Submodular Inequalities for the Path Structures of the Capacitated Fixed-Charge Network Flow Problems

By

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A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Engineering – Industrial Engineering and Operations Research in the Graduate Division of the University of California, Berkeley

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Abstract

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Capacitated fixed-charge network flow problems (CFCNF) are used to model a variety of problems in telecommunication, facility location, production planning and supply chain management. We model CFCNF as a linear mixed-integer program and study the polyhedral structure of various path networks.

We investigate capacitated path substructures and derive strong and easy-to-compute path cover and path pack inequalities. These inequalities are based on an explicit characterization of the submodular inequalities through a fast computation of parametric minimum cuts on a path, and they generalize the well-known flow cover and flow pack inequalities for the single-node relaxations of fixed-charge flow models. Computational results demonstrate the effectiveness of the inequalities when used as cuts in a branch-and-cut algorithm.

Moreover, we consider single item lot-sizing problems with backlogging and inventory bounds and fixed costs (LSBIB). Using the underlying fixed-charge network structure of LSBIB, we derive explicit path pack inequalities that are proposed in this thesis. These inequalities are generalizations of the valid inequalities proposed by Atamtürk and Küçükyavuz (2005) for lot-sizing problems under the existence of backlogging. Furthermore, we propose extensions of these inequalities where the binary variables for inventory and backlogging are lifted. Finally, we present computational results suggest that show the effectiveness of both path pack and extended path pack inequalities when used in a branch-and-cut algorithm.
This thesis is dedicated to my parents who loved and supported me through it all.
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Chapter 1

Introduction

Given a graph with demand/supply nodes and arcs with upper bounds on flow, the capacitated fixed-charge network flow problem (CFCNF) aims to find a subset of arcs such that the flow balance on each node is conserved and flow capacities are not exceeded. In this thesis, we model CFCNF as a mixed-integer program. We propose valid inequalities and analyze their strength both theoretically and computationally.

The research in this thesis has roots in the seminal paper by Wolsey (1989). In this paper, the author proposes submodular inequalities (see Section 1.4) that are valid for any CFCNF. While submodular inequalities are quite general and make no assumptions on the network structure, computing their coefficients require solving many maximum flow problems. Due to their implicit coefficients, using them in a branch-and-cut algorithm requires a significant amount of computational effort. As a result, the strength of these inequalities have not been analyzed for general network structures.

In this thesis, we focus on path networks and propose a linear-time algorithm to express submodular inequalities explicitly. All of the chapters are concluded with an extensive computational study. In Chapter 2, we study on simple paths and propose path pack inequalities. Then, we generalize path pack inequalities by a simultaneous lifting procedure. In Chapter 3, we generalize the simple path by adding arcs \((j+1,j)\) to the consecutive nodes in the path. In addition to the path cover inequalities for these path structures, we propose a second class of submodular inequalities and refer to them as path pack inequalities. We show that path cover and path pack inequalities reduce to flow cover and flow pack inequalities when the underlying network consists of a single node. In Chapter 4, we extend the research of Atamtürk and Küçükyavuz (2005) on lot-sizing problems with inventory bounds by considering backlogging arcs. Using the path structure of this problem, we propose eleven classes of path pack inequalities parametrically. Then, we extend these inequalities by incorporating the binary fixed-charge variables associated with inventory and backlogging arcs. The computational results prove that path cover and path pack inequalities and their extensions are quite useful while solving fixed-charge network flow problems.
In the remainder of this chapter, we give a brief background information on polyhedral analysis using the definitions from Wolsey and Nemhauser (1999) and Wolsey (1998). Following the basics in Section 1.1, we introduce a mixed-integer programming formulation for the capacitated fixed-charge networks in Section 1.2. In Sections 1.3 and 1.4, we introduce the existing flow cover, flow pack and submodular inequalities which are referred to often in this thesis. In Section 1.5, we go over the lifting process of valid inequalities.

1.1 Polyhedral analysis terminology

We first start by describing mixed-integer programming (MIP) as

$$\max\{cx + hy : Ax + Gy \leq b, \ x \in \mathbb{Z}^n, \ y \in \mathbb{R}^p\}$$

where $\mathbb{Z}^n$ is the set of integer $n$-dimensional vectors and $\mathbb{R}^p$ is the set of real $p$-dimensional vectors. In a MIP $A, G$ and $b$ are the parameter set and $x, y$ are the decision variables. The feasible set of MIP is defined by

$$S = \{x \in \mathbb{Z}^n, \ y \in \mathbb{R}^p : Ax + Gy \leq b\}$$

and we call the set

$$S_{LP} = \{x \in \mathbb{R}^n, \ y \in \mathbb{R}^p : Ax + Gy \leq b\}$$

the linear programming (LP) relaxation of the feasible set $S$.

Following definitions are directly cited from Wolsey and Nemhauser (1999).

**Definition** Convex hull of $S$ is the set of all points that are convex combinations of points in $S$.

**Definition** A polyhedron $P \subseteq \mathbb{R}^n$ is the set of points that satisfy a number of linear inequalities; that is, $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $(A, b)$ is an $m \times (n + 1)$ matrix.

**Definition** $x \in P$ is an extreme point of $P$ if there do not exist $x^1, x^2 \in P$, $x^1 \neq x^2$ such that $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.

**Definition** A set of points $x^1, \ldots, x^k \in \mathbb{R}^n$ is affinely independent if the unique solution of $\sum_{i=1}^{k} \alpha_i x^i = 0$, $\sum_{i=1}^{k} \alpha_i = 0$ is $\alpha_i = 0$ for $i = 1, \ldots, k$.

**Definition** A polyhedron $P$ is of dimension $k$ if the maximum number of affinely independent points in $P$ is $k + 1$ and it is represented by $\dim(P) = k$.

**Definition** A polyhedron $P \subseteq \mathbb{R}^n$ is full-dimensional if $\dim(P) = n$. 
Definition The inequality \( \pi x \leq \pi_0 \) (or \( (\pi, \pi_0) \)) is called a valid inequality for \( P \) if it is satisfied by all points in \( P \).

Definition If \( (\pi, \pi_0) \) is a valid inequality for \( P \), and \( F = \{ x \in P : \pi x = \pi_0 \} \), \( F \) is called a face of \( P \).

Definition A face \( F \) of the polyhedron \( P \) is called a facet of \( P \) if \( \dim(F) = \dim(P) - 1 \). The inequality describing \( F \) is then a facet-defining inequality or a facet.

1.2 Capacitated fixed-charge network flow problems

Let \( G = (N, A) \) be a directed graph with nodes \( N \) and arcs \( A \) and \( |N| = n, |A| = a \). Let \( E^+_j = \{(i, j) \in A : i \in N\} \) and \( E^-_j = \{(j, i) \in A : i \in N\} \) for all \( j \in N \). Let the demand value at node \( j \in N \) be \( d_j \) (where \( d_j < 0 \) indicates \( j \) is a supply node). Define the binary variable \( x_t \) to be 1 if arc \( t \in A \) is used and the variable \( y_t \) to be the flow through arc \( t \), which has a capacity of \( c_t \).

The mathematical programming representation of capacitated fixed-charge network problems can be represented by the following feasibility set:

\[
\begin{align*}
\sum_{t \in E^+_j} y_t - \sum_{t \in E^-_j} y_t &\leq d_j, \quad j \in N, \\
0 &\leq y_t \leq c_t x_t, \quad t \in A, \\
x_t &\in \{0, 1\}, \quad t \in A.
\end{align*}
\]

We represent the feasible set of the capacitated fixed charge network flow problem on underlying network \( G \) by \( \mathcal{P}_G \).

1.3 Flow cover and flow pack inequalities

In this section, we introduce the well known flow cover inequalities of Padberg et al. (1985) and flow pack inequalities of Atamtürk (2001). Consider a single-node capacitated fixed charge network flow problem with the feasible set

\[
\mathcal{P} = \{ y \in \mathbb{R}^n_+, x \in \mathbb{R}^n : \sum_{t \in E^+_S} y_t - \sum_{t \in E^-_S} y_t \leq d, \ y_t \leq c_t x_t \text{ for } t \in E^+ \cup E^- \}.
\]

See Figure (1.1) for a graph representation of \( \mathcal{P} \). Let \( S^+ \subseteq E^+ \), \( S^- \subseteq E^- \) and \( L^- \subseteq E^- \setminus S^- \). The set pair \( (S^+, S^-) \) is called a flow cover if \( \sum_{t \in S^+} c_t - \sum_{t \in S^-} c_t = d + \lambda \) and \( \lambda > 0 \). Similarly, the pair \( (S^+, S^-) \) is called a flow pack if \( \sum_{t \in S^+} c_t - \sum_{t \in S^-} c_t = d - \mu \) and \( \mu > 0 \).
Figure 1.1: A single node network

Let \((S^+, S^-)\) be a flow cover, then the inequality

\[
\sum_{t \in S^+} (y_t + (c_t - \lambda)^+ (1 - x_t)) \leq d + \sum_{t \in S^-} c_t + \sum_{t \in L^-} \lambda x_t + \sum_{t \in E^- \setminus (L^- \cup S^-)} y_t \tag{1.1}
\]

is valid for \(\mathcal{P}\). Inequality (1.1) is referred to as a generalized flow cover inequality. van Roy and Wolsey (1986) show that if \(d > 0\), \(\max_{t \in S^+} c_t > \lambda\) and \(c_t > \lambda\) for \(t \in L^-\) and \(S^- = \emptyset\), then inequality (1.1) is a facet of the convex hull of \(\mathcal{P}\).

Let \((S^+, S^-)\) be a flow pack and let \(L^+ \subseteq E^+ \setminus S^+\), then the inequality

\[
\sum_{t \in S^+} y_t + \sum_{t \in L^+} (y_t - \min \{c_t, \mu\} x_t) \leq c_t + \sum_{t \in S^-} (c_t - \mu)(1 - x_t) + \sum_{t \in E^- \setminus S^-} y_t \tag{1.2}
\]

is valid for \(\mathcal{P}\) and is referred to as a flow pack inequality.

### 1.4 Submodular inequalities

**Definition** A set function \(v\) on a set \(S = \{1, \ldots, s\}\) is submodular if

\[
v(A) + v(B) \geq v(A \cup B) + v(A \cap B)
\]

for all \(A, B \subseteq S\).

**Proposition 1.1.** Let \(\rho_j(A) = v(A \cup \{j\}) - v(A)\). The function \(v\) is submodular if and only if

\[
\rho_j(A) \geq \rho_j(B)
\]

for all \(A \subseteq B \subseteq S \setminus \{j\}\).

Next, we summarize some of the main results in Wolsey (1989). Let \(G(V, A)\) be a capacitated fixed-charge network and let \(N \subseteq V\) be a subset of nodes and define \(E^+ = \{(i, j) \in A : i \notin N, j \in N\}\), \(E^- = \{(i, j) \in A : i \in N, j \notin N\}\) and let \(E := E^+ \cup E^-\).

Let \(S^+ \subseteq E^+\) and \(L^- \subseteq E^-\). Define the set function \(v(S^+, L^-)\) (or \(v(C)\) in short
notation) by the following optimization problem:

\[ v(S^+, L^-) = \max \sum_{t \in E} a_t y_t \]

s.t. \[ \sum_{t \in E_j^+} y_t - \sum_{t \in E_j^-} y_t \leq d_j, \quad j \in N, \]

\[ 0 \leq y_t \leq c_t, \quad t \in E, \]

\[ y_t = 0, \quad t \in (S^+ \setminus S^+) \cup L^-, \]

where \( a_t \in \{0, 1\} \) for \( t \in E^+ \) and \( a_t \in \{0, -1\} \) for \( t \in E^- \). Let \( \rho_t(S^+, L^-) \) be

\[ \rho_t(S^+ \setminus \{t\}, L^-) = v(S^+, L^-) - v(S^+ \setminus \{t\}, L^-) \]

for \( t \in S^+ \) and

\[ \rho_t(S^+, L^- \setminus \{t\}) = v(S^+, L^-) - v(S^+, L^- \setminus \{t\}) \]

for \( t \in L^- \). For convenience, we represent the sets \( (S^+, L^-) \) as a single let:

\[ C := S^+ \cup L^- \]

and

\[ \rho_t(C) := v(C \cup \{t\}) - v(C). \]

Wolsey (1989) proves that \( v(S^+, L^-) \) is submodular on the set \( E \) and shows that the inequalities

\[
\sum_{t \in E} a_t y_t \leq v(C) - \sum_{t \in S^+} \rho_t(E \setminus \{t\})(1 - x_t) - \sum_{t \in L^-} \rho_t(E \setminus \{t\})x_t + \sum_{t \in E^+ \setminus S^+} \rho_t(C)x_t + \sum_{t \in E^- \setminus L^-} \rho_t(C)(1 - x_t)
\]

and

\[
\sum_{t \in E} a_t y_t \leq v(C) - \sum_{t \in S^+} \rho_t(C \setminus \{t\})(1 - x_t) - \sum_{t \in L^-} \rho_t(C \setminus \{t\})x_t + \sum_{t \in N^+ \setminus S^+} \rho_t(\emptyset)x_t + \sum_{t \in E^- \setminus L^-} \rho_t(\emptyset)(1 - x_t)
\]

are valid for the capacitated fixed-charge network feasible set \( \mathcal{P}_G \).
1.5 Lifting valid inequalities

Given a valid inequality \((\pi, \pi_0)\) for polyhedron \(P\), lifting is the procedure of extending this inequality to higher dimensions. The notion of lifting was first introduced by Gomory (1969) and then generalized and formally described in Wolsey (1976), Zemel (1978) and Balas and Zemel (1978). Lifting proved to be very useful in solving integer programs more efficiently; however, in most cases, the coefficients of the lifted valid inequalities depend on the sequence of the variables being lifted. Obtaining the lifting coefficients require solving a separate optimization problem. As a result, sequence dependence creates a computational burden for the lifting procedure. For binary integer programs, Wolsey (1977) showed that if the lifting function is super-additive then the coefficients are independent from the sequence of lifting. This result was generalized for mixed binary integer programs by Gu (1994) and Gu et al. (2000), and for general mixed integer programs by Atamtürk (2004).

Gu et al. (1999) provide the analytical form of the lifting function of flow cover inequalities, introduce a valid super additive lifting function and present computational results to show the effectiveness of lifting. In the next section, we introduce a procedure to obtain a valid super additive lifting function for submodular path inequalities. Next, we describe the general lifting procedure for mixed binary integer programs.

Let \(X = \{x \in \mathbb{B}^p, y \in \mathbb{R}^q : Ax + Gy \leq d\; l_j \leq y_j \leq u_j, \; j \in J\}\) be the feasible set of a mixed binary integer program where \(\mathbb{B} = \{0, 1\}\) and index sets \(I\) and \(J\) contain binary and continuous variables respectively. Size of \(d\) is \(m \times 1\), \(A = \{a_{ij}\}_{i=1,\ldots,m, \; j=1,\ldots,p}\) and \(G = \{g_{ij}\}_{i=1,\ldots,m, \; j=1,\ldots,q}\). Let \(C^i, \; i = 0, \ldots, t\) be a partition of the variables \(I \cup J\). We define a restriction of \(X\) by setting a subset of the variables to one of their bound values \(b_j\). Note that, \(b_j \in \{0, 1\}\) for \(j \in I\) and \(b_j \in \{l_j, u_j\}\) for \(j \in J\). Let the restricted feasible set associated with partition \(j\) be represented as \(X^j\).

\[
X^0 = \{(x, y) \in X : x_j = b_j, \; j \in I \setminus C^0, \; y_j = b_j, \; j \in J \setminus C^0\}.
\]

Let

\[
\sum_{j \in I \setminus C^0} \alpha_j x_j + \sum_{j \in J \setminus C^0} \beta_j y_j \leq \pi \tag{1.3}
\]

be a valid inequality for \(X^0\). We would like to extend this valid inequality by adding the variables in \((I \cup J) \setminus C^0\) and obtain an inequality of the form

\[
\sum_{j \in I \setminus C^0} \alpha_j x_j + \sum_{j \in J \setminus C^0} \beta_j y_j + \sum_{j \in I \setminus C^0} \alpha_j (x_j - b_j^x) + \sum_{j \in J \setminus C^0} \beta_j (y_j - b_j^y) \leq \pi.
\]

that is valid for \(X\).

In order to reach this inequality, lifting is carried out by steps. At each step, a par-
tion of variables $C^i$ are lifted simultaneously. Unfortunately, the resulting coefficients are usually dependent on the sequence in which these partitions are lifted. Without loss of generality, we assume variables in $C^i$ are lifted in $i$th order. Then the valid inequality obtained after $i$th lifting sequence is valid for the restricted feasible set

$$X^i = \{(x, y) \in X : x_j = b_j^x, \ y_j = b_j^y, \ j \in J \setminus (\cup_{k=0}^{i-1}C^k)\}.$$ 

Given the valid inequality for set $X^{i-1}$:

$$\sum_{j \in I \cap C^0} \alpha_j x_j + \sum_{j \in J \cap C^0} \beta_j y_j + \sum_{j \in I \setminus (\cup_{k=1}^{i-1}C^k)} \alpha_j (x_j - b_j^x) + \sum_{j \in J \setminus (\cup_{k=1}^{i-1}C^k)} \beta_j (y_j - b_j^y) \leq \pi$$

the lifting problem for set $C^i$ aims to find coefficients $(\alpha_j, \beta_j), \ j \in C^i$ such that

$$\sum_{j \in I \cap C^0} \alpha_j x_j + \sum_{j \in J \cap C^0} \beta_j y_j + \sum_{j \in I \setminus (\cup_{k=1}^{i-1}C^k)} \alpha_j (x_j - b_j^x) + \sum_{j \in J \setminus (\cup_{k=1}^{i-1}C^k)} \beta_j (y_j - b_j^y) \leq \pi \ (1.4)$$

is valid for $X^i$. We also define the $i$th lifting function associated with inequality as:

$$f_i(z) = \min \ \pi - \left[ \sum_{j \in I \cap C^0} \alpha_j x_j + \sum_{j \in J \cap C^0} \beta_j y_j \right.\left. + \sum_{j \in I \setminus (\cup_{k=1}^{i-1}C^k)} \alpha_j (x_j - b_j^x) + \sum_{j \in J \setminus (\cup_{k=1}^{i-1}C^k)} \beta_j (y_j - b_j^y) \right]$$

s.t. $\sum_{j \in I \setminus (\cup_{k=1}^{i-1}C^k)} a_{lj} (x_j - b_j^x) + \sum_{j \in J \setminus (\cup_{k=1}^{i-1}C^k)} g_{lj} (y_j - b_j^y) \leq d_l^i - z_l, \ l = 1, \ldots, m, $

where $d_l^i = d_l - \sum_{j \in I \cap C^0} a_{lj} b_j^x - \sum_{j \in J \cap C^0} g_{lj} b_j^y$ and $z \in Z^i$ with

$$Z^i = \left\{ z \in \mathbb{R}^m : \exists (x, y) \in X^i : \sum_{j \in I \cap C^0} a_{lj} (x_j - b_j) + \sum_{j \in J \cap C^0} g_{lj} (y_j - b_j) = z_l \ l = 1, \ldots, m \right.$$ 

$$\sum_{j \in I \setminus (\cup_{k=0}^{i-1}C^k)} a_{lj} (x_j - b_j^x) + \sum_{j \in J \setminus (\cup_{k=0}^{i-1}C^k)} g_{lj} (y_j - b_j^y) \leq d_l^i - z_l \ l = 1, \ldots, m \right\}.$$
Let $Z = Z^1 \times \ldots \times Z^t$.

**Proposition 1.2** (Gu et al. (1999)). Inequality (1.4) is valid for $X^i$ for a choice of $(\alpha_j, \beta_j), \ j \in C^i$ if and only if $h_i(z) \leq f_i(z)$ for any $z \in Z^i$ where

$$h_i(z) = \max \sum_{j \in I \cap C^i} \alpha_j (x_j - b_j) + \sum_{j \in J \cap C^i} \beta_j (y_j - b_j)$$

s.t. $\sum_{j \in I \cap C^i} a_{lj} (x_j - b_j) + \sum_{j \in J \cap C^i} g_{lj} (y_j - b_j) = z_l \ l = 1, \ldots, m$

$x_j \in \{0, 1\} \ j \in I \cap C^i$

$l_j \leq y_j \leq u_j \ j \in J \cap C^i$.

**Definition** Lifting function $f(z)$ of the valid inequality (1.3) is defined as $f(z) = f_1(z)$ for all $z \in Z$.

**Definition** Lifting of valid inequality (1.3) is sequence independent if $f(z) = f_i(z)$ for $i = 2, \ldots, t$.

**Definition** A function $f$ is super-additive on $Z$ if

$$f(z_1) + f(z_2) \leq f(z_1 + z_2)$$

for all $z_1, z_2, z_1 + z_2 \in Z$.

It is shown in the literature that, if $f(z)$ is super additive on $Z$, then lifting is sequence independent. Unfortunately, super-additivity of $f(z)$ is very uncommon.

**Definition** Function $g$ is called valid super additive lifting function if it is super additive and $g(z) \leq f(z)$ for all $z \in Z$.

**Proposition 1.3.** If $g$ is a valid super additive lifting function and $(\alpha_j, \beta_j)$ for $j \in C^i$ ensure that $h_i(z) \leq g(z)$ for all $z \in Z$ and $i = 1, \ldots, t$, then inequality (1.4) is valid for $X$. 
Chapter 2

Submodular Path Inequalities for the Capacitated Fixed-Charge Network Flow Problem

Given a directed graph with demand and supply on the nodes, and capacity, fixed and variable cost of flow on the arcs, the capacitated fixed-charge network flow (CFNF) problem is to choose a subset of the arcs and route the flow on the chosen arcs while satisfying the supply, demand and capacity constraints, so that the sum of fixed and variable costs is minimized. There are numerous polyhedral studies on the fixed-charge network flow problem. However, few give explicit valid inequalities that simultaneously make use of the path substructures of the network as well as the arc capacities, which is the goal of the current chapter.

For the uncapacitated fixed-charge network flow problem, van Roy and Wolsey (1985) give flow path inequalities that are based on path substructures. Rardin and Wolsey (1993) introduce a new family of dicut inequalities and show that they describe the projection of extended multicommodity formulation onto the original variables of fixed-charge network flow problem. Ortega and Wolsey (2003) present a computational study on the performance of path and cut-set (dicut) inequalities. For the capacitated fixed-charge network flow problem, almost all known valid inequalities are based on single-node relaxations. Padberg et al. (1985), van Roy and Wolsey (1986) and Gu et al. (1999) give flow cover, generalized flow cover and lifted flow cover inequalities. Stallaert (1997) introduces a complementary class of flow cover inequalities; Atamtürk (2001) describes lifted flow pack inequalities. Both uncapacitated path inequalities and capacitated flow cover inequalities are highly valuable in solving a host of practical problems and are part of the suite of cutting planes implemented in modern mixed-integer programming solvers.

The path structure arises naturally in network models of the lot-sizing problem. Atamtürk and Muñoz (2004) introduce valid inequalities for the capacitated lot-sizing
problems with infinite inventory capacities. Atamtürk and Küçükayvuz (2005) give
valid inequalities for the lot-sizing problems with finite inventory and infinite pro-
duction capacities. Van Vyve (2013) introduces valid inequalities for uncapacitated
fixed charge transportation problems. Van Vyve and Ortega (2004) and Gade and
Küçükayvuz (2011) give valid inequalities and extended formulations for uncapacitated
lot-sizing with fixed charges on stocks.

In this chapter we consider a generic path relaxation, with supply and/or demand
nodes and capacities on all incoming and outgoing arcs. First, by exploiting the path
substructure of the network, we provide an explicit description of the submodular
inequalities introduced by Wolsey (1998). An important consequence of the explicit
derivation is that the coefficients of these submodular path inequalities can be computed
efficiently. In particular, we show that all coefficients of an inequality can be computed
by solving max-flow/min-cut problems over the path in linear time. For a path with a
single node, the inequalities reduce to the flow cover inequalities introduced by Padberg
et al. (1985). We give sufficient and necessary facet-defining conditions. Then we
generalize the inequalities further by superadditive lifting using an approximate multi-
dimensional lifting function. Finally, we demonstrate the effectiveness of the proposed
inequalities when used as cuts in a branch-and-cut algorithm.

Outline

The remainder of this chapter is organized as follows: In Section 2.1, we describe the
CFNF problem on a path, its formulation and the assumptions we make. In Section 2.2,
we review the submodular inequalities of Wolsey (1998) and introduce the submodular
path inequalities. In Section 2.3, we generalize the submodular path inequalities by
superadditive lifting. In Section 2.4, we present some computational results depicting
the effectiveness of the valid inequalities proposed.

2.1 Capacitated fixed-charge network flow on a path

Let $G = (N', A)$ be a directed path with nodes $N'$ partitioned into sets $N$ and $\{s_N, t_N\}$,
where $s_N$ and $t_N$ are source and sink nodes and $V$ are the path nodes (see Figure 2.1).
Let $N = \{k, \ldots, \ell\}$ and the path arcs be $I := \{(k, k + 1), (k + 1, k + 2), \ldots, (\ell - 1, \ell)\}$. Define $a := |A|$ and $n := |N|$. Without loss of generality, we assume that node $k$ has
an incoming path arc $(k - 1, k)$ and node $\ell$ has an outgoing path arc $(\ell, \ell + 1)$ with
zero capacity (represented by dotted lines in Figure 2.1).

For each node $j \in N$, let $E_j^+ = \{(s_N, i) \in A : i = j\}$ and $E_j^- = \{(i, t_N) \in A : i = j\}$. Let the union of such sets be denoted by $E^+ := \{(s_N, i) \in A : i \in N\}$ and
$E^- := \{(i, t_N) \in A : i \in N\}$. Finally, let $E := E^+ \cup E^-$ with $e := |E|$.

In this chapter, we consider simple directed paths where sets $E_j^+$ and $E_j^-$ for all
$j \in N$ are mutually exclusive. In other words, the arcs in $I$ form the unique directed
path from node $k$ to node $\ell$. For any graph, if need be, one can construct such a relaxation by duplicating arcs $t \in E^+_i \cap E^-_j$ as $t^+ \in E^+_i$ and $t^- \in E^-_j$ for $i, j \in N$.

Let the demand at node $j \in N$ be $d_j$. We call a node $j \in N$ a demand node if $d_j > 0$ and a supply node if $d_j < 0$. Define the binary variable $x_t$ to be 1 if arc $t \in E$ is open, 0 otherwise; and the variable $y_t$ to be the flow through arc $t$, with capacity of $c_t$. We refer to the flow of path arc $(j, j + 1)$ as $i_j$, and denote the corresponding capacity by $u_j$. We let $[k, \ell] := \{t \in Z : k \leq t \leq \ell\}$ and let $d(T) = \sum_{j \in T} d_j$ for a given $T \subseteq V$. For notational convenience, we refer to $d([k, \ell])$ as $d_{k\ell}$. Let $c(S) = \sum_{t \in S} c_t$, $y(S) = \sum_{t \in S} y_t$, $x(S) = \sum_{t \in S} x_t$ and $(a)^+ = \max\{0, a\}$. Moreover, let us denote the convex hull of a formulation $P$ as $\text{conv}(P)$.

Let the path-set relaxation $(\mathcal{P}_G)$ of CFNF problem on $G$ be

\begin{align*}
  i_{j-1} + y(E^+_j) - y(E^-_j) - i_j &\leq d_j, & j &\in N, \quad (2.1) \\
  0 \leq i_j &\leq u_j, & j &\in N, \quad (2.2) \\
  0 \leq y_t &\leq c_t x_t, & t &\in E, \quad (2.3) \\
  x_t &\in \{0, 1\}, & t &\in E. \quad (2.4)
\end{align*}

In order to avoid trivial cases we make the following assumptions:

(A.1) $u_j > 0$ for $j \in I$ (otherwise, $i_t = 0$ and arc $t$ can be removed from the graph),

(A.2) $c_t > 0$ for $t \in E$ (otherwise $y_t = 0$ and arc $t$ can be removed from the graph and the problem decomposes into two subproblems defined on two simple paths),

(A.3) for all $t \in E$, $G' = (V, A \setminus \{t\})$ with corresponding feasible set $\mathcal{P}_{G'} \neq \emptyset$ (otherwise $x_t = 1$).

It follows from the assumptions that the dimension of $\mathcal{P}_G$ is $e + a$ (i.e., $\mathcal{P}_G$ is full dimensional).
2.2 Submodular path inequalities

In this section, we review the submodular inequalities of Wolsey (1989) and derive their explicit form for the path networks. Let \( S^+ \subset E^+ \) and \( L^- \subset E^- \). Sets \( S^+ \) and \( L^- \) can be partitioned into sets \( S_j^+ \) and \( L_j^- \) for \( j = k, \ldots, \ell \) where \( S_j^+ = S^+ \cap E_j^+ \) and \( L_j^- = L^- \cap E_j^- \).

Now, we consider the following formulation introduced by Wolsey (1989) for the path structure \( G \):

\[
\begin{align*}
 v(S^+, L^-) &= \max \sum_{t \in E} a_t y_t \quad \text{(2.5)} \\
 \text{s.t.} \quad &i_{j-1} + y(E_j^+) - y(E_j^-) - i_j \leq d_j, \quad j \in N, \quad \text{(2.6)} \\
 &0 \leq i_j \leq u_j, \quad j \in I, \quad \text{(2.7)} \\
 & (P') \quad 0 \leq y_t \leq c_t, \quad t \in E, \quad \text{(2.8)} \\
 & y_t = 0, \quad t \in (E^+ \setminus S^+) \cup L^-, \quad \text{(2.9)}
\end{align*}
\]

where \( a_t \in \{0, 1\} \) for \( t \in E^+ \) and \( a_t \in \{0, -1\} \) for \( t \in E^- \). Define the sets \( K^+ := \{t \in E^+: a_t = 1\} \) and \( K^- := \{t \in E^-: a_t = 0\} \). In this section, we choose \( K^+ = S^+ \) and \( K^- = L^- \). We make the following additional assumption:

\textbf{(A.4)} The sets \( S^+ \) and \( L^- \) are selected such that the formulation \( P' \) is feasible.

**Proposition 2.1** (Wolsey (1989)). The set function \( v : 2^{|E^+|} \times 2^{|E^-|} \rightarrow \mathbb{R} \) defined through (2.5)-(2.9) is submodular.

It follows from Corollary 8 of Wolsey (1989) that the submodular inequality

\[
\sum_{t \in S^+} (y_t + \rho_t(S^+ \setminus \{t\}, L^-)(1 - x_t)) + \sum_{t \in L^-} \rho_t(S^+, L^- \setminus \{t\}) x_t \leq v(S^+, L^-) + \sum_{t \in E^- \setminus L^-} y_t, \quad \text{(2.10)}
\]

where \( \rho_t(S^+ \setminus \{t\}, L^-) = v(S^+ \setminus \{t\}, L^-) - v(S^+, L^- \setminus \{t\}) \), \( t \in S^+ \)

and \( \rho_t(S^+, L^- \setminus \{t\}) = v(S^+, L^-) - v(S^+, L^- \setminus \{t\}) \), \( t \in L^- \)

is valid for \( P_G \). Observe that inequality (2.10) requires computing \( \rho_t(S^+ \setminus \{t\}, L^-) \) for \( t \in S^+ \) and \( \rho_t(S^+, L^- \setminus \{t\}) \) for \( t \in L^- \), which can be done by solving \(|S^+ \cup L^-| + 1 \) optimization problems. Next, we investigate the structure of \( P' \) in order to obtain the explicit form of submodular inequalities.

**Proposition 2.2.** The optimization problem \( P' \) is equivalent to a maximum flow problem from source \( s_N \) to sink \( t_N \).
Proof. By Proposition A.1 and Remark A.1 in Appendix A, the decision variables \( y_t \) for \( t \in E^- \) such that \( a_t = -1 \) can be assumed to be zero in the optimization problem \( P' \). Then, the problem defined by (2.5)-(2.9) reduces to:

\[
\max \{ y(S^+) : i_{j-1} + y(S_j^+) - y(L_j^-) - i_j = b_j, \ j \in N, \\
0 \leq b_j \leq d_j, \ j \in N, \ (2.7) - (2.9) \},
\]

where \( b_j \) represents the flow on the dummy demand arc of node \( j \).

There are numerous algorithms for solving maximum flow problems for general graphs in polynomial time. However, one can solve the maximum flow problem on path graphs in linear time. In Figure 2.2, we present three different arc sets that we solve the maximum flow on.

**Definition** If \( v(S^+, L^-) = d_{k\ell} \), then \( S^+ \) is called a path cover.

**Observation 1.** Let \( S^+ \) be a path cover. A minimum cut associated with \( v(S^+, L^-) \) is defined by the source and sink partitions \( \{s_N, k, \ldots, \ell\} \) and \( \{t_N\} \). Informally, this minimum cut passes below the path (see Figure 2.3a for a representation).

Obtaining an explicit form of inequality (2.10) requires finding the optimal objective value \( v(S^+, L^-) \) as well as the values \( v(S^+ \setminus \{t\}, L^-) \) for all \( t \in S^+ \) and \( v(S^+, L^- \setminus \{t\}) \) for all \( t \in L^- \). By definition 2.2, \( v(S^+, L^-) \) is equal to \( d_{k\ell} \) if \( S^+ \) is a path cover. In the next lemma, we identify minimum cuts associated with values \( v(S^+ \setminus \{t\}, L^-) \) for \( t \in S^+ \) and \( v(S^+, L^- \setminus \{t\}) \) for \( t \in L^- \).

**Lemma 2.1.** Let \( S^+ \) be a path cover. At least one minimum cut associated with \( v(S^+ \setminus \{t\}, L^-) \) for \( t \in S^+ \) and \( v(S^+, L^- \setminus \{t\}) \) is defined by one of the following source-sink partitions:

(i) \( \{s_N, k, \ldots, \ell\} \) and \( \{t_N\} \) (informally, minimum cut is the same as \( v(S^+, L^-) \); see Figure 2.3a),
(ii) \( \{ s_N, k, \ldots, 1b \} \) and \( \{ 1b + 1, \ldots, \ell, t_N \} \) for some \( 1b \in [k - 1, j - 1] \) (informally, minimum cut goes above the path at most once through an arc \((1b, 1b + 1)\); see Figure 2.3b),

(iii) \( \{ s_N, k, \ldots, 1b, rb, \ldots, \ell \} \) and \( \{ 1b + 1, \ldots, rb - 1 \} \) for some \( rb \in [j + 1, \ell + 1] \) (informally, if minimum cut goes above the path, then it goes below the path at most once through an arc \((rb - 1, rb)\); see Figure 2.3c).

**Proof.** For \( t \in S^+_j \) observe that dropping an arc \( t \in S^+_j \) may cause a decrease in the maximum flow from \( s_N \) to \( t_N \) (see the change from Figure 2.2a to Figure 2.2b). If there is a positive decrease in the maximum flow, then at least one minimum cut has to pass above node \( j \). Any cut that does not pass above node \( j \) has a value greater than or equal to \( d_{k^*} \) due to the assumption that the value of minimum cut before dropping arc \( t \) is equal to \( d_{k^*} \). If there is no decrease in the maximum flow, then minimum cut will remain unchanged with a value \( d_{k^*} \).

Now suppose that a minimum cut after dropping arc \( t \) goes above the path through more than one path arc. We refer to the nodes that are below this minimum cut as \([1b_1 + 1, rb_1 - 1] \) and \([1b_2 + 1, rb_2 - 1] \) for \( rb_2 > 1b_2 \geq rb_1 > 1b_1 \). Suppose \( j \not\in [1b_p + 1, rb_p - 1] \) for some \( p = 1, 2 \). For a minimum cut to go above the path at path arc \((1b_p, 1b_p + 1)\), it is required to have \( u_{1b_p} + \sum_{i \in [1b_p + 1, rb_p - 1]} c(S^+_i) < d_{1b_p + 1, rb_p - 1} \). However, the assumption of \( v(S^+, L^-) = d_{k^*} \) implies that \( u_{1b_p} + \sum_{i \in [1b_p + 1, rb_p - 1]} c(S^+_i) \geq d_{1b_p + 1, rb_p - 1} \). Therefore, we reach a contradiction.

The proof follows the same steps for \( t \in L^- \) (see the change from Figure 2.2a to Figure 2.2c). \( \square \)

Let \( \lambda_j \) represent the total excess flow capacity that can be sent through arcs in \( I \) and \( S^+ \) to node \( j \) after satisfying all the demands \( d_i, i \in N \).

**Definition** Let \( S^+ \) be a path cover. The *excess flow capacity* \( \lambda_j \) is the difference between the value of a minimum cut where node \( j \) is included in the sink partition (informally, a minimum cut that passes through arcs in \( S^+_j \)) and \( d_{k^*} \). The excess values \( \lambda_j \) can be calculated by

\[
\lambda_j = \min_{(1b, rb) : k - 1 \leq 1b < j < (rb) \leq \ell + 1} \{ d_{k(1b)} + u_{(1b)} + c(\cup_{j = 1b + 1}^rb - 1 S^+_j) + d_{(rb)\ell} \} - d_{k^*}. \tag{2.11}
\]

**Remark 2.1.** If \( S^+ \) is a path cover, then \( \lambda_j \geq 0 \) for all \( j \in N \).

**Proposition 2.3.** Let \( S^+ \) be a path cover. Then,

\[
\rho_t(S^+ \setminus \{ t \}, L^-) = (c_t - \lambda_j)^+, \quad t \in S^+,
\]

\[
\rho_t(S^+, L^- \setminus \{ t \}) = -\min\{c_t, \lambda_j\}, \quad t \in L^-.
\]
Proof. First, notice that the value \( v(S^+, L^-) \) will change if and only if at least one of the flow balance constraints (2.6) can no longer be satisfied at equality. Let \( t \in S^+_j \) and suppose \( c_t \leq \lambda_j \). Then, the excess flow that can be sent to node \( j \) will cover any flow that will be lost due to cancellation of arc \( t \). Otherwise, all the excess capacity will be used up to satisfy demand \( d_j \) and the remaining unsatisfied demand will cause a decrease of \( (c_t - \lambda_j) \) in the objective function (2.5). Similarly, let \( t \in L^-_j \). Dropping \( t \) from \( L^-_j \) corresponds to adding an arc from node \( j \) to \( t_N \). Now, we can potentially increase the objective function value by pushing more flow through \( S^+ \). The largest flow that can be pushed out of node \( j \) is \( \lambda_j \), when merged with the capacity constraint of arc \( t \), the increase in the objective function value becomes \( \min\{\lambda_j, c_t\} \).

Definition 2.2 and Proposition 2.3 lead to the following explicit description of the submodular inequalities for path substructures.

**Corollary 2.4.** Let \( S^+ \) be a path cover, then inequality (2.10) can be stated explicitly as

\[
y(S^+) + \sum_{j \in N} \sum_{t \in S^+_j} (c_t - \lambda_j)^+(1 - x_t) \leq d_{k\ell} + \sum_{j \in N} \sum_{t \in L^-_j} \min\{\lambda_j, c_t\} x_t + y(E^- \setminus L^-) \quad (2.12)
\]

We refer to inequality (2.12) as the *submodular path inequality*. Note that if \( c_t \leq \lambda_j \) for some \( t \in L^-_j, j \in N \), then excluding arc \( t \) from the set \( L^- \) provides a stronger submodular path inequality since \( y_t \leq c_t x_t \).
Remark 2.2. Observe that the flow cover inequalities of Padberg et al. (1985) are special cases of the submodular path inequalities (2.12). Suppose the path consists of a single node $N = \{j\}$ with demand $d := d_j > 0$. Then the $S^+ \subseteq E^+_j$ is a path cover if $\lambda := c(S^+) - d > 0$ and the resulting submodular path inequality

$$y(S^+) + \sum_{t \in S^+} (c_t - \lambda)^+(1 - x_t) \leq d + \lambda x(L^-) + y(E^- \setminus L^-)$$

is the flow cover inequality.

Example 1. Consider the path consisting of the three nodes in Figure 2.4. Suppose we select path $N = \{a, b, c\}$ and $S^+ = \{1, 2, 3\}$, $L^- = \{4, 5\}$. Then, $\lambda_a = 5$, $\lambda_b = 12$ and $\lambda_c = 21$ and the resulting submodular path inequality is

$$y_1 + y_2 + y_3 + 5(1 - x_1) + 8(1 - x_2) + 4(1 - x_3) \leq 23 + 12x_4 + 5x_5. \quad (2.13)$$

For $N = \{b\}$, $S^+ = \{2\}$ and $L^- = \{4\}$, we have $\lambda = 10$ and the corresponding submodular path inequality is

$$y_2 + 10(1 - x_2) \leq 10 + 10x_2 + i_2, \quad (2.14)$$

which is the flow cover inequality for node $b$ with the same set selection $(S^+, L^-)$.

Notice that calculating $\lambda_j$ for some $j \in N$ using Equation (2.11) has a complexity of $O(|S^+| + |N|)$. However, we can compute all $\lambda_j$ for $j \in N$ in the same complexity as well by using the minimum cut descriptions in Lemma 2.1. In particular, we compute all values $\lambda_j$, $j \in N$ simultaneously by using both a forward and a backward recursive formula. In a forward pass, we calculate, for all $j \in [k, \ell]$, the minimum cut value $\alpha_j$ for nodes $[k, j]$ that goes through $S^+_j$. In a backward pass, we calculate, for all $j \in [k, \ell]$, the minimum cut value $\beta_j$ that goes through $S^+_j$ for nodes $[j, \ell]$. Then,

$$\lambda_j = \alpha_j + \beta_j - c(S^+_j) - d_{k\ell},$$

Figure 2.4: Path set example.
where \( \alpha_j \) for \( j \in [k, \ell] \) is
\[
\alpha_j = \min\{\alpha_{j-1}, d_{k(j-1)} + u_{j-1}\} + c(S^+_j),
\]
with \( \alpha_{k-1} = 0 \) and \( \beta_j \) for \( j \in [k, \ell] \) is
\[
\beta_j = \min\{\beta_{j+1}, d_{(j+1)\ell}\} + c(S^+_j),
\]
with \( \beta_{\ell+1} = 0 \).

**Proposition 2.5.** All values \( \lambda_j \), for \( j \in N \), can be computed in \( O(|S^+| + |V|) \) time.

Next, we provide necessary and sufficient conditions under which the submodular path inequalities are facet defining for \( \text{conv}(\mathcal{P}_G) \).

**Theorem 2.6.** For \( S^+ \subseteq E^+ \) and \( L^- \subseteq E^- \), let \( S^+ \) be a path cover for \( V \). Then, the following conditions are necessary for inequality (2.12) to be facet-defining for \( \text{conv}(\mathcal{P}_G) \).

1. \( \rho_t(S^+ \setminus \{t\}, L^-) < c_t \) for all \( t \in S^+ \),
2. \( \rho_t(S^+, L^- \setminus \{t\}) > -c_t \) for all \( t \in L^- \),
3. if \( L^- = \emptyset \), then \( \max_{t \in S^+} \rho_t(S^+ \setminus \{t\}, L^-) > 0 \),
4. for \( p := \max\{j \in [k, \ell] : S^+_j \cup L^-_j \neq \emptyset\} \), at least one of the following holds:
   (i) \( c(S^+_p) < d_{p\ell} \) or
   (ii) \( \max_{t \in S^+_p} \rho_t(S^+ \setminus \{t\}, L^-) > 0 \) (i.e., \( \max_{t \in S^+_p} c_t > \lambda_t \)) or
   (iii) \( L^-_p \neq \emptyset \),
5. if \( \alpha_j = d_{k,j-1} + u_{j-1} + c(S^+_j) \) for some \( j \in [k + 1, \ell] \), then \( \cup_{i=j}^{\ell} S^+_i \) is not a path cover for \( \bar{V} = \{j, \ldots, \ell\} \).

**Proof.** See Appendix B.1.

**Theorem 2.7.** For \( S^+ \subseteq E^+ \), \( L^- \subseteq E^- \), let \( S^+ \) be a path cover for \( V \). If \( d_j \geq 0 \), for all \( j \in N \), then the necessary conditions in Theorem 2.6 along with the condition \( \max_{t \in S^+_j} c_t > \lambda_j \) for \( j \in N \) are sufficient for inequality (2.12) to be facet-defining for \( \text{conv}(\mathcal{P}_G) \).

**Proof.** See Appendix B.2.
2.3 Lifting submodular path inequalities

In this section, we first generalize the submodular path inequalities and then give a sequence independent lifting procedure a la Gu et al. (1999). Let $S^+ \subseteq E^+$ and $L^- \subseteq E^-$ be defined as in Section 2.2. We select a new subset of outgoing arcs $S^- \subseteq E^- \setminus L^-$. In this section, we let $K^+ = S^+$ and $K^- := L^- \cup S^-$. In other words, we select the objective function coefficients

$$a_t = \begin{cases} 
1, & t \in S^+, \\
-1, & t \in E^- \setminus (S^- \cup L^-), \\
0, & \text{otherwise.}
\end{cases}$$

For convenience, let $L^- := E^- \setminus (S^- \cup L^-)$.

**Definition** The pair $(S^+, S^-)$ is called a **generalized path cover** if $v(S^+, L^-) = d_{k\ell} + c(S^-)$.

For a generalized path cover $(S^+, S^-)$ the optimization problem (2.5)-(2.9) and inequality (2.10) lead to the **generalized submodular path inequality**

$$y(S^+) + \sum_{j \in N} \sum_{t \in S^+_j} (c_t - \lambda_j)^+(1 - x_t) \leq d_{k\ell} + c(S^-) + \sum_{j \in N} \lambda_j x(L^-_j) + y(L^-), \quad (2.15)$$

where

$$\lambda_j = \alpha_j + \beta_j - c(S^+_j) - d_{k\ell} - c(S^-_j)$$

and $\alpha_j$ for $j \in [k, \ell]$ is

$$\alpha_j = \min\{\alpha_{j-1}, d_{k(j-1)} + c(\bigcup_{i=k}^{j-1} S^-_i) + u_{j-1}\} + c(S^+_j),$$

with $\alpha_{k-1} = 0$ and $\beta_j$ for $j \in [k, \ell]$ is

$$\beta_j = \min\{\beta_{j+1}, d_{(j+1)\ell} + c(\bigcup_{i=j+1}^{\ell} S^-_i)\} + c(S^+_j),$$

with $\beta_{\ell+1} = 0$.

**Example 2.** Consider the path $N = \{a, b, c\}$ in Figure 2.5. For $S^+ = \{1, 2, 3, 4\}$, $L^- = \{8\}$ and $S^- = \{5, 6, 7\}$, we have $\lambda_a = 3, \lambda_b = 6$ and $\lambda_c = 9$. Then, the corresponding generalized submodular path inequality is

$$y_1 + y_2 + y_3 + y_4 + 7(1 - x_1) + (1 - x_2) + (1 - x_4) \leq 21 + 9x_8. \quad (2.16)$$
In order to strengthen inequality (2.15), we lift it by introducing variables $x_t$, $t \in S^-$ simultaneously into the submodular path inequality:

$$y(S^+) + \sum_{j \in N} \sum_{t \in S^+_j} (c_t - \lambda_j)^+(1 - x_t) \leq d_{k\ell} + \sum_{t \in S^-} (c_t + \theta_t(1 - x_t))$$

$$+ \sum_{j \in N} \lambda_j x(L^-_j) + y(L^-). \quad (2.17)$$

Let $K$ be the set of variables used for simultaneous lifting. We select at most one arc from $S^-_j$ for each $j \in N$ to include in $K$. Denote the arcs in $K$ using the subscript $t_j := S^-_j \cap K$ and let $K = \{t_k, \ldots, t_\ell\}$. For convenience, we refer to the capacity, flow and fixed-charge variables of arc $t_j$ as $c_j$, $y_j$ and $x_j$ respectively.

**Proposition 2.8** (Gu et al. (1999)). Let the function $f(z)$ be defined by (2.18)-(2.22), $h(z)$ be defined by (2.23)-(2.26), and set $Z$ be defined by (2.27). If the lifting coefficients $\theta$ are selected such that $h(z) \leq f(z)$ for any $z \in Z$, and $f(z)$ is superadditive, then inequality (2.17) is valid for $P_G$.

The function $f(z)$ is called the lifting function for inequality (2.17) and is defined as

$$f(z) = \min \ G(x, y) \quad (2.18)$$

s.t. $i_{j-1} + y(S^+_j) - y(E^-_j \setminus S^-_j) - i_j \leq d_j + c(S^-_j) - z_j, \quad \forall j \in N, \quad (2.19)$

$0 \leq i_j \leq u_j, \quad \forall j \in N, \quad (2.20)$

$0 \leq y_t \leq c_t x_t, \quad \forall t \in E, \quad (2.21)$

$x_t \in \{0, 1\}, \quad \forall t \in E, \quad (2.22)$

where

$$G(x, y) := d_{k\ell} + c(S^-) + \sum_{j \in N} \lambda_j x(L^-_j) + y(L^-) - y(S^+) - \sum_{j \in N} \sum_{t \in S^+_j} (c_t - \lambda_j)^+(1 - x_t)$$
is the difference between the right and the left hand sides of inequality (2.15). We call function \( h(z) \) the dual lifting function and it is defined as

\[
h(z) := \max_{t_j \in K} \sum_{t_j \in K} \theta_j(x_j - 1)
\]

\[(2.23)\]

\[\text{s.t. } y_j - c_j = -z_j, \quad t_j \in K, \quad (2.24)\]

\[0 \leq y_j \leq c_j x_j, \quad t_j \in K, \quad (2.25)\]

\[x_j \in \{0, 1\}, \quad t_j \in K. \quad (2.26)\]

Finally, the feasible region of the vector \( z \) is:

\[Z = \{z \in \mathbb{R}^v : \exists (x, y, i) : y_j - c_j = -z_j, \quad t_j \in K, \quad y_t = c_t, \quad t \in S^- \setminus K, \quad (2.19) - (2.22)\}. \quad (2.27)\]

**Definition** (Gu et al. (1999)) A function \( \Phi(z) \) is a superadditive valid lifting function of \( f(z) \) if \( \Phi(z) \leq f(z) \) for all \( z \in Z \) and is superadditive.

**Proposition 2.9** (Gu et al. (1999)). Let \( \Phi(z) \) be a superadditive valid lifting function. If the lifting coefficients \( \theta \) satisfy \( h(z) \leq \Phi(z) \) for any \( z \in Z \), then inequality (2.17) is valid for \( PG \).

### 2.3.1 Superadditive valid lifting function

In this section, we provide a superadditive valid lifting function for \( f(z) \). We tackle the multidimensionality of \( f(z) \) by decomposing it with respect to \( j \in N \). Notice that the path arc flows \( i_j, j = k, \ldots, \ell - 1 \) are the only terms binding constraints (2.19) together. Therefore, we first duplicate the path arc flow variables, \( i_j, j = k, \ldots, \ell - 1 \) into \( i_j^1 \) and \( i_j^2 \). Then, we add a constraint that enforce them to be equal. This procedure leads to the following formulation

\[
f(z) = \min \quad G(x, y)
\]

\[\text{s.t. } i_{j-1}^1 + y(S_j^+) - y(E_j^- \setminus S_j^-) - i_j^2 \leq d_j + c(S_j^-) - z_j, \quad j \in N, \quad (2.28)\]

\[0 \leq i_j^s \leq u_j, \quad j \in N, \quad s = 1, 2, \quad (2.29)\]

\[i_j^2 = i_j^1, \quad j \in N \setminus \{\ell\}, \quad (2.30)\]

\[(2.21) - (2.22).\]

**A Lagrangian relaxation**

Now, constraints (2.30) are the only constraints that bind the subproblems for each node \( j \in N \) together. We employ a Lagrangian relaxation w.r.t. (2.30) to decompose the problem:
\[ f^L(z, \mu) = G(x, y) + \sum_{j=k}^{\ell-1} \mu_j (i_j^1 - i_j^2) \]

subject to \( (2.21) - (2.22), (2.28) - (2.29) \).

Notice that the relaxed formulation above is now decomposable

\[ f^L(z, \mu) = \sum_{j=k}^{\ell} f_j^L(z_j, \mu_{j-1}, \mu_j), \]

where

\[ f_j^L(z_j, \mu_{j-1}, \mu_j) = \min \ H_j(x, y) \]

subject to \( i_{j-1}^1 + y(S_j^+) - y(E_j^+ \setminus S_j^-) - i_j^2 \leq d_j + c(S_j^-) - z_j \), \( 0 \leq y_t \leq c_t x_t \), \( t \in E_j^+ \cup E_j^- \), \( x_t \in \{0, 1\} \), \( t \in E_j^+ \cup E_j^- \), \( i_{j-1}^1 \leq u_{j-1} \), \( i_j^2 \leq u_j \).

and

\[ H_j(x, y) := d_j + c(S_j^-) + \lambda_j x(L_j^-) + y(L_j^-) - y(S_j^+) - \sum_{t \in S_j^+} (c_t - \lambda_j)^+ (1 - x_t) + \mu_j i_j^2 + \mu_{j-1} i_{j-1}^1. \]

From Lagrangian duality, \( f^L(z, \mu) \leq f(z) \) for any \( \mu \). We pick \( \mu_i \) to be 1 for all \( i \in [k, \ell] \) and \( \mu_{k-1} = 0 \) for simplicity. Therefore, in the remainder of this chapter, we refer to \( f_j^L(z_j, \mu_{j-1}, \mu_j) \) as \( f_j^L(z_j) \).

Let \( S_j^{++} = \{ t \in S_j^+ : c_t > \lambda_j \} \), let \( S_j^{++} \cup L_j^- := \{v_1, v_2, \ldots, v_r\} \) where \( c_{v_i} \geq c_{v_{i+1}} \) and \( r_j := |S_j^{++} \cup L_j^-| \). In formulation (2.31)-(2.36), we observe three conditions that hold in at least one of its optimal solutions. First, \( y_t \) for \( t \in E_j^+ \setminus S_j^+ \) do not appear either in \( H_j(x, y) \) or in constraint (2.32) and therefore, can be assumed to be zero. Second, if \( t \in S_j^{++} \), then either \( y_t > c_t - \lambda_j \) or \( y_t = 0 \). Third, if \( t \in L_j^- \), then either \( y_t = 0 \) or \( y_t = c_t \).

While optimizing \( f_j^L(z_j) \), we first set \( y_t = c_t \) for all \( t \in S_j^{++} \) and \( y_t = 0 \) for all \( t \in L_j^- \). If the flow balance constraint (2.32) is violated, then we decrease the flow of arcs in \( S_j^{++} \) and increase the flow of variables in \( L_j^- \) in the order of \( v_1, v_2, \ldots, v_r \).

Let \( mp_j = \min_{t \in S_j^{++}} \{ c_t \} \) and \( p_j \) be the largest index such that \( c_{v_{p_j}} = mp_j \), \( M_{j,i} = \sum_{k=1}^{i} c_{v_k} \) and \( M_{j,0} = 0 \). Let \( m_j = u_{j-1} + u_j + c(S_j^+ \setminus S_j^{++}) + c(L_j^-) \), \( ml_j = \min \{ \lambda_j, m_j \} \) and \( \varphi_{j,i} = \max \{ 0, c_{v_{p_j + i}} - (mp_j - \lambda_j) - ml_j \} \) for \( i = p_j, \ldots, r_j - 1 \).
We may assume that the path arc \((j - 1, j)\) is in \(S^+_j \setminus S^{++}_j\) and path arc \((j, j + 1)\) is in \(L_j^-\) since they appear in formulation (2.31)-(2.36) with same coefficients. The variables \(c_t - y_t\) for \(t \in L_j^-\) (thus \(u_j - i^2_j\) included) and \(y_t\) for \(t \in S^+_j \setminus S^{++}_j\) (thus \(i^1_{j-1}\) included) appear together in formulation (2.31)-(2.36). Therefore, without loss of generality, we can merge these variables into a new variable \(w\) and assume that the corresponding arcs are always open:

\[
w = i^1_{j-1} + (u_j - i^2_j) + y(S^+_j \setminus S^{++}_j) + c(L_j^-) - y(L_j^-).
\]

Then, the formulation (2.31)-(2.36) can be simplified as:

\[
f^L_j(z_j) = \min \quad \hat{d}_j - w - c(L^-_j) + \lambda_j x(L^-_j) - y(S^+_{j+1}) - \sum_{t \in S^{++}_j} (c_t - \lambda_j)(1 - x_t) \quad (2.37)
\]

\[
s.t. \quad w + y(S^+_{j+1}) + c(L^-_j) - y(L^-_j) \leq \hat{d}_j - z_j, \quad (2.38)
\]

\[
0 \leq y_t \leq c_t x_t, \quad t \in S^{++}_j \cup L^-_j, \quad (2.39)
\]

\[
x_t \in \{0, 1\}, \quad t \in S^{++}_j \cup L^-_j, \quad (2.40)
\]

\[
0 \leq w \leq m_j, \quad (2.41)
\]

where \(\hat{d}_j = d_j + u_j + c(E^-_j)\). Moreover, letting

\[
\lambda_{j} := u_{j-1} + c(S^+) - (d_j + c(S^-_j)), \quad j \in N \quad (2.42)
\]

we observe that

\[
\hat{d}_j = m_j + M_{j,r_j} - \lambda_j.
\]

**Theorem 2.10.** The function \(f^L_j(z)\) can be expressed as:

\[
f^L_j(z) = \begin{cases}
  z + i \lambda_j - M_{j,i}, & \text{if } M_{j,i} - \lambda_j \leq z \leq M_{j,i} - \lambda_j + \lambda_j, \quad i \in [0, p_j - 1] \\
  (i + 1) \lambda_j - \lambda_j, & \text{if } M_{j,i} + \lambda_j - \lambda_j \leq z \leq M_{j,i+1} - \lambda_j, \quad i \in [0, p_j - 1] \\
  z + (i + 1) \lambda_j - M_{j,i} - \varphi_{j,i} - m_{l_j}, & \text{if } M_{j,i} - \lambda_j \leq z \leq M_{j,i} - \lambda_j + m_{l_j}, \quad i \in [p_j, r_j - 1] \\
  z + (i + 1) \lambda_j - \lambda_j, & \text{if } M_{j,i} - \lambda_j + m_{l_j} < z \leq M_{j,i} - \lambda_j + m_{l_j} + \varphi_{j,i}, \quad i \in [p_j, r_j - 1] \\
  z - M_{j,r_j} + r_j \lambda_j, & \text{if } M_{j,r_j} - \lambda_j \leq z \leq \hat{d}_j.
\end{cases}
\]

**Proof.** See Appendix B.3. \( \square \)

**Remark 2.3.** If \(\lambda_j = \lambda_j\), then \(f^L_j(z)\) is the same as the lifting function of the flow cover inequality provided in Gu et al. (1999).
Remark 2.4. In equation (2.42), the incoming flow from up-stream nodes \( \{j, \ldots, j-1\} \) is \( i_{j-1} = u_{j-1} \) and there is no flow being pulled from down-stream nodes \( \{j+1, \ldots, \ell\} \). Therefore, \( \bar{\lambda}_j \geq \lambda_j \) and the domain of \( f^L_j(z) \) includes \([0, \infty)\).

The function \( f^L_j(z_j) \) is not necessarily superadditive and does not satisfy the conditions in Proposition 2.9. Therefore, we construct a lower-bound on \( f^L_j \) that is superadditive on \( \mathbb{R}_+ \cup \{0\} \). Gu et al. (1999) propose a slightly different version of the following superadditive valid lifting function for flow cover inequalities:

\[
\psi_j(z) = \begin{cases} 
  z + i\lambda_j - M_{j,i} & M_{j,i} - \bar{\lambda}_j \leq z \leq M_{j,i} - \bar{\lambda}_j + \lambda_j, \quad i \in [0, p_j - 1] \\
  (i+1)\lambda_j - \bar{\lambda}_j & M_{j,i} + \lambda_j - \bar{\lambda}_j \leq z \leq M_{j,i+1} - \bar{\lambda}_j, \quad i \in [0, p_j - 1] \\
  z + i\lambda_j - M_{j,i} & M_{j,i} - \bar{\lambda}_j \leq z \leq M_{j,i} - \bar{\lambda}_j + ml_j + \varphi_{j,i}, \quad i \in [p_j, r_j - 1] \\
  (i+1)\lambda_j - \bar{\lambda}_j & M_{j,i} - \bar{\lambda}_j + ml_j + \varphi_{j,i} < z \leq M_{j,i+1} - \bar{\lambda}_j, \quad i \in [p_j, r_j - 1] \\
  z - M_{j,r_j} + r_j\lambda_j & M_{j,r_j} - \bar{\lambda}_j \leq z \leq \hat{d}_j.
\end{cases}
\]

(2.44)

If the path consists of a single node as in flow cover inequalities, then \( \psi_j(z) \) is superadditive. However, when there are multiple nodes in \( V \), the superadditivity of both \( f^L_j \) and \( \psi_j \) depends on where the \( z = 0 \) case lies. Therefore, we examine the function \( f^L_j \) case by case. Let the index \( T_j \) be \( T_j = \min\{1 \leq i \leq r_j : M_{j,i} - \bar{\lambda}_j \geq 0\} \) if \( M_{j,r_j} - \bar{\lambda}_j \geq 0 \), and \( T_j = r_j \) otherwise.

Theorem 2.11. The function \( \psi_j(z) \leq f^L_j(z) \) is superadditive if \( T_j \lambda_j - \bar{\lambda}_j \leq 0 \).

Proof. See Appendix B.4.

Note that if \( T_j \lambda_j - \bar{\lambda}_j > 0 \), then \( \psi_j \) is not superadditive since \( \psi_j(0) > 0 \). Under this case, we obtain a different superadditive lower bound. Using Lemma 2.2, we find the largest convex lower-bound of \( f^L_j(z) \), \( \phi_j(z) \), such that \( \phi_j(0) \leq 0 \) is the superadditive lower estimation.

Lemma 2.2 (Hille and Phillips (1957) p. 237). A convex function \( \phi : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R} \) with \( \phi(0) \leq 0 \) is superadditive.

The generic form of the largest convex lower-bound of \( f^L_j(z) \) that has a non-positive value at zero is

\[
\phi_j(z) = \begin{cases} 
  -\frac{\lambda_j}{M_{j,r_j} - \bar{\lambda}_j} z + \Gamma, & 0 \leq z \leq M_{j,r_j} - \bar{\lambda}_j, \\
  \lambda_j \left( z - M_{j,i} + \bar{\lambda}_j \right) + i\lambda_j - \bar{\lambda}_j, & M_{j,i} - \bar{\lambda}_j \leq z \leq M_{j,i+1} - \bar{\lambda}_j, \quad i \in [\tau, r_j - 1] \\
  z - M_{j,r_j} + r_j\lambda_j, & M_{j,r_j} - \bar{\lambda}_j \leq z \leq \hat{d}_j.
\end{cases}
\]

(2.45)

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where \( \Gamma \) is the value of \( \phi_j(0) = \min\{0, f^L_j(0)\} \) and \( \tau \) is the smallest index that ensures the slope of linear function connecting points \((0, \Gamma)\) and \((M_j, \tau - \bar{\lambda}_j, \phi_j(M_j, \tau - \bar{\lambda}_j))\) is the smallest compared to the slope of next linear pieces of \( \phi_j \). The values of \( \Gamma \) and \( \tau \) depend on the parameters of the problem (see Appendix B.6 for explicit descriptions).

Let \( \Phi_j(z) \) be the superadditive lower bound of \( f^L_j \). In summary, \( \Phi_j(z) \) takes two different forms and it has the following description:

\[
\Phi_j(z) = \begin{cases} 
\psi_j(z) & \text{if } T_j \lambda_j - \bar{\lambda}_j \leq 0, \\
\phi_j(z) & \text{otherwise.}
\end{cases}
\]

For each node \( j \in N \), the lower bounding function \( \Phi_j(z_j) \) is superadditive. Therefore, \( \sum_{j=k}^{\ell} \Phi_j(z_j) \) is superadditive as well. Furthermore, since inequality (2.15) is valid for \( P_G, f(0) \geq 0 \). On the contrary, the approximation \( \sum_{j=k}^{\ell} \Phi_j(0) \) can be less than zero. Then, we can construct a tighter superadditive valid lifting function \( \Phi(z) \) by:

\[
\Phi(z) = \max \left\{ \sum_{j=k}^{\ell} \Phi_j(z_j), 0 \right\} \leq f(z).
\]

Note that since \( \sum_{j=k}^{\ell} \Phi_j(z_j) \) is superadditive, \( \max\{\sum_{j=k}^{\ell} \Phi_j(z_j), 0\} \) is superadditive as well, due to Observation 5 in Appendix B.5. **Example 2 (cont).** Recall that path \( N = \{a, b, c\} \) for the set selection \( S^+ = \{1, 2, 3, 4\} \), \( L^- = \emptyset \) and \( S^- = \{5, 6, 7\} \) have \( \lambda_a = 3, \lambda_b = 6 \) and \( \lambda_c = 9 \). However note that the upper bounds \( \bar{\lambda}_a = 3, \bar{\lambda}_b = 11 \) and \( \bar{\lambda}_c = 11 \). The sets \( S^+ \cup L^- \) are \( S_a^+ \cup L_a^- = \{1\}, S_b^+ \cup L_b^- = \{2\} \) and \( S_c^+ \cup L_c^- = \{4, 8\} \) with \( v_{a,1} = 1, v_{b,1} = 2 \) and \( v_{c,1} = 8, v_{c,2} = 4 \) since \( c_8 > c_4 \). The cardinalities of sets \( S_j^+ \cup L_j^- \) are \( r_a = r_b = 1 \) and \( r_c = 2 \) and the cumulative capacities are \( M_{a,v_{a,1}} = 10, M_{b,v_{b,1}} = 7 \) and \( M_{c,v_{c,1}} = 12, M_{c,v_{c,2}} = 12 + 10 = 22 \). We also have \( p_a = 1, p_b = 1 \) and \( p_c = 2 \). The values \( m_j \) for \( j = a, b, c \) are \( m_a = 8, m_b = 8 + 8 + 3 = 19 \) and \( m_c = 8 \). Finally, the modified demand values are \( \hat{d}_a = \hat{d}_b = 2 + 8 + 5 = 15 \) and \( \hat{d}_c = 2 + 5 + 12 = 19 \).

Let us now examine the decomposed functions \( f^L_j(z) \) for Example 2. Using the values calculated in Section 2.3.1, we observe that

\[
\begin{align*}
 f^L_a(z) &= \begin{cases} 
 0 & 0 \leq z \leq 7 \\
 z - 7 & 7 \leq z \leq 15
\end{cases}, \\
 f^L_b(z) &= \begin{cases} 
 -2 & 0 \leq z \leq 1 \\
 z - 3 & 1 \leq z \leq 10 \\
 7 & 10 \leq z \leq 11 \\
 z - 4 & 11 \leq z \leq 19.
\end{cases}
\end{align*}
\]
Note that $T_a = T_b = T_c = 1$ and $T_a \lambda_a - \bar{\lambda}_a = 0 \leq 0$, $T_b \lambda_b - \bar{\lambda}_b = -5 \leq 0$ and $T_c \lambda_c - \bar{\lambda}_c = -2 \leq 0$. Therefore, $\psi_a(z)$, $\psi_b(z)$ and $\psi_c(z)$ are all superadditive. We also observe that under this parameter set, $f^L_j(z) = \psi_j(z)$ for $j = a, b, c$. We then have $\Phi_j(z) = \psi_j(z)$ for $j = a, b, c$. \hfill \Box

### 2.3.2 Dual lifting function

In this section, we investigate the explicit form of the dual lifting function $h(z)$. First, notice that the optimization problem defined by (2.24)-(2.26) is decomposable with respect to each arc $t_j \in K$. Therefore, we write it as a sum of single dimensional functions $h_j(z_j)$:

$$h(z) = \sum_{t_j \in K} h_j(z_j),$$

where

$$h_j(z_j) = \max \theta_j (x_j - 1)$$

$$\text{s.t. } y_j - c_j = -z_j,$$

$$0 \leq y_j \leq c_j x_j,$$

$$x_j \in \{0, 1\}.$$

**Remark 2.5.** The function $h(z)$ is decomposable with respect to $j \in N$ as well, by the construction of the lifting set $K$.

Functions $h_j(z_j)$ are the same as the dual lifting function of single node flow cover inequalities. Therefore, we directly use the result of Gu et al. (1999) to analytically evaluate it.

**Theorem 2.12** (Gu et al. (1999) Theorem 4).

$$h_j(z_j) = \begin{cases} 0 & 0 \leq z_j < c_j, \\ -\theta_j & z_j = c_j, \end{cases}$$

with $\theta_j \leq 0$.

### 2.3.3 Lifting coefficients

Using Proposition 2.9, we construct the lifting coefficients $\theta$ such that

$$\sum_{j=k}^\ell h_j(z_j) \leq \max \left\{ \sum_{j=k}^\ell \Phi_j(z_j), 0 \right\}.$$  \hfill (2.46)
Theorem 2.12 shows that the function takes a nonzero value only if \( z_j = c_j \). Observe that if \( z_j \neq c_j \) for all \( j \in N \), then inequality (2.46) implies \( 0 \leq \max \{ \sum_{j=k}^{k} \Phi_j(z_j), 0 \} \) and does not provide any information on the lifting coefficients \( \theta \). Let us now examine the case where \( z_i \neq c_i \) for \( i \neq j \) and \( z_j = c_j \) for some \( j \in N \). Using Theorem 2.12, inequality (2.46) can be written as

\[
\sum_{j=k}^{t} h_j(z_j) = -\theta_j \leq \max \{ 0, \Phi_j(c_j) + \sum_{m \neq j} \Phi_k(z_m) \}, \quad z : z_j = c_j, z_m \neq c_m, m \neq j,
\]

for each \( j \in N \). Inequality (2.47) implies that \(-\theta_j\) should be less than or equal to all possible values of the right hand side, equivalently:

\[
-\theta_j \leq \min_{z_j = c_j, z_m \neq c_m, m \neq j} \left\{ \max \left\{ 0, \Phi_j(c_j) + \sum_{m \neq j} \Phi_m(z_m) \right\} \right\}. \quad (2.48)
\]

Since \( \Phi_i(z_i) \) is a non-decreasing function, inequality (2.48) can be simplified as

\[
-\theta_j \leq \max \left\{ 0, \Phi_j(c_j) + \sum_{m \neq j} \Phi_m(0) \right\}. \quad (2.49)
\]

Let us now examine the vector \( z \) where \( z_i = c_i \) and \( z_j = c_j \) and \( z_m \neq c_m \) for any \( m \neq i \) and \( m \neq j \). Then, using Theorem 2.12, inequality (2.46) becomes:

\[
-\theta_j - \theta_i \leq \max \left\{ 0, \Phi_i(c_i) + \Phi_j(c_j) + \sum_{m \neq i,j} \Phi_m(z_m) \right\}, \forall z : z_i = c_i, z_j = c_j, z_m \neq c_m, m \neq i, j. \quad (2.50)
\]

Since \( \Phi_m(z_m) \) is a non-decreasing function, we can replace the right hand side of inequality (2.50) with its smallest value:

\[
-\theta_j - \theta_i \leq \max \left\{ 0, \Phi_i(c_i) + \Phi_j(c_j) + \sum_{m \neq i,j} \Phi_m(0) \right\}. \quad (2.51)
\]

**Proposition 2.13.** Inequality (2.51) provides a smaller lower bound on \( \theta_i + \theta_j \) than bounds of \( \theta_i \) and \( \theta_j \) in inequality (2.49) summed. Equivalently,

\[
\max \left\{ 0, \Phi_i(c_i) + \Phi_j(c_j) + \sum_{m \neq i,j} \Phi_m(0) \right\} \geq \max \left\{ 0, \Phi_j(c_j) + \sum_{m \neq j} \Phi_m(0) \right\} + \max \left\{ 0, \Phi_i(c_i) + \sum_{m \neq i} \Phi_m(0) \right\}. \quad (2.52)
\]
Proof. Recall that \( \Phi_j(0) \leq 0 \) and \( \Phi_j(z_j) \) is a non-decreasing function. Hence,

\[
\Phi_i(c_i) + \Phi_j(c_j) + \sum_{m \neq i,j} \Phi_m(0) \geq \Phi_j(c_j) + \sum_{m \neq j} \Phi_m(0),
\]

\[
\Phi_i(c_i) + \Phi_j(c_j) + \sum_{m \neq i,j} \Phi_m(0) \geq \Phi_i(c_i) + \sum_{m \neq i} \Phi_m(0),
\]

\[
\Phi_i(c_i) + \Phi_j(c_j) + \sum_{m \neq i,j} \Phi_m(0) \geq \Phi_j(c_j) + \sum_{m \neq j} \Phi_m(0) + \Phi_i(c_i) + \sum_{m \neq i} \Phi_m(0).
\]

Proposition 2.13 implies that the constraints provided in (2.49) are tighter than the constraints provided by (2.51). Using the same argument, one can easily show that as the number of components of vector \( z \) such that \( z_j = c_j \) is increased, the constraints enforced on the coefficients \( \theta \) by inequality (2.46) get weaker.

Theorem 2.14. Lifted generalized submodular path inequality (2.17) where \( \theta_j = -\max\{0, \Phi_j(c_j) + \sum_{k \neq j} \Phi_k(0)\} \) is valid for \( P_G \).

Proof. Selecting \( \theta_j \) as the lower bound given in (2.49) guarantees that inequality (2.46) is satisfied, due to the tightness argument in Proposition 2.13. Then, inequality (2.17) is valid for the feasibility set \( P_G \) from Proposition 2.9.

Example 2 continued: We now calculate the lifting coefficients for arcs in \( S^- = \{5,6,7\} \). From Theorem 2.14, we have \( \theta_5 = -\max\{0, \Phi_5(5) + \Phi_6(0) + \Phi_7(0)\} = -\max\{0, 0 - 1 - 2\} = 0, \theta_6 = -\max\{0, \Phi_6(0) + \Phi_5(5) + \Phi_7(0)\} = -\max\{0, 0 + 4 - 2\} = -2 \) and \( \theta_7 = -\max\{0, \Phi_7(0) + \Phi_6(0) + \Phi_5(5)\} = -\max\{0, 0 - 1 + 2\} = -1. \) Then, we lift inequality (2.16) using Theorem 2.14 and obtain

\[
y_1 + y_2 + y_3 + y_4 + 7(1 - x_1) + (1 - x_2) + (1 - x_4) \leq 21 - 2(1 - x_6) - (1 - x_7) + 9x_8.
\]
nonnegative demands, which is typical in production planning problems. Given fixed cost \( f_t \) and variable cost \( p_t \) associated with an arc \( t \in E \), and variable and fixed inventory holding costs \( h_j \) and \( s_j \) for a path arc \( j \), we solve the following mixed integer optimization problem:

\[
\min \sum_{t \in E} (f_t x_t + p_t y_t) + \sum_{j \in N} (h_j i_j + s_j z_j)
\]

s.t. \( i_{j-1} + y(E^+_j) - y(E^-_j) - i_j = d_j \), \( j \in N \),

\( 0 \leq y_t \leq c_t x_t \), \( t \in E \),

\( 0 \leq i_j \leq u_j z_j \), \( j \in N \),

\( x_t \in \{0, 1\} \), \( t \in E \),

\( z_j \in \{0, 1\} \), \( j \in N \),

where \( y_t \) and \( i_j \) are the flow values on a non-path arc \( t \in E \) and a path arc \( j \in N \) respectively, and \( x_t \) and \( z_j \) are the binary variables representing whether these arcs are being used or not.

**Finding violated submodular path inequalities**

Given a path and a feasible solution \((x^*, y^*, i^*)\) to the LP relaxation with fractional \( x^* \), the separation problem aims to find sets \( S^+, L^- \) and \( S^- \) that maximize the violation

\[
\sum_{t \in S^+} \left[ y_t^* + (c_t - \lambda_j)^+(1 - x_t^*) \right] - d_{kt} - c(S^-) - \sum_{t \in L^-} \lambda_j x_t^* - \sum_{t \in L^-} y_t^*.
\]

We use the knapsack relaxation based heuristic separation strategy described in (Wolsey and Nemhauser, 1999, pg. 500) for flow cover inequalities to choose sets \( S^+ \) and \( S^- \) with a knapsack capacity \( d_{kt} \). Then, we add arc \( t \in E^- \setminus S^- \) to \( L^- \) if \( \lambda_j x_t^* < y_t^* \) and \( \lambda_j < c_t \).

**Instance Generation**

The data we used for computational experiments has the following properties: There are \( v \) nodes on the network and each node has a single non-path incoming arc and a single non-path outgoing arc. Demand is drawn from integer uniform between 1 and 19. The parameter \( c \in \{2, 3, 4, 5\} \) controls the tightness of arc capacities compared to the average demand of all nodes and \( f \in \{100, 200, 500, 1000\} \) controls how large the fixed costs are compared to inventory holding costs. The capacities of non-path incoming, non-path outgoing and path arcs are drawn from integer uniform with bounds \([0.75 \times c \times \bar{d}, 1.25 \times c \times \bar{d}]\), \( 2 \times [0.75 \times c \times \bar{d}, 1.25 \times c \times \bar{d}] \) and \([0.75 \times c \times \bar{d}, c \times \bar{d}]\) respectively where \( \bar{d} \) is the average demand. Similarly, fixed costs of non-path incoming and non-path outgoing and path arcs are generated from integer uniform with bounds
\[0.9 \times f \times \tilde{h}, 1.1 \times f \times \tilde{h}, 0.2 \times f \times \tilde{h}, 0.5 \times f \times \tilde{h}\] respectively where \(\tilde{h}\) is the average inventory holding cost. Path arcs have fixed and variable costs of \(f\) and 10, respectively. Variable ordering costs of non-path incoming and outgoing arcs are generated from integer uniform at the interval \([81, 119]\) and \(-1 \times [160, 240]\). Five random instances are generated for each \((v, c, f)\) combination.

**Preprocessing**

In order to tighten the capacities of an instance we apply the following preprocessing steps for all arcs that have positive variable costs.

1) \(\bar{u}_j = \min\{u_j, (d_{j+1})^+ + u_{j+1} + c(E_{j+1}^-)\}, \) for \(j \in N\)

2) \(\bar{c}_t = \min\{c_t, (d_j)^+ + \bar{u}_j + c(E_j^+)\} \) for all \(t \in E_j^+, j \in N\).

We report a number of performance measures in the following tables. Let \(z_{\text{INIT}}\) be the optimal objective function value of the LP relaxation with no valid inequalities added, \(z_{\text{ROOT}}\) be the optimal objective function value of the LP relaxation after adding the violated cuts at the root node. Moreover, let \(z_{\text{UB}}\) be the objective function value of the best feasible solution found within the time/memory limit. We report the percentage gap improvement \((\text{gap imp} = 100 \times \frac{z_{\text{ROOT}} - z_{\text{INIT}}}{z_{\text{BEST}} - z_{\text{INIT}}})\), the initial gap \((\text{init gap} = 100 \times \frac{z_{\text{UB}} - z_{\text{INIT}}}{z_{\text{UB}}})\), and the root gap \((\text{root gap} = 100 \times \frac{z_{\text{UB}} - z_{\text{ROOT}}}{z_{\text{UB}}})\). Moreover, we report the number of violated cuts added \((\text{cuts})\), the number of branch and bound nodes explored \((\text{nodes})\) and the elapsed time for solving an instance \((\text{time})\) in seconds. If the instance is not solved within the time/memory limit, then we report in parentheses the average percentage gap between the best lower bound and the best integer solution \(\text{endgap}\) and the number of instances with a positive end gap \(\text{unsolv}\). We round all the values to the nearest integer except for percent gaps. Each row reports the average values for \text{init gap, gap imp, cuts, nodes, time and endgap} over five instances. The user cuts are added only at the root node of the branch and bound tree.

In Table 2.1, we compare submodular path inequalities (2.12) \((\text{columns spi})\) with lifted generalized submodular path inequalities (2.17) \((\text{columns lspi})\) for instances with 50 nodes. To see first the effect of the proposed inequalities alone, in this experiment, all built-in CPLEX cuts are disabled. We enumerated all paths in the graph and for each path we used the separation heuristic explained above to find violated submodular path inequalities. Observe that both \text{spi} and \text{lspi} perform better for smaller \(f\) values. These results in the table suggest that submodular path inequalities alone reduce the integrality gap on average by 61% and that the generalized submodular path inequalities improve the gap reduction by another 15% on average. Furthermore, we observe that the number of nodes, the average solution times and the number of unsolved instances decrease substantially. Table 2.1 shows the positive impact of submodular path inequalities and their generalizations very clearly. In the remainder of computational study, we use \text{lspi} for the experiments.
Table 2.1: Comparison of submodular path inequalities (spi) and lifted generalized submodular path inequalities (lspi), n = 50.

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<th>gap</th>
<th>imp %</th>
<th>cuts</th>
<th>nodes</th>
<th>time (endgap:unslv)</th>
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<td>366</td>
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</table>
Although enumerating all the paths for a given network shows the full potential of the inequalities, it is too time consuming except for small cases. Therefore, in Table 2.2, we examine the effect of path size on the computational effectiveness of Lspi. We use the same instances \((n = 50)\) and separation heuristic as in Table 2.1. In columns \(= 1\), we select all paths of size 1 whereas in columns \(\leq k\), we enumerate all paths of sizes \(1, 2, \ldots, k\). Note that column \(n\) corresponds to enumerating all simple paths of all sizes. The results in Table 2.2 underline the diminishing rate of returns with path size. Clearly, we find more violated inequalities with longer paths; however, the increase in the gap improvement diminishes. The tradeoff between searching for more paths and solution time and quality is most visible in the time column: the elapsed time first decreases and then increases with increasing efforts for finding paths. Therefore, in the remainder of the experiments, in order to find a balance between time spent for finding paths and the solution quality, for each given extreme point solution, we stop searching for paths if we fail to find violated inequalities for four consecutive path sizes.
Table 2.2: Effect of path length on the performance of lspi.

<table>
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<tr>
<th>c</th>
<th>f</th>
<th>init gap</th>
<th>gap imp %</th>
<th>cuts</th>
<th>time</th>
</tr>
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<td>= 1 ≤ 2 ≤ 3 ≤ 4 ≤ n</td>
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<td>142 14 7 7 21</td>
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<td>1944 895 479 511 617</td>
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<td>169 263 328 383 490</td>
<td>1291 1147 1114 1029 1272</td>
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</tr>
</tbody>
</table>

Average 27.8 65.5 73.8 75.5 76.0 76.2 126 210 262 295 366 636 408 271 269 292
We next compare \texttt{lspi} with lifted single node flow cover (fc) and uncapacitated path inequalities (uc). The results with the lifted flow cover inequalities are reported in Table 3.3 under columns fc and are equivalent to inequality (2.17) where the path size is equal to one. Under columns uc, we report the uncapacitated path inequalities of van Roy and Wolsey (1985) that are of the form

\[ y(S^+) \leq \sum_{j=k}^{\ell} d_{j\ell} x(S^+_j) + y(E^-), \]  

where \( N = [k, \ell] \) is the path and \( S^+ \subseteq E^+ \). We enumerate all paths in a given instance and given a fractional solution \((x^*, y^*)\) we use a simple comparison to select set \( S^+ \) as \( S^+_j = \{ t \in E^+_j : y^*_t > d_{j\ell} x^*_t \} \) where \( S^+_j = S^+ \cap E^+_j \).

We observe in Table 3.3 that \texttt{lspi} helps to reduce the integrality gap by additional 11% and 45% on average compared to fc and uc, respectively. However for some instances, where both \( f \) and \( c \) are large, uc performs better than both \texttt{lspi} and fc. A positive effect of larger \( c \) on the performance of uc is expected since these cases are closer to being uncapacitated.

Although Table 3.3 clearly shows the positive impact of exploiting the path structure together with the arc capacities, in the final computational experiment, we test the marginal contribution of \texttt{lspi} over CPLEX fixed-charge network cuts, namely flow cover, flow path and multi-commodity flow cuts. In Table 2.4, we present the results of the experiments where we use both \texttt{lspi} and CPLEX flow cover, flow path and MCF cuts under columns \texttt{lspix}, and the results with only CPLEX flow cover, flow path and MCF cuts under columns \texttt{cpx}. The positive effect of \texttt{lspi} is most apparent for smaller \( f \) values. On average, gap improvement increases by 3% when \texttt{lspi} is used. The presence of \texttt{lspi} decreases the number of branch and bound nodes explored by 50% and increases the number of instances that is solved to optimality within time/memory limit. Under the \texttt{cuts} column we present total number of cuts flow, flow path and MCF cuts added. On average, 74% of all the cuts added in \texttt{lspix} are user cuts (i.e., \texttt{lspi}). Furthermore, we observe that total number of flow, flow path and MCF cuts added decreases by 38% on average under the presence of \texttt{lspi}. Although, we observe that \texttt{lspix} took a few more seconds to terminate the algorithm on average, in 56% of the instances \texttt{lspi} terminated the algorithm faster than \texttt{cpx}.
Table 2.3: Comparison of lifted submodular path inequalities (lspi) with lifted flow cover inequalities (fc) and uncapacitated path inequalities (uc).

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<th></th>
<th>gap imp</th>
<th>cuts</th>
<th>nodes</th>
<th>time (endgap:unsolv)</th>
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<td>fc</td>
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Average 27.8 76.2 65.5 30.8 365 126 710 1019538 4484825 5109269 284 (0.0:0) 636 (0.4:1) 1342 (1.0:2)
Table 2.4: Experiments with CPLEX flow cover, flow path and MCF cuts.

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<th>cuts</th>
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Chapter 3

Path Cover and Path Pack
Inequalities for the Capacitated Fixed-Charge Network Flow Problem

Given a directed multigraph with demand or supply on the nodes, and capacity, fixed and variable cost of flow on the arcs, the capacitated fixed-charge network flow (CFNF) problem is to choose a subset of the arcs and route the flow on the chosen arcs while satisfying the supply, demand and capacity constraints, so that the sum of fixed and variable costs is minimized.

There are numerous polyhedral studies on the fixed-charge network flow problem. In a seminal paper Wolsey (1989) introduces the so-called submodular inequalities, which subsume almost all valid inequalities known for capacitated fixed-charge networks. Although the submodular inequalities are very general, their coefficients are defined implicitly through value functions. In this chapter, we give explicit valid inequalities that simultaneously make use of the path substructures of the network as well as the arc capacities.


For the capacitated fixed-charge network flow problem, almost all known valid inequalities are based on single-node relaxations. Padberg et al. (1985), van Roy and Wolsey (1986), Gu et al. (1999) give flow cover, generalized flow cover and lifted flow cover inequalities. Stallaert (1997) introduces a complementary class to generalized
flow cover inequalities and Atamtürk (2001) describes lifted flow pack inequalities. Atamtürk et al. (2016) give generalizations of flow cover inequalities from three partitions. Both uncapacitated path inequalities and capacitated flow cover and flow pack inequalities are highly valuable in solving a host of practical problems and are part of the suite of cutting planes implemented in modern mixed-integer programming solvers.


Contributions

In this chapter we consider a generic path relaxation, with supply and/or demand nodes and capacities on incoming and outgoing arcs. By exploiting the path substructure of the network and introducing notions of path cover and path pack we provide two explicitly-described subclasses of the submodular inequalities. The most important consequence of the explicit derivation is that the coefficients of the submodular inequalities on a path can be computed efficiently. In particular, we show that the coefficients of these inequalities can be computed by solving max-flow/min-cut problems parametrically over the path. Moreover, we show that all of the coefficients can be computed with a single linear-time algorithm. For a path with a single node, the inequalities reduce to the well-known flow cover and flow pack inequalities. In addition, the path cover and path pack inequalities dominate flow cover and flow pack inequalities for the corresponding single node relaxation of a path obtained by merging the path into a single node. We give necessary and sufficient facet-defining conditions. Finally, we report on computational experiments demonstrating the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm.

Outline

The remainder of this chapter is organized as follows: In Section 3.1, we describe the capacitated fixed-charge flow problem on a path, its formulation and the assumptions we make. In Section 3.2, we review the submodular inequalities, discuss their
computation on a path, and introduce two explicit subclasses: path cover inequalities and path pack inequalities. In Section 3.3, we analyze sufficient and necessary facet-defining conditions. In Section 3.4, we present computational experiments showing the effectiveness of the path cover and path pack inequalities compared to other network inequalities.

### 3.1 Capacitated fixed-charge network flow on a path

Let $G = (N', A)$ be a directed multigraph with nodes $N'$ and arcs $A$. Let $s_N$ and $t_N$ be the source and the sink nodes of $G$. Let $N := N' \setminus \{s_N, t_N\}$. Without loss of generality, we label $N := \{1, \ldots, n\}$ such that a directed forward path arc exists from node $i$ to node $i + 1$ and a directed backward path arc exists from node $i + 1$ to node $i$ for each node $i = 1, \ldots, n - 1$ (see Figure 3.1 for an illustration). In Remarks 3.1 and 3.2, we discuss how to obtain a “path” graph $G$ from a more general directed multigraph.

Let $E^+ = \{(i, j) \in A : i = s_N, j \in N\}$ and $E^- = \{(i, j) \in A : i \in N, j = t_N\}$. Moreover, let us partition the sets $E^+$ and $E^-$ such that $E^+_k = \{(i, j) \in A : i \notin N, j = k\}$ and $E^-_k = \{(i, j) \in A : i = k, j \notin N\}$ for $k \in N$. We refer to the arcs in $E^+$ and $E^-$ as non-path arcs. Finally, let $E := E^+ \cup E^-$ be the set of all non-path arcs. For convenience, we generalize this set notation scheme. Given an arbitrary subset of non-path arcs $Y \subseteq E$, let $Y_j^+ = Y \cap E_j^+$ and $Y_j^- = Y \cap E_j^-$. 

**Remark 3.1.** Given a directed multigraph $G = (\tilde{N}, \tilde{A})$ with nodes $\tilde{N}$, arcs $\tilde{A}$ and a path that passes through nodes $N$, we can construct $G$ as described above by letting $E^+ = \{(i, j) \in \tilde{A} : i \in \tilde{N} \setminus N, j \in N\}$ and $E^- = \{(i, j) \in \tilde{A} : i \in N, j \in \tilde{N} \setminus N\}$ and letting all the arcs in $E^+$ be the outgoing arcs from a dummy source $s_N$ and all the arcs in $E^-$ to be incoming to a dummy sink $t_N$.

**Remark 3.2.** If there is an arc $t = (i, j)$ from node $i \in N$ to $j \in N$, where $|i - j| > 1$, then we construct a relaxation by removing arc $t$, and replacing it with two arcs $t^- \in E_i^-$ and $t^+ \in E_j^+$. If there are multiple arcs from node $i$ to node $j$, one can repeat the same procedure.

Throughout the chapter, we use the following notation: Let $[k, j] = \{k, k+1, \ldots, j\}$ if $k \leq j$ and $\emptyset$ otherwise, $c(S) = \sum_{t \in S} c_t$, $y(S) = \sum_{t \in S} y_t$, $(a)^+ = \max\{0, a\}$ and $d_{kj} = \sum_{i=k}^j d_i$ if $j \geq k$ and 0 otherwise. Moreover, let $\dim(A)$ denote the dimension of a polyhedron $A$ and $\text{conv}(S)$ be the convex hull of a set $S$.

