## Title
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# PATH CONNECTEDNESS AND ENTROPY DENSITY OF THE SPACE OF ERGODIC HYPERBOLIC MEASURES

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Abstract. We show that the space of hyperbolic ergodic measures of a given index supported on an isolated homoclinic class is path connected and entropy dense provided that any two hyperbolic periodic points in this class are homoclinically related. As a corollary we obtain that the closure of this space is also path connected.

## 1. Introduction

In this paper we consider homoclinic classes of periodic points for $C^{1+\alpha}$ diffeomorphisms of compact manifolds and we discuss two properties of the space of invariant measures supported on them and equipped with the week$^*$-topology – connectedness and entropy density of the subspace of hyperbolic ergodic measures. The study of connectedness of the latter space was initiated by Sigmund in a short article [27]. He established path connectedness of this space in the case of transitive topological Markov shifts and as a corollary, of Axiom $A$ diffeomorphisms. Sigmund's idea was to show first that any two periodic measures (i.e., invariant atomic measures on periodic points) can be connected by a continuous path of ergodic measures and second that if one of the two periodic measures lies in a small neighborhood of another one, then the whole path can be chosen to lie in this neighborhood. In order to carry out the first step Sigmund shows that any periodic measure can be approximated by Markov measure and that any two Markov measures can be connected by a path of Markov measures. We use Sigmund's idea in our proof of Theorem 1.1.

A different approach to Sigmund's theorem is to show that ergodic measures on a transitive topological Markov shift are dense in the space of all invariant measures. Since the latter space is a simplex and ergodic measures are its extremal points, it means that this space is the Poulsen simplex (which is unique up to a homeomorphism). The desired result now follows from a complete description of the Poulsen simplex given in [23] (see also [14]). We use this approach to prove our Theorem 1.2.

Since Sigmund's work the interest to the study of connectedness of the space of hyperbolic ergodic measures has somehow been lost,[1] and only recently it has regain attention. In [15] Gogolev and Tahzibi, motivated by their study of existence of non-hyperbolic invariant measures, raised a question of whether the space of ergodic

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[1]At the time of writing this paper there is no single reference to the paper by Sigmund [27] in *MathSciNet*.

measures invariant under some partially hyperbolic systems is path connected. This includes, in particular, the famous example by Shub and Wilkinson [26]. Some results on connectedness and other topological properties of the space of invariant measures were obtained in [7, 14].

All known proofs of connectedness of the space of invariant measures are based on approximating invariant measures by either measures supported on periodic orbits or Markov ergodic measures supported on invariant horseshoes. It is therefore natural to ask whether such approximations can be arranged to also ensure convergence of entropies. If this is possible, the space of approximants is called *entropy dense*. Some results in this direction were obtained in [18]. We stress that approximating *hyperbolic ergodic* measures with positive entropy by "nice" measures supported on invariant horseshoes so that the convergence of entropies is also guaranteed, was first done by Katok in [20] (see also [2, 21]). We use this result in the proof of our Theorem 1.4 where we approximate also some hyperbolic ergodic measures with zero entropy as well as non-ergodic measures.

We shall now state our results. Consider a $C^{1+\alpha}$-diffeomorphism $f : M \to M$ of a compact smooth manifold $M$. Let $p \in M$ be a hyperbolic periodic point. We say that a hyperbolic periodic point $q \in M$ is homoclinically related to $p$ and write $q \sim p$ if the stable manifold of $q$ has transversal intersections with the unstable manifold of $p$ and vise versa. We denote by $\mathcal{H}(p)$ the homoclinic class associated with the point $p$, that is the closure of the set of hyperbolic periodic points homoclinically related to $p$.

A basic hyperbolic set gives the simplest example of a homoclinic class, but in general the set $\mathcal{H}(p)$ can have a much more complicated structure and dynamical properties. In particular, it can contain non-hyperbolic periodic points, and it can support hon-hyperbolic measures in a robust way, see [3, 4, 9, 22].[2] Moreover – and this is of importance for us in this paper – there may exist hyperbolic periodic points in $\mathcal{H}(p)$ that are *not* homoclinically related to $p$, see [9, 12]. Besides, it can happen that periodic orbits outside the homoclinic class $\mathcal{H}(p)$ accumulate to $\mathcal{H}(p)$; for example, this is part of the Newhouse phenomena, and also occurs in the family of standard maps, see [13, 16]. We wish to avoid both of these complications, and we therefore, impose the following crucial requirements on the homoclinic class $\mathcal{H}(p)$. By the index $s(p)$ of a hyperbolic periodic point $p$ we mean the dimension of the invariant unstable manifold of $p$.

(H1) *For any periodic hyperbolic point $q \in \mathcal{H}(p)$ with $s(q) = s(p)$ we have $q \sim p$.*
(H2) *The homoclinic class $\mathcal{H}(p)$ is isolated.*

We stress that these requirements do hold in many interesting cases, see examples in Section 2. In particular, Condition (H2) holds if the map $f$ has only one homoclinic class. This is the case in Examples 1 and 2 in Section 2. We also note a result in [7] that is somewhat related to Condition (H2): if the map $f$ admits a dominated splitting of index $s$, then a linear combination of hyperbolic ergodic measures of index $s$ can be approximated by a sequence of hyperbolic ergodic measures of index $s$ if and only if their homoclinic classes coincide.

The space of all invariant ergodic measures supported on $\mathcal{H}(p)$ can be extremely reach and contain hyperbolic measures with different number of positive Lyapunov exponents as well as non-hyperbolic measures. We denote by $\mathcal{M}_p$ the space of all

[2] It is conjectured that existence of non-hyperbolic ergodic measures is a characteristic property of non-hyperbolic homoclinic classes, see [3, 9].

hyperbolic invariant measures supported on $\mathcal{H}(p)$ for which the number of positive Lyapunov exponents at almost every point is exactly $s(p)$. Further, we denote by $\mathcal{M}_p^e$ the space of all hyperbolic ergodic measures with $s(p)$ positive Lyapunov exponents which are supported on $\mathcal{H}(p)$. We assume that the space $\mathcal{M}_p$ is equipped with the weak*-topology.

**Theorem 1.1.** *Under Conditions (H1) and (H2) the space $\mathcal{M}_p^e$ is path connected.*

Notice that without Conditions (H1) and (H2) the conclusion of Theorem 1.1 may fail, see Subsection 2.2.

It follows immediately from Theorem 1.1 that the closure of $\mathcal{M}_p$ is connected. In fact, a stronger statement holds.

**Theorem 1.2.** *Under Conditions (H1) and (H2) the closure of the space $\mathcal{M}_p^e$ is path connected.*

We shall now discuss the entropy density of the space $\mathcal{M}_p^e$.

**Definition 1.3.** *A subset $S \subseteq \mathcal{M}_p$ is entropy dense in $\mathcal{M}_p$ if for any $\mu \in \mathcal{M}_p$ there exists a sequence of measures $\{\xi_n\}_{n\in\mathbb{N}} \subset S$ such that $\xi_n \to \mu$ and $h_{\xi_n} \to h_\mu$ as $n \to \infty$.*

**Theorem 1.4.** *Under Conditions (H1) and (H2) the space $\mathcal{M}_p^e$ is entropy dense in $\mathcal{M}_p$.*

## 2. Examples

In this section we present some examples that illustrate importance of Conditions (H1) and (H2).

### 2.1. **Homoclinic classes satisfying Conditions (H1) and (H2).**

We describe a class of diffeomorphisms with partially hyperbolic attractors for which the attractor is the homoclinic class of any of its periodic point and which satisfies Conditions (H1) and (H2). We follow [8]. Let $f$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold $M$ and $\Lambda$ a topological attractor for $f$. This means that there is an open set $U \subset M$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(U)$. We assume that $\Lambda$ is a partially hyperbolic set for $f$, that is for every $x \in \Lambda$ there is an invariant splitting of the tangent space $T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$ into stable $E^s(x)$, central $E^c(x)$ and unstable $E^u(x)$ subspaces such that $\|df|v\| < \lambda_1\|v\|$ for every $v \in E^s(x)$, $\lambda_2\|v\| < \|df|v\| < \lambda_3\|v\|$ for every $v \in E^c(x)$, $\|df|v\| > \lambda_4\|v\|$ for every $v \in E^u(x)$, where

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$$

are some constants and $\lambda_1 < 1$, $\lambda_4 > 1$.

If $\Lambda$ is a partially hyperbolic attractor for $f$, then for every $x \in \Lambda$ one can construct a global unstable leaf $W^u(x)$ through $x$ such that $T_yW^u(x) = E^u(y)$ for every $y \in W^u(x)$. It is easy to see that for every $x \in \Lambda$ we have $W^u(x) \subset \Lambda$ and that $f(W^u(x)) = W^u(f(x))$. Moreover, the collection of all global unstable leaves $W^u(x)$ forms a continuous lamination of $\Lambda$ with smooth leaves, and if $\Lambda = M$, then it is a continuous foliation of $M$ with smooth leaves.

An invariant measure $\mu$ on $\Lambda$ is called a $u$-measure if the conditional measures it generates on the unstable leaves $W^u(x)$ are equivalent to the leaf volume on $W^u(x)$ induced by the Riemannian metric (see [24]). Following [8] we say that a $u$-measure

$\mu$ has negative central exponents on an invariant subset $A \subset \Lambda$ of positive measure if for every $x \in A$ and $v \in T_x E^c(x)$ the Lyapunov exponent $\chi(x, v) < 0$.

We consider the following requirement on the map $f|\Lambda$:

(D) for every $x \in \Lambda$ the positive semi-trajectory of the global unstable leaf $W^u(x)$ is dense in $\Lambda$ that is

$$\overline{\bigcup_{n\geq 0} f^n(W^u(x))} = \overline{\bigcup_{n\geq 0} W^u(f^n(x))} = \Lambda.$$

It is shown in [8] that if $\mu$ is a $u$-measure on $\Lambda$ and if $f$ satisfies Condition (D), then 1) $\mu$ has negative central exponents at almost every point $x \in \Lambda$; 2) $\mu$ is the unique SRB-measure for $f$ supported on the whole $\Lambda$; and 3) the basin of attraction for $\mu$ coincides with the open set $U$.

It is easy to see that in this case

(1) the attractor $\Lambda$ is the homoclinic class of every of its hyperbolic periodic points of index equal to the dimension of the unstable leaves; these periodic orbits are dense in $\Lambda$;
(2) the homoclinic class satisfies Conditions (H1) and (H2), and hence Theorems 1.1, 1.2, and 1.4 are applicable.

Condition (D) clearly holds if the unstable lamination is minimal, i.e., every leaf of the lamination is dense in $\Lambda$.

Let $f_0$ be a partially hyperbolic diffeomorphism which is either 1) a skew product with the map in the base being a topologically transitive Anosov diffeomorphism or 2) the time-1 map of an Anosov flow. If $f$ is a small perturbation of $f_0$ then $f$ is partially hyperbolic and by [19], the central distribution of $f$ is integrable. Furthermore, the central leaves are compact in the first case and there are compact leaves in the second case. It is shown in [8] that $f$ has minimal unstable foliation provided there exists a compact periodic central leaf $\mathcal{C}$ (i.e., $f^\ell(\mathcal{C}) = \mathcal{C}$ for some $\ell \geq 1$) for which the restriction $f^\ell|\mathcal{C}$ is a minimal transformation.

Furthermore, it follows from the results in [1] that starting from a volume preserving partially hyperbolic diffeomorphism $f_0$ with one-dimensional central subspace, it is possible to construct a $C^2$ volume preserving diffeomorphism $f$ which is arbitrarily $C^1$-close to $f_0$ and has negative central exponents on a set of positive volume. Moreover, if $\mathcal{C}$ is a compact periodic central leaf, then $f$ can be arranged to coincide with $f_0$ in a small neighborhood of the trajectory of $\mathcal{C}$.

We now consider the two particular examples.

**Example 1.** Consider the time-1 map $f_0$ of the geodesic flow on a compact surface of negative curvature. Clearly, $f_0$ is partially hyperbolic and has a dense set of compact periodic central leaves. It follows from what was said above that there is a volume preserving perturbation $f$ of $f_0$ such that

(1) $f$ is of class $C^2$ and is arbitrary close to $f_0$ in the $C^1$-topology;
(2) $f$ is a partially hyperbolic diffeomorphism with one-dimensional central subspace;
(3) there exists a central leaf $\mathcal{C}$ such that the restriction $f^\ell|\mathcal{C}$ is a minimal transformation (here $\ell$ is the period of the leaf);
(4) $f$ has negative central exponents on a set of positive volume;
(5) the unstable foliation for $f$ is minimal and hence, satisfies Condition (D).

We conclude that in this example the whole manifold is the homoclinic class of every hyperbolic periodic point of index one and that this class satisfies Conditions (H1) and (H2).

**Example 2.** Consider the map $f_0 = A \times R$ of the 3-torus $T^3 = T^2 \times T^1$ where $A$ is a linear Anosov automorphism of the 2-torus $T^2$ and $R$ is an irrational rotation of the circle $T^1$. It follows from what was said above that there is a volume preserving perturbation $f$ of $f_0$ such that the properties (1) – (5) in the previous example hold, and hence the unique homoclinic class satisfies Conditions (H1) and (H2).

**Remark 2.1.** It was shown in [5] that the set of partially hyperbolic diffeomorphisms with one dimensional central direction contains an open and dense subset of diffeomorphisms with minimal unstable foliation. However, in our examples we use preservation of volume to ensure negative central Lyapunov exponents on a set of positive volume, so we cannot immediately apply the result in [5] to obtain an open set of systems for which Conditions (H1) and (H2) hold, compare with Problem 7.25 from [6].

**Remark 2.2.** In both Examples 1 and 2 the map possesses a non-hyperbolic ergodic invariant measure (e.g. supported on the compact periodic leaf). We believe that in these examples presence of non-hyperbolic ergodic invariant measures is persistent under small perturbations. Indeed, since the central subspace is one dimensional, the central Lyapunov exponent with respect to a given ergodic measure is an integral of a continuous function (i.e., log of the expansion rate along the central subspace) over this measure, existence of periodic points of different indices combined with (presumable) connectedness of the space of ergodic measures should imply existence of a non-hyperbolic invariant ergodic measure. See [3, 4, 11, 17] for the related results and discussion.

2.2. **Homoclinic classes that do not satisfy Conditions (H1) and (H2).** There is an example of an invariant set for a partially hyperbolic map with one dimensional central subspace which is a homoclinic class containing two non-homoclinically related hyperbiolic periodic orbits, hence, not satisfying Condition (H1), see [9, 12, 10]. Moreover, the space of hyperbolic ergodic measures supported on this homoclinic class is not connected due to the fact that the set of all central Lyapunov exponents is split into two disjoint closed intervals, see Remark 5.2 in [10].

Condition (H2) ensures that the horseshoes and periodic orbits that we use to approximate a given hyperbolic ergodic measure do belong to the initial homoclinic class. We do not know whether given a not necessarily isolated homoclinic class, every hyperbolic ergodic invariant measure supported on this homoclinic class can always be approximated in such a way.

## 3. Proofs

The space $\mathcal{M}$ of all probability Borel measures on $M$ equipped with the weak$^*$-topology is metrizable with the distance $d_{\mathcal{M}}$ given by

$$d_{\mathcal{M}}(\mu, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \int f_k \, d\mu - \int f_k \, d\nu \right|, \tag{1}$$

where $\{f_k\}_{k\in\mathbb{N}}$ is a dense subset in the unit ball in $C^0(M)$. While the distance defined in this way depends on the choice of the subset $\{f_k\}_{k\in\mathbb{N}}$, the topology it generates does not. We will choose the functions $f_k$ to be smooth.

*Proof of Theorem 1.1.* By a periodic hyperbolic measure $\mu_q$ we mean an atomic ergodic measure equidistributed on a hyperbolic periodic orbit of $q$.

**Lemma 3.1.** *Let $q_1, q_2 \in \mathcal{H}(p)$ be hyperbolic periodic points with the same index as $p$. Then the periodic hyperbolic measures $\mu_{q_1}$ and $\mu_{q_2}$ can be connected in $\mathcal{M}_p$ by a continuous path.*

*Proof of Lemma 3.1.* By Condition (H1) the points $q_1$ and $q_2$ are homoclinically related. By the Smale-Birkhoff theorem, there is a locally maximal invariant hyperbolic horseshoe $\Lambda$ that contains both $q_1$ and $q_2$. Lemma 3.1 now follows from the results by Sigmund, see [27]. □

We wish to approximate a given hyperbolic measure by periodic measures. There are several results in this direction, see, for example, [2, Theorem 15.4.7]. However, we need some specific properties of such approximations that are stated in the following lemma.

**Lemma 3.2.** *For any hyperbolic ergodic measure $\mu$ and for any $\varepsilon > 0$ the following statements hold:*

(1) *There exists a periodic hyperbolic measure $\mu_q$ such that $d_{\mathcal{M}}(\mu_q, \mu) < \varepsilon$;*
(2) *There exists $\delta > 0$ such that for any periodic hyperbolic measures $\mu_{q_1}$ and $\mu_{q_2}$ in the $\delta$-neighborhood of the measure $\mu$ there exists a continuous path $\{\nu_t\}_{t\in[0,1]} \subset \mathcal{M}_p$ with $\nu_0 = \mu_{q_1}$, $\nu_1 = \mu_{q_2}$ and such that $d_{\mathcal{M}}(\nu_t, \mu) < \varepsilon$ for all $t \in [0, 1]$.*

*Proof of Lemma 3.2.* Let $\mathcal{R}$ be the set of all Lyapunov-Perron regular points, and for each $\ell \geq 1$ let $\mathcal{R}_\ell$ be the regular set. There exists $\ell \in \mathbb{N}$ such that $\mu(\mathcal{R}_\ell) > 0$. Fix $\varepsilon > 0$. For a $\mu$-generic point $x \in \mathcal{R}_\ell$, by Birkhoff's Ergodic Theorem, there exists $N \in \mathbb{N}$ such that for any $n > N$

$$d_{\mathcal{M}}\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu\right) < \frac{\varepsilon}{2}. \tag{2}$$

Choose $L \in \mathbb{N}$ such that $\sum_{k=L+1}^{\infty}\frac{1}{2^k} < \frac{\varepsilon}{4}$. For the dense collection of smooth functions $\{f_k\}$ from the definition (1) of the distance $d_{\mathcal{M}}$, denote by $C = C(\varepsilon)$ the common Lipschitz constant of the functions $\{f_1, \dots, f_L\}$. Let us now choose $\delta > 0$ such that $C\delta < \frac{\varepsilon}{4}$. By [2, Lemma 15.1.2], there exists $n > N$ and a periodic point $y \in M$ of period $n$ such that $\mathrm{dist}_M(f^k(x), f^k(y)) < \delta$ for all $k = 0, \dots, n-1$. We have

$$\begin{aligned}
&d_{\mathcal{M}}\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(y)}\right)\\
&\leq \sum_{k=1}^{L}\frac{1}{2^k}\left|\int f_k d\left(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}\right) - \int f_k d\left(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(y)}\right)\right| + \sum_{k=L+1}^{\infty}\frac{1}{2^k}\\
&\leq C\delta + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.
\end{aligned} \tag{3}$$

The first statement of the lemma now follows from (2) and (3). □

We now compete the proof of Theorem 1.1. Let $\eta$ and $\widetilde{\eta} \in \mathcal{M}_p$ be two hyperbolic ergodic measures. By Lemma 3.2, there are sequences of periodic hyperbolic measures $\{\mu_{q_k}\}_{k\in\mathbb{N}}$ and $\{\mu_{\widetilde{q}_k}\}_{k\in\mathbb{N}}$ such that $\mu_{q_k} \to \eta$, $\mu_{\widetilde{q}_k} \to \widetilde{\eta}$. By Condition (H2), the homoclinic class $\mathcal{H}(p)$ is isolated, and hence, the points $q_k$ and $\widetilde{q}_k$ belong to $\mathcal{H}(p)$ for large $k$. By Lemma 3.1, there is a path $\{\nu_t\}_{t\in\left[\frac{1}{3},\frac{2}{3}\right]}$ in $\mathcal{M}_p$ that connects $\mu_{q_1}$ and $\mu_{\widetilde{q}_1}$, that is, $\nu_{\frac{1}{3}} = \mu_{q_1}$ and $\nu_{\frac{2}{3}} = \mu_{\widetilde{q}_1}$. By Lemma 3.2, for any $k \in \mathbb{N}$ there are paths

$$\{\nu_t\}_{t\in\left[\frac{1}{3^{k+1}},\frac{1}{3^k}\right]} \text{ and } \{\nu_t\}_{t\in\left[1-\frac{1}{3^k},1-\frac{1}{3^{k+1}}\right]}$$

in $\mathcal{M}_p$ that connect measures $\mu_{q_k}$, $\mu_{q_{k+1}}$ and measures $\mu_{\widetilde{q}_k}$, $\mu_{\widetilde{q}_{k+1}}$, respectively. Moreover, by Lemma 3.2, the path $\{\nu_t\}_{t\in[0,1]}$ given by the above choices and such that $\nu_0 = \eta$, $\nu_1 = \widetilde{\eta}$ is continuous. □

*Proof of Theorem 1.4.* Given a (not necessarily ergodic) measure $\mu \in \mathcal{M}$, by the ergodic decomposition, there exists a measure $\nu$ on the space $\mathcal{M}_p^e$ such that

$$\mu = \int t\, d\nu(t) \quad\text{and}\quad h_\mu = \int h_t d\nu(t).$$

It follows that for any $\varepsilon > 0$ there are measures $t_1, \dots, t_N \in \mathcal{M}_p^e$ and positive coefficients $\alpha_1, \dots, \alpha_N$ such that

$$d_{\mathcal{M}_p}\left(\mu, \sum_{k=1}^{N} \alpha_k t_k\right) < \varepsilon \quad\text{and}\quad \left| h_\mu - \sum_{k=1}^{N} \alpha_k h_{t_k} \right| < \varepsilon. \tag{4}$$

Given a hyperbolic ergodic measure $t$ with $h_t > 0$, there exist a sequence of invariant horseshoes $\Lambda_n$ and a sequence of ergodic measures $\{\nu_n\}_{n\in\mathbb{N}}$ supported on $\Lambda_n$ such that $\nu_n \to t$ and $h_{\nu_n} \to h_t$ as $n \to \infty$, see, for example, Corollary 15.6.2 in [2]. One can ensure in the construction of these horseshoes that $\Lambda_n \subseteq \mathcal{H}(p)$ and that $\nu_n$ are the measures of maximal entropy and hence, Markov measures, see also [25].

