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Triple Cup Products in Heegaard Floer Homology

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Abstract of the Dissertation

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Professor Ciprian Manolescu, Chair

Manolescu and Ozsváth have recently developed a formula for calculating the Heegaard Floer homologies of integral surgery on a link. We use their link surgery formula to give a complete calculation of $HF^\infty(Y, \mathfrak{s}; \mathbb{Z}/2\mathbb{Z})$ for a torsion Spin^c structure $\mathfrak{s}$ on any closed, orientable three-manifold Y in terms of the cup product structure on its integral cohomology ring.

The dissertation of Tye Lidman is approved.

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To the sunny California skies...

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CHAPTER 1

Introduction

1.1 Background

The study of geometric topology has always relied heavily on the use of invariants built from all areas of mathematics. Over the last few decades, such invariants have been appearing non-stop to study questions in low-dimensions. Most notably, the Poincaré conjecture in dimension three was recently solved by Perelman using the Ricci flow [Per02, Per03]; there is still no known proof of this result using purely topological tools. Furthermore, gauge theory, which relies heavily on geometric input from a manifold, has been used to construct homeomorphic manifolds with exotic smooth structures [Don87] or with no smooth structures at all [Don83].

Many of these new tools are being developed to arise as a categorification of an older invariant - a suped-up version which contains the original information of the invariant, but carries a richer structure. A toy example of this is singular homology. It is clear that the singular homology of a topological space contains all of the information that its Euler characteristic sees. However, singular homology has many additional properties where the Euler characteristic lies flat, such as gradings, functoriality, and exact sequences, all of which have proved to be extremely useful.

In the 1980s, Casson developed an invariant of integer homology three-spheres, which was an integral lift of the classical Rohlin invariant (see [AM90]). This was built from $SU(2)$-representations of the fundamental group and was able to answer many open problems, including the construction of a four-manifold which does not admit a triangulation. Later, Taubes was able to rephrase the Casson invariant in terms of gauge theory - a certain

count of flat connections on a trivial $SU(2)$-bundle over the three-manifold, or equivalently, critical points of the Chern-Simons functional [Tau90]. An extremely important categorification of the Casson invariant was given by Floer to construct a graded abelian group, I_*, called instanton homology [Flo88]. The homology theory is roughly defined by taking as its generators critical points of the Chern-Simons functional and has differential determined by counting gradient flowlines of this functional. These groups can be seen as capturing the infinite-dimensional Morse theory of the space of connections with respect to this functional. Not only did the Euler characteristic recapture the Casson invariant (modulo a factor of 2), the theory also has an additional property that cobordisms induce homomorphisms between the instanton homology groups; this can be used to give invariants of closed four-manifolds, which are the Donaldson invariants. The Morse-theoretic framework that instanton homology is built on has led to a sequence of further categorifications of classical invariants, proving astonishing new results.

In the following decade, the Seiberg-Witten equations became the next source of low-dimensional invariants arising from gauge theory. The Seiberg-Witten invariants proved to be particularly effective for constructing examples of four-manifolds with infinitely many smooth structures (see, for instance, [FS98, Par02, Sza96]). However, it was an extremely difficult task to construct a homology theory for three-manifolds analogous to instanton Floer homology. Many different attempts were made [CW03, Man03, Mar96], but the most recent construction of Kronheimer and Mrowka [KM07] has become the most widely used.

In particular, using this theory, Kronheimer, Mrowka, Ozsváth, and Szabó were able to prove a Dehn surgery characterization of the unknot: if $S^3_{p/q}(K)$ is orientation-preserving homeomorphic to $L(p,q)$, then K is the unknot [KMO07]. On the other hand, a heavy drawback of Seiberg-Witten theory is that it is extremely difficult to do calculations with. In light of this, Ozsváth and Szabó constructed a homology theory which they hoped to model the Seiberg-Witten theory, but be much more computable [OS04d]. This approach uses symplectic geometry instead of gauge theory, but still utilizes the Morse-theoretic methods developed by Floer. They defined the Heegaard Floer homology of a three-manifold roughly by the Lagrangian Floer homology of certain tori in the symmetric product of a Heegaard

surface.

Heegaard Floer homology associates to a closed, connected, oriented three-manifold, Y, and a Spin^c structure, $\mathfrak{s}$, a collection of abelian groups, $HF^+(Y,\mathfrak{s})$, $HF^-(Y,\mathfrak{s})$, $HF^\infty(Y,\mathfrak{s})$, and $\widehat{HF}(Y,\mathfrak{s})$. Furthermore, Spin^c cobordisms induce maps between the associated Heegaard Floer groups, analogous to the functoriality of instanton Floer homology mentioned above [OS06b]. In fact, Heegaard Floer homology can be seen as a categorification of Turaev torsion [OS04c] and has recently been shown to be isomorphic to the monopole Floer homology developed by Kronheimer and Mrowka [CGH11, KLT10a, KLT10b, KLT10c, KLT11, Tau10]. Additional Heegaard Floer invariants have been developed for other structures, including knots [OS04b, Ras03], contact structures [OS05a], and manifolds with boundary [Juh06, LOT08].

The primary appeal of working with Heegaard Floer homology is its ease of computation. While the definition of Heegaard Floer homology requires solving partial differential equations on manifolds, the theory has been shown to be algorithmically computable [MOT09, SW10]. In addition, there are many other tools to aid with the calculations, including exact triangles and spectral sequences.

One particularly useful tool for computation is the integer surgery formula for knots [OS08b]. Given a nullhomologous knot K in a three-manifold Y, the integer surgery formula gives a method of computing the Heegaard Floer homology of $Y_n(K)$, the manifold obtained by $n/1$-surgery on K. Manolescu and Ozsváth generalized this to the link surgery formula [MO10]. This formula associates a hypercube of chain complexes to a framed nullhomologous link (L,Λ) in Y whose homology calculates the Heegaard Floer homology of $Y_\Lambda(L)$.

A hypercube of chain complexes is naturally equipped with a filtration that we call the ε-filtration; this induces a spectral sequence which is helpful for calculating the homology of the total complex. For example, Ozsváth and Szabó first used this principle to construct a spectral sequence from $\widetilde{Kh}(L;\mathbb{F})$, the reduced Khovanov homology of a link in S^3, to the Heegaard Floer homology of the double-branched cover of the mirror of this link, $\widehat{HF}(\Sigma_2(\overline{L});\mathbb{F})$ [OS05b]. Here, and for the rest of this thesis, $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$. This has immediate applications,

including showing that double-branched covers of quasi-alternating links cannot admit co-orientable taut foliations. As the link surgery formula is proved using $\mathbb{F}$-coefficients, we will use these throughout as well.

1.2 The Main Theorem

Our main goal is to use the link surgery formula and associated spectral sequence to calculate the relatively-graded $\mathbb{F}[U,U^{-1}]$-module $HF^\infty(Y,\mathfrak{s};\mathbb{F})$ for torsion Spin^c structures $\mathfrak{s}$.

For closed three-manifolds, the Heegaard Floer chain complexes come in many flavors: $\widehat{CF}$, CF^+, CF^-, and CF^∞. These flavors are all in fact derived from CF^∞, the complex defined over the base ring $\mathbb{Z}[U,U^{-1}]$, by some action on the chain level; for example, $\widehat{CF}$ is given by setting the variable U equal to 0. Therefore, having an understanding of the homology, HF^∞, provides foundational information for the other flavors. One useful fact is that the $\mathbb{Z}$-rank of $\widehat{HF}(Y)$ is always bounded below by the $\mathbb{Z}[U,U^{-1}]$-rank of $HF^\infty(Y)$. We will use HF° when we do not want to specify a flavor.

For a torsion Spin^c structure $\mathfrak{s}$, the group $HF^\infty(Y,\mathfrak{s})$ is the least complicated of all the Heegaard Floer flavors - in fact, Ozsváth and Szabó conjectured in [OS03c] that it should be determined (modulo gradings) by the integral cohomology ring. While $HF^\infty(Y,\mathfrak{s})$ comes equipped with a very powerful $\mathbb{Q}$-grading, we will not calculate this. We first prove the following.

Proposition 1.2.1. *Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be torsion Spin^c structures on Y_1 and Y_2 respectively. If $H^*(Y_1;\mathbb{Z}) \cong H^*(Y_2;\mathbb{Z})$, then $HF^\infty(Y_1,\mathfrak{s}_1;\mathbb{F}) \cong HF^\infty(Y_2,\mathfrak{s}_2;\mathbb{F})$ as relatively-graded $\mathbb{F}[U,U^{-1}]$ modules.*

In [OS04c], Ozsváth and Szabó calculate $HF^\infty(Y,\mathfrak{s})$ for all Y with $b_1(Y) \leq 2$ and all Spin^c structures $\mathfrak{s}$. They are also able to calculate HF^∞ for arbitrary three-manifolds equipped with torsion Spin^c structures when working instead with certain twisted coefficients. This gives a universal coefficients spectral sequence such that for torsion Spin^c structures, $\mathfrak{s}_0$, the E_3 term is given by $\Lambda^*(H^1(Y;\mathbb{Z}))\otimes\mathbb{Z}[U,U^{-1}]$ and converges to $HF^\infty(Y,\mathfrak{s}_0)$. They then

conjecture that the d_3 differential is determined by the cup product structure and that all higher differentials vanish (see Conjecture 4.10 of [OS03c]). More specifically, the conjectured d_3 differential is given by the map

$$\partial_Y^\infty(\alpha \otimes U^j) = \iota_{\mu_Y}(\alpha) \otimes U^{j-1},$$

where μ_Y is the *integral triple cup product form*, the three-form on $H^1(Y;\mathbb{Z})$ defined by

$$\mu_Y(a \wedge b \wedge c) = \langle a \smile b \smile c, [Y] \rangle.$$

In [Mar08], Mark studies the complex $C_*^\infty(Y)$ which has chain groups $\Lambda^*(H^1(Y;\mathbb{Z})) \otimes \mathbb{Z}[U,U^{-1}]$ and differential ∂_Y^∞. The homology of this complex, *cup homology*, is denoted HC^∞. Therefore, for all practical purposes, the conjecture of Ozsváth and Szabó can be rephrased as establishing an isomorphism $HC^\infty(Y) \cong HF^\infty(Y,\mathfrak{s})$ for torsion Spin^c structures.

From now on all of our Heegaard Floer homologies will be calculated with coefficients in $\mathbb{F}$, unless mentioned otherwise. Furthermore, we will use cup homology with $\mathbb{F}$-coefficients. In this case, we are using $C_*^\infty(Y) \otimes \mathbb{F}[U,U^{-1}]$ (calculate integral triple cup products and then reduce mod 2, as opposed to using mod 2 triple cup products). We will omit this from the notation - the coefficients for Heegaard Floer homology and for cup homology are always assumed to be $\mathbb{F}$ unless stated otherwise. We are now ready to state the main theorem of the thesis.

Theorem 1. *If $\mathfrak{s}$ is a torsion Spin^c structure on Y, the relatively-graded $\mathbb{F}[U,U^{-1}]$-modules $HF_*^\infty(Y,\mathfrak{s})$ and $HC_*^\infty(Y)$ are isomorphic. Thus, $HF^\infty(Y,\mathfrak{s})$ agrees with Conjecture 4.10 of [OS03c] mod 2.*

Remark 1.2.2. It is also known in monopole Floer homology that for torsion Spin^c structures, $HC^\infty(Y;\mathbb{Q}) \cong \overline{HM}(Y,\mathfrak{s};\mathbb{Q})$ (see Section IX of [KM07]); furthermore the Main Theorem of Kutluhan, Lee, and Taubes [KLT10a, KLT10b, KLT10c, KLT11] shows $\overline{HM}(Y,\mathfrak{s};\mathbb{Z}) \cong HF^\infty(Y,\mathfrak{s};\mathbb{Z})$. Thus, Theorem 1 is already known with $\mathbb{Q}$-coefficients.

Remark 1.2.3. While it is tempting to try to calculate HF^∞ for non-torsion Spin^c structures, the methods we use cannot be extended to this case. This is because we will actually need

to work with a completed ring, $\mathbb{F}[[U, U^{-1}]$, in order to apply the link surgery formula. It turns out that for non-torsion Spin^c structures, completion causes HF^∞ to vanish [MO10].

1.3 Applications of the Main Theorem

In [Lee05], Lee constructs a spectral sequence to prove that $\dim \widetilde{Kh}(L) \geq 2^{|L|-1}$, where $|L|$ is the number of components of L. We instead use the spectral sequence relating Khovanov homology to the Heegaard Floer homology of the double-branched cover mentioned above to obtain a different bound.

Theorem 1.3.1. *Let L be a link in S^3. Then,*

$$\dim \widetilde{Kh}(L; \mathbb{F}) \geq 2^{b_1(\Sigma_2(L))} \cdot |\operatorname{Tor} H_1(\Sigma_2(L); \mathbb{Z})|.$$

Another strength of Heegaard Floer homology is its TQFT-like structure. Given a Spin^c cobordism $(W^4, \mathfrak{t})$ from $(Y_1, \mathfrak{s}_1)$ to $(Y_2, \mathfrak{s}_2)$, Ozsváth and Szabó construct a map $F^\circ_{W,\mathfrak{t}} : HF^\circ(Y_1, \mathfrak{s}_1) \to HF^\circ(Y_2, \mathfrak{s})$; furthermore, the absolute shift in grading can be determined by classical invariants of $(W, \mathfrak{t})$: the signature, Euler characteristic, and $\langle c_1(\mathfrak{t})^2, [W] \rangle$ [OS06b]. We would like to use Theorem 1 to study these cobordism maps. Since $HF^\infty(S^3)$ is a free $\mathbb{F}[U, U^{-1}]$-module of rank 1, it suffices to know whether or not this map is 0 to completely understand the map (it turns out that HF^∞ is always a free $\mathbb{F}[U, U^{-1}]$-module for torsion Spin^c structures).

Recall that W is a *2-handlebody* if it is obtained by attaching 2-handles to a four-ball; such a W will be simply-connected and will have $H_2(W; \mathbb{Z})$ torsion free. If W is a manifold with connected boundary, Y, we may remove a ball from W to obtain a cobordism from S^3 to Y, which we still denote by W.

Theorem 1.3.2. *Suppose that W is a 2-handlebody. Let $\mathfrak{t}$ be a Spin^c structure on W which restricts to a torsion Spin^c structure, $\mathfrak{s}$, on Y. The induced map $F^\infty_{W,\mathfrak{t}} : HF^\infty(S^3) \to HF^\infty(Y, \mathfrak{s})$ is non-zero if and only $b_2^+ = 0$ and $\mu_Y \equiv 0$ (mod 2).*

Finally, we recall the fifth flavor of Heegaard Floer homology, $HF_{red}(Y, \mathfrak{s})$. This is defined by $\ker\{U^d : HF^+(Y, \mathfrak{s}) \to HF^+(Y, \mathfrak{s})\}$, for sufficiently large d (this will always stabilize).

HF_{red} is particularly important for studying closed four-manifolds with Heegaard Floer homology; the way that the closed four-manifold invariants are constructed is by cutting X^4 into two cobordisms $X = X_1 \cup_Y X_2$, such that the maps $F^-_{X_1,\mathfrak{t}_1}$ and $F^+_{X_2,\mathfrak{t}_2}$ each factor through $HF_{red}(Y)$. The composition of these are used to define the *mixed invariants* of X, which are conjecturally the same as the Seiberg-Witten invariants.

Thus, it is useful to know when cobordisms induce maps that factor through HF_{red}. In [OS06b], it is shown that if a cobordism has $b_2^+ > 0$, then this will always happen. Furthermore, Ozsváth and Szabó show that factoring through HF_{red} is equivalent to the map on HF^∞ vanishing. Therefore, we have:

Corollary 1.3.3. *Suppose that W is a 2-handlebody. Let $\mathfrak{t}$ be a Spin^c structure on W which restricts to a torsion Spin^c structure $\mathfrak{s}$ on Y. The map induced by $(W, \mathfrak{t})$ factors through $HF_{red}(Y, \mathfrak{s})$ (either from $HF^-(S^3)$ to $HF^-(Y)$ or from $HF^+(Y)$ to $HF^+(S^3)$) if and only if $b_2^+ > 0$ or μ_Y does not vanish identically (mod 2).*

1.4 Outline

At this point, we give a brief outline of how the proof of Theorem 1 goes. By the theory of surgery-equivalences of three-manifolds due to Cochran, Gerges, and Orr, if two three-manifolds have isomorphic triple cup product forms, then they can be related by a certain sequence of surgeries. We show that such surgeries cannot affect HF^∞. This will give an easy proof of Proposition 1.2.1.

We use this surgery-invariance to reduce Theorem 1 to a special class of manifolds - those which are presented as 0-surgery on an algebraically split link in S^3. Since Theorem 1 has already been computed by Ozsváth and Szabó for $b_1 \leq 2$ in [OS04c], we begin with $b_1(Y) = 3$. In this case, the special class of manifolds, which includes $\#^3_{i=1} S^2 \times S^1$ and $\mathbb{T}^3$, is easily described; all of these manifolds are 0-surgery on a knot generalizing the Borromean knot in $\#^2_{i=1} S^2 \times S^1$. The Heegaard Floer homology of these manifolds can be explicitly calculated with the link surgery formula (in fact, just the original integer surgery formula for knots).

For the case of higher b_1, we apply the link surgery formula to a general 0-framed, algebraically split link in S^3. The approach is to analyze the spectral sequence converging to HF^∞ arising from a certain filtration on the link surgery formula. The first four pages can be explicitly calculated based on the previously established information about manifolds with $b_1 \leq 3$. It turns out that the E_4 page is exactly the conjectured cup homology groups. The proof that the higher differentials vanish is completed by an induction argument on the "complexity" of the algebraically split link.

The thesis is organized as follows. In Chapter 2, we give a brief introduction to Heegaard Floer homology and link Floer homology which summarizes the material from [OS04d] and [OS08a] that we will need. We will also review the multi-pointed Heegaard diagrams of [MOS09]. Readers familiar with link Floer homology should definitely skip this part. In Chapter 3, we mention a few simple facts from homological algebra that will be useful. Furthermore, we set up the requisite machinery on hypercubes of chain complexes to review the link surgery formula. The link surgery formula is then summarized in Chapter 4. We discuss the special properties of the link surgery formula for the ∞ flavor of Heegaard Floer homology in Chapter 5. In Chapter 6, we recall the notions of surgery equivalences of three-manifolds and use this to prove Proposition 1.2.1. Chapter 7 illustrates the method of composing knots and calculates HF^∞ for three-manifolds with $b_1 = 3$. We then use the link surgery formula in Chapter 8 to complete the proof of Theorem 1. The two applications are proved in Chapter 9. The final chapter contains some brief concluding remarks.

We remark that the proofs of Proposition 1.2.1 and the $b_1 = 3$ and 4 cases of Theorem 1 are given in [Lid10]. The remainder of the proof of Theorem 1 can be found in [Lid11].

CHAPTER 2

A Review of Heegaard Floer Theory

2.1 Heegaard Floer Homology for Three-Manifolds

Let Y be a closed, connected, oriented three-manifold equipped with a Spin^c structure $\mathfrak{s}$. It is the goal of this section to introduce the groups $HF^\circ(Y,\mathfrak{s})$, where $\circ$ is any of $+,-,\infty,\widehat{\ }$, introduced by Ozsváth and Szabó in [OS04d]. As opposed to the traditional construction with $\mathbb{Z}[U]$ coefficients, we will instead work over $\mathbb{F}[[U]]$. For this reason, we will use the notation $\mathbf{HF}^\circ$ to indicate that we are working with $\mathbb{F}[[U]]$-coefficients; when we want to revert back to $\mathbb{Z}[U]$ or $\mathbb{F}[U]$ instead, this will simply be denoted HF° and we'll point out which one of these we are using. The reason why we work over this different base ring is that the link surgery formula of Manolescu and Ozsváth is constructed over this ring. We will see in Chapter 5.1 that one can recover the uncompleted mod 2 Heegaard Floer homology from $\mathbf{HF}^\infty$, for torsion Spin^c structures.

Begin with a self-indexing Morse function, $f : Y \to \mathbb{R}$, with one minimum (index 0) and one maximum (index 3). Let Σ_g denote $f^{-1}(3/2)$, a closed connected surface of genus g equal to the number of index 1 critical points (equivalently index 2 critical points) of f. The decomposition $Y = f^{-1}([0,3/2]) \cup f^{-1}([3/2,3])$ gives a Heegaard splitting for Y (a decomposition of Y into two solid handlebodies). We obtain g-tuples of simple closed curves $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^g$ and $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^g$ by intersecting the Heegaard surface with the ascending manifolds of the index 1 critical points (for the α_i) and the descending manifolds of the index two critical points (for the β_i). Finally, we choose a flow line from the index 3 critical point to the index 0 critical point. This flow line intersects Σ at exactly one point, w.

2.1.1 The Heegaard Floer Chain Complex

The idea is to construct the Heegaard Floer homology groups as the Lagrangian Floer homology of tori, $\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_g$, inside of $\mathrm{Sym}^g(\Sigma)$, the g-fold symmetric product of Σ. By some perturbation of f, we can choose these tori to intersect transversely; since these half-dimensional tori are compact, this intersection will be a finite set of points.

We first define the chain groups

$$\mathbf{CF}^\infty(\Sigma, \alpha, \beta, w) = \mathbb{F}[[U, U^{-1}] \cdot \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle.$$

Note that this is a finitely-generated $\mathbb{F}[[U, U^{-1}]$-vector space. We now need to define the differential. Fix $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. First the formula, then the explanation of all the terms.

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi) = 1} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{n_w(\phi)} \mathbf{y}.$$

Here, ϕ is a *Whitney disk*: a homotopy class of maps from $\mathbb{D}^2$ into $\mathrm{Sym}^g(\Sigma)$ such that $\phi(-i) = \mathbf{x}$, $\phi(i) = \mathbf{y}$, $\phi(\{z : |z| = 1, \mathrm{Re}(z) \geq 0\}) \subset \mathbb{T}_\alpha$, and $\phi(\{z : |z| = 1, \mathrm{Re}(z) \leq 0\}) \subset \mathbb{T}_\beta$ (ϕ is called a *Whitney disk*). We use $\pi_2(\mathbf{x}, \mathbf{y})$ to denote the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$.

It is important to note that $\mathrm{Sym}^g(\Sigma)$ can be given a symplectic structure such that the $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian [Per08] - this is not the original construction of Heegaard Floer homology, but it is cleaner to exposit this way. We fix a one-parameter family of almost complex structures J_s compatible with respect to this symplectic structure. We can ask which representatives of the homotopy class of ϕ are pseudoholomorphic with respect to J_s in an appropriate sense. Denote by $\mathcal{M}(\phi)$ the moduli space of these holomorphic curves, which is a manifold for a generic choice of family J_s. Note that a non-constant holomorphic Whitney disk admits a natural free $\mathbb{R}$-action by biholomorphically identifying the twice-punctured unit disk with an infinite strip in the complex plane and applying a translation. When the dimension of this moduli space is 1 (this is the condition $\mu(\phi) = 1$), quotienting by $\mathbb{R}$ leaves a finite number of points for generic families J_s by Gromov

compactness [OS04d]. This is precisely what we count in $\#(\mathcal{M}(\phi)/\mathbb{R})$. Since we are working mod 2, we don't need to worry about orienting the moduli spaces to count these points with sign like in the integral case. Finally, $n_w(\phi)$ is the intersection number of the map ϕ with the (complex) codimension 1 submanifold $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$. It is important to note that if ϕ is holomorphic, this intersection is always non-negative, since we are intersecting two almost-complex submanifolds.

Again, for a generic choice of J_s, we have that $\partial^2 = 0$. We may take the homology of this complex, which we denote by $\mathbf{HF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$. While the action of $\mathbb{F}[[U, U^{-1}]$ on the resulting homology is clear, we point out without details that this can be extended to a $\Lambda^*(H^1(Y;\mathbb{Z})) \otimes \mathbb{F}[[U, U^{-1}]$-module structure.

Remark 2.1.1. In order to properly define Heegaard Floer homology, one needs an additional hypothesis on the Heegaard diagram known as *admissibility* (we will only require weak admissibility). This technical condition will not often be a concern in this thesis. Therefore, we will not bother to define it and instead state heuristically that diagrams can be made admissible by isotoping the $\boldsymbol{\alpha}$ curves to introduce more intersections with the $\boldsymbol{\beta}$ curves.

2.1.2 Spin^c Structures and Gradings

Ozsváth and Szabó construct a map $\mathfrak{s}_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$ such that $\pi_2(\mathbf{x}, \mathbf{y})$ is non-empty if and only if $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}_w(\mathbf{y})$ are associated to the same Spin^c structure. This implies that $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ splits into a direct sum of complexes $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s})$, with associated homologies $\mathbf{HF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s})$, for each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

Let $\mathfrak{d} = \gcd_{\sigma \in H_2(Y;\mathbb{Z})} \langle c_1(\mathfrak{s}), \sigma \rangle$. We can equip $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s})$ with a relative $\mathbb{Z}/\mathfrak{d}$-grading, defined by

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_w(\phi),$$

where ϕ is any element of $\pi_2(\mathbf{x}, \mathbf{y})$. For this reason, the element U is endowed with grading -2, so that our differential can lower grading by 1. In particular, if $c_1(\mathfrak{s})$ is torsion, then we obtain a relative $\mathbb{Z}$-grading. In general, we will abuse notation and say that $\mathfrak{s}$ is *torsion* if $c_1(\mathfrak{s})$ is torsion in $H^2(Y;\mathbb{Z})$.

One is often interested in other related flavors of Heegaard Floer homology. First, there is $\mathbf{CF}^-$ which is obtained by considering the subcomplex generated by elements with non-negative powers of U. Then, there is $\mathbf{CF}^+$ which is the quotient complex $\mathbf{CF}^\infty/\mathbf{CF}^-$. One finally can consider $\widehat{\mathbf{CF}} = \mathbf{CF}^-/U \cdot \mathbf{CF}^-$. The homologies of these complexes are denoted $\mathbf{HF}^-$, $\mathbf{HF}^+$, and $\widehat{\mathbf{HF}}$ respectively. We will in general be focusing solely on $\mathbf{HF}^\infty$. When we do not want to refer to a specific flavor of Heegaard Floer homology, we will use the notation $\mathbf{HF}^\circ$.

2.1.3 Invariance

While we have successfully defined the Heegaard Floer homology groups for a given Heegaard diagram, it's not in any way clear that this was independent of the choices made. In order to prove that the resulting homologies are always isomorphic, we first need to know how to relate Heegaard diagrams presenting the same manifold.

Let's begin with a diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$. It is clear that if we isotope the $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ curves then the resulting manifolds these diagrams represent are diffeomorphic. Furthermore, if we choose two curves in $\boldsymbol{\alpha}$, say α_1 and α_2, we can form a new Heegaard diagram $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}, w)$ by replacing α_2 with α_2', where α_1, α_2, and α_2' bound a pair of pants in Σ. This move is called a *handleslide*. Standard handle calculus (see [GS99]) shows that the resulting manifolds will be diffeomorphic. Finally, given a three-manifold, we can connect-sum with S^3 and that will not change the diffeomorphism-type of the manifold. Therefore, we can take a Heegaard diagram for S^3, denoted $(\mathbb{T}^2, \alpha_0, \beta_0)$, where α_0 and β_0 intersect transversely in a single point, and concatenate this with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ to obtain a new splitting $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$, where Σ' is diffeomorphic to $\Sigma \# \mathbb{T}^2$. This move is called a *stabilization*. It turns out that these are the only moves one needs.

Theorem 2.1.2 (Singer, Theorem 8 of [Sin33])**.** *Suppose that* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ *are Heegaard diagrams for* Y*. The diagrams are diffeomorphic if and only if they are related by a sequence of isotopies, handleslides, and (de)stabilizations.*

One can extend this to pointed Heegaard diagrams by requiring that these isotopies,

handleslides, and stabilizations stay away from the basepoint.

Theorem 2.1.3 (Osvaáth-Szabó [OS04d])**.** *The isomorphism-type of the* $\mathbb{F}[[U]]$*-modules* $\mathbf{HF}^\circ(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s})$ *are independent of the choices made (Morse function, almost complex structure, etc).*

Remark 2.1.4. In light of Theorem 2.1.3, we will simply use the notation $\mathbf{HF}^\circ(Y, \mathfrak{s})$ to either refer to the Heegaard Floer homology for some specified pointed Heegaard diagram or its isomorphism-type.

Remark 2.1.5. There is additional structure on the Heegaard Floer homology groups. More specifically, the $\mathbb{F}[[U]]$-module structure can be extended to a $\Lambda^*(H^1(Y;\mathbb{Z}))\otimes\mathbb{F}[[U]]$-module structure [OS04d]. We will not define it as we will only need to know that it exists.

We will not give a proof of Theorem 2.1.3; however, we will discuss some of the techniques associated with the proof, as the tools to show invariance (and the theorem itself) are an essential part of the construction of the link surgery formula.

First of all, the independence of the choice of almost complex structure is essentially a theorem about Lagrangian Floer homologies, and the case of Heegaard Floer homology is proved in Theorem 6.1 of [OS04d]. We will ignore the argument for stabilization, since we will end up assuming our diagrams are sufficiently stabilized to begin with.

Given an isotopy or a handleslide from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ to $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}, w)$, we would like to study the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}')$. It is not difficult to see that this represents $\#_{i=1}^g S^2\times S^1$. We will see later in Remark 2.1.14 that $\mathbf{HF}^-(\#_{i=1}^g S^2\times S^1, \mathfrak{s}_0) \cong \Lambda^*(H^1(\#_{i=1}^g S^2\times S^1;\mathbb{Z}))\otimes\mathbb{F}[[U]]$, where $\mathfrak{s}_0$ is the unique torsion Spin^c structure on $\#_{i=1}^g S^2\times S^1$. In particular, we can pick a cycle $\theta \in \mathbf{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}', w, \mathfrak{s}_0)$, which represents the unique generator sitting in the highest $\mathbb{Z}$-grading. We can similarly use this to construct corresponding cycles for the other flavors as well. We can define a map $\Phi : \mathbf{HF}^\circ(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s}) \to \mathbf{HF}^\circ(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}, w, \mathfrak{s})$ by

$$\Phi^{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(\mathbf{x}) = \sum_{\mathbf{y}} \sum_{\psi\in\pi_2(\mathbf{x},\theta,\mathbf{y}),\mu(\psi)=0} \#(\mathcal{M}(\psi))\cdot U^{n_w(\psi)}\mathbf{y},$$

where we are now counting holomorphic triangles in the triple-diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\beta}, w)$ - each side of the triangle is on a Lagrangian torus determined by a set of attaching curves and

the corners are sent to $\mathbf{x}$, θ, and $\mathbf{y}$ in a clockwise-manner. We now count Maslov index zero triangles, where each edge of the triangle is mapped to one of $\mathbb{T}_\alpha$, $\mathbb{T}_{\alpha'}$, or $\mathbb{T}_\beta$.

Remark 2.1.6. More generally, given a diagram $(\Sigma, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n, w)$, one can count holomorphic n-gons in $\mathrm{Sym}^g(\Sigma)$ with boundary on the $\mathbb{T}_{\alpha_i}$. Counting holomorphic polygons is a very useful tool. For example, it is not true that $\Phi^{\boldsymbol{\alpha}',\boldsymbol{\alpha}''} \circ \Phi^{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = \Phi^{\boldsymbol{\alpha},\boldsymbol{\alpha}''}$ on the chain level for three sets of curves $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$, and $\boldsymbol{\alpha}''$. However, these two maps are chain homotopic via a homotopy which counts quadrilaterals in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\alpha}'', \boldsymbol{\beta}, w)$. We will rely on this fact heavily when constructing the link surgery formula.

2.1.4 Basic Examples and Properties

At this point, we discuss a few useful computations and basic properties of Heegaard Floer homology that will come in handy later. For notational convenience, we will often just point out the case of $\mathbf{HF}^\infty$. We will say that $\mathbf{CF}^+$ or $\mathbf{HF}^+$ is "free" if it is a direct sum of $\mathbb{F}[[U, U^{-1}]/\mathbb{F}[[U]]$.

It is well-known that the homology of the product of two topological spaces satisfies a Künneth formula. The analogous result holds for Heegaard Floer homology when we replace products with connect-sums.

Theorem 2.1.7 (Ozsváth-Szabó, Theorem 6.2 of [OS04c])**.** $\mathbf{HF}^\infty(Y_1, \mathfrak{s}_1) \otimes \mathbf{HF}^\infty(Y_2, \mathfrak{s}_2) \cong \mathbf{HF}^\infty(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$.

For the next proposition, we remark that there exists a spectral sequence with E_1 page $\widehat{\mathbf{HF}}(Y, \mathfrak{s}) \otimes \mathbb{F}[[U, U^{-1}]$ converging to $\mathbf{HF}^\infty(Y, \mathfrak{s})$ coming from a filtration on $\mathbf{CF}^\infty(Y, \mathfrak{s})$ determined by $n_w(\phi)$.

Proposition 2.1.8 (Ozsváth-Szabó, see Proposition 5.1 of [OS04c])**.** *For all Spin^c structures $\mathfrak{s}$ on a rational homology sphere Y, we have that $\chi(\widehat{\mathbf{HF}}(Y, \mathfrak{s})) = 1$. Therefore, $\mathbf{HF}^\infty(Y, \mathfrak{s})$ is non-trivial, as are all other flavors.*

Remark 2.1.9. Theorem 1 will prove that $\mathbf{HF}^\infty(Y, \mathfrak{s})$ is non-trivial for torsion Spin^c structures on manifolds with $b_1(Y) > 0$, even though the Euler characteristic is 0 [OS04c].

Example 2.1.10 (Lens Spaces). Choose the standard genus 1 Heegaard splitting for $L(p,q)$. In this case, there is a Heegaard diagram $(\mathbb{T}^2, \boldsymbol{\alpha}, \boldsymbol{\beta})$, such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ intersect transversely in p points. Therefore, $\mathbf{CF}^\circ(L(p,q))$ is free of rank p. Each $\mathbf{HF}^\circ(L(p,q),\mathfrak{s})$ must be non-trivial by Proposition 2.1.8. Since there are exactly p Spin^c structures (all torsion), each $\mathbf{HF}^\circ(L(p,q),\mathfrak{s})$ must be free of rank 1. This calculation also covers the case of S^3 when $p = 1$.

Corollary 2.1.11. *For any p, $\mathbf{HF}^\infty(Y,\mathfrak{s}) \cong \mathbf{HF}^\infty(Y\#L(p,1),\mathfrak{s}\#\mathfrak{s}_p)$ for any $\mathfrak{s}_p$ on $L(p,1)$.*

Remark 2.1.12. This is actually true for all flavors, but one has to be slightly more careful with the statement of the Künneth formula above.

Example 2.1.13 ($S^2 \times S^1$). It is easy to construct a Heegaard diagram for $S^2 \times S^1$, by simply taking $\Sigma = \mathbb{T}^2$ and $\alpha = \beta$ any essential curve on $\mathbb{T}^2$. However, we need to make α and β intersect transversely. We push β mostly off of α, except for a pair of transverse intersection points, $\mathbf{x}$ and $\mathbf{y}$. Therefore, $\mathbf{CF}^\circ$ will have rank 2. There are two Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. Each of these two homotopy classes has a unique holomorphic representative (after quotienting out by the $\mathbb{R}$-action) and thus has Maslov index 1. These cancel out mod 2 and thus $\partial \equiv 0$. Therefore, $\mathbf{HF}^\circ$ is again free of rank 2. It turns out that this is supported in the unique torsion Spin^c structure, $\mathfrak{s}_0$.

Remark 2.1.14. By applying the appropriate rephrasing of Theorem 2.1.7 for general flavors and Example 2.1.13, we see that $\mathbf{HF}^\infty(\#_{i=1}^n S^2 \times S^1, \#_{i=1}^n \mathfrak{s}_0) \cong \Lambda^*(H^1(\#_{i=1}^n S^2 \times S^1;\mathbb{Z})) \otimes \mathbb{F}[[U,U^{-1}]$, and similarly for the other flavors.

It turns out the above calculations basically did not depend on the flavor we were working with. This is absolutely not true in general. The most powerful invariants are $\mathbf{HF}^+$ and $\mathbf{HF}^-$ which are essentially dual to each other. The next most information comes from $\widehat{\mathbf{HF}}$. Finally, $\mathbf{HF}^\infty$ is the simplest in structure. We give some examples of the potential distinctions the various flavors can make.

Example 2.1.15. In [OS03a], the Heegaard Floer homologies of the Brieskorn sphere $\Sigma(2,3,7)$

(+1-surgery on the left-handed trefoil) are calculated to be

$$\mathbf{HF}^\circ(\Sigma(2,3,7)) \cong \begin{cases} \mathbb{F}^3 & \text{if } \circ = \widehat{\ } \\ \mathbb{F}[[U]] \oplus \mathbb{F} & \text{if } \circ = - \\ \mathbb{F}[[U,U^{-1}]/\mathbb{F}[[U]] \oplus \mathbb{F} & \text{if } \circ = + \\ \mathbb{F}[[U,U^{-1}] & \text{if } \circ = \infty \end{cases}.$$

On the other hand, in [OS04c] it is shown that there exists a torsion Spin^c structure, $\mathfrak{s}_0$, such that the Heegaard Floer homology of +19-surgery on the $(-2,7)$ torus knot, which in their notation is $Y_{3,19}$, is given by

$$\mathbf{HF}^\circ(Y_{3,19},\mathfrak{s}_0) \cong \begin{cases} \mathbb{F}^3 & \text{if } \circ = \widehat{\ } \\ \mathbb{F}[[U]] \oplus \mathbb{F}[[U]]/U^2 & \text{if } \circ = - \\ \mathbb{F}[[U,U^{-1}]/\mathbb{F}[[U]] \oplus \mathbb{F}[[U]]/U^2 & \text{if } \circ = + \\ \mathbb{F}[[U,U^{-1}] & \text{if } \circ = \infty \end{cases}.$$

Comparing these with Example 2.1.10 illustrates the hierarchy of structure in flavors of Heegaard Floer homology.

2.1.4.1 Universal Coefficients

While the goal of this thesis is to calculate $HF^\infty(Y,\mathfrak{s};\mathbb{F})$ for all torsion Spin^c structures, it turns out that this can be easily done if one uses the right coefficients. In particular, if one uses the appropriate choice of twisted coefficients for Heegaard Floer homology (which we will not define), the resulting calculation is straightforward.

Theorem 2.1.16 (Ozsváth-Szabó, Theorem 10.12 of [OS04c])**.** *For each torsion Spin^c structure $\mathfrak{s}$ on Y, there exists a coefficient system such that the Heegaard Floer homology with twisted coefficients, $\underline{HF^\infty}(Y,\mathfrak{s};\mathbb{Z})$, is isomorphic to $\mathbb{Z}[U,U^{-1}]$ as $\mathbb{Z}[U,U^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[H^1(Y;\mathbb{Z})]$-modules, where $H^1(Y;\mathbb{Z})$-acts trivially.*

Corollary 2.1.17. *If $\mathfrak{s}$ is torsion, there is a universal coefficients spectral sequence with E_2 term $\Lambda^*(H^1(Y;\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}[U,U^{-1}]$ converging to $HF^\infty(Y,\mathfrak{s};\mathbb{Z})$.*

We refer the reader to Proposition 16 of [Mar08] for more details on the construction of this spectral sequence. We remark that this universal coefficients spectral sequence works exactly the same when $\mathbb{Z}[U, U^{-1}]$-coefficients are replaced by $\mathbb{F}[[U, U^{-1}]$-coefficients. This spectral sequence will be studied in more detail in Chapter 7.1.

2.1.5 More Properties

Sometimes it is not reasonable to use a Heegaard diagram to calculate Heegaard Floer homologies. Conveniently enough, Heegaard Floer homology has many tools for calculations which allow one to avoid working directly with a Heegaard diagram. Many of these tools will be used throughout the thesis. We briefly mention a few such results.

One of the most fundamental tools in Heegaard Floer homology is the surgery exact triangle. This in fact was originally developed by Floer in the case of instanton homology [Flo90].

Theorem 2.1.18 (Ozsváth-Szabó, Theorem 9.16 of [OS04c])**.** *Suppose that K is nullhomologous in Y. Then there is a long exact sequence*

$$\ldots \to \widehat{\mathbf{HF}}(Y) \to \widehat{\mathbf{HF}}(Y_0(K)) \to \widehat{\mathbf{HF}}(Y_1(K)) \to \widehat{\mathbf{HF}}(Y) \ldots$$

This exact triangle underlies the idea behind the surgery formulas for knots and links that we will use.

As we saw, $\mathbf{HF}^\circ(Y, \mathfrak{s})$ can be equipped with a relative $\mathbb{Z}$-grading if $\mathfrak{s}$ is torsion. However, since the grading is relative, one cannot compare the gradings of elements in different Spin^c structures, let alone those coming from different manifolds. Using the functoriality of Heegaard Floer homology, Ozsváth and Szabó were able to consistently lift the relative-grading to a $\mathbb{Q}$-valued grading [OS06b]. While we do not give the construction of the grading, we do point out (and will use) that it exists.

Calculating the Heegaard Floer homology of $\mathbb{T}^3$ is extremely difficult to attempt from the definitions given here. However, with the help of the surgery exact triangle, the absolute grading, and the universal coefficients spectral sequence, this can still be computed. This,

in some sense, will be the starting point for our calculation of HF^∞.

Theorem 2.1.19 (Ozsváth-Szabó, Proposition 1.9 of [OS03a]). *If $\mathfrak{s}_0$ is the unique torsion Spin^c structure on $\mathbb{T}^3$, then*

$$\mathbf{HF}^\infty(\mathbb{T}^3, \mathfrak{s}_0) \cong (\Lambda^1(H^1(\mathbb{T}^3;\mathbb{Z})) \oplus \Lambda^2(H^1(\mathbb{T}^3;\mathbb{Z}))) \otimes \mathbb{F}[[U, U^{-1}].$$

The analogous statement holds for the other flavors.

2.1.6 Applications of Heegaard Floer Theory

To convince the reader to press forward in this thesis, we will mention a few important applications that have come out of Heegaard Floer theory. While the case of the unknot in the following theorem was dealt with using monopole Floer homology, the argument works exactly the same in Heegaard Floer homology. The rest was done with Heegaard Floer homology.

Theorem 2.1.20 (Kronheimer-Mrowka-Ozsváth-Szabó, Theorem 1.1 of [KMO07] and Ozsváth-Szabó, Theorems 1.1-1.2 of [OS06a]). *Let K be the unknot, figure-eight knot, or a trefoil. If K' is a knot in S^3 such that $S^3_r(K') \cong +S^3_r(K)$, then $K' = K$.*

Knowing something about the Heegaard Floer homology often allows one to deduce valuable information about the topology of a three-manifold.

Theorem 2.1.21 (Ozsváth-Szabó, Theorem 1.4 of [OS04a]). *If Y is a rational homology sphere and $\dim \widehat{\mathbf{HF}}(Y) = |H_1(Y;\mathbb{Z})|$, then Y cannot admit a co-orientable taut foliation.*

Previous to Heegaard Floer homology, it was very difficult to find examples of hyperbolic manifolds which do not admit co-orientable taut foliations. The major known results at the time were found in [CD03] and [RSS03]. In light of Theorems 2.1.18 and 2.1.21, it is easy to construct infinite classes of such manifolds.

Corollary 2.1.22. *Let K be a hyperbolic knot which admits a positive lens space surgery (many examples of these exist and are conjecturally classified by Berge [Ber]). Then, $S^3_r(K)$ does not admit a co-orientable taut foliation for sufficiently large $r \in \mathbb{Q}$.*

In another important application, Lisca and Stipsicz gave a complete classification of tight contact structures on Seifert fibered rational homology spheres using contact invariants arising in Heegaard Floer homology [LS07].

2.2 Heegaard Floer Invariants for Links

Given a nullhomologous link L in Y, there is an analogous notion of a Heegaard Floer complex for L. We will review this construction.

2.2.1 The Knot Floer Complex

Let's begin with the case of an oriented, nullhomologous knot K in Y. Suppose that the Morse function we choose on Y to obtain our Heegaard splitting has the property that K is the union of two distinct flows between the index 3 and index 0 critical points. We will denote their intersections with Σ as z and w. We can put a filtration on $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ which takes into account this additional point z and thus records information about K. This was first carried out independently by Ozsváth-Szabó [OS04b] and Rasmussen [Ras03]. The tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ is called a *doubly-pointed Heegaard diagram for K in Y*. It turns out that such a diagram always exists for any K in Y. We work with a fixed torsion Spin^c structure, $\mathfrak{s}$, on Y throughout.

The $\mathbb{Z}$-filtration we place on $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s})$ is called the *Alexander filtration* and is determined by

$$A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi) \text{ for } \phi \in \pi_2(\mathbf{x}, \mathbf{y}).$$

Furthermore, we impose the condition that U lowers Alexander filtration by 1. We can make this Alexander grading absolute by requiring that it be symmetric about 0 (the number of intersection points with $A(\mathbf{x}) = i$ is the same as the number of those with grading $-i$).

This naturally defines a filtered chain complex, whose filtered chain homotopy type is an invariant of K by arguments similar to those mentioned for Theorem 2.1.3 [OS04b]. We also have the other flavors in analogy with Heegaard Floer homology for three-manifolds.

As the Alexander filtration induces a spectral sequence, we study the (E_0, d_0) complex. The E_0 page is $\mathbf{CF}^\circ$ on the level of chain groups, but the differential now only counts pseudoholomorphic disks which also have $n_z(\phi) = 0$. The homology of the (E_0, d_0) complex, which is the E_1 page of this spectral sequence, is called *knot Floer homology*. The Alexander filtration actually extends to an honest relative $\mathbb{Z}$-grading on the knot Floer homology groups (the homogeneous elements in fixed filtration level form a subcomplex). In this subsection, we focus on $\widehat{\mathbf{HFK}}$, the case where $\circ = \widehat{\ }$. Knot Floer homology is a bi-graded group, $\widehat{\mathbf{HFK}}_*(Y, K, s)$, where $*$ refers to the Maslov grading and s is the Alexander grading.

If one constructs the bi-graded Euler characteristic for a knot $K \subset S^3$,

$$\sum_{s\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} (-1)^i t^s \dim \widehat{\mathbf{HFK}}_i(S^3, K, s),$$

Ozsváth and Szabó show in [OS04b] that this is the classical Alexander polynomial of K. In particular, we have thus said that knot Floer homology categorifies the Alexander polynomial. However, it contains much more information than the Alexander polynomial. We mention a few key results about knot Floer homology.

Recall that the Alexander polynomial cannot detect the unknot, as there are non-trivial knots with $\Delta_K(t) = 1$. It turns out that the additional structure of the Heegaard Floer homology groups can in fact detect the unknot.

Theorem 2.2.1 (Ozsváth-Szabó, Theorem 1.2 of [OS04a])**.** *If $K \subset S^3$, then the genus of K, $g(K)$, is given by* $\max_{s\geq 0}\{\widehat{\mathbf{HFK}}(S^3, K, s; \mathbb{Z}) \neq 0\}$.

While the Alexander polynomial of a fibered knot is monic, the converse is not always true. On the other hand, the categorified statement is an if and only if.

Theorem 2.2.2 (Ni, Theorem 1.1 of [Ni07] and Ghiggini, Theorem 1.4 of [Ghi08])**.** *If K is a nullhomologous knot in Y such that $Y - K$ is irreducible, then K is a fibered knot if and only if $\widehat{HFK}(Y, K, g(K); \mathbb{Z}) \cong \mathbb{Z}$.*

More generally, we remark that Heegaard Floer homology is particularly adept at identifying surfaces in three-manifolds. Knot Floer homology also gives a new proof of the classical Milnor conjecture, originally proved by Kronheimer and Mrowka using gauge theory.

Theorem 2.2.3 (Ozsváth-Szabó, Corollary 1.7 of [OS03b]). *The slice genus of the* (p, q)*-torus knot is* $(p-1)(q-1)/2$.

It turns out that the knot Floer complex is very well suited for studying the Heegaard Floer homology of Dehn surgeries on K. This filtered chain complex will essentially be the starting point for constructing the link surgery formula. Before discussing this, we will first construct the corresponding Floer complex for links.

2.2.2 The Link Floer Complex

Before carrying out the general construction of the link Floer complex, we must extend our understanding of what it means to be a Heegaard diagram. Throughout this section, we will work with a fixed oriented, nullhomologous link $L = K_1 \cup \ldots \cup K_\ell$ in Y. We will discuss the notion of a general multi-pointed Heegaard diagram from [MOS09].

In our construction of a Heegaard diagram, we began with a self-indexing Morse function with only one index 0 and one index 3 critical point. However, we would like to be able to allow for more index 0 and index 3 critical points. The idea is that for a link L, each component should correspond to two gradient flow lines, as in the case of doubly-pointed Heegaard diagrams. However, since components are disjoint, they cannot all intersect at the same index 0/3 critical points. Therefore, we must allow Morse functions that have at least one index 0/3 pair for each component of L. This prompts the following definition.

Definition 2.2.4. A *multi-pointed Heegaard diagram for* L *in* Y is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ each consist of $g + k - 1$ disjoint simple closed curves that span a g-dimensional subspace of $H_1(\Sigma; \mathbb{R})$, where there are k basepoints of type w and ℓ basepoints of type z (and $k \geq \ell$). We label the components of $\Sigma - \boldsymbol{\alpha}$ by $A_1, \ldots, A_k$ and the components of $\Sigma - \boldsymbol{\beta}$ by $B_1, \ldots, B_k$. We ask that w_i is in $A_i \cap B_i$. Furthermore, there must be a permutation σ of the B_i's, for $1 \leq i \leq \ell$, such that each z_i is in $A_i \cap B_{\sigma(i)}$. Finally, we require that connecting w_i to z_i in the handlebody spanned by the $\boldsymbol{\alpha}$ curves and then z_i to w_i in the $\boldsymbol{\beta}$-handlebody gives a knot isotopic to the link component K_i and that the union of these components recovers L.

Remark 2.2.5. There is a more general form of multi-pointed Heegaard diagram given in [MOS09], which allows us to have multiple z and w basepoints to represent K_i. In our case, the $w_{\ell+1}, \dots, w_k$ represent extra basepoints which are not associated with any component. When working with the link surgery formula, these will be remnants of components of a larger link that used to be accounted for. These, however, contain no information about L (or any link that came before since we got rid of the associated z basepoint).

We can now construct the link Floer complex, $\mathbf{CFL}^\infty$, analogous to the knot Floer complex. However, this complex has many filtrations - one for each component. Again we restrict to intersection points corresponding to the fixed torsion Spin^c structure on Y. The chain groups of $\mathbf{CFL}^\infty$ will be freely generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, but this time over the larger ring $\mathbb{F}[[U_1, \dots, U_k, U_1^{-1}, \dots, U_k^{-1}]$. The differential will record the intersections of holomorphic disks with the basepoint w_i by a power of U_i. We can write this explicitly as

$$\partial(x) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi) = 1} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{n_{w_1}(\phi)} \cdot \dots \cdot U_k^{n_{w_k}(\phi)} \mathbf{y}.$$

For each component K_i, we have its induced Alexander filtration:

$$A_i(x) - A_i(y) = n_{z_i}(\phi) - n_{w_i}(\phi).$$

This gives a $\mathbb{Z}^\ell$-filtration on $\mathbf{CFL}^\infty$.

As in the case of knots, we can pin down the absolute Alexander gradings by requiring that they are centered about 0. Similarly, we can construct $\widehat{\mathbf{CFL}}$ by choosing some U_i to set to 0 (just one, not all of them!).

As the reader should expect at this point, Ozsváth and Szabó show in [OS08a] that the $\mathbb{Z}^\ell$-filtered chain homotopy type of $\mathbf{CFL}^\infty$ is independent of the multi-pointed Heegaard diagram chosen. It is important to note that their proof shows that all of the U_i must act the same up to chain homotopy. This guarantees that the $\mathbb{Z}^\ell$-filtered chain homotopy type of $\widehat{\mathbf{CFL}}$ is also an invariant.

While one can take an appropriate homology of this object, the so-called *link Floer homology*, we will not be interested in working with this. Instead, we focus on the multi-

filtered chain homotopy type of $\mathbf{CFL}^\infty$. This is the object that we will use to build the link surgery formula to calculate the Heegaard Floer homology of integer surgeries on L.

CHAPTER 3

Requisite Homological Algebra

We now introduce the relevant preliminaries in homological algebra in order to construct and utilize the link surgery formula. The material on hypercubes of chain complexes was developed in [MO10]. The rest can be found in a standard homological algebra textbook such as [Wei94].

3.1 Mapping Cones

Suppose that $f : (V, \partial_V) \to (W, \partial_W)$ is a chain map between chain complexes which are finitely-generated over a field of characteristic 2 (so we can ignore signs). Construct the *mapping cone* of f, denoted $Cone(f)$, which has chain groups $[V]_1 \oplus W$ (we shift the grading of V up by 1) and differential

$$\partial(v, w) = (\partial_V(v), f(v) + \partial_W(w)).$$

Fact 3.1.1. Using the obvious long exact sequence in homology, we see that

$$\dim H_*(Cone(f)) = \dim H_*(V, \partial_V) + \dim H_*(W, \partial_W) - 2 \operatorname{rk} f_*,$$

where this is an equality for the *total* dimensions of the homologies, not for individual gradings (although an analogous formula can obviously be worked out).

While a straightforward linear algebra exercise, the following will be needed for the final step of the proof of Theorem 1

Lemma 3.1.2. *Suppose V_1, V_2, W_1, and W_2 are isomorphic finite-dimensional vector spaces over a field of characteristic 2, each equipped with the differential $\partial \equiv 0$. Let $F_{i,j} : V_i \to W_j$*

and define $\Theta : V_1 \oplus V_2 \to W_1 \oplus W_2$ *by*

$$\Theta(v_1, v_2) = (F_{1,1}(v_1) + F_{2,1}(v_2), F_{1,2}(v_1) + F_{2,2}(v_2)).$$

Furthermore, suppose that $F_{2,2}$ *is such a quasi-isomorphism between* V_2 *and* W_2 *(or equivalently, an invertible map). Then, the total dimension of* $H_*(Cone(\Theta))$ *is equal to the total dimension of* $H_*(Cone(F_{1,1} - F_{2,1}F_{2,2}^{-1}F_{1,2}))$.

Proof. By Fact 3.1.1, the homology of $Cone(\Theta)$ has total dimension given by

$$\dim(V_1 \oplus V_2) + \dim(W_1 \oplus W_2) - 2\,\mathrm{rk}\,\Theta.$$

We study the matrix $\Theta = \begin{pmatrix} F_{1,1} & F_{2,1} \\ F_{1,2} & F_{2,2} \end{pmatrix}$. It is easy to see that this matrix has the same dimension as

$$X = \begin{pmatrix} F_{1,1} - F_{2,1}F_{2,2}^{-1}F_{1,2} & 0 \\ F_{1,2} & F_{2,2} \end{pmatrix},$$

which is the sum of the dimensions of each of $F_{1,1} - F_{2,1}F_{2,2}^{-1}F_{1,2}$ and $F_{2,2}$, the latter of which is $\dim V_2$. Now, we have

$$\begin{aligned} \dim H_*(Cone(\Theta)) &= \dim V_1 + \dim W_1 + 2\dim V_2 - 2(\mathrm{rk}(F_{1,1} - F_{2,1}F_{2,2}^{-1}F_{1,2}) + \dim V_2) \\ &= \dim V_1 + \dim W_1 - 2\,\mathrm{rk}(F_{1,1} - F_{2,1}F_{2,2}^{-1}F_{1,2}) \\ &= \dim H_*(Cone(F_{1,1} - F_{2,1}F_{2,2}^{-1}F_{1,2})). \end{aligned}$$

□

3.2 Hypercubes of Chain Complexes

Definition 3.2.1. An *n-dimensional hyperbox of size* $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ is the following subset of $\mathbb{N}^n$

$$\mathbb{E}(\mathbf{d}) = \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) |\ 0 \leq \varepsilon_i \leq d_i\}.$$

If $\mathbf{d} = (1, \ldots, 1)$, then $\mathbb{E}(\mathbf{d})$ is a *hypercube*. The *length* of ε, $\|\varepsilon\|$, is given by $\sum_i \varepsilon_i$. The elements of $\mathbb{E}(\mathbf{d})$ are called *vertices*.

There is a natural partial order on $\mathbb{E}(d)$ given by $\varepsilon \leq \varepsilon'$ if and only if $\varepsilon_i \leq \varepsilon'_i$ for all i. Two vertices in the hyperbox are *neighbors* if they differ by an element of $\{0, 1\}^n$. The

important example to keep in mind is the n-dimensional hypercube determined by the set of sublinks of an n-component link L. We may identify the sublinks, M, of L with the vertices of $\{0,1\}^n$ by $\varepsilon(M) = (\varepsilon(M)_1, \ldots, \varepsilon(M)_n)$, where

$$\varepsilon(M)_i = \begin{cases} 1 & \text{if } K_i \subset M, \\ 0 & \text{if } K_i \not\subset M. \end{cases}$$

Definition 3.2.2. An *n-dimensional hyperbox of chain complexes of size* $\mathbf{d}$ is a collection of chain complexes $(\mathcal{C}^\varepsilon_*, D^{\mathbf{0}}_\varepsilon)$ for each $\varepsilon \in \mathbb{E}(\mathbf{d})$ equipped with additional operators $D^{\varepsilon'}_\varepsilon : \mathcal{C}^\varepsilon_* \to \mathcal{C}^{\varepsilon+\varepsilon'}_{*+\|\varepsilon'\|-1}$, for $\varepsilon' \neq \mathbf{0}$ in $\{0,1\}^n$; the operators are assumed to be 0 if $\varepsilon + \varepsilon'$ is no longer in the hyperbox. For each ε, these operators are required to satisfy the following relation for all $\varepsilon' \in \{0,1\}^n$:

$$\sum_{\gamma \leq \varepsilon'} D^{\varepsilon'-\gamma}_{\varepsilon+\gamma} \circ D^\gamma_\varepsilon = 0. \tag{3.1}$$

The way to think of this is that the $D^{\varepsilon'}$ are chain maps when $\|\varepsilon'\| = 1$ and chain homotopies for $\|\varepsilon'\| = 2$. The higher maps are chain homotopies of chain homotopies, etc.

We can construct a single chain complex if the hyperbox of chain complexes is a hypercube. This is called the *total complex*, defined by

$$(\mathcal{C}_* = \bigoplus_\varepsilon \mathcal{C}^\varepsilon_{*+\|\varepsilon\|}, D = \sum_{\varepsilon,\varepsilon'} D^\varepsilon_{\varepsilon'}).$$

We will omit the subscript notation from the D from now on, where it will just be assumed that the map is 0 if the relevant domains and ranges do not match up. Furthermore, we use the notation ∂ for D^0 at any vertex of the hypercube.

Given a filtered chain complex $(\mathcal{C}, \mathcal{F})$, we denote the ith page of the associated spectral sequence by $E_i(\mathcal{C}, \mathcal{F})$, or just $E_i(\mathcal{C})$ if the filtration is clear. We will use d^C_k or d_k to denote the kth differential in the spectral sequence associated to this filtration. The *depth* of a filtered complex is the largest difference in the filtration levels of two non-zero elements. If k is greater than the depth of the filtration, the kth differential in the spectral sequence, d_k, must vanish.

Fact 3.2.3. Given a filtered chain map $f : (\mathcal{C}_1, \mathcal{F}_1) \longrightarrow (\mathcal{C}_2, \mathcal{F}_2)$, for each $i \geq 0$ there exist filtrations $\mathcal{F}(i)$ on $Cone(f)$ such that $E_i(Cone(f), \mathcal{F}(i)) \cong Cone(f_i)$, where f_i is the induced map from $E_i(\mathcal{C}_1, \mathcal{F}_1)$ to $E_i(\mathcal{C}_2, \mathcal{F}_2)$. This tells us that over a field, the dimension of f_∞ is equal to the rank of f_*, the induced map on homology. More generally, if some f_i induces isomorphisms on the E_i pages, then all subsequent f_r are isomorphisms for $r \geq i$. This is because a bijective chain map, in this case $f_i : (E_i(\mathcal{C}_1), d_i) \longrightarrow (E_i(\mathcal{C}_2), d_i)$, is always a quasi-isomorphism, so f_{i+1} is an isomorphism. In this case, $E_{i+1}(Cone(f), \mathcal{F}(i))$ is acyclic and thus $Cone(f)$ is acyclic. In particular, f is a quasi-isomorphism. We will heavily rely on this fact.

Definition 3.2.4. Let $\mathcal{C}$ be an n-dimensional hypercube of chain complexes. The *ε-filtration* on $\mathcal{C}$ is defined by

$$\mathcal{F}(x) = n - \|\varepsilon\| \text{ for } x \in \mathcal{C}^\varepsilon.$$

The spectral sequence induced by this filtration is called the *ε-spectral sequence*.

Note that the induced spectral sequence from the ε-filtration has depth n and thus all differentials d_k vanish for $k > n$.

Definition 3.2.5. Define an *ε-filtered quasi-isomorphism* to be an ε-filtered chain map (up to an overall absolute shift) between the total complexes of two hypercubes of chain complexes which induces quasi-isomorphisms on the (E_0, d_0) pages of the respective ε-spectral sequences. It is necessarily a quasi-isomorphism on the total complexes by Fact 3.2.3.

Remark 3.2.6. When working with a hypercube of chain complexes, the filtration will be assumed to be the ε-filtration unless mentioned otherwise. In this setting, there is always a canonical isomorphism between $(E_0(\mathcal{C}), d_0)$ and $(\mathcal{C}, \partial)$, so we will often not distinguish between the two.

CHAPTER 4

Review of the Link Surgery Formula

We now give a brief overview of the link surgery formula of Manolescu and Ozsváth [MO10]. Although we discuss the basics, a working understanding of their paper is really necessary to follow all of the details in this thesis.

Their machine takes as input a framed link (L, Λ) in a three-manifold Y and outputs a hypercube of chain complexes; the homology of the total complex is isomorphic to $\mathbf{HF}^\infty(Y_\Lambda(L))$. While we only work with $\mathbf{HF}^\infty$ in this thesis, their surgery formula is done for all flavors of Heegaard Floer homology. Manolescu and Ozsváth only prove the link surgery formula for integer homology spheres, but we mark that this holds for a nullhomologous link in *any* three-manifold Y, as long as one restricts all of the complexes to account for only torsion Spin^c structures on Y. For convenience, we will only describe the link surgery formula for integer homology spheres Y, as the changes we need to make when we generalize this will in fact be easier done than said.

In order to explain the link surgery formula, there is a very large amount of notation and formalism required simply to state the theorem. Therefore, we will first give a complete description in the case that L is a knot to give a more concrete set-up. Then we will give the general formulation. As my training was originally as a probabilist, we will use $x \vee y$ to denote $\max\{x, y\}$.

4.1 The Surgery Formula for Knots

We begin with an oriented knot, $K \subset Y$. We will restate (without proof) a well-known formula for $\mathbf{HF}^\infty(Y_n(K))$ (compare with Theorem 1.1 of [OS08b]). While the notation here

may seem cumbersome and excessive, it will provide a useful foundation for the construction of the general link surgery formula.

Begin with a doubly-pointed Heegaard diagram $\mathcal{H}^K = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ for K in Y, and a Heegaard diagram for Y, $\mathcal{H}^\emptyset = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', w')$, such that the diagrams have the same underlying surface. Let's suppose for simplicity that after removing the basepoint z from $\mathcal{H}^K$, we can relate the resulting diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ to $\mathcal{H}^\emptyset$ by a sequence of handleslides and isotopies that avoid the basepoints (this means there are no (de)stabilizations necessary); this gives a sequence of Heegaard diagrams $\mathcal{H}^{K,+K}$. We define $\mathcal{H}^{K,-K}$ analogously for the removal of w.

First, define $\mathbb{H}(K) = \mathbb{Z}$ and $\overline{\mathbb{H}}(K) = \mathbb{H}(K) \cup \{-\infty, +\infty\}$; also, let $\mathbb{H}(\emptyset) = 0$ and $\overline{\mathbb{H}}(\emptyset) = 0 \cup \{-\infty, +\infty\}$. Note that there are two sublinks of K, namely K and $\emptyset$. If we write K or $+K$, we will mean the knot with the induced orientation from K; $-K$ will refer to the reversed orientation. Fix $s \in \overline{\mathbb{H}}(K)$ and an oriented sublink, $\vec{M} \subset K$. Define

$$p^{\vec{M}}(s) = \begin{cases} +\infty & \text{if } \vec{M} = +K, \\ -\infty & \text{if } \vec{M} = -K, \\ s & \text{if } M = \emptyset. \end{cases}$$

Similarly, define $\psi^{\vec{M}}(s) = +\infty$ if $\vec{M} = \pm K$ and set $\psi^\emptyset(s) = s$.

We want to construct two complexes for each $s \in \mathbb{H}(K)$, namely one for $\mathcal{H}^\emptyset$ and one for $\mathcal{H}^K$. The first complex is given by

$$\mathfrak{A}^\infty(\mathcal{H}^\emptyset, p^K(s)) = \mathfrak{A}^\infty(\mathcal{H}^\emptyset, +\infty) = \mathbf{CF}^\infty(\mathcal{H}^\emptyset).$$

The complex for $\mathcal{H}^K$ will be more complicated and will actually depend on s. Using the Alexander filtration coming from K we will define a new complex $\mathfrak{A}^\infty(\mathcal{H}^K, s)$. This has the same chain groups as $\mathbf{CF}^\infty(\mathcal{H}^K)$: the free module over $\mathbb{F}[[U, U^{-1}]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential is now twisted by the Alexander filtration:

$$\partial \mathbf{x} = \mathcal{D}^\emptyset \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi) = 1} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{E_s(\phi)} \mathbf{y},$$

where

$$E_s(\phi) = (A(\mathbf{x}) - s) \vee 0 - (A(\mathbf{y}) - s) \vee 0 + n_w(\phi).$$

Notice that as s becomes very positive (respectively negative), ∂ is only counting w (respectively z) and this complex is precisely $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ (respectively $\mathbf{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$).

We would like a way to relate the different complexes we have constructed for $+K$, $-K$, and $\emptyset$. Define the *inclusions* $\mathcal{I}_s^{\pm K} : \mathfrak{A}^\infty(\mathcal{H}^K, s) \to \mathfrak{A}^\infty(\mathcal{H}^K, p^{\pm K}(s))$ by

$$\mathcal{I}_s^{\pm K}(\mathbf{x}) = U^{(\pm(A(\mathbf{x})-s))\vee 0}\mathbf{x}.$$

This essentially corresponds to removing z or w from the Heegaard diagram, since s is sent to $\pm\infty$. We set $\mathcal{I}_s^\emptyset$ to simply be the identity. Recall that we had sequences of Heegaard diagrams, $\mathcal{H}^{K,\pm K}$, relating $\mathcal{H}^K$ with z or w removed to $\mathcal{H}^\emptyset$. Each isotopy or handleslide induces a chain map between Floer complexes by counting triangles as mentioned in the discussion after Theorem 2.1.3. Composing the induced chain maps induced by this sequence of moves results in the *destabilization* maps

$$\mathcal{D}^{\pm K}_{p^{\pm K}(s)} : \mathfrak{A}^\infty(\mathcal{H}^K, p^{\pm K}(s)) \to \mathfrak{A}^\infty(\mathcal{H}^\emptyset, \psi^{\pm K}(p^{\pm K}(s))).$$

The composition $\mathcal{D}^{\vec{M}}_{p^{\vec{M}}(s)} \circ \mathcal{I}_s^{\vec{M}}$ is denoted $\Phi_s^{\vec{M}}$; finally, we have $\Phi_s^\emptyset(\mathbf{x}) = \partial(\mathbf{x})$.

Consider the following complex composed of all the smaller complexes we have built up:

$$\mathcal{C}^\infty(\mathcal{H}, n) = \prod_{s \in \mathbb{H}(K)} (\mathfrak{A}^\infty(\mathcal{H}^K, s) \oplus \mathfrak{A}^\infty(\mathcal{H}^\emptyset, \psi^K(s)))$$

with differential given by

$$\mathcal{D}^\infty(s, \mathbf{x}) = (s + n, \Phi^{-K}_{\psi^M(s)}(\mathbf{x})) + (s, \Phi^{+K}_{\psi^M(s)}(\mathbf{x})) + (s, \Phi^\emptyset_{\psi^M(s)}(\mathbf{x})),$$

for $\mathbf{x} \in \mathfrak{A}^\infty(\mathcal{H}^{K-M}, \psi^M(s))$. The s in the first component is simply serving as an index to help indicate the domain and range of the differential. Here we are using the convention that $\Phi_s^{\pm K}(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathfrak{A}^\infty(\mathcal{H}^\emptyset, \psi^K(s))$ (in other words, if $\mathbf{x}$ is not in the domain). We now state the integer surgery formula for knots in this framework.

Theorem 4.1.1 (Ozsváth-Szabó, c.f. Theorem 1.1 of [OS08b])**.** *The homology of the complex $\mathcal{C}^\infty(\mathcal{H}, n)$ is isomorphic to $\mathbf{HF}^\infty(Y_n(K))$.*

Remark 4.1.2. The result of Ozsváth and Szabó proved this for HF^+ and $\widehat{HF}$ instead of $\mathbf{HF}^\infty$. In fact, this cannot be done for HF^∞; the completed coefficients are necessary.

4.2 The General Construction

We now generalize the construction above to arbitrary framed links. For simplicity, we will always assume that the Heegaard diagrams for links we work with have exactly one z basepoint for each component, but may (and will) have additional w basepoints in the diagram not on any component of the link. Furthermore, we require that for the component K_i, both z_i and w_i are the basepoints on this component. As mentioned, we will ignore all details about admissibility of the Heegaard diagrams (see Section 4 of [MO10]).

The starting point will be an oriented link $\vec{L}$ in Y with components $K_1, \dots, K_\ell$ and a framing Λ telling us how to perform surgery on L. The framing Λ will be given as the linking matrix for (L, Λ); diagonal entries are the surgery coefficients and the off-diagonal entries are the pairwise linking numbers of the components. Note that we may think of the row-vectors Λ_i as elements in $H_1(Y - L)$. When we are considering oriented sublinks, $\vec{M}$ will refer to an arbitrary orientation, while M with no vector decoration will indicate that M has the orientation induced by L.

Remark 4.2.1. In what follows, it will sometimes be necessary to readjust the indexing of the basepoints and components at various steps of the link surgery formula to keep notation consisten. Again, it is easier to just ignore the issue and have the reader guess how this should be dealt with rigorously rather than to introduce *colorings* as defined in Section 4 of [MO10].

4.2.1 Spin^c Structures

As is common in Heegaard Floer theory, we want to see where the Spin^c structures appear in our theory. It will be necessary to also relate the relative Spin^c structures defined on $Y - L$ to those on $Y - M$ for sublinks $M \subset L$.

Define the affine space $\mathbb{H}(L) = \bigoplus_{i=1}^{\ell} \mathbb{H}(L)_i$, where

$$\mathbb{H}(L)_i = \frac{\mathrm{lk}(K_i, L - K_i)}{2} + \mathbb{Z}.$$

We extend these lattices to $\overline{\mathbb{H}}(L)_i = \mathbb{H}(L)_i \cup \{+\infty, -\infty\}$ and $\overline{\mathbb{H}}(L) = \oplus_{i=1}^{\ell} \overline{\mathbb{H}}(L)_i$. It is not hard to see that as lattices $\mathrm{Spin}^c(Y_\Lambda(L)) \cong \mathbb{H}(L)/\Lambda$ (where $/\Lambda$ means quotienting out by the action of each row-vector, Λ_i, of Λ); it turns out that such an identification can be made explicitly. Therefore, we will often refer to Spin^c structures on $Y_\Lambda(L)$ as equivalence classes $[\mathbf{s}]$.

Let $I_+(\vec{L}, \vec{M})$ be the set of indices of components of M which are consistently oriented with L. The remaining indices of components of M form $I_-(\vec{L}, \vec{M})$. We define the maps $p_i^{\vec{M}} : \mathbb{H}(L)_i \to \mathbb{H}(L)_i$ by

$$p_i^{\vec{M}}(s_i) = \begin{cases} +\infty & \text{if } i \in I_+(\vec{L}, \vec{M}), \\ -\infty & \text{if } i \in I_-(\vec{L}, \vec{M}), \\ s_i & \text{otherwise.} \end{cases}$$

We can then apply the restriction map $p^{\vec{M}}(\mathbf{s}) = (p_1^{\vec{M}}(s_1), \ldots, p_n^{\vec{M}}(s_\ell))$. This will allow us to remove the components of M, but still keep track of Spin^c structures consistently.

By viewing $\mathbb{H}(L)$ as an affine space over $H_1(Y - L)$ we can define the map $\psi^{\vec{M}} : \mathbb{H}(L) \to \mathbb{H}(L - M)$ by $\psi^{\vec{M}}(\mathbf{s}) = \mathbf{s} - [\vec{M}]/2$. In other words, we ignore the components of $\mathbf{s}$ coming from $\vec{M}$, but we must change the remaining components based on their linking with the components of M. We extend $\psi^{\vec{M}}$ to go from $\overline{\mathbb{H}}(L)$ to $\overline{\mathbb{H}}(L - M)$ in the obvious way.

4.2.2 The $\mathfrak{A}^\infty$-Complexes

For each $\mathbf{s}$, we can define a new Heegaard Floer complex. Begin with a multi-pointed Heegaard diagram for L, $\mathcal{H}^L$, with $n \geq \ell$ basepoints of type w. For each $\mathbf{s_0} \in \overline{\mathbb{H}}(L)$ and each $M \subset L$, we will define the complex $\mathfrak{A}^\infty(\mathcal{H}^{L-M}, \psi^M(\mathbf{s_0}))$. For notation, set $\mathbf{s} = \psi^M(\mathbf{s_0})$. The chain groups will all be the same, freely generated over $\mathbb{F}[[U_1, \ldots, U_n, U_1^{-1}, \ldots, U_n^{-1}]$ by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential

$$D^0 = \partial : \mathfrak{A}^\infty(\mathcal{H}^{L-M}, \mathbf{s}) \to \mathfrak{A}^\infty(\mathcal{H}^{L-M}, \mathbf{s}),$$

is given by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x},\mathbf{y}), \mu(\phi)=1} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{E^1_{s_1}(\phi)} \cdot \ldots \cdot U_n^{E^n_{s_n}(\phi)} \mathbf{y},$$

where

$$E^i_{s_i}(\phi) = \begin{cases} (A_i(\mathbf{x}) - s_i) \vee 0 - (A_i(\mathbf{y}) - s_i) \vee 0 + n_{w_i}(\phi) & \text{if } 1 \leq i \leq \ell, \\ n_{w_i}(\phi) & \text{otherwise.} \end{cases}$$

If s_i is sufficiently positive (respectively negative), then these counts are again just $n_{w_i}(\phi)$ (respectively $n_{z_i}(\phi)$); also, we must use the obvious conventions for subtracting and adding $\pm\infty$ for consistency when $s_i = \pm\infty$. Therefore, setting some s_i to $+\infty$ (respectively $-\infty$) is the same thing as forgetting the ith component of the link and having an additional basepoint w_i (respectively z_i). We think of z_i as functioning as a w_i basepoint, since there is no information about K_i anymore.

4.2.3 Complete Systems

Let $(\Sigma, \boldsymbol{\alpha}, \mathbf{w}, \mathbf{z})$ be a Heegaard diagram for a handlebody, with basepoints $\mathbf{w} = \{w_1, \ldots, w_n\}$ and $\mathbf{z} = \{z_1, \ldots, z_\ell\}$ on $\Sigma - \boldsymbol{\alpha}$.

We will assume that all bipartition functions (Section 6.2 of [MO10]) send everything to $\boldsymbol{\beta}$, so we will not worry about defining α-hyperboxes or keeping track of bipartition functions. We will ultimately work with a basic system, so this assumption will not be a problem (see Section 6.7 of [MO10]). Furthermore, we will always be using maximal colorings (Section 4.1 of [MO10]), so we also won't have to worry about colorings either.

Definition 4.2.2. An *empty β-hyperbox of size* $\mathbf{d}$ is a collection of $\boldsymbol{\beta}$ multi-curves, $\{\boldsymbol{\beta}^\varepsilon\}_{\varepsilon \in \mathbb{E}(\mathbf{d})}$, which can be pairwise-related by sequences of isotopies and handleslides in $\Sigma - \mathbf{w} - \mathbf{z}$. A *filling* of an empty β-hyperbox $\mathcal{H}$ is a choice of elements $\Theta_{\varepsilon,\varepsilon'} \in \mathfrak{A}^\infty(\mathbb{T}_{\beta_\varepsilon}, \mathbb{T}_{\beta_{\varepsilon'}}, \mathbf{0})$ for any neighbors $\varepsilon < \varepsilon'$. These elements must satisfy equation (50) in [MO10]: summing over the polygon maps associated to each possible sequence $\Theta_{\varepsilon_1,\varepsilon_2}, \Theta_{\varepsilon_2,\varepsilon_3}, \ldots \Theta_{\varepsilon_{m-1},\varepsilon_m}$ in the Heegaard multi-diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{\varepsilon_1}, \ldots, \boldsymbol{\beta}_{\varepsilon_m})$ is identically 0. If $\|\varepsilon - \varepsilon'\| = 1$, then $\Theta_{\varepsilon,\varepsilon'}$ must also correspond to a cycle generating the top-dimensional homology group of $\mathfrak{A}^-(\mathbb{T}_{\beta_\varepsilon}, \mathbb{T}_{\beta_{\varepsilon'}}, \mathbf{0})$

(see Remark 2.1.14). A *hyperbox of Heegaard diagrams for L* is simply an empty $\boldsymbol{\beta}$-hyperbox equipped with a choice of filling such that each $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_\varepsilon, \mathbf{w}, \mathbf{z})$ is a Heegaard diagram for L.

Remark 4.2.3. Given a fixed $\mathbf{s} \in \mathbb{H}(L)$, we can create a hyperbox of chain complexes from a hyperbox of Heegaard diagrams for L as follows: for each $\varepsilon \in \mathbb{E}(\mathbf{d})$ we set $(\mathcal{C}_{\mathbf{s}}^{\varepsilon(M)}, D^0)$ to be $\mathfrak{A}^\infty(\mathbb{T}_\alpha, \mathbb{T}_{\beta_{\varepsilon(M)}}, \psi^M(\mathbf{s}))$. If $\|\varepsilon' - \varepsilon\| = 1$, then the chain map $D_\varepsilon^{\varepsilon'-\varepsilon}$ consists of counting triangles in the Heegaard triple $(\boldsymbol{\alpha}, \boldsymbol{\beta}_\varepsilon, \boldsymbol{\beta}_{\varepsilon'})$ with fixed generator $\Theta_{\varepsilon,\varepsilon'}$. The higher homotopies are defined similarly; we sum up the corresponding holomorphic polygon counts over the associated sequence of Θ elements in the Heegaard multi-diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_\varepsilon, \ldots, \boldsymbol{\beta}_{\varepsilon'})$.

In Lemma 6.6 of [MO10], Manolescu and Ozsváth show that any empty $\boldsymbol{\beta}$-hyperbox admits a filling and thus every empty $\boldsymbol{\beta}$-hyperbox can be made into a hyperbox of Heegaard diagrams.

Given an m-component sublink $\vec{M} \subset L' \subset L$ and a hyperbox of Heegaard diagrams $\mathcal{H}$ for L', we construct a hyperbox of Heegaard diagrams $r_{\vec{M}}(\mathcal{H})$ for $L' - M$. This is defined as follows. Remove z_i from each Heegaard diagram in $\mathcal{H}$ if $i \in I_+(\vec{L}, \vec{M})$; if $i \in I_-(\vec{L}, \vec{M})$, instead remove w_i and relabel the z_i as w_i.

Definition 4.2.4. A *hyperbox for the pair* $(\vec{L}', \vec{M})$ is an m-dimensional hyperbox of Heegaard diagrams, $\mathcal{H}^{\vec{L}',\vec{M}}$, for $\vec{L}' - M$.

Let's study some special cases. If $M = \emptyset$, then a 0-dimensional hyperbox for the pair $(\vec{L}', \emptyset)$ is a single Heegaard diagram, which we denote by $\mathcal{H}^{\vec{L}'}$. If M is a single component K, then we have $\mathcal{H}^{\vec{L}',\pm K}$ is a one-dimensional hyperbox, or in other words, a finite sequence of Heegaard diagrams. For the integer surgeries formula, this related $\mathcal{H}^K$ with z or w removed to $\mathcal{H}^\emptyset$; this is exactly the idea that we would like to keep in mind. The hyperbox of Heegaard diagrams is going to tell us how to define the maps analogous to $\mathcal{D}^{\pm K}$ for arbitrary sublinks $\vec{M}$.

Given a sublink, $M' \subset M$, we will exhibit a hyperbox for $(\vec{L} - M', \vec{M} - M')$ inside of $\mathcal{H}^{\vec{L},\vec{M}}$. Suppose that $\mathcal{H}^{\vec{L},\vec{M}}$ has size $\mathbf{d}$. The hyperbox for $(\vec{L} - M', \vec{M} - M')$, which we denote $\mathcal{H}^{\vec{L},\vec{M}}(M', M)$, is given by the sub-hyperbox with specified corners $\mathbf{d} \cdot \varepsilon(M')$ and $\mathbf{d} \cdot \varepsilon(M)$ (here we are doing componentwise multiplication).

For knots, we simply pointed out that for large $|s|$, $\mathfrak{A}^\infty(\mathcal{H}^K, s)$ behaves as though there is either no z or no w basepoint and can be compared to $\mathfrak{A}^\infty(\mathcal{H}^\emptyset, \psi^K(s))$. We now state the analogous requirement for comparing hyperboxes with certain basepoints removed.

Two hyperboxes of Heegaard diagrams are *compatible* if $\mathcal{H}^{\vec{L},\vec{M}}(\emptyset, M') \cong r_{\vec{M}-M'}(\mathcal{H}^{\vec{L},\vec{M}'})$ for each M' a sublink consistently oriented with $\vec{M} \subset L$. Similarly, $\mathcal{H}^{\vec{L},\vec{M}}$ and $\mathcal{H}^{\vec{L}-M',\vec{M}-M'}$ are compatible if $\mathcal{H}^{\vec{L},\vec{M}}(M', M) \cong \mathcal{H}^{\vec{L}-M',\vec{M}-M'}$. Here, the relation '$\cong$' means that the hyperboxes of Heegaard diagrams are related by a single isotopy. In other words, there is a single isotopy of Σ not passing any curves over basepoints, independent of ε, which takes the Heegaard diagram at the vertex ε on one hyperbox to the Heegaard diagram at the vertex ε on the other.

Definition 4.2.5. A *complete system of hyperboxes of Heegaard diagrams for* L, $\mathcal{H}$, is a collection of hyperboxes, $\mathcal{H}^{\vec{L}',\vec{M}}$, one for each pair $(\vec{L}', \vec{M})$, such that for any sublink $\vec{M}' \subset M$ with orientation induced by $\vec{M}$, the hyperbox $\mathcal{H}^{\vec{L}',\vec{M}}$ is compatible with both $\mathcal{H}^{\vec{L}'-M',\vec{M}-M'}$ and $\mathcal{H}^{\vec{L}',\vec{M}'}$.

Manolescu and Ozsváth construct complete systems of hyperboxes for any oriented link in Y in Section 6.7 of [MO10].

Remark 4.2.6. There is an additional technical condition that must be satisfied to be a complete system in the sense of Manolescu and Ozsváth (Definition 6.27 in [MO10]): it essentially says that the paths traced out by the basepoints on the Heegaard surfaces while passing between the different isotopies of diagrams in the hyperboxes must be homotopic to minimal segments of the components of the link that they sit on - these are part of what are called a *good set of trajectories*. This will mostly not be an issue when we are working with basic systems (see Section 4.3). We will only make a brief remark in passing about this in Chapter 7.3 when we need to check this condition.

4.2.4 The Link Surgery Formula

Given a complete system of hyperboxes of Heegaard diagrams, $\mathcal{H}$, for an ℓ-component link L, we would like to turn

$$\mathcal{C}^\infty(\mathcal{H},\Lambda)=\prod_{\mathbf{s}\in\mathbb{H}(L)}\sum_{M\subset L}\mathfrak{A}^\infty(\mathcal{H}^{L-M},\psi^M(\mathbf{s}))$$

into an ℓ-dimensional hypercube of chain complexes, generalizing the construction for knots. Somehow, this should extend the construction of a hypercube of chain complexes in Remark 4.2.3. We will set the chain complex at the vertex $\varepsilon(M)$ to be

$$\mathcal{C}^{\varepsilon(M)}=\prod_{s\in\mathbb{H}(L)}\mathfrak{A}^\infty(\mathcal{H}^{L-M},\psi^M(\mathbf{s}))$$

with the differential given by the product of the component-wise differentials. While these chain complexes do not depend on Λ, the higher D^ε's that we will ultimately construct will depend heavily on this choice.

We now want to generalize the maps $\Phi^{\pm K}$ relating the $\mathfrak{A}^\infty$-complexes. We will construct a map from $\mathfrak{A}^\infty(\mathcal{H}^M,\mathbf{s})$ to $\mathfrak{A}^\infty(\mathcal{H}^{M-M'},\psi^{\vec{M}'}(\mathbf{s}))$ for each $\vec{M}'\subset M$ and $\mathbf{s}\in\mathbb{H}(M)$. The first step is to remove the appropriate z or w basepoints in $\mathcal{H}^M$ by $r_{\vec{M}'}$. The result is a Heegaard diagram for M'. This action corresponded to $\mathcal{I}$ in the integer surgery formula for knots.

We can define the inclusions for a sublink $\vec{M}$

$$\mathcal{I}^{\vec{M}}_{\mathbf{s}}:\mathfrak{A}^\infty(\mathcal{H}^{L'},\mathbf{s})\to\mathfrak{A}^\infty(\mathcal{H}^{L'},p^{\vec{M}}(\mathbf{s}))$$

by

$$\mathcal{I}^{\vec{M}}_{\mathbf{s}}(\mathbf{x})=\prod_{i\in I_+(L,\vec{M})}U_i^{(A_i(\mathbf{x})-s_i)\vee 0}\prod_{j\in I_-(L,\vec{M})}U_j^{(s_j-A_j(\mathbf{x}))\vee 0}\mathbf{x}.$$

It is easy to check that these are in fact chain maps. Note that this map is only defined if s_i is not $\pm\infty$ when $i\in I_\mp(L,\vec{M})$; this issue will not arise in the complex we construct.

We would now like to define the destabilizations,

$$\mathcal{D}^{\vec{M}}_{p^{\vec{M}}(\mathbf{s})}:\mathfrak{A}^\infty(\mathcal{H}^{L'},p^{\vec{M}}(\mathbf{s}))\to\mathfrak{A}^\infty(\mathcal{H}^{L'-M},\psi^{\vec{M}}(p^{\vec{M}}(\mathbf{s}))),$$

which are generalizations of the $\mathcal{D}^{\pm K}$. We first identify $r_{\vec{M}}(\mathcal{H}^{L'})$ with its corresponding vertex in $\mathcal{H}^{L',\vec{M}}$, $\mathcal{H}^{L',\vec{M}}_{(0,\dots,0)}$, by compatibility. The compatibility induces a map from $\mathfrak{A}^{\infty}(\mathcal{H}^{L'}, p^{\vec{M}}(\mathbf{s}))$ to $\mathfrak{A}^{\infty}(\mathcal{H}^{L',\vec{M}}_{(0,\dots,0)}, \psi^{\vec{M}}(p^{\vec{M}}(\mathbf{s})))$, by counting holomorphic triangles coming from the isotopy of Heegaard diagrams; this map differs from the usual triangle-counting map defined in Heegaard Floer theory mentioned in Section 2.1.3 in that it counts intersections with the basepoints by $E^i_{\mathbf{s_i}}(\phi)$ instead of $n_{w_i}(\phi)$.

Remark 4.2.7. It is shown in Theorem 4.10 of [MO10] that destabilizing by a single component is a (grading-preserving) chain homotopy equivalence.

For simplicity, we first assume that $\mathcal{H}^{L',\vec{M}}$ is in fact a hypercube. The idea is that each way of traversing the edges of the hypercube gives a sequence of isotopies and handleslides from $\mathcal{H}^{L',\vec{M}}_{(0,\dots,0)}$ to $\mathcal{H}^{L',\vec{M}}_{(1,\dots,1)}$. The destabilization map $\mathcal{D}^{\vec{M}}$ will measure the failure of the induced triangle maps to commute. The filling of an empty $\boldsymbol{\beta}$-hyperbox gives the desired map from $\mathfrak{A}^{\infty}(\mathcal{H}^{L',\vec{M}}_{(0,\dots,0)}, p^{\vec{M}}(\mathbf{s}))$ to $\mathfrak{A}^{\infty}(\mathcal{H}^{L',\vec{M}}_{(1,\dots,1)}, p^{\vec{M}}(\mathbf{s}))$ by counts of holomorphic polygons. Again, the counts are twisted by $E^i_s(\phi)$ as opposed to the usual $n_{w_i}(\phi)$.

If $\mathcal{H}^{L',\vec{M}}$ is not a hypercube, but instead has size $\mathbf{d}$, then applying the above maps, $\mathcal{D}^{\vec{M}}$, will go to $\mathcal{H}^{L',\vec{M}}_{(1,\dots,1)}$, which will not be $\mathcal{H}^{L'-M}$. Instead, we must do what is called *compression* to arrive at $\mathcal{H}^{L',\vec{M}}_{\mathbf{d}\cdot(1,\dots,1)} = \mathcal{H}^{L'-M}$. If $\vec{M} = \pm K_i$ for a knot K_i, then we would like a map $\tilde{\mathcal{D}}^{\pm K_i}$ which goes from $\mathcal{H}^{L',\pm K_i}_0$ to $\mathcal{H}^{L',\pm K_i}_{\mathbf{d}}$. As discussed, there exist maps which count triangles from $\mathfrak{A}^{\infty}(\mathcal{H}^{L',\pm K_i}_j, p^{\pm K_i}(\mathbf{s}))$ to $\mathfrak{A}^{\infty}(\mathcal{H}^{L',\pm K_i}_{j+1}, p^{\pm K_i}(\mathbf{s}))$ that are induced by the Heegaard moves relating these Heegaard diagrams. In this case we would simply take the map $\tilde{\mathcal{D}}^{\pm K_i}$ to be the composition of the d_i triangle-counting maps, where d_i is the ith component of the size vector $\mathbf{d}$.

In fact, compression will produce a hypercube of chain complexes, $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}}^{\varepsilon})$. The vertices, $(\tilde{\mathcal{C}}^{\varepsilon}, \tilde{\mathcal{D}}^{\mathbf{0}})$ for $\varepsilon \in \{0,1\}^{\ell}$, will be given by $(\mathcal{C}^{\mathbf{d}\cdot\varepsilon}, \mathcal{D}^{\mathbf{0}})$. As mentioned, $\|\varepsilon - \varepsilon'\| = 1$ and $\varepsilon \geq \varepsilon'$ in the compressed hypercube, $\tilde{\mathcal{D}}^{\varepsilon-\varepsilon'}$ is the composition of the triangle-counting edge maps coming from the original hyperbox. However, in general one cannot just take a composition of the maps from the original hyperbox (a composition of chain homotopies is not a chain homotopy for the compositions). For illustration, we define the appropriately compressed

map for a size $(2,1)$ hyperbox and refer the interested reader to Section 3.2 in [MO10] for the general case.

Example 4.2.8. Consider a hyperbox of chain complexes, $\mathcal{C}$, of size $(2,1)$. We can turn this into a hypercube of chain complexes, $\tilde{\mathcal{C}}$, as follows. Take $\tilde{\mathcal{C}}^{\varepsilon_1,\varepsilon_2} = \mathcal{C}^{2\varepsilon_1,\varepsilon_2}$ and keep $\tilde{D}^{0,0} = D^{0,0}$. In other words, the vertices of the compressed hypercube are given by the corners of the original hyperbox. The map $\tilde{D}^{1,0}$ is given by $D^{1,0} \circ D^{1,0}$, while $\tilde{D}^{0,1} = D^{0,1}$. So far we have not done anything different from above, but $\tilde{D}^{1,1}$ will have to be more complicated. A standard exercise in homological algebra shows that the correct choice for $\tilde{D}^{1,1}$ is

$$D^{1,0} \circ D^{1,1} + D^{1,1} \circ D^{1,0}.$$

In the present setting, $D^{1,0}$ and $D^{0,1}$ represent triangle-counting maps, while $D^{1,1}$ counts holomorphic rectangles.

Once the correct map from $\mathfrak{A}^\infty(\mathcal{H}^{L',\vec{M}}_{\mathbf{d}\cdot\varepsilon(\emptyset)}, p^{\vec{M}}(\mathbf{s}))$ to $\mathfrak{A}^\infty(\mathcal{H}^{L',\vec{M}}_{\mathbf{d}\cdot\varepsilon(M)}, p^{\vec{M}}(\mathbf{s}))$ is defined, we simply apply our identification of this final Heegaard diagram with $\mathcal{H}^{L'-M}$ to get one last triangle counting map, again by compatibility. The composition of these maps is the destabilization $\mathcal{D}^{\vec{M}}_{p^{\vec{M}}(\mathbf{s})}$.

For each sublink, $\vec{M}$, define the map $\Phi^{\vec{M}}_{\mathbf{s}} = \mathcal{D}^{\vec{M}}_{p^{\vec{M}}(\mathbf{s})} \circ \mathcal{I}^{\vec{M}}_{\mathbf{s}}$. The differential D^∞ on $\mathcal{C}^\infty(\mathcal{H},\Lambda)$ is now given by

$$D^\infty(\mathbf{s},\mathbf{x}) = \sum_{\vec{N}\subset L-M} (\mathbf{s} + \Lambda_{L,\vec{N}}, \Phi^{\vec{N}}_{\psi^{M\cup\vec{N}}(\mathbf{s})}(\mathbf{x})).$$

Here, $\mathbf{x} \in \mathfrak{A}^\infty(\mathcal{H}^{L-M}, \psi^M(s))$ and $\Lambda_{L,\vec{N}} = \sum_{i\in I_-(L,\vec{N})} \Lambda_i$. Note that the sum is over all possible *oriented* sublinks of $L-M$.

Manolescu and Ozsváth prove that this is indeed the total complex of a hypercube of chain complexes, where

$$(D^\infty)^{\varepsilon(N)}(\mathbf{s},\mathbf{x}) = \sum_{\vec{N}} (\mathbf{s} + \Lambda_{L,\vec{N}}, \Phi^{\vec{N}}_{\psi^{M\cup\vec{N}}(\mathbf{s})}(\mathbf{x})),$$

now summing over orientations of the fixed sublink N.

Recall that we identified $\mathrm{Spin}^c(Y_\Lambda(L))$ with $\mathbb{H}(L)/H(L,\Lambda)$; we denote the equivalence class of $\mathbf{s}$ in $\mathbb{H}(L)/H(L,\Lambda)$ by $[\mathbf{s}]$. It is important to note that $[\mathbf{s}+\Lambda_{L,\vec{N}}] = [\mathbf{s}]$ for any $\vec{N} \subset L$.

Therefore, $\mathcal{C}^\infty(\mathcal{H},\Lambda)$ splits over Spin^c structures on $Y_\Lambda(L)$; we denote the subcomplex for the Spin^c structure $[\mathbf{s}]$ by $\mathcal{C}^\infty(\mathcal{H},\Lambda,[\mathbf{s}])$. At this point we remark that $\mathcal{C}^\infty(\mathcal{H},\Lambda,[\mathbf{s}])$ can be equipped with a $\mathbb{Z}/\mathfrak{d}\mathbb{Z}$-grading which we will not define (see Section 7.4 in [MO10]). We will only need to make use of its existence in what follows.

After much work, we are now finally ready to state the *link surgery formula*!

Theorem 4.2.9 (Manolescu-Ozsváth, Theorem 1.1 of [MO10])**.** *Consider a complete system of hyperboxes, $\mathcal{H}$, for $\vec{L}\subset Y$ and a framing Λ. Given a Spin^c structure $\mathfrak{s}$ on $Y_\Lambda(L)$ corresponding to $[\mathbf{s}]\in\mathbb{H}(L)/\Lambda$, there is a relatively $\mathbb{Z}/\mathfrak{d}\mathbb{Z}$-graded $\mathbb{F}[[U,U^{-1}]$-vector space isomorphism*

$$\mathbf{HF}^\infty_*(Y_\Lambda(L),\mathfrak{s})\cong H_*(\mathcal{C}^\infty(\mathcal{H},\Lambda,[\mathbf{s}]),D^\infty),$$

where $\mathfrak{d}$ is the usual divisibility of the Spin^c structure.

Remark 4.2.10. While the link surgery formula is defined over a ring with many formal variables U_i, the theorem implies that they become equal in homology. We will see later that this also implies that the U_i also become equal in the homology of the $\mathfrak{A}^\infty$-complexes.

Note that in the case of a torsion Spin^c structure, the grading is a relative $\mathbb{Z}$-grading.

Definition 4.2.11. The *(link) surgery formula* for a framed link (L,Λ) will be the hypercube of chain complexes $(\mathcal{C}^\infty(\mathcal{H},\Lambda),(D^\infty)^\varepsilon)$ for some complete system of hyperboxes. A *vertex complex* in the link surgery formula is a single complex $\mathfrak{A}^\infty(\mathcal{H}^{L'},\psi^{L-L'}(\mathbf{s}))$, denoted $\mathcal{C}^{\varepsilon(L')}_{\mathbf{s}}$.

Often, we will also refer to the total complex of the link surgery formula as the link surgery formula as well.

Remark 4.2.12. Theorem 4.2.9 implies that the homology of the surgery formula is in fact independent of all of the choices made along the way.

4.3 Basic Systems

So far we have not required a specific complete system of hyperboxes. The complete system that we will work with is a *basic system* of hyperboxes. Instead of recalling the construction,

we will review only the properties we need and refer the reader to Section 6.7 in [MO10] for more details. Most importantly, in this section Manolescu and Ozsváth show that there exists a basic system for every link.

Basic systems have the property that if $\vec{M}'$ has the induced orientation of $\vec{M}$ for a sublink $M' \subset M$, then $\mathcal{H}^{\vec{M},\vec{M}'}$ consists of a single Heegaard diagram obtained from $\mathcal{H}^{\vec{M}}$ by removing the z basepoints corresponding to components of M'. Let L' be a sublink with $\vec{M} \subset \vec{L}'$ and an additional component K_j not contained in M. By compatibility of the complete system, the size of $\mathcal{H}^{\vec{L}',\vec{M}\cup+K_j}$ has $d_j = 0$.

Lemma 4.3.1. *Is a basic system, if $\vec{M}$ has at least two components, one of which is compatibly oriented with L, then $\Phi^{\vec{M}}$ vanishes.*

Proof. Let $\vec{M}'$ be a nonempty sublink of $L - K_i$, where K_i is consistently oriented with L. We will show that $\mathcal{D}^{+K_i\cup\vec{M}'}$ vanishes. Since $\Phi^{+K_i\cup\vec{M}'} = \mathcal{D}^{+K_i\cup\vec{M}'} \circ \mathcal{I}^{+K_i\cup\vec{M}'}$, this will prove the lemma.

Let's study destabilization maps more carefully. Destabilizing a link of k components is given by compression, or in other words, playing the kth standard symphony for some hypercubical collection (see Section 3 of [MO10]); if one of the edges in the hyperbox that we are summing over has length 0, the sum over algebra elements in the hypercubical collection when playing the song will be empty, if $k \geq 2$. This is true because when $k \geq 2$, the kth standard symphony contains a harmony with the element i at least once. According to the definition of playing a song, and thus in compression, in order for there to be nonzero terms in the formula, the number of harmonies that contain i must be at most d_i; however, we have established that this is 0. Therefore, $\mathcal{D}^{+K_i\cup\vec{M}'}$ must be 0. □

CHAPTER 5

Why is the Surgery Formula Special for $\mathbf{HF}^\infty$?

5.1 Quick Preliminaries

It is important to understand what the effect of working with the completed $\mathbb{F}[[U, U^{-1}]$ coefficients is. First, we establish the hopelessness of calculating $HF^\infty(Y, \mathfrak{s})$ for non-torsion Spin^c structures using the link surgery formula, due to this completion in $\mathbf{HF}^\infty$.

Proposition 5.1.1 (Manolescu-Ozsváth, Equation 4 of [MO10])**.** *Let $\mathfrak{s}$ be a non-torsion Spin^c structure. Then $\mathbf{HF}^\infty(Y, \mathfrak{s})$ is trivial.*

We do remark that (uncompleted) $HF^\infty(Y, \mathfrak{s})$ can definitely be non-trivial for non-torsion Spin^c structures, so we are missing out on some information by completing with respect to U. However, in the case of torsion Spin^c structures, we are not!

Proposition 5.1.2 (Manolescu-Ozsváth, page 7 of [MO10])**.** *Let $\mathfrak{s}$ be a torsion Spin^c structure. Then, we can recover $HF^\infty(Y, \mathfrak{s})$ (with mod 2 coefficients) from $\mathbf{HF}^\infty(Y, \mathfrak{s})$.*

Proof. Note that $\mathbb{F}[[U, U^{-1}]$ is flat over $\mathbb{F}[U, U^{-1}]$ (see, for example, Lemma 4.9 of [Lid10]). Therefore,

$$\mathbf{HF}^\infty(Y, \mathfrak{s}) \cong HF^\infty(Y, \mathfrak{s}) \otimes \mathbb{F}[[U, U^{-1}].$$

Since $\mathfrak{s}$ is torsion, $HF^\infty(Y, \mathfrak{s})$ has a relative $\mathbb{Z}$-grading. This in fact forces $HF^\infty(Y, \mathfrak{s})$ to be a free $\mathbb{F}[U, U^{-1}]$-module. The result now follows since $\mathbb{F}[[U, U^{-1}]$ is a field. □

5.2 The ∞ in $\mathbf{HF}^\infty$

Let's study some special properties of the link surgery formula which are unique to the infinity flavor. Fix a complete system $\mathcal{H}$ for the framed link (L, Λ). It turns out that for a fixed Y, all vertex complexes are quasi-isomorphic.

Proposition 5.2.1. *For a component K, The inclusion maps, $\mathcal{I}_s^{\pm K}$, are quasi-isomorphisms which preserve the relative Maslov grading of the vertex complexes (and thus lowers the grading by 1 in the surgery formula). Therefore, $\Phi_{\mathbf{s}}^{\pm K} : \mathfrak{A}^\infty(\mathcal{H}^L, \mathbf{s}) \to \mathfrak{A}^\infty(\mathcal{H}^{L-K}, \psi^{\pm K}(\mathbf{s}))$, is also a quasi-isomorphism.*

Proof. We have already discussed that the inclusions are chain maps. Multiplication by U_i is an automorphism of $\mathbb{F}[[U_1, \dots, U_n, U_1^{-1}, \dots, U_n^{-1}]$ and thus the inclusions give bijective chain maps. Such a map is always a quasi-isomorphism. Finally, the preservation of the relative grading is shown in Section 7.1 of [MO10].

As mentioned in Remark 4.2.7, the destabilizations maps, $\mathcal{D}^{\pm K}$, also induce grading-preserving quasi-isomorphisms. This proves the second statement. □

Remark 5.2.2. The reason why Proposition 5.2.1 will not hold for the other flavors is that the inclusion maps $\mathcal{I}$ will simply not be quasi-isomorphisms, as multiplication by U is not an isomorphism for $\mathbb{F}[[U]]$ or $\mathbb{F}[[U, U^{-1}]/\mathbb{F}[[U]]$. Since the inclusions encode the information coming from the link we are performing surgery on (they see the induced filtrations coming from $\mathbf{CFL}^\infty$), the infinity flavor will not retain much geometric information about the link. We will see that most of the information will be contained in the Milnor triple linking invariants.

We will use $\mathcal{C}$ as shorthand for $\mathcal{C}^\infty(\mathcal{H}, \Lambda)$, or possibly the subcomplex corresponding to a torsion Spin^c structure on $Y_\Lambda(L)$ when this will not cause confusion.

Lemma 5.2.3. *Suppose that L is a nullhomologous link in Y. For any $L' \subset L$, the homology of the complex $\mathcal{C}_{\mathbf{s}}^{\varepsilon(L')} = \mathfrak{A}^\infty(\mathcal{H}^{L-L'}, \mathbf{s})$ is isomorphic to $\mathbf{HF}^\infty(Y)$ as $\mathbb{F}[[U, U^{-1}]$ vector spaces. In particular, all of the U_i become equal in homology.*

Proof. By applying $\Phi^{\pm K_j}$ for all components $K_j \subset L - L'$, we obtain that $\mathfrak{A}^\infty(\mathcal{H}^{L-L'}, \mathbf{s})$ is quasi-isomorphic to $\mathfrak{A}^\infty(\mathcal{H}^\emptyset, +\infty) = \mathbf{CF}^\infty(\mathcal{H}^\emptyset)$ by Proposition 5.2.1. Note that $\mathcal{H}^\emptyset$ is a multi-pointed Heegaard diagram for Y, so all of the U_i act equally in $\mathbf{HF}^\infty(\mathcal{H}^\emptyset)$ (see, for example, Proposition 2.5 in [MOS09]). Therefore, we have that $H_*(\mathfrak{A}^\infty(\mathcal{H}^\emptyset, +\infty)) \cong \mathbf{HF}^\infty(Y)$ as $\mathbb{F}[[U, U^{-1}]$ vector spaces. □

However, we can generalize this for certain classes of links. The links that we will study all share a particular property that we would like to focus on.

Definition 5.2.4. A link $L = K_1 \cup \ldots \cup K_\ell$ is *algebraically split* if $lk(K_i, K_j) = 0$ for all i, j.

Suppose that L is algebraically split. Consider a face, F, of any dimension in $\{0,1\}^\ell$. Let L_F be the sublink

$$L_F = \{K_i \subset L : \text{ there exist } \varepsilon, \varepsilon' \in F \text{ with } \varepsilon_i = 0, \varepsilon'_i = 1\}.$$

Define $\mathbb{H}(L, \Lambda|_{L_F})$ to be the sublattice of $\mathbb{H}(L)$ generated by the Λ_i for which K_i is a component of L_F. Let's construct the following module

$$\mathcal{C}_F = \prod_{\mathbf{s} \in \mathbb{H}(L, \Lambda|_{L_F})} \sum_{\varepsilon(M) \in F} \mathfrak{A}^\infty(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})).$$

This is naturally a chain complex, even if it is not a sub- or quotient-complex. This is because any such face-module is the result of a sequence of subcomplexes of quotient-complexes of subcomplexes etc. Choose a component K_j that is not in L_F such that $\varepsilon_j = 0$ for all $\varepsilon \in F$. We can construct a new face of the same dimension, F_j, given by

$$F_j = \{\varepsilon + \varepsilon(K_j) : \varepsilon \in F\}.$$

This gives a new face complex $\mathcal{C}_{F_j}$. It is clear that this also inherits the structure of a hypercube of chain complexes which can be naturally equipped with the ε-filtration.

Lemma 5.2.5. *With the previous notation, the face complexes $\mathcal{C}_F$ and $\mathcal{C}_{F_j}$ are ε-filtered quasi-isomorphic.*

Proof. We now study the map

$$\Gamma^{+K_j} = \prod_{\mathbf{s}\in\mathbb{H}(L,\Lambda|_{L_F})} \Gamma_{\mathbf{s}}^{+K_j} = \prod_{\mathbf{s}\in\mathbb{H}(L,\Lambda|_{L_F})} \sum_{\vec{L'}\subset L_F} \Phi^{+K_j\cup\vec{L'}}_{\psi^{L'}(\mathbf{s})} : \mathcal{C}_F \longrightarrow \mathcal{C}_{F_j}.$$

This is a chain map by construction (essentially the relation for a hypercube of chain complexes). Consider the filtration on the mapping cone of Γ^{+K_j} given by $\mathcal{F}_j(x) = -\sum_{i\neq j}\varepsilon_i$. The only components that preserve the filtration level will be ∂ and Φ^{K_j}. Since the maps Φ^{K_j} are quasi-isomorphisms by Proposition 5.2.1, we obtain quasi-isomorphisms on the (E_0, d_0) pages of the ε-spectral sequence and the proof is complete. □

Lemma 5.2.5 does not imply that all associated face complexes of the same dimension in $\mathcal{C}^\infty(\mathcal{H},\Lambda)$ are quasi-isomorphic. This does, however, tell us how to relate face complexes to the complexes corresponding to surgery on certain sublinks. Fix a complete system $\mathcal{H}$ and a sublink $L' \subset L$. There is naturally a complete system $\mathcal{H}|_{L'}$ given by only considering the hyperboxes of Heegaard diagrams for pairs (L'', M), where $L'' \subset L'$. Furthermore, this can naturally be identified with a subcomplex of $\mathcal{C}^\infty(\mathcal{H},\Lambda)$ by the following (see Section 11.1 of [MO10] for details).

Proposition 5.2.6 (Manolescu-Ozsváth)**.** *Given a complete system $\mathcal{H}$ for L and a sublink L', there is a natural identification of $\mathcal{C}^\infty(\mathcal{H}|_{L'}, \Lambda|_{L'})$ with the subcomplex of $\mathcal{C}^\infty(\mathcal{H},\Lambda)$ given by C_F, where F is the face of $\{0,1\}^\ell$ consisting of all $\varepsilon((L-L')\cup M)$ for $M \subset L'$. This identification is an ε-filtered quasi-isomorphism.*

Remark 5.2.7. Lemmas 5.2.3 and 5.2.5 and Proposition 5.2.6 will be used throughout this thesis, with a strong emphasis on the ε-filteredness of the identifications.

CHAPTER 6

Surgery Equivalence, Cohomology Rings, and HF^∞

6.1 Triple Cup Products and Surgery Equivalence

In this chapter, we give the necessary background on the triple cup product form and cup homology that we will need. Mark in fact first studied cup homology explicitly in order to better understand HF^∞ and Ozsváth and Szabó's conjecture about its form [Mar08].

Definition 6.1.1. The *triple cup product form*, μ_Y, for a closed, oriented, connected three-manifold Y is the three form on $H^1(Y;\mathbb{Z})$

$$\mu_Y(x^1 \wedge x^2 \wedge x^3) = \langle x^1 \smile x^2 \smile x^3, [Y]\rangle.$$

Since μ_Y is an odd-degree form, $\iota_{\mu_Y} \circ \iota_{\mu_Y} = 0$, where ι is contraction. We can use this to construct a homology theory.

Definition 6.1.2. The *cup homology of* Y, denoted $HC^\infty(Y)$, is the homology of the *cup complex* - the chain complex with chain groups

$$C^\infty(Y) = \Lambda^*(H^1(Y;\mathbb{Z})) \otimes \mathbb{F}[[U, U^{-1}]$$

and differential

$$\partial_Y^\infty(\alpha \otimes U^j) = \iota_{\mu_Y}(\alpha) \otimes U^{j-1}.$$

Here, $deg(U) = -2$, so that ∂_Y^∞ lowers gradings by 1.

Convention 6.1.3. In our version of cup homology, we are taking triple cup products in integral cohomology and reducing mod 2. Cup homology was originally defined over $\mathbb{Z}[U, U^{-1}]$

instead of $\mathbb{F}[[U, U^{-1}]$; the mod 2, completed coefficients are what we need, however, to compare to the homology of the link surgery formula. At this point we also introduce shorthand notation $\Lambda^*_{\mathbb{F}/U} = \Lambda^*(H^1(Y;\mathbb{Z})) \otimes \mathbb{F}[[U, U^{-1}]$ for the cup complex chain groups.

We first begin with some simple examples.

Example 6.1.4. It is clear that $\mu_Y = -\mu_{-Y}$. Therefore, $HC^\infty(Y) \cong HC^\infty(-Y)$.

Example 6.1.5. If Y has $b_1(Y) \leq 2$, then μ_Y must be identically 0. In particular, $HC^\infty(Y) \cong \Lambda^*_{\mathbb{F}/U}$.

Therefore, the calculations of HF^∞ of Ozsváth and Szabó from Theorem 6.2.8 establishes Theorem 1 for all three-manifolds with $b_1 \leq 2$.

Example 6.1.6 ($Y = \mathbb{T}^3$). Choose a basis x^1, x^2, x^3 for $H^1(\mathbb{T}^3;\mathbb{Z})$ such that $x^1 \smile x^2 \smile x^3$ generates $H^3(\mathbb{T}^3;\mathbb{Z})$. We therefore have $\mu_{\mathbb{T}^3}(x^1 \wedge x^2 \wedge x^3) = 1$. Thus, $HC^\infty(\mathbb{T}^3) \cong (\Lambda^1 \oplus \Lambda^2) \otimes \mathbb{F}[[U, U^{-1}]$.

Given the triple cup product forms for Y_1 and Y_2, it is easy to construct the triple cup product form for $Y_1 \# Y_2$:

$$\mu_{Y_1 \# Y_2} = \mu_{Y_1} \otimes 1 + 1 \otimes \mu_{Y_2}.$$

This quickly leads to a Künneth formula for cup homology.

Proposition 6.1.7 (Mark, Theorem 2 of [Mar08])**.** $HC^\infty(Y_1 \# Y_2) \cong HC^\infty(Y_1) \otimes_{\mathbb{F}[[U,U^{-1}]} HC^\infty(Y_2)$.

We can therefore conclude that the operation of connect-summing with a lens space does not affect the integral triple cup product form or the isomorphism type of the cup homology (which we also saw in Corollary 2.1.11 does not affect $\mathbf{HF}^\infty$). Note that connect-sums with lens spaces correspond to performing a non-zero surgery on a trivial knot inside of an embedded B^3. Cochran, Gerges, and Orr have completely understood the effects of non-zero nullhomologous surgeries on the integral triple cup product form, extending the above observation [CGO00]. We will only state the portions of their theorems that will be used in the proof.

Definition 6.1.8. We will say that Y_1 and Y_2 are *surgery equivalent* (respectively *rationally surgery equivalent*) if they can be related by a sequence of ± 1-surgeries (respectively non-zero surgeries, which can be chosen to be integral) on nullhomologous (respectively rationally nullhomologous) knots.

Theorem 6.1.9 (Cochran-Gerges-Orr, Corollary 3.5 of [CGO00]). *If $H_1(Y_1;\mathbb{Z}) \cong H_1(Y_2;\mathbb{Z})$ is torsion-free, then Y_1 and Y_2 have isomorphic integral triple cup product forms if and only if they are surgery equivalent.*

Theorem 6.1.10 (Cochran-Gerges-Orr, Theorem 5.1 of [CGO00]). *Y_1 and Y_2 have isomorphic integral triple cup product forms if and only if they are rationally surgery equivalent.*

Therefore, if Y_1 and Y_2 are rationally surgery equivalent, then $HC^\infty(Y_1)$ and $HC^\infty(Y_2)$ are isomorphic.

Observation 6.1.11. If Y_1 and Y_2 have isomorphic integral cohomology rings, then the integral triple cup product form of Y_1 is isomorphic to that of Y_2 or $-Y_2$ and thus are rationally surgery equivalent. We do have to work with an orientation - it's not necessarily true that Y_2 and $-Y_2$ are surgery equivalent.

In the case of $H_1 \cong \mathbb{Z}^3$ we can explicitly write down the possible surgery equivalence classes.

Theorem 6.1.12 (Cochran-Gerges-Orr, Example 3.3 of [CGO00]). *If $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}^3$, there exists a unique $n \geq 0$ such that Y is surgery equivalent to the manifold M_n with Kirby diagram shown in Figure 6.1.*

We will call the component that spirals n times Z_n. It is useful to note that $M_0 = \#_{i=1}^3 S^2 \times S^1$ and $M_1 = \mathbb{T}^3$. Furthermore, the triple cup product form on M_n has multiplicity n. Therefore, since we are reducing mod 2, $HC^\infty(M_{2n}) \cong HC^\infty(\#_{i=1}^3 S^2 \times S^1)$, which has dimension 8, and $HC^\infty(M_{2n+1}) \cong HC^\infty(\mathbb{T}^3)$, which has dimension 6.

Remark 6.1.13. Given an algebraically split link L in S^3 and a sublink L', the components of $L - L'$ are nullhomologous in any surgery on L'. This in particular implies that $H_1(S^3_{\mathbf{0}}(L);\mathbb{Z}) \cong \mathbb{Z}^\ell$ and is thus torsion-free.

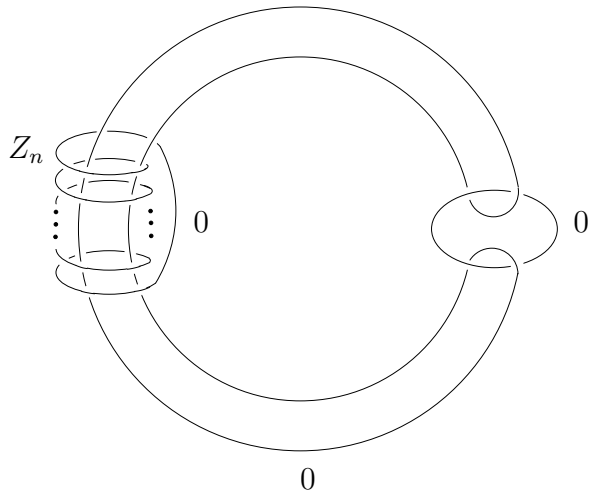


Figure 6.1: Surgery presentation of M_n

In fact, there is an explicit way to produce a 3-manifold with $H_1(Y) \cong \mathbb{Z}^\ell$ in each surgery equivalence class by a construction similar to the M_n. In order to do this, we must recall the connection between the triple cup product form and Milnor triple linking invariants. We will not give the definition of the Milnor triple linking invariants here, but the necessary background can be found in [Coc90], for instance. There will also never be any indeterminacy in our Milnor triple linking invariants, since we are working with algebraically split links.

Given a link $L = K_1 \cup \ldots K_\ell$, we will use the notation $\bar{\mu}_L(i,j,k)$ to denote the Milnor triple linking invariants of $K_i \cup K_j \cup K_k$. Note that this is independent of the ambient link containing $K_i \cup K_j \cup K_k$ as the Milnor triple linking invariants only depend on this three component sublink (see, for example, [Mur66]).

Remark 6.1.14. Given an algebraically split link, L, the Hom-duals of the meridians of the K_i, denoted x^i, form a basis for $H^1(S^3_{\mathbf{0}}(L); \mathbb{Z})$.

The following is due to Turaev; we state only the portion of the theorem which we need.

Theorem 6.1.15 (Lemma 4.2 of [Tur84])**.** *Let L be an algebraically split link in S^3 and let x^i be the basis for H^1 given by the Hom-duals of the meridians of the components of L. Then*

$$\mu_{S^3_{\mathbf{0}}(L)}(x^{i_1} \wedge x^{i_2} \wedge x^{i_3}) = \bar{\mu}_L(i_1, i_2, i_3).$$

We will make the distinction clear between when the Milnor triple-linking invariants are taken in $\mathbb{Z}$ and when they are taken mod 2 (which we will need to compare to Heegaard Floer homology) if it affects the validity of the statements.

We can now construct a three-manifold with arbitrary triple cup product form simply by taking 0-surgery on any algebraically split link with the corresponding Milnor triple linking invariants. This is not as difficult as it sounds at first: one simply constructs a link L with the property that each three-component link $K_i \cup K_j \cup K_k$ with $\bar{\mu}_L(i,j,k)$ equal to the desired triple cup value, say $\mu_{i,j,k}$ - that just means choosing $K_i \cup K_j \cup K_k$ to be isotopic to the link used for the surgery presentation of $M_{\mu_{i,j,k}}$ in Figure 6.1.

We saw in Theorem 6.1.10 that cup homology is unchanged by rational surgery equivalence. We would like to prove a similar statement for $\mathbf{HF}^\infty$. Once proven, we can reduce the calculation to manifolds obtained by 0-surgery on algebraically split links, which we will see to be a much easier class of spaces to work with.

6.2 $\mathbf{HF}^\infty$ is Determined by the Cohomology Ring

With the framework of surgery equivalences of three-manifolds set up, we will be able to prove that $\mathbf{HF}^\infty$ is completely determined by the integral cohomology ring of Y (essentially the rational surgery equivalence type) for torsion Spin^c structures. Rather than calculating the effects on $\mathbf{HF}^\infty$ of the associated sequence of surgeries via the rational surgeries formula of Ozsváth and Szabó [OS11], we will instead do more work on the topological level so that we only need to understand the behavior of HF^∞ for surgeries on nullhomologous knots.

6.2.1 Nullhomologous Surgeries and $\mathbf{HF}^\infty$

Convention 6.2.1. Given Y_1 and Y_2, we say they have the *same* $\mathbf{HF}^\infty$ if $\mathbf{HF}^\infty(Y_1, \mathfrak{s}_1) \cong \mathbf{HF}^\infty(Y_2, \mathfrak{s}_2)$ for each pair of torsion Spin^c structures on Y_1 and Y_2.

We will make use of the following proposition, made as an observation in Section 4.1 of [OS03c]. We give a proof of this to get some practice with the link surgery formula, as these

techniques will be used throughout. For notation, we set $\Psi^K_{\mathbf{s}} = \Phi^{+K}_{\mathbf{s}} + \Phi^{-K}_{\mathbf{s}}$.

Proposition 6.2.2 (Ozsváth-Szabó)**.** *Fix a torsion Spin^c structure $\mathfrak{s}_0$ on Y and a nonzero integer n. Let $\mathfrak{s}_K \in Spin^c(Y_n(K))$ agree with $\mathfrak{s}_0$ on $Y - K$. Then we have that $\mathbf{HF}^\infty(Y, \mathfrak{s}_0)$ and $\mathbf{HF}^\infty(Y_n(K), \mathfrak{s}_K)$ are isomorphic.*

Proof. Again, for notational convenience, we assume that Y is an integer homology sphere. Let $\mathcal{H}$ be a complete system for K in Y. Furthermore, we work with $n > 0$; the proof for $n < 0$ is essentially the same. Fix a Spin^c structure, $\mathfrak{s}_K$, that agrees with $\mathfrak{s}_0$ on $Y - K$. The idea is to show that for some s, $H_*(\mathfrak{A}^\infty(\mathcal{H}^K, s)) \cong \mathbf{HF}^\infty(Y_n(K), \mathfrak{s}_K)$. Since Proposition 5.2.1 implies that $H_*(\mathfrak{A}^\infty(\mathcal{H}^K, s))$ is isomorphic to $\mathbf{HF}^\infty(Y, \mathfrak{s}_0)$, this will complete the proof.

Fix an s whose mod n equivalence class, $[s]$, corresponds to $\mathfrak{s}_K$. Recall that Theorem 4.2.9 tells us $H_*(Cone(\Psi^K_n, [s])) \cong \mathbf{HF}^\infty(Y_n(K), \mathfrak{s}_K)$. Consider the subcomplex of $Cone(\Psi^K_n, [s])$ given by

$$C_{>s} = \prod_{\substack{s'>s \\ s' \equiv s \pmod n}} \mathfrak{A}^\infty(\mathcal{H}^K, s) \oplus \prod_{\substack{s'>s \\ s' \equiv s \pmod n}} \mathfrak{A}^\infty(\mathcal{H}^\emptyset).$$

We claim that this complex is acyclic. Equip $C_{>s}$ with the filtration $\mathcal{F}_>((s', \mathbf{x})) = -s'$ for $\mathbf{x} \in \mathfrak{A}^\infty(\mathcal{H}^K, s')$ or $(s', \mathbf{x}) \in \mathfrak{A}^\infty(\mathcal{H}^\emptyset)$. The only components of the differential that do not lower the filtration level are the vertex differentials ∂ and the map Φ^{+K}. Therefore, the associated graded splits as a product of complexes of the form

$$(\mathfrak{A}^\infty(\mathcal{H}^K, s'), \partial) \xrightarrow{\Phi^K_{s'}} (\mathfrak{A}^\infty(\mathcal{H}^\emptyset), \partial).$$

By Proposition 5.2.1, these are all acyclic. Therefore, $C_{>s}$ is acyclic as well.

Construct the subcomplex

$$C_{<s} = \prod_{\substack{s' \le s-n \\ s' \equiv s \pmod n}} \mathfrak{A}^\infty(\mathcal{H}^K, s') \oplus \prod_{\substack{s' \le s \\ s' \equiv s \pmod n}} \mathfrak{A}^\infty(\mathcal{H}^\emptyset).$$

Note that if we take $Cone(\Psi^K_n, [s])$ and remove $C_{<s}$ and $C_{>s}$, we are left solely with the vertex complex $\mathfrak{A}^\infty(\mathcal{H}^K, s)$, since there can be only one element of $\mathbb{H}(K)$ in the interval $(s - n, s]$ that corresponds to $\mathfrak{s}_K$. Thus, the proof will be complete if we can show that $C_{<s}$

is also acyclic. This follows by the same argument as before, except now we use the filtration

$$\mathcal{F}_<(x) = \begin{cases} s' & \text{if } x \in \mathfrak{A}^\infty(\mathcal{H}^K, s') \\ s' - n & \text{if } x \in \mathfrak{A}^\infty(\mathcal{H}^\emptyset). \end{cases}$$

This time the associated graded splits into the complexes

$$(\mathfrak{A}^\infty(\mathcal{H}^K, s'), \partial) \xrightarrow{\Phi_{s'}^{-K}} (\mathfrak{A}^\infty(\mathcal{H}^\emptyset), \partial).$$

Again, by Proposition 5.2.1, these are acyclic. Thus, $C_{<s}$ is acyclic. □

Corollary 6.2.3. *If Y_1 and Y_2 are surgery equivalent, and the torsion Spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ agree away from the surgery region, then $\mathbf{HF}^\infty(Y_1, \mathfrak{s}_1)$ is isomorphic to $\mathbf{HF}^\infty(Y_2, \mathfrak{s}_2)$ as Spin^c-graded $\mathbb{F}[[U, U^{-1}]$-vector spaces. In particular, if $H_1(Y_1; \mathbb{Z})$ is torsion-free, then Y_1 and Y_2 have the same $\mathbf{HF}^\infty$.*

6.2.2 Eliminating Torsion

While, it's obviously not true that every three-manifold is presented by 0-surgery on an algebraically split link (consider $\mathbb{R}P^3$ for instance), we can prove a weaker statement which is sufficient for us. Since $H_1(S^3_{\mathbf{0}}(L); \mathbb{Z})$ is torsion-free, denote the unique torsion Spin^c structure on $S^3_{\mathbf{0}}(L)$ by $\mathfrak{s}_0$.

Proposition 6.2.4. *For each three-manifold Y, there exists an algebraically split L such that $\mu_Y \cong \mu_{S^3_{\mathbf{0}}(L)}$ and $\mathbf{HF}^\infty(Y, \mathfrak{s}) \cong \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L), \mathfrak{s}_0)$ are isomorphic for all torsion $\mathfrak{s}$.*

First, we need an algebraic lemma to change our surgery presentations around appropriately; the proof can be found at the end of [Lid10].

Lemma 6.2.5 (Manolescu)**.** *Let Y be a closed, oriented 3-manifold. There exist finitely many integers $m_1, \ldots, m_k$, all greater than 1, such that there exists an algebraically split surgery presentation in S^3 for $Y \# L(m_1, 1) \# \ldots \# L(m_k, 1)$.*

Proof of Proposition 6.2.4. By applying Lemma 6.2.5, we may connect-sum Y with the necessary lens spaces such that the resulting manifold is presented by $S^3_\Lambda(L_*)$, where L_* is an

algebraically split link. Connect sums with lens spaces do not change the integral triple cup product form (Proposition 6.1.7) or $\mathbf{HF}^\infty$ (Corollary 2.1.11). Since each surgery in the presentation will now be performed on a nullhomologous knot, Proposition 6.1.10 (respectively Corollary 6.2.3) shows that the triple cup product form (respectively $\mathbf{HF}^\infty$) of $S^3_\Lambda(L_*)$ will be the same as the 3-manifold obtained by surgery on the sublink of L_* consisting of components that are 0-framed. This sublink is the desired L. □

Proof of Proposition 1.2.1. Theorem 6.1.9, Proposition 6.1.10, Proposition 6.2.2, and Proposition 6.2.4 prove that the integral triple cup product form determines $\mathbf{HF}^\infty$ for any torsion Spin^c structure. A little more work allows the statement for the integral cohomology ring.

Note that if we apply Proposition 6.2.4 to both Y_1 and Y_2, then the manifolds produced by this proposition, $S^3_{\mathbf{0}}(L_1)$ and $S^3_{\mathbf{0}}(L_2)$, will also have isomorphic cohomology rings. Furthermore, we have not affected the integral triple cup product forms or $\mathbf{HF}^\infty$ (except for the number of torsion Spin^c structures, all of which yield isomorphic $\mathbf{HF}^\infty$). Thus, we may assume Y_1 and Y_2 do not have torsion in H_1. Either the triple cup product form of Y_1 is isomorphic to that of Y_2 or that of $-Y_2$. If Y_2, then Y_1 is surgery equivalent to Y_2 and we are done. On the other hand, if Y_1 and $-Y_2$ have isomorphic triple cup product forms, then we can apply Corollary 3.8 of [CGO00] to see that Y_2 is surgery equivalent to $-Y_2$, since H_1 is torsion-free. This completes the proof. □

Remark 6.2.6. In light of the work of this chapter, for the rest of the proof of Theorem 1 we will assume that Y is presented as 0-surgery on an algebraically split link unless otherwise. We will not hesitate to replace one manifold that we are trying to check Theorem 1 on with one which has isomorphic HC^∞ and the same $\mathbf{HF}^\infty$ to make a calculation more convenient; this will not always be pointed out to the reader because it will become a common occurrence. We have seen that if two links algebraically split links L and L' have the same Milnor triple linking invariants, the corresponding 0-surgeries will have the same $\mathbf{HF}^\infty$ and isomorphic HC^∞; therefore, we will analogously switch L out for L' as well.

Convention 6.2.7. When we are studying $S^3_{\mathbf{0}}(L)$ for algebraically split links, all components will have framing 0, so we will not distinguish between L and the surgered manifold, or

between $b_1(S^3_{\mathbf{0}}(L))$ and the number of components of L. Thus, we will make statements like surgery equivalent links to mean that the manifolds obtained by 0-surgery on each link are surgery equivalent. Also, $\mathfrak{s}_0$ will always refer to the unique torsion Spin^c structure on $S^3_{\mathbf{0}}(L)$.

As we are about to begin our journey towards the proof of Theorem 1 in the next chapter, we kick things off with what was originally known about $\mathbf{HF}^\infty$ for three-manifolds with small first Betti number. For exercise with the link surgery formula, we encourage the reader to try to prove the following using the link surgery formula.

Theorem 6.2.8 (Ozsváth-Szabó, Theorem 10.1 of [OS04c])**.** *If* $b_1(Y) \leq 2$*, then* $\mathbf{HF}^\infty(Y, \mathfrak{s}) \cong \Lambda^*(H^1(Y;\mathbb{Z})) \otimes \mathbb{F}[[U, U^{-1}]$ *for torsion* $\mathfrak{s}$*. In particular,* $\dim \mathbf{HF}^\infty(Y, \mathfrak{s}) = 2^{b_1(Y)}$.

CHAPTER 7

The Case of $b_1 = 3$

It is the goal of this chapter to do the case of $b_1 = 3$. In order to establish Theorem 1 on manifolds with $b_1(Y) = 3$, the work of the previous chapter shows that it suffices to verify it on each of the manifolds M_n in Figure 6.1. Before doing this, we must discuss in more detail the universal coefficients spectral sequence for HF^∞ mentioned in Corollary 2.1.17.

Remark 7.0.9. We will see in Chapter 8 that the ε-spectral sequence associated to the link surgery formula essentially mimics the universal coefficients spectral sequence. For this reason, using the universal coefficients spectral sequence is not necessary, but it will help to motivate where some of the calculations in the next chapter come from.

7.1 Universal Coefficients for $\mathbf{HF}^\infty$

In particular, we must analyze the gradings in this spectral sequence. More specifically, the universal coefficients spectral sequence for $HF^\infty(Y, \mathfrak{s})$, for $\mathfrak{s}$ torsion and $\mathbb{Z}$-coefficients, identifies $E_2^{i,*}$, for i even, with $\Lambda^*(H^1(Y;\mathbb{Z}))$; this spectral sequence respects the $\mathbb{Z}[U]$-module structure, as multiplication by U induces an isomorphism between $E_2^{i,*}$ and $E_2^{i-2,*}$. Furthermore, $E_2^{i,*}$ vanishes for odd i. This implies that $d_k : E_k^{i,j} \longrightarrow E_k^{i+k-1,j-k}$ automatically vanishes if k is even and thus the E_2 and E_3 pages are isomorphic. This gives the identification of E_3 with $C_*^\infty(Y)$ as chain groups. The difficult part is analyzing the differentials in this spectral sequence. Again, we transfer this over to $\mathbf{HF}^\infty$ by replacing everything with $\mathbb{F}[[U, U^{-1}]$.

The easiest case for us to study is M_0, which is simply $\#_{i=1}^3 S^2 \times S^1$. We saw in Remark 2.1.14 that $\mathbf{HF}^\infty$ has dimension 8, which is exactly the dimension of $\Lambda^*(H^1(M_0;\mathbb{Z}))$.

Therefore, all of the differentials d_3 and higher in the universal coefficients spectral sequence must vanish. Note that this corresponds exactly to what we expect for Theorem 1, since $\mu_{M_0} = 0$.

7.2 Example: $\mathbb{T}^3$

The main goal of this section is to understand the simplest nontrivial example, M_1. From Figure 6.1, we can represent M_n by 0-surgery on the knot Z_n in $S^2 \times S^1 \# S^2 \times S^1$ and therefore will apply the link surgery formula. For M_1, this in fact gives 0-surgery on the Borromean rings, which is $\mathbb{T}^3$. As mentioned, Ozsváth and Szabó showed $\mathbf{HF}^\infty(\mathbb{T}^3)$ has dimension 6 (Theorem 2.1.19). Analyzing this result via the surgery formula will allow us to deduce valuable information for the remaining M_n. But first, let us specialize to the case of $b_1 = 3$ for the universal coefficients spectral sequence.

Let's study the differentials $d_k : E_k^{i,j} \longrightarrow E_k^{i+k-1,j-k}$. Since each $E_2^{i,j}$ is a copy of $\Lambda^j(H^1(Y;\mathbb{Z})) \otimes \mathbb{F}[[U,U^{-1}]$, the $E_2 \cong E_3$ page is supported entirely in the region $0 \leq j \leq b_1(Y)$. Therefore, for $b_1 = 3$ the spectral sequence must collapse after d_3. In fact, the only possibly nontrivial component of d_3 maps from $\Lambda^3 \otimes \mathbb{F}[[U,U^{-1}]$ to $\Lambda^0 \otimes \mathbb{F}[[U,U^{-1}]$, each of which has dimension 1. Thus, calculating d_3 for $b_1 = 3$ is equivalent to finding $\mathbf{HF}^\infty$. If $\dim \mathbf{HF}^\infty = 8$, then $d_3 \equiv 0$, and if $\dim \mathbf{HF}^\infty = 6$, then $d_3(\phi^1 \wedge \phi^2 \wedge \phi^3) = 1$, for a basis ϕ^1, ϕ^2, ϕ^3 of $H^1(Y;\mathbb{F})$. It is clear that these are the only possibilities.

For $\mathbb{T}^3$, Theorem 1 predicts that $d_3 : \Lambda^3_{\mathbb{F}/U} \longrightarrow \Lambda^0_{\mathbb{F}/U}$ should be nonzero, since $\mu_{\mathbb{T}^3}$ is non-trivial. This is exactly true, since $\dim \mathbf{HF}^\infty(\mathbb{T}^3, \mathfrak{s}_0) = 6$. We would like to see what conditions this forces on the link surgery formula.

For 0-surgery on a knot, the only $s \in \mathbb{H}(K)$ that we have to consider is $s = 0$ (since this corresponds to the torsion Spin^c structure). Furthermore, the case of $s = \pm\infty$ will be made clear by the Heegaard diagram we are working with (whether it has a z or a w). Therefore, we will often omit the s-notation from the surgery formula in this chapter.

We fix a basic system of hyperboxes of Heegard diagrams $\mathcal{H}$ for K in $S^2 \times S^1 \# S^2 \times S^1$. We

use the notation $\mathfrak{A}_w^\infty$, $\mathfrak{A}_z^\infty$, $\mathfrak{A}_{z,w}^\infty$ for $\mathfrak{A}^\infty(r_{+K}(\mathcal{H}^K), \mathfrak{s}_0)$, $\mathfrak{A}^\infty(r_{-K}(\mathcal{H}^K), \mathfrak{s}_0)$, and $\mathfrak{A}^\infty(\mathcal{H}^K, \mathfrak{s}_0, 0)$ respectively (where $\mathfrak{s}_0$ means restricting the generators in the link surgery formula to those that correspond to the unique torsion Spin^c structure on $S^2 \times S^1 \# S^2 \times S^1$ - this is the necessary step to apply the link surgery formula to manifolds with $b_1 > 0$). We can study the maps on homology induced by the inclusion maps, $\mathcal{I}_*^{+K} : H_*(\mathfrak{A}_{z,w}^\infty) \to H_*(\mathfrak{A}_w^\infty)$. There is also the analogous map $\mathcal{I}_*^{-K}$. Finally, there is the induced destabilization map, $\mathcal{D}_*^{-K} : H_*(\mathfrak{A}_z^\infty) \to H_*(\mathfrak{A}_w^\infty)$. Since we are using a basic system, we can make natural identifications between the intersection points in $\mathcal{H}^K$, $r_{+K}(\mathcal{H}^K)$ and $r_{-K}(\mathcal{H}^K)$. Therefore, the inclusion maps can be written purely in terms of multiplication by powers of U, and $\mathcal{D}^{-K}$ is computed from handleslides and isotopies taking $r_{-K}(\mathcal{H}^K)$ to $r_{+K}(\mathcal{H}^K)$ (transforming the diagram with w removed to the diagram with z removed). Even better, the map $\mathcal{D}_*^{+K}$ is given by the identity.

Theorem 4.2.9 tells us to calculate the homology of the mapping cone of

$$\Psi^K = (\Phi^{+K} + \Phi^{-K}) : \mathfrak{A}_{z,w}^\infty \to \mathfrak{A}_w^\infty$$

to recover $\mathbf{HF}^\infty(\mathbb{T}^3)$. However, since $\mathbb{F}[[U, U^{-1}]$ is a field, we really only need to know the rank of $\Phi_*^{+K} + \Phi_*^{-K}$. By applying Lemma 5.2.3 to $S^2 \times S^1 \# S^2 \times S^1$, we have

$$H_*(\mathfrak{A}_{z,w}^\infty) \cong H_*(\mathfrak{A}_z^\infty) \cong H_*(\mathfrak{A}_w^\infty) \cong \mathbf{HF}^\infty(S^2 \times S^1 \# S^2 \times S^1) \cong \Lambda^*_{\mathbb{F}/U}.$$

The universal coefficients spectral sequence guarantees that for any nullhomologous knot in $S^2 \times S^1 \# S^2 \times S^1$, the map Ψ_*^K can only be rank 0 or 1 (corresponding to the dimension of $\mathbf{HF}^\infty$ being 6 or 8).

Knowing the dimension of $\mathbf{HF}^\infty(\mathbb{T}^3)$, we would like to understand the map $\mathcal{D}_*^{-Z_1} : H_*(\mathfrak{A}_z^\infty) \to H_*(\mathfrak{A}_w^\infty)$ in detail.

The best way to understand the calculation is via matrix representatives, so we must pick appropriate bases for $H_*(\mathfrak{A}_{z,w}^\infty), H_*(\mathfrak{A}_z^\infty)$, and $H_*(\mathfrak{A}_w^\infty)$. The idea is to pick a basis for one of these three complexes and then push it over using maps which are easy to understand. Define the map $\Theta^K : \mathfrak{A}_z^\infty \longrightarrow \mathfrak{A}_w^\infty$ by $\Theta^K(\mathbf{x}) = U^{A(\mathbf{x})}\mathbf{x}$, where $A(\mathbf{x})$ is the Alexander filtration coming from K.

Proposition 7.2.1. $\Theta^K \circ \mathcal{I}^{-K} = \mathcal{I}^{+K}$.

Proof. Add the powers of U together. □

Proposition 7.2.1 shows that Θ^K must be a chain map and, like the inclusions and destabilizations, this is a quasi-isomorphism.

Since we are working with a torsion Spin^c structure on $S^2 \times S^1 \# S^2 \times S^1$, there exists an absolute $\mathbb{Q}$-grading on $\mathbf{CF}^\infty(S^2 \times S^1 \# S^2 \times S^1, \mathfrak{s}_0)$ which we can induce on $\mathfrak{A}_w$ and $\mathfrak{A}_z$. We recall from Remark 4.2.7 that these gradings are preserved under $\mathcal{D}^{\pm K}$. We can therefore study the effects of Θ^K on absolute gradings.

Lemma 7.2.2. Θ^K *preserves the absolute Maslov grading.*

Proof. We study $\Phi^K + \Phi^{-K} = (\Theta^K + \mathcal{D}^{-K}) \circ \mathcal{I}^{-K}$ on the chain level. Since this is a component of the differential in the link surgery formula, the map has homogeneous degree. We know that the bijective chain map $\mathcal{I}^{-K}$ preserves relative gradings by Proposition 5.2.1. Factoring out $\mathcal{I}^{-K}$ shows $\Theta^K + \mathcal{D}^{-K}$ must be a homogeneous map. However, as $\mathcal{D}^{-K}$ preserves the absolute grading, Θ^K must preserve absolute gradings as well. □

We can choose ordered $\mathbb{F}$-bases (x_1, x_2) for $H_0(\mathfrak{A}_z^\infty)$ and (y_1, y_2) for $H_1(\mathfrak{A}_z^\infty)$ (elements in grading 0 and grading 1 respectively). The key to this choice is that the pairs live in adjacent Maslov gradings. This clearly gives an ordered $\mathbb{F}[[U, U^{-1}]$-basis for the entire module. Furthermore, we use Θ_*^K to push this basis over to $H_*(\mathfrak{A}_w^\infty)$ to obtain a basis with the same properties. By Remark 4.2.7, $\mathcal{D}_*^{-K}$ is represented by a matrix (we keep the same ordering between the bases) of the form

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} \qquad a, b, c, d, e, f, g, h \in \mathbb{F}.$$

Choose a basis for $H_*(\mathfrak{A}_{z,w}^\infty)$ such that $\mathcal{I}_*^{-K}$ can be represented by the identity. The next thing that we would like to understand is the matrix representative for $\mathcal{I}_*^{+K}$.