The capacitated fixed-charge network flow problem on a path can be formulated as a mixed-integer optimization problem. Let $d_j$ be the demand at node $j \in N$. We call a node $j \in N$ a demand node if $d_j \geq 0$ and a supply node if $d_j \leq 0$. Let the flow on forward path arc $(j, j+1)$ be represented by $i_j$ with an upper bound $u_j$ for $j \in N \setminus \{n\}$. Similarly, let the flow on backward path arc $(j + 1, j)$ be represented by $r_j$ with an upper bound $b_j$ for $j \in N \setminus \{n\}$. Let $y_t$ be the amount of flow on arc $t \in E$ with an
upper bound $c_t$. Define binary variable $x_t$ to be 1 if $y_t > 0$, and zero otherwise for all $t \in E$. An arc $t$ is closed if $x_t = 0$ and open if $x_t = 1$. Moreover, let $f_t$ be the fixed cost and $p_t$ be the unit flow cost of arc $t$. Similarly, let $h_j$ and $g_j$ be the costs of unit flow, on forward and backward arcs $(j, j+1)$ and $(j+1, j)$ respectively for $j \in N \setminus \{n\}$. Then, the problem is formulated as

$$\begin{align*}
\min & \sum_{t \in E} (f_t x_t + p_t y_t) + \sum_{j \in N} (h_j i_j + g_j r_j) \\
\text{s. t.} & \quad i_{j-1} - r_{j-1} + y(E^+_j) - y(E^-_j) - i_j + r_j = d_j, \quad j \in N, \\
& \quad 0 \leq y_t \leq c_t x_t, \quad t \in E, \\
& \quad 0 \leq i_j \leq u_j, \quad j \in N, \\
& \quad 0 \leq r_j \leq b_j, \quad j \in N, \\
& \quad x_t \in \{0, 1\}, \quad t \in E, \\
& \quad i_0 = i_n = r_0 = r_n = 0.
\end{align*}$$

(F3.1)

Let $\mathcal{P}$ be the set of feasible solutions of (F3.1). Figure 3.1 shows an example network representation of (F3.1).

Throughout we make the following assumptions on (F3.1):

(A.1) The set $\mathcal{P}_t = \{(x, y, i, r) \in \mathcal{P} : x_t = 0\} \neq \emptyset$ for all $t \in E$,

(A.2) $c_t > 0$, $u_j > 0$, $b_j > 0$ for all $t \in E$ and $j \in N$,

(A.3) $c_t \leq d_{1n} + c(E^-)$ for all $t \in E^+$,

(A.4) $c_t \leq b_{j-1} + u_j + (d_j)^+ + c(E_j^-)$, for all $j \in N, t \in E^+_j$,

(A.5) $c_t \leq b_j + u_{j-1} + (-d_j)^+ + c(E_j^+)$ for all $j \in N, t \in E^-_j$. 

39
Assumptions (A.1)–(A.2) ensure that \( \dim(\text{conv}(P)) = 2|E| + |N| - 2 \). If (A.1) does not hold for some \( t \in E \), then \( x_t = 1 \) for all points in \( P \). Similarly, if (A.2) does not hold, the flow on such an arc can be fixed to zero. Finally, assumptions (A.3)–(A.5) are without loss of generality. An upper bound on \( y_t \) can be obtained directly from the flow balance equalities (3.1b) by using the upper and lower bounds of the other flow variables that appear in the same constraint. As a result, the flow values on arcs \( t \in E \) cannot exceed the capacities implied by (A.3)–(A.5).

Next, we review the submodular inequalities that are valid for any capacitated fixed-charge network flow problem. Furthermore, using the path structure, we provide an \( O(|E| + |N|) \) time algorithm to compute their coefficients.

### 3.2 Submodular inequalities on paths

Let \( S^+ \subseteq E^+ \) and \( L^- \subseteq E^- \). Wolsey (1989) shows that the value function of the following optimization problem is submodular:

\[
v(S^+, L^-) = \max \sum_{t \in E} a_t y_t
\]

s. t. \( i_{j-1} - r_{j-1} + y(E^+_j) - y(E^-_j) - i_j + r_j \leq d_j, \quad j \in N, \) (3.2b)

\( 0 \leq i_j \leq u_j, \quad j \in N, \) (3.2c)

\( 0 \leq r_j \leq b_j, \quad j \in N, \) (3.2d)

(F3.2) \( 0 \leq y_t \leq c_t, \quad t \in E, \) (3.2e)

\( i_0 = i_n = r_0 = r_n = 0, \) (3.2f)

\( y_t = 0, \quad t \in (E^+ \setminus S^+) \cup L^-, \) (3.2g)

where \( a_t \in \{0, 1\} \) for \( t \in E^+ \) and \( a_t \in \{0, -1\} \) for \( t \in E^- \). Let \( Q \) denote the set of feasible solutions of (F3.2).

We call the sets \( S^+ \) and \( L^- \) that are used in the definition of \( v(S^+, L^-) \) the objective sets. For ease of notation, we also represent the objective sets as \( C := S^+ \cup L^- \).

Following this notation, let \( v(C) := v(S^+, L^-), v(C \setminus \{t\}) = v(S^+ \setminus \{t\}, L^-) \) for \( t \in S^+ \) and \( v(C \setminus \{t\}) = v(S^+, L^- \setminus \{t\}) \) for \( t \in L^- \). Similarly, let \( v(C \cup \{t\}) = v(S^+ \cup \{t\}, L^-) \), for \( t \in S^+ \) and \( v(C \cup \{t\}) = v(S^+, L^- \cup \{t\}) \) for \( t \in L^- \). Moreover, let

\[
\rho_t(C) = v(C \cup \{t\}) - v(C)
\]

be the marginal contribution of adding an arc \( t \) to \( C \) with respect to the value function \( v \). Wolsey (1989) shows that the following inequalities are valid for \( P \):

\[
\sum_{t \in E} a_t y_t + \sum_{t \in C} \rho_t(C \setminus \{t\})(1 - x_t) \leq v(C) + \sum_{t \in E \setminus C} \rho_t(0)x_t,
\] (3.3)
\[
\sum_{t \in E} a_t y_t + \sum_{t \in C} \rho_t (E \setminus \{t\}) (1 - \bar{x}_t) \leq v(C) + \sum_{t \in E \setminus C} \rho_t (C) \bar{x}_t,
\]

(3.4)

where the variable \( \bar{x}_t \) is defined as

\[
\bar{x}_t = \begin{cases} 
  x_t, & t \in E^+ \\
  1 - x_t, & t \in E^-.
\end{cases}
\]

In fact, inequalities (3.3) and (3.4) are also valid for fixed-charge network flow formulations where the flow balance constraints (3.1b) are replaced with constraints (3.2b). However, in this chapter, we focus on formulations with flow balance equalities (3.1b).

We refer to submodular inequalities (3.3) and (3.4) derived for path structures as path inequalities. In this chapter, we consider sets \( S^+ \) and \( L^- \) such that (F3.2) is feasible for all objective sets \( C \) and \( C \setminus \{t\} \) for all \( t \in C \).

### 3.2.1 Equivalence to the maximum flow problem

Define sets \( K^+ \) and \( K^- \) such that the coefficients of the objective function (3.2a) are:

\[
a_t = \begin{cases} 
  1, & t \in K^+ \\
  -1, & t \in K^- \\
  0, & \text{otherwise},
\end{cases}
\]

(3.5)

where \( S^+ \subseteq K^+ \subseteq E^+ \) and \( K^- \subseteq E^- \setminus L^- \). We refer to the sets \( K^+ \) and \( K^- \) as coefficient sets. Let the set of arcs with zero coefficients in (3.2a) be represented by \( \bar{K}^- = E^+ \setminus K^+ \) and \( \bar{K}^+ = E^- \setminus K^- \). Given a selection of coefficients as described in (3.5), we claim that (F3.2) can be transformed to a maximum flow problem. We first show this result assuming \( d_j \geq 0 \) for all \( j \in N \). Then, in Appendix C, we show that the nonnegativity of demand is without loss of generality for the derivation of the inequalities.

**Proposition 3.1.** Let \( S^+ \subseteq E^+ \) and \( L^- \subseteq E^- \) be the objective sets in (F3.2) and let \( Y \) be the nonempty set of optimal solutions of (F3.2). If \( d_j \geq 0 \) for all \( j \in N \), then there exists at least one optimal solution \( (y^*, r^*, i^*) \in Y \) such that \( y^*_t = 0 \) for \( t \in \bar{K}^+ \cup K^- \cup L^- \).

**Proof.** Observe that \( y^*_t = 0 \) for all \( t \in E^+ \setminus S^+ \), due to constraints (3.2g). Since \( \bar{K}^+ \subseteq E^+ \setminus S^+ \), \( y^*_t = 0 \), for \( t \in \bar{K}^+ \) from feasibility of (F3.2). Similarly, \( y^*_t = 0 \) for all \( t \in L^- \) by constraints (3.2g).

Now suppose that, \( y^*_t = \epsilon > 0 \) for some \( t \in \bar{K}^- \) (i.e., \( a_t = -1 \) for arc \( t \) in (F3.2)). Let the slack value at constraint (3.2b) for node \( j \) be

\[
s_j = d_j - [i^*_{j-1} - r^*_{j-1} + y^*(E^+_j) - y^*(E^-_j \setminus \{t\})] - y^*_t - i^*_j + r^*_j.
\]
If \( s_j \geq \epsilon \), then decreasing \( y_t^* \) by \( \epsilon \) both improves the objective function value and conserves the feasibility of flow balance inequality (3.2b) for node \( j \), since \( s_j - \epsilon \geq 0 \).

If \( s_j < \epsilon \), then decreasing \( y_t^* \) by \( \epsilon \) violates flow balance inequality since \( s_j < \epsilon < 0 \). In this case, there must exist a simple directed path \( P \) from either the source node \( s_N \) or a node \( k \in N \setminus \{j\} \) to node \( j \) where all arcs have at least a flow of \( (\epsilon - s_j) \). This is guaranteed because, \( s_j < \epsilon \) implies that, without the outgoing arc \( t \), there is more incoming flow to node \( j \) than outgoing. Then, notice that decreasing the flow on arc \( t \) and all arcs in path \( P \) by \( \epsilon - s_j \) conserves feasibility. Moreover, the objective function value either remains the same or increases, because decreasing \( y_t \) by \( \epsilon - s_j \) increases the objective function value by \( \epsilon - s_j \) and the decreasing the flow on arcs in \( P \) decreases it by at most \( \epsilon - s_j \). At the end of this transformation, the slack value \( s_j \) does not change, however; the flow at arc \( t \) is now \( y_t^* = s_j \) which is equivalent to the first case that is discussed above. As a result, we obtain a new solution to (F3.2) where \( y_t^* = 0 \) and the objective value is at least as large.

\[ \square \]

**Proposition 3.2.** If \( d_j \geq 0 \) for all \( j \in N \), then (F3.2) is equivalent to a maximum flow problem from source \( s_N \) to sink \( t_N \) on graph \( G \).

**Proof.** At the optimal solution of problem (F3.2) with objective set \( (S^+, L^-) \), the decision variables \( y_t \), for \( t \in (E^+ \setminus S^+) \cup K^- \cup L^- \) can be assumed to be zero due to Proposition 3.1 and constraints (3.2g). Then, these variables can be dropped from (F3.2) since the value \( v(S^+, L^-) \) does not depend on them and formulation (F3.2) reduces to

\[
v(S^+, L^-) = \max \left\{ y(S^+) : i_{j-1} - r_{j-1} + y(S_j^+) - y(K_j^-) - i_j + r_j \leq d_j, \quad j \in N, \right.\]

\[
\left. (3.2c) - (3.2f) \right\}. \tag{3.6}
\]

Now, we reformulate (3.6) by representing the left hand side of the flow balance constraint by a new nonnegative decision variable \( z_j \) that has an upper bound of \( d_j \) for each \( j \in N \):

\[
\max \left\{ y(S^+) : i_{j-1} - r_{j-1} + y(S_j^+) - y(K_j^-) - i_j + r_j = z_j, \quad j \in N, \right.\]

\[
\left. 0 \leq z_j \leq d_j, \quad j \in N, \quad (3.2c) - (3.2f) \right\}.
\]

The formulation above is equivalent the maximum flow formulation from the source node \( s_N \) to the sink node \( t_N \) for the path structures we are considering in this chapter.

\[ \square \]

Under the assumption that \( d_j \geq 0 \) for all \( j \in N \), Proposition 3.1 and Proposition 3.2 together show that the optimal objective function value \( v(S^+, L^-) \) can be computed by solving a maximum flow problem from source \( s_N \) to sink \( t_N \). We generalize this
result in Appendix C for node sets \( N \) such that \( d_j < 0 \) for some \( j \in N \). As a result, obtaining the explicit coefficients of submodular inequalities (3.3) and (3.4) reduces to solving \(|E| + 1\) maximum flow problems. For a general underlying graph, solving \(|E| + 1\) maximum flow problems would take \( O(|E|^2|N|) \) time (e.g., see King et al. (1994)), where \(|E|\) and \(|N|\) are the number of arcs and nodes, respectively. In the following subsection, by utilizing the equivalence of maximum flow and minimum cuts and the path structure, we show that all coefficients of (3.3) and (3.4) can be obtained in \( O(|E| + |N|) \) time using dynamic programming.

### 3.2.2 Computing the coefficients of the submodular inequalities

Throughout the chapter, we use minimum cut arguments to find the explicit coefficients of inequalities (3.3) and (3.4). Figure 3.2a illustrates an example where \( N = [1, 5] \), \( E^+ = [1, 5] \), \( E^- = [6, 10] \) and in Figure 3.2b, we give an example of an \( s_N - t_N \) cut for \( S^+ = \{2, 4, 5\} \), \( L^- = \{10\} \) and \( \bar{K}^- = \{7, 9, 10\} \). The dashed line in Figure 3.2b represents a cut that corresponds to the partition \( \{s_N, 2, 5\} \) and \( \{t_N, 1, 3, 4\} \) with a value of \( b_1 + d_2 + c_7 + u_2 + c_4 + b_4 + d_5 \). Moreover, we say that a cut passes below node \( j \) if \( j \) is in the source partition and passes above node \( j \) if \( j \) is in the sink partition.

Let \( \alpha_j^u \) and \( \alpha_j^d \) be the minimum value of a cut on nodes \([1, j]\) that passes above and below node \( j \), respectively. Similarly, let \( \beta_j^u \) and \( \beta_j^d \) be the minimum values of cuts on nodes \([j, n]\) that passes above and below node \( j \) respectively. Finally, let

\[
S^- = E^- \setminus (K^- \cup L^-),
\]
where $K^{-}$ is defined in (3.5). Recall that $S^+$ and $L^{-}$ are the given objective sets. Given the notation introduced above, all of the arcs in sets $S^-$ and $L^{-}$ have a coefficient zero in (F3.2). Therefore, dropping an arc from $L^{-}$ is equivalent to adding that arc to $S^-$. We compute $\alpha_j^{\{u,d\}}$ by a forward recursion and $\beta_j^{\{u,d\}}$ by a backward recursion:

\begin{align}
\alpha_j^{u} &= \min \{ \alpha_{j-1}^{d} + u_{j-1}, \alpha_{j-1}^{u} \} + c(S_{j}^{+}) \\
\alpha_j^{d} &= \min \{ \alpha_{j-1}^{u} + b_{j-1}, \alpha_{j-1}^{d} \} + d_j + c(S_{j}^{-}),
\end{align}

where $\alpha_0^{u} = \alpha_0^{d} = 0$ and

\begin{align}
\beta_j^{u} &= \min \{ \beta_{j+1}^{u}, \beta_{j+1}^{d} + b_{j} \} + c(S_{j}^{+}) \\
\beta_j^{d} &= \min \{ \beta_{j+1}^{u} + u_{j}, \beta_{j+1}^{d} \} + d_j + c(S_{j}^{-}),
\end{align}

where $\beta_{n+1}^{u} = \beta_{n+1}^{d} = 0$.

Let $m_j^{u}$ and $m_j^{d}$ be the values of minimum cuts for nodes $[1,n]$ that pass above and below node $j$, respectively. Notice that

\begin{equation}
m_j^{u} = \alpha_j^{u} + \beta_j^{u} - c(S_{j}^{+})
\end{equation}

and

\begin{equation}
m_j^{d} = \alpha_j^{d} + \beta_j^{d} - d_j - c(S_{j}^{-}).
\end{equation}

For convenience, let

\begin{equation}
m_j := \min \{ m_j^{u}, m_j^{d} \}.
\end{equation}

Notice that $m_j$ is the minimum of the minimum cut values that passes above and below node $j$. Since the minimum cut corresponding to $v(C)$ has to pass either above or below node $j$, $m_j$ is equal to $v(C)$ for all $j \in N$. As a result, the minimum cut (or maximum flow) value for the objective set $C = S^+ \cup L^-$ is

\begin{equation}
v(C) = m_1 = \cdots = m_n.
\end{equation}

**Proposition 3.3.** All values $m_j$, for $j \in N$, can be computed in $O(|E| + |N|)$ time.

Obtaining the explicit coefficients of inequalities (3.3) and (3.4) also requires finding $v(C \setminus \{t\})$ for $t \in C$ and $v(C \cup \{t\})$ for $t \not\in C$ in addition to $v(C)$. It is important to note that we do not need to solve the recursions above repeatedly. Once the values $m_j^{u}$ and $m_j^{d}$ are obtained for the set $C$, the marginals $\rho_t(C \setminus \{t\})$ and $\rho_t(C)$ can be found in $O(1)$ time for each $t \in E$.

We use the following observation while computing the marginal values $\rho_t(C)$ and $\rho_t(C \setminus \{t\})$ as a function of $m_j^{u}$ and $m_j^{d}$ for $t \in E_j^+ \cup E_j^-$ and $j \in N$.

**Observation 2.** Let $c \geq 0$ and $d := (b - a)^+$, then,
1. \( \min\{a + c, b\} - \min\{a, b\} = \min\{c, d\} \),

2. \( \min\{a, b\} - \min\{a, b - c\} = (c - d)^+ \).

In the remainder of this section, we give a linear-time algorithm to compute the coefficients \( \rho_t \) for inequalities (3.3) and (3.4) explicitly for paths.

**Coefficients of inequality (3.3): Path cover inequalities**

Let \( S^+ \) and \( L^- \) be the objective sets in (F3.2) and \( S^- \subseteq E^- \setminus L^- \). We select the coefficient sets in (3.5) as \( K^+ = S^+ \) and \( K^- = E^- \setminus (L^- \cup S^-) \) to obtain the explicit form of inequality (3.3). As a result, the set definition of \( S^- = E^- \setminus (K^- \cup L^-) \) is conserved.

**Definition** Let the coefficient sets in (3.5) be selected as above and \( (S^+, L^-) \) be the objective set. The set \( (S^+, S^-) \) is called a *path cover* for the node set \( N \) if

\[
\nu(S^+, L^-) = d_{1n} + c(S^-).
\]

If we assume that the set \( (S^+, S^-) \) is a path cover for \( N \), then by definition,

\[
\nu(C) = m_1 = \cdots = m_n = d_{1n} + c(S^-)
\]

in inequality (3.3). After obtaining the values \( m_j^u \) and \( m_j^d \) for a node \( j \in N \) using recursions in (4.3)–(4.6), it is trivial to find the minimum cut value after dropping an arc \( t \) from \( S_j^+ \):

\[
\nu(C \setminus \{t\}) = \min\{m_j^u - c_t, m_j^d\}, \quad t \in S_j^+, \quad j \in N.
\]

Similarly, dropping an arc \( t \in L_j^- \) results in the minimum cut value:

\[
\nu(C \setminus \{t\}) = \min\{m_j^u, m_j^d + c_t\}, \quad t \in L_j^-, \quad j \in N.
\]

Using Observation 2, we obtain the marginal values

\[
\rho_t(C \setminus \{t\}) = (c_t - \lambda_j)^+, \quad t \in S_j^+, \quad j \in N
\]

and

\[
\rho_t(C \setminus \{t\}) = \min\{\lambda_j, c_t\}, \quad t \in L_j^-, \quad j \in N
\]

where

\[
\lambda_j = (m_j^u - m_j^d)^+, \quad j \in N.
\]

On the other hand, all the coefficients \( \rho_t(\emptyset) = 0 \) for arcs \( t \in E \setminus C \). First, notice that, for \( t \in E^+ \setminus S^+ \), \( \nu(\{t\}) = 0 \), because the coefficient \( a_t = 0 \) for \( t \in E^+ \setminus S^+ \).
Furthermore, \( v(\{t\}) = 0 \) for \( t \in E^- \setminus L^- \), since all incoming arcs would be closed for an objective set \((\emptyset, \{t\})\). As a result, inequality (3.3) for the objective set \((S^+, L^-)\) can be written as

\[
y(S^+) + \sum_{j \in N} \sum_{t \in S^+_j} (c_t - \lambda_j)^+ (1 - x_t) \leq d_{1n} + c(S^-) \\
+ \sum_{j \in N} \sum_{t \in L^-_j} \min \{c_t, \lambda_j\} x_t + y(E^- \setminus (L^- \cup S^-)). \quad (3.14)
\]

We refer to inequalities (3.14) as \textit{path cover inequalities}.

\textbf{Remark 3.3.} Observe that for a path consisting of a single node \( N = \{j\} \) with demand \( d := d_j > 0 \), the path cover inequalities (3.14) reduce to the flow cover inequalities (Padberg et al., 1985, van Roy and Wolsey, 1986). Suppose that the path consists of a single node \( N = \{j\} \) with demand \( d := d_j > 0 \). Let \( S^+ \subseteq E^+ \) and \( S^- \subseteq E^- \). The set \((S^+, S^-)\) is a flow cover if \( \lambda := c(S^+) - d - c(S^-) > 0 \) and the resulting path cover inequality

\[
y(S^+) + \sum_{t \in S^+} (c_t - \lambda)^+ (1 - x_t) \leq d + c(S^-) + \lambda x(L^-) + y(E^- \setminus L^-) \quad (3.15)
\]

is a flow cover inequality.

\textbf{Proposition 3.4.} Let \((S^+, S^-)\) be a path cover for the node set \( N \). The path cover inequality for node set \( N \) is at least as strong as the flow cover inequality for the single node relaxation obtained by merging the nodes in \( N \).

\textit{Proof.} Flow cover and path cover inequalities differ in the coefficients of variables \( x_t \) for \( t \in S^+ \) and \( t \in L^- \). Therefore, we compare the values \( \lambda_j, j \in N \) of path cover inequalities (3.14) to the value \( \lambda \) of flow cover inequalities (3.15) and show that \( \lambda_j \leq \lambda \), for all \( j \in N \). The merging of node set \( N \) in graph \( G \) is equivalent to relaxing the values \( u_j \) and \( b_j \) to be infinite for \( j \in [1, n-1] \). As a result, the value of the minimum cut that goes above the merged node is \( \bar{m}^u = c(S^+) \) and the value of the minimum cut that goes below the merged node is \( \bar{m}^d = d_{1n} + c(S^-) \). Now, observe that the recursions in (4.3)–(4.6) imply that the minimum cut values for the original graph \( G \) are smaller:

\[
m^u_j = \alpha^u_j + \beta^u_j - c(S^+_j) \leq c(S^+) = \bar{m}^u
\]

and

\[
m^d_j = \alpha^d_j + \beta^d_j - d_j - c(S^-_j) \leq d_{1n} + c(S^-) = \bar{m}^d
\]

for all \( j \in N \). Recall that the coefficient for the flow cover inequality is \( \lambda = (\bar{m}^u - \bar{m}^d)^+ \) and the coefficients for path cover inequality are \( \lambda_j = (m^u_j - m^d_j)^+ \) for \( j \in N \). The fact that \((S^+, S^-)\) is a path cover implies that \( m^d_j = d_{1n} + c(S^-) \) for all \( j \in N \).
Since $\bar{m} = m_j^d$ and $m_j^u \leq \bar{m}^u$ for all $j \in N$, we observe that $\lambda_j \leq \lambda$ for all $j \in N$. Consequently, the path cover inequality (3.14) is at least as strong as the flow cover inequality (3.15).

Example 3.1. Consider the lot-sizing instance in Figure 3.3 where $N = [1, 4], S^+ = \{2, 3\}$, $L^- = \emptyset$. Observe that $m_1^u = 45$, $m_1^d = 40$, $m_2^u = 65$, $m_2^d = 40$, $m_3^u = 60$, $m_3^d = 40$, and $m_4^u = 45$, $m_4^d = 40$. Then, $\lambda_1 = 5$, $\lambda_2 = 25$, $\lambda_3 = 20$, and $\lambda_4 = 5$ leading to coefficients 10 and 10 for $(1 - x_2)$ and $(1 - x_3)$, respectively. Furthermore, the maximum flow values are $v(C) = 40$, $v(C \setminus \{2\}) = 30$, and $v(C \setminus \{3\}) = 30$. Then, the resulting path cover inequality (3.14) is

$$y_2 + y_3 + 10(1 - x_2) + 10(1 - x_3) \leq 40,$$  
and it is facet-defining for $\text{conv}(P)$ as will be shown in Section 3.3. Now, consider the relaxation obtained by merging the nodes in $[1, 4]$ into a single node with incoming arcs $\{1, 2, 3, 4\}$ and demand $d = 40$. As a result, the flow cover inequalities can be applied to the merged node set. The excess value for the set $S^+ = \{2, 3\}$ is $\lambda = c(S^+) - d = 25$. Then, the resulting flow cover inequality (3.15) is

$$y_2 + y_3 + 10(1 - x_2) + 5(1 - x_3) \leq 40,$$
and it is weaker than the path cover inequality (3.16).

**Coefficients of inequality (3.4): Path pack inequalities**

Let $S^+$ and $L^-$ be the objective sets in (F3.2) and let $S^- \subseteq E^- \setminus L^-$. We select the coefficient sets in (3.5) as $K^+ = E^+$ and $K^- = E^- \setminus (S^- \cup L^-)$ to obtain the explicit form of inequality (3.4). As a result, the set definition of $S^- = E^- \setminus (K^- \cup L^-)$ is conserved.

**Definition** Let the coefficients in (3.5) be selected as above and $(S^+, L^-)$ be the objective set. The set $(S^+, S^-)$ is called a **path pack** for node set $N$ if

$$v(S^+, L^-) = c(S^+).$$
For inequality (3.4), we assume that the set \((S^+, S^-)\) is a path pack for \(N\) and \(L^- = \emptyset\) for simplicity. Now, we need to compute the values of \(v(C)\), \(v(E)\), \(v(E \setminus \{t\})\) for \(t \in C\) and \(v(C \cup \{t\})\) for \(t \in E \setminus C\). The value of \(v(C \cup \{t\})\) can be obtained using the values \(m^u_j\) and \(m^d_j\) that are given by recursions (4.3)–(4.6). Then,

\[
v(C \cup \{t\}) = \min\{m^u_j + c_t, m^d_j\}, \quad t \in E_j^+ \setminus S_j^+, \quad j \in N
\]

and

\[
v(C \cup \{t\}) = \min\{m^u_j, m^d_j + c_t\}, \quad t \in S_j^-, \quad j \in N.
\]

Then, using Observation 2, we compute the marginal values

\[
\rho_t(C) = \min\{c_t, \mu_j\}, \quad t \in E_j^+ \setminus S_j^+, \quad j \in N
\]

and

\[
\rho_t(C) = (c_t - \mu_j)^+, \quad t \in S_j^-, \quad j \in N
\]

where

\[
\mu_j = (m^d_j - m^u_j)^+, \quad j \in N.
\]

Next, we compute the values \(v(E)\) and \(v(E \setminus \{t\})\) for \(t \in C\). The feasibility of (F3.1) implies that \((E^+, \emptyset)\) is a path cover for \(N\).

By Assumption (A.1), \((E^+ \setminus \{t\}, \emptyset)\) is also a path cover for \(N\) for each \(t \in S^+\). Then

\[
v(E) = v(E \setminus \{t\}) = d_{1t},
\]

and

\[
\rho_t(E \setminus \{t\}) = 0, \quad t \in S^+ \cup L^-.
\]

Then, inequality (3.4) can be explicitly written as

\[
y(S^+) + \sum_{j \in N} \sum_{t \in E_j^+ \setminus S_j^+} (y_t - \min\{c_t, \mu_j\} x_t) \leq c(S^+)
\]

\[
+ y(E^- \setminus S^-) - \sum_{j \in N} \sum_{t \in S_j^-} (c_t - \mu_j)^+(1 - x_t). \quad (3.17)
\]

We refer to inequalities (3.17) as path pack inequalities.

**Remark 3.4.** Observe that for a path consisting of a single node \(N = \{j\}\) with demand \(d := d_j > 0\), the path pack inequalities (3.17), reduce to the flow pack inequalities (Atamtürk, 2001). Let \((S^+, S^-)\) be a flow pack and \(\mu := d - c(S^+) + c(S^-) > 0\). Moreover, the maximum flow that can be sent through \(S^+\) for demand \(d\) and arcs in \(S^-\) is \(c(S^+)\). Then, the value function \(v(S^+) = c(S^+)\) and the resulting path pack inequality
\[
y(S^+) + \sum_{t \in E^+ \setminus S^+} (y_t - \min\{c_t, \mu\} x_t) \leq c(S^+)
\]
\[
+ y(E^- \setminus S^-) - \sum_{t \in S^-} (c_t - \mu)^+ (1 - x_t)
\]
\[(3.18)\]
is equivalent to the flow pack inequality.

**Proposition 3.5.** Let \((S^+, S^-)\) be a path pack for the node set \(N\). The path pack inequality for \(N\) is at least as strong as the flow pack inequality for the single node relaxation obtained by merging the nodes in \(N\).

**Proof.** The proof is similar to that of Proposition 3.4. Flow pack and path pack inequalities only differ in the coefficients of variables \(x_t\) for \(t \in E^+ \setminus S^+\) and \(t \in S^-\). Therefore, we compare the values \(\mu_j, j \in N\) of path pack inequalities (3.17) to the value \(\mu\) of flow pack inequalities (3.18) and show that \(\mu_j \leq \mu\) for all \(j \in N\). For the single node relaxation, the values of the minimum cuts that pass above and below the merged node are \(\bar{m}^u = c(S^+)\) and \(\bar{m}^d = d_{1n} + c(S^-)\), respectively. The recursions in (4.3)–(4.6) imply that

\[
m_j^u = \alpha_j^u + \beta_j^u - c(S_j^+) \leq c(S^+) = \bar{m}^u
\]

and

\[
m_j^d = \alpha_j^d + \beta_j^d - d_j - c(S_j^-) \leq d_{1n} + c(S^-) = \bar{m}^d.
\]
The coefficient for flow pack inequality is \(\mu = (\bar{m}^d - \bar{m}^u)^+\) and for path pack inequality \(\mu_j = (m_j^d - m_j^u)^+\). Since \((S^+, S^-)\) is a path pack, the minimum cut passes above all nodes in \(N\) and \(m_j^d = c(S^+)\) for all \(j \in N\). As a result, \(m_j^u = \bar{m}^u\) for all \(j \in N\) and \(m_j^d \leq \bar{m}^d\). Then, observe that the values

\[
\mu_j \leq \mu, \quad j \in N.
\]

□

**Example 1** (continued). Recall the lot-sizing instance with backlogging given in Figure 3.3. Let the node set \(N = [1, 4]\) with \(E^- = \emptyset\) and \(S^+ = \{3\}\). Then, \(m_1^u = 30, m_1^d = 40, m_2^u = 30, m_2^d = 40, m_3^u = 30, m_3^d = 30, m_4^u = 30, m_4^d = 30\), leading to \(\mu_1 = 10, \mu_2 = 10, \mu_3 = 0\) and \(\mu_4 = 0\). Moreover, the maximum flow values are \(v(C) = 30, v(C \cup \{1\}) = 40, v(C \cup \{2\}) = 40, v(C \cup \{4\}) = 30, v(E) = 40\), and \(v(E \setminus \{3\}) = 40\). Then the resulting path pack inequality (3.17) is

\[
y_1 + y_2 + y_3 + y_4 \leq 30 + 10x_1 + 10x_2
\]
\[(3.19)\]
and it is facet-defining for \(\text{conv}(P)\) as will be shown in Section 3.3. Now, suppose that the nodes in \([1, 4]\) are merged into a single node with incoming arcs \(\{1, 2, 3, 4\}\)
and demand \( d = 40 \). For the same set \( S^+ \), we get \( \mu = 40 - 30 = 10 \). Then, the corresponding flow pack inequality (3.18) is

\[
y_1 + y_2 + y_3 + y_4 \leq 30 + 10x_1 + 10x_2 + 10x_4,
\]

which is weaker than the path pack inequality (3.19).

\[\square\]

**Proposition 3.6.** If \( |E^+ \setminus S^+| \leq 1 \) and \( S^- = \emptyset \), then inequalities (3.14) and (3.17) are equivalent.

**Proof.** If \( E^+ \setminus S^+ = \emptyset \) and \( S^- = \emptyset \), then it is easy to see that the coefficients of inequality (3.17) are the same as (3.14). Moreover, if \( |E^+ \setminus S^+| = 1 \) (and wlog \( E^+ \setminus S^+ = \{j\} \)), then the resulting inequality (3.17) is

\[
y(E^+) - y(E^-) \leq v(C) + \rho_j(C)x_j
\]

\[
= v(C) + (v(C \cup \{j\}) - v(C))x_j
\]

\[
= v(C \cup \{j\}) - \rho_j(C)(1 - x_j),
\]

which is equivalent to path cover inequality (3.14) with the objective set \((E^+, \emptyset)\). \[\square\]

### 3.3 The strength of the path cover and pack inequalities

The capacities of the forward and the backward path arcs play an important role in finding the coefficients of the path cover and pack inequalities (3.14) and (3.17). Recall that \( K^+ \) and \( K^- \) are the coefficient sets in (3.5), \((S^+, L^-)\) is the objective set for (F3.2) and \( S^- = E^- \setminus (K^- \cup L^-) \).

**Definition** A node \( j \in N \) is called **backward independent** for set \((S^+, S^-)\) if

\[
\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+),
\]

or

\[
\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-).
\]

**Definition** A node \( j \in N \) is called **forward independent** for set \((S^+, S^-)\) if

\[
\beta_j^u = \beta_{j+1}^d + b_j + c(S_j^+),
\]

or

\[
\beta_j^d = \beta_{j+1}^u + u_j + d_j + c(S_j^-).
\]
Intuitively, backward independence of node \( j \in N \) implies that the minimum cut either passes through the forward path arc \((j - 1, j)\) or through the backward path arc \((j, j - 1)\). Similarly, forward independence of node \( j \in N \) implies that the minimum cut either passes through the forward path arc \((j, j + 1)\) or through the backward path arc \((j + 1, j)\). In Lemmas 3.1 and 3.2 below, we further explain how forward and backward independence affect the coefficients of path cover and pack inequalities.

First, let \( S^+_j = \bigcup_{i=j}^{N} S^+_i \), \( S^-_j = \bigcup_{i=j}^{N} S^-_i \) and \( L_{jk} = \bigcup_{i=j}^{k} L^-_i \) if \( j \leq k \), and \( \emptyset \) otherwise.

**Lemma 3.1.** If a node \( j \in N \) is backward independent for set \((S^+, S^-)\), then the values \( \lambda_j \) and \( \mu_j \) do not depend on the sets \( S^+_{1j-1} \), \( S^-_{1j-1} \) and the value \( d_{1j-1} \).

**Proof.** If a node \( j \) is backward independent, then either \( \alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j) \) or \( \alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_j + c(S^-_j) \). If \( \alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j) \), then the equality in (4.3) implies \( \alpha^u_{j-1} + u_{j-1} \leq \alpha^u_{j-1} \). As a result, the equality in (4.4) gives \( \alpha^d_j = \alpha^d_{j-1} + d_j + c(S^-_j) \).

Following the definitions in (4.7)–(4.8), the difference

\[
 w_j := m^u_j - m^d_j 
\]

is \( \beta^u_j - \beta^d_j + u_{j-1} \) which only depends on sets \( S^+_k \) and \( S^-_k \) for \( k \in [j, n] \), the value \( d_{jn} \) and the capacity of the forward path arc \((j - 1, j)\).

If \( \alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_j + c(S^-_j) \), then the equality in (4.4) implies \( \alpha^u_{j-1} + b_{j-1} \leq \alpha^u_{j-1} \). As a result, the equality in (4.3) gives \( \alpha^u_j = \alpha^u_{j-1} + c(S^+_j) \). Then, the difference \( w_j = \beta^u_j - \beta^d_j - b_{j-1} \) which only depends on sets \( S^+_k \) and \( S^-_k \) for \( k \in [j, n] \), the value \( d_{jn} \) and the capacity of the backward path arc \((j, j - 1)\).

Since the values \( \lambda_j \) and \( \mu_j \) are defined as \( (w_j)^+ \) and \( (-w_j)^+ \) respectively, the result follows. \( \square \)

**Remark 3.5.** Let \( w_j := m^u_j - m^d_j \). If a node \( j \in N \) is backward independent for a set \((S^+, S^-)\), then we observe the following: (1) If \( \alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j) \), then

\[
 w_j = \beta^u_j - \beta^d_j + u_{j-1},
\]

and (2) if \( \alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_j + c(S^-_j) \), then

\[
 w_j = \beta^u_j - \beta^d_j - b_{j-1}.
\]

**Lemma 3.2.** If a node \( j \in N \) is forward independent for set \((S^+, S^-)\), then the values \( \lambda_j \) and \( \mu_j \) do not depend on the sets \( S^+_{j+1n} \), \( S^-_{j+1n} \) and the value \( d_{j+1n} \).

**Proof.** The forward independence implies either \( \beta^u_j = \beta^d_{j+1} + b_j + c(S^+_j) \) and \( \beta^d_j = \beta^d_{j+1} + d_j + c(S^-_j) \) or \( \beta^u_j = \beta^u_{j+1} + c(S^+_j) \) and \( \beta^d_j = \beta^d_{j+1} + u_j + d_j + c(S^-_j) \). Then, the difference \( w_j = m^u_j - m^d_j \) is either \( \alpha^u_j + \alpha^d_j + b_j \) or \( \alpha^u_j + \alpha^d_j - u_j \) and in both cases, it is independent of the sets \( S^+_k, S^-_k \) for \( k \in [1, j - 1] \) and the value \( d_{1j-1} \). \( \square \)
**Remark 3.6.** Let $w_j := m_{j}^u - m_{j}^d$. If a node $j \in N$ is forward independent for a set $(S^+, S^-)$, then we observe the following: (1) If $\beta_{j}^u = \beta_{j+1}^d + b_j + c(S_j^+)$, then
\[ w_j = \alpha_{j}^u - \alpha_{j}^d + b_j, \]
and (2) if $\beta_{j}^d = \beta_{j+1}^u + u_j + d_j + c(S_j^-)$, then
\[ w_j = \alpha_{j}^u - \alpha_{j}^d - u_j. \]

**Corollary 3.7.** If a node $j \in N$ is backward independent for set $(S^+, S^-)$, then the values $\lambda_k$ and $\mu_k$ for $k \in [j, n]$ are also independent of the sets $S_{ij-1}^+$, $S_{ij-1}^-$ and the value $d_{ij-1}$. Similarly, if a node $j \in N$ is forward independent for set $(S^+, S^-)$, then the values $\lambda_k$ and $\mu_k$ for $k \in [1, j]$ are also independent of the sets $S_{j+1n}^+$, $S_{j+1n}^-$ and the value $d_{j+1n}$.

**Proof.** The proof follows from recursions (4.3)–(4.6). If a node $j$ is backward independent, we write $\alpha_{j+1}^u$ and $\alpha_{j+1}^d$ in terms of $\alpha_{j}^u$ and $\alpha_{j}^d$ and observe that the difference $w_{j+1} = m_{j+1}^u - m_{j+1}^d$ does not depend on $\alpha_{j}^u$ nor $\alpha_{j}^d$ which implies independence of sets $S_{ij-1}^+$, $S_{ij-1}^-$ and the value $d_{ij-1}$. We can repeat the same argument for $w_j$, $j \in [j+2, n]$ to show independence.

We show the same result for forward independence by writing $\beta_{j-1}^u$ and $\beta_{j-1}^d$ in terms of $\beta_{j}^u$ and $\beta_{j}^d$, we observe that $w_j$ does not depend on $\beta_{j}^u$ nor $\beta_{j}^d$. Then, it is clear that $w_{j-1}$ is also independent of the sets $S_{j+1n}^+$, $S_{j+1n}^-$ and the value $d_{j+1n}$. We can repeat the same argument for $w_j$, $j \in [1, j-1]$ to show independence. \hfill \Box

Proving the necessary facet conditions frequently requires a partition of the node set $N$ into two disjoint sets. Suppose, $N$ is partitioned into $N_1 = [1, j - 1]$ and $N_2 = [j, n]$ for some $j \in N$. Let $E_{N1}$ and $E_{N2}$ be the set of non-path arcs associated with node sets $N_1$ and $N_2$. We consider the forward and backward path arcs $(j-1, j)$ and $(j, j-1)$ to be in the set of non-path arcs $E_{N1}$ and $E_{N2}$ since the node $j-1 \in N_i$ and $j \notin N_i$ for $i = 1, 2$. In particular, $E_{N1}^+ := (j-1, j) \cup E_{1j-1}^+$, $E_{N1}^- := (j-1, j) \cup E_{1j-1}^-$ and $E_{N2}^+ := (j-1, j) \cup E_{jn}^+$, $E_{N2}^- := (j-1, j) \cup E_{jn}^-$, where $E_{k\ell}^+$ and $E_{k\ell}^-$ are defined as $\bigcup_{i=k}^{\ell} E_i^+$ and $\bigcup_{i=k}^{\ell} E_i^-$ if $k \leq \ell$ respectively, and as the empty set otherwise. Since the path arcs for $N$ do not have associated fixed-charge variables, one can assume that there exists auxiliary binary variables $\bar{x}_k = 1$ for $k \in \{(j-1, j), (j, j-1)\}$. Moreover, we partition the sets $S^+, S^-$ and $L^-$ into $S_{N1}^+ \supseteq S_{1j-1}^+$, $S_{N1}^- \supseteq S_{1j-1}^-$, $L_{N1}^- := L_{1j-1}^-$ and $S_{N2}^+ \supseteq S_{jn}^+$, $S_{N2}^- \supseteq S_{jn}^-$, $L_{N2}^- := L_{jn}^-$. Then, let $v_1$ and $v_2$ be the value functions defined in (F3.2) for the node sets $N_1$ and $N_2$ and the objective sets $(S_{N1}^+, L_{N1}^-)$ and $(S_{N2}^+, L_{N2}^-)$. Moreover, let $\alpha_{j}^u$, $\alpha_{j}^d$, $\beta_{j}^u$ and $\beta_{j}^d$ be defined for $j \in N$ in recursions (4.3)–(4.6) for the set $(S^+, S^-)$ and recall that $S^- = E^- \setminus (K^- \cup L^-)$.

**Lemma 3.3.** Let $(S^+, L^-)$ be the objective set for the node set for $N = [1, n]$. If
\[ \alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j) \text{ or } \beta^u_j = \beta^d_{j-1} + b_{j-1} + c(S^-_{j-1}), \text{ then} \]
\[ v(S^+, L^-) = v_1(S^+_{N_1}, L^-_{N_1}) + v_2(S^+_{N_2}, L^-_{N_2}), \]
where \( N_1 = [1, j-1], N_2 = [j, n] \) and the arc sets are \( S^+_{N_1} = (j, j-1) \cup S^+_{1j-1}, \]
\( S^+_{N_2} = (j-1, j) \cup S^+_{jn}, S^-_{N_1} = S^-_{1j-1}, S^-_{N_2} = S^-_{jn}. \)

**Proof.** See Appendix D.1.

**Lemma 3.4.** Let \((S^+, L^-)\) be the objective set for the node set for \( N = [1, n] \). If
\[ \alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_{j-1} + c(S^-_{i-j}) \text{ or } \beta^d_j = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1}), \text{ then} \]
\[ v(S^+, L^-) = v_1(S^+_{N_1}, L^-_{N_1}) + v_2(S^+_{N_2}, L^-_{N_2}), \]
where \( N_1 = [1, j-1], N_2 = [j, n] \) and the arc sets are \( S^+_{N_1} = S^+_{1j-1}, S^+_{N_2} = S^+_{jn}, \]
\( S^-_{N_1} = (j-1, j) \cup S^-_{1j-1}, S^-_{N_2} = (j, j-1) \cup S^-_{jn}. \)

**Proof.** See Appendix D.2.

**Lemma 3.6.** Let \((S^+, L^-)\) be the objective set for the node set for \( N = [1, n] \). If
\[ \alpha^d_j = \alpha^u_{j-1} + u_{j-1} + c(S^+_{j}) \text{ and } \beta^d_j = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1}), \text{ then} \]
\[ v(S^+, L^-) = v_1(S^+_{N_1}, L^-_{N_1}) + v_2(S^+_{N_2}, L^-_{N_2}), \]
where \( N_1 = [1, j-1], N_2 = [j, n] \) and the arc sets are \( S^+_{N_1} = S^+_{1j-1}, S^+_{N_2} = (j-1, j) \cup S^+_{jn}, \]
\( S^-_{N_1} = S^-_{1j-1}, S^-_{N_2} = (j, j-1) \cup S^-_{jn}. \)

**Proof.** See Appendix D.3.

In the remainder of this section, we give necessary and sufficient conditions for path cover and pack inequalities (3.14) and (3.17) to be facet-defining for the convex hull of \( \mathcal{P} \).

**Theorem 3.8.** Let \( N = [1, n], \) and \( d_j \geq 0 \) for all \( j \in N \). If \( L^- = \emptyset \) and the set \((S^+, S^-)\) is a path cover for \( N \), then the following conditions are necessary for path cover inequality (3.14) to be facet-defining for \( \text{conv}(\mathcal{P}) \):
(i) \( \rho_t(C \setminus \{t\}) < c_t \), for all \( t \in C \),

(ii) \( \max_{t \in S^+} \rho_t(C \setminus \{t\}) > 0 \),

(iii) if a node \( j \in [2, n] \) is forward independent for set \((S^+, S^-)\), then node \( j - 1 \) is not backward independent for set \((S^+, S^-)\),

(iv) if a node \( j \in [1, n - 1] \) is backward independent for set \((S^+, S^-)\), then node \( j + 1 \) is not forward independent for set \((S^+, S^-)\),

(v) if \( \max_{t \in S^+} (c_t - \lambda_i) > 0 \) for \( i = p, \ldots, n \) for some \( p \in [2, n] \), then the node \( p - 1 \) is not forward independent for \((S^+, S^-)\),

(vi) if \( \max_{t \in S^+} (c_t - \lambda_i) > 0 \) for \( i = 1, \ldots, q \) for some \( q \in [1, n - 1] \), then the node \( q + 1 \) is not backward independent for \((S^+, S^-)\).

**Proof.**

(i) If for some \( t' \in S^+ \), \( \rho_{t'}(C \setminus \{t'\}) \geq c_{t'} \), then the path cover inequality with the objective set \( S^+ \setminus \{t'\} \) summed with \( y_{t'} \leq c_{t'}x_{t'} \) results in an inequality at least as strong. Rewriting the path cover inequality using the objective set \( S^+ \), we obtain

\[
\sum_{t \in S^+ \setminus \{t'\}} (y_t + \rho_t(S^+ \setminus \{t\})(1 - x_t)) + y_{t'} \leq v(S^+) - \rho_{t'}(S^+ \setminus \{t'\})(1 - x_{t'}) + y(E^+ \setminus S^-) \]

\[
= v(S^+ \setminus \{t'\}) + \rho_{t'}(S^+ \setminus \{t'\})x_{t'} + y(E^+ \setminus S^-).
\]

Now, consider summing the path cover inequality for the objective set \( S^+ \setminus \{t'\} \)

\[
\sum_{t \in S^+ \setminus \{t'\}} (y_t + \rho_t(S^+ \setminus \{t, t'\})(1 - x_t)) \leq v(S \setminus \{t'\}) + y(E^+ \setminus S^-),
\]

and \( y_{t'} \leq c_{t'}x_{t'} \). The resulting inequality dominates inequality (3.3) because \( \rho_t(S^+ \setminus \{t\}) \leq \rho_t(S^+ \setminus \{t, t'\}) \), from the submodularity of the set function \( v \). If the assumption of \( L^- = \emptyset \) is dropped, this condition extends for arcs \( t \in L^- \) as \( \rho_t(C \setminus \{t\}) > -c_t \) with a similar proof.

(ii) If \( L^- = \emptyset \) and \( \max_{t \in S^+} \rho_t(C \setminus \{t\}) = 0 \), then summing flow balance equalities (3.1b) for all nodes \( j \in N \) gives an inequality at least as strong.

(iii) Suppose a node \( j \) is forward independent for \((S^+, S^-)\) and the node \( j - 1 \) is backward independent for \((S^+, S^-)\) for some \( j \in [2, n] \). Lemmas 3.3–3.6 show that the nodes \( N \) and the arcs \( C = S^+ \cup L^- \) can be partitioned into \( N_1 = [1, j - 1] \), \( N_2 = [j, n] \) and \( C_1, C_2 \) such that the sum of the minimum cut values for \( N_1, N_2 \) is equal to the minimum cut for \( N \). From Remarks 3.5 and 3.6 and Corollary 3.7, it is easy to see that \( \lambda_i \) for \( i \in N \) will not change by the partition procedures.
described in Lemmas 3.3–3.6. We examine the four cases for node \( j - 1 \) to be forward independent and node \( j \) to be backward independent for the set \((S^+, S^-)\).

(a) Suppose \( \alpha_j^u = \alpha_j^d + u_j - 1 + c(S_j^+) \) and \( \beta_j^{-} = \beta_j^d + b_j - 1 + c(S_{j-1}^+) \). Consider the partition procedure described in Lemma 3.3, where \( S_{N_1}^+ = (j, j - 1) \cup S_{j-1}^+ \), \( S_{N_2}^+ = (j - 1, j) \cup S_{jn}^+ \), \( S_{N_1}^- = S_{ij-1}^- \), \( S_{N_2}^- = S_{jn}^- \). Then, the path cover inequalities for nodes \( N_1 \) and \( N_2 \)

\[
\sum_{i=1}^{j-1} \sum_{t \in S_i^+} y_t + (c_t - \lambda_i)^+(1 - x_t) \leq v_1(S_{N_1}^+) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-) + i_{j-1}
\]

and

\[
\sum_{i=j}^{n} \sum_{t \in S_i^+} y_t + (c_t - \lambda_i)^+(1 - x_t) \leq v_2(S_{N_2}^+) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-) + r_{j-1}
\]

summed gives

\[
\sum_{i=1}^{n} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_i)^+(1 - x_t)) \leq v(S^+) + y(E^- \setminus S^-),
\]

which is the path cover inequality for \( N \) with the objective set \( S^+ \).

(b) Suppose \( \alpha_j^d = \alpha_j^d + u_j - 1 + d_j - 1 + c(S_j^-) \) and \( \beta_j^{-} = \beta_j^d + u_j - 1 + d_j - 1 + c(S_{j-1}^-) \). Consider the partition described in Lemma 3.4, where \( S_{N_1}^+ = S_{ij-1}^+ \), \( S_{N_2}^+ = S_{jn}^+ \), \( S_{N_1}^- = (j - 1, j) \cup S_{ij-1}^- \), \( S_{N_2}^- = (j, j - 1) \cup S_{jn}^- \). The path cover inequalities for nodes \( N_1 \) and \( N_2 \)

\[
\sum_{i=1}^{j-1} \sum_{t \in S_i^+} y_t + (c_t - \lambda_i)^+(1 - x_t) \leq v_1(S_{N_1}^+) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-)
\]

and

\[
\sum_{i=j}^{n} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_i)^+(1 - x_t)) \leq v_2(S_{N_2}^+) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-).
\]

summed gives the path cover inequality for nodes \( N \) and arcs \( C \).

(c) Suppose \( \alpha_j^u = \alpha_j^d + u_j - 1 + c(S_j^+) \) and \( \beta_j^{-} = \beta_j^d + u_j - 1 + d_j - 1 + c(S_{j-1}^-) \). Consider the partition described in Lemma 3.5, where \( S_{N_1}^+ = S_{ij-1}^+ \), \( S_{N_2}^+ = (j - 1, j) \cup S_{jn}^+ \), \( S_{N_1}^- = S_{ij-1}^- \), \( S_{N_2}^- = (j, j - 1) \cup S_{jn}^- \). The path cover inequalities
for nodes $N_1$ and $N_2$

$$\sum_{i=1}^{j-1} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_t)^+(1 - x_t)) \leq v_1(S_{N1}^+) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-) + i_{j-1}$$

and

$$i_{j-1} + \sum_{i=j}^{n} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_t)^+(1 - x_t)) \leq v_2(S_{N2}^+) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-).$$

summed gives the path cover inequality for nodes $N$ and arcs $C$.

(d) Suppose $a_j^d = a_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$ and $\beta_{j-1}^u = \beta_{j-1}^d + b_{j-1} + c(S_{j-1}^+)$.

Consider the partition described in Lemma 3.6, where $S_{N1}^+ = (j, j-1) \cup S_{1j-1}^+$, $S_{N2}^+ = S_{jn}^+$, $S_{N1}^- = (j-1, j) \cup S_{1j-1}^-$, $S_{N2}^- = S_{jn}^-$. The path cover inequalities for nodes $N_1$ and $N_2$

$$r_{j-1} + \sum_{i=1}^{j-1} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_t)^+(1 - x_t)) \leq v_1(S_{N1}^+) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-)$$

and

$$\sum_{i=j}^{n} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_t)^+(1 - x_t)) \leq v_2(S_{N2}^+) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-) + r_{j-1}.$$

summed gives the path cover inequality for nodes $N$ and arcs $C$.

(iv) The same argument for condition (iii) above also proves the desired result here.

(v) Suppose $(c_t - \lambda_t)^+ = 0$ for all $t \in S_i^+$ and $i \in [p, n]$ and the node $p-1$ is forward independent for some $p \in [2, n]$. Then, we partition the node set $N = [1, n]$ into $N_1 = [1, p-1]$ and $N_2 = [p, n]$. We follow Lemma 3.3 if $\beta_{p-1}^u = \beta_{p-1}^d + b_{p-1} + c(S_{p-1}^+)$ and follow Lemma 3.4 if $\beta_{p-1}^d = \beta_{p-1}^u + u_{p-1} + d_{p-1} + c(S_{p-1}^-)$ to define $S_{N1}^+$, $S_{N1}^-$, $S_{N2}^+ = S_{N2}^-$, and $S_{N2}^-$. Remark 3.6 along with the partition procedure described in Lemma 3.3 or 3.4 implies that $\lambda_i$ will remain unchanged for $i \in N_1$. The path cover inequality for nodes $N$ and arcs $C$ is

$$y(S^+) + \sum_{i=1}^{p-1} \sum_{t \in S_i^+} (c_t - \lambda_t)^+(1 - x_t) \leq v(S^+) + y(E^- \setminus S^-).$$

If $\beta_{p-1}^u = \beta_{p-1}^d + b_{p-1} + c(S_{p-1}^+)$, then the path cover inequality for nodes $N_1$ and

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arcs $S^+_{N_1}, S^-_{N_1}$ described in Lemma 3.3 is

$$r_{p-1} + \sum_{i=1}^{p-1} \sum_{t \in S^+_i} (y_t + (c_t - \lambda_t)^+(1 - x_t)) \leq v(S^+_{N_1}) + \sum_{i=1}^{p-1} y(E_i^- \setminus S^-_i) + i_{p-1}. $$

Moreover, let $\bar{m}^u_p$ and $\bar{m}^d_p$ be the values of minimum cut that goes above and below node $p$ for the node set $N_2$ and arcs $S^+_{N_2}, S^-_{N_2}$ and observe that

$$\bar{m}^u_p = \beta^u_p + u_{p-1} \quad \text{and} \quad \bar{m}^d_p = \beta^d_p.$$ 

Then, comparing the difference $\bar{\lambda}_p := (\bar{m}^u_p - \bar{m}^d_p)^+ = (\beta^u_p - \beta^d_p + u_{p-1})^+$ to $\lambda_p = (m^u_p - m^d_p)^+ = (\beta^u_p - \beta^d_p + \alpha^u_p - \alpha^d_p + c(S^+_p) - d_p - c(S^-_p))^+$, we observe that $\bar{\lambda}_p \leq \lambda_p$ since $\alpha^u_p - \alpha^d_p + c(S^+_p) - d_p - c(S^-_p) \leq u_{p-1}$ from (4.3)–(4.4). Since $(c_t - \lambda_t)^+ = 0$, then $(c_t - \bar{\lambda}_t)^+ = 0$ as well. Using the same technique, it is easy to observe that $\bar{\lambda}_i \geq \lambda_i$ for $i \in [p + 1, n]$ as well. As a result, the path cover inequality for $N_2$ with sets $S^+_{N_2}, S^-_{N_2}$ is

$$i_{p-1} + \sum_{i=p}^n y(S^+_i) \leq v(S^+_{N_2}) + \sum_{i=p}^n y(E_i^- \setminus S^-_i) + r_{p-1}. $$

The path cover inequalities for $N_1, S^+_{N_1}, S^-_{N_1}$ and for $N_2, S^+_{N_2}, S^-_{N_2}$ summed gives the path cover inequality for $N, S^+, S^-$. Similarly, if $\beta^d_{p-1} = \beta^u_{p-1} + d_{p-1} + c(S^-_{p-1})$, the proof follows very similarly to the previous argument using Lemma 3.4. Letting $\bar{m}^u_p$ and $\bar{m}^d_p$ be the values of minimum cut that goes above and below node $p$ for the node set $N_2$ and arcs $S^+_{N_2}, S^-_{N_2}$, we get

$$\bar{m}^u_p = \beta^u_p \quad \text{and} \quad \bar{m}^d_p + b_{p-1} = \beta^d_p$$

under this case. Now, notice that $\alpha^u_p - \alpha^d_p + c(S^+_p) - d_p - c(S^-_p) \geq -b_{p-1}$ from (4.3)–(4.4), which leads to $\bar{\lambda}_p \leq \lambda_p$. Then the proof follows same as above.

(vi) The proof is similar to that of the necessary condition (v). We use Lemmas 3.5 and 3.6 and Remark 3.6 to partition the node set $N$ and arcs $S^+, S^-$ into node sets $N_1 = [1, q]$ and $N_2 = [q + 1, n]$ for $q \in [2, n]$ and arcs $S^+_{N_1}, S^-_{N_1}$ and $S^+_{N_2}, S^-_{N_2}$. Next, we check the values of minimum cut that goes above and below node $q$ for the node set $N_1$ and arcs $S^+_{N_1}, S^-_{N_1}$. Then, observing $-u_q \leq \beta^u_q - \alpha^d_q + c(S^+_q) - d_q - c(S^-_q) \leq b_q$ from (4.5)–(4.6), it is easy to show that the coefficients $x_t$ for $t \in S^+_{N_1}$ are equal to zero in the path cover inequality for node set $N_1$. As a result, the path cover inequalities for $N_1, S^+_{N_1}, S^-_{N_1}$ and for $N_2, S^+_{N_2}, S^-_{N_2}$ summed gives the path cover inequality for $N, S^+, S^-$.  

\[\square\]
Remark 3.7. If the node set $N$ consists of a single node, then the conditions (i) and (ii) of Theorem 3.8 reduce to the sufficient facet conditions of flow cover inequalities (Padberg et al., 1985, Theorem 4), (van Roy and Wolsey, 1986, Theorem 6). In this setting, conditions (iii)-(vi) are no longer relevant.

Theorem 3.9. Let $N = [1, n]$, $E^- = \emptyset$, $d_j > 0$ and $|E_j^+| = 1$, for all $j \in N$ and let the set $S^+$ be a path cover. The necessary conditions in Theorem 3.8 along with

(i) $(c_t - \lambda_j)^+ > 0$ for all $t \in S_j^+$, $j \in N$,

(ii) $(c_t - \lambda_j)^+ < c(E^+ \setminus S^+)$ for all $t \in S_j^+$, $j \in N$

are sufficient for path cover inequality (3.14) to be facet-defining for $\text{conv}(\mathcal{P})$.

Proof. Recall that $\text{dim}(\text{conv}(\mathcal{P})) = 2|E| + n - 2$. In this proof, we provide $2|E| + n - 2$ affinely independent points that lie on the face $F$

$$F = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{r}) \in \mathcal{P} : y(S^+) + \sum_{t \in S^+} (c_t - \lambda_j)^+(1 - x_t) = d_{1n} \right\}.$$

First, we provide Algorithm 1 which outputs an initial feasible solution $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{r})$, where all the arcs in $S^+$ have non-zero flow. Let $\tilde{d}_j$ be the effective demand on node $j$, that is, the sum of $d_j$ and the minimal amount of flow that needs to be sent from the arcs in $S_j^+$ to ensure $\nu(S^+) = d_{1n}$. In Algorithm 1, we perform a backward pass and a forward pass on the nodes in $N$. This procedure is carried out to obtain the minimal amounts of flow on the forward and backward path arcs to satisfy the demands. For each node $j \in N$, these minimal outgoing flow values added to the demand $d_j$ give the effective demand $\tilde{d}_j$.

Algorithm 1 ensures that at most one of the path arcs $(j - 1, j)$ and $(j, j - 1)$ have non-zero flow for all $j \in [2, n]$. Moreover, note that sufficient condition (i) ensures that all the arcs in $S^+$ have nonzero flow. Moreover, for at least one node $i \in N$, it is guaranteed that $c(S_i^+) > \tilde{d}_i$. Otherwise, $\rho_t(C) = c_t$ for all $t \in S^+$ which contradicts the necessary condition (i). Necessary conditions (iii) and (iv) ensure that $\tilde{t}_j < u_j$ and $\tilde{r}_j < b_j$ for all $j = 1, \ldots, n - 1$. Let

$$e := \arg \max_{i \in N} \{ c(S_i^+) - \tilde{d}_i \}$$

be the node with the largest excess capacity. Also let $\mathbf{1}_j$ be the unit vector with 1 at position $j$.

Next, we give $2|S^+|$ affinely independent points represented by $\bar{w}^t = (\mathbf{x}^t, \mathbf{y}^t, \mathbf{i}^t, \mathbf{r}^t)$ and $\tilde{w}^t = (\mathbf{x}^t, \mathbf{y}^t, \tilde{i}^t, \tilde{r}^t)$ for $t \in S^+$:

(i) Select $\bar{w}^e = (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{r})$ given by Algorithm 1. Let $\varepsilon > 0$ be a sufficiently small value. We define $\bar{w}^t$ for $e \neq t \in S^+$ as $\mathbf{y}^t = \mathbf{y}^e + \varepsilon \mathbf{1}_e - \varepsilon \mathbf{1}_t$, $\mathbf{x}^t = \mathbf{x}^e$. If $t < e$, then $\tilde{i}^t = \mathbf{i}^e$ and $\tilde{r}_j^t = \tilde{r}_j^e + \varepsilon$ for $j < t$ and for $t \geq e$, $\tilde{r}_j^t = \tilde{r}_j^e + \varepsilon$ for $t \leq j < e$.
In this class of affinely independent solutions, we close the arcs in $S^+$ one at a time and open all the arcs in $E^+ \setminus S^+$. Let $\mathbf{x}^t = \mathbf{x} - 1_t + \sum_{j \in E^+ \setminus S^+} 1_j$. Next, we send an additional $\mathbf{y}_j - (c_t - \lambda_j)^+$ amount of flow from the arcs in $S^+ \setminus \{t\}$. This is a feasible operation because $v(C \setminus \{t\}) = d_{1n} - (c_t - \lambda_j)^+$. Let $(\mathbf{y}^*, \mathbf{r}^*, \mathbf{x}^*)$ be the optimal solution of (F3.2) corresponding to $v(S^+ \setminus \{t\})$. Then let, $\mathbf{y}^j_t = y^j_{t_j}$ for $j \in S^+ \setminus \{t\}$. Since $v(C \setminus \{t\}) < d_{1n}$, additional flow must be sent through nodes in $E^+ \setminus S^+$ to satisfy flow balance equations (3.1b). This is also a feasible operation, because of assumption (A.1). Then, the forward and backward path flows $\mathbf{i}^t$ and $\mathbf{r}^t$ are calculated using the flow balance equations.

In the next set of solutions, we give $2|E^+ \setminus S^+| - 1$ affinely independent points represented by $\mathbf{w}^t = (\mathbf{x}^t, \mathbf{y}^t, \mathbf{i}^t, \mathbf{r}^t)$ and $\mathbf{w}^t = (\mathbf{x}^t, \mathbf{y}^t, \mathbf{i}^t, \mathbf{r}^t)$ for $t \in E^+ \setminus S^+$.

(iii) Starting with solution $\mathbf{w}^e$, we open arcs in $E^+ \setminus S^+$, one by one. $\mathbf{y}^t = \mathbf{y}^e$, $\mathbf{x}^t = \mathbf{x}^e + 1_t$, $\mathbf{i}^t = \mathbf{i}^e$, $\mathbf{r}^t = \mathbf{r}^e$.

(iv) If $|E^+ \setminus S^+| \geq 2$, then we can send a sufficiently small $\varepsilon > 0$ amount of flow from arc $t \in E^+ \setminus S^+$ to $t \neq k \in E^+ \setminus S^+$. Let this set of affinely independent points be represented by $\mathbf{w}^t$ for $t \in E^+ \setminus S^+$. While generating $\mathbf{w}^t$, we start with the solution $\mathbf{w}^e$, where the non-path arc in $S^+_e$ is closed. The feasibility of this operation is guaranteed by the sufficiency conditions (ii) and necessary conditions (iii) and (iv).

(a) If $\mathbf{y}^e_i = c_t$, then there exists at least one arc $t \neq m \in E^+ \setminus S^+$ such that $0 \leq \mathbf{y}^e_m < c_m$ due to sufficiency assumption (ii), then for each $t \in E^+ \setminus S^+$
such that $\tilde{y}_t^e = c_t$, let $\tilde{y}^t = \tilde{y}^e - \varepsilon 1_t + \varepsilon 1_m$, $\tilde{x}^t = \tilde{x}^e$. If $t < m$, then $\tilde{i}^t = \tilde{i}^e$ and $\tilde{r}^t = \tilde{r}^e + \varepsilon \sum_{i=t}^{m-1} 1_i$. If $t > m$, then $\tilde{i}^t = \tilde{i}^e + \varepsilon \sum_{i=t}^{m-1} 1_i$ and $\tilde{r}^t = \tilde{r}^e$.

(b) If $\tilde{y}_t^e < c_t$ and there exists at least one arc $t \neq m \in E^+ \setminus S^+$ such that $\tilde{y}_m^e = 0$, then the same point described in (a) is feasible.

(c) If $\tilde{y}_t^e < c_t$ and there exists at least one arc $t \neq m \in E^+ \setminus S^+$ such that $\tilde{y}_m^e = c_m$, then, we send $\varepsilon$ amount of flow from $t$ to $m$, $\tilde{y}^t = \tilde{y}^e + \varepsilon 1_t - \varepsilon 1_m$, $\tilde{x}^t = \tilde{x}^e$. If $t < m$, then $\tilde{i}^t = \tilde{i}^e + \varepsilon \sum_{i=t}^{m-1} 1_i$ and $\tilde{r}^t = \tilde{r}^e$. If $t > m$, then $\tilde{i}^t = \tilde{i}^e$ and $\tilde{r}^t = \tilde{r}^e + \varepsilon \sum_{i=m}^{t-1} 1_i$.

Finally, we give $n - 1$ points that perturb the flow on the forward path arcs $(j,j + 1)$ for $j = 1, \ldots, n - 1$ represented by $\tilde{w}^j = (\tilde{x}^j, \tilde{y}^j, \tilde{i}^j, \tilde{r}^j)$. Let $k = \min\{i \in N : S_i^+ \neq \emptyset\}$ and $\ell = \max\{i \in N : S_i^+ \neq \emptyset\}$. The solution given by Algorithm 1 guarantees $i_j < u_j$ and $r_j < b_j$ for $j = 1, \ldots, n - 1$ due to necessary conditions (iii) and (iv).

(v) For $j = 1, \ldots, n - 1$, we send an additional $\varepsilon$ amount of flow from the forward path arc $(j,j + 1)$ and the backward path arc $(j + 1, j)$. Formally, the solution $\tilde{w}^j$ can be obtained by: $\tilde{y}^j = \tilde{y}^e$, $\tilde{x}^j = \tilde{x}^e$, $\tilde{i}^j = \tilde{i}^e + \varepsilon 1_j$ and $\tilde{r}^j = \tilde{r}^e + \varepsilon 1_j$.

$\Box$

Next, we identify conditions under which path pack inequality (3.17) is facet-defining for $\text{conv}(\mathcal{P})$.

**Theorem 3.10.** Let $N = [1,n]$, $d_j \geq 0$ for all $j \in N$, let the set $(S^+, S^-)$ be a path pack and $L^- = \emptyset$. The following conditions are necessary for path pack inequality (3.17) to be facet-defining for $\text{conv}(\mathcal{P})$:

(i) $\rho_j(S^+) < c_j$, for all $j \in E^+ \setminus S^+$,

(ii) $\max_{t \in S^-} \rho_t(C) > 0$,

(iii) if a node $j \in [2,n]$ is forward independent for set $(S^+, S^-)$, then node $j - 1$ is not backward independent for set $(S^+, S^-)$,

(iv) if a node $j \in [1,n - 1]$ is backward independent for set $(S^+, S^-)$, then node $j + 1$ is not forward independent for set $(S^+, S^-)$,

(v) if $\max_{t \in E^+ \setminus S_i^+} \rho_t(C) = 0$ and $\max_{t \in S_i^-} \rho_t(C) = 0$ for $i = p, \ldots, n$ for some $p \in [2,n]$, then the node $p - 1$ is not forward independent for $(S^+, S^-)$,

(vi) if $\max_{t \in E^+ \setminus S_i^+} \rho_t(C) = 0$ and $\max_{t \in S_i^-} \rho_t(C) = 0$ for $i = 1, \ldots, q$ for some $q \in [1, n - 1]$, then the node $q + 1$ is not backward independent for $(S^+, S^-)$.
Proof. (i) Suppose that for some \(k \in E^+ \setminus S^+\), \(\rho_k(S^+) = c_k\). Then, recall the implicit form of path pack inequality (3.17) is

\[
y(E^+ \setminus \{k\}) + y_k + \sum_{t \in S^-} \rho_t(S^+)(1 - x_t) \leq v(S^+) + \sum_{k \neq t \in E^+ \setminus S^+} \rho_t(S^+)x_t + c_kx_k + y(E^- \setminus S^-).\]

Now, if we select \(a_k = 0\) in (F3.2), then the coefficients of \(x_k\) and \(y_k\) become zero and summing the path cover inequality

\[
y(E^+ \setminus \{k\}) + \sum_{t \in S^-} \rho_t(S^+)(1 - x_t) \leq v(S^+) + \sum_{k \neq t \in E^+ \setminus S^+} \rho_t(S^+)x_t + y(E^- \setminus S^-).\]

with \(y_k \leq c_kx_k\) gives the first path cover inequality.

(ii) Suppose that \(\rho_j(S^+) = 0\) for all \(j \in S^-\). Then the path pack inequality is

\[
y(E^+) \leq v(S^+) + \sum_{t \in E^+ \setminus S^+} \rho_t(S^+)x_t + y(E^- \setminus (L^- \cup S^-)),\]

where \(L^- = \emptyset\). If an arc \(j\) is dropped from \(S^-\) and added to \(L^-\), then \(v(S^+) = v(S^+ \cup \{j\})\) since \(\rho_j(S^+) = 0\) for \(j \in S^-\). Consequently, the path pack inequality with \(S^- = S^- \setminus \{j\}\) and \(L^- = \{j\}\)

\[
y(E^+) + \sum_{t \in S^-} \rho_t(S^+ \cup \{j\})(1 - x_t) \leq v(S^+) + \sum_{t \in E^+ \setminus S^+} \rho_t(S^+ \cup \{j\})x_t + y(E^- \setminus (L^- \cup S^-))\]

But since \(0 \leq \rho_t(S^+ \cup \{j\}) \leq \rho_t(S^+)\) from submodularity of \(v\) and \(\rho_t(S^+) = 0\) for all \(t \in S^-\), we observe that the path pack inequality above reduces to

\[
y(E^+) \leq v(S^+) + \sum_{t \in E^+ \setminus S^+} \rho_t(S^+ \cup \{j\})x_t + y(E^- \setminus (L^- \cup S^-))\]

and it is at least as strong as the first pack inequality for \(S^+, S^-\) and \(L^- = \emptyset\).

(iii)–(iv) We repeat the same argument of the proof of condition (iii) of Theorem 3.8. Suppose a node \(j\) is forward independent for \((S^+, S^-)\) and the node \(j - 1\) is backward independent for \((S^+, S^-)\) for some \(j \in [2, n]\). Lemmas 3.3–3.6 show that the nodes \(N\) and the arcs \(C = S^+ \cup L^-\) can be partitioned into \(N_1 = [1, j-1]\), \(N_2 = [j, n]\) and \(C_1, C_2\) such that the sum of the minimum cut values for \(N_1, N_2\) is equal to the minimum cut for \(N\). From Remarks 3.5 and 3.6 and Corollary 3.7, it is easy to see that \(\mu_i\) for \(i \in N\) will not change by the partition procedures described in Lemmas 3.3–3.6. We examine the four cases for node \(j - 1\) to be forward independent and node \(j\) to be backward independent for the set \((S^+, S^-)\).
For ease of notation, let

\[ Q_{jk}^+ := \sum_{i=j}^{k} \sum_{t \in E_i^+ \setminus S_i^+} (y_t - \min\{\mu_i, c_t\} x_t) \]

and

\[ Q_{jk}^- := \sum_{i=j}^{k} \sum_{t \in S_i^-} (c_t - \mu_i)^+(1 - x_t) \]

for \( j \leq k \) and \( j \in N; k \in N \) (and zero if \( j > k \)), where the values \( \mu_i \) are the coefficients that appear in the path pack inequality (3.17). As a result, the path pack inequality can be written as

\[ y(S^+) + Q_{1n}^+ \leq v(C) + Q_{1n}^- + y(E^+ \setminus S^-). \tag{3.20} \]

(a) Suppose \( \alpha_j^d = \alpha_j^d + u_{j-1} + c(S_j^+) \) and \( \beta_j^u = \beta_j^u + b_{j-1} + c(S_j^{-1}) \). Consider the partition procedure described in Lemma 3.3, where \( S_{N1}^+ = (j, j-1) \cup S_{1j-1}^+, S_{N2}^+ = (j-1, j) \cup S_{jn}^+ \), \( S_{N1}^- = S_{1j-1}^-, S_{N2}^- = S_{jn}^- \). Then, the path pack inequalities for nodes \( N_1 \) is

\[ r_{j-1} + y(S_{1j-1}^+) + Q_{1j-1}^+ \leq v_1(C_1) + Q_{1j-1}^- + y(E_{1j-1}^- \setminus S_{1j-1}^-) + i_{j-1}. \tag{3.21} \]

Similarly, the path pack inequality for \( N_2 \) is

\[ i_{j-1} + y(S_{jn}^+) + Q_{jn}^+ \leq v_2(C_2) + Q_{jn}^- + y(E_{jn}^- \setminus S_{jn}^-) + r_{j-1}. \tag{3.22} \]

Inequalities (3.21)–(3.22) summed gives the path pack inequality (3.20).

(b) Suppose \( \alpha_j^d = \alpha_j^d - u_{j-1} + d_j + c(S_j^-) \) and \( \beta_j^u = \beta_j^u + u_{j-1} + d_j + c(S_j^{-1}) \). Consider the partition described in Lemma 3.4, where \( S_{N1}^+ = S_{1j-1}^+, S_{N2}^+ = S_{jn}^+, S_{N1}^- = (j-1, j) \cup S_{1j-1}^-, S_{N2}^- = (j, j-1) \cup S_{jn}^- \). The submodular inequality (3.4) for nodes \( N_1 \) where the objective coefficients of (F3.2) are selected as \( a_t = 1 \) for \( t \in E_{1j-1}^+ \), \( a_t = 0 \) for \( t = (j, j-1) \), \( a_t = -1 \) for \( t \in E_{N1}^- \setminus S_{N1}^- \) and \( a_t = 0 \) for \( t \in S_{N1}^- \) is

\[ y(S_{1j-1}^+) + \sum_{t \in S_{N1}^-} k_t (1 - x_t) + Q_{1j-1}^+ \leq v_1(C_1) - Q_{1j-1}^- + y(E_{1j-1}^- \setminus S_{1j-1}^-), \tag{3.23} \]

where \( k_t \) for \( t \in S_{N1}^- \) are some nonnegative coefficients. Similarly, the submodular inequality (3.4) for nodes \( N_2 \), where the objective coefficients of (F3.2) are selected as \( a_t = 1 \) for \( t \in E_{jn}^+ \), \( a_t = 0 \) for \( t = (j-1, j) \), \( a_t = -1 \)
for \( t \in E_{N_2}^- \setminus S_{N_2}^- \) and \( a_t = 0 \) for \( t \in S_{N_2}^- \) is

\[
y(S_{jn}^+) + \sum_{t \in S_{N_2}^+} k_t (1 - x_t) + Q_{jn}^+ \leq v_2(C_2) - Q_{jn}^- + y(E_{jn}^- \setminus S_{jn}^-), \tag{3.24}
\]

where \( k_t \) for \( t \in S_{N_2}^+ \) are some nonnegative coefficients. The sum of inequalities (3.23)–(3.24) is at least as strong as the path pack inequality (3.20).

(c) Suppose \( \alpha_j^a = \alpha_{j-1}^d + u_{j-1} + c(S_j^+) \) and \( \beta_{j-1}^d = \beta_j^a + u_{j-1} + d_{j-1} + c(S_{j-1}^-) \).
Consider the partition described in Lemma 3.5, where \( S_{N_1}^+ = S_{1j-1}^+ \), \( S_{N_2}^+ = (j-1,j) \cup S_{jn}^+ \), \( S_{N_1}^- = S_{1j-1}^- \), \( S_{N_2}^- = (j,j-1) \cup S_{jn}^- \). The submodular inequality (3.4) for nodes \( N_1 \) where the objective coefficients of (F3.2) are selected as \( a_t = 1 \) for \( t \in E_{1j-1}^+ \), \( a_t = 0 \) for \( t = (j,j-1) \), \( a_t = -1 \) for \( t \in E_{N_1}^- \setminus S_{N_1}^- \) and \( a_t = 0 \) for \( t \in S_{N_1}^- \) is

\[
y(S_{1j-1}^+ - 1) + \sum_{t \in S_{N_1}^+} k_t (1 - x_t) + Q_{1j-1}^+ \leq v_1(C_1) - Q_{1j-1}^- + y(E_{1j-1}^- \setminus S_{1j-1}^-) + i_{j-1}, \tag{3.25}
\]

where \( k_t \) for \( t \in S_{N_1}^+ \) are some nonnegative coefficients. The path pack inequality for \( N_2 \) is

\[
i_{j-1} + y(S_{jn}^+) + Q_{jn}^+ \leq v_2(C_2) + Q_{jn}^- + y(E_{jn}^- \setminus S_{jn}^-). \tag{3.26}
\]

The sum of inequalities (3.25)–(3.26) is at least as strong as inequality (3.20).

(d) Suppose \( \alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-) \) and \( \beta_{j-1}^u = \beta_j^d + b_{j-1} + c(S_{j-1}^+) \).
Consider the partition described in Lemma 3.6, where \( S_{N_1}^+ = (j-1,j) \cup S_{1j-1}^+ \), \( S_{N_2}^+ = S_{jn}^+ \), \( S_{N_1}^- = (j-1,j) \cup S_{1j-1}^- \), \( S_{N_2}^- = S_{jn}^- \). The path pack inequalities for nodes \( N_1 \) is

\[
r_{j-1} + y(S_{1j-1}^+) + Q_{1j-1}^+ \leq v_1(C_1) - Q_{1j-1}^- + y(E_{1j-1}^- \setminus S_{1j-1}^-). \tag{3.27}
\]

The submodular inequality (3.4) for nodes \( N_2 \) where the objective coefficients of (F3.2) are selected as \( a_t = 1 \) for \( t \in E_{jn}^+ \), \( a_t = 0 \) for \( t = (j-1,j) \), \( a_t = -1 \) for \( t \in E_{N_2}^- \setminus S_{N_2}^- \) and \( a_t = 0 \) for \( t \in S_{N_2}^- \) is

\[
y(S_{jn}^+) + \sum_{t \in S_{N_2}^+} k_t (1 - x_t) + Q_{jn}^+ \leq v_2(C_2) - Q_{jn}^- + y(E_{jn}^- \setminus S_{jn}^-) + r_{j-1}, \tag{3.28}
\]

where \( k_t \) for \( t \in S_{N_2}^+ \) are some nonnegative coefficients. The sum of inequalities (3.27)–(3.28) is at least as strong as the path pack inequality (3.20).
Suppose $(c_t - \mu_i)^+ = 0$ for all $t \in S^{-i}$ and $i \in [p, n]$ and the node $p - 1$ is forward independent. Then, we partition the node set $N = [1, n]$ into $N_1 = [1, p - 1]$ and $N_2 = [p, n]$. We follow Lemma 3.3 if $\beta_{p-1}^u = \beta_p^u + b_{p-1} + c(S_{p-1}^r)$ and follow Lemma 3.4 if $\beta_{p-1}^d = \beta_p^d + u_{p-1} + d_{p-1} + c(S_{p-1}^r)$ to define $S^+_{N_1}, S^-_{N_1}, S^+_{N_2}$ and $S^-_{N_2}$. Remark 3.6 along with the partition procedure described in Lemma 3.3 or 3.4 implies that $\mu_i$ will remain unchanged for $i \in N_1$.

If $\beta_{p-1}^u = \beta_p^d + b_{p-1} + c(S_{p-1}^r)$, then the coefficients $\mu_i$ of the path pack inequality for nodes $N_1$ and arcs $S^+_{N_1}, S^-_{N_1}$ described in Lemma 3.3 is the same as the coefficients of the path pack inequality for nodes $N$ and arcs $S^+, S^-$. Moreover, let $\tilde{m}_p^u$ and $\tilde{m}_p^d$ be the values of minimum cut that goes above and below node $p$ for the node set $N_2$ and arcs $S^+_{N_2}, S^-_{N_2}$ and observe that

$$\tilde{m}_p^u = \beta_p^u + u_{p-1} \quad \text{and} \quad \tilde{m}_p^d = \beta_p^d.$$

Then, comparing the difference $\bar{\mu}_p := (\tilde{m}_p^d - \tilde{m}_p^u)^+ = (\beta_p^d - \beta_p^u - u_{p-1})^+$ to $\mu_p = (m_p^d - m_p^u)^+ = (\beta_p^d - \beta_p^u + \alpha_p^d - \alpha_p^u - c(S_p^r) + d_p + c(S_p^r))^+$, we observe that $\bar{\mu}_p \geq \mu_p$ since $\alpha_p^d - \alpha_p^u - c(S_p^r) + d_p + c(S_p^r) \geq -u_{p-1}$ from (4.3)-(4.4). Since $(c_t - \mu_p)^+ = 0$, then $(c_t - \bar{\mu}_p)^+ = 0$ as well. Using the same technique, it is easy to observe that $\bar{\mu}_i \geq \mu_i$ for $i \in [p + 1, n]$ as well. As a result, the path pack inequality for $N_2$ with sets $S^+_{N_2}, S^-_{N_2}$, summed with the path pack inequality for nodes $N_1$ and arcs $S^+_{N_1}, S^-_{N_1}$ give the path pack inequality for nodes $N$ and arc $S^+, S^-$. Similarly, if $\beta_{p-1}^d = \beta_p^u + u_{p-1} + d_{p-1} + c(S_{p-1}^r)$, the proof follows very similarly to the previous argument using Lemma 3.4. Letting $\tilde{m}_p^u$ and $\tilde{m}_p^d$ be the values of minimum cut that goes above and below node $p$ for the node set $N_2$ and arcs $S^+_{N_2}, S^-_{N_2}$, we get

$$\tilde{m}_p^u = \beta_p^u \quad \text{and} \quad \tilde{m}_p^d + b_{p-1} = \beta_p^d$$

under this case. Now, notice that $\alpha_p^d - \alpha_p^u - c(S_p^r) + d_p + c(S_p^r) \leq u_{p-1}$ from (4.3)-(4.4), which leads to $\bar{\mu}_p \geq \mu_p$. Then the proof follows same as above.

(vi) The proof is similar to that of the necessary condition (v) above. We use Lemmas 3.5–3.6 and Remark 3.6 to partition the node set $N$ and arcs $S^+, S^-$ into node sets $N_1 = [1, q]$ and $N_2 = [q + 1, n]$ and arcs $S^+_{N_1}, S^-_{N_1}$ and $S^+_{N_2}, S^-_{N_2}$. Next, we check the values of minimum cut that goes above and below node $q$ for the node set $N_1$ and arcs $S^+_{N_1}, S^-_{N_1}$. Then, observing $-b_q \leq \beta_q^d - \alpha_q^u - c(S_q^r) + d_q + c(S_q^r) \leq u_q$ from (4.5)-(4.6), it is easy to see that the coefficients of $x_t$ for $t \in S_{N_1}^r$ and $t \in E_{N_1} \setminus S_{N_1}^r$ are equal to zero in the path pack inequality for node set $N_1$. As a result, the path pack inequalities for $N_1, S^+_{N_1}, S^-_{N_1}$ and for $N_2, S^+_{N_2}, S^-_{N_2}$ summed gives the path pack inequality for $N, S^+, S^-$. □
Remark 3.8. If the node set $N$ consists of a single node, then the conditions (i) and (ii) of Theorem 3.10 reduce to the necessary and sufficient facet conditions of flow pack inequalities (Atamtürk, 2001, Proposition 1). In this setting, conditions (iii)–(vi) are no longer relevant.

Theorem 3.11. Let $N = [1, n]$, $E^- = \emptyset$, $d_j > 0$ and $|E_j^+| = 1$, for all $j \in N$ and let the objective set $S^+$ be a path pack for $N$. The necessary conditions in Theorem 3.10 along with

(i) for each $j \in E^+ \setminus S^+$, either $S^+ \cup \{j\}$ is a path cover for $N$ or $\rho_j(S^+) = 0$,

(ii) for each $t \in S^+$, there exists $j_t \in E^+ \setminus S^+$ such that $\sum S^+ \setminus \{t\} \cup \{j_t\}$ is a path cover for $N$,

(iii) for each $j \in [1, n - 1]$, there exists $k_j \in E^+ \setminus S^+$ such that the set $S^+ \cup \{k_j\}$ is a path cover and neither node $j$ is backward independent nor node $j + 1$ is forward independent for the set $S^+ \cup \{k_j\}$

are sufficient for path pack inequality (3.17) to be facet-defining for $\text{conv}(\mathcal{P})$.

Proof. We provide $2|E| + n - 2$ affinely independent points that lie on the face:

$$F = \left\{ (x, y, i, r) \in \mathcal{P} : y(S^+) + \sum_{t \in E^+ \setminus S^+} (y_t - \min \{\mu_j, c_t\} x_t) = c(S^+) \right\}.$$ 

Let $(y^*, i^*, r^*) \in Q$ be an optimal solution to (F3.2). Since $S^+$ is a path pack and $E^- = \emptyset$, $v(S^+) = c(S^+)$. Then, notice that $y_t^* = c_t$ for all $t \in S^+$. Moreover, let $e$ be the arc with largest capacity in $S^+$, $\varepsilon > 0$ be a sufficiently small value and $1_j$ be the unit vector with 1 at position $j$. First, we give $2|E^+ \setminus S^+|$ affinely independent points represented by $\tilde{z}^t = (x^t, y^t, i^t, r^t)$ and $\hat{z}^t = (\hat{x}^t, \hat{y}^t, \hat{i}^t, \hat{r}^t)$ for $t \in E^+ \setminus S^+$.

(i) Let $t \in E^+ \setminus S^+$, where $S^+ \cup \{t\}$ is a path cover for $N$. The solution $\tilde{z}^t$ has arcs in $S^+ \cup \{t\}$ open, $x_j^t = 1$ for $j \in S^+ \cup \{t\}$, 0 otherwise, $y_j^t = y_j^*$ for $j \in S^+$ and $\rho_j(S^+) = 0$ otherwise. The forward and backward path arc flow values $i_j^t$ and $\hat{i}_j^t$ can then be calculated using flow balance equalities (3.1b) where at most one of them can be nonzero for each $j \in N$. Sufficiency condition (i) guarantees the feasibility of $\tilde{z}^t$.

(ii) Let $t \in E^+ \setminus S^+$, where $\rho_j(S^+) = 0$ and let $t \neq \ell \in E^+ \setminus S^+$, where $S^+ \cup \{\ell\}$ is a path cover for $N$. The solution $\tilde{z}^t$ has arcs in $S^+ \cup \{t, \ell\}$ open, $x_j^t = 1$ for $j \in S^+ \cup \{t, \ell\}$, and 0 otherwise, $y_j^t = y_j^*$ for $j \in S^+$, $y_j^\ell = 0$, $\rho_j(S^+) = 0$ otherwise. The forward and backward path arc flow values $\hat{i}_j^t$ and $\hat{i}_j^t$ can then be calculated using flow balance equalities (3.1b) where at most one of them can be nonzero for each $j \in N$. Sufficiency condition (i) guarantees the feasibility of $\hat{z}^t$. 

65
(iii) The necessary condition (i) ensures that \( \rho_i(S^+) < c_i \), therefore \( \bar{y}_i^t < c_i \). In solution \( \bar{z}^t \), starting with \( \bar{z}_i^t \), we send a flow of \( \varepsilon \) from arc \( t \in E^+ \setminus S^+ \) to \( e \in S^+ \). Let \( \bar{y}_i^t = y_i^t + \varepsilon 1_i - \varepsilon 1_e \) and \( \bar{x}_i^t = \bar{x}_i \). If \( e < t \), then \( \bar{r}_i^t = r_i^t + \varepsilon \sum_{i=t}^{t-1} 1_i \), \( \bar{t}_i^t = \bar{t}_i^t \) and if \( e > t \), then \( \bar{r}_i^t = \bar{r}_i^t + \varepsilon \sum_{i=t}^{t-1} 1_i \), \( \bar{r}_i^t = \bar{r}_i^t \).

Next, we give \( 2|S^+| - 1 \) affinely independent feasible points \( \bar{z}^t \) and \( \bar{z}^t \) corresponding to \( t \in S^+ \) that are on the face \( F \). Let \( k \) be the arc in \( E^+ \setminus S^+ \) with largest capacity.

(iv) In the feasible solutions \( \bar{z}^t \) for \( e \neq t \in S^+ \), we open arcs in \( S^+ \cup \{k\} \) and send an \( \varepsilon \) flow from arc \( k \) to arc \( t \). Let \( \bar{y}_i^t = y_i^k + \varepsilon 1_k - \varepsilon 1_t \) and \( \bar{x}_i^t = \bar{x}_i^k \). If \( t < k \), then \( \bar{r}_i^t = \bar{r}_i^k + \varepsilon \sum_{i=t}^{k-1} 1_i \), \( \bar{t}_i^t = \bar{t}_i^k \) and if \( t > k \), then \( \bar{r}_i^t = \bar{r}_i^k + \varepsilon \sum_{i=t}^{k-1} 1_i \), \( \bar{r}_i^t = \bar{r}_i^k \).

(v) In the solutions \( \bar{z}^t \) for \( t \in S^+ \), we close arc \( t \) and open arc \( j_t \in E^+ \setminus S^+ \) that is introduced in the sufficient condition (ii). Then, \( \bar{x}_j^t = 1 \) if \( j \in S^+ \setminus \{t\} \) and if \( j = j_t \) and \( \bar{x}_j^t = 0 \) otherwise. From sufficient condition (ii), there exists \( \bar{y}_j^t \) values that satisfy the flow balance equalities (3.1b). Moreover, these \( \bar{y}_j^t \) values satisfy inequality (3.17) at equality since both \( S^+ \cup \{j_t\} \) and \( S^+ \setminus \{t\} \cup \{j_t\} \) are path covers for \( N \). Then, the forward and backward path arc flows are found using flow balance equalities where at most one of \( \bar{t}_j^t \) and \( \bar{r}_j^t \) are nonzero for each \( j \in N \).

Finally, we give \( n - 1 \) points \( \bar{z}^j \) corresponding to forward and backward path arcs connecting nodes \( j \) and \( j + 1 \).

(vi) In the solution set \( \bar{z}^j \) for \( j = 1, \ldots, n - 1 \), starting with solution \( \bar{z}^{k_j} \), where \( k_j \) is introduced in the sufficient condition (iii), we send a flow of \( \varepsilon \) from both forward path arc \( (j-1, j) \) and backward path arc \( (j, j-1) \). Since the sufficiency condition (iii) ensures that \( \bar{r}_{j}^{k_j} < b_j \) and \( \bar{r}_{j}^{k_j} < u_j \), the operation is feasible. Let \( \bar{y}_j^k = \bar{y}_j^{k_j} \), \( \bar{x}_j^k = \bar{x}_j^{k_j} \), \( \bar{t}_j^k = \bar{t}_j^{k_j} + \varepsilon 1_j \) and \( \bar{r}_j^k = \bar{r}_j^{k_j} + \varepsilon 1_j \).

\( \Box \)

### 3.4 Computational study

We test the effectiveness of path cover and path pack inequalities (3.14) and (3.17) by embedding them in a branch-and-cut framework. The experiments are run on a Linux workstation with 2.93 GHz Intel®Core™ i7 CPU and 8 GB of RAM with 1 hour time limit and 1 GB memory limit. The branch-and-cut algorithm is implemented in C++ using IBM’s Concert Technology of CPLEX (version 12.5). The experiments are run with one hour limit on elapsed time and 1 GB limit on memory usage. The number of threads is set to one and the dynamic search is disabled. We also turn off heuristics and preprocessing as the purpose is to see the impact of the inequalities by themselves.
Instance generation

We use a capacitated lot-sizing model with backlogging, where constraints (3.1b) reduce to:

\[ i_{j-1} - r_{j-1} + y_j - i_j + r_j = d_j, \quad j \in N. \]

Let \( n \) be the total number of time periods and \( f \) be the ratio of the fixed cost to the variable cost associated with a non-path arc. The parameter \( c \) controls how large the non-path arc capacities are with respect to average demand. All parameters are generated from a discrete uniform distribution. The demand for each node is drawn from the range \([0, 30]\) and non-path arc capacities are drawn from the range \([0, 75]\). Let \( d \) be the average demand over all time periods. Forward and backward path arc capacities are drawn from \([1, 25d]\) and \([0, 8d]\), respectively. The variable costs \( p_t, h_t \) and \( g_t \) are drawn from the ranges \([1, 10]\), \([1, 10]\) and \([1, 20]\) respectively. Finally, fixed costs \( f_t \) are set equal to \( f \times p_t \). Using these parameters, we generate five random instances for each combination of \( n \in \{50, 100, 150\} \), \( f \in \{100, 200, 500, 1000\} \) and \( c \in \{2, 5, 10\} \).

Finding violated inequalities

Given a feasible solution \((x^*, y^*, i^*, r^*)\) to a linear programming (LP) relaxation of (F3.1), the separation problem aims to find sets \( S^+ \) and \( L^- \) that maximize the difference

\[
y^*(S^+) - y^*(E^+ \setminus S^+) - \sum_{t \in S^+} (c_t - \lambda_j)^+ (1 - x^*_t) - \sum_{t \in L^-} (\min\{\lambda_j, c_t\}) x^*_t - d_{1n} - c(S^-)
\]

for path cover inequality (3.14) and sets \( S^+ \) and \( S^- \) that maximize

\[
y^*(S^+) - y^*(E^+ \setminus S^+) - \sum_{t \in E^+ \setminus S^+} \min\{c_t, \mu_j\} x^*_t + \sum_{t \in S^-} (c_t - \mu_j)^+ (1 - x_t) - c(S^+)
\]

for path pack inequality (3.17). We use the knapsack relaxation based heuristic separation strategy described in Wolsey and Nemhauser (1999, pg. 500) for flow cover inequalities to choose the objective set \( S^+ \) with a knapsack capacity \( d_{1n} \). Using \( S^+ \), we obtain the values \( \lambda_j \) and \( \mu_j \) for each \( j \in N \) and let \( S^- = \emptyset \) for path cover and path pack inequalities (3.14) and (3.17). For path cover inequalities (3.14), we add an arc \( t \in E^- \) to \( L^- \) if \( \lambda_j x^*_t < y^*_t \) and \( \lambda_j < c_t \). We repeat the separation process for all subsets \([k, \ell] \subseteq [1, n] \).

Results

We report multiple performance measures. Let \( z_{\text{INIT}} \) be the objective function value of the initial LP relaxation and \( z_{\text{ROOT}} \) be the objective function value of the LP relaxation
Table 3.1: Effect of path size on the performance.

<table>
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<tr>
<th>n</th>
<th>f</th>
<th>c</th>
<th>p = 1</th>
<th>p ≤ 5</th>
<th>p ≤ 0.5×n</th>
<th>p ≤ n</th>
</tr>
</thead>
<tbody>
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<td>gap</td>
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<td>cuts</td>
<td>gap</td>
<td>imp</td>
<td>cuts</td>
</tr>
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<td>361</td>
<td>124</td>
</tr>
<tr>
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<td>50%</td>
<td>44</td>
<td>99%</td>
<td>182</td>
<td>102</td>
</tr>
</tbody>
</table>

Average: 45.5 36% 21 92% 51% 365 157 97% 52% 1699 206 97% 52% 1894 206

after all the valid inequalities added. Moreover, let $z_{UB}$ be the objective function value of the best feasible solution found within time/memory limit among all experiments for an instance. Let $\text{init gap} = 100 \times \frac{z_{UB} - z_{UB}}{z_{UB}}$, $\text{root gap} = 100 \times \frac{z_{UB} - z_{ROOT}}{z_{UB}}$. We compute the improvement of the relaxation due to adding valid inequalities as $\text{gap imp} = 100 \times \frac{\text{init gap} - \text{root gap}}{\text{init gap}}$. We also measure the optimality gap at termination as $\text{end gap} = \frac{z_{UB} - z_{LB}}{z_{UB}}$, where $z_{LB}$ is the value of the best lower bound given by CPLEX. We report the average number of valid inequalities added at the root node under column $\text{cuts}$, average elapsed time in seconds under $\text{time}$, average number of branch-and-bound nodes explored under $\text{nodes}$. If there are instances that are not solved to optimality within the time/memory limit, we report the average end gap and the number of unsolved instances under $\text{unsolv}$ next to $\text{time}$ results. All numbers except initial gap, end gap and time are rounded to the nearest integers.

In Tables 3.1, 3.2 and 3.3, we present the performance with the path cover (3.14) and path pack (3.17) inequalities under columns $\text{spi}$. To understand how the forward and backward path arc capacities affect the computational performance, we also apply them to the single node relaxations obtained by merging a path into a single node, where the capacities of forward and backward path arcs within a path are considered to be infinite. In this case, the path inequalities reduce to the flow cover and flow pack inequalities. These results are presented under columns $\text{mspi}$.

In Table 3.1, we focus on the impact of path size on the gap improvement of the path cover and path pack inequalities for instances with $n = 50$. In the columns under $p = 1$, we obtain the same results for both $\text{mspi}$ and $\text{spi}$ since the paths are singleton nodes. We present these results under $(m)$spi. In columns $p ≤ q$, we add valid inequalities for paths of size $1, \ldots, q$ and observe that as the path size increases, the gap improvement of the path inequalities increase rapidly. On average 97% of the initial gap is closed as longer paths are used. On the other hand, flow cover and pack inequalities from merged paths reduce about half of the initial gap. These results underline the importance of exploiting path arc capacities for strengthening the formulations. We also observe that the increase in gap improvement diminishes as path size grows. We choose a conservative maximum path size limit of $0.75 \times n$ for the
experiments reported in Tables 3.2, 3.3 and 3.4.

In Table 3.2, we investigate the computational performance of path cover and path pack inequalities independently. We present the results for path cover inequalities under columns titled cov, for path pack inequalities under pac and for both of them under the columns titled spi. On average, path cover and path pack inequalities independently close the gap by 63% and 53%, respectively. However, when used together, the gap improvement is 96%, which shows that the two classes of inequalities complement each other very well.

In Table 3.3, we present other performance measures as well for instances with 50, 100, and 150 nodes. We observe that the forward and backward path arc capacities have a large impact on the performance level of the path cover and pack inequalities. Compared to flow cover and pack inequalities added from merged paths, path cover and path pack inequalities reduce the number of nodes and solution times by orders of magnitude. This is mainly due to better integrality gap improvement (50% vs 95% on average).

In Table 3.4, we examine the incremental effect of path cover and path pack inequalities over the fixed-charge network cuts of CPLEX, namely flow cover, flow path and multi-commodity flow cuts. Under cpx, we present the performance of flow cover, flow path and multi-commodity flow cuts added by CPLEX and under cpx_spi, we add path cover and path pack inequalities addition to these cuts. We observe that with the addition of path cover and pack inequalities, the gap improvement increases from 86% to 95%. The number of branch and bound nodes explored is reduced about 900 times. Moreover, with path cover and path pack inequalities the average elapsed time is reduced to almost half and the total number of unsolved instances reduces from 13 to 6 out of 180 instances.

Tables 3.1, 3.2, 3.3 and 3.4 show that submodular path inequalities are quite effective in tackling lot-sizing problems with finite arc capacities. When added to the LP relaxation, they improve the optimality gap by 95% and the number of branch and bound nodes explored decreases by a factor of 1000. In conclusion, our computational experiments indicate that the use of path cover and path pack inequalities is beneficial in improving the performance of the branch-and-cut algorithms.
Table 3.2: Effect of path cover (cov) and path pack (pac) inequalities when used separately and together (spi).

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<th>pac</th>
<th>spi</th>
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Average: 45.5 63% 53% 96% 1357 6751 18 984 136 1297 0.9 0.6 0.3

Table 3.3: Comparison of path inequalities applied to paths (spi) versus applied to merged paths (mspi).

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<th>mspi</th>
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<th>mspi</th>
<th>spi</th>
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<th>time (endgap:unslvd)</th>
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Table 3.4: Effectiveness of the path inequalities when used together with CPLEX’s network cuts.

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**Average:** 45.0 | 95% | 86% | 1213 | 1087194 | 185.1 (0.0:6) | 320.2 (0.2:13)
Chapter 4

Path Pack Inequalities for Lot-sizing Problems with Backlogging and Inventory Bounds

Given the upper-bounds of inventory holding and backlogging values for each time period, the lot-sizing with backlogging and inventory bounds (LSBIB) problem aims to find a production schedule that minimizes the total cost of fixed and unit production, inventory holding and backlogging. In this chapter, we give valid inequalities for single item LSBIB that utilizes inventory and backlogging capacities as well as the underlying path structure of the lot-sizing problems.

There is a vast literature on lot-sizing problems. We refer the reader to Pochet and Wolsey (2006) for a detailed review of different lot-sizing problems. Moreover, Brahimi et al. (2006) survey single item and Karimi et al. (2003) overviews capacitated lot-sizing problems.


For the capacitated lot-sizing with backlogging, Van Vyve (2006) gives a linear extended formulation for the constant production capacity case with infinite inventory and backlog capacities. Zhong et al. (2016) and Chu et al. (2013) give polynomial time algorithms for lot-sizing problems with infinite production and finite inventory/backlogging capacities. Constantino (2000) considers an extended multi-item capacitated lot-sizing framework with start-up costs and infinite inventory and back-
logging capacities. The author provides a cutting plane algorithm and a class of valid inequalities. Wu et al. (2011) give two formulations for multi-item capacitated lot-sizing with backlogging and analyze the tightness of the lower bounds they provide.


Lot-sizing problems are also special cases of fixed-charge network flow problems. For a single node relaxation of capacitated fixed-charge network problems, Padberg et al. (1985), van Roy and Wolsey (1986) and Gu et al. (1999) give flow cover, generalized flow cover and lifted flow cover inequalities. For a general fixed-charge network, Wolsey (1989) introduces submodular inequalities, however these inequalities have implicit coefficients. Atamturk et al. (2017) derive submodular path inequalities which are explicit description of submodular inequalities for a simple path sub-structure. In this work, we extend submodular path inequalities by considering backlogging arcs between consecutive nodes in a path. We show that all coefficients of these inequalities can be computed in linear time using maximum flow minimum cut equivalency. We give necessary and sufficient facet-defining condition. Finally, we show the effectiveness of these inequalities when used in a branch-and-cut algorithm.

Outline: The remainder of the chapter is organized as follows: In Section 4.1, we give the mathematical programming formulation and the underlying fixed-charge network flow problem of LSBIB. In Section 4.2, we introduce eleven classes of path pack inequalities that are valid for LSBIB. In Section 4.3, we lift these inequalities by incorporating the fixed charge variables of inventory and backlog arcs. Finally, in Section 4.4, we present some computational results that shows the effectiveness of both path pack inequalities introduced in Section 4.2 and lifted path pack inequalities introduced in Section 4.3.

4.1 Lot-sizing Problems with Inventory and Backlogging Bounds

We provide a mixed-integer programming formulation for single-item lot-sizing problem with inventory and backlogging capacities. Let $N = \{1, \ldots, n\}$ be the set of time periods and $d_t$ be the demand at time $t \in N$. Without loss of generality, we assume time period $i$ comes before time period $j$ if and only if $i < j$. Let $y_t$, $i_t$ and $r_t$ be the
production, inventory and backlogging units at time period \( t \) with upper bounds \( c_t, u_t \) and \( b_t \) respectively. In this chapter, we assume that \( c_t \) is a very large value for all \( t \in N \). Let the binary variable \( x_t \) be 1 if there is production at time \( t \), 0 otherwise and let \( f_t^p \) represent the fixed cost of production at time \( t \). Let \( z_t \) and \( q_t \) be the binary variables that are equal to 1 if there is a non-zero amount of inventory held and a non-zero amount backlogged at time \( t \), respectively. The fixed costs associated with \( z_t \) and \( q_t \) are represented by \( f_t^i \) and \( f_t^b \), respectively. Let the unit cost of production, inventory holding and backlogging be represented by parameters \( v_t^p \), \( v_t^i \) and \( v_t^b \), respectively. The single-item lot-sizing problem with inventory and backlogging upper-bounds can be represented as the mathematical optimization problem:

\[
\begin{align*}
\min & \quad \sum_{t \in V} f_t^p x_t + f_t^i z_t + f_t^b q_t + v_t^p y_t + v_t^i i_t + v_t^b r_t \\
\text{s. t.} & \quad i_{t-1} - r_{t-1} + y_t - i_t + r_t = d_t, \quad t \in N, \quad (4.1a) \\
& \quad 0 \leq y_t \leq c_t x_t, \quad t \in N, \quad (4.1b) \\
& \quad 0 \leq i_t \leq u_t z_t, \quad t \in N, \quad (4.1c) \\
& \quad 0 \leq r_t \leq b_t q_t, \quad t \in N, \quad (4.1d) \\
& \quad x_t \in \{0, 1\}, \quad t \in N, \quad (4.1e) \\
& \quad z_t \in \{0, 1\}, \quad t \in N, \quad (4.1f) \\
& \quad q_t \in \{0, 1\}, \quad t \in N, \quad (4.1g) \\
& \quad i_0 = i_n = r_0 = r_n = 0. \quad (4.1h)
\end{align*}
\]

where \( u_0 = b_0 = u_n = b_n = 0 \). Moreover, let \( \mathcal{P} \) be the set of feasible solutions of formulation 4.1.

Throughout the chapter, we use the following notation: Let \([k, j] := \{k, k+1, \ldots, j\}\) if \( j \geq k \) and \( \emptyset \) if \( j < k \), \((a)^+ := \max \{0, a\}\) and \( d_{kj} = \sum_{i=k}^{j} d_i \) if \( j \geq k \) and 0 otherwise. Moreover, let dim(\( A \)) denote the dimension of a polyhedron \( A \) and convex hull of a set \( S \).

We make some assumptions on the parameter set of the problem:

1. \( u_j + u_{t-1} \geq d_t, \quad t \in N \). In other words, the feasible set \( F_t = \{x \in \mathcal{P} : y_t = 0\} \) is non-empty for all \( t \in N \),
2. \( u_j > 0 \) and \( b_j > 0 \) for all \( j \in N \backslash \{n\} \),
3. \( d_j \geq 0, \) for \( j \in N \),
4. \( u_{j-1} \leq d_j + u_j, \) for \( j \in N \),
5. \( b_j \leq b_{j-1} + d_{j-1}, \) for \( j \in N \).

Assumptions (1) and (2) and enforced to ensure that \( \dim(\text{conv}(\mathcal{P})) = 5n - 2 \). Notice that, if assumption (1) does not hold for some \( t \in N \), then \( x_t = 1 \) in all feasible
solutions of LSBIB. Similarly, if $b_j = 0$ (or $u_j = 0$), then $r_j = 0$ ($i_j = 0$) at all feasible solutions of LSBIB which would reduce the dimension of $\text{conv}(\mathcal{P})$. Assumptions (3), (4) and (5) significantly reduce the number of potential minimum cuts in a given a set of production arcs that are open which directly reduces the complexity of computing submodular inequality coefficients. The utilization of these assumptions will be further explained in Section 4.2.

There are $6n - 4$ decision variables in LSBIB. However, using constraints (4.1i), any feasible point in LSBIB can be described by $(y, i, x, z, q, r_1)$, where $y$ and $x$ are $n$-dimensional and $i$, $z$ and $q$ are $n - 1$-dimensional vectors. Consequently, we observe that $\dim(\text{conv}(\mathcal{P})) = 5n - 2$.

One can represent the optimization problem LSBIB using a fixed-charge network where time periods are vertices and demand, production, inventory and backlog units are arcs as in Figure 4.1. Throughout the chapter, we refer to production arcs $(s_N, j)$ as “arc $j$” for convenience. Let $A$ be the union of all production, inventory and backlog arcs and let $N' := N \cup \{s_N, t_N\}$.

![Fixed-charge network representation of a path.](image)

Next, we review the path pack inequalities introduced in Atamturk et al. (2017) that are valid for any capacitated-path graph with fixed-charge variables.

### 4.2 Path pack inequalities

Let $[k, \ell]$ be a subset of consecutive time periods, where $k \geq 1$ and $\ell \leq n$. Let $E^+_{k\ell} := \{(i, j) \in A : i \notin [k, \ell], j \in [k, \ell]\}$. In other words, $E^+_{k\ell}$ is the set of production arcs $[k, \ell]$, the inventory arc $(k - 1, k)$ and the backlog arc $(\ell + 1, \ell)$. Similarly, let $E^-_{k\ell} := \{(i, j) \in A : i \in [k, \ell], j \notin [k, \ell]\}$. For the graph structure of lot-sizing problems the set $E^-_{k\ell}$ is equivalent to $\{(k, k - 1), (\ell, \ell + 1)\}$. 

75
Let $S^+ \subseteq E^+_kt$, $L^+ \subseteq E^+_kt \setminus S^+$ and $S^- \subseteq E^-kt$. We refer to the subsets $(S^+, L^+, S^-)$ as the \textit{objective sets}. For all $j \in [k, \ell]$, define $S^+_j = \{(j-1, j), (s_N, j)\} \cap E^+_kt$, $L^+_j = \{(j-1, j), (s_N, j)\} \cap E^+_kt$ and $S^-_j = \{(j, j-1), (j, j+1)\} \cap E^-kt$.

In Atamturk et al. (2017), the authors introduce path pack inequalities which are derived from submodular inequalities introduced by Wolsey (1989). Each coefficient that appears in a submodular inequality is computed by solving a maximum flow problem. Letting the value function $v$ be computed using the optimization problem

$$v(S^+, S^-) = \max \sum_{t \in S^+ \cup L^+} y_t - \sum_{t \in E^+_kt \setminus S^-} y_t$$

s. t. $i_{j-1} - r_{j-1} + y_j - i_j + r_j \leq d_j, \quad j \in [k, \ell],$

$0 \leq i_j \leq u_j, \quad j \in [k, \ell],$

$0 \leq r_j \leq b_j, \quad j \in [k, \ell],$

$0 \leq y_t \leq c_t, \quad t \in [k, \ell],$

$\alpha_j^u = \min\{\alpha_j^{u, -1}, \alpha_j^{u, 1}\} + c(S^+_j), \quad j \in [k, \ell], \quad (4.3)$

$\alpha_j^d = \min\{\alpha_j^{d, -1}, \alpha_j^{d, 1}\} + d_j + c(S^-_j), \quad j \in [k, \ell], \quad (4.4)$
where $\alpha_{k-1}^u = \alpha_{k-1}^d = 0$ and

\begin{align}
\beta_j^u &= \min\{\beta_{j+1}^u, \beta_{j+1}^d + b_j\} + c(S_j^+), \quad j \in [k, \ell], \\
\beta_j^d &= \min\{\beta_{j+1}^u + u_j, \beta_{j+1}^d + d_j + c(S_j^-), \quad j \in [k, \ell],
\end{align}

(4.5)

(4.6)

where $\beta_{\ell+1}^u = \beta_{\ell+1}^d = 0$ and for some arc set $S$ the function $c$ is defined as

\[ c(S) := \sum_{t \in S \cap [k, \ell]} c_t + \sum_{t \in S \cap \{(k-1, k), (\ell, \ell+1)\}} u_t + \sum_{t \in S \cap \{(k, k-1), (\ell+1, \ell)\}} b_t. \]

Let $m_j^u$ and $m_j^d$ be the values of minimum cuts for nodes $[k, \ell]$ that pass above and below node $j$, respectively. Notice that

\[ m_j^u = \alpha_j^u + \beta_j^u - c(S_j^+), \quad j \in [k, \ell] \]

(4.7)

and

\[ m_j^d = \alpha_j^d + \beta_j^d - d_j - c(S_j^-), \quad j \in [k, \ell]. \]

(4.8)

For convenience, let

\[ m_j := \min\{m_j^u, m_j^d\}, \quad j \in [k, \ell]. \]

Then, the path pack inequalities which are valid for LSBIB can be expressed as

\[ y(S^+ \cup L^+) \leq m_k + \sum_{j=k}^\ell \sum_{t \in L_j^+} \mu_j x_t + y(E_{k\ell}^- \setminus S^-), \]

(4.9)

where

\[ \mu_j := (m_j^d - m_j^u)^+, \quad j \in [k, \ell] \]

and for some arc set $C$, the function $y$ is defined as

\[ y(C) := \sum_{t \in C \cap [k, \ell]} y_t + \sum_{t \in C \cap \{(k-1, k), (\ell, \ell+1)\}} i_t + \sum_{t \in C \cap \{(k, k-1), (\ell+1, \ell)\}} r_t. \]

### 4.2.1 Explicit inequalities for LSBIB

In this section, we give path pack inequalities where the coefficients are computed explicitly for LSBIB problems.

**Proposition 4.1.** The inequality

\[ i_{k-1} + \sum_{t \in L^+} y_t \leq u_{k-1} + \sum_{j \in L^+} \left( \min\{d_{kj} + u_j - u_{k-1}, d_{k\ell} - u_{k-1}\} \right)^+ x_j + i_\ell + r_{k-1} \]

(4.10)
is valid for LSBIB.

**Proof.** Inequality (4.10) is equivalent to inequality (4.9), where \( S^+ = \{(k - 1, k)\} \), \( S^- = \emptyset \) and \( L^+ \subseteq [k, \ell] \). We compute \( \alpha^u_j, \alpha^d_j, \beta^u_j \) and \( \beta^d_j \) using the recursive equations (4.3)–(4.6). Due to assumption (4), we observe that

\[
\alpha^u_j = u_{k-1}, \quad \forall j \in [k, \ell].
\] (4.11)

Similarly, due to assumption (1), we observe that

\[
\alpha^d_j = d_{kj}, \quad \forall j \in [k, \ell].
\] (4.12)

Furthermore, using assumption (4), it is easy to see that

\[
\beta^u_j = \begin{cases} 
    u_{k-1} & \text{if } j = k, \\
    0 & \text{if } j \in [k + 1, \ell].
\end{cases}
\] (4.13)

and

\[
\beta^d_j = \min\{u_j, d_{j+1, \ell}\} + d_j, \quad \forall j \in [k, \ell].
\] (4.14)

As a result, we obtain the minimum cut values that pass above and below node \( j \) as

\[
m_j^u = u_{k-1}
\]

and

\[
m_j^d = d_{kj} + \min\{u_j, d_{j+1, \ell}\}
\]

respectively, for each \( j \in [k, \ell] \). Recall that the value of the overall minimum cut can be computed as

\[
v(S^+) = \min\{m_j^u, m_j^d\}
\]

using any \( j \in [k, \ell] \). Now, suppose we would like to compute \( \rho_j(S^+) = v(S^+ \cup \{j\}) - v(S^+) \). Since production arcs are assumed to be uncapacitated (i.e., very large), the minimum cut has to pass below node \( j \) when the production arc \( j \) is open. Consequently,

\[
v(S^+ \cup \{j\}) = m_j^d
\]

and \( \rho_j(S^+) = \max\{m_j^d - m_j^u, 0\} \). Using the values computed above, we observe that for each \( j \in L^+ \),

\[
\rho_j(S^+) = \left( \min\{d_{kj} + u_j - u_{k-1}, d_{k\ell} - u_{k-1}\} \right)^+.
\]
Proposition 4.2. The inequality
\begin{equation}
 r_{\ell} + \sum_{t \in L^+} y_t \leq b_{\ell} + \sum_{j \in L^+} \left( \min \{d_{j \ell} + b_{j-1} - b_{\ell}, d_{k \ell} - b_{\ell} \} \right)^+ x_j + i_{\ell} + r_{k-1},
\end{equation}
is valid for LSBIB.

Proof. Inequality (4.15) is equivalent to inequality (4.9), where $S^+ = \{(\ell + 1, \ell)\}$, $S^- = \emptyset$ and $L^+ \subseteq [k, \ell]$. The proof follows very similar to that of proposition 4.1.

Proposition 4.3. The inequality
\begin{equation}
 i_{k-1} + \sum_{t \in L^+} y_t \leq u_{k-1} + \sum_{j \in L^+} (d_{kj} + u_j - u_{k-1}) x_j + r_{k-1}
\end{equation}
is valid for LSBIB.

Proof. Inequality (4.16) is equivalent to inequality (4.9), where $S^+ = \{(k - 1, k)\}$, $S^- = \{((\ell, \ell + 1)\}$ and $L^+ \subseteq [k, \ell]$.

Proposition 4.4. The inequality
\begin{equation}
 i_{k-1} + \sum_{t \in L^+} y_t \leq u_{k-1} + \sum_{j \in L^+} \left( \min \{b_{k-1} + d_{kj-1}, u_{k-1} + b_{j-1} \} + u_j + d_j - u_{k-1} \right)^+ x_j
\end{equation}
is valid for LSBIB.

Proof. Inequality (4.17) is equivalent to inequality (4.9), where $S^+ = \{(k - 1, k)\}$, $S^- = \{(k, k-1), (\ell, \ell + 1)\}$ and $L^+ \subseteq [k, \ell]$.

Remark 4.1. If the backlogging upper bounds are set to zero, then this problem reduces to the lot-sizing problem studied in Atamtürk and Küçükyavuz (2005). If the backlog arc capacities are zero, then inequality (4.17) reduces to inequality (5) of Atamtürk and Küçükyavuz (2005).

Proposition 4.5. The inequality
\begin{equation}
 i_{k-1} + \sum_{t \in L^+} y_t \leq u_{k-1} + \sum_{t \in L^+} \rho_t x_j + i_{\ell},
\end{equation}
where
\[ \rho_t = \left( \min \{u_j + d_{kj} + b_{k-1} - u_{k-1}, u_j + d_j + b_{j-1}, d_{k \ell} + b_{k-1} - u_{k-1}, d_{j \ell} + b_{j-1} \} \right)^+ \]
for all $t \in L^+$, is valid for LSBIB.
Proof. Inequality (4.18) is equivalent to inequality (4.9), where \( S^+ = \{(k-1,k)\}, \ S^- = \{(k,k-1)\} \) and \( L^+ \subseteq [k, \ell] \).

Remark 4.2. If the backlog arc capacities are zero, then inequality (4.18) reduces to inequality (3) of Atamtürk and Küçükyavuz (2005).

Proposition 4.6. The inequality
\[
 r_\ell + \sum_{t \in L^+} y_t \leq b_\ell + \sum_{j \in L^+} (d_{j,\ell} + b_{j-1} - b_\ell) x_j + i_\ell, \tag{4.19}
\]
is valid for LSBIB.

Proof. Inequality (4.19) is equivalent to inequality (4.9), where \( S^+ = \{((\ell+1,\ell)\}, \ S^- = \{((k,k-1)\} \) and \( L^+ \subseteq [k, \ell] \).

Proposition 4.7. The inequality
\[
 r_\ell + \sum_{t \in L^+} y_t \leq b_\ell + \sum_{j \in L^+} \left( \min\{b_\ell + u_j, d_{j+1,\ell} + u_\ell\} + d_j + b_{j-1} - b_\ell \right)^+ x_j, \tag{4.20}
\]
is valid for LSBIB.

Proof. Inequality (4.20) is equivalent to inequality (4.9), where \( S^+ = \{((\ell+1,\ell)\}, \ S^- = \{((k,k-1),((\ell,\ell+1)\} \) and \( L^+ \subseteq [k, \ell] \).

Proposition 4.8. The inequality
\[
 r_\ell + \sum_{j \in L^+} y_j \leq b_\ell + \sum_{j \in L^+} \rho_j x_j + r_{k-1}, \tag{4.21}
\]
where
\[
 \rho_j = \left( \min\{u_j + b_{j-1} + d_j, u_j + d_{k,j}, d_{j,\ell} + u_\ell + b_{j-1} - b_\ell, d_{k,\ell} + u_\ell - b_\ell \} \right)^+
\]
for all \( j \in L^+ \), is valid for LSBIB.

Proof. Inequality (4.21) is equivalent to inequality (4.9), where \( S^+ = \{((\ell+1,\ell)\}, \ S^- = \{((\ell,\ell+1)\} \) and \( L^+ \subseteq [k, \ell] \).

Proposition 4.9. The inequality
\[
 i_{k-1} + r_\ell + \sum_{j \in L^+} y_j \leq \min\{u_{k-1} + b_\ell, b_{k-1} + d_{k,\ell} + u_\ell\} + \sum_{j \in L^+} \rho_j x_j, \tag{4.22}
\]
where
\[
 \rho_j = \left( \min\{d_{k,j-1} + b_{k-1} + u_{k-1} + b_{j-1}\} + \min\{b_\ell + u_j, d_{j+1,\ell} + u_\ell\} + d_j - u_{k-1} - b_\ell \right)^+
\]
for all \( j \in L^+ \), is valid for LSBIB.

**Proof.** Inequality (4.22) is equivalent to inequality (4.9), where \( S^+ = \{(k - 1, k), (\ell + 1, \ell)\} \), \( S^- = \{(k, k - 1), (\ell, \ell + 1)\} \) and \( L^+ \subseteq [k, \ell] \). 

**Proposition 4.10.** The inequality

\[
i_{k-1} + r_\ell + \sum_{j \in L^+} y_j \leq \min\{u_{k-1} + b_\ell, d_{k\ell} + u_\ell\} + \sum_{j \in L^+} \rho_j x_j + r_{k-1},
\tag{4.23}
\]

where

\[
\rho_j = \left( \min\{b_\ell + u_j, d_{j+1,\ell} + u_\ell\} + d_{kj} - u_{k-1} - b_\ell \right)^+
\]

for all \( j \in L^+ \), is valid for LSBIB.

**Proof.** Inequality (4.23) is equivalent to inequality (4.9), where \( S^+ = \{(k - 1, k), (\ell + 1, \ell)\} \), \( S^- = \{(\ell, \ell + 1)\} \) and \( L^+ \subseteq [k, \ell] \).

**Proposition 4.11.** The inequality

\[
i_{k-1} + r_\ell + \sum_{j \in L^+} y_j \leq \min\{u_{k-1} + b_\ell, d_{k\ell} + b_{k-1}\} + \sum_{j \in L^+} \rho_j x_j + i_\ell,
\tag{4.24}
\]

where

\[
\rho_j = \left( \min\{d_{kj-1} + b_{k-1}, u_{k-1} + b_{j-1}\} + d_{j\ell} - u_{k-1} - b_\ell \right)^+
\]

for all \( j \in L^+ \), is valid for LSBIB.

**Proof.** Inequality (4.24) is equivalent to inequality (4.9), where \( S^+ = \{(k - 1, k), (\ell + 1, \ell)\} \), \( S^- = \{(k, k - 1)\} \) and \( L^+ \subseteq [k, \ell] \).

**Remark 4.3.** Selecting \( S^+ = \{(k - 1, k), (\ell + 1, \ell)\} \) and \( S^- = \emptyset \) gives a weak submodular inequality. Due to assumptions (1), (4) and (5) we know that \( v(\{(k - 1, k), (\ell + 1, \ell)\}) = d_{k\ell} \). Notice that, for this set selection, the sum of constraints in (F2) implies that \( d_{k\ell} \) is the largest value that the function \( v \) can take. As a result, \( \rho_j(S^+) = 0 \) for all \( j \in L^+ \).

**Remark 4.4.** In inequalities (4.10), (4.16), (4.17), (4.18), (4.15), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24), if \( \rho_j(S^+) \geq u_j + b_{j-1} + d_j \) for any \( j \in L^+ \), then the inequality is weak.

**Proof.** Suppose for an arc \( j \in L^+ \), \( \rho_j(S^+) \geq u_j + b_{j-1} + d_j \). Then, the path pack inequality where arc \( j \) is dropped from \( L^+ \) summed with \( y_j \leq (u_j + b_{j-1} + d_j)x_j \) gives an inequality at least as strong as the initial path pack inequality.
4.2.2 Finding violated inequalities

In this section, we discuss how to find inequalities (4.10), (4.16), (4.17), (4.18), (4.15), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24) violated by a given point \( (\bar{x}, \bar{y}, \bar{s}, \bar{r}, \bar{q}, \bar{z}) \in \mathbb{R}^6 \). For a given \( [k, \ell] \subseteq [1, n] \), and the sets \( S^+ \) and \( S^- \), the set \( L^+ \) that maximizes the violation of these inequalities can be found in linear time. A production arc \( t \in [k, \ell] \) is added to \( L^+ \) if and only if

\[
\bar{y}_t > \rho_t \bar{x}_t,
\]

where \( \rho_t \) is the coefficient of \( x_t \) if \( t \) is added to \( L^+ \).

In inequalities (4.10), (4.15), (4.16), (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24), we assume that all of the inventory and backlog arcs are open. In the next section, we extend these inequalities by incorporating the fixed charge variables of inventory and backlog arcs.

4.3 Inventory and backlog fixed charge variables

Let \( H = \{h_1, \ldots, h_{|H|}\} \) be a subset of inventory arc indices in increasing order. In this convention, \( h_i = j \) refers to the inventory arc \( (j, j + 1) \). For an inventory arc \( (j, j + 1) \) we define \( m(j) = \min\{t \in L^+ \cup \{(\ell + 1) : t > j\} \} \). Moreover, let \( M \supseteq \cup_{i=1}^{p} m(h_i) \).

Alternatively, we represent this set as \( M = \{m_1, \ldots, m_{|M|}\} \). For convenience, let \( m_0 := k, m_{|M|+1} := \ell \) and \( h_0 = k, h_{|H|+1} = \ell \). Finally, let \( L(t) = L^+ \cap [t + 1, \ell] \).

**Proposition 4.12.** The inequality

\[
i_{k-1} + \sum_{j \in L^+} y_j \leq u_{k-1} + \sum_{j \in L^+} \rho_j x_j + i_{\ell} + r_{k-1} + \sum_{j \in M} \delta_j q_{j-1},
\]

where

\[
\rho_j = \begin{cases}
\min\{d_{kj} + u_j - u_{k-1}, d_{k\ell} - u_{k-1}, u_j + d_j, d_{j\ell}\} & \text{if } j \in L^+, \\
\min\{d_{kj} + u_j - u_{k-1}, d_{k\ell} - u_{k-1}, u_j + d_j, d_{j\ell}\} + \min\{(d_{kj-1} - u_{k-1})^+, d_{m_i, j-1}\} & \text{if } m_i \leq j \leq m_{i+1}, i = 0, \ldots, |M|
\end{cases}
\]

and

\[
\delta_m = \min\{b_{m_i-1}, d_{m_i-1, m_i-1}\}, \quad \forall \ m_i \in M
\]

is valid for LSBIB.

**Proof.** Let \( (x^*, y^*, i^*, r^*, q^*, z^*) \) be a feasible solution to LSBIB. We derive the coeffi-
coefficients $\rho_j$ and $\delta_j$ are derived from the submodular function $v$ where

$$
\delta_{m_j} = v(\{m_j\}|q_{m_{i-1}} = 0, \forall i \in [1, j-1]) - v(\{m_j\}|q_{m_{i-1}} = 0, \forall i \in [1, j])
$$

(4.28)

and

$$
\rho_j = \rho_j(S^+|q_{i-1} = 0, \forall i \in M).
$$

(4.29)

For feasible solutions where $q^*_t = 0$ for all $t \in M$, inequality (4.25) is equivalent to submodular inequality (4.9) with $S^+ = \{(k-1, k)\}$ for the lot-sizing graph where the backlog arcs $(t, t-1)$ for $t \in M$ have capacities of zero. Consequently, inequality (4.25) is valid when $q^*_t = 0$ for all $t \in M$.

Now, let $\hat{L}^+ = \{j \in L^+: x_j^* = 1\}$, $\hat{M} := \{t \in M : q^*_t = 1\}$ and suppose $\hat{M} \neq \emptyset$. From the definition of $v$, with the objective $i_{k-1} + \sum_{t \in L^+} y_t - i_{\ell} - r_{k-1}$, it is clear that

$$
i^*_{k-1} + \sum_{j \in L^+} y_j^* - i^*_{\ell} - r^*_{k-1} \leq v(\{s\} \cup \hat{L}^+|q_{t-1} = 0, \forall t \in M \setminus \hat{M}).
$$

Without loss of generality, let $\hat{M} = \{\hat{m}_1, \ldots, \hat{m}_{|\hat{M}|}\}$. First, we observe that

$$
v(\{s\}) + \sum_{t \in L^+} \rho_t(\{s\}|q_{j-1} = 0, \forall j \in M) \geq v(\{s\} \cup \hat{L}|q_{j-1} = 0, \forall j \in M).
$$

To obtain the inequality above, we use the submodularity of the function $v$ and the fact that $v(\{s\}) = v(\{s\}|q_{j-1} = 0, \forall j \in M)$. This is because maximum flow sent by the inventory arc $\{k-1, k\}$ is $u_{k-1}$ and only depends on the inventory arcs of the path and not on backlog arcs.

In the remainder of the proof, we use induction to show that the right hand side of (4.25) at this solution has the following relationship

$$
u_{k-1} + \sum_{j \in L^+} \rho_j + \sum_{t \in M} \delta_t \geq v(\{s\} \cup \hat{L}^+|q_{t-1} = 0, \forall t \in M \setminus \hat{M}).
$$

First, we make some observations on the values $\delta_j$. The flow sent on a production arc $j$ while computing $v(\{j\})$ flows on three directions: (i) to satisfy the demand $d_j$, (ii) through the backlog arc $(j, j-1)$ and (iii) through the inventory arc $(j, j+1)$. If the backlog arc $(j, j-1)$ is closed, then the flow corresponding to $v(\{j\}|q_{j-1} = 0)$, consists of the latter two components. As a result,

$$
v(\{j\}) = v(\{j\}|q_{j-1} = 0) + v(\{j\}|z_j = 0) - d_j.
$$

This logic extends to the $\delta_{m_j}$ values as well. The definition in (4.28) is equivalent to the maximum change in the maximum flow value sent on arc $m_j$ when the backlog arc
which shows that the induction holds.

In the base case, let \( \hat{m}_{|\hat{M}|} = m_t \) for some \( m_t \in M \) and suppose that all backlog arcs \((j, j - 1)\) for \( j \in M \) are closed. When the backlog arc \((m_t, m_t - 1)\) is opened, the maximum flow that can be sent on it through the production arc \( m_t \) is equal to \( \delta_{m_t} \).

Due to assumption \((1)\), among the production arcs \([m_t, \ell]\), the production arc \( m_t \) can send the largest flow on the backlog arc \((m_t, m_t - 1)\). Consequently,

\[
v\{s\} \cup \hat{L}|q_{i-1} = 0, \forall j \in M \setminus \{m_t\}\} - v\{s\} \cup \hat{L}|q_{j-1} = 0, \forall j \in M \}
\leq v\{m_t\}|q_{m_{t-1}} = 0, \forall i \in [1, t-1]\} - v\{m_t\}|q_{m_{t-2}} = 0, \forall i \in [1, t]\} \quad (4.30)
\]

which proves the base case of the induction.

At level \( j \) of the induction, assuming that

\[
v\{s\} \cup \hat{L}|q_{i-1} = 0, \forall j \in M \setminus \{m_{j+2}, \ldots, \hat{m}_{|\hat{M}|}\} + \delta_{\hat{m}_{j+1}}
\geq v\{s\} \cup \hat{L}|q_{i-1} = 0, \forall j \in M \setminus \{\hat{m}_{j+1}, \ldots, \hat{m}_{|\hat{M}|}\}
\]

holds we show

\[
v\{s\} \cup \hat{L}|q_{i-1} = 0, \forall i \in M \setminus \{\hat{m}_{j+1}, \ldots, \hat{m}_{|\hat{M}|}\} + \delta_{\hat{m}_j}
\geq v\{s\} \cup \hat{L}|q_{i-1} = 0, \forall i \in M \setminus \{\hat{m}_j, \ldots, \hat{m}_{|\hat{M}|}\}.
\]

Using the same logic as the base case, we know that among the production arcs \([m_j, \ell]\) the arc \( \hat{m}_j \) can send the largest flow on the backlog arc \((\hat{m}_j, \hat{m}_j - 1)\). Let \( m_t = \hat{m}_j \) for \( m_t \in M \), then

\[
v\{s\} \cup \hat{L}|q_{i-1} = 0, \forall i \in M \setminus \{\hat{m}_{j+1}, \ldots, \hat{m}_{|\hat{M}|}\} + \delta_{\hat{m}_j}
\leq v\{\hat{m}_j\}|q_{m_{i-1}} = 0, \forall i \in [1, l-1]\} - v\{\hat{m}_j\}|q_{m_{i-2}} = 0, \forall i \in [1, l]\} \quad (4.31)
\]

which shows that the induction holds.

Consequently, we show that inequality \((4.25)\) is valid since

\[
i^*_k - \sum_{j \in \hat{L}^+} y^*_j - i_\ell - r_{k-1} \leq v\{s\} \cup \hat{L}^+|q_{t-1} = 0, \forall t \in M \setminus \hat{M}\}
\leq u_{k-1} + \sum_{j \in \hat{L}^+} \rho_j + \sum_{t \in M} \delta_t.
\]

\[\square\]
Proposition 4.13. The inequality
\[ i_{k-1} + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j) \leq u_{k-1} + \sum_{j \in L^+} \rho_j x_j + i_\ell + r_{k-1} + \sum_{j \in M} \delta_j q_{j-1}, \tag{4.32} \]
where \( \rho_j \) for \( j \in L^+ \) and \( \delta_j \) for \( j \in M \) are defined as in (4.26) and (4.27) respectively and
\[
\gamma_{h_i} = \begin{cases} 
\min\{ (u_{k-1} - d_{kh_i})^+, d_{h_i+1,h_i+1} \} & \text{if } m(h_i) = m(h_{i+1}), \\
\min\{ (u_{k-1} - d_{kh_i})^+, d_{h_i+1,m(h_i)-1} \} & \text{if } m(h_i) < m(h_{i+1})
\end{cases} \quad \tag{4.33}
\]
is valid for LSBIB.

Proof. The coefficients of \( \gamma_{h_i} \) can be written in terms of the functions \( v \) and \( \rho \).
\[
\gamma_{h_i} = \rho_s(L(h_i)|z_{h_j} = 0, \forall j \in [i+1, \tau], q_{j-1} = 0, \forall j \in M) \\
- \rho_s(L(h_i)|z_{h_j} = 0, \forall j \in [i, \tau], q_{j-1} = 0, \forall j \in M).
\]
Similarly, the definitions for \( \delta_j \) and \( \rho_j \) are provided in (4.28) and (4.29).

Similar to the lot-sizing without backlog arcs, these coefficients depend on the order of variables in sets \( H \) and \( M \). In this chapter, we assume that the elements of \( H \) and \( M \) are in increasing order.

Let \((x^*, y^*, i^*, r^*, q^*, z^*)\) be a feasible solution to LSBIB. For feasible solutions where \( z_i^* = 1 \) for all \( t \in [k, \ell] \), inequality (4.32) is equivalent to (4.25) and valid for LSBIB.

Let \( \hat{L}^+ := \{ j \in L^+ : x_j^* = 1 \} \), \( \hat{H} := \{ j \in H : z_j^* = 0 \} \) and \( \hat{M} := \{ j \in M : q_{j-1}^* = 1 \} \). Recall that \( S^+ = \{(k - 1, k)\} \) and let us represent the inventory arc \((k - 1, k)\) with index \( s \). From the definition of the function \( v \) with objective \( i_{k-1} + \sum_{j \in L^+} y_j - i_\ell - r_{k-1} \), we know that
\[
i_{k-1}^* + \sum_{j \in L^+} y_j^* - i_\ell^* - r_{k-1}^* \leq v(\{s\} \cup \hat{L}^+|q_{j-1} = 0, \forall j \in M \setminus \hat{M}, z_j = 0, \forall j \in \hat{H}).
\]
Let
\[
R := u_{k-1} + \sum_{j \in L^+} \rho_j + \sum_{t \in \hat{M}} \delta_t - \sum_{t \in \hat{H}} \gamma_t
\]
be the right hand side of inequality (4.32) at this solution. In this proof, we show that
\[
R \geq v(\{s\} \cup \hat{L}^+|q_{j-1} = 0, \forall j \in M \setminus \hat{M}, z_j = 0, \forall j \in \hat{H}).
\]

The first part of the proof follows similar to that of Proposition E.3. For notational convenience let
\[
\rho_s(L(t)|z_t = 0, q_{j-1} = 0, \forall j \in M) = \rho_s(L(t)|z_t = 0, M)
\]
and
\[ v(S^+|z_t = 0, q_{j-1} = 0, \forall j \in M) = v(S^+|z_t = 0, M). \]

Let \( h_{\text{max}} = \max\{j \in \hat{H}\} \). Then, using submodularity and the path structure of LSBIB, we observe the following

\[
v(\{s\}) - \rho_s(L(h_{\text{max}})|z_{h_{\text{max}} + 1} = 0, M) + \rho_s(L(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \\
+ \sum_{j \in \hat{L}(h_{\text{max}})} \rho_j(\{s\}|M) \tag{4.34}
\]

\[
\geq v(\{s\} \cup \hat{L}(h_{\text{max}})|M) - \rho_s(L(h_{\text{max}})|z_{h_{\text{max}} + 1} = 0, M) \\
+ \rho_s(L(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \tag{4.35}
\]

\[
\geq v(\{s\} \cup \hat{L}(h_{\text{max}})|M) - \rho_s(\hat{L}(h_{\text{max}})|z_{h_{\text{max}} + 1} = 0, M) \\
+ \rho_s(L(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \tag{4.36}
\]

\[
\geq v(\{s\} \cup \hat{L}(h_{\text{max}})|M) - \rho_s(\hat{L}(h_{\text{max}})|M) + \rho_s(L(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \tag{4.37}
\]

\[
= v(\hat{L}(h_{\text{max}})|M) + \rho_s(L(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \tag{4.38}
\]

\[
= v(\{s\} \cup \hat{L}(h_{\text{max}})|z_{h_{\text{max}}} = 0, M). \tag{4.39}
\]

Inequality (4.35) is obtained using the submodularity of the function \( v \) and the fact that
\[ v(\{s\}) = v(\{s\}|M) = u_{k-1}. \]

Inequality (4.37) holds since for any \( h_j \in H \) and \( h_i \geq h_j \),
\[ \rho_s(L(h_j)|z_{h_i} = 0, M) \leq \rho_s(L(h_j)|M) \]
due to the path structure of LSBIB. Similarly, we reach the equality of (4.37) and (4.38) since \( v(\hat{L}(h_{\text{max}})|M) = v(\hat{L}(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \). Using the inequality (4.34)–(4.39), we observe that

\[
R - \sum_{t \in M} \delta_t \geq v(\{s\} \cup \hat{L}(h_{\text{max}})|z_{h_{\text{max}}} = 0, M) \\
- \sum_{i \in \hat{T} \setminus \{h_{\text{max}}\}} \left( \rho_s(L(h_i)|z_{h_{i+1}} = 0, M) - \rho_s(L(h_i)|z_{h_i} = 0, M) \right) \\
+ \sum_{i \in \hat{L}^+ \setminus \hat{L}(h_{\text{max}})} \rho_i(\{s\}|M).
\]

For the rest of the proof, we use an induction approach. First, we introduce some notation for simplification: let \( \hat{H}(h_{j}) = \hat{H} \cap \{h_{j}, \ldots, h_r\} \) and let \( v(C|\hat{H}(h_j), M) := v(C|z_i = 0, \forall i \in \hat{H}(h_j), q_{i-1} = 0, \forall i \in M) \) and \( \rho_l(C|\hat{H}(h_j), M) := \rho_l(C|z_i = 0, \forall i \in \hat{H}(h_j), q_{i-1} = 0, \forall i \in M) \).
Assuming that the following is true

\[ R - \sum_{t \in \mathcal{M}} \delta_t \geq v(\{s\} \cup \hat{L}(h_{j+1})|\hat{H}(h_{j+1}), M) \]

\[ - \sum_{i \in \hat{H}[h_{j+1}, h_v]} (\rho_s(L(h_i)|z_{h_i+1} = 0, M) - \rho_s(L(h_i)|z_{h_i} = 0, M)) \]

\[ + \sum_{i \in \hat{L}^+ \setminus \hat{L}(h_{j+1})} \rho_i(\{s\}|M) \]

for iteration \( j + 1 \), we extend the result for \( h_j \). First, we make the following observation

\[ \rho_i(\{s\}|M) \geq \rho_i(\{s\}|z_{h_i+1} = 0, M) = \rho_i(\{s\}|z_{h_i} = 0, \forall i \in [j + 1, \tau], M) \]

for any \( i \in L^+ \setminus L(h_{j+1}) \). Then,

\[ v(\{s\} \cup \hat{L}(h_{j+1})|\hat{H}(h_{j+1}), M) - \rho_s(L(h_j)|h_{j+1} = 0) + \rho_s(L(h_j)|z_{h_j} = 0, M) \]

\[ + \sum_{i \in \hat{L}(h_j) \setminus \hat{L}(h_{j+1})} \rho_i(\{s\}|M) \quad (4.40) \]

\[ \geq v(\{s\} \cup \hat{L}(h_j)|\hat{H}(h_{j+1}), M) - \rho_s(L(h_j)|h_{j+1} = 0, M) \]

\[ + \rho_s(L(h_j)|z_{h_j} = 0, M) \quad (4.41) \]

\[ \geq v(\hat{L}(h_j)|\hat{H}(h_{j+1}), M) + \rho_s(L(h_j)|z_{h_j} = 0, M) \]

\[ = v(\{s\} \cup \hat{L}(h_j)|\hat{H}(h_j), M). \quad (4.42) \]

Inequalities (4.41)–(4.42) are obtained from the observation above and submodularity of the function \( v \) and the equality of (4.42) to (4.43) is due to the path structure of LSBIB. As a result of the induction, we have proved that

\[ R \geq v(\{s\} \cup \hat{L} | \hat{H}, M) + \sum_{t \in \mathcal{M}} \delta_t. \]

In the second step of the proof, we show that

\[ v(\{s\} \cup \hat{L} | \hat{H}, M) + \sum_{t \in \mathcal{M}} \delta_t \geq v(\{s\} \cup \hat{L} | \hat{H}, M \setminus \hat{M}). \quad (4.44) \]

In the proof of Proposition 4.12, we discuss that \( \delta_j \) only depends on the backlog arcs \( \{(k + 1, k), \ldots, (j, j - 1)\} \) and not on the inventory arcs. Consequently, we observe that

\[ \delta_{m_j} = v(\{m_j\}|q_{m_i-1} = 0, \forall i \in [1, j - 1]) - v(\{m_j\}|q_{m_i-1} = 0, \forall i \in [1, j]) \]

\[ = v(\{m_j\}|q_{m_i-1} = 0, \forall i \in [1, j - 1], \hat{H}) - v(\{m_j\}|q_{m_i-1} = 0, \forall i \in [1, j], \hat{H}). \]
Then, showing the relationship in (4.44) is a repetition of the proof of Proposition 4.12 for a lot-sizing network where the upper-bounds of inventory arcs in $\hat{H}$ are all zero. As a result, we conclude that

$$i_{k-1}^* + \sum_{j \in L^+} y_j - i_t - r_{k-1} \leq v(\{s\} \cup \hat{L}^+|q_{j-1} = 0, \forall j \in \hat{M}, z_j = 0, \forall j \in \hat{H}) \leq R$$

and inequality (4.32) is valid for LSBIB.

**Proposition 4.14.** Inequality (4.32) can be strengthened by partitioning the set $M$ into two subsets $M_1$ and $M_2$ where $M_1 = \{t \in M : \delta_t < b_{t-1}\}$ and $M_2 = M \setminus M_1$. Then,

$$i_{k-1} + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j(1 - z_j) \leq u_{k-1} + \sum_{j \in L^+} \rho_j x_j + i_t + r_{k-1} + \sum_{j \in M_1} \delta_j q_{j-1} + \sum_{j \in M_2} r_{j-1},$$

where $\rho_j$ for $j \in L^+$, $\delta_j$ for $j \in M$ and $\gamma_j$ for $j \in H$ are defined in (4.26), (4.27) and (4.33), respectively, is also valid for LSBIB.

**Proof.** We use a simple reformulation argument to show validity of inequality (4.45). Suppose we break a backlog arc $(t, t-1)$ into two clone arcs: one arc incoming to node $t-1$ and one arc outgoing from node $t$. Then, we force the flows on these clone arcs to be equal to each other by adding non-anticipativity constraints. Now, consider the submodular function $v$ where the objective function coefficients of the outgoing backlog clones are $-1$ while the coefficients of the incoming backlog clones are 0. To obtain inequality (4.45), the backlog arcs $(j, j-1)$ for all $j \in M_2$ are cloned. Since the coefficients of $x_t$ for $t \in L^+$ assume that the backlog arcs $(j, j-1)$ for all $j \in M$ are closed, the coefficients for $t \in L^+$ as inequalities (4.25) and (4.32).

Let $(x^*, y^*, i^*, r^*, q^*, z^*)$ be a feasible solution to LSBIB and let $\hat{L}^+ := \{j \in L^+ : x^*_j = 1\}$, $\hat{H} := \{j \in H : z^*_j = 0\}$ and $\hat{M} := \{j \in M : q^*_{j-1} = 1\}$. Then, using the definition of $v$ with the objective function $i_{k-1} + \sum_{j \in L^+} y_j - \sum_{j \in M_2} r_{j-1} - i_t - r_{k-1}$, we observe that

$$i_{k-1}^* + \sum_{j \in L^+} y_j^* - i_t^* - r_{k-1}^* - \sum_{j \in M_2} r_{j-1}^* \leq v(\{s\} \cup \hat{L}^+|q_{j-1} = 0, \forall j \in \hat{M}, z_j = 0, \forall j \in \hat{H}).$$

As a result, it suffices to show that

$$u_{k-1} + \sum_{j \in L} \rho_j + \sum_{j \in M \cap M_1} \delta_j - \sum_{j \in \hat{H}} \gamma_j \geq v(\{s\} \cup \hat{L}^+|q_{j-1} = 0, \forall j \in \hat{M}, z_j = 0, \forall j \in \hat{H}).$$

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to prove the validity of inequality (4.45). We omit the remainder of the proof since it follows very similarly to the proof of Proposition 4.13.

**Proposition 4.15.** The inequality

\[ i_{k-1} + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j) \leq u_{k-1} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in M} \delta_j q_{j-1} + r_{k-1} \]  

(4.46)

where \( \delta_j \) for \( j \in M \) and \( \gamma_j \) for \( j \in H \) are defined as in (4.27) and (4.33), respectively and

\[ \rho_j = \begin{cases} 
\min\{d_{kj} + u_j - u_{k-1}, u_j + d_j\} & \text{if } j \in M, \\
\min\{d_{kj} + u_j - u_{k-1}, u_j + d_j\} + \min\{(d_{k,j-1} - u_{k-1})^+, d_{m_i,j-1}\} & \text{if } m_i < j < m_{i+1}, i = 0, \ldots, |M| \\
\end{cases} \]

is valid for LSBIB.

**Proof.** The proof follows very similar to that of Proposition 4.32 with the exception that the function \( v \) is computed using the objective function \( i_{k-1} + \sum_{j \in L^+} y_j - r_{k-1} \).

Let \( \bar{d}_{ij} = d_{ij} + b_{k-1} \) if \( i = k \) and \( d_{ij} \) otherwise.

**Proposition 4.16.** The inequality

\[ i_{k-1} + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j) \leq u_{k-1} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in M} \bar{\delta}_j q_{j-1} \]  

(4.47)

where \( \gamma_j \) for \( j \in H \) is defined as in (4.33),

\[ \rho_j = \begin{cases} 
\min\{b_{k-1} + d_{kj} + u_j - u_{k-1}, u_j + d_j\} & \text{if } j \in M, \\
\min\{b_{k-1} + d_{kj} + u_j - u_{k-1}, u_j + d_j\} \\
+ \min\{(b_{k-1} + d_{k,j-1} - u_{k-1})^+, \bar{d}_{m_i,j-1}\} & \text{if } m_i < j < m_{i+1}, i = 0, \ldots, |M| \\
\end{cases} \]

and

\[ \bar{\delta}_{m_i} = \min\{b_{m_i-1}, \bar{d}_{m_i-1,m_i-1}\}, \quad \forall \ m_i \in M \]  

(4.48)

is valid for LSBIB.

**Proof.** The proof follows very similar to that of Proposition 4.32 with the exception that the function \( v \) is computed using the objective function \( i_{k-1} + \sum_{j \in L^+} y_j \).

**Remark 4.5.** If \( M = L^+ \) and the backlog arc capacities are zero, then inequality (4.47) reduces to inequality (19) of Atamtürk and Küçükyavuz (2005).
Proposition 4.17. The inequality

\[ i_{k-1} + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j) \leq u_{k-1} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in M} \tilde{\delta}_j q_{j-1} + i_t, \quad (4.49) \]

where \( \gamma_j \) for \( j \in H \) and \( \tilde{\delta}_j \) for \( j \in M \) are defined as in (4.33) and (4.48), respectively and

\[ \rho_j = \begin{cases} 
\min\{b_{k-1} + d_{kj} + u_j - u_{k-1}, b_{k-1} + d_{kt} - u_{k-1}, u_j + d_j, d_{jt}\} & \text{if } j \in M, \\
\min\{b_{k-1} + d_{kj} + u_j - u_{k-1}, b_{k-1} + d_{kt} - u_{k-1}, u_j + d_j, d_{jt}\} + \min\{(b_{k-1} + d_{k,j-1} - u_{k-1})^+, \bar{d}_{m,j-1}\} & \text{if } m_i < j < m_{i+1}, i = 0, \ldots, |M| 
\end{cases} \]

is valid for LSBIB.

Proof. The proof follows very similar to that of Proposition 4.32 with the exception that the function \( v \) is computed using the objective function \( i_{k-1} + \sum_{j \in L^+} y_j - i_t. \) \hfill \Box

Remark 4.6. If \( M = L^+ \) and the backlog arc capacities are zero, then inequality (4.49) reduces to inequality (9) of Atamtürk and Küçükyavuz (2005).

Remark 4.7. The procedure described in Proposition 4.14 can be applied to inequalities (4.46), (4.47), (4.49) by partitioning the set \( M \) into two subsets:

\[ M_1 = \{ t \in M : \delta_t < b_{t-1} \} \]

and \( M_2 = M \setminus M_1. \) If \( j \in M_1, \) then the backlog arc \( (j, j - 1) \) appears in the right hand side of these inequalities as \( \delta_j q_{j-1} \) and as \( r_{j-1}, \) otherwise. Notice that, any partition of \( M \) into \( M_1 \) and \( M_2 \) would conserve the feasibility.

The partition \( M_1 \) and \( M_2 \) that maximizes the violation of these inequalities by a given point \((\bar{x}, \bar{y}, \bar{i}, \bar{r}, \bar{q}, \bar{z}) \in \mathbb{R}^{6n}\) can be found in linear time. Simply, add a backlog arc \((j, j - 1)\) to \( M_2 \) if and only if

\[ \bar{r}_{j-1} < \delta_j \bar{x}_{j-1}. \]

Let \( \hat{d}_i = (d_{i,t} - b_t)^+ \) for \( i \geq m_{|M|} \) and \( d_i \) otherwise and let \( \hat{d}_{ij} = \sum_{t=i}^j d_t \) if \( i \leq j \) and 0 otherwise. Moreover, let \( d'_{ij} = \hat{d}_{ij} + b_{k-1} \) if \( i = k \) and \( \hat{d}_{ij} \) otherwise.

Proposition 4.18. The inequality

\[ i_{k-1} + r_t + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j) \leq \min\{u_{k-1} + b_t, b_{k-1} + d_{kt} + u_t\} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in M} \tilde{\delta}_j q_{j-1}, \quad (4.50) \]
where
\[
\gamma_{h_i} = \begin{cases} 
\min\{(u_{k-1} - \hat{d}_{kh_i})^+, \hat{d}_{h_{i+1}h_{i+1}}\} & \text{if } m(h_i) = m(h_{i+1}), \\
\min\{(u_{k-1} - \hat{d}_{kh_i})^+, \hat{d}_{h_{i+1}m(h_i)-1}\} & \text{if } m(h_i) < m(h_{i+1})
\end{cases}
\] (4.51)

\[
\rho_j = \begin{cases} 
\min\{b_{k-1} + \hat{d}_{kj} + u_j - u_{k-1}, u_j + \hat{d}_j\} & \text{if } j \in M, \\
\min\{b_{k-1} + \hat{d}_{kj} + u_j - u_{k-1}, u_j + \hat{d}_j\} + \min\{(b_{k-1} + \hat{d}_{kj-1} - u_{k-1})^+, d_{m_ij-1}^m\} & \text{if } m_i < j < m_{i+1}, i = 0, \ldots, |M|
\end{cases}
\]

and
\[
\hat{\delta}_{m_i} = \min\{b_{m_i-1}, d'_{m_i-1,m_i-1}\}, \quad \forall m_i \in M
\] (4.52)
is valid for LSBIB.

**Proof.** In this inequality, the cover set is \(S^+ = \{(k-1, k), (\ell+1, \ell)\}\) and for convenience, let the indices \(s_1\) and \(s_2\) represent arcs \((k-1, k)\) and \((\ell+1, \ell)\), respectively. The objective function used to compute \(v\) is \(i_{k-1} + r^{\ell} + \sum_{j \in L^+} y_j\). The coefficients \(\rho_j\), \(\gamma_j\) and \(\hat{\delta}_j\) are derived from the submodular function \(v\) where
\[
\rho_j = \rho_j(S^+|q_{i-1} = 0, \forall i \in M),
\] (4.53)

\[
\gamma_{h_i} = \rho_{s_1}(L(h_i) \cup \{s_2\}|z_{h_j} = 0, \forall j \in [i+1, \tau], q_{j-1} = 0, \forall j \in M) - \rho_{s_1}(L(h_i) \cup \{s_2\}|z_{h_j} = 0, \forall j \in [i, \tau], q_{j-1} = 0, \forall j \in M)
\]

and
\[
\hat{\delta}_{m_i} = v(\{m_j, s_2\}|q_{m_i-1} = 0, \forall i \in [1, j-1]) - v(\{m_j, s_2\}|q_{m_i-1} = 0, \forall i \in [1, j]).
\] (4.54)

Let \((x^*, y^*, i^*, r^*, q^*, z^*)\) be a feasible solution to LSBIB. Let \(\hat{L}^+ := \{j \in L^+: x^*_j = 1\}\), \(\hat{H} := \{j \in H : z^*_j = 0\}\) and \(\hat{M} := \{j \in M : q^*_{j-1} = 1\}\). From the definition of the function \(v\), we know that
\[
\hat{i}_{k-1}^* + \hat{r}_\ell^* + \sum_{j \in \hat{L}^+} y_j^* \leq v(\{s_1, s_2\} \cup \hat{L}^+|q_{j-1} = 0, \forall j \in M \setminus \hat{M}, z_j = 0, \forall j \in \hat{H}).
\]

Let
\[
R := \min\{u_{k-1} + b_\ell, b_{k-1} + d_{k\ell} + u_\ell\} + \sum_{j \in L^+} \rho_j + \sum_{t \in \hat{M}} \delta_t - \sum_{t \in \hat{H}} \gamma_t
\]
be the right hand side of inequality (4.50) at this solution. Moreover, in this inequality

$$v(S^+) = \min \{ u_{k-1} + b_\ell, b_{k-1} + d_{k \ell} + u_\ell \}.$$ 

The rest of the proof is equivalent to the proof of Proposition (4.16), where the upper bound of the backlog arc ($\ell + 1$, $\ell$) is saturated into the nodes in $[m|M|, \ell]$. In other words, the demand values are taken as $\hat{d}_j$ instead of $d_j$. 

**Proposition 4.19.** The inequality

$$i_{k-1} + r_\ell + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j)$$

$$\leq \min \{ u_{k-1} + b_\ell, d_{k \ell} + u_\ell \} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in M} \delta_j q_{j-1} + r_{k-1}, \quad (4.55)$$

where $\gamma_j$ for $j \in H$ is defined as in (4.51) and

$$\rho_j = \begin{cases} 
\min \{ \hat{d}_{kj} + u_j - u_{k-1}, u_j + \hat{d}_j \} & \text{if } j \in M, \\
\min \{ \hat{d}_{kj} + u_j - u_{k-1}, u_j + \hat{d}_j \} + \min \{ (\hat{d}_{k,j-1} - u_{k-1})^+, \hat{d}_{m,j-1} \} & \text{if } m_i < j < m_{i+1}, i = 0, \ldots, |M| 
\end{cases}$$

and

$$\hat{\delta}_{m_i} = \min \{ b_{m_i-1}, \hat{d}_{m_i-1,m_i-1} \}, \quad \forall m_i \in M \quad (4.56)$$

is valid for LSBIB.

**Proof.** The proof follows very similar to that of Proposition 4.18 with the exception that the function $\nu$ is computed using the objective function $i_{k-1} + r_\ell + \sum_{j \in L^+} y_j - r_{k-1}$. 

**Proposition 4.20.** The inequality

$$i_{k-1} + r_\ell + \sum_{j \in L^+} y_j + \sum_{j \in H} \gamma_j (1 - z_j)$$

$$\leq \min \{ u_{k-1} + b_\ell, d_{k \ell} + b_{k-1} \} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in M} \delta_j q_{j-1} + i_\ell, \quad (4.57)$$

where $\gamma_j$ for $j \in H$ and $\hat{\delta}_j$ for $j \in M$ are defined as in (4.51) and (4.56), respectively and

$$\rho_j = \begin{cases} 
\min \{ b_{k-1} + \hat{d}_{kj} + u_j - u_{k-1}, b_{k-1} + \hat{d}_{k \ell} - u_{k-1}, u_j + \hat{d}_j, \hat{d}_j \} & \text{if } j \in M, \\
\min \{ b_{k-1} + \hat{d}_{kj} + u_j - u_{k-1}, b_{k-1} + \hat{d}_{k \ell} - u_{k-1}, u_j + \hat{d}_j, \hat{d}_j \} + \min \{ (b_{k-1} + \hat{d}_{k,j-1} - u_{k-1})^+, d''_{m,j-1} \} & \text{if } m_i < j < m_{i+1}, i = 0, \ldots, |M| 
\end{cases}$$
The proof follows very similar to that of Proposition 4.18 with the exception that the function \( v \) is computed using the objective function \( i_{k-1} + r_\ell + \sum_{j \in L^+} y_j - i_\ell \). 

**Remark 4.8.** The procedure described in Remark 4.7 and Proposition 4.14 can be applied to inequalities (4.50), (4.55), (4.57) by partitioning the set \( M \) into two subsets:

\[ M_1 = \{ t \in M : \delta t < b_{t-1} \} \]

and \( M_2 = M \setminus M_1 \). If \( j \in M_1 \), then the backlog arc \((j, j-1)\) appears in the right hand side of these inequalities as \( \delta_j q_{j-1} \) and as \( r_{j-1} \), otherwise. Notice that, any partition of \( M \) into \( M_1 \) and \( M_2 \) would conserve the feasibility.

The partition \( M_1 \) and \( M_2 \) that maximizes the violation of these inequalities by a given point \((\bar{x}, \bar{y}, \bar{t}, \bar{r}, \bar{q}, \bar{z}) \in \mathbb{R}^{6n}\) can be found in linear time. Simply, add a backlog arc \((j, j-1)\) to \( M_2 \) if and only if

\[ \bar{r}_{j-1} < \delta_j \bar{x}_{j-1}. \]

Let \( T = \{ t_1, \ldots, t_{|T|} \} \) be a subset of backlog arc indices in an increasing order. In this convention, \( t_i = j \) refers to the backlog arc \((j + 1, j)\). For a backlog arc \((j + 1, j)\), we define \( n(j) = \max\{ t \in L^+ \cup \{ (k - 1) \} : t \leq j \} \). Moreover, let \( K = \bigcup_{i=1}^r n(t_i) \). Alternatively, we represent this set as \( K = \{ k_1, \ldots, k_{|K|} \} \). For convenience, let \( k_0 = k, k_{|K|+1} = \ell \) and \( t_0 = k, t_{|T|+1} = \ell \). Finally, let \( \bar{L}(t) = L^+ \cap [k, t] \).

**Proposition 4.21.** The inequality

\[ r_\ell + \sum_{l \in L^+} y_l + \sum_{j \in T} \phi_j (1 - q_j) \leq b_\ell + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j + i_\ell + r_{k-1}, \quad (4.58) \]

where

\[
\rho_j = \begin{cases} 
\min\{d_{j}\ell + b_{j-1} - b_{\ell}, d_{k\ell} - b_{\ell}, d_j + b_{j-1}, d_{kj}\} & \text{if } j \in K, \\
\min\{d_{j}\ell + b_{j-1} - b_{\ell}, d_{k\ell} - b_{\ell}, d_j + b_{j-1}, d_{kj}\} + \min\{(d_{j+1}\ell - b_{\ell})^+, d_{j+1,k_{i+1}}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases}
\]

\[
\phi_{ti} = \begin{cases} 
\min\{b_{\ell} - d_{ti+1,\ell}^+, d_{ti+1,ti}\} & \text{if } n(t_{i-1}) = n(t_i), \\
\min\{b_{\ell} - d_{ti+1,\ell}^+, d_{n(t_i)+1,ti}\} & \text{if } n(t_{i-1}) < n(t_i)
\end{cases}
\]

and

\[ \psi_{ki} = \min\{u_{ki}, d_{ki+1,k_{i+1}}\}, \quad \forall \ k_i \in K \]
is valid for LSBIB.

Proof. In this inequality, the cover set is $S^+ = \{(\ell + 1, \ell)\}$ and for convenience, let the index $s$ represent the backlog arc $(\ell + 1, \ell)$. In this inequality, the objective function used to compute $v$ is $r_\ell + \sum_{j \in L^+} y_j - i_\ell - r_{k-1}$. The coefficients $\rho_j$, $\phi_j$ and $\psi_j$ are derived from the submodular function $v$ where

$$
\rho_j = \rho_j(S^+ | z_i = 0, \forall i \in K),
$$

(4.60)

$$
\phi_{t_j} = \rho_s(\bar{L}(t_j) \cup \{s\}) | q_{t_i} = 0, \forall i \in [1, j - 1], z_i = 0, \forall i \in K
- \rho_s(\bar{L}(t_j) | q_{t_i} = 0, \forall i \in [1, j], z_i = 0, \forall i \in K)
$$

and

$$
\psi_{k_j} = v(\{k_j\} | z_{k_i} = 0, \forall i \in [j + 1, |K|]) - v(\{k_j\} | z_{k_i} = 0, \forall i \in [j, |K|]).
$$

(4.61)

Inequality (4.58) is equivalent to inequality (4.32) for the mirrored image of the node set $[k, \ell]$. More formally, consider renaming nodes $k, k+1, \ldots, \ell$ as $\bar{\ell}, \bar{\ell} - 1, \ldots, \bar{k}$, respectively. Moreover, rename the backlogging variables $r_\ell, r_{\ell-1}, \ldots, r_{k-1}$ as $\bar{r}_{k-1}, \bar{i}_k, \ldots, \bar{i}_\ell$ and rename the inventory variables $i_{k-1}, i_k, \ldots, i_\ell$ as $\bar{i}_\ell, \bar{r}_{\ell-1}, \ldots, \bar{r}_{k-1}$, respectively. Then, inequality (4.32) for the node set $[\bar{k}, \bar{\ell}]$, where $S^+ = \{(\bar{k} - 1, \bar{k})\}$ is equivalent to inequality (4.58), where $S^+ = \{(\ell + 1, \ell)\}$. Consequently, the proof of this proposition is a symmetric version of the proof of Proposition 4.32. □

**Proposition 4.22.** The inequality

$$
r_\ell + \sum_{t \in L^+} y_t + \sum_{j \in T} \phi_j (1 - q_j) \leq b_\ell + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j + i_\ell,
$$

(4.62)

where $\phi_j$ is defined as in (4.59),

$$
\rho_j = \begin{cases} 
\min \{d_{j, \ell} + b_{j-1} - b_\ell, d_j + b_{j-1}\} & \text{if } j \in K, \\
\min \{d_{j, \ell} + b_{j-1} - b_\ell, d_j + b_{j-1}\} + \min \{d_{j+1, \ell} - b_\ell, d_{j+1, k_i+1}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases}
$$

and

$$
\psi_{k_i} = \min \{u_{k_i}, d_{k_i+1, k_{i+1}}\}, \quad \forall i \in K
$$

is valid for LSBIB.

Proof. The proof follows very similar to that of Proposition 4.22 with the exception that
the function $v$ is computed using the objective function $r_\ell + \sum_{j \in L^+} y_j - i_\ell$. Moreover, notice that inequality (4.62) is symmetric to inequality (4.46).

**Proposition 4.23.** The inequality

$$r_\ell + \sum_{t \in L^+} y_t + \sum_{j \in T} \phi_j (1 - q_j) \leq b_\ell + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j,$$

(4.63)

where $\phi_j$ is defined as in (4.59),

$$\rho_j = \begin{cases} 
\min\{u_\ell + d_{j_\ell} + b_{j-1} - b_\ell, d_j + b_{j-1}\} & \text{if } j \in K, \\
\min\{u_\ell + d_{j_\ell} + b_{j-1} - b_\ell, d_j + b_{j-1}\} \\
+ \min\{(u_\ell + d_{j+1,\ell} - b_\ell)^+, \tilde{d}_{j+1,k_{i+1}}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases}$$

and

$$\psi_{k_i} = \min\{u_{k_i}, \tilde{d}_{k_i+1,k_{i+1}}\}, \quad \forall k_i \in K$$

with $\tilde{d}_{ij} = d_{ij} + u_\ell$ if $j = \ell$ and $d_{ij}$ otherwise, is valid for LSIB.

**Proof.** The proof follows very similar to that of Proposition 4.22 with the exception that the function $v$ is computed using the objective function $r_\ell + \sum_{j \in L^+} y_j$. Moreover, notice that inequality (4.63) is symmetric to inequality (4.47).

**Proposition 4.24.** The inequality

$$r_\ell + \sum_{t \in L^+} y_t + \sum_{j \in T} \phi_j (1 - q_j) \leq b_\ell + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j + r_{k-1},$$

(4.64)

where $\phi_j$ is defined as in (4.59),

$$\rho_j = \begin{cases} 
\min\{u_\ell + d_{j_\ell} + b_{j-1} - b_\ell, u_\ell + d_{k_\ell} - b_\ell, d_j + b_{j-1}, d_{kj}\} & \text{if } j \in K, \\
\min\{u_\ell + d_{j_\ell} + b_{j-1} - b_\ell, u_\ell + d_{k_\ell} - b_\ell, d_j + b_{j-1}, d_{kj}\} \\
+ \min\{(u_\ell + d_{j+1,\ell} - b_\ell)^+, \tilde{d}_{j+1,k_{i+1}}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases}$$

and

$$\psi_{k_i} = \min\{u_{k_i}, \tilde{d}_{k_i+1,k_{i+1}}\}, \quad \forall k_i \in K$$

with $\tilde{d}_{ij} = d_{ij} + u_\ell$ if $j = \ell$ and $d_{ij}$ otherwise, is valid for LSIB.

**Proof.** The proof follows very similar to that of Proposition 4.22 with the exception that the function $v$ is computed using the objective function $r_\ell + \sum_{j \in L^+} y_j - r_{k-1}$. Moreover, notice that inequality (4.64) is symmetric to inequality (4.49).
Remark 4.9. The procedure described in Remark 4.7 and Proposition 4.14 can be applied to inequalities (4.58), (4.62), (4.63) and (4.64) by partitioning the set $K$ into two subsets:

$$K_1 = \{ t \in K : \psi_t < u_t \}$$

and $K_2 = K \setminus K_1$. If $j \in K_1$, then the inventory arc $(j, j + 1)$ appears in the right hand side of these inequalities as $\psi_j z_j$ and as $i_j$, otherwise. Notice that, any partition of $K$ into $K_1$ and $K_2$ would conserve the feasibility.

The partition $K_1$ and $K_2$ that maximizes the violation of these inequalities by a given point $(\bar{x}, \bar{y}, \bar{i}, \bar{r}, \bar{q}, \bar{z}) \in \mathbb{R}^m$ can be found in linear time. Simply, add an inventory arc $(j, j + 1)$ to $K_2$ if and only if

$$\bar{i}_j < \psi_j \bar{z}_j.$$ 

\[ \square \]

Let $\tilde{d}_i = (d_{ki} - u_{k-1})^+$ for $i \leq k_1$ and $d_i$ otherwise. Moreover, let $\tilde{d}_{ij} = \sum_{t=i}^j \tilde{d}_t$ if $i \leq j$ and $0$ otherwise.

**Proposition 4.25.** The inequality

$$i_{k-1} + r_\ell + \sum_{j \in L^+} y_j + \sum_{j \in T} \phi_j (1 - q_j)$$

$$\leq \min\{u_{k-1} + b_\ell, b_{k-1} + d_{k\ell} + u_\ell\} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j,$$  (4.65)

where

$$\rho_j = \begin{cases} 
\min\{u_\ell + \tilde{d}_j \ell + b_{j-1} - b_\ell, \tilde{d}_j + b_{j-1}\} & \text{if } j \in K, \\
\min\{u_\ell + \tilde{d}_j \ell + b_{j-1} - b_\ell, \tilde{d}_j + b_{j-1}\} \\
+ \min\{(u_\ell + \tilde{d}_{j+1, \ell} - b_\ell)^+, \tilde{d}_{j+1, k_i+1}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases}$$

$$\phi_{t_i} = \begin{cases} 
\min\{(b_\ell - \tilde{d}_{t_i+1, \ell})^+, \tilde{d}_{t_i+1, t_i}\} & \text{if } n(t_{i-1}) = n(t_i), \\
\min\{(b_\ell - \tilde{d}_{t_i+1, \ell})^+, \tilde{d}_{n(t_i)+1, t_i}\} & \text{if } n(t_{i-1}) < n(t_i)
\end{cases}$$  (4.66)

and

$$\psi_{k_i} = \min\{u_{k_i}, \tilde{d}_{k_i+1, k_i+1}\}, \quad \forall \ k_i \in K$$

with $\tilde{d}_{ij} = \tilde{d}_{ij} + u_\ell$ if $j = \ell$ and $\tilde{d}_{ij}$ otherwise, is valid for LSBIB.

**Proof.** In this inequality, the cover set is $S^+ = \{(k-1, k), (\ell+1, \ell)\}$ and for convenience, let the indices $s_1$ and $s_2$ represent arcs $(k-1, k)$ and $(\ell+1, \ell)$, respectively. Here, the objective function used to compute $v$ is $i_{k-1} + r_\ell + \sum_{j \in L} y_j$. The coefficients $\rho_j$, $\phi_j$
and \( \psi_j \) are derived from the submodular function \( v \) where

\[
\rho_j = \rho_j(S^+|z_i = 0, \forall i \in K),
\]

\[
\phi_{t_j} = \rho_{s_2}(L(t_j) \cup \{s_1\}|q_{t_i} = 0, \forall i \in [1, j - 1], z_i = 0, \forall i \in K)
- \rho_{s_2}(L(t_j) \cup \{s_1\}|q_{t_i} = 0, \forall i \in [1, j], z_i = 0, \forall i \in K)
\]

and

\[
\psi_{k_j} = v(\{k_j, s_1\}|z_{k_i} = 0, \forall i \in [j + 1, |K|]) - v(\{k_j, s_1\}|z_{k_j} = 0, \forall i \in [j, |K|]).
\]

Inequality (4.65) is symmetrical to inequality (4.50). Moreover, the coefficients \( \phi_j \) for \( j \in T \), \( \rho_j \) for \( j \in L^+ \) and \( \psi_j \) for \( j \in K \) are equivalent to the coefficients of inequality (4.63), where the demands at nodes \([k, k_1]\) are saturated with the upper-bound of the inventory arc \((k - 1, k)\). Consequently, the proof follows very closely to that of Proposition 4.18.

**Proposition 4.26.** The inequality

\[
i_{k-1} + r_\ell + \sum_{j \in L^+} y_j + \sum_{j \in T} \phi_j(1 - q_j) \\
\leq \min\{u_{k-1} + b_\ell, d_{k\ell} + u_\ell\} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j + r_{k-1}, \quad (4.67)
\]

where \( \phi_j \) for \( j \in T \) is defined as in (4.66),

\[
\rho_j = \begin{cases} 
\min\{u_\ell + \tilde{d}_{j\ell} + b_{j-1} - b_\ell, u_\ell + \tilde{d}_{k\ell} - b_\ell, \tilde{d}_j + b_{j-1}, \tilde{d}_{k_j}\} & \text{if } j \in K, \\
\min\{u_\ell + \tilde{d}_{j\ell} + b_{j-1} - b_\ell, u_\ell + \tilde{d}_{k\ell} - b_\ell, \tilde{d}_j + b_{j-1}, \tilde{d}_{k_j}\} \\
+ \min\{(u_\ell + \tilde{d}_{j+1,\ell} - b_\ell)^+, \tilde{d}_{j+1, k_{i+1}}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases}
\]

and

\[
\psi_{k_i} = \min\{u_{k_i}, \tilde{d}_{k_{i+1, k_{i+1}}}\}, \quad \forall k_i \in K
\]

with \( \tilde{d}_{ij} = \tilde{d}_{ij} + u_\ell \) if \( j = \ell \) and \( \tilde{d}_{ij} \) otherwise, is valid for LSBIB.

**Proof.** The proof follows very similar to that of Proposition 4.25 with the exception that the function \( v \) is computed using the objective function \( i_{k-1} + r_\ell + \sum_{j \in L^+} y_j - r_{k-1}. \)

**Proposition 4.27.** The inequality

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\[ i_{k-1} + r_\ell + \sum_{j \in L^+} y_j + \sum_{j \in P} \phi_j(1 - q_j) \leq \min\{u_{k-1} + b_\ell, d_{k\ell} + b_{k-1}\} + \sum_{j \in L^+} \rho_j x_j + \sum_{j \in K} \psi_j z_j + i_\ell, \quad (4.68) \]

where \( \phi_j \) for \( j \in T \) is defined as in (4.66),

\[ \rho_j = \begin{cases} 
\min\{\tilde{d}_{j\ell} + b_{j-1} - b_\ell, \tilde{d}_j + b_{j-1}\} & \text{if } j \in K, \\
\min\{\tilde{d}_{j\ell} + b_{j-1} - b_\ell, \tilde{d}_j + b_{j-1}\} + \min\{(\tilde{d}_{j+1,\ell} - b_\ell)^+, \tilde{d}_{j+1,k_{i+1}}\} & \text{if } k_i < j < k_{i+1}, i = 0, \ldots, |K|,
\end{cases} \]

and

\[ \psi_{k_i} = \min\{u_{k_i}, \tilde{d}_{k_{i+1},k_{i+1}}\}, \quad \forall k_i \in K \]

is valid for LSBIB.

**Proof.** The proof follows very similar to that of Proposition 4.25 with the exception that the function \( v \) is computed using the objective function \( i_{k-1} + r_\ell + \sum_{j \in L^+} y_j - i_\ell \). \( \square \)

**Remark 4.10.** The partition procedure described in Remark 4.9 can be extended to inequalities (4.65), (4.67) and (4.68). \( \square \)

### 4.3.1 Strength of lifted inequalities

In this section, we investigate the conditions under which some of these valid inequalities are facet defining for the convex hull of \( \mathcal{P} \). Let \([k, \ell] \subseteq [1, n], S^+ \subseteq E^+, \]
\( L^+ \subseteq E^+ \setminus S^+ \) and \( S^- \subseteq E^- \). For convenience, we introduce some notation in this section. The inventory arc \((k - 1, k)\) is presented by index \( s \). Let the elements of the set \( L^+ \) be represented by \( \{l_1, \ldots, l_L\} \) and let \( \varepsilon > 0 \) be a sufficiently small value. We also define the following indices:

\[ p = \min\{j \in L^+: u_{k-1} \leq d_{kj}\}, \]
\[ \hat{t} = \min\{j \in [k, \ell + 1] \setminus L^+\} \]

and

\[ \tilde{t} = \max\{j \in L^+: j \leq p\}. \]

Moreover, let \( e_j, f_j, g_j, h_j, k_j \) and \( m_j \) be the unit vectors corresponding to variables \( y_j, i_j, r_j, x_j, z_j \) and \( q_j \) respectively.

**Theorem 4.28.** For a node set \([k, \ell]\), let \( H \subseteq [k - 1, \ell - 1], M = L^+ \) in Proposition 4.13. Recall that in \( S^+ = \{(k - 1, k)\} \) and \( S^- = \emptyset \) in inequality (4.32). If \( d_j > 0 \)
for all \( j \in [k, \ell] \), then the conditions below are sufficient for inequality (4.32) to be facet-defining for \( \text{conv}(\mathcal{P}) \):

(i) \((k - 1) \cup \{ [k, p - 1] \cap L^+ \} \subseteq H; \)

(ii) for all \( j \in L^+, \rho_j(\{s\}) > 0; \)

(iii) \( u_i > d_{i+1, \ell}; \)

(iv) \( u_{l_1} \geq d_{l_1+1, \ell}; \)

(v) if \( u_{k-1} = d_{k,p-1} \), then \( l_1 < p \), otherwise \( l_1 \leq p; \)

(vi) \( \hat{t} \in [k, p] \), \( u_i \geq d_{i+1, \ell} \) and \( b_{t-1} \geq d_{k_1, -1}; \)

(vii) if \( \ell < n \), then \( u_{l_1} \geq d_{l_1+1, \ell+1}; \)

(viii) for all \( l_j \in L^+ \), \( b_{l_j-1} > d_{l_j-1, l_j-1}. \)

\textbf{Proof.} We provide \( 5n - 2 \) affinely independent points that lie on the face:

\[
F = \{ (y, i, r, x, z, q) \in \mathcal{P} : \quad i_{k-1} + \sum_{t \in L^+} (y_t - \rho_j x_j) + \sum_{t \in H} \gamma_j (1 - z_j) - i_\ell - r_{k-1} - \sum_{j \in L^+} \delta_j q_{j-1} = u_{k-1} \}.
\]

Consider the following base values for variables \((y, i, r, x, z, q)\) of LSBIB described below:

\[
i_j = \begin{cases} 0, & j \in [1, k - 2] \cup [\ell, n], \\
u_{k-1}, & j = k - 1, \\
(u_{k-1} - d_{kj})^+, & j \in [k, l - 1], \end{cases} \quad y_j = \begin{cases} d_j, & j \in [1, k - 2] \cup [\ell + 1, n], \\
d_{k-1} + u_{k-1}, & j = k - 1, \\
0, & j \in [k, \ell], \end{cases}
\]

\[
z_j = \begin{cases} 0, & j \in [1, k - 2] \cup [\ell, n], \\
1, & j \in [k - 1, l - 1], \end{cases} \quad x_j = \begin{cases} 1, & j \in [1, k - 1] \cup [\ell + 1, n], \\
0, & j \in [k, \ell], \end{cases}
\]

\[
r_j = 0, & j \in [1, n], \quad q_j^* = 0, & j \in [1, n].
\]

Let \( w_0 \) be the vector representation of the solution above:

\[
w_0 = \sum_{j \in [1, k - 2]} (d_j e_j + h_j) + \sum_{j \in [\ell + 1, n]} (d_j e_j + h_j) + (d_{k-1} + u_{k-1})e_{k-1} + h_{k-1} + \sum_{j \in [k, \ell - 1]} k_j + \sum_{j \in [k - 1, p - 1]} (u_{k-1} - d_{kj})f_j.
\]

Since \( u_{k-1} < d_{kk} \), \( w_0 \) is not feasible for LSBIB. In the remainder of the proof we perturb \( w_0 \) in various ways to obtain \( 5n - 2 \) affinely independent points.

First, we give \( 2|L^+| \) feasible points represented by \( w_j \) and \( \tilde{w}_j \) for \( j \in L^+. \)
(i) For $j \in L^+$: If $j \leq p$, then
\[
w_j = w_0 + h_j + (d_{k\ell} - u_{k-1})e_j + \sum_{t \in [j, p-1]} (d_{t+1, \ell} - (u_{k-1} - d_{k\ell}))f_t + \sum_{t \in [p, \ell]} d_{t+1, \ell}f_t;
\]
otherwise let
\[
w_j = w_0 + h_j + h_j + d_{j\ell}e_j + (d_{k\ell - 1} - u_{k-1})e_j + \sum_{t \in [i, p-1]} (d_{t+1, j-1} - (u_{k-1} - d_{k\ell}))f_t + \sum_{t \in [p, j-1]} d_{t+1, j-1}f_t + \sum_{t \in [j, \ell]} d_{t+1, \ell}f_t,
\]
where $\hat{t}$ is defined as in condition (vi).

(ii) For $j \in L^+$:
\[
\bar{w}_j = \begin{cases} \ w_j + \varepsilon e_j + \varepsilon \sum_{t \in [j, \ell]} f_t - \varepsilon e_{t+1} + k_t, & \text{if } j \geq \hat{t}, \\
\ w_j + \varepsilon e_j - \varepsilon e_{k-1} - \varepsilon \sum_{t \in [k-1, j-1]} f_t, & \text{otherwise.}
\end{cases}
\]

Next, we give $2 \times |[k, \ell] \setminus L^+|$ affinely independent feasible points $w_j$ and $\bar{w}_j$ corresponding to $j \in [k, \ell] \setminus L^+$ that are on the face $F$.

(iii) $w_j = w_i + h_j$ and

(iv) First, recall that $\hat{t} \leq p$ from condition (vi). Then, the next set of solutions corresponding to $j \in [k, \ell] \setminus L^+$ can be summarized by:
\[
\bar{w}_j = \begin{cases} \ w_0 + h_j + (d_{k\ell} - u_{k-1})e_j + \sum_{t \in [j, p-1]} (d_{t+1, \ell} - (u_{k-1} - d_{k\ell}))f_t + \sum_{t \in [p, \ell]} d_{t+1, \ell}f_t & \text{if } j \leq p, \\
\ \bar{w}_i + h_j - \varepsilon e_i + \varepsilon e_j - \varepsilon \sum_{t \in [i, j-1]} f_t & \text{if } j > p.
\end{cases}
\]

Next, we give $2[[1, k-1]]$ points corresponding to production arcs $j \in [1, k-1]$.

(v) If $j > 1$, then $w_j = w_i - h_j - d_j e_j + d_j e_{j-1} + d_j f_{j-1} + k_{j-1}$ and if $j = 1$, then $w_j = w_i - h_j - d_j e_j + d_j e_{j+1} + d_j g_j + m_j$.

(vi) If $j < k-1$, then $\bar{w}_j = w_i + \varepsilon e_j - \varepsilon e_{j+1} + k_j + \varepsilon f_j$ and if $j = k-1$ and $k > 2$, then $\bar{w}_j = w_i + \varepsilon e_j - \varepsilon e_{j-1} + m_{j-1} + \varepsilon g_{j-1}$. Finally, if $k = 2$, then $\bar{w}_{k-1} = w_i - \varepsilon e_{k-1} + \varepsilon e_i - \varepsilon \sum_{t \in [k-1, \ell-1]} k_t$. Note that, if $k = 1$, then this set of points are irrelevant.

Now, we give $2[[\ell + 1, n]]$ points corresponding to production arcs $j \in [\ell + 1, n]$.

(vii) if $j = \ell + 1$, then $w_j = w_{l\ell} - h_j - d_j e_j + d_j e_{l\ell} + d_j \sum_{t \in [l, \ell]} f_t + k_{j}$ and otherwise $w_j = w_{l\ell} - h_j - d_j e_j + d_j e_{j-1} + d_j f_{j-1} + k_{j-1}$.
(viii) if $j = \ell + 1$, then $\bar{w}_j = w_{tL} + \varepsilon e_{tL} - \varepsilon e_j + \varepsilon \sum_{t \in [tL, \ell]} f_t + k_\ell$ and if $j > \ell + 1$, then $\bar{w}_j = w_{tL} + \varepsilon e_j - \varepsilon e_{j+1} + \varepsilon f_j + \varepsilon k_j$.

Next, we give 3$|1, k - 2| + 3|\ell, n - 1|$ points corresponding to inventory arcs $(j, j + 1)$ and backlog arcs $(j + 1, j)$ for $j \in [1, k - 2] \cup [\ell, n - 1]$ represented by $\bar{w}_j, \hat{w}_j$ and $\check{w}_j$. Then, for $j \in [1, k - 2] \cup [\ell, n - 1]$, let

(ix) $\hat{w}_j = w_{t_1} + k_j$,

(x) $\hat{w}_j = w_{t_1} + m_j$ and

(xi) if $j \neq \ell$, then let $\check{w}_j = w_{t_1} + k_j + m_j + \varepsilon f_j + \varepsilon g_j$, otherwise let

$$\check{w}_j = \bar{w}_i - \sum_{t \in [\ell, \ell - 1]} \varepsilon f_t - \varepsilon e_t + \varepsilon g_t + m_t.$$

Next, we give points on $F$ that correspond to the inventory arcs in $[k - 1, \ell]$. We represent these points by $\hat{w}_j$ for $(j, j + 1)$, for $j \in [k - 1, \ell - 1]$. Let $H = \{h_1, \ldots, h_{|H|}\} \subseteq [k - 1, p - 1]$. Without loss of generality, we assume $h_{|H|} \leq p - 1$ since $\gamma_{h_j} = 0$ if $h_j \geq p$.

(xii) If $j \in [p, \ell - 1]$, then let

$$\hat{w}_j = \sum_{t \in [1, k - 2] \cup [\ell + 1, n]} (d_t e_t + h_t) + (d_{kj} - u_{k-1}) e_t + h_t + \sum_{t \in [k - 1, \ell - 1] \setminus \{j\}} k_t + (d_{k-1} + u_{k-1}) e_{k-1} + h_{k-1} + d_{j+1, \ell} e_{j+1} + \sum_{t \in [k - 1, \ell - 1]} d_{t, j-1} f_t + \sum_{t \in [j, \ell - 1]} d_{t+1, \ell} f_t$$

Recall that $m(j) = \min\{t \in L^+ : t \geq j\}$. If $j \in [k - 1, p - 1]$, $j \in H$ and $m(j) \leq \ell$, then

$$\hat{w}_j = \sum_{t \in [1, k - 2] \cup [\ell + 1, n]} (d_t e_t + h_t) + d_{k-1, j} e_{k-1} + h_{k-1} + \sum_{t \in [k - 1, j - 1]} (d_{t+1, j} f_t + k_t) + \sum_{t \in [j, m(j) - 1]} (d_t e_t + h_t) + h_{m(j)} + d_{m(j), \ell} e_{m(j)} + \sum_{t \in [j + 1, \ell - 1]} k_t + \sum_{t \in [m(j), \ell - 1]} d_{t+1, \ell} f_t.$$
Finally, consider the case where \( j \in [k-1, p-1] \) and \( j \notin H \) and let \( h_L = \max\{t \in H : t < j\} \). If, \( j \in [h_L + 1, m(h_L) - 1] \), then, let

\[
\bar{w}_j = \bar{w}_{h_L} - k_j.
\]

Note that, \( j \in [h_L + 1, m(h_L) - 1] \) always holds, since \( m(h_L) \in H \) if \( m(h_L) \leq p - 1 \) due to condition (i). Consequently, if \( j \notin [h_L + 1, m(h_L) - 1] \), then we would reach a contradiction with the definition of \( h_L \).

Next, we give points on \( F \) that correspond to the backlog arcs in \([k, \ell] \). We represent these points by \( \hat{w}_j \) for arcs \((j + 1, j)\), where \( j \in [k - 1, \ell - 1] \).

(xii) If \( j + 1 \notin L^+ \), then consider the points

\[
\hat{w}_j = w_t + m_j.
\]

For \( j \in L^+ \), we use indices \( l_j \). Let \( l_0 := k - 1 \) and \( j = 1, \ldots, |L^+| \). If \( l_{j-1} \in [p, \ell - 1] \), then let

\[
\hat{w}_{l_j} = \sum_{t \in [1, k-2] \cup [\ell + 1, n]} (d_t e_t + h_t) + (d_{k, l_j-1} - u_{k-1}) e_t + h_t + \sum_{t \in [k-1, \ell-1] \setminus \{l_j-1\}} k_t +
\]

\[
(d_{k-1} + u_{k-1}) e_{k-1} + h_{k-1} + \sum_{t \in [k, \ell-1]} (u_{k-1} - d_{kt}) f_t +
\]

\[
\sum_{t \in [l_{j-1}, k-1]} d_{t, l_{j-1}} f_t + \sum_{t \in [l_{j-1} + 1, l_{j-1}]} (d_{t, l_{j-1}} g_t + m_t) + \sum_{t \in [l_{j-1}, \ell-1]} d_{t+1, l_j} f_t.
\]

If \( l_{j-1} \in [k - 1, p - 1] \), then

\[
\hat{w}_{l_j} = \sum_{t \in [1, k-2] \cup [\ell + 1, n]} (d_t e_t + h_t) + d_{k-1, l_{j-1}} e_{k-1} + h_{l_j} + \sum_{t \in [k-1, \ell-1] \setminus \{l_j-1\}} k_t +
\]

\[
\sum_{t \in [k-1, l_{j-1} - 1]} d_{t+1, l_{j-1}} f_t + \sum_{t \in [l_{j-1} + 1, l_j - 1]} (d_{t, l_{j-1}} g_t + m_t) + \sum_{t \in [l_j, \ell-1]} d_{t+1, l_j} f_t.
\]

Note that, \( l_{j-1} \in H \) due to sufficiency condition (i).

Now, we give the set of points \( \hat{w}_j \) where \( j \in [k - 1, \ell - 1] \):

(xiii) If \( j + 1 \notin L^+ \), then consider the points:

\[
\hat{w}_j = w_t + m_j + \varepsilon g_j + \varepsilon f_j.
\]

For \( j \in L^+ \), we use indices \( l_j \), where \( j = 1, \ldots, |L^+| \) as in the definitions of \( \hat{w}_j \). If \( l_{j-1} \in [p, \ell - 1] \), then define
\[ \hat{w}_{l_j} = \sum_{t \in [1, k-2] \cup \{l+1, n\}} (d_t e_t + h_t) + (d_{k,l_j-1} - u_{k-1}) e_t + h_t + \sum_{t \in [k-1, l] \setminus \{l_{j-1}\}} k_t + \sum_{t \in [k-1, l-1] \setminus \{l_{j-1}\}} (d_{k,l_{j-1}} - \varepsilon) e_{k-1} + h_{k-1} + (d_{l_{j-1}+1, t} + \varepsilon) e_{l_j} + \sum_{t \in [k, t-1]} (u_{k-1} - d_k - \varepsilon) f_t + \sum_{t \in [l, l_{j-1}]} (d_{l_{j-1}} - \varepsilon) f_t + \sum_{t \in [l_{j-1}+1, l_{j-1}]} ((d_{l_{j-1}} + \varepsilon) g_t + m_t) + \sum_{t \in [l_{j-1}+1, l_{j-1}]} d_{l_{j-1}+1} f_t. \]

Suppose that, \( l_{j-1} \in [k-1, p-1] \). Due to the sufficiency condition (i), we assume \( l_{j-1} \in H \). Then, consider the solution:

\[ \hat{w}_{l_j} = \sum_{t \in [1, k-2] \cup \{l+1, n\}} (d_t e_t + h_t) + (d_{k-1, l_{j-1}} - \varepsilon) e_{k-1} + h_{l_j} + (d_{l_{j-1}+1, t} + \varepsilon) e_{l_j} + \sum_{t \in [k, t-1]} k_t + \sum_{t \in [k, t-1] \setminus \{l_{j-1}\}} (d_{l_{j-1}} - \varepsilon) f_t + \sum_{t \in [l, l_{j-1}]} ((d_{l_{j-1}} + \varepsilon) g_t + m_t) + \sum_{t \in [l, l_{j-1}]} d_{l_{j-1}+1} f_t. \]

So far, we provided \( 5n - 3 \) affinely independent points. Let \( \hat{w}_0 \) be the last affinely independent point that is on face \( F \):

\[ \hat{w}_0 = \hat{w}_{l_t} + \varepsilon e_{l_t} + \varepsilon \sum_{t \in [k-1, l_{t-1}]} g_t + m_{k-1}. \]

\[ \square \]

In the next theorem, we provide sufficient conditions for inequality (4.58). First, we define the following indices:

\[ q = \max\{j \in L^+: b_t \leq d_{j+1}\}, \]

\[ \ell = \max\{j \in [k-1, \ell] \setminus L^+\} \]

and

\[ \ell' = \min\{j \in L^+: j \geq q\}. \]

**Theorem 4.29.** For a node set \([k, \ell]\), let \( T \subseteq [k, \ell], K = L^+ \) in Proposition 4.21. Recall that in \( S^+ = \{(\ell + 1, \ell)\} \) and \( S^- = \emptyset \) in inequality (4.58). If \( d_j > 0 \) for all \( j \in [k, \ell] \), then the conditions below are sufficient for inequality (4.58) to be facet-defining for \( \text{conv}(\mathcal{P}) \):

(i) \( (\ell + 1, \ell) \cup \{[q+1, \ell] \cap L^+\} \subseteq H \);

(ii) for all \( j \in L^+, \rho_j(\{(\ell + 1, \ell)\}) > 0 \);
(iii) \(b_{v-1} > d_{k,v-1}\);

(iv) \(b_{l-1} \geq d_{k,l-1}\);

(v) if \(b_\ell = d_{q+1,\ell}\), then \(l_L > q\), otherwise \(l_L \geqq\);

(vi) \(\ell \in [q, \ell]\), \(b_{\ell-1} \geq d_{k,\ell-1}\) and \(u_\ell \geq d_{\ell+1,\ell}\);

(vii) if \(k > 1\), then \(b_{k-1} \geq d_{k-1,\ell-1}\);

(viii) for all \(l_j \in L^+\), \(u_{l_j} > d_{l_j+1, \ell+1}\).

Proof. Inequality (4.58) can be considered as a mirror image of inequality (4.32), where the ordering of nodes \([1, n]\) is changed to \([q, \ldots, 1]\) and inventory and backlog arcs are flipped. Consequently, the \(5n-2\) affinely independent points provided in Theorem 4.28 can be transformed in the same fashion so that they are on the face induced by inequality (4.58).

4.3.2 Finding violated inequalities

In this section, we discuss how to find lifted path pack inequalities introduced in Section 4.3 that are violated by a given point \((x, y, s, r, q, z) \in \mathbb{R}^{6n}\). For a given \([k, \ell] \subseteq [1, n]\), and the sets \(S^+\) and \(S^-\), the set \(L^+\) that maximizes the violation of these inequalities can be found in \(O(n^2)\) time. We explain this procedure using inequality (4.32), where the set \(M = L^+\) for simplification. We rearrange this inequality such that for a given \([k, \ell]\) selection and the fractional solution, the right hand side is a constant value:

\[
\sum_{j \in L^+} (y_j - \rho_j x_j - \delta_j q_{j-1}) + \sum_{j \in H} \gamma_j (1 - z_j) \leq u_{k-1} + i_\ell + r_{k-1} - i_{k-1}.
\]

The separation problem aims to find the sets \(L^+\) and \(H\) that maximize the left hand side of the inequality above. In Atamtürk and Küçükyavuz (2005), the authors model this problem as a longest path algorithm. We follow a similar approach here.

We construct a graph consisting of \(O(n^2)\) nodes and \(O(n^3)\) arcs. Let the nodes be represented by pairs such as \([i', j]\) and \([i', j']\). The first item of the pair keeps track of the last time period that was selected to be in \(L^+\). If the path passes by the node \([i', j]\), then the arc \((j, j + 1)\) is selected to be in \(H\) and the last production arc in \(L^+\) is \(i'\). Similarly, if the path passes by the node \([i', j']\), then the production arc \(j' \in L^+\) and the previously chosen item in \(L^+\) is \(i'\). Note that, tracking \(i'\) now allows us to compute the coefficient \(\delta_{j'}\). More formally the nodes of this temporary graph can be listed as follows:

1) \([i', j] : i' \in [k - 1, \ell - 1], j \in [i' + 1, \ell - 1]\),

2) \([i', j'] : i' \in [k - 1, \ell - 1], j' \in [i' + 1, \ell]\),
3) \([k-1]:\) select the inventory arc \((k-1,k)\) to be in \(H,\)

4) \([(k-1)']:\) arc \((k-1,k)\) is not in \(H,\)

5) \([0]:\) dummy source node.

Similarly, the arcs connecting the nodes described above are as follows:

1) \([i',j] \rightarrow [i',t],\) for \(i' \in [k-1, \ell - 2], j \in [i' + 1, \ell - 2] \) and \(t \in [j + 1, \ell - 1]\) with cost

\[
(1 - \bar{z}_{j}) \times \left[ \min \{(u_{k-1} - d_{kj})^+, d_{j+1,t}\} \right],
\]

2) \([i',j'] \rightarrow [j',t'],\) for \(i' \in [k-1, \ell - 1], j' \in [i' + 1, \ell] \) and \(t' \in [j' + 1, \ell]\) with cost

\[
\bar{y}_{j'} + \min \{d_{kj'} + u_{j'} - u_{k-1}, d_{k\ell} - u_{k-1}, u_{j'} + d_{j',t'} \} \times \bar{x}_{j'}
- \min \{\bar{r}_{j'-1}, \bar{q}_{j'-1} \times \min \{b_{j'-1}, d_{i'j'-1}\}\},
\]

3) \([i',j] \rightarrow [i',t'],\) for \(i' \in [k-1, \ell - 2], j \in [i' + 1, \ell - 2] \) and \(t' \in [i', \ell]\) with cost

\[
(1 - \bar{z}_{j}) \times \left[ \min \{(u_{k-1} - d_{kj})^+, d_{j+1,t'-1}\} \right],
\]

4) \([i',j] \rightarrow [j',t],\) for \(i' \in [k-1, \ell - 1], j' \in [i' + 1, \ell] \) and \(t \in [j' + 1, \ell - 1]\) with cost

\[
\bar{y}_{j'} + \min \{d_{kj'} + u_{j'} - u_{k-1}, d_{k\ell} - u_{k-1}, u_{j'} + d_{j',t} \} \times \bar{x}_{j'}
- \min \{\bar{r}_{j'-1}, \bar{q}_{j'-1} \times \min \{b_{j'-1}, d_{i'j'-1}\}\},
\]

5) \([k-1] \rightarrow [(k-1)',j],\) for \(j \in [k, \ell - 1]\) with cost

\[
(1 - \bar{z}_{k-1}) \times \left[ \min \{u_{k-1}, d_{kj}\} \right],
\]

6) \([k-1] \rightarrow [(k-1)',j'],\) for \(j \in [k, \ell]\) with cost

\[
(1 - \bar{z}_{k-1}) \times \left[ \min \{u_{k-1}, d_{k,j'-1}\} \right],
\]

7) \([(k-1)'] \rightarrow [(k-1)',j'],\) for \(j' \in [k, \ell]\) with cost zero,

8) \([(k-1)'] \rightarrow [(k-1)',j],\) for \(j \in [k, \ell - 1]\) with cost zero,

9) \([0] \rightarrow [k-1]\) with cost zero,

10) \([0] \rightarrow [(k-1)']\) with cost zero.
Consequently the longest path from the dummy source node [0] gives the sets \( L^+ \) and \( H \) that maximize the violation of inequality (4.32) by a given fractional solution \((\tilde{x}, \tilde{y}, \tilde{s}, \tilde{r}, \tilde{q}, \tilde{z}) \in \mathbb{R}^m\). Since the graph described here is a directed acyclic graph consisting of \( O(n^3) \) arcs and \( O(n^2) \) nodes, the longest path can be found in \( O(n^3) \) time. This structure can be repeated for all extended path pack inequalities where the costs are updated.

### 4.4 Computational Study

We test the effectiveness of the cuts introduced in this chapter by embedding them in a branch-and-cut framework. The experiments are ran on a Linux workstation with 3.60 GHz Intel® Xeon® CPU E5-1650 and 32 GB of RAM with 1 hour time limit and 1 GB memory limit. The branch-and-cut algorithm is implemented in C++ using IBM’s Concert Technology of CPLEX (version 12.5). The experiments are ran with one hour limit on elapsed time and 1 GB limit on memory usage. The number of threads is set to one and the dynamic search is disabled. We also turn off heuristics and preprocessing as the purpose is to see the impact of the inequalities by themselves.

### Instance Generation

Let \( n \) be the total number of time periods and \( f \) be the ratio of the fixed cost to the variable cost associated with a production arc. The parameter \( c \) controls how large the production arc capacities are with respect to average demand. All parameters are generated from a discrete uniform distribution. The demand for each node is drawn from the range [1, 30]. Let \( \bar{d} \) be the average demand over all time periods. Inventory and backlogging upper-bounds are drawn from \([1.0 \times \bar{d}, 2.0 \times \bar{d}]\) and \([0.3 \times \bar{d}, 0.8 \times \bar{d}]\), respectively. The variable costs \( v^p_t \), \( v^i_t \) and \( v^b_t \) are drawn from the ranges \([1, 10]\), \([1, 10]\) and \([1, 20]\), respectively. Finally, fixed ordering costs \( f^p_t \) are set equal to \( f \times v^p_t \), fixed inventory holding cost is \( f^i_t = f \), for all \( t \in [1, n] \) and fixed backlogging cost is set to \( f^b_t = 2 \times f \), for all \( t \in [1, n] \). Using these parameters, we generate five random instances for each combination of \( n \in \{50, 100, 150, 200\} \), \( f \in \{100, 200, 500, 1000\} \) and \( c \in \{2, 5, 10\} \).

### Results

We report multiple performance measures. Let \( z_{\text{INIT}} \) be the objective function value of the initial LP relaxation and \( z_{\text{ROOT}} \) be the objective function value of the LP relaxation after all the valid inequalities added. Moreover, let \( z_{\text{UB}} \) be the objective function value of the best feasible solution found within time/memory limit among all experiments for an instance. Let \( \text{init gap} = 100 \times \frac{z_{\text{UB}} - z_{\text{INIT}}}{z_{\text{UB}}} \), \( \text{root gap} = 100 \times \frac{z_{\text{UB}} - z_{\text{ROOT}}}{z_{\text{UB}}} \). We compute the improvement of the relaxation due to adding valid inequalities as \( \text{gap} \)
imp = $100 \times \frac{\text{init gap} - \text{root gap}}{\text{init gap}}$. We also measure the optimality gap at termination as:
\[
\text{end gap} = \frac{z_{UB} - z_{LB}}{z_{UB}},
\]
where $z_{LB}$ is the value of the best lower bound given by CPLEX. We report the average number of valid inequalities added at the root node under column cuts, average elapsed time in seconds under time, average number of branch-and-bound nodes explored under nodes. If there are instances that are not solved to optimality within the time/memory limit, we report the the end gap averaged over unsolved instances under end gap and the number of unsolved instances under unslvd next to time results. All numbers except initial gap, end gap and time are rounded to the nearest integers.

In Table 4.1, we present the results for path pack inequalities introduced in Section 4.2 under columns spi. Under columns lspi, we show results for both path pack inequalities and lifted path pack inequalities introduced in Section 4.3. For comparison, under base columns, we present the results for experiments where no valid inequalities are added. Consequently, in these experiments, the gap improvement and the number of cuts added are zero for all instances. We observe that the introduction of the inventory and backlog fixed charge variables to path pack inequalities, the average gap improvement increases by 6%. Moreover, the number of branch-and-bound nodes explored decreases by 90%. The average elapsed time is more than double without the lifted inequalities. All of the base experiments terminated early due to the 1GB memory limit for instances with $n = 100$. As a result, we observe a lower average elapsed time with a 22% average optimality gap.

In Table 4.2, we investigate the classes of inequalities introduced in Sections 4.2 and 4.3 deeper. Under lspi, we present the same results as in Table 4.1 where all classes of inequalities are used. Under columns $[i, j]$, we use a only a certain subset of inequalities. In this naming scheme, $i \in \{1, 2, 3\}$ and $j \in \{0, 1, 2, 3\}$, where $i = 1$ implies that the set is $S^+ = \{(k - 1, k)\}$, $i = 2$ implies $S^+ = \{((\ell+1), \ell)\}$, $i = 3$ implies $S^+ = \{(k - 1, k), (\ell + 1, \ell)\}$ and similarly, $j = 0$ implies $S^- = \emptyset$, $j = 1$ implies $S^- = \{(k, k-1)\}$, $j = 2$ implies $S^- = \{((\ell+1), \ell)\}$ and $j = 3$ implies $S^- = \{(k, k-1), ((\ell+1), \ell)\}$. More specifically in columns $[1, 0]$, inequalities (4.10) and (4.32), in $[1, 1]$, inequalities (4.18) and (4.49), in $[1, 2]$, inequalities (4.16) and (4.46), in $[1, 3]$, inequalities (4.17) and (4.47), in $[2, 0]$, inequalities (4.15) and (4.58), in $[2, 1]$, inequalities (4.19) and (4.62), in $[2, 2]$, inequalities (4.21) and (4.64), in $[2, 3]$, inequalities (4.20) and (4.63), in $[3, 1]$, inequalities (4.24), (4.57) and (4.68), in $[3, 2]$, inequalities (4.23), (4.55) and (4.67) and finally in $[3, 3]$, inequalities (4.22), (4.50) and (4.65) are used. We observe that each $(S^+, S^-)$ selection perform relatively well on their own, however when all cuts used simultaneously, the performance is dominating both in terms of time and gap improvement. For this instance set, column $[1, 0]$ (the set selection $S^+ = \{(k - 1, k)\}$ and $S^- = \emptyset$) give the best performance and in comparison, its symmetrical case $[2, 0]$ (the selection $S^+ = \{((\ell + 1), \ell)\}$ and $S^+ = \emptyset$) performs worse. This difference in performance is because the backlog arc capacities are generated to be smaller and it reflects on the effect of the cuts.
In Table 4.3, we investigate the marginal contribution of path pack inequalities of Sections 4.2 and 4.3 when added on top of CPLEX’s network cuts: multi-commodity flow (MCF), flow cover and flow path inequalities. Under the columns titled `lspix`, we report results for computations where both path pack inequalities and CPLEX’s network cuts are used. Similarly, under `cpx` columns, we report results for computations where only CPLEX’s network cuts are used. In this table, the columns `cuts` report the total number of cuts added to the branch-and-bound tree averaged over instances. On average, about 82% of the cuts added under `lspix` column are path pack inequalities. When path pack inequalities are used in addition to CPLEX’s network cuts, the gap improvement increases by 7%, the number of branch-and-bound nodes explored decreases by four orders of magnitude and the average time spent decreases by three orders of magnitude. Moreover, all of the instances were solved to optimality within our time/memory limits when path pack inequalities were used.

Table 4.1: Performance of lifted submodular path inequalities.

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Average: 87.9 94% 88% 16878 327093 11314830 8925 6509 720 (0.2,1) 1674 (1.1,4) 642 (22.3,3)
Table 4.2: Performance of each class of lspi broken down.

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**Average** 93% 88% 80% 66% 65% 77% 70% 79% 71% 76% 87% 78%
Table 4.3: Marginal contribution of path pack inequalities to CPLEX’s network cuts.

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Chapter 5

Summary of Thesis and Conclusions

In this thesis, we study various fixed-charge networks by formulating them as linear mixed-integer programs. We propose different classes of valid inequalities and show their strength both theoretically and computationally. These valid inequalities are derived from submodular inequalities that are initially proposed by Wolsey (1989).

After giving a brief introduction and some preliminary definitions in Chapter 1, we focus on simple directed path structures of capacitated fixed-charge networks in Chapter 2. We show how to efficiently compute an explicit form of the submodular inequalities. We refer to these explicitly expressed submodular inequalities as path cover inequalities. Furthermore, we obtain a generalized class through a simultaneous lifting procedure. In Chapter 3, we focus on slightly more general paths where consecutive nodes $j$ and $j+1$ are connected through a forward path arc $(j, j+1)$ and a backward path arc $(j+1, j)$. In this chapter, we give two explicit descriptions of submodular inequalities and refer them as path cover and path pack inequalities.

We provide necessary and sufficient conditions under which the inequalities introduced in Chapters 2 and 3 are facet defining for the convex hull of the feasible solutions. Moreover, we present extensive computational results that show the effectiveness of these inequalities when used in a branch-and-cut framework.

In Chapter 4, we study single item lot-sizing problems with backlogging and inventory bounds. This class of the lot-sizing problem is a special case of the path structure studied in Chapter 3. Since we assume that the production arcs are uncapacitated, the coefficients of path pack inequalities were obtained parametrically. Using different arc set selections, we give eleven classes of path pack inequalities. Then, we incorporate the binary variables of inventory and backlog arcs to path pack inequalities using a lifting procedure and present computational results.
Bibliography


Appendices
Appendix A

Equivalency of $P'$ to the maximum flow problem

Recall that $K^+ = \{ t \in E^+ : a_t = 1 \}$ and $K^- = \{ t \in E^- : a_t = 0 \}$, where the values $a_t$ are the coefficients of arcs in $E$ in the objective function of $P'$.

Proposition A.1. If $d_j \geq 0$ for all $j \in V$, then there exists an optimal solution to the optimization problem $P'$ where $y_t = 0$ for all $t \in E^- \setminus K^-$. 

Proof. Given an optimal solution $(y^*, i^*)$ to $P'$, for notational convenience, let $\bar{Y}_{j^-} := \sum_{t \in E_j^-} y_t^*$ and $\bar{Y}_{j^+} := \sum_{t \in E_j^+} y_t^*$ for $j \in V$. Let

$$\epsilon_j' = d_j - [i_{j-1} + \bar{Y}_{j^+}] + [\bar{Y}_{j^-} + i_j]$$

be the slack of flow balance constraint (2.6) of node $j$. Moreover, let $p$ be some node in $V$ with $\bar{Y}_p^- = \epsilon > 0$. Observe that if $\epsilon_p' > 0$, then decreasing the outgoing flow $\bar{Y}_p^-$ by $\epsilon_p'$ increases the objective function value by $\epsilon_p'$ while preserving feasibility. This operation provides contradiction with the assumption of optimality. Therefore, we assume that $\epsilon_j' = 0$ for nodes $j$ with $\bar{Y}_j^- > 0$.

Then, applying the procedure described in Algorithm 2 to the optimal solution $(y^*, i^*)$, we find an optimal solution where $\bar{y}_t = 0$ for all $t \in E^- \setminus K^-$. 

Remark A.1. If $d_j < 0$ for some $j \in V$, one can represent the supply amount as a dummy arc in $E_j^+$ with a fixed flow and capacity of $-d_j$ and set the modified demand of node $j$ to be $d_j = 0$.

If the path $V$ has nodes $j$ with $d_j < 0$, then using Remark A.1, we convert it into a path where $d_j \geq 0$ for all $j \in V$. After the modification of $V$, we note that the dummy supply arcs will always be open, therefore, they will always be included in the set $S^+$. Let us call the constraints of fixed flow value on the dummy supply arc fixed-flow constraints for the sake of conciseness. Notice that the formulation $P'$ becomes a
Algorithm 2

\[(y^*, i^*)\): an optimal solution to \(P'\).
\[ J = \{k \leq j \leq \ell : \bar{Y}_j^- > 0\}. \]
for \( p \in J \) do
\[ \epsilon := Y_p^-, \bar{Y}_p^- = 0 \]
for \( j = p \) to \( k \) do
\[ \Delta = \min\{\epsilon, \bar{Y}_j^+\} \]
\[ \bar{Y}_j^+ = \bar{Y}_j^+ - \Delta \]
\[ i_{j-1} = i_{j-1} - (\epsilon - \Delta) \]
\[ \epsilon = \epsilon - \Delta \]
end for
end for

relaxation for the modified path because of the missing fixed-flow constraints. In the next proposition, we prove that the optimal objective function value does not change by adding/dropping the fixed-flow constraints to \(P'\).

**Proposition A.2.** Suppose \(d_j < 0\) for some \(j \in V\). Let \(G'\) be the modification of graph \(G\) using Remark A.1. If the optimization problem \(P'\) for \(G'\) along with the fixed-flow constraints is feasible, then the optimal objective function value does not change by adding the fixed-flow constraints to \(P'\).

**Proof.** We need to show that \(P'\) has an optimal solution where the dummy supply arcs has flows at their capacities. Notice that the modification using Remark A.1 makes Proposition A.1 applicable to the modified graph \(G'\). Therefore, there exists an optimal solution \((y^*, i^*)\) to \(P'\) where \(y_t^* = 0\) for \(t \in E^+ \setminus K^-\). Let \(p \in S_j^+\) represent the index of the dummy supply arc with \(c_p = -d_j\). If \(y_p^* < c_p\), then satisfying the fixed-flow constraints will require pushing flow through the arcs in \(E^- \setminus K^-\). We use Algorithm 3 in order to obtain an optimal solution with \(y_p^* = c_p\). Note that each arc in \(E_j^- \setminus K^-\) affect the objective function value and constraints of \(P'\) the same way, therefore we merge these outgoing arcs into one in Algorithm 3. We represent the merged flow and capacity by \(\bar{Y}_j^- = \sum_{t \in E_j^- \setminus K^-} y_t^*\) and \(\bar{C}_j = c(E_j^- \setminus K^-)\) for \(j \in V\).

Proposition A.2 shows that adding the fixed-flow constraints to \(P'\) formulated for a modified a path via Remark A.1 does not change the optimal objective function value. Consequently, without loss of generality, we assume that \(d_j \geq 0\) for all \(j \in V\). Then, using the optimality condition in Proposition A.1, the variables \(y_t\) for \(t \in E^- \setminus K^-\) can be dropped from the formulation \(P'\). As a result, without loss of generality, all arcs in \(S^+\) are leaving a dummy source node \(s_V\), the outgoing arcs \(K^- \setminus L^-\) are incoming to a dummy sink node \(t_V\) and the demands values are represented as arcs with capacity \(d_j\), incoming to \(t_V\) from node \(j \in V\) (see Figure 2.2 for a representation).
Algorithm 3

\( \mathcal{J} \): Set of supply nodes in \( V \) where the nodes are sorted with respect to their order in \( V \).

\((y^*, i^*)\): the optimal solution to \( \mathcal{P}' \) with \( y_t = 0 \) for all \( t \in E^- \).

for \( q \in \mathcal{J} \) do

\( \Delta = c_p - y_p^* \)

for \( j = q \) to \( \ell \) do

\( \bar{Y}_j^- = \bar{Y}_j^- + \min\{\bar{C}_j - \bar{Y}_j^-, \Delta\} \)

\( \bar{Y}_j^- = \Delta - \min\{\bar{C}_j - \bar{Y}_j^-, \Delta\} \)

\( i_j^* = i_j^* + \Delta \)

end for

if \( \Delta > 0 \) then

The problem \( \mathcal{P}' \) along with fixed-flow constraints is infeasible

end if

end for
Appendix B

Proofs from Chapter 2

B.1 Proof of Theorem 2.6

1. If for some $t' \in S^+$, $\rho_v(S^+ \setminus \{t\}, L^-) \geq c_v$, then removing $t'$ from $S^+$ in inequality (2.10) results in an inequality at least as strong. To see this, let $S' = S^+ \setminus \{t'\}$.

Rewriting inequality (2.10), we obtain

$$
\sum_{t \in S'} (y_t + \rho_t(S^+ \setminus \{t\}, L^-) (1 - x_t)) + y_v + \sum_{t \in L^-} \rho_t(S^+, L^- \setminus \{t\}) x_t
\leq v(S^+, L^-) - \rho_v(S^+ \setminus \{t'\}, L^-) + \rho_v(S^+ \setminus \{t'\}, L^-) x_v + y(E^- \setminus L^-)
= v(S', L^-) + \rho_v(S^+ \setminus \{t'\}, L^-) x_v + y(E^- \setminus L^-).
$$

Consider adding the submodular inequality

$$
\sum_{t \in S'} (y_t + \rho_t(S' \setminus \{t\}, L^-) (1 - x_t)) + \sum_{t \in L^-} \rho_t(S', L^- \setminus \{t\}) x_t \leq v(S', L^-) + y(E^- \setminus L^-),
$$

and $y_v \leq c_v x_v$. The resulting inequality dominates inequality (2.10). This follows because $\rho_t(S' \setminus \{t\}, L^-) \geq \rho_t(S^+ \setminus \{t\}, L^-)$, from the definition of submodularity of the set function $v$.

2. If for some $t \in L^-$, $\rho_t(S^+, L^- \setminus \{t\}) \leq -c_t$, then summing the submodular inequality obtained by removing $t$ from $L^-$, and the inequality $y_t \leq c_t x_t$ results in an inequality at least as strong.

3. If $L^- = \emptyset$ and $\max_{t \in S^+} \rho_t(S^+ \setminus \{t\}, L^-) = 0$, then summing flow balance inequalities (2.1) gives an inequality at least as strong.

4. Suppose at node $p$ we have $c(S^+_p) \geq d_p$, $\max_{t \in S^+_p} \rho_t(S^+ \setminus \{t\}, L^-) = 0$ and $L^-_p = \emptyset$. 

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Then, observe that the submodular path inequality for path $V$ is
\[
\sum_{j=k}^{p} \sum_{t \in S_j^+} (y_t + (c_t - \lambda_j)^+(1 - x_t)) + \sum_{j=p}^{\ell} y(S_j) \\
\leq d_{k\ell} + \sum_{j=k}^{p} (\lambda_j x(L_j^-) + y(E_j^- \setminus L_j^-)) + \sum_{j=p}^{\ell} y(E_j^-). \quad (B.1)
\]
Since $c(S_p^+) \geq d_{p\ell}$, nodes $\{p, \ldots, \ell\}$ do not pull any flow through nodes $\{k, \ldots, p-1\}$ which implies that they do not have any effect on the excess values $\lambda_j$ for $j \in [k, p-1]$. The submodular path inequality for path $V = [k, p-1]$
\[
\sum_{j=k}^{p-1} \sum_{t \in S_j^+} (y_t + (c_t - \lambda_j)^+(1 - x_t)) \\
\leq d_{k,p-1} + i_{p-1} + \sum_{j=k}^{p-1} (\lambda_j x(L_j^-) + y(E_j^- \setminus L_j^-))
\]
summed with the flow balance inequalities (2.1) for nodes $j \in [p, \ell]$
\[
i_{p-1} + \sum_{j=p}^{\ell} (y(E_j^+) - y(E_j^-)) \leq d_{p\ell}
\]
gives an inequality at least as strong as inequality (B.1).

5. Suppose for some node $k < j \leq \ell$, $\alpha_j = d_{k,j-1} + u_{j-1} + c(S_j^+)$ and $\cup_{i=j}^{\ell} S_i^+$ is a path cover for nodes $j, \ldots, \ell$. Let $\lambda_i, i \in V$ be the excess values computed for path $V$. We consider dividing path $V = [k, \ell]$ into two sub-paths where $V_1 = [k, j-1]$ and $V_2 = [j, \ell]$ and identify submodular path inequalities for $V_1$ and $V_2$.

Dropping nodes $j, \ldots, \ell$ from path $V$ does not change the excess values $\lambda_i$ for $i \in V_1$ because $\cup_{i=j}^{\ell} S_i^+$ is a path cover for $V_2$. In other words, nodes in $V_2$ do not pull any flow from nodes in $V_1$ and therefore, they do not affect the excess values $\lambda_i$ for $i \in V_1$. Then, submodular path inequality for path $V_1 = [k, j-1]$ is
\[
\sum_{i \in V_1} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_i)^+(1 - x_t)) \\
\leq d_{k,j-1} + i_{j-1} + \sum_{i \in V_1} (\lambda_i x(L_i^-) + y(L_i^-)).
\]

Now, we consider the submodular path inequality defined for $V_2$. If the path arc $(j-1, j)$ with capacity $u_{j-1}$ is added to $S_j^+$, then $\lambda_i$ values for $j \in V_2$ remain unchanged because $\alpha_j = d_{k,j-1} + u_{j-1} + c(S_j^+)$ implies that the path arc $(j-1, j)$ with capacity $u_{j-1}$ provides a bottleneck for sending a flow from node $k$ towards node $j$. In other words, the excess value that can be carried to node $j$ from nodes in $V_1$ is at least as large as the capacity of the path arc $(j-1, j)$, $u_{j-1}$. Therefore, the submodular path inequality defined for path $V_2 = [j, \ell]$ is
\[
i_{j-1} + \sum_{i \in V_2} \sum_{t \in S_i^+} (y_t + (c_t - \lambda_i)^+(1 - x_t)) \\
\leq d_{j\ell} + \sum_{i \in V_2} (\lambda_i x(L_i^-) + y(L_i^-)).
\]
As a result, submodular path inequalities defined for path $V_1$ and $V_2$ summed give an inequality at least as strong as the submodular path inequality defined for path $V$.

\[ \square \]

### B.2 Proof of Theorem 2.7

Let us first introduce the effective demand $\tilde{d}_j$ and effective path flow $\tilde{i}_j$ at each node of the path $V$. From node $\ell$ to node $k$ we first calculate:

$$
\tilde{d}_{j-1} = d_{j-1} + (\tilde{d}_j - c(S_j^+))^{+} \quad \text{and} \quad \tilde{i}_{j-1} = (\tilde{d}_j - c(S_j^+))^{+}
$$

with $\tilde{d}_\ell = d_\ell$ and $\tilde{i}_\ell = 0$. Then, we reassign effective demand $\tilde{d}_j$ to be $\tilde{d}_j = \min\{c(S_j^+), d_j\}$.

We also define effective remaining capacity of path arcs as $\tilde{u}_j = u_j - \tilde{i}_j$. Note that $\tilde{i}_j \leq u_j$ due to the assumption of $S^+$ defining a path cover. Using this notation, we have

$$
\lambda_j = \min\{\tilde{u}_{j-1}, \lambda_{j-1}\} + c(S_j^+) - \tilde{d}_j.
$$

Furthermore, we observe that $\tilde{u}_j > 0$ because $\tilde{u}_j = 0$ implies $\lambda_{j+1} = 0$ which contradicts with the first necessary condition in Theorem 2.6. We represent the feasible point $p$ as $z^p = (y^p, x^p, i^p)$ or $u^p = (y^p, x^p, i^p)$. Define set $K = \{ t \in S^+ : t = \arg\max_{t \in S_j^+} c_t, \quad j \in [k, \ell] \}$. First we describe the feasible point $\tilde{z} = (\tilde{y}, \tilde{x}, \tilde{i})$ where

$$
\begin{align*}
(\tilde{y}_t, \tilde{x}_t) &= (c_t - (\lambda_j - \min\{\tilde{u}_{j-1}, \lambda_{j-1}\})^{+}, 1), & t \in S_j^+ \cap K, \\
(\tilde{y}_t, \tilde{x}_t) &= (c_t, 1), & t \in S^+ \setminus K, \\
(\tilde{y}_t, \tilde{x}_t) &= (0, 0), & t \in A \setminus (S^+ \cup I), \\
\tilde{i}_j &= \tilde{i}_j, & j \in V.
\end{align*}
$$

The feasible point $\tilde{z}$ satisfies submodular path inequality at equality since $\sum_{t \in S_j^+} \tilde{y}_t = \tilde{d}_j$ for all $j \in V$ and $d_{k\ell} = \sum_{j=k}^{\ell} \tilde{d}_j$. Without loss of generality, let us call the arc with maximum capacity in $S_k^+$ arc 1. Recall that $c_1 > \lambda_k$. Let $\epsilon > 0$ be a sufficiently small value. We first introduce the following 2$|S^+|$ affinely independent points, where first $|S^+|$ points are represented as $\bar{z}$ and second half are represented by $\hat{z}$.

- i) Let $\bar{z}^1 = \bar{z}$ and let $t \in S_j^+ \setminus \{1\}$. Point $\bar{z}^t$ is obtained by sending extra flow amount $\epsilon$ from arc 1 and $\epsilon$ less from arc $t$. Then, $\bar{z}^t$ is described by $\bar{y}^1_t = \bar{y}^1 + \epsilon, \bar{y}^t_t = \bar{y}^1_t - \epsilon, \bar{y}^m_m = \bar{y}^m_m$ for $m \in A \setminus (I \cup \{1, t\}), \bar{i}_s = \bar{i}_s + \epsilon$ for $s = k, \ldots, j-1$ and $\bar{x}^t = \bar{x}^1$.

- ii) Let $t \in S_j^+$, point $\hat{z}^t$ is obtained by closing arc $t$. In order to satisfy inequality (2.10) at equality, we need to send $\hat{y}^1_t - (c_t - \lambda_j)^+$ extra amount of flow from
Let

We now send a \textit{flow} of $y$ next we describe 2

Point 2.10

Next, we define the \textit{affinely independent points} $\tilde{w}^t$ and $\hat{w}^t$ described for arcs $t \in L^-$. 

v) Let $\tilde{w}^t$ is obtained by opening the arc $t \in L^- \tilde{t}$ and setting $\tilde{x}^t = 1$. Starting with feasible point $\tilde{z}$, we send an extra flow of $\lambda_j$ from arcs in $K \cap S^+$ for $i = k, \ldots, j$.

This operation conserves feasibility and satisfies the inequality (2.10) at equality because $v(S^+, L^- \setminus \{t\}, S^-)$ is feasible with a value of $d_{kt} - \lambda_j$. Extra incoming flow of $\lambda_j$ is sent from outgoing arc $t$ resulting $\tilde{y}_t = \lambda_j$. Let $\tilde{x}_m = 1$ for $m \in S^+ \cup \{t\}$ and 0 otherwise. At points $\tilde{w}$, the flow balance constraints hold at equality. Therefore, $\tilde{y} = (\tilde{y}_j - \sum_{m \in S^+} \tilde{y}_m - d_j)^+$. 

vi) Let $\hat{w}^t$ have similar properties to $\tilde{w}^t$ and let $t \in L^- \hat{t}$. The second set of feasible points have $\hat{x}^t = \tilde{x}^t$, $\hat{y} = \tilde{y}$, $\hat{y}_m = \tilde{y}_m$ for $m \neq t$ and $\hat{y}_t = \lambda_j + \epsilon$.

We now construct the \textit{affinely independent points} corresponding to arcs $t \in E^- \setminus L^-$. 

vii) Starting with feasible point $\tilde{z}$, we obtain point $w$ by opening arc $t \in E^- \setminus L^-$. Let $y' = \tilde{y}$, $x'_m = \tilde{x}_m$ for $m \neq t$ and $x'_t = 1$, $i' = \tilde{i}$. 

viii) Let $t \in L^- \hat{t}$. Starting with $\tilde{z}$, we send an extra flow of $\epsilon$ from arc 1, $y'_m = \tilde{y}_1 + \epsilon$, $y'_m = \epsilon$ and $y'_m = \tilde{y}_m$ for $m \neq t, m \neq 1$. Let $x'' = x^t$ and $i''_s = i_s + \epsilon$ for $s = k, \ldots, j - 1$.

Affinely independent points corresponding to path arcs are represented by $v^j$ and defined below.

ix) Prior to giving the feasible points, we first define a new notion called \textit{minimal cover blocks}. Suppose we have a subpath $B \subseteq M$ with cardinality $|B| = b$ and nodes $B = \{n_1, \ldots, n_b\}$, where without loss of generality $n_1$ and $n_b$ are the first and the last nodes of subpath $B$ respectively. We say that $B$ is a \textit{minimal cover block} if (i) $\cup_{j \in B} S^+_j$ is a path cover for $B$, (ii) $\cup_{j \in B \setminus \{n_1\}} S^+_j$ is not a path cover for $B \setminus \{n_1\}$ and (iii) if $b > 1$, then $S^+_b$ is not a cover for $\{n_b\}$. Notice that $V$ can be partitioned into a number of minimal cover blocks $B_1, \ldots, B_q$ where $V = \cup_{i=1}^q B_i$. 

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Let $n^i_1$ be the first node of block $B_i$ and $m^i = K \cap S^i_{n^i_1}$ be the arc with largest capacity incoming to node $n^i_1$. Moreover, let us call nodes in $B_i = \{n^i_1, \ldots, n^i_b\}$. While finding the point associated with path arc incoming to node $j$ where $j \in B_i$, we start from the feasible point where arc $m^i$ is closed, $z^{m^i}$. For each path arc $p = n^i_1 -1, \ldots, n^i_b -1$, send an extra flow of $\epsilon$ from path arcs $s = p, \ldots, n^i_b -1$. We, then, describe point $v^j$ as: $x^j = \hat{x}^m$, $y^j = \hat{y}^m$ and $v^j_i = v^m_i + \epsilon$ for $s = p, \ldots, n^i_b -1$.

\[
\text{We do the proof case by case. For each interval, we construct a solution that follows the optimality conditions provided in Section 2.3. We show the feasibility of the solution explicitly only under Case 1, since the procedure is the same for all the cases.}
\]

\textbf{Case 1:} $M_{j,i} - \bar{l}_j \leq z \leq M_{j,i} - \bar{\lambda}_j + \lambda_j$, \quad $i \in [1, p_j -1]$.

If $v_l \in S_{j^+} \setminus \{v_{p_j}\}$, then $y_{v_l} = 0$ for $l \leq i$ and $y_{v_l} = c_{v_l}$ for $l > i$, $y_{v_{p_j}} = c_{v_{p_j}} - z + M_{j,i} - \lambda_j$. If $v_l \in L_{j^-}$, then $y_{v_l} = c_{v_l}$ for $l \leq i$, $y_{v_l} = 0$ for $l > i$ and $w = m_j$. Because of the boundaries of $z$ we have,

\[c_{v_{p_j}} - \lambda_j \leq y_{v_{p_j}} \leq c_{v_{p_j}}.\]

Therefore, this solution agrees with the optimality conditions described above.

The solution is also feasible because constraint (2.38) is:

\[w + \sum_{t \in S_{j^+}} y_t + \sum_{t \in L_{j^-}} (c_t - y_t) = m_j + M_{j,r_j} - M_{j,i} - z + M_{j,i} - \bar{\lambda}_j = \hat{d}_j - z.\]

The objective function value of this solution is:

\[
\hat{d}_j - m_j + \sum_{t \in L_{j^-}} \lambda_j x_t + \sum_{t \in S_{j^+}} \lambda_j (1 - x_t) - \sum_{t \in S_{j^+}} (y_t + c_t (1 - x_t)) - \sum_{t \in L_{j^-}} c_t = \hat{d}_j - m_j + i \lambda_j - M_{j,r_j} - z - \bar{l}_j + \bar{\lambda}_j = z + i \lambda_j - M_{j,i}
\]

\textbf{Case 2:} $M_{j,i} + \lambda_j - \bar{\lambda}_j \leq z \leq M_{j,i+1} - \bar{\lambda}_j$, \quad $i \in [0, p_j -1]$.

If $v_l \in S_{j^+}$, then $y_{v_l} = 0$ for $l \leq i+1$, $y_{v_l} = c_{v_l}$ for $l > i+1$. If $v_l \in L_{j^-}$, then $y_{v_l} = c_{v_l}$ for $l \leq i+1$ and $y_{v_l} = 0$ for $l > i+1$ and $w = m_j$. 

\[\square\]

\textbf{B.3 Proof of Theorem 2.10}

We do the proof case by case. For each interval, we construct a solution that follows the optimality conditions provided in Section 2.3. We show the feasibility of the solution explicitly only under Case 1, since the procedure is the same for all the cases.
Case 3: $M_{j,i} - \bar{\lambda}_j \leq z \leq M_{j,i} - \bar{\lambda}_j + ml_j, \quad i \in [p_j, r_j - 1].$

If $v_l \in S_j^{++}$, then $y_{v_l} = 0$. If $v_l \in L_j^-$, then $y_{v_l} = c_{v_l}$ for $l \leq i$, $y_{v_l} = 0$ for $l > i$ and $w = m_j + M_{j,i} - \bar{\lambda}_j - z$.

Case 4: $M_{j,i} - \bar{\lambda}_j + ml_j < z \leq M_{j,i} - \bar{\lambda}_j + ml_j + \varphi_{j,i}, \quad i \in [p_j, r_j - 1].$

If $v_l \in S_j^{++} \setminus \{v_{p_j}\}$, then $y_{v_l} = 0$, $y_{v_{p_j}} = M_{j,i+1} - \bar{\lambda}_j - z$. If $v_l \in L_j^-$, then $y_{v_l} = c_{v_l}$ for $l \leq i + 1$ and $y_{v_l} = 0$ for $l > i + 1$ and $w = m_j$.

Case 5: $M_{j,i} - \bar{\lambda}_j + ml_j + \varphi_{j,i} \leq z \leq M_{j,i+1} - \bar{\lambda}_j, \quad i \in [p_j, r_j - 1].$

If $v_l \in S_j^{++}$, then $y_{v_l} = 0$. If $v_l \in L_j^-$, then $y_{v_l} = c_{v_l}$ for $l \leq i + 1$ and $y_{v_l} = 0$ for $l > i + 1$ and $w = m_j$.

Case 6: $M_{j,r} - \bar{\lambda}_j \leq z \leq \hat{d}_j$.

If $v_l \in S_j^{++}$, then $y_{v_l} = 0$. If $v_l \in L_j^-$, then $y_{v_l} = c_{v_l}$ and $w = m_j + M_{j,r_j} - \bar{\lambda}_j - z$.

□

**B.4 Proof of Theorem 2.11**

Figures (B.1a) and (B.1b) represent the two different forms that $\psi_j(z)$ can take if $T_j\lambda_j - \bar{\lambda}_j \leq 0$. 

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Figure B.1: Two different representations of the functions $\bar{\phi}(z)$ and $\phi_j(z)$.

In Figures B.1a and B.1b, the function $\bar{\psi}(z)$ is a superadditive function that conforms to the description in Theorem 13 of Gu et al. (1999). Then, using shifting down argument in Observation 3 in Appendix B.5 and changing a nonnegative piece of the function with non-positive values argument in Observation 4 that is also in Appendix B.5, one can easily verify that $\psi_j(z)$ is superadditive when $T_j \lambda_j - \bar{\lambda}_j \leq 0$.\hfill $\Box$

B.5 Observations on superadditive functions

Observation 3. If function $f(z)$ is superadditive for $z \in Z$ and $K \leq 0$, then $g(z) = K + f(z)$ is also superadditive for $z \in Z$. In other words, shifting a superadditive function downwards preserves superadditivity.

Proof. Let $z_1, z_2$ and $z_1 + z_2 \in Z$. We know that $f(z_1) + f(z_2) \leq f(z_1 + z_2)$. Since $K \leq 0$, summing left side with $2 \times K$ and right side with $K$, ensures that the following inequality holds:

$$K + f(z_1) + K + f(z_2) \leq K + f(z_1 + z_2)$$
\[ \implies g(z_1) + g(z_2) \leq g(z_1 + z_2). \]

Note that if a superadditive function is summed with a positive constant \( K \), then we need to ensure that

\[ K + \max \{ g(z_1) + g(z_2) - g(z_1 + z_2) : z_1, z_2, z_1 + z_2 \in Z \} \leq 0 \]

in order to conclude that the resulting function is superadditive as well.

**Observation 4.** Let \( f(x) \) be superadditive and non-positive on \( 0 \leq x \leq k \) for some \( k \geq 0 \) and \( g(x) \geq 0 \) be non-decreasing and superadditive on \( x \geq k \). Let \( f(k) = g(k) = 0 \). Then, the function

\[ \phi(x) = \begin{cases} f(x) & 0 \leq x \leq k \\ g(x) & x \geq k \end{cases} \]

is superadditive.

**Proof.** There are four cases we examine:

1. **Case 1:** \( 0 \leq x_1 \leq k \), \( 0 \leq x_2 \leq k \) and \( x_1 + x_2 \leq k \). Then,

\[ \phi(x_1) + \phi(x_2) - \phi(x_1 + x_2) = f(x_1) + f(x_2) - f(x_1 + x_2) \leq 0 \]

because \( f \) is superadditive on \([0, k]\).

2. **Case 2:** \( 0 \leq x_1 \leq k \), \( 0 \leq x_2 \leq k \) and \( x_1 + x_2 > k \). Then,

\[ \phi(x_1) + \phi(x_2) - \phi(x_1 + x_2) = f(x_1) + f(x_2) - g(x_1 + x_2) \leq 0 \]

because \( f(x_1) \leq 0 \), \( f(x_2) \leq 0 \) and \( g(x_1 + x_2) \geq 0 \).

3. **Case 3:** \( 0 \leq x_1 \leq k \), \( x_2 > k \) and \( x_1 + x_2 > k \). Then,

\[ \phi(x_1) + \phi(x_2) - \phi(x_1 + x_2) = f(x_1) + g(x_2) - g(x_1 + x_2) \leq 0 \]

since \( g(x_2) - g(x_1 + x_2) \leq 0 \) and \( f(x_1) \leq 0 \).

4. **Case 4:** \( x_1 > k \), \( x_2 > k \) and \( x_1 + x_2 > k \). Then,

\[ \phi(x_1) + \phi(x_2) - \phi(x_1 + x_2) = g(x_1) + g(x_2) - g(x_1 + x_2) \leq 0 \]

due to the superadditivity of the function \( g \).
Observation 5. If \( f(x) : \mathbb{R}^n \to \mathbb{R}^m \) is a superadditive non-decreasing function on \( \mathbb{R}_+^n \), then \( g(x) = \max\{0, f(x)\} \) is a superadditive function as well.

Proof. We would like to show that

\[
\max\{f(x_1 + x_2), 0\} \leq \max\{f(x_1), 0\} + \max\{f(x_2), 0\}.
\]  

(B.2)

Without loss of generality, we assume \( x_1 \leq x_2 \), then, there are three cases we need to examine:

Case 1: \( f(x_1 + x_2) \leq 0 \): We have \( f(x_1) \leq 0 \) and \( f(x_2) \leq 0 \) since \( f \) is non-decreasing. Then, both sides of inequality (B.2) are zero.

Case 2: \( f(x_1 + x_2) > 0, f(x_1) \leq 0 \) and \( f(x_2) \geq 0 \): Inequality (B.2) becomes \( f(x_1 + x_2) \geq f(x_2) \) which holds since \( f \) is non-decreasing.

Case 3: \( f(x_1 + x_2) > 0, f(x_1) \geq 0 \) and \( f(x_2) \geq 0 \): Inequality (B.2) becomes \( f(x_1 + x_2) \geq f(x_1) + f(x_2) \) which holds due to superadditivity of \( f \).

B.6 Convex lower-bound of \( f^L_j(z) \)

Largest convex lower-bound function of \( f^L_j(z) \) with a nonpositive value at origin is a piecewise linear function with the following generic form:

\[
\phi_j(z) = \begin{cases} 
\frac{\lambda_j - \bar{\lambda}_j - \Gamma}{M_{j,\tau}\lambda_j} z + \Gamma & 0 \leq z \leq M_{j,\tau} - \bar{\lambda}_j \\
\frac{\lambda_j}{c_{i+1}} (z - M_{j,i} + \bar{\lambda}_j) + i\lambda_j - \bar{\lambda}_j & M_{j,i} - \bar{\lambda}_j \leq z \leq M_{j,i+1} - \bar{\lambda}_j \\
z - M_{j,r_j} + r_j\lambda_j & M_{j,r_j} - \bar{\lambda}_j \leq z \leq \hat{d}_j.
\end{cases}
\]

Recall that index \( T_j \) is defined as \( \min\{1 \leq i \leq r_j : M_{j,i} - \bar{\lambda}_j \geq 0\} \) if \( M_{j,r_j} - \bar{\lambda}_j \geq 0 \), and \( r_j \) otherwise. Moreover, for notational convenience we let \( c_{v_{r_j+1}} := \lambda_j \). Next, we find values of \( \Gamma \) and \( \tau \) explicitly for cases where \( T_j \lambda_j - \bar{\lambda}_j > 0 \).

Case 1: \( T_j \leq p_j \).

Case 1.1: \( M_{j,\tau-1} - \bar{\lambda}_j + \lambda_j \leq 0 \).

In this case \( f^L_j(z) \) has the form in Figure B.2.
We select index $\tau$ to be the minimum such that the piecewise linear function has an increasing slope. Then,

$$\tau = \min \left\{ T_j \leq i \leq r_j : \frac{i\lambda_j - \bar{\lambda}_j}{M_{j,i} - \bar{\lambda}_j} \leq \frac{\lambda_j}{c_{v_{i+1}}} \right\}$$

and $\Gamma = 0$.

**Case 1.2:** $M_{j,T_{j-1}} - \bar{\lambda}_j + \lambda_j > 0$.

The function $f_j^L(z)$ under this conditions has the form in Figure B.3.
envelope. We have

$$
\tau = \min \left\{ T_j \leq i \leq r_j : \frac{M_{j,i-1} - \bar{\lambda}_j + (i - T_j + 1)\lambda_j}{M_{j,i} - \lambda_j} \leq \frac{\lambda_j}{c_{v_{i+1}}} \right\}
$$

and $\Gamma = (T_j - 1)\lambda_j - M_{j,T_j-1}$.

**Case 2:** $T_j > p_j$.

**Case 2.1:** $M_{j,T_j-1} - \bar{\lambda}_j + m_j + \varphi_j;T_j-1 \leq 0$ This is very similar to Case 1.1. Please see Figure B.4 for a representation of $f_j^L(z)$ and it’s convex lower bound.

![Representation of Case 2.1](image)

Figure B.4: Representation of Case 2.1.

Values of $\tau$ and $\Gamma$ are the same as in Case 1.1.

**Case 2.2:** $M_{j,T_j-1} - \bar{\lambda}_j + m_j + \varphi_j;T_j-1 > 0$.

The closed form of convex lower bound depends on the following two sub cases.

**Case 2.2.1:** $M_{j,T_j-1} - \bar{\lambda}_j + m_j \leq 0$.

Please see Figure B.5 for a representation of this case.
The largest convex lower bound that satisfies $\phi_j(0) \leq 0$ is the convex envelope of $f^L_j(z)$. We have

$$k = \min \left\{ T_j \leq i \leq r_j : \frac{M_j, T_j - \bar{\lambda}_j + m_l_j + \varphi_j, T_j - 1 + (i - T_j)\lambda_j}{M_j, i - \bar{\lambda}_j} \leq \frac{\lambda_j}{c_{i+1}} \right\}$$

and $a = T_j\lambda_j - \varphi_j, T_j - 1 - m_l_j - M_j, T_j - 1$.

**Case 2.2.2:** $M_j, T_j - 1 - \bar{\lambda}_j + m_l_j > 0$.

The representation of $f^L_j(z)$ under this case can be seen in Figure B.6. The convex envelope of the function satisfies $\phi_j(0) \leq 0$. Therefore, it is superadditive. Values of $\tau$ and $\Gamma$ are the same as in Case 1.2.
Appendix C

Equivalency of (F3.2) to the maximum flow problem

In Section 3.2, we showed the maximum flow equivalency of \( v(S^+, L^-) \) under the assumption that \( d_j \geq 0 \) for all \( j \in N \). In this section, we generalize the equivalency for the paths where \( d_j < 0 \) for some \( j \in N \).

**Observation 6.** If \( d_j < 0 \) for some \( j \in N \), one can represent the supply amount as a dummy arc incoming to node \( j \) (i.e., added to \( E_j^+ \)) with a fixed flow and capacity of \(-d_j\) and set the modified demand of node \( j \) to be \( d_j = 0 \).

Given the node set \( N \) with at least one supply node, let \( T(N) \) be the transformed path using Observation 6. Transformation \( T \) ensures that the dummy supply arcs are always open. As a result, they are always in the set \( S^+ \). We refer to the additional constraints that fix the flow to the supply value on dummy supply arcs as fixed-flow constraints. Notice that, \( v(S^+, L^-) \) computed for \( T(N) \) does not take fixed-flow constraints into account. In the next proposition, for a path structure, we show that there exists at least one optimal solution to (F3.2) such that the fixed-flow constraints are satisfied.

**Proposition C.1.** Suppose that \( d_j < 0 \) for some \( j \in N \). If (F3.2) for the node set \( N \) is feasible, then it has at least one optimal solution that satisfies the fixed-flow constraints.

**Proof.** We need to show that \( v(S^+, L^-) \) has an optimal solution where the flow at the dummy supply arcs is equal to the supply values. The transformation \( T \) makes Proposition 3.1 applicable to the modified path \( T(N) \). Let \( \mathcal{Y} \) be the set of optimal solutions of (F3.2). Then, there exists a solution \((y^*, i^*, r^*) \in \mathcal{Y} \) where \( y_t^* = 0 \) for \( t \in E^- \setminus (S^- \cup L^-) \). Let \( p \in S^+_j \) represent the index of the dummy supply arc with \( c_p = -d_j \). If \( y_p^* < c_p \), then satisfying the fixed-flow constraints require pushing flow through the arcs in \( E^- \setminus L^- \). We use Algorithm 4 to construct an optimal solution with \( y_p^* = c_p \).
Note that each arc in $E_k^- \setminus L_k^-$ for $k \in N$ appear in (F3.2) with the same coefficients, therefore we merge these outgoing arcs into one in Algorithm 4. We represent the merged flow and capacity by $\tilde{Y}_k^- = \sum_{t \in E_k^- \setminus (S_k^- \cup L_k^-)} y_t^*$ and $\tilde{C}_k = c(E_k^- \setminus (S_k^- \cup L_k^-))$ for $k \in N$.

\begin{algorithm}
\begin{algorithmic}
\State $J$: Set of supply nodes in $N$ where the nodes are sorted with respect to their order in $N$.
\State $(y^*, i^*, r^*) \in J$: $y_t^* = 0$ for all $t \in E^-$. 
\For{$q \in J$} 
\State Let $p$ be the dummy supply arc in $S_q^+$ 
\State $\Delta = c_p - y_p^*$ 
\For{$j = q$ to $n$} 
\State $\tilde{Y}_j^- = \tilde{Y}_j^- + \min\{\tilde{C}_j - \tilde{Y}_j^-, \Delta\}$ 
\State $\Delta = \Delta - \min\{\tilde{C}_j - \tilde{Y}_j^-, \Delta\}$ 
\State $i_j^* = i_j^* + \Delta$ 
\If{$i_j^* > u_j$} 
\State $\Delta = i_j^* - u_j$ 
\State $i_j^* = u_j$ 
\State Let $k := j$ 
\State break inner loop 
\EndIf 
\EndFor 
\If{$\Delta > 0$} 
\For{$j = k$ to $1$} 
\State $\tilde{Y}_j^- = \tilde{Y}_j^- + \min\{\tilde{C}_j - \tilde{Y}_j^-, \Delta\}$ 
\State $\Delta = \Delta - \min\{\tilde{C}_j - \tilde{Y}_j^-, \Delta\}$ 
\State $r_j^* = r_j^* + \Delta$ 
\If{$r_j^* > b_j$} 
\State $\Delta = r_j^* - b_j$ 
\State break inner loop 
\EndIf 
\EndFor 
\EndIf 
\EndIf 
\EndFor 
\end{algorithmic}
\end{algorithm}

Proposition C.1 shows that, under the presence of supply nodes, transformation $T$ both captures the graph’s structure and does not affect (F3.2)’s validity. As a result, Propositions 3.1 and 3.2 become relevant to the transformed path and submodular path inequalities (3.14) and (3.17) are also valid for paths where $d_j < 0$ for some $j \in N$. 

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Appendix D

Proofs from Chapter 3

D.1 Proof of Lemma 3.3

Recall that $C = S^+ \cup L^-$ and let $C_1 = S^+_N \cup L^-_N$ and $C_2 = S^+_N \cup L^-_N$. In (3.13), we showed that the value of the minimum cut is

$$v(C) = m_i = \min \{ \alpha^u_i + \beta^u_i - c(S^+_i), \alpha^d_i + \beta^d_i - d_i - c(S^-_i) \}$$

for all $i \in N$. For node set $N_1$ and the arc set $C_1$, the value of the minimum cut is

$$v_1(C_1) = \min \{ \alpha^{u}_{j-1} + b_{j-1}, \alpha^{d}_{j-1} \}.$$  

This is because of three observations: (1) the values $\alpha^{u,d}_i$ for $i \in [1, j-2]$ are the same for the node sets $N_1$ and $N$, (2) for the arc set $C_1$ the set $S^+_j$ now includes the backward path arc $(j, j-1)$ and (3) node $j-1$ is the last node of the first path.

Similarly, for node set $N_2$ and the arc set $C_2$, the value of the minimum cut is

$$v_2(C_2) = \min \{ \beta^{u}_j + u_{j-1}, \beta^{d}_j \}.$$  

For nodes $N_2$ and the arc set $C_2$, (1) the values $\beta^{u,d}_i$ for $i \in [j+1, n]$ are the same for the node sets $N_2$ and $N$, (2) for the arc set $C_2$ the set $S^+_j$ now includes the forward path arc $(j-1, j)$ and (3) node $j$ is the first node of the second path.

Now, if $\alpha^u_j = \alpha^{u}_{j-1} + u_{j-1} + c(S^+_j)$, then $\alpha^d_j = \alpha^{d}_{j-1} + d_j + c(S^-_j)$ from equations in (4.3)--(4.4). Then, rewriting $v(C) = m_j$ and $v_1(C_1)$ in terms of $\alpha^{d}_{j-1}$:

$$v(C) = \alpha^{d}_{j-1} + \min \{ \beta^{u}_j + u_{j-1}, \beta^{d}_j \}$$

and

$$v_1(C_1) = \alpha^{d}_{j-1}.$$  

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the
assumption for the value of $\alpha_j^u$. Similarly, if $\beta_{j-1}^u = \beta_j^d + b_{j-1} + c(S_{j-1}^+)$, then $\beta_{j-1}^{d} = \beta_j^d + d_{j-1} + c(S_{j-1}^-)$ from equations in (4.5)–(4.6). Then, rewriting $v(C) = m_{j-1}$ and $v_2(C_2)$ in terms of $\beta_j^d$:

$$v(C) = \beta_j^d + \min\{\alpha_{j-1}^u + b_{j-1}, \alpha_{j-1}^d\}$$

and

$$v_2(C_2) = \beta_j^d.$$ 

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the value of $\beta_{j-1}^u$.

### D.2 Proof of Lemma 3.4

The proof follows closely to that of Lemma 3.3. Let $C = S^+ \cup L^-$, $C_1 = S_{N_1}^+ \cup L_{N_1}^-$ and $C_2 = S_{N_2}^+ \cup L_{N_2}^-$. For node set $N_1$ and the arc set $C_1$, the value of the minimum cut is

$$v_1(C_1) = \min\{\alpha_{j-1}^u, \alpha_{j-1}^d + u_{j-1}\},$$

where $u_{j-1}$ is added because $c(S_{N_1}^-) = c(S_{j-1}^-) + u_{j-1}$. Similarly, for node set $N_2$ and the arc set $C_2$, the value of the minimum cut is

$$v_2(C_2) = \min\{\beta_{j-1}^u, \beta_{j-1}^d + b_{j-1}\},$$

where $b_{j-1}$ is added because $c(S_{N_2}^-) = c(S_{j-1}^-) + b_{j-1}$.

Now, if $\alpha_{j-1}^d = \alpha_{j-1}^u + b_{j-1} + d_{j-1} + c(S_{j-1}^-)$, then $\alpha_{j-1}^u = \alpha_{j-1}^u + c(S_{j-1}^+)$ from equations in (4.3)–(4.4). Then, rewriting $v(C) = m_{j-1}$ and $v_1(C_1)$ in terms of $\alpha_{j-1}^u$:

$$v(C) = \alpha_{j-1}^u + \min\{\beta_{j-1}^u, \beta_{j-1}^d + b_{j-1}\}$$

and

$$v_1(C_1) = \alpha_{j-1}^u.$$ 

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the value of $\alpha_{j-1}^d$.

Similarly, if $\beta_{j-1}^d = \beta_{j-1}^u + u_{j-1} + d_{j-1} + c(S_{j-1}^-)$, then $\beta_{j-1}^u = \beta_{j-1}^u + c(S_{j-1}^+)$ from equations in (4.5)–(4.6). Then, rewriting $v(C) = m_{j-1}$ and $v_2(C_2)$ in terms of $\beta_{j-1}^u$:

$$v(C) = \beta_{j-1}^u + \min\{\alpha_{j-1}^u, \alpha_{j-1}^d + u_{j-1}\}$$

and

$$v_2(C_2) = \beta_{j-1}^u.$$ 

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the
assumption for the value of $\beta_{j-1}^d$.

### D.3 Proof of Lemma 3.5

The proof follows closely to that of Lemmas 3.3 and 3.4. Let $C = S^+ \cup L^-$, $C_1 = S^+_{N_1} \cup L^-_{N_1}$ and $C_2 = S^+_{N_2} \cup L^-_{N_2}$. For node set $N_1$ and the arc set $C_1$, the value of the minimum cut is

$$ v_1(C_1) = \min\{\alpha_{j-1}^u, \alpha_{j-1}^d\} $$

and for node set $N_2$ and the arc set $C_2$, the value of the minimum cut is

$$ v_2(C_2) = \min\{\beta_j^u + u_{j-1}, \beta_j^d + b_{j-1}\}. $$

Now, if $\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+)$ and $\beta_j^d = \beta_{j-1}^u + u_{j-1} + c(S_j^-)$, then $\alpha_j^d = \alpha_{j-1}^d + d_j + c(S_j^-)$ and $\beta_j^u = \beta_{j-1}^u + c(S_j^+)$. Then, rewriting $v(C) = m_j, v_1(C_1)$ and $v_2(C_2)$:

$$ v(C) = \alpha_{j-1}^d + \min\{u_{j-1} + \beta_j^u, \beta_j^d\} = \alpha_{j-1}^d + u_{j-1} + \beta_j^u, $$

$$ v_1(C_1) = \alpha_{j-1}^d \text{ and } v_2(C_2) = \beta_j^u + u_{j-1}. $$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the values of $\alpha_j^u$ and $\beta_j^d$.

### D.4 Proof of Lemma 3.6

The proof follows closely to that of Lemmas 3.3 and 3.4. Let $C = S^+ \cup L^-$, $C_1 = S^+_{\tilde{N}_1} \cup L^-_{\tilde{N}_1}$ and $C_2 = S^+_{\tilde{N}_2} \cup L^-_{\tilde{N}_2}$. For node set $N_1$ and the arc set $C_1$, the value of the minimum cut is

$$ v_1(C_1) = \min\{\alpha_{j-1}^u + b_{j-1}, \alpha_{j-1}^d + u_{j-1}\} $$

and for node set $N_2$ and the arc set $C_2$, the value of the minimum cut is

$$ v_2(C_2) = \min\{\beta_j^u, \beta_j^d\}. $$

Now, if $\alpha_j^u = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$ and $\beta_j^u = \beta_{j-1}^u + b_{j-1} + c(S_j^+)$, then $\alpha_j^u = \alpha_{j-1}^u + c(S_j^+)$ and $\beta_j^u = \beta_{j-1}^u + d_j + c(S_j^-)$. Then, rewriting $v(C) = m_j, v_1(C_1)$ and $v_2(C_2)$:

$$ v(C) = \alpha_{j-1}^u + \min\{\beta_j^u, \beta_j^d + b_{j-1}\} = \alpha_{j-1}^u + \beta_j^d + b_{j-1}, $$

$$ v_1(C_1) = \alpha_{j-1}^u + b_{j-1} \text{ and } v_2(C_2) = \beta_j^d. $$
As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the values of $\alpha_j^d$ and $\beta_{j-1}^u$. 
Appendix E

Lot Sizing with Inventory Bounds and Fixed Costs

In this section, we show that the inequalities by Atamtürk and Küçükaydın (2005) is a special case of (4.9). Recall that they formulate the lot-sizing with bounded inventory (LSBI) problem as the following optimization problem:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} \left( f_t x_t + p_t y_t + g_t z_t + h_t i_t \right) \\
\text{s.t.} & \quad i_{t-1} + y_t - i_t = d_t, \quad t \in [1, n], \\
& \quad 0 \leq i_t \leq u_t z_t, \quad t \in [1, n] \\
& \quad 0 \leq y_t \leq (d_t + u_t)x_t, \quad t \in [1, n], \\
& \quad y \in \mathbb{R}^n, \quad x \in \{0, 1\}^n, \\
& \quad i \in \mathbb{R}^{n+1}, \quad z \in \{0, 1\}^{n+1}.
\end{align*}
\]

(LSBI)

In the first set of inequalities, they give valid inequalities for LSBI, when \( z_t = 1 \), for all \( t \in [1,n] \). Let \([k,\ell] \subseteq [1,n]\) be a subset of nodes and \( L \subseteq [k,\ell] \) be a subset of production arcs. Then the inequalities

\[
\begin{align*}
i_{k-1} + \sum_{t \in L^+} y_t & \leq u_{k-1} + \sum_{t \in L^+} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{\ell\ell}\} x_t + i_{\ell}, & (E.1) \\
i_{k-1} + \sum_{t \in L^+} y_t & \leq u_{k-1} + \sum_{t \in L^+} (d_{kt} - u_{k-1} + u_t) x_t & (E.2)
\end{align*}
\]

are valid for LSBI.

**Proposition E.1.** Inequality (E.1) is weaker than inequality (4.9) for LSBI, where \( S^+ = \{(k-1,k)\} \).

**Proof.** Let \( p = \min\{t \in [k,\ell] : u_{k-1} < d_{kt}\} \). Due to the assumption of \( u_{t-1} \leq d_t + u_t \),
for all $t \in [1,n]$, we notice that $v(S^t) = u_{k-1}$. Moreover, notice that the maximum flow that can be sent by a production arc $t$ is $\min\{d_{kt}, d_t + u_t\}$. If $t \leq p$, then $v(S^+ \cup \{t\}) = d_{kt-1} + \min\{u_t + d_t, d_{kt}\}$ and if $t > p$, then $v(S^+ \cup \{t\}) = u_{k-1} + \min\{u_t + d_t, d_{kt}\}$. Using the definition of $p$, one can write $v(S^+ \cup \{t\})$ in a compact form as $\min\{u_{k-1} + \min\{u_t + d_t, d_{kt}\}, d_{kt-1} + \min\{u_t + d_t, d_{kt}\}\}$. After some algebraic manipulation, we observe that

$$\rho_t(S^+) = \left(\min\{d_{kt} + u_t - u_{k-1}, d_{kt} - u_{k-1}, u_t + d_t, d_{kt} + u_t\}\right)^+$$

for each $t \in L^+$. \hfill \Box

**Proposition E.2.** Inequality (E.2) is weaker than inequality (4.9) for LSBI, where $S^+ = \{(k - 1, k)\}$.

**Proof.** Proof follows similar to that of Proposition E. Since $a_t = 0$ for the inventory arc $(\ell, \ell + 1)$, the effective demand at node $\ell$ becomes $d_{\ell} + u_\ell$. This leads to

$$\rho_t(S^+) = \left(\min\{d_{kt} + u_t - u_{k-1}, d_{kt} + u_{k-1}, u_t + d_t, d_{kt} + u_t\}\right)^+.$$ 

From the assumption of $u_{t-1} \leq d_t + u_t$, for all $t \in [1,n]$, we know that $d_{kt} + u_t - u_{k-1} \leq d_{kt} + u_{k-1}$ and $u_t + d_t \leq d_{kt} + u_t$. Then, for each $t \in L^+$,

$$\rho_t(S^+) = \left(\min\{d_{kt} + u_t - u_{k-1}, d_t + u_t\}\right)^+.$$ 

\hfill \Box

In the second set of inequalities, they introduce inventory fixed charge variables to inequalities (E.1) and (E.2). Recall that $p = \min\{t \in [k, \ell] : u_{k-1} < d_{kt}\}$. For $1 \leq k \leq \ell \leq n$ such that $u_{k-1} \leq d_{kt}$, let $L \subseteq [k, \ell]$ and $T = \{t_1, t_2, \ldots, t_r\} \subseteq [k-1, p-1]$. For $j \in T$, let $s(j) = \min\{L \cup \{\ell + 1\} : t > j\}$. Then the inequalities

$$i_{k-1} + \sum_{t \in L^+} y_t + \sum_{t \in T} \gamma_t (1 - z_t) \leq u_{k-1} + \sum_{t \in L^+} \min\{d_{kt} + u_t - u_{k-1}, d_{kt} - u_{k-1}, d_{kt} + u_t\} x_t + i_\ell,$$ 

(E.3)

$$i_{k-1} + \sum_{t \in L^+} y_t + \sum_{t \in T} \gamma_t (1 - z_t) \leq u_{k-1} + \sum_{t \in L^+} (d_{kt} - u_{k-1} + u_t) x_t,$$ 

(E.4)

where

$$\gamma_{t_j} = \begin{cases} 
    u_{k-1} - d_{kt}, & \text{if } j = \tau \text{ and } s(t_j) > p, \\
    d_{t(j+1)(j+1)} & \text{if } j < \tau \text{ and } s(t_j) = s(t_{j+1}), \\
    d_{t(j+1)(s(t_j)-1)} & \text{if } (j < \tau \text{ and } s(t_j) < s(t_{j+1})) \text{ or } (j = \tau \text{ and } s(t_j) \leq p) 
\end{cases}$$

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are valid for LSBI.

**Proposition E.3.** Let \( s \) represent the inventory arc \((k - 1, k)\) and let \( t \in [k - 1, p - 1] \). For \( S = \{s\} \) and \( T = \{(t, t + 1)\} \), the inequality

\[
i_{k-1} + \sum_{j \in L^+} y_j + (\rho_s(L(t)) - \rho_s(L(t)|z_t = 0))(1 - z_t) \leq v(S) + \sum_{j \in L^+} \rho_j(S)x_j \tag{E.5}
\]

where \( L(t) = L^+ \cap [t + 1, \ell] \) is equivalent to inequality (E.3) and is valid for LSBI.

**Proof.** Let \((y^*, i^*, x^*, z^*)\) be a feasible solution of LSBI and let \( \hat{L}_t = L^+ \setminus L(t) \). We show validity of inequality (E.5), under two cases: \( z^*_t = 1 \) and \( z^*_t = 0 \). When \( z^*_t = 1 \), inequality (E.5) is equivalent to inequality (4.9) and is valid for LSBI.

Now, suppose \( z^*_t = 0 \). Inequality (E.5) becomes

\[
i_{k-1}^* - i_t^* + \sum_{j \in L^+} y_j^* \leq v(S) - \rho_s(L(t)) + \rho_s(L(t)|z_t = 0) + \sum_{j \in L^+} \rho_j(S), \tag{E.6}
\]

where \( \hat{L}_t = \{t \in L^+ : x_t = 1\} \). Moreover, let \( \hat{L}(t) = \hat{L}_t \cap L(t) \). From the definition of the function \( v \), when \( z^*_t = 0 \), it is guaranteed that

\[
i_{k-1}^* - i_t^* + \sum_{j \in L^+} y_j^* \leq v(S \cup \hat{L}^+|z_t = 0).
\]

Moreover, using the structure of the path we observe

\[
\rho_s(L(t)|z_t = 0) = \rho_s(\hat{L}(t)|z_t = 0)
\]

and

\[
v(S \cup \hat{L}^+|z_t = 0) = \rho_s(\hat{L}(t)|z_t = 0) + v(\hat{L}(t)|z_t = 0)
\]

since \( t \leq p - 1 \). Then, we make the following observation about the right hand side of inequality (E.6):

\[
v(S) - \rho_s(L(t)) + \rho_s(L(t)|z_t = 0) + \sum_{j \in L^+} \rho_j(S)
\]

\[
\geq v(S \cup \hat{L}(t)) - \rho_s(L(t)) + \rho_s(L(t)|z_t = 0) + \sum_{j \in L(t) \cap \hat{L}^+} \rho_j(S)
\]

\[
\geq v(S \cup \hat{L}(t)) - \rho_s(\hat{L}(t)) + \rho_s(L(t)|z_t = 0) + \sum_{j \in L(t) \cap \hat{L}^+} \rho_j(S)
\]

\[
= v(\hat{L}(t)) + \rho_s(L(t)|z_t = 0) + \sum_{j \in \hat{L}(t) \cap \hat{L}^+} \rho_j(S)
\]

\[
= v(S \cup \hat{L}^+|z_t = 0) + \sum_{j \in \hat{L}(t) \cap \hat{L}^+} \rho_j(S).
\]
In the first and second inequalities, we use submodularity of the function \( v \) and in the last equality, we use the structure of the path. As a result, 

\[
\tilde{i}_k^* - \tilde{i}_\ell^* + \sum_{j \in L^+} y_j^* \leq v(\{s\} \cup \hat{L}^+ | z_\ell = 0) \\
\leq v(\{s\}) - \rho_s(L(t)) + \rho_s(L(t)|z_\ell = 0) + \sum_{j \in L^+} \rho_j(S)
\]

and we conclude that inequality (E.5) is valid for LSBI when \( z^*_i = 0 \). \( \square \)

As it is pointed out in Atamtürk (2004), the coefficients \( \gamma_t \) are sequence-dependent. Let the sequence of inventory arcs to be lifted be represented as \( T = \{t_1, \ldots, t_\tau\} \). Then, one can represent the coefficients as:

\[
\gamma_{t_j} = \rho_s(L(t_j)|t_i = 0, \forall i \in [j+1, \tau]) - \rho_s(L(t_j)|t_i = 0, \forall i \in [j, \tau])
\]

**Proposition E.4.** Let \( t_i, t_j \in T \) for some \( i \neq j \). If \( t_i > t_j \) and \( i < j \), then \( \gamma_{t_i} = 0 \).

**Proof.** Recall that the coefficient \( \gamma_{t_i} \) can be represented as \( \rho_s(L(t_i)|t_l = 0, \forall l \in [i+1, \tau]) - \rho_s(L(t_i)|t_l = 0, \forall l \in [i, \tau]) \). Without loss of generality, let \( j := \arg \min_{l \in [i+1, \tau]} \{t_l\} \) and suppose \( t_j < t_i \). Since \( t_j < t_i \), we know that \( \rho_s(L(t_i)|t_l = 0, \forall l \in [i, \tau]) = d_{kt_j} \) and \( \rho_s(L(t_i)|t_l = 0, \forall l \in [i, \tau]) = d_{kt_j} \). Consequently, the coefficient of \( z_{t_i} \) is \( \gamma_{t_i} = 0 \). \( \square \)

Due to Proposition E.4, we assume that \( t_i < t_j \) for all \( t_i, t_j \in T \) and \( i < j \). In other words, we assume that \( T \) is an increasing set.

**Remark E.1.** For LSBI, the coefficients

\[
\rho_s(L(t_i)|t_l = 0, \forall l \in [i, \tau]) = \rho_s(L(t_i)|t_i = 0)
\]

and

\[
\rho_s(L(t_i)|t_l = 0, \forall l \in [i+1, \tau]) = \rho_s(L(t_i)|t_i+1 = 0).
\]

In other words, the change in maximum flow by adding the inventory arc \((k-1, k)\) to \( L(t_i) \) only depends on the smallest (left-most) inventory arc \((j, j+1)\) that is closed.

Note that the results shown in Propositions E.3, E.4 and Remark E.1 are not general for all network structures and depend heavily on the path structure of LSBI.

**Proposition E.5.** Let \( s \) represent the inventory arc \((k-1, k)\) and let \( t \in [k-1, p-1] \). For \( S = \{s\} \) and \( T = \{(t, t+1)\} \), the inequality

\[
i_{k-1} + \sum_{j \in L^+} y_j + \sum_{t \in T} (\rho_s(L(t)|t_{j+1} = 0) - \rho_s(L(t)|t_j = 0)) (1 - z_t) \leq v(S) + \sum_{j \in L^+} \rho_j(S)x_j
\]

(E.7)
is equivalent to inequality (E.3) and is valid for LSBI.

Proof. Let \((y^*, i^*, x^*, z^*)\) be a feasible solution of LSBI. If \(z^*_t = 1\) for all \(t \in T\), then inequality (E.7) is equivalent to (4.9) and is valid for LSBI. Suppose \(z^*_t = 0\) for some \(t \in T\). Let \(\hat{T} = \{t \in T : z^*_t = 0\}\). From the definition of the function \(v\), we observe

\[
i^*_k - i^*_t + \sum_{j \in \hat{L}^+} y^*_j \leq v(\{s\} \cup \hat{L}^+ | z_i = 0, \forall i \in \hat{T}).
\]

Inequality (E.7) when \(z_i = 0\) for \(i \in \hat{T}\) becomes

\[
i_k - 1 + \sum_{j \in \hat{L}^+} y_j \leq v(S) - \sum_{j \in \hat{T}} (\rho_s(L(j) | t_{j+1} = 0) - \rho_s(L(j) | t_j = 0)) + \sum_{j \in \hat{L}^+} \rho_j(S). \tag{E.8}
\]

Let \(R\) be the right hand side of inequality (E.8) and let \(t_{max} = \max\{j \in \hat{T}\}\). Then, using submodularity and the path structure of LSBI, we observe the following

\[
v(\{s\}) - \rho_s(L(t_{max}) | z_{t_{max} + 1} = 0) + \rho_s(L(t_{max}) | z_{t_{max}} = 0) + \sum_{j \in L(t_{max})} \rho_j(S) \tag{E.9}
\]

\[
\geq v(\{s\} \cup \hat{L}(t_{max})) - \rho_s(L(t_{max}) | z_{t_{max} + 1} = 0) + \rho_s(L(t_{max}) | z_{t_{max}} = 0) \tag{E.10}
\]

\[
\geq v(\{s\} \cup \hat{L}(t_{max})) - \rho_s(\hat{L}(t_{max}) | z_{t_{max} + 1} = 0) + \rho_s(L(t_{max}) | z_{t_{max}} = 0) \tag{E.11}
\]

\[
\geq v(\{s\} \cup \hat{L}(t_{max})) - \rho_s(\hat{L}(t_{max})) + \rho_s(L(t_{max}) | z_{t_{max}} = 0) \tag{E.12}
\]

\[
= v(\hat{L}(t_{max})) + \rho_s(L(t_{max}) | z_{t_{max}} = 0) \tag{E.13}
\]

\[
= v(\{s\} \cup \hat{L}(t_{max}) | z_{t_{max}} = 0). \tag{E.14}
\]

The inequality (E.12) holds since for any \(t_j \in T\) and \(t_i \geq t_j\),

\[
\rho_s(L(t_j) | z_{t_i} = 0) \leq \rho_s(L(t_j))
\]

due to the path structure of LSBI. Similarly, we reach the equality of (E.13) and (E.14) since \(v(\hat{L}(t_{max})) = v(\hat{L}(t_{max}) | z_{t_{max}} = 0)\). Using the inequality (E.10)–(E.14), we observe that

\[
R \geq v(\{s\} \cup \hat{L}(t_{max}) | z_{t_{max}} = 0) - \sum_{i \in \hat{T} \setminus \{t_{max}\}} (\rho_s(L(t_i) | t_{i+1} = 0) - \rho_s(L(t_i) | t_i = 0)) + \sum_{i \in \hat{L}^+ \setminus L(t_{max})} \rho_i(S).
\]

For the rest of the proof, we use an induction logic. First, we introduce some notation for simplification: let \(\hat{T}(t_j) = \hat{T} \cap \{t_j, \ldots, t_r\}\) and let \(v(C | \hat{T}(t_j)) := v(C | z_i = \)

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0, ∀i ∈ \( \hat{T}(t_j) \) and \( \rho_i(C|\hat{T}(t_j)) := \rho_i(C|z_i = 0, \forall i ∈ \hat{T}(t_j)) \).

Assuming that the following is true

\[
R \geq v(\{s\} \cup \hat{L}(t_{j+1})|\hat{T}(t_{j+1})) - \sum_{i \in T \setminus [t_{j+1}, \tau]} (\rho_s(L(t_i)|z_{t_{i+1}} = 0) - \rho_s(L(t_i)|z_{t_i} = 0)) + \sum_{i \in L^+ \setminus (L(t_{j+1}) \cup \hat{L}(t_j))} \rho_i(S)
\]

for iteration \( j + 1 \), we extend the result for \( t_j \). First, we make the following observation

\[
\rho_i(S) \geq \rho_i(S|z_{t_{j+1}} = 0) = \rho_i(S|z_{t_i} = 0, \forall i \in [j + 1, \tau])
\]

for any \( i \in L^+ \setminus L(t_{j+1}) \). Then,

\[
v(\{s\} \cup \hat{L}(t_{j+1})|\hat{T}(t_{j+1})) - \rho_s(L(t_j)|t_{j+1} = 0) + \rho_s(L(t_j)|z_{t_j} = 0) + \sum_{i \in L(t_j) \setminus \hat{L}(t_j)} \rho_i(S)
\]

\[
\geq v(\{s\} \cup \hat{L}(t_j)|\hat{T}(t_{j+1})) - \rho_s(L(t_j)|t_{j+1} = 0) + \rho_s(L(t_j)|z_{t_j} = 0)
\]

\[\text{(E.15)}\]

\[
\geq v(\hat{L}(t_j)|\hat{T}(t_{j+1})| + \rho_s(L(t_j)|z_{t_j} = 0)
\]

\[\text{(E.16)}\]

\[= v(\{s\} \cup \hat{L}(t_j)|\hat{T}(t_j)).\]

\[\text{(E.17)}\]

Inequalities (E.15)–(E.16) are obtained from the observation above and submodularity of the function \( v \) and the equality of (E.16) to (E.17) is due to the path structure of LSBI. As a result of the induction, we have proved that

\[ R \geq v(\{s\} \cup \hat{L}|\hat{T}). \]

Then, combining with the definition of the function \( v \), we conclude that

\[ i^*_k - i^*_k + \sum_{j \in L^+} y^*_j \leq v(\{s\} \cup \hat{L}^+|\hat{T}) \leq R \]

and inequality (E.7) is valid for LSBI.

\[\square\]

**Remark E.2.** In Atamtürk and Küçükyavuz (2005), they select \( T \subseteq [k - 1, p - 1] \) since closing any inventory arc \( j \geq p \) does not change \( v(S^+) \) nor \( \rho_t(S^+) \) for any \( t \in L^+ \). As a result, the lifting coefficient of such an inventory arc is zero.

**Example E.1.** Consider the example in Figure E.1. Inequality (4.9) for the node set \([2, 5]\) and the arc sets \( S^+ = \{(1, 2)\} \) and \( L^+ = \{2, 4, 5\} \) is

\[ i_1 + y_2 + y_4 + y_5 \leq 40 + 14x_2 + 14x_4 + 14x_5 + i_5, \]

which is equivalent to inequality (E.1) for the same subsets of nodes and arcs.
Note that $p = 5$ since $d_{24} \leq 40$ and $d_{25} > 40$. Selecting $T = \{(2, 3)\}$, we get the lifting coefficient of 13. We obtain this value by the difference of $\rho_{(1,2)}(\{4, 5\}) = 25$ and $\rho_{(1,2)}(\{4, 5\}|z_2 = 0) = 12$. Consequently,

\[ i_1 + y_2 + y_4 + y_5 + 13(1 - z_2) \leq 40 + 14x_2 + 14x_4 + 14x_5 + i_5, \]

is also valid for LSBI.