In the case when $h_t = 0$ the measure $t$ can be approximated by a periodic hyperbolic measure supported on an orbit of a hyperbolic periodic point $q$. By Condition (H2), the homoclinic class $\mathcal{H}(p)$ is isolated and therefore, $q \in \mathcal{H}(p)$. The first return map to a small neighborhood of $q$ contains a horseshoe of topological dimension 2 that belongs to $\mathcal{H}(p)$. If the neighborhood is taken sufficiently small, the topological entropy of the map restricted to the whole horseshoe will become arbitrarily small. It follows that any Markov measure on this horseshoe must have small entropy and be close to the periodic measure supported on the orbit of the periodic point $q$ (notice that the support of this Markov measure does not have to be close to the orbit of $q$).

It follows from what was said above that for each ergodic measure $t_k$ we can associate a topologically transitive invariant horseshoe $\Lambda_k$ and a Markov measure $\nu_k$ supported on $\Lambda_k$ such that for every $k = 1, \dots, N$ we have

$$d_{\mathcal{M}_p}(t_k, \nu_k) < \frac{\varepsilon}{N} \quad\text{and}\quad |h_{t_k} - h_{\nu_k}| < \frac{\varepsilon}{N}. \tag{5}$$

Notice that all horseshoes $\Lambda_k$ have the same index $s(p)$ and that they are homoclinically related. This implies that there exists a topologically transitive horseshoe $\Lambda \subset \mathcal{H}(p)$ that contains all $\Lambda_k$.

The Markov measure $\nu_k$ is constructed with respect to a Markov partition of $\Lambda_k$ that we denote by $\xi_k$. There exists a Markov partition $\xi$ of $\Lambda$ such that its restriction on each $\Lambda_k$ is a refinement of $\xi_k$. The measure $\sum_{k=1}^{N} \alpha_k \nu_k$ is a Markov measure on $\Lambda$ with respect to the partition $\xi$. Notice that Markov measures as well as their entropies depend continuously on their stochastic matrices. Therefore, given an arbitrarily (not necessarily ergodic) Markov measure, one can produce its small perturbation which is an ergodic Markov measure whose entropy is close to the entropy of the unperturbed one. This gives the required approximation of the measure $\sum_{k=1}^{N}\alpha_k\nu_k$, which by (4) and (5) is close to the initial measure $\mu$. $\square$

*Proof of Theorem 1.2.* Notice that $\mathcal{M}_p$ is a convex set. Indeed, if $\mu_1,\mu_2\in\mathcal{M}_p$, then for any $t\in[0,1]$ the measure $t\mu_1+(1-t)\mu_2$ is hyperbolic with the same number of positive Lyapunov exponents and hence, it belongs to $\mathcal{M}_p$. It follows from Theorem 1.4 that

$$\mathcal{M}_p^e\subseteq\mathcal{M}_p\subseteq\overline{\mathcal{M}_p^e}.$$

This means that $\overline{\mathcal{M}_p}=\overline{\mathcal{M}_p^e}$, and since $\overline{\mathcal{M}_p}$ is a simplex, it must be a Poulson simplex. The desired result follows now from [23] (see also [14]). $\square$

## References


[1] A. Baraviera, C. Bonatti, Removing zero Lyapunov exponents, *Ergodic Theory Dyn. Syst.*, **23:6** (2003), 16551670.

[2] L. Barreira, Ya. Pesin, Nonuniform hyperbolicity: Dynamics of systems with nonzero Lyapunov exponents. *Encyclopedia of Mathematics and its Applications*, 115. Cambridge University Press, Cambridge, 2007. xiv+513 pp.

[3] J. Bochi, Ch. Bonatti, L. Diaz, Robust criterion for the existence of nonhyperbolic ergodic measures, preprint (arXiv:1502.06535).

[4] Ch. Bonatti, L. Diaz, A. Gorodetski, Non-hyperbolic ergodic measures with large support, *Nonlinearity* **23** (2010), 687–705.

[5] Ch. Bonatti, L. Diaz, R. Ures, Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms, *J. Inst. Math.* , **1** (2002), 513–541.

[6] Ch. Bonatti, M. Viana, Dynamics beyond uniform hyperbolicity, *Encyclopaedia of Mathematical Sciences (Mathematical Physics)*, **102**, Springer Verlag, Berlin, 2005, xviii+384 pp.

[7] Ch. Bonatti, K. Gelfert, Dominated Pesin theory: convex sum of hyperbolic measures, preprint (arXiv:1503.05901).

[8] K. Burns, D. Dolgopyat, Ya. Pesin, M. Pollicott, Stable ergodicity of partially hyperbolic attractors with nonzero exponents, *Journal of Modern Dynamics*, **2:1** (2008) 1–19.

[9] L. Diaz, K. Gelfert, Porcupine-like horseshoes: Transitivity, Lyapunov spectrum, and phase transitions, *Fund. Math.* **216** (2012), 55–100.

[10] L. Diaz, K. Gelfert, M. Rams, Abundant rich phase transitions in step-skew products, *Nonlinearity* **27** (2014), 2255–2280.

[11] L. Diaz, A. Gorodetski, Non-hyperbolic ergodic measures for non-hyperbolic homoclinic classes, *Ergodic Theory Dynam. Systems* **29** (2009), 1479–1513.

[12] L. Diaz, V. Horita, I. Rios, M. Sambarino, Destroying horseshoes via heterodimensional cycles: generating bifurcations inside homoclinic classes, *Ergodic Theory Dynam. Systems* **29** (2009), 433–474.

[13] P. Duarte, Plenty of elliptic islands for the standard family of area preserving maps, *Ann. Inst. H. Poincar Anal. Non Lineaire* **11** (1994), 359–409.

[14] K. Gelfert, D. Kwietniak, The (Poulsen) simplex of invariant measures, preprint (arXiv:1404.0456).

[15] A. Gogolev, A. Tahzibi, Center Lyapunov exponents in partially hyperbolic dynamics, preprint (arXiv:1310.1985).

[16] A. Gorodetski, On stochastic sea of the standard map, *Comm. Math. Phys.* **309** (2012), 155–192.

[17] A. Gorodetski, Yu. Ilyashenko, V. Kleptsyn, M. Nalsky, Nonremovability of zero Lyapunov exponents. (Russian) *Funktsional. Anal. i Prilozhen.* **39** (2005), no. 1, 27–38; translation in *Funct. Anal. Appl.* **39** (2005), no. 1, 21–30.

[18] H. Föllmer, S. Orey, Large deviations for the empirical field of a Gibbs measure, *Ann. Probab.* **16** (1988), 961–977.

[19] M. Hirsch, C. M. Shub, Invariant Manifolds, *Springer Lecture Notes on Mathematics*, **583**, Springer-Verlag, Berlin-New York, 1977.

[20] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.* **51** (1980), 137–173.

[21] A. Katok, B. Hasselblatt, Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza, *Encyclopedia of Mathematics and its Applications* **54**, Cambridge University Press, Cambridge, 1995, xviii+802 pp.

[22] V. Kleptsyn, M. Nalskii, Stability of the existence of nonhyperbolic measures for $C^1$-diffeomorphisms. (Russian) Funktsional. Anal. i Prilozhen. 41 (2007), no. 4, 30–45, 96; translation in Funct. Anal. Appl. 41 (2007), no. 4, 271–283.

[23] J. Lindenstrauss, G. Olsen, Y. Sternfeld, The Poulsen simplex, *Annales de l'Institut Fourier*, **28** (1978), 91–114.

[24] Ya. Pesin, Ya. Sinai, Gibbs measures for partially hyperbolic attractors, *Ergodic Theory and Dyn. Syst.*, **2: 3-4** (1982) 417–438.

[25] F. Sanchez-Salas, Variational principles and approximation of dynamical indicators for systems with nonuniformly hyperbolic behavior, preprint (arXiv:1303.5010).

[26] M. Shub, A. Wilkinson, Pathological foliations and removable zero exponents, *Invent. Math.* **139** (2000), no. 3, 495–508.

[27] K. Sigmund, On the connectedness of ergodic systems, *Manuscripta Math.* **22** (1977), 27–32.

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