Lemma 7.2.3. *With respect to these bases, $\mathcal{I}_*^{+K}$ is represented by the identity.*

Proof. Because the representative for $\mathcal{I}_*^{-K}$ is the identity, Proposition 7.2.1 guarantees $\mathcal{I}_*^K$ and Θ_*^K will be represented by the same matrix. However, we know that Θ_*^K is represented by the identity by construction. □

We now specialize to the case of $K = Z_1$. Consider the collection of matrices

$$X = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

Proposition 7.2.4. *The map $\mathcal{D}_*^{-Z_1}$ is represented by a matrix in X.*

Proof. As discussed, the rank of $\Phi_*^{+Z_1} + \Phi_*^{-Z_1}$ must be precisely 1 in order for $\mathbf{HF}^\infty(\mathbb{T}^3, \mathfrak{s}_0)$ to be dimension 6. Since $\Phi_*^{Z_1} + \Phi_*^{-Z_1}$ is represented by

$$\begin{pmatrix} a+1 & b & 0 & 0 \\ c & d+1 & 0 & 0 \\ 0 & 0 & e+1 & f \\ 0 & 0 & g & h+1 \end{pmatrix},$$

exactly three of the two-by-two blocks must be identically 0 and the other must have rank 1. It is easy to check that each of the matrices in X have this property. We now rule out all other possibilities. Either $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ must be the identity. Without loss of generality, we assume $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now, the possible blocks $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F})$ that

don't appear in matrices in X are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Direct calculation shows that $\Phi_*^{+Z_1} + \Phi_*^{-Z_1}$ would have either rank 0 or rank 2 in any of these cases, which would be a contradiction. Repeating the arguments with the top-left and bottom-right blocks switched discounts all other matrices not in X. □

Remark 7.2.5. We note that Proposition 7.2.4 does not apply to every nullhomologous knot in $S^2 \times S^1 \# S^2 \times S^1$. Doing 0-surgery on the split unknot, Z_0, to get $\#_{i=1}^3 S^2 \times S^1$, which has dimension 8, shows that $\Phi_*^{+Z_0} = \Phi_*^{-Z_0}$. This in fact means that after these choices of bases, $\mathcal{D}_*^{-Z_0}$ must be the identity.

After choosing bases analogously, it remains to analyze $\mathcal{D}_*^{-Z_n}$ to yield the calculation for M_n ($n \geq 2$). To do this, we rephrase the computation as an iteration of what we've done for $\mathbb{T}^3$ using a technique we call *composing knots*.

7.3 Composing Knots and the Calculation for M_n

Recall that given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, any two points on $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ determine a knot, K, in Y. Now, suppose there are instead 3 distinct points, z, u, and w. Then the pairs of basepoints, (z, u), (u, w), and (z, w), determine three knots. We want to consider Heegaard diagrams containing this information. We will ignore concerns with orientations, since these will not arise in our setting. Finally, knots will always be nullhomologous.

Definition 7.3.1. A *Heegaard diagram for* (K, K_1, K_2) in Y is a Heegaard diagram for Y, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, equipped with 3 distinct basepoints z, u, and w in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ such that (z, w), (z, u), and (u, w) determine K, K_1, and K_2 respectively.

Proposition 7.3.2. *Given a Heegaard diagram for* (K, K_1, K_2)*, we have* $\mathcal{D}_*^{-K} = \mathcal{D}_*^{-K_2} \circ \mathcal{D}_*^{-K_1}$.

Proof. $\mathcal{D}^{-K_1}$ is induced by a sequence of Heegaard moves taking $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u)$ and $\mathcal{D}^{-K_2}$ comes from a sequence of moves from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$. Therefore, the

composition of isotopies and handleslides goes from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ and is chain homotopic to $\mathcal{D}^{-K}$. □

Remark 7.3.3. In this setup, the concatenation of a good set of trajectories from z to u and from u to w gives a good set of trajectories from z to w, so we do not need to worry about the technical condition mentioned in Remark 4.2.6. This is what guarantees that the composition of $\mathcal{D}^{-K_1}$ and $\mathcal{D}^{-K_2}$ is actually chain homotopic to $\mathcal{D}^{-K}$.

Thus, since most of the complexity in the knot surgery formula for $\mathbf{HF}^\infty$ comes from the map $\mathcal{D}_*^{-K}$, having a Heegaard diagram for (K, K_1, K_2) and an understanding of each $\mathcal{D}_*^{-K_i}$ should make the computation more manageable. This is the approach we will use for the rest of the chapter. However, we must first establish that such diagrams exist and derive a way of relating this information to the M_n.

Lemma 7.3.4. *Suppose K_1 and K_2 are knots in Y where $K_1 \cap K_2$ is an embedded connected interval. Then if $K = \overline{(K_1 \cup K_2) - K_1 \cap K_2}$ (see Figure 7.1), there exists a Heegaard diagram for (K, K_1, K_2).*

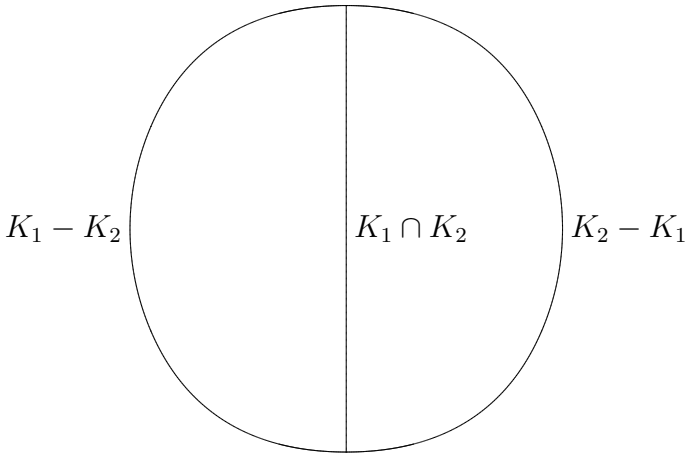


Figure 7.1: Each simple cycle corresponds to a knot

Proof. The idea follows the construction of Heegaard diagrams for knots in [OS06c]. Begin with a self-indexing Morse function, $h : S^3 \longrightarrow [0, 3]$, with exactly two critical points. Note that traversing a flow from index 0 to index 3 and then another flow in "reverse" gives a knot. Thus, three flow lines give three knots as in the statement of the lemma (see Figure 7.2).

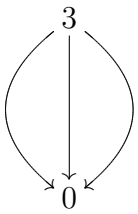


Figure 7.2: Three flow lines forming three knots

Choose a small neighborhood, U, of three flow lines between the two points. Identify a neighborhood, N, of $K_1 \cup K_2$ in Y with U such that each knot gets mapped to the union of two of the three flows. We will now use h to refer to the induced Morse function on N, with index 0 and index 3 critical points, p and q. Extend h to a Morse function f on all of Y such that it is still self-indexing. If there were no other index 0 or index 3 critical points, then we could construct the desired Heegaard diagram for (K, K_1, K_2) simply by choosing the three basepoints to be where the three flow lines pass through the Heegaard surface, $f^{-1}(3/2)$. The idea is to cancel any critical points of index 0 or 3 outside of N, without affecting $f|_N$.

If such critical points exist, we rescale the Morse function in a neighborhood of p and q so as to not affect the critical points, but make $h(p) = -\epsilon$ and $h(q) = 3 + \epsilon$ (and thus the same for f). Now, remove the balls $\{f > 3 + \epsilon/2\}$ and $\{f < -\epsilon/2\}$ around the index 0 and index 3 critical points from N, to obtain a cobordism $W : S^2 \longrightarrow S^2$. In the terminology of [Mil65], this is a self-indexing Morse function on the triad (W, S^2, S^2). Since each manifold in the triad is connected, we know that for each index 0 critical point, there is a corresponding index 1 with a single flow line traveling to the index 0. This pair can be canceled such that the Morse function will not be changed outside of a neighborhood of the flow line between them. We want to see that by perhaps choosing a smaller neighborhood, N', of the knots inside of N, this flow line does not hit N'. This must be the case because if no such neighborhood existed, by compactness, this flow line would have to intersect K_1 or K_2. But these are flows of f themselves, so the two lines cannot intersect.

Hence, we can alter f to remove the index 0/1 pair without affecting $f|_{N'}$. By repeating this argument and an analogous one for index 2/3 pairs, we can remove all of the critical

points of index 0 and 3 in W in this fashion. This says, after rescaling the function on the neighborhoods of p and q back to their original values, the new Morse function is self-indexing on Y with exactly one index 0 and one index 3 critical point, and furthermore, still agrees with h when restricted to a small enough neighborhood of the knots. This is exactly what we want to give the desired Heegaard diagram. □

Remark 7.3.5. One can introduce isotopies to this diagram such that the three resulting doubly-pointed Heegaard diagrams are all admissible.

Consider the link in the Kirby diagram for M_n, Figure 6.1. Since Z_n is the knot which we will apply the surgery formula to, we would like a way to decompose Z_n and apply Lemma 7.3.4.

Proposition 7.3.6. *For each n, there exists a Heegaard diagram for (Z_n, Z_1, Z_{n-1}) in $S^2 \times S^1 \# S^2 \times S^1$.*

Proof. Let us first study Figure 7.3. Here we have attached an arc to Z_n at two points (the large black dots). This creates two additional knots as follows. Note that one can travel two different paths from the bottom attachment point to the top attachment point; we may either wind in an upward spiral once around the two vertical strands or follow the path that begins by winding downward $n-1$ times. Beginning at the top attachment point, following the attaching arc to the bottom point, and finally traversing one of the two winding paths back to the top point gives either Z_1 or Z_{n-1}. We are now in the position to apply Lemma 7.3.4 to Z_n, Z_1, and Z_{n-1}. □

When applying the surgery formula for $\mathbb{T}^3$, it was critical to use the map Θ^K_* to make all of the inclusions consistently identity matrices. The following lemma will allow us to do this consistently for triples (K, K_1, K_2).

Lemma 7.3.7. *Consider a Heegaard diagram for (K, K_1, K_2). Then $\Theta^K = \Theta^{K_2} \circ \Theta^{K_1}$.*

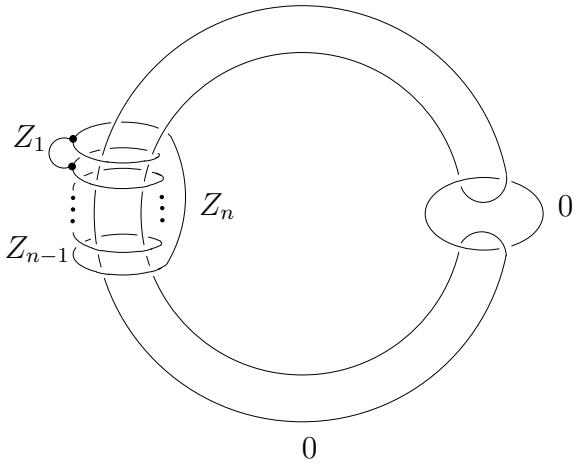


Figure 7.3: Splitting of Z_n into Z_{n-1} and Z_1

Proof. The Alexander gradings for the three knots in the diagram satisfy

$$\begin{aligned} A_K(x) - A_K(y) &= n_z(\phi) - n_w(\phi) \\ &= n_z(\phi) - n_u(\phi) + n_u(\phi) - n_w(\phi) \\ &= A_{K_1}(x) - A_{K_1}(y) + A_{K_2}(x) - A_{K_2}(y) \end{aligned}$$

for each $\phi \in \pi_2(x, y)$. Therefore, the relative Alexander grading for K is the sum of the relative Alexander gradings for K_1 and K_2. Thus, the absolute Alexander grading for K is the sum of the absolute Alexander gradings for K_1 and K_2 plus an additional constant. Therefore, $\Theta^K = U^\ell \cdot \Theta^{K_2} \circ \Theta^{K_1}$, for some $\ell \in \mathbb{Z}$. By Proposition 7.2.2, the Θ maps preserve absolute Maslov gradings, so we know that $\ell = 0$. □

Fix a Heegaard diagram for (Z_n, Z_1, Z_{n-1}) as given by Proposition 7.3.6. We now will choose the proper bases as in the $\mathbb{T}^3$ example. Figure 7.4 will provide a useful visual reference for the following proposition.

Proposition 7.3.8. *Following Section 7.2, choose bases for $H_*(\mathfrak{A}^\infty_{z,u})$, $H_*(\mathfrak{A}^\infty_z)$, and $H_*(\mathfrak{A}^\infty_u)$, such that the inclusions and $\Theta^{Z_1}_*$ are given by the identity and the map $\mathcal{D}^{-Z_1}_*$ is a matrix in X. Now, choose bases for $H_*(\mathfrak{A}^\infty_w)$ and $H_*(\mathfrak{A}^\infty_{u,w})$ such that the inclusions and $\Theta^{Z_{n-1}}_*$ are the identity. Then, there exists a basis for $H_*(\mathfrak{A}^\infty_{z,w})$ such that $\mathcal{I}^{-Z_n}_*$, $\mathcal{I}^{Z_n}_*$, and $\Theta^{Z_n}_*$ are given by the identity.*

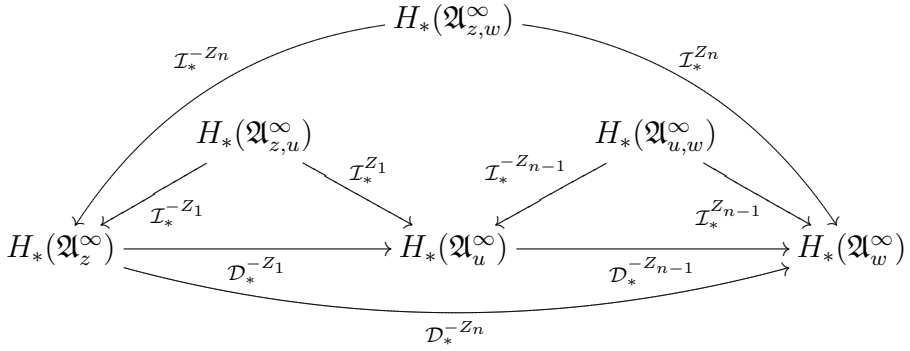


Figure 7.4: The setup that appears in Proposition 7.3.8

Proof. Clearly we can fix a basis for $H_*(\mathfrak{A}^\infty_{z,w})$ such that $\mathcal{I}_*^{-Z_n}$ is the identity. Now, we combine the fact that $\mathcal{I}_*^{Z_n} = \Theta_*^{Z_n} \circ \mathcal{I}_*^{-Z_n}$ with $\Theta_*^{Z_n} = \Theta_*^{Z_{n-1}} \circ \Theta_*^{Z_1} = I$ (Lemma 7.3.7), to get the required result. □

Remark 7.3.9. These constructions could be generalized to any number of basepoints, but we only need three basepoints for our purposes.

Although $\mathcal{D}_*^{-Z_{n-1}}$ is not necessarily represented by an element of X in this diagram, we do know that it comes in the form of $A \oplus B$ for $A, B \in GL_2(\mathbb{F})$, since $\mathcal{D}_*^{-Z_{n-1}}$ preserves absolute gradings.

Remark 7.3.10. While the individual matrix representations may seem to depend on the choice of Heegaard diagram, if $\mathcal{D}_*^{-K} = I$, this is independent of the diagram as long as the bases are chosen such that $\mathcal{I}_*^K = \Theta_*^K = I$. A similar statement based on the work of Section 7.2 can be made about $\mathcal{D}_*^{-K}$ being in X regardless of diagram.

We are now ready for the calculation of the maps $\mathcal{D}_*^{-Z_n}$ for all n.

Theorem 7.3.11. *Begin with a diagram for (Z_{2n+1}, Z_1, Z_{2n}). After a choice of bases given by Proposition 7.3.8 we have that $\mathcal{D}_*^{-Z_{2n}}$ is the identity and $\mathcal{D}_*^{-Z_{2n+1}}$ is a matrix in X for all $n \geq 0$.*

Proof. For $n = 0$, we know that the map $\mathcal{D}_*^{-Z_0}$ must be the identity in order to have $\dim \mathbf{HF}^\infty(\#_{i=1}^3 S^2 \times S^1) = 8$. Similarly, from our computation for $\mathbb{T}^3$, we have seen that $\mathcal{D}_*^{-Z_1}$ is in X. Thus, the base case is established.

For the induction step, note that as soon as $\mathcal{D}_*^{-Z_{2n}}$ is the identity, we can compose with $\mathcal{D}_*^{-Z_1}$ to get that $\mathcal{D}_*^{-Z_{2n+1}}$ is of type X. Thus, we only need to find $\mathcal{D}_*^{-Z_{2n}}$.

By hypothesis, $\mathcal{D}_*^{-Z_{2n-1}} \in X$. The first case we consider is if $\mathcal{D}_*^{-Z_1}$ and $\mathcal{D}_*^{-Z_{2n-1}}$ were to be represented by two different elements of X when considering bases chosen for (Z_{2n}, Z_1, Z_{2n-1}). If this were to happen, then the product of the matrices, which gives a representative for $\Phi_*^{-Z_{2n}}$, has the property that its sum with the identity, $\Phi_*^{Z_{2n}}$, has rank at least 2. However, this is impossible by the dimension bounds coming from the universal coefficients spectral sequence. Therefore, both $\mathcal{D}_*^{-Z_{2n-1}}$ and $\mathcal{D}_*^{-Z_1}$ are represented by the same matrix. But, every element of X squares to the identity. $\mathcal{D}_*^{-Z_{2n}}$ must then be the identity. □

We are now ready to do the base case for Theorem 1.

Theorem 7.3.12. *Theorem 1 holds for M_n for each $n \geq 0$. In particular,* $\dim \mathbf{HF}^\infty(M_n) = 6$ *if n is odd and* 8 *if n is even.*

Proof. We apply Theorem 7.3.11 to see that the rank of $\Phi_*^{-Z_{2n}} + \Phi_*^{Z_{2n}}$ is equal to that of $\Phi_*^{-Z_0} + \Phi_*^{Z_0}$. Therefore, $\mathbf{HF}^\infty(M_{2n}, \mathfrak{s}_0)$ and $\mathbf{HF}^\infty(M_0, \mathfrak{s}_0)$ are isomorphic. Similarly, we see that $\Phi_*^{-Z_{2n+1}} + \Phi_*^{+Z_{2n+1}}$ and $\Phi_*^{-Z_1} + \Phi_*^{+Z_1}$ have the same rank. Thus, $\mathbf{HF}^\infty(M_{2n+1}, \mathfrak{s}_0) \cong \mathbf{HF}^\infty(M_1, \mathfrak{s}_0)$. Similarly, we have that $\mu_{M_{2n}} \cong \mu_{M_0}$ and $\mu_{M_{2n+1}} \cong \mu_{M_1}$, where we have reduced the integral triple cup product form mod 2. Therefore, $HC^\infty(M_{2n}) \cong HC^\infty(M_0)$ and $HC^\infty(M_{2n+1}) \cong HC^\infty(M_1)$.

As we have already seen that Theorem 1 holds for $M_0 = \#_{i=1}^3 S^2 \times S^1$ and $M_1 = \mathbb{T}^3$, the proof is complete. □

CHAPTER 8

Manifolds with Higher b_1

We have now reached the meat of the proof. It remains to prove that for an algebraically split link L in S^3, if $b_1(S^3_{\mathbf{0}}(L)) \geq 4$ (equivalently $|L| \geq 4$), then $\mathbf{HF}^\infty(S^3_{\mathbf{0}}(L)) \cong HC^\infty(S^3_{\mathbf{0}}(L))$. While we are ultimately going to induct on b_1, we will in fact have to induct on a complexity of algebraically split links for a fixed b_1. The induction on this complexity will be very similar to the method of composing knots. For this chapter, we will always assume that our links are in S^3.

Here is an outline of the rest of the proof of Theorem 1, which consists primarily of two different steps. The main idea is to use the link surgery formula to calculate the Heegaard Floer homology of $S^3_{\mathbf{0}}(L)$. The first step uses the standard 0-framed, algebraically split surgery presentation and analyzes the first few pages of the ε-spectral sequence for the link surgery formula. In fact, as we travel through this spectral sequence, which converges to $E_\infty \cong \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L))$, we will actually see that $(E_3, d_3) \cong (C^\infty(S^3_{\mathbf{0}}(L)), \partial^\infty_{S^3_{\mathbf{0}}(L)})$. This spectral sequence will mimic the behavior of the universal coefficients spectral sequence.

The second step of the proof is thus to show that the higher differentials in the ε-spectral sequence vanish. In order to do this, we actually use a different presentation of $S^3_{\mathbf{0}}(L)$ where we can see a component of L, decomposed into two knots as in the method of composing knots. This presentation allows us to write the link surgery formula as an amalgamation of two link surgery formulas for links which are inductively "less complex". Knowing that the higher differentials vanish for these simpler links will allow us to conclude that the higher differentials vanish for L as well.

8.1 The First Two Differentials in the ε-Spectral Sequence

Now, let $\mathcal{H}$ be a basic system for $L = K_1 \cup K_2 \cup \ldots \cup K_\ell$, an algebraically split link in S^3 with ℓ components (we will allow $\ell < 4$ as well). We will denote by $Y = S^3_{\mathbf{0}}(L)$; we also denote the framing by Λ_0, which is simply the 0-matrix, since the pairwise linking is 0. We can then induce the ε-filtration on $\mathcal{C} = \mathcal{C}^\infty(\mathcal{H}, \Lambda_0, \mathbf{0})$, the subcomplex corresponding to the link surgery formula for unique torsion Spin^c structure on Y. Note that the equivalence class of elements in $\mathbb{H}(L)$ for each Spin^c structure on Y has only one element. Furthermore, we will suppress the orientations of Y and L, as this can be seen to have no effect on any of the calculations.

Remark 8.1.1. The complex $\mathcal{C}^\infty(\mathcal{H}, \Lambda_0, \mathbf{0})$ has only one $\mathfrak{A}^\infty$-complex at each vertex of $\{0,1\}^\ell$; in fact, by Lemma 5.2.3, we can see the E_1 page must have dimension 2^ℓ over $\mathbb{F}[[U, U^{-1}]$. For comparison, recall that $\mathrm{rk}_{\mathbb{Z}} \Lambda^*(H^1(Y;\mathbb{Z})) = 2^\ell$.

With this notation set, we are ready to study the first pages of the ε-spectral sequence.

Proposition 8.1.2. *The d_1 and d_2 differentials in the ε-spectral sequence for $\mathcal{C}$ vanish.*

Proof. We prove this by induction on ℓ. In $\mathcal{C}$ there can only be one possible value of $\psi^M(\mathbf{s})$ (modulo ∞'s) for each sublink, namely $\mathbf{0} \in \mathbb{H}(L - M)$; therefore, we will omit this from the notation. Let us first show that $d_1 \equiv 0$.

As a warm-up, if $\ell = 0$, the depth of the ε-filtration is 0. Therefore, we must have that $d_1 = 0$. Now, for $\ell = 1$, we see that

$$\mathbf{HF}^\infty(Y) \cong \mathbb{F}[[U, U^{-1}] \oplus \mathbb{F}[[U, U^{-1}]$$

by Theorem 6.2.8. Again,

$$E_1 \cong H_*(\mathcal{C}^0, \partial) \oplus H_*(\mathcal{C}^1, \partial) \cong \mathbb{F}[[U, U^{-1}] \oplus \mathbb{F}[[U, U^{-1}],$$

so $d_1 = 0$.

Suppose d_1 vanishes for any link with ℓ components and let L have $(\ell + 1)$-components. We use the notation d_1^i to represent the component of the differential d_1 which maps from

$E_1(\mathcal{C})^\varepsilon$ to $E_1(\mathcal{C})^{\varepsilon+\varepsilon(K_i)}$ (not the d_1 differential on $\mathcal{C}_i$). Now, for some $j \neq i$, let's consider the subcomplex $\mathcal{C}_j = \bigoplus_{\varepsilon_j=1} \mathcal{C}^\varepsilon$, which calculates $\mathbf{HF}^\infty$ for 0-surgery on the ℓ-component sublink $L' = L - K_j$ by Proposition 5.2.6. Note that the inclusion of $\mathcal{C}_j$ into $\mathcal{C}$ is ε-filtered. It is easy to see that the map from $E_1(\mathcal{C}_j)$ to $E_1(\mathcal{C})$ is injective.

By our induction hypothesis, d_1 is identically 0 for the complex associated to an ℓ-component sublink and thus $d_1^{\mathcal{C}_j} = 0$ by Proposition 5.2.6. Therefore, $d_1^i|_{\mathcal{C}_j} = 0$. We now want to see that d_1^i vanishes on the quotient complex

$$E_1(\mathcal{C}/\mathcal{C}_j) = \bigoplus_{\varepsilon_j=0} E_1(\mathcal{C})^\varepsilon.$$

Since d_1^i has no nonzero component from $E_1(\mathcal{C}/\mathcal{C}_j)$ to $E_1(\mathcal{C}_j)$, this will show that d_1^i is identically 0 everywhere. We can in fact identify $\mathcal{C}/\mathcal{C}_j$ with $\mathcal{C}_j$, simply by applying the ε-filtered quasi-isomorphism

$$\Gamma^{+K_j} = \sum_{\vec{M} \subset L - K_j} \Phi^{+K_j \cup \vec{M}}$$

from Lemma 5.2.5. Therefore, d_1^i is 0 on $E_1(\mathcal{C}/\mathcal{C}_j)$. Repeating this argument for various i and j, we obtain $d_1 \equiv 0$.

In fact, we can repeat this argument to prove that d_2 is identically 0 as well. For $\ell = 0$ and 1, this is trivial simply by the depth of the filtration. Thus, we begin our analysis with $\ell = 2$. As before, from Theorem 6.2.8 we have that $\mathbf{HF}^\infty(Y)$ has dimension 4 over $\mathbb{F}[[U, U^{-1}]$. However, we know that the total dimension of the E_1 page must in fact be $2^\ell = 4$. Therefore, d_2 must vanish.

Now, for the induction step, we first notice that $E_2 \cong E_1$ by the previous argument that $d_1 = 0$. Therefore, we have the same injectivity properties on the E_2 pages coming from the inclusion of the faces $\mathcal{C}_j$. We obtain $d_2 = 0$ on $\mathcal{C}_j$ again by including the corresponding complex for the sublink $L - K_j$, which has vanishing d_2 by induction. For $\mathcal{C}/\mathcal{C}_j$, we have a statement similar to the d_1 case, which is that $d_2^{i_1,i_2} = 0$ for $i_1, i_2 \neq j$, where $d_2^{i_1,i_2}$ is the component of d_2 which maps from $E_2(\mathcal{C})^\varepsilon$ to $E_2(\mathcal{C})^{\varepsilon+\varepsilon(K_{i_1})+\varepsilon(K_{i_2})}$. This follows by again identifying $\mathcal{C}/\mathcal{C}_j$ with $\mathcal{C}_j$ via Γ^{+K_j}. After doing this for different values of j, we see that $d_2^{i_1,i_2} = 0$ for all pairs (i_1, i_2) with $i_1 \neq i_2$. This shows d_2 vanishes. $\square$

8.2 The d_3 Differential

By Proposition 8.1.2, we have a natural isomorphism $E_1 \cong E_2 \cong E_3$. However, this still does not yet look like the exterior algebra found in C^∞_*. This next lemma creates the image we desire. To motivate this, recall that the E_1 term has dimension 2^ℓ, which is exactly the total dimension of the exterior algebra of an ℓ-dimensional vector space. We use $\Lambda^*_{\mathbb{F}/U}(L)$ to denote $\Lambda^*_{\mathbb{F}/U}(S^3_{\mathbf{0}}(L)) \cong \Lambda^*(\mathbb{Z}^\ell) \otimes \mathbb{F}[[U, U^{-1}]$.

We must take the word 'natural' with a grain of salt. As the link surgery formula only establishes a relative grading on $\mathcal{C}^\infty(\mathcal{H}, \Lambda_0, \mathbf{0})$, we simply fix a choice of absolute grading on this complex. We will keep everything fixed with respect to this choice of absolute grading, in the sense that inclusions of complexes corresponding to sublinks should respect this absolute grading. In this sense the identifications we will make will be 'natural' with respect to these inclusions.

Proposition 8.2.1. *There is a natural identification* $E_1(\mathcal{C}) \cong E_2(\mathcal{C}) \cong E_3(\mathcal{C}) \cong \Lambda^*_{\mathbb{F}/U}(Y)$.

Proof. Because $d_1 = d_2 = 0$, we need only make the identification with the exterior algebra for the E_1 term. Recall from Lemma 5.2.3 that each vertex complex in $\mathcal{C}$ has homology isomorphic to $\mathbf{HF}^\infty(S^3) \cong \mathbb{F}[[U, U^{-1}]$ (there is only one $\mathbf{s}$ associated to each $\mathcal{C}^\varepsilon$ by our choice of link and Spin^c structure). Therefore, the term $E_1^p(\mathcal{C})$, the subspace of $E_1(\mathcal{C})$ consisting of elements in filtration level p, of the ε-spectral sequence will simply be $\binom{\ell}{p}$ copies of $\mathbf{HF}^\infty(S^3)$.

Choose a basis $\{x^i\}$ for H^1 such that x^i corresponds to the Hom-dual of the meridian of K_i. First, identify 1 in $\Lambda^*_{\mathbb{F}/U}(L)$ with the generator θ_1 of $H_*(\mathcal{C}^{(1,\dots,1)}, \partial) \cong \mathbf{HF}^\infty(S^3)$ with fixed absolute grading 0. Let $\theta_{K_i} \in H_*(\mathcal{C}^{\varepsilon(K)}, \partial)$ be given by $(\Phi_*^{+K_i})^{-1}(\theta_1)$. We identify θ_{K_i} with x^i. This process can be repeated for any sublink, by applying $(\Phi_*^{+K_i})^{-1}$ to θ_1 for each component K_i in the sublink M to obtain the element θ_M. We associate θ_M to the corresponding wedge of x^i, where we include x^i if and only if $K_i \subset M$. By Equation (3.1), the various $\Phi_*^{+K_i}$ commute; therefore, we see the order does not matter in this construction.

Note that each exterior algebra element lives in the filtration level corresponding to the number of times we have applied $(\Phi_*^{+K})^{-1}$. Since the destabilization maps lower the relative

grading on $\mathcal{C}$ by 1 (these are components of the differential), we can see that each θ_{K_i} has grading 1 and wedge product is additive on grading; furthermore, U still has grading -2. This therefore establishes the relatively-graded isomorphism of $E_1(\mathcal{C})$ with $\Lambda^*_{\mathbb{F}/U}(L)$. □

We have not needed much about the complete system $\mathcal{H}$ until now. At this point, we will make use of the fact that we are working with a basic system. Recall that if f is a filtered chain map, we denote the induced map between the E_k pages as f_k.

Remark 8.2.2. By Lemma 4.3.1, we have that $\Gamma^{+K_i} = \Phi^{+K_i}$ in a basic system. Therefore, all of the identifications of Proposition 8.2.1 are equivariant with respect to the maps Γ^{+K_i} used in Lemma 5.2.5. In particular, we have seen that the map $\Gamma_k^{+K_i} = \Phi_k^{+K_i}$ is contraction by $[K_i]$ (the dual of x^i) on $E_k(\mathcal{C})^\varepsilon$ for all $1 \leq k \leq 3$. However, since Γ^{+K_i} preserves the ε-filtration levels exactly (up to absolute shifts), this implies that $\Gamma_k^{+K_i}$ will be the map induced by contraction by $[K_i]$ for *all* k. Under this identification, we will say contraction by $[K_i]$ even when it is the induced map in higher pages of the ε-spectral sequence. This is a very important fact that we will use.

Lemma 8.2.3. *Under these identifications,* $d_3 : \Lambda^i_{\mathbb{F}/U}(L) \otimes U^j \to \Lambda^{i-3}_{\mathbb{F}/U}(L) \otimes U^{j-1}$.

Proof. Recall that d_3 lowers filtration level by 3, but it only lowers grading by 1. In other words, d_3 will take $\alpha \otimes 1$, for $\alpha \in \Lambda^i \otimes 1$, to $\beta \in \Lambda^{i-3} \otimes U^k$ for some k. By our identifications, the grading will be lowered by $3 + 2k$; therefore, k must be -1. This shows

$$d_3 : \Lambda^i \otimes U^j \to \Lambda^{i-3} \otimes U^{j-1}.$$ □

Proposition 8.2.4. *Under these identifications, the* d_3 *differential is given by*

$$\alpha \otimes U^j \mapsto \iota_{\mu_Y}(\alpha) \otimes U^{j-1}. \tag{8.1}$$

Proof. We will use the same argument as in Proposition 8.1.2 to calculate d_3: study how it acts on faces and add up the components.

By Proposition 8.2.1, we have identified the E_3 page with $\Lambda^*_{\mathbb{F}/U}$ by associating $H_*(\mathcal{C}^\varepsilon, \partial)$ with $\mathrm{span}_{\mathbb{F}[[U,U^{-1}]}\{x^{i_1} \wedge \ldots \wedge x^{i_k}\}$, where $\varepsilon_{i_m} = 0$ for $1 \leq m \leq k = \|\varepsilon\|$. We now can easily

see how the subcomplexes and quotient complexes given by the faces of the hypercube fit into this picture via subspaces and quotient spaces of $\Lambda^*_{\mathbb{F}/U}$. Let's prove that d_3 is given by Equation (8.1).

Again, we use the notation $d_3^{i_1,i_2,i_3}$ for the components of the d_3 differential that map from $E_3(\mathcal{C})^{\varepsilon}$ to $E_3(\mathcal{C})^{\varepsilon+\varepsilon(K_{i_1})+\varepsilon(K_{i_2})+\varepsilon(K_{i_3})}$ and let x^{i_1} be the basis for $H^1(Y)$ given by Hom-duals of the meridians of K_i. Consider the three-form, μ_{i_1,i_2,i_3}, on $H^1(Y)$ given by $\mu_{i_1,i_2,i_3}(x^{k_1} \wedge x^{k_2} \wedge x^{k_3}) = \bar{\mu}_{K_{i_1}\cup K_{i_2}\cup K_{i_3}}(K_{i_1}, K_{i_2}, K_{i_3}) \pmod 2$ if $\{i_1, i_2, i_3\} = \{k_1, k_2, k_3\}$ and 0 otherwise. We claim that

$$\mu_Y = \sum_{i_1<i_2<i_3} \mu_{i_1,i_2,i_3} = \sum_{i_1<i_2<i_3} \mu_{S^3_{\mathbf{0}}(K_{i_1}\cup K_{i_2}\cup K_{i_3})}. \tag{8.2}$$

The value of μ_Y on $x^{i_1} \wedge x^{i_2} \wedge x^{i_3}$ is given by $\bar{\mu}_L(K_{i_1}, K_{i_2}, K_{i_3})$ by Theorem 6.1.15. As mentioned earlier, since $K_{i_1} \cup K_{i_2} \cup K_{i_3}$ is a sublink of L, the Milnor triple linking invariants agree for these three indices, and the claim is shown.

A similar argument using Lemma 5.2.5 as in Proposition 8.1.2 in conjunction with Remark 8.2.2 shows that $d_3^{i_1,i_2,i_3}$ will be given by interior multiplying by μ_Y if it behaves this way for links with 3 components; this again follows by the injectivity of the subcomplex inclusions on the E_3 terms. Thus, it suffices to establish that for $\ell = 3$, the d_3 differential is given by $\iota_{\mu_Y} \otimes U^{-1}$.

By Theorem 7.3.12,

$$\dim \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L)) = 8 - 2 \cdot \langle x^1 \smile x^2 \smile x^3, [Y] \rangle.$$

Proposition 8.1.2 established that the E_3 page has dimension 8. By the depth of the filtration, all differentials after d_3 must vanish. Furthermore, d_3 can only be nonzero on E_3^3. Each of E_3^3 and E_3^0 has dimension 1, generated by $x^1 \wedge x^2 \wedge x^3$ and 1 respectively in $\Lambda^*_{\mathbb{F}/U}$. Therefore, we may conclude that d_3 sends $x^1 \wedge x^2 \wedge x^3 \otimes U^j$ to $\langle x^1 \smile x^2 \smile x^3, [Y] \rangle \cdot 1 \otimes U^{j-1}$. This shows that d_3 is exactly what we want for $b_1 = 3$, completing the proof. □

We have therefore completed the first step in our two step plan. In particular, we have shown that the E_4 page of the spectral sequence is isomorphic to HC^∞. Since

$$\dim HC^\infty = \dim E_4 \geq \dim E_\infty = \dim \mathbf{HF}^\infty,$$

we have successfully proved half of Theorem 1! We now set out to prove the opposite inequality.

Remark 8.2.5. For $k = 2$ and 3, the (E_k, d_k) pages of the ε-spectral sequence have been chain complex isomorphic (not just quasi-isomorphic) to the conjectured form of the corresponding pages of the universal coefficients spectral sequence (the d_3 differential is the only unknown piece of these first pages). Ozsváth and Szabó conjecture all higher differentials vanish in the universal coefficients spectral sequence, and this is what we will prove for the ε-spectral sequence.

8.3 Composing Knots and Complexities of Links

In order to prove Theorem 1, it is necessary to proceed inductively. However, we must induct on something more complicated than simply b_1. We will assume that Theorem 1 holds for $b_1 \leq \ell - 1$, or equivalently, that the higher differentials in the ε-spectral sequence vanish. Note that this is automatic for $\ell \leq 4$. From here, we induct on the set of algebraically split links with ℓ components.

Recall that if the triple-cup product forms of two manifolds obtained by 0-surgery on an algebraically split link are isomorphic (or equivalently, if some associated algebraically split links have the same Milnor triple-linking invariants), then they are in fact surgery equivalent.

We let $L_1 \coprod L_2$ indicate that the two links are separated by an embedded 2-sphere (and both links will always be nonempty when using this notation). Begin with an ℓ-component algebraically split link L.

Example 8.3.1. Suppose that $\bar{\mu}_L(1,2,3) = n$ and all $\bar{\mu}_L$ vanish (in $\mathbb{Z}$) for all other triples of indices. Let $L' = K_1 \cup K_2 \cup K_3$. Then L has the same Milnor triple linking invariants as $L' \coprod (L - L')$. Theorem 2.1.7 gives

$$\mathbf{HF}^\infty(S^3_{\mathbf{0}}(L)) \cong \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L')) \otimes \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L - L')).$$

We have also seen that the analogous formula for HC^∞ also holds (Proposition 6.1.7). Therefore, given Theorem 1 for 0-surgery on all links with at most $\ell - 1$ components, these

connect-sum formulas prove that Theorem 1 holds for 0-surgery on L as well.

With this example in mind, we define a complexity of L with the idea that a reduction in complexity makes the link closer, in spirit, to being geometrically split. This complexity is given by

$$c(L) = \#\{(i,j,k) : \bar{\mu}_L(i,j,k) \neq 0,\ 1 \leq i < j < k \leq \ell\}. \tag{8.3}$$

If $c(L) \leq 1$, then we have seen that L is surgery equivalent to a geometrically split link. We can also understand what happens for higher c-complexity links.

Lemma 8.3.2. *For any algebraically split L, then either L is surgery equivalent to some $L_1 \coprod L_2$ or there exists some component K_i which has $\bar{\mu}_L(i,j,k)$ nonzero (in $\mathbb{Z}$) for at least two different pairs (j,k) (there is some slight reordering on the triples of indices that may be necessary for this to make sense, but this is just a notational concern).*

Proof. Fix a component K_i. If $\bar{\mu}_L(i,j,k) = 0$ for all j and k, then L is surgery equivalent to $K_i \coprod (L - K_i)$. Therefore, assume there are j and k such that $\bar{\mu}_L(i,j,k) \neq 0$. Now, if no other $\bar{\mu}_L(i,\cdot,\cdot)$, $\bar{\mu}_L(j,\cdot,\cdot)$, or $\bar{\mu}_L(k,\cdot,\cdot)$ are nonzero, then L is surgery equivalent to $L' \coprod (L - L')$, where $L' = K_1 \cup K_2 \cup K_3$. Otherwise, we are in the alternate condition. □

For a fixed b_1 we will induct on the c-complexity (we also have assumed inductively that Theorem 1 holds for smaller c-complexity). In light of this, we are really only concerned with links with c-complexity at least 2 which are not surgery equivalent to geometrically split links; to deal with the geometrically split cases, we simply apply our connect-sum formulas and our inductive knowledge for smaller b_1.

First we motivate why we are inducting on link complexities: the main step to finishing the proof of Theorem 1 will be the following.

Theorem 8.3.3. *Suppose $K = K' \# K''$ is a component of L. If Theorem 1 holds for $(L - K) \cup K'$ and $(L - K) \cup K''$, then it will also hold for L.*

In light of Lemma 8.3.2 and our goal of Theorem 8.3.3, we have the following proposition.

Proposition 8.3.4. *Suppose $c(L) \geq 2$ and that L is not surgery equivalent to any $L_1 \coprod L_2$. Let K_r be a component with at least two different pairs (s,t) such that $\bar{\mu}_L(r,s,t)$ are nonzero. Then, there is an ordered, ℓ-component link $\tilde{L}$ with the following two properties. First, $\bar{\mu}_L(i,j,k) = \bar{\mu}_{\tilde{L}}(i,j,k)$ for all i,j,k. Second, there is a knot $K \subset \tilde{L}$ which we can express as a band-sum $K'\#K''$ such that $c((\tilde{L}-K)\cup K') < c(L)$ and $c((\tilde{L}-K)\cup K'') < c(L)$.*

Before proving Theorem 8.3.3 or Proposition 8.3.4, we will see how to complete the proof of Theorem 1.

Proof of Theorem 1. For fixed $b_1 = \ell$, we induct on c. As discussed, we can apply connect-sum formulas to reduce to the case where $c(L) \geq 2$ and L does not decompose as two geometrically split links. Apply Proposition 8.3.4 to replace L by $\tilde{L}$. Since $\bar{\mu}_L(i,j,k) = \bar{\mu}_{\tilde{L}}(i,j,k)$, we know that L and $\tilde{L}$ will be surgery equivalent. It now suffices to prove Theorem 1 for $\tilde{L}$.

Decompose the component K as $K'\#K''$. Since $(\tilde{L}-K)\cup K'$ and $(\tilde{L}-K)\cup K''$ have strictly smaller c-values, Theorem 1 holds for each of these. Apply Theorem 8.3.3 to complete the induction on c. □

We now recall a helpful theorem of Cochran describing the $\bar{\mu}$-invariants of connect sums.

Theorem 8.3.5. *(Theorem 8.13 of [Coc90]) Suppose L and L' are n-component links that are separated by an embedded 2-sphere and satisfy $\bar{\mu}_L(J) = \bar{\mu}_{L'}(J) = 0$ for multi-indices J of length at most n. Construct $L\#L'$ by connecting each pair of components K_i and K_i' with a band that passes through the separating sphere exactly once. Then $\bar{\mu}_{L\#L'}(I) = \bar{\mu}_L(I) + \bar{\mu}_{L'}(I)$ for any multi-index I of length at most $n+1$.*

In particular, given two algebraically split links L and L', we have that $\bar{\mu}_{L\#L'}(i,j,k) = \bar{\mu}_L(i,j,k) + \bar{\mu}_{L'}(i,j,k)$.

Proof of Proposition 8.3.4. By hypothesis and Lemma 8.3.2, we may consider two distinct pairs of indices (j_1,k_1) and (j_2,k_2) such that $\bar{\mu}_L(r,j_1,k_1)$ and $\bar{\mu}_L(r,j_2,k_2)$ are nonzero for L. Construct an ℓ-component algebraically split link L' with an ordering on the components such that

$$\bar{\mu}_{L'}(a,b,c) = \begin{cases} \bar{\mu}_L(a,b,c) & \text{if } (a,b,c) \neq (r,j_2,k_2), \\ 0 & \text{if } (a,b,c) = (r,j_2,k_2). \end{cases}$$

Such a link can be explicitly constructed by repeated applications of "Borromean braiding" (see Corollary 3.5 of [CGO00] for more details).

Next, isotope a small arc from both K'_{j_2} and K'_{k_2} (components of L') out and away from the rest of the diagram for L', and isotope the arc from K'_{j_2} such that it creates $\bar{\mu}_L(r,j_2,k_2)$ twists. We now take an unknot, U, and thread it through the twists of K'_{j_2} and through K'_{k_2} as in Figure 8.1. By construction, $\bar{\mu}_{U \cup K'_{j_2} \cup K'_{k_2}}(U, K'_{j_2}, K'_{k_2}) = \bar{\mu}_L(r,j_2,k_2)$.

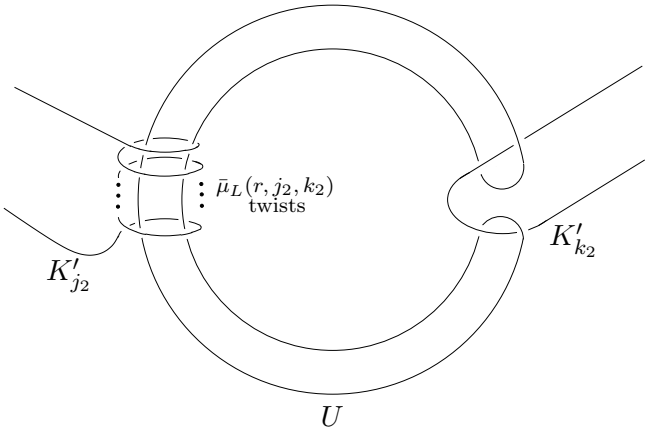


Figure 8.1: Threading the unknot to recreate $\bar{\mu}_L(r,j_2,k_2)$

We will choose K' to be K'_r in L' and $K'' = U$. From this we can see

$$\bar{\mu}_{K'' \cup (L'-K')}(K'', K'_{j_2}, K'_{k_2}) = \bar{\mu}_L(r,j_2,k_2)$$

and all other $\bar{\mu}_{K'' \cup (L'-K')}(K'', \cdot, \cdot)$ vanish. We now want to see that the band sum $K = K' \# K''$ yields a link $\tilde{L} = K \cup (L' - K')$ with the same Milnor triple linking numbers as L.

By Theorem 8.3.5, it suffices to prove that $\tilde{L}$ can be constructed by connecting two geometrically split ℓ-component links by bands between pairs of components which intersect the separating 2-sphere exactly once such that the sums of the $\bar{\mu}$-invariants for these two links add up to $\bar{\mu}_L$.

We choose our two links as follows. The first link will be L'. The other link is an ℓ-component link, L^*, consisting of two split sublinks: a three-component sublink with $\bar{\mu}_{L^*}(r, j_2, k_2) = \bar{\mu}_L(r, j_2, k_2)$ and an $(\ell - 3)$-component unlink, so all other Milnor triple linking numbers vanish. Clearly the values of $\bar{\mu}$ add up as expected and Figure 8.2 demonstrates how we can connect them to obtain $\tilde{L}$ with $K = K' \# K''$. Note that L^*_r is what creates K'' in $\tilde{L}$.

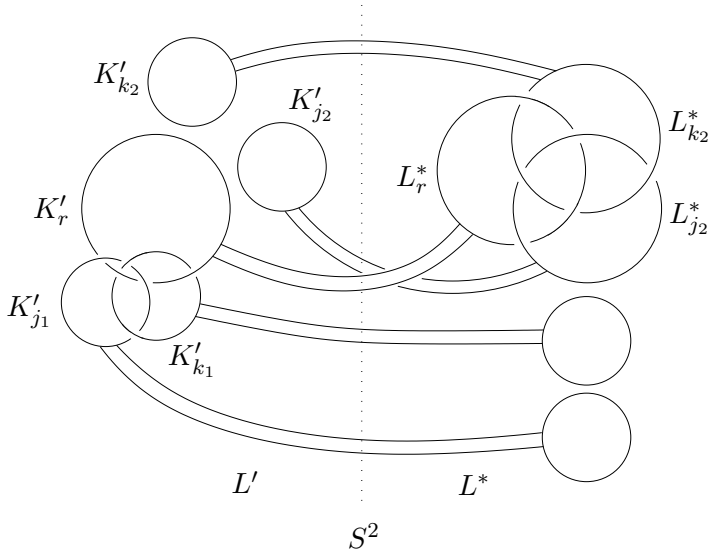


Figure 8.2: Expressing $\tilde{L}$ as the connect-sum of L' and L^*

By construction, both $c(K' \cup (\tilde{L} - K))$, which equals $c(L')$, and $c(K'' \cup (\tilde{L} - K))$ are strictly less than $c(L)$. This completes the proof. □

8.4 Chopping Down the Link Surgery Formula

Recall that our goal is to prove Theorem 8.3.3. We now construct a complex which contains all of the Heegaard Floer information of K' and K'' simultaneously, where we have identified $K = K' \# K''$ as the component to reduce complexity at from Proposition 8.3.4. With this we will be able to use our inductive knowledge for K' and K'' to produce the desired result.

The way that this is done is via a standard Kirby calculus trick (see, for example, [Sav99]); we express 0-surgery on K as 0-surgery on three components: K', K'', and an unknot U geometrically linking each once as shown in Figure 8.3. Note that if we remove U from our link, the result is algebraically split.

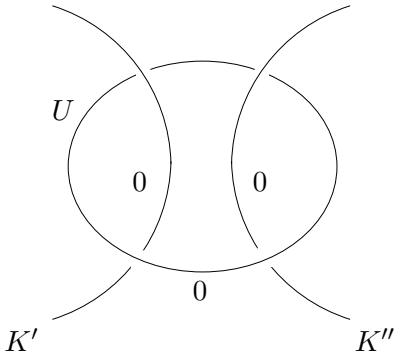


Figure 8.3: An equivalent diagram for 0-surgery on $K'\#K''$

Thus, if our original link L (and thus $\tilde{L}$) has ℓ components, then the corresponding link that we would like to study will have $\ell + 2$. As further abuse of notation we will now call this link L, since 0-surgery results in the same manifold. The framing Λ will change as well due to the algebraic linking that has been introduced. Reorder the components in such a way that K', K'', and U are the first, second, and third components (K_1, K_2, and K_3) respectively. This three-component sublink will arise often, so we will refer to it as W. We see that $\Lambda_1 = \Lambda_2 = (0, 0, 1, 0, \ldots, 0)$ and $\Lambda_3 = (1, 1, 0, \ldots, 0)$. Therefore, the equivalence class in $\mathbb{H}(L)$ corresponding to the torsion Spin^c structure $\mathfrak{s}_0$ will be a 2-dimensional lattice spanned by Λ_1 and Λ_3; in fact, $\mathfrak{s}_0 = [(\frac{1}{2}, \frac{1}{2}, 1, 0, \ldots, 0)]$. We again pick a basic system $\mathcal{H}$ for L. Since there is non-trivial linking in our link L, the complex $\mathcal{C}^\infty(\mathcal{H}, \Lambda, [(\frac{1}{2}, \frac{1}{2}, 1, 0, \ldots, 0)])$ will be infinitely-generated. Again, the link surgery formula tells us that the homology of this complex will calculate $\mathbf{HF}^\infty$.

Using arguments similar to those in Proposition 6.2.2, we will induce the appropriate filtrations and remove acyclic complexes to cut down the size of $\mathcal{C}^\infty(\mathcal{H}, \Lambda, [(\frac{1}{2}, \frac{1}{2}, 1, 0, \ldots, 0)])$ to a smaller finite-dimensional object. Before continuing, we remark that the reader in-

terested in the details of this section should first try to read the example in Section 8.2 of [MO10], where $\mathbf{HF}^-$ for surgeries on the Hopf link are calculated via the link surgery formula; the arguments here will be similar. Let's also recall the simplified notation used in that computation. We let $\varepsilon_1\varepsilon_2\dots\varepsilon_{\ell+2_{\mathbf{s}}}$ represent the complex $\mathfrak{A}^\infty(\mathcal{H}^{L-M},\psi^M(\mathbf{s}))$ where $\varepsilon_1\dots\varepsilon_{\ell+2}=\varepsilon(M)$. To shorten notation further in our setting, we will use $\varepsilon_1\varepsilon_2\varepsilon_3*_{(s_1,s_2,s_3)}$ to denote the hypercube of chain complexes at $(s_1,s_2,s_3,0,\dots,0)$ with ε_1, ε_2, ε_3 fixed, but all remaining ε_i free. We are setting the last components of $\mathbf{s}$ to be 0 since this corresponds to choosing the unique torsion Spin^c structure on $S^3_{\mathbf{0}}(L-W)$. For example, by Proposition 5.2.6, $111*_{(s_1,s_2,s_3)}$ is in fact the surgery formula for the unique torsion Spin^c structure on $S^3_{\mathbf{0}}(L-W)$. We will often omit the $\mathbf{s}$ from the Φ maps.

From now on, $\{s_i>r\}$ will refer to the complex generated by all $\varepsilon_1\varepsilon_2\varepsilon_3*_{(s_1,s_2,s_3)}$, where $s_i>r$; note that (s_1,s_2,s_3) is recording an index in $\mathbb{H}(W)$, which doesn't depend on whether a component K_i has been destabilized and thus we do not apply the relevant ψ maps. Similar notation, such as $\{s_1>r,\varepsilon_2=0\}$, is also clear.

Proposition 8.4.1. *The link surgery complex, $\mathcal{C}=\mathcal{C}^\infty(\mathcal{H},\Lambda,[(\frac{1}{2},\frac{1}{2},1,\dots,0)])$, for 0-surgery on all components of L with Spin^c structure $\mathfrak{s}_0$ is quasi-isomorphic to*

$$001*_{(\frac{1}{2},\frac{1}{2},1)} \xrightarrow{\Gamma^{+K_1}} 101*_{(\frac{1}{2},\frac{1}{2},1)} \xleftarrow{\Gamma^{-K_1}} 001*_{(\frac{1}{2},\frac{1}{2},0)}$$
$$\Gamma^{+K_2}\searrow \qquad\qquad \swarrow\Gamma^{-K_2}$$
$$011*_{(\frac{1}{2},\frac{1}{2},1)}$$

Proof. Induce the filtration on $\mathcal{C}$ defined by

$$\mathcal{F}_3(\mathbf{x})=-(s_1+\sum_{i\neq 3}\varepsilon_i)$$

for $x\in\varepsilon_1\varepsilon_2\varepsilon_3*_{(s_1,s_2,s_3)}$. The components of the differential that preserve filtration level are given by ∂ and Φ^{+K_3}. Consider the subcomplex $\{s_1>\frac{1}{2}\}$. The associated graded with respect to the $\mathcal{F}_3$-filtration restricted to this subcomplex splits as a product of two-step complexes of the form

$$(\varepsilon_1\varepsilon_2 0\varepsilon_4\dots\varepsilon_{\ell+2_{\mathbf{s}}},\partial)\xrightarrow{\Phi^{+K_3}}(\varepsilon_1\varepsilon_2 1\varepsilon_4\dots\varepsilon_{\ell+2_{\mathbf{s}}},\partial).$$

Since the maps Φ^{+K_3} are quasi-isomorphisms by Proposition 5.2.1, we have that the associated graded on $\{s_1 > \frac{1}{2}\}$ is acyclic. By Fact 3.2.3, this implies that all of $\{s_1 > \frac{1}{2}\}$ is acyclic as well. Therefore, $\mathcal{C}$ is quasi-isomorphic to the quotient complex $\mathcal{C}/\{s_1 > \frac{1}{2}\}$, which is simply $\{s_1 \leq \frac{1}{2}\}$. We then induce a similar filtration,

$$\mathcal{G}_3(\mathbf{x}) = s_1 - \sum_i \varepsilon_i.$$

The differentials preserving this filtration level will now be ∂ and Φ^{-K_3}. We consider the subcomplex, $\mathcal{C}'$, of $\{s_1 \leq \frac{1}{2}\}$ defined by

$$\{s_1 < \frac{1}{2}, \varepsilon_3 = 0\} \oplus \{s_1 \leq \frac{1}{2}, \varepsilon_3 = 1\}$$

(this is everything except $\{s_1 = \frac{1}{2}, \varepsilon_3 = 0\}$). The associated graded for $\mathcal{G}_3$ restricted to this subcomplex now splits as a product of complexes of the form

$$(\varepsilon_1\varepsilon_2 0 \varepsilon_4 \dots \varepsilon_{l+2_{\mathbf{s}}}, \partial) \xrightarrow{\Phi^{-K_3}} (\varepsilon_1\varepsilon_2 1 \varepsilon_4 \dots \varepsilon_{l+2_{(\mathbf{s}+\Lambda_3)}}, \partial).$$

Similarly, since the maps Φ^{-K_3} are quasi-isomorphisms, $\mathcal{C}'$ is acyclic. We are content to remove this and study only the remaining terms, namely $\{s_1 = \frac{1}{2}, \varepsilon_3 = 0\}$. It is best to visualize the remaining terms, which we think of as the remains after collapsing the link surgery formula in the Λ_3-direction, via Figure 8.4.

We can further reduce this complex in a similar way: by collapsing in the Λ_1-direction (which also happens to be the Λ_2-direction). Consider the filtration,

$$\mathcal{F}_1(\mathbf{x}) = -(s_3 + \sum_{i \neq 1} \varepsilon_i),$$

on the subcomplex $\{s_3 > 1\}$ of $\{s_1 = \frac{1}{2}, \varepsilon_3 = 0\}$. The associated graded for $\mathcal{F}_1$ on this subcomplex splits as a product of complexes of the form

$$(0\varepsilon_2 0 \varepsilon_4 \dots \varepsilon_{l+2_{\mathbf{s}}}, \partial) \xrightarrow{\Phi^{+K_1}} (1\varepsilon_2 0 \varepsilon_4 \dots \varepsilon_{l+2_{\mathbf{s}}}, \partial).$$

This complex is acyclic, as the Φ^{+K_1} are quasi-isomorphisms. After removing this subcomplex, we are left with $\{s_3 \leq 1\}$. We must tread carefully to chop the remaining complex down further. Consider the filtration,

$$\mathcal{F}_2(\mathbf{x}) = s_3 - 2\varepsilon_1 - \sum_{i \neq 1} \varepsilon_i.$$

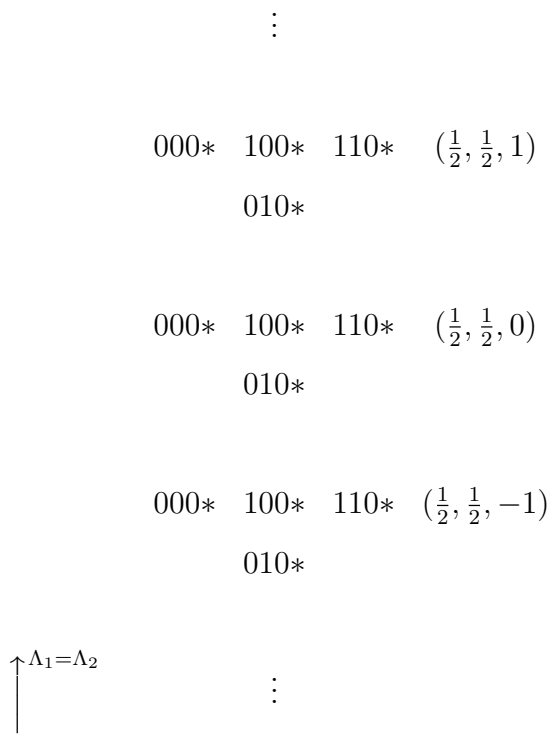


Figure 8.4: The complex $\{s_1 = \frac{1}{2}, \varepsilon_3 = 0\}$

This odd-looking filtration is defined such that Φ^{-K_1} lowers the filtration level, but Φ^{-K_2} does not, even though $\Lambda_1 = \Lambda_2$. We now study the subcomplex

$$\{s_3 = 1, \varepsilon_1 = \varepsilon_2 = 1\} \oplus \{s_3 = 0, \varepsilon_1 + \varepsilon_2 \geq 1\} \oplus \{s_3 \leq -1\}.$$

This subcomplex is best seen by the boxed elements in Figure 8.5 (where all of the terms that the vertical ellipses are representing should also be boxed).

The associated graded splits into a product of complexes analogous to the ones defined previously. Since Φ^{-K_2} is a quasi-isomorphism, we may again remove this acyclic complex in our study.

We can now see that the remaining complex is the same as the one in the statement of the proposition, except for the fact that each of the ε_3 is 0 instead of 1. However, we can simply apply Γ^{+K_3} to the remaining complex; by Lemma 5.2.5, this map is an ε-filtered quasi-isomorphism to the complex in the statement of the proposition. □

Remark 8.4.2. By Proposition 5.2.6, we have an identification of the complex $\varepsilon_1\varepsilon_2\varepsilon_3*_{(\frac{1}{2},\frac{1}{2},1)}$

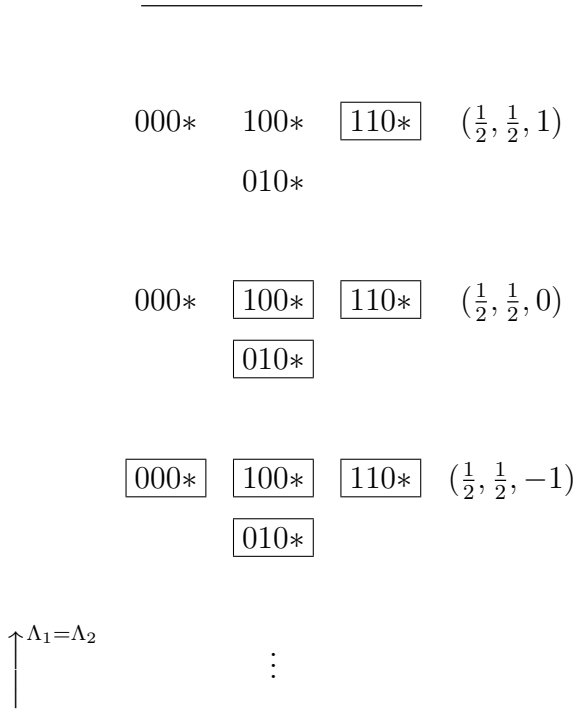


Figure 8.5: The boxed terms form the final acyclic complex

with $\varepsilon_1\varepsilon_2*_{(0,0)}$, where $\varepsilon_1\varepsilon_2*$ refers to the chain complex sitting inside of the surgery formula for $L - K_3$ with fixed ε_1 and ε_2 and all other ε_i free. This identification respects all of the $\Gamma^{\pm K_i}$ maps as well, so from now on we will replace the $\varepsilon_1\varepsilon_2\varepsilon_3*$ terms with $\varepsilon_1\varepsilon_2*$. We will still use $\Gamma^{\pm K_i}$ to denote the corresponding maps between the complexes for the restricted complete systems for $L - K_3$. Furthermore, we will omit the $\mathbf{s}$ from the ε-complexes for notation.

8.5 The $\Gamma^{\pm K_i}$ Maps

Fix any algebraically split link M. There is a natural inclusion $S^3 - M \to S^3 - M'$ for $M' \subset M$, inducing

$$H^1(S^3_{\mathbf{0}}(M'); \mathbb{Z}) \hookrightarrow H^1(S^3_{\mathbf{0}}(M); \mathbb{Z}),$$

which acts as inclusion on the basis of Hom-duals to meridians of components of M'. We will use this fact implicitly in what follows.

Recall from the previous section that the link $L - K_3 = K_1 \cup K_2 \cup L'$ that we are focusing on, which is again algebraically split. Note that we have done away with x^3. Let's study the ε-filtration on the link surgery formula for $K_1 \cup K_2 \cup L'$ as well as on each face complex $\varepsilon_1\varepsilon_2*$. Since $\Gamma_3^{+K_i}$ is contraction by $[K_i]$ by Remark 8.2.2, we have identified $E_3(00*)$ with $x^1 \wedge x^2 \wedge \Lambda^*_{\mathbb{F}/U}(L')$. There is a similar identification of $E_3(01*)$ (respectively $E_3(10*)$) with $x^1 \wedge \Lambda^*_{\mathbb{F}/U}(L')$ (respectively $x^2 \wedge \Lambda^*_{\mathbb{F}/U}(L')$).

From Proposition 8.2.4, we can explicitly identify the d_3 differential for the ε-spectral sequence on $\varepsilon_1\varepsilon_2*$, which we denote $d_3^{\varepsilon_1\varepsilon_2}$. It is exactly contraction by triple cup products that *do not* use x^1 or x^2; more precisely,

$$d_3^{\varepsilon_1\varepsilon_2} = \sum_{3<r<s<t} \iota_{\mu_{r,s,t}} \otimes U^{-1},$$

where $\mu_{r,s,t}$ is the three-form used in Equation (8.2), but for the link $L - K_3$. In particular, we can identify $d_3^{\varepsilon_1\varepsilon_2}$ with $\partial^\infty_{S^3_{\mathbf{0}}(L')}$. Since $b_1(S^3_{\mathbf{0}}(L'))$ is one less than $b_1(S^3_{\mathbf{0}}(L))$, we may apply our induction hypothesis to see that the ε-spectral sequence for $\varepsilon_1\varepsilon_2*$ collapses after we take homology with respect to the d_3 differential (since each face complex is ε-filtered

quasi-isomorphic to the link surgery formula for L'). Thus, we can identify $E_\infty(\varepsilon_1\varepsilon_2*)$ with $HC^\infty(S^3_{\mathbf{0}}(L'))$ wedged with each x^i such that $\varepsilon_i = 0$.

On the other hand, there is a map

$$d_3^{K_i} = \sum_{j,k} \iota_{\mu_{i,j,k}} \otimes U^{-1},$$

which has domain $\{\varepsilon_i = 0\}$ and range $\{\varepsilon_i = 1\}$. This is contraction by the triple cup products that *do* use x^1 or x^2.

Remark 8.5.1. Consider the complex

$$(x^i \wedge \Lambda^*_{\mathbb{F}/U}(L'), x^i \wedge \partial^\infty_{S^3_{\mathbf{0}}(L')}) \xrightarrow{d_3^{K_i}} (\Lambda^*_{\mathbb{F}/U}(L'), \partial^\infty_{S^3_{\mathbf{0}}(L')}),$$

where $i = 1$ or 2. Note that this is exactly $C^\infty(S^3_{\mathbf{0}}(K_i \cup L'))$. In particular, $d_3^{K_i}$ is a chain map and induces a map from $x^1 \wedge HC^\infty(S^3_{\mathbf{0}}(L'))$ to $HC^\infty(S^3_{\mathbf{0}}(L'))$.

Using this decomposition, we can explicitly compute $\Gamma_\infty^{-K_i}$.

Lemma 8.5.2. *Under the identifications with $\Lambda^*_{\mathbb{F}/U}(K_1 \cup K_2 \cup L')$, the map $\Gamma_\infty^{-K_i}$ is given by $\iota_{[K_i]} + (d_3^{K_i})_*$, where $(d_3^{K_i})_*$ is the induced map from $x^i \wedge HC^\infty(S^3_{\mathbf{0}}(L'))$ to $HC^\infty(S^3_{\mathbf{0}}(L'))$.*

Proof. Without loss of generality, $i = 1$. First, apply Γ^{+K_2} to

$$(00* \xrightarrow{\Gamma^{+K_1}+\Gamma^{-K_1}} 10*)$$

to obtain

$$(01* \xrightarrow{\Gamma^{+K_1}+\Gamma^{-K_1}} 11*).$$

By Remark 8.2.2, it therefore suffices to establish the lemma instead for this complex. However, by Proposition 5.2.6, this can be identified with the surgery formula for the link $K_1 \cup L'$. Under our identifications,

$$(E_3(01*), d_3^{01}) \xrightarrow{\Gamma_3^{+K_1}+\Gamma_3^{-K_1}} (E_3(11*), d_3^{11})$$

is precisely

$$(x^1 \wedge \Lambda^*_{\mathbb{F}/U}(L'), x^1 \wedge \partial^\infty_{S^3_{\mathbf{0}}(L')}) \xrightarrow{d_3^{K_1}} (\Lambda^*_{\mathbb{F}/U}(L'), \partial^\infty_{S^3_{\mathbf{0}}(L')}).$$

Furthermore, after taking homology of the E_3 pages,

$$(d_3^{K_1})_* = \Gamma_4^{+K_1} + \Gamma_4^{-K_1}.$$

Since $c(K_1 \cup L') < c(L)$, we may apply our induction hypothesis to see that the higher differentials in the ε-spectral sequence for $K_1 \cup L'$ vanish. Therefore, all of the higher pages in these spectral sequences, E_i (including $i = \infty$) are canonically isomorphic to E_4. This in fact says that

$$(d_3^{K_1})_* = \Gamma_\infty^{+K_1} + \Gamma_\infty^{-K_1}.$$

However, by Remark 8.2.2,

$$\Gamma_\infty^{+K_1} = \Gamma_4^{+K_1} = \iota_{[K_1]}.$$

This now completes the proof. □

8.6 The Final Calculation

We will now complete the proof of Theorem 8.3.3. As discussed in Section 8.3, this will also complete the proof of Theorem 1.

Proof of Theorem 8.3.3. Recall that by Proposition 8.4.1 and Remark 8.4.2, $\mathbf{HF}^\infty(S^3_{\mathbf{0}}(L))$ is calculated by the homology of the complex

$$\begin{array}{ccccc} 00* & \xrightarrow{\Gamma^{+K_1}} & 10* & \xleftarrow{\Gamma^{-K_1}} & 00*\,. \\ & \searrow^{\Gamma^{+K_2}} & & \swarrow_{\Gamma^{-K_2}} & \\ & & 01* & & \end{array} \tag{8.4}$$

Note that the complex (8.4) is quasi-isomorphic to

$$\begin{array}{ccccc} E_\infty(00*) & \xrightarrow{\Gamma_\infty^{+K_1}} & E_\infty(10*) & \xleftarrow{\Gamma_\infty^{-K_1}} & E_\infty(00*) \\ & \searrow^{\Gamma_\infty^{+K_2}} & & \swarrow_{\Gamma_\infty^{-K_2}} & \\ & & E_\infty(01*) & & \end{array} \tag{8.5}$$

by Fact 3.2.3. Note that the map $\Gamma_\infty^{+K_i}$ is invertible, as the induced map on the E_3 pages is a bijection (or because Γ^{+K_i} is an ε-filtered quasi-isomorphism). Therefore, we finally can

apply the mysterious Lemma 3.1.2 from Chapter 3. This gives an isomorphism between the homology of (8.5) and the homology of

$$E_\infty(00*) \xrightarrow{\Gamma_\infty^{-K_1}+\Gamma_\infty^{+K_1}\circ(\Gamma_\infty^{+K_2})^{-1}\circ\Gamma_\infty^{-K_2}} E_\infty(10*). \tag{8.6}$$

By Remark 8.2.2 and Lemma 8.5.2, Equation (8.6) is quasi-isomorphic to the complex

$$(x^1 \wedge x^2 \wedge \Lambda^*_{\mathbb{F}/U}(L'), d_3^{L'}) \xrightarrow{d_3^{K_1}+\iota_{[K_1]}\circ(x^2\wedge\cdot)\circ d_3^{K_2}} (x^2 \wedge \Lambda^*_{\mathbb{F}/U}(L'), d_3^{L'}). \tag{8.7}$$

Recall $\tilde{L} = K \cup L'$, the link we obtained from Proposition 8.3.4 before the Kirby calculus, and recall that $S^3_{\mathbf{0}}(\tilde{L}) = S^3_{\mathbf{0}}(L)$. Let $x^{1\#2}$ denote the Hom-dual of the meridian of K in $H^1(S^3_{\mathbf{0}}(\tilde{L}); \mathbb{Z})$. There is a natural identification of $H^1(S^3_{\mathbf{0}}(\tilde{L}); \mathbb{Z})$ with each $H^1(S^3_{\mathbf{0}}(K_i \cup L'); \mathbb{Z})$, given by identifying $x^{1\#2}$ with x^i for $i = 1$ or 2, and fixing x^j for $j > 2$.

By the construction of $K = K_1 \# K_2$ in Proposition 8.3.4, Equation (8.7) is quasi-isomorphic to

$$(x^{1\#2} \wedge x^2 \wedge \Lambda^*_{\mathbb{F}/U}(L'), d_3^{L'}) \xrightarrow{d_3^K} (x^2 \wedge \Lambda^*_{\mathbb{F}/U}(L'), d_3^{L'}).$$

After contracting by x^2, this complex is exactly $(C^\infty(S^3_{\mathbf{0}}(\tilde{L})), \partial^\infty_{S^3_{\mathbf{0}}(\tilde{L})})$. This completes the proof, since we have shown that the homology of this complex also computes $\mathbf{HF}^\infty(S^3_{\mathbf{0}}(L))$.

□

CHAPTER 9

Applications

Recall that $HF^\infty(Y)$ (everything uncompleted in this chapter is with $\mathbb{F}$-coefficients) is said to be *standard* if $HF^\infty(Y,\mathfrak{s})$ is isomorphic to $\Lambda^*(H^1(Y;\mathbb{Z}))\otimes\mathbb{F}[U,U^{-1}]$ for each torsion Spin^c structure $\mathfrak{s}$ [OS03a]. By Theorem 1, this is equivalent to the integral triple cup product form not vanishing identically mod 2. We will abuse notation and say that $\mathbf{HF}^\infty(Y)$ or $\mathbf{HF}^\infty(Y,\mathfrak{s})$ is standard (for $\mathfrak{s}$ torsion) - it is clear what these imply.

9.1 New Lower Bounds for Khovanov Homology

We first recall a classical result about branched covers of spheres.

Lemma 9.1.1 (Hirsch-Neumann, c.f. Lemma of [HN75])**.** *If Y is the double-branched cover of a link in a rational homology sphere, then $\mu_Y \equiv 0$.*

Therefore, if Y is the double-branched cover of a link in a rational homology sphere, $HF^\infty(Y)$ is standard. This leads to the following application to Khovanov homology. For a quick introduction to Khovanov homology, we refer the reader to [Bar02].

Proof of Theorem 1.3.1. Lemma 9.1.1 implies that for each torsion Spin^c structure $\mathfrak{s}$ on $\Sigma_2(L)$,

$$\mathrm{rk}_{\mathbb{F}[U,U^{-1}]}\, HF^\infty(\Sigma_2(L),\mathfrak{s}) = 2^{b_1(\Sigma_2(L))}.$$

From the definition of $\widehat{HF}$, it is a simple exercise to see that

$$\dim_{\mathbb{F}}\widehat{HF}(\Sigma_2(L),\mathfrak{s}) \geq \mathrm{rk}_{\mathbb{F}[U,U^{-1}]}\, HF^\infty(\Sigma_2(L),\mathfrak{s}).$$

Summing over all Spin^c structures, we obtain

$$\dim_{\mathbb{F}} \widehat{HF}(\Sigma_2(L)) \geq 2^{b_1(\Sigma_2(L))} \cdot |\mathrm{Tor}(H_1(\Sigma_2(L)))|. \tag{9.1}$$

We apply the spectral sequence of Ozsváth-Szabó (Theorem 1.1 of [OS05b]), which has $E_2 \cong \widetilde{Kh}(L;\mathbb{F})$ and $E_\infty \cong \widehat{HF}(\Sigma_2(\overline{L}))$. Since $\Sigma_2(\overline{L}) \cong -\Sigma_2(L)$ and the dimension of $\widehat{HF}$ is unaffected by orientation reversal, the result follows from Equation (9.1). □

Remark 9.1.2. Ozsváth and Szabó were aware of Theorem 1.3.1 whenever HF^∞ was known to be standard. In particular, they pointed out the desired inequality when $det(L) \neq 0$ in Corollary 1.2 of [OS05b].

Remark 9.1.3. This theorem could also be proved in the following (overkill) way. As mentioned in the introduction, one may combine the isomorphism of Kutluhan, Lee, and Taubes between monopole Floer homology and Heegaard Floer homology with the calculations of $\overline{HM}$ by Kronheimer and Mrowka to see that $HF^\infty(Y,\mathfrak{s};\mathbb{Q}) \cong HC^\infty(Y;\mathbb{Q})$ for $\mathfrak{s}$ torsion. If Y is a double-branched cover (and thus has trivial μ_Y from Lemma 9.1.1),

$$\begin{aligned}\mathrm{rk}_{\mathbb{F}[U,U^{-1}]} HF^\infty(Y) &\geq \mathrm{rk}_{\mathbb{Q}[U,U^{-1}]} HF^\infty(Y;\mathbb{Q}) \\ &\geq 2^{b_1(Y)}|\mathrm{Tor}H_1(Y;\mathbb{Z})|.\end{aligned}$$

We can therefore still repeat the above arguments to obtain the desired results.

9.2 Induced Maps on HF^∞ for 2-Handlebodies

One of the most important aspects of Heegaard Floer theory is that it yields invariants of smooth four-manifolds. If $(W,\mathfrak{t})$ is a Spin^c cobordism from $(Y_1,\mathfrak{s}_1)$ to $(Y_2,\mathfrak{s}_2)$, then it is shown in [OS06b] that there are induced homogeneous $\mathbb{F}[U]$-module maps $F^\circ_{W,\mathfrak{t}} : HF^\circ(Y_1,\mathfrak{s}_1) \to HF^\circ(Y_2,\mathfrak{s}_2)$, whose absolute shift in grading can be determined by classical invariants of $(W,\mathfrak{t})$ (the signature, Euler characteristic, and $\langle c_1(\mathfrak{t})^2,[W]\rangle$). If $\partial W = Y$ has only one component, we can remove a small ball to obtain a cobordism from S^3 to Y, which we will also denote by W. In this case, since $HF^\infty(S^3) \cong \mathbb{F}[U,U^{-1}]$, understanding $F^\infty_{W,\mathfrak{t}}$ is

completely determined by whether or not it is non-zero (along with knowing its classical invariants to determine the shift in degree).

Recall that W is a *2-handlebody* if it is obtained by attaching only 2-handles to a four-ball; such a W will be simply-connected and will have $H_2(W;\mathbb{Z})$ torsion-free. We will prove Theorem 1.3.2 using $\mathbb{F}[[U,U^{-1}]$-coefficients, but like for Theorem 1, all of our constructions will be sufficient to prove the results for $\mathbb{F}[U,U^{-1}]$-coefficients.

Proof of Theorem 1.3.2. Suppose that $b_2^+(W) > 0$. In this case, Lemma 8.2 of [OS06b] implies that the map $\mathbf{F}^\infty_{W,\mathfrak{t}}$ will always vanish identically. Therefore, we assume that $b_2^+(W) = 0$. Let $b_2 = \ell$.

Suppose that $H_2(W;\mathbb{Z}) \cong \mathbb{Z}^m$. We choose a presentation for W by attaching 2-handles along a framed link $L = K_1 \cup \dots K_\ell$. We first would like to study the intersection form of W. Let $Z = \{x \in H_2(W;\mathbb{Z}) | Q_W(x,y) = 0 \text{ for all } y \in H_2(W;\mathbb{Z})\}$. Then, we have $H_2(W;\mathbb{Z}) = Z \oplus N$, where $Q_W|_Z = 0$ and $Q_W|_N$ is negative-definite. Suppose that $\operatorname{rk} Z = k$. Let L' denote the sublink of L formed by the first k components. After a sequence of handleslides, corresponding to a change of basis for $H_2(W;\mathbb{Z})$, we may assume that L' is 0-framed and algebraically split - in other words, the 2-handlebody obtained by attaching 0-framed 2-handles along L' has intersection form $Q_W|_Z$. We also have that the components K_i for $i > k$ have (algebraically) trivial linking with each of the components of L'. Furthermore, the 2-handlebody obtained by attaching 2-handles to the remaining K_i will have intersection form $Q_W|_N$.

First, assume that μ_Y is identically 0. The idea is to break W into two pieces: W_1, which is the 2-handlebody obtained by attaching handles to S^3 along L', and W_2 which is the 2-handlebody obtained by attaching handles along $K_{k+1}, \dots, K_\ell$ in $S^3_{\mathbf{0}}(L')$. Because $Q_W|_N$ is negative-definite, Proposition 6.1.10 implies that $HC^\infty(Y) \cong HC^\infty(S^3_{\mathbf{0}}(L'))$. Therefore, $\mathbf{HF}^\infty(Y,\mathfrak{s}) \cong \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L'))$ is standard.

We claim that the map induced by W_1 is non-zero for any Spin^c structure on W_1 which restricts to the unique torsion Spin^c structure on $S^3_{\mathbf{0}}(L')$. To see this, suppose

$$(W_K, \mathfrak{t}) : (M, \mathfrak{t}|_M) \to (M_0(K), \mathfrak{t}|_{M_0(K)})$$

is a Spin^c cobordism obtained by attaching a 0-framed 2-handle along a nullhomologous knot K in M, where $\mathfrak{t}|_M$ and $\mathfrak{t}|_{M_0(K)}$ are torsion. Proposition 9.3 of [OS03a] states that $\mathbf{F}^\infty_{W_K,\mathfrak{t}}$ is an injection if $\mathbf{HF}^\infty(M_0(K))$ is standard. Because $\mu_{S^3_{\mathbf{0}}(L')}$ vanishes, we have that $\mu_{S^3_{\mathbf{0}}(L'')}$ vanishes for any $L'' \subset L'$, and $\mathbf{HF}^\infty(S^3_{\mathbf{0}}(L''))$ is standard for all $L'' \subset L'$. In particular, the sequence of 2-handle attachments

$$S^3 \to S^3_0(K_1) \to S^3_{\mathbf{0}}(K_1 \cup K_2) \to \ldots \to S^3_{\mathbf{0}}(L')$$

induces injections on $\mathbf{HF}^\infty$.

Because $\mathbf{HF}^\infty(S^3_{\mathbf{0}}(L')) \cong \mathbf{HF}^\infty(Y,\mathfrak{s})$ is standard, we have that any negative-definite Spin^c cobordism from $(S^3_{\mathbf{0}}(L'),\mathfrak{s}_0)$ to $(Y,\mathfrak{s})$ must induce an isomorphism on $\mathbf{HF}^\infty$ by Proposition 9.4 of [OS03a]. A simple Mayer-Vietoris argument shows that $\mathfrak{t}$ is determined by its restrictions to W_1 and W_2. Therefore, we may apply the gluing theorem for cobordism maps (Theorem 3.4 of [OS06b]) to see that $\mathbf{F}^\infty_{W,\mathfrak{t}} = \mathbf{F}^\infty_{W_2,\mathfrak{t}|_{W_2}} \circ \mathbf{F}^\infty_{W_1,\mathfrak{t}|_{W_1}}$. This proves that $F^\infty_{W,\mathfrak{t}}$ is non-zero.

We now prove the converse. Fix a basic system $\mathcal{H}$ for L. Suppose that μ_Y is not identically 0 (mod 2). We therefore have that $\mu_{S^3_{\mathbf{0}}(L')}$ is non-vanishing. Let's choose three components, K_{i_1}, K_{i_2}, and K_{i_3}, of L' which have $\mu_{S^3_{\mathbf{0}}(L')}(x^{i_1} \wedge x^{i_2} \wedge x^{i_3}) \neq 0$. We study the ε-spectral sequence for the link surgery formula on $S^3_{\mathbf{0}}(L)$ restricted to the torsion Spin^c structure. In particular, we look at the E_3 page. Let $\varepsilon^\star$ be the element of $\{0,1\}^k$ with $\varepsilon_{i_1} = \varepsilon_{i_2} = \varepsilon_{i_3} = 0$ and all other $\varepsilon_i = 1$. We have that $d_3|_{\mathcal{C}^{\varepsilon^*}}$ is non-zero, because the corresponding Milnor triple linking number is non-zero (this follows from the proof of Proposition 8.2.4). In particular, the bottom level of the ε-filtration, $\mathbf{1_0}$, does not survive to the E_4 page. Thus, the inclusion of $\mathbf{1_0}$ into $\mathcal{C}^\infty(\mathcal{H},\Lambda_0,\mathbf{0})$ induces the 0-map on the respective E_4 pages.

Let's study a similar map: the map on homology induced by this inclusion, which goes from $H_*(\mathbf{1_0})$ into $H_*(\mathcal{C}^\infty(\mathcal{H}|_{L'},\Lambda_0))$. By Theorem 11.1 of [MO10], this corresponds to the induced map $\mathbf{F}^\infty_{W_1,\mathfrak{t}|_{W_1}} : \mathbf{HF}^\infty(S^3) \to \mathbf{HF}^\infty(S^3_{\mathbf{0}}(L'))$ ($\mathfrak{t}|_{W_1}$ is the unique Spin^c structure on W_1 which restricts to be torsion on $S^3_{\mathbf{0}}(L')$). Since the proof of Theorem 1 showed that $E_4 \cong E_\infty$ for the ε-spectral sequence, we have that the inclusion of $\mathbf{1_0}$ induces 0 on the E_∞ page of the spectral sequence. Therefore, the induced map on homology must also be 0 by Fact 3.2.3.

Again, applying the gluing theorem for W_1 and W_2 shows that the map $\mathbf{F}^\infty_{W,\mathfrak{t}}$ is zero. $\square$

CHAPTER 10

Concluding Remarks

Theorem 1.3.2 can easily be extended to a larger class of cobordisms, including the presence of certain 1- and 3-handles. It seems likely that with some more work, *all* of the maps on HF^∞ coming from cobordisms with Spin^c structures that restrict to be torsion on the boundary could be computed in terms of basic topological information. Modulo the absolute $\mathbb{Q}$-gradings, this would give a complete understanding of the $\mathbf{HF}^\infty$ package.

Proposition 9.3 of [OS03a] allows one to relate the action of $\Lambda^*(H^1(Y;\mathbb{Z}))$ on $HF^\infty(Y,\mathfrak{s})$ to certain cobordism maps. Thus, combining Theorem 1 with Theorem 1.3.2, one can compute part (if not all) of the action of $\Lambda^*(H^1(Y;\mathbb{Z}))$ on $HF^\infty(Y,\mathfrak{s})$.

Finally, the most obvious question to ask is whether these calculations can be extended to $\mathbb{Z}$-coefficients. The major obstruction to this is proving a version of the link surgery formula over $\mathbb{Z}$, which means being able to understand the orientations of the relevant moduli spaces of holomorphic polygons; this would also require working out signs for the homological algebra used in hypercubes of chain complexes, especially for compression. Currently, these are both difficult problems.

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