Title
Dualities and Topological Field Theories from Twisted Geometries

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Publication Date
2012

Peer reviewed|Thesis/dissertation

Dualities and Topological Field Theories from Twisted Geometries

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

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Spring 2012

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Abstract

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I will present three studies of string theory on twisted geometries.

In the first calculation included in this dissertation we use gauge/gravity duality to study the Coulomb branch of an unusual type of nonlocal field theory, called Puff Field Theory. On the gravity side, this theory is given in terms of D3-branes in type IIB string theory with a geometric twist. While the field theory description, available in the IR limit, is a deformation of Yang-Mills gauge theory by an order seven operator which we here compute.

In the rest of this disertation we explore $\mathcal{N}=4$ super Yang-Mills (SYM) theory compactied on a circle with S-duality and R-symmetry twists that preserve $\mathcal{N}=6$ supersymmetry in $2+1D$. It was shown that abelian theory on a flat manifold gives Chern-Simons theory in the low-energy limit and here we are interested in the non-abelian counterpart. To that end, we introduce external static supersymmetric quark and anti-quark sources into the theory and calculate the Witten Index of the resulting Hilbert space of ground states on a two-torus. Using these results we compute the action of simple Wilson loops on the Hilbert space of ground states without sources. In some cases we find disagreement between our results for the Wilson loop eigenvalues and previous conjectures about a connection with Chern-Simons theory.

The last result discussed in this dissertation demonstrates a connection between gravitational Chern-Simons theory and $\mathcal{N}=4$ four-dimensional SYM theory compactified on a circle twisted by S-duality where the remaining three-manifold is not flat starting with the explicit geometric realization of S-duality in terms of (2, 0) theory.

To the universe,

for being so wonderful,

and twisted.

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Acknowledgments

I want to thank my advisor, Ori Ganor, for being very kind and supportive throughout our work together. And especially for taking me on, as his student, in a difficult moment in my life. Only we truly know I would not be receiving my PhD if it were not for him. It was so wonderful to work with someone so excited about our research and so patient to explain things over and over again. Ori, thank you, I have learned so much from you.

I would also like to thank all my other wonderful teachers. I am very lucky to have had more amazing teachers than I could fit the names of on this page, so let me spare you such a list. I will remember you and all the things you taught me as long as I possibly can. Knowledge is what I value the most in life — thank you for teaching me.

I am very grateful to all my collaborators on the projects that made it into this dissertation, in part or in full. I am extending my most sincere thanks to Yoon Pyo Hong, Hai Siong Tan, Shannon McCurdy and Nesty Torres Chicon — working with you was so much more satisfying than working alone, thank you.

I want to thank all my close friends. Thank you for accompanying me in the happiness and in the sadness. Thank you for being there. I hope you know how much you mean to me.

I want to thank all the people I dated and without whose love I would have felt so lost during gradschool. I am so happy to have shared this life with you.

I want to thank all my roommates while apologizing for how difficult it was to dig me out from under my books at times. I know I was not always a fun person to live with so thank you for sticking with me.

I also want to thank the Langton Labs community for providing me with another home whenever I needed it and with a shop for all the projects that my Space Ship would not contain. Your science minded people were my other family during gradschool and I am so grateful for having them.

And last, though certainly not least, I want to thank my real family, well, mostly for giving me space when I wanted to work and for having a healthy sense of humor about this physics I do and don't know how to tell them about.

Part I

Insights from Geometric Twists

Chapter 1

An Introduction to Twists

This thesis will present three studies of string theory on twisted geometries that can be constructed as orbifolds, yielding new and interesting results. Thus I introduce this concept first, as an attempt of setting a common theme for my dissertation.

Starting with a manifold M with a discrete symmetry group G we can construct an orbifold M/G via modding out by this group [2, 3, 4]. This means that each point $x \in M$ is identified with its orbit under the action of the group (we think of x and $y = gx$ as the same point for every $g \in G$, as well as their tangent spaces and so on). If M has no fixed points under the action of G, the quotient space M/G is a smooth manifold. On the other hand, fixed points result in orbifold singularities rendering M/G not a manifold. In certain cases it is known how to repair the singularities by removing these points and gluing in manifolds with the appropriate boundary. This is called blowing up and resolving the singularities.

For a simple example, one can consider the real line, $\mathbb{R}$, with coordinate x. It has an infinite discrete symmetry translating $x \to x + 2\pi R$ with no fixed points so that $\mathbb{R}/\mathbb{Z}$ is a smooth manifold — the circle. There is also a discrete reflection symmetry $x \to -x$ of the real line. This symmetry, however, has a fixed point $x = 0$ so that the orbifold $\mathbb{R}/\mathbb{Z}_2$ is the half line starting at an orbifold singularity.

Many mathematical properties follow from the definition of the orbifold. For example, the fundamental group of an orbifold is G/F where F is the subgroup generated by elements that have a fixed point.

Orbifolds are interesting in the context of string theory, since they provide a space of string compactifications that are richer than tori but still have exactly solvable sigma models [5, 6]. We can construct a string theory on the orbifold M/G by gauging a symmetry group G. To do this we must know the action of G both on the space-time and on the gauge degrees of freedom. The symmetry group, in fact, does not have to have a geometric interpretation but it can act only on the internal degrees of freedom.

The orbifold construction indicates that we should keep the states and operators invariant under the symmetry group. But that is not the whole story for a closed string theory. A string theory on an orbifold also includes additional string states with periodicity condition twisted by G. This means that for an element g of G we include a twisted sector of strings closed only up to the action of g. Thus the procedure of modding out by G both reduces the number of states because the states must be invariant under G, but it also increases the number of states because of these novel twisted sectors (which correspond to conjugacy classes of G).

Such twisted sectors would not appear in the quantum mechanics of a point particle on an orbifold which is well described only by G-invariant wave functions. But twisted closed string states are allowed (they also must be projected onto a G-invariant subspace) and are actually required for modular invariance.

Strings propagate smoothly over the orbifold singularities of classical geometry. But the strings in twisted sectors are localized at the fixed points as the aperiodicity forbids any center of mass coordinate or momentum. If the fixed points of G are isolated, expansion about strings sitting at different fixed points leