Weinberg, Phys. Rev. 133, B1318 (1964).
7. H. Epstein, J. Math. Phys. 8, 750 (1967). Epstein proved lemma 2 in the case $n = 1$. His method can easily be generalized to arbitrary n .
8. E. Goursat, Differential Equations (Dover Publications, Inc., New York, 1959).
9. P. Hartman, Ordinary Differential Equations (John Wiley and Sons, Inc., New York, 1964).
10. This observation is also made in Ref. 7.
11. H. Epstein, V. Glaser, and A. Martin, Commun. Math. Phys. 13, 257 (1969). In the case $n = 1$, $s = 0$, it is shown here that the functions $f^i(x)$ have compact support. The precise region of support is not determined.
12. J. Schwartz, J. Math. Phys. 2, 271 (1961).
13. P. Carruthers, Phys. Rev. Letters 18, 353 (1967).
14. B. Zumino and D. Zwanziger, Phys. Rev. 164, 1959 (1967).
15. H. Araki, J. Math. Phys. 2, 267 (1961).
16. R. Haag, Phys. Rev. 112, 669 (1958).
17. D. Ruelle, Helv. Phys. Acta 35, 147 (1962).
18. Another perhaps more direct way of proving this fact is to use Eq. (25) and theorem A3.2 of Ref. 11.
19. Quasi-local operators can be defined as those operators which satisfy a space-like cluster property given by Haag in Ref. 5. See also Eq. (A.1) in the Appendix with $s = 0$.
20. K. Hepp in 1965 Brandeis University Summer Institute in Theoretical Physics, Axiomatic Field Theory, M. Chretien and S. Deser, Eds. (Gordon and Breach Science Publishers, Inc., New York, 1966), Vol. I.
21. K. Hepp, Commun. Math. Phys. 1, 95 (1965).

LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

TECHNICAL INFORMATION DIVISION
LAWRENCE RADIATION LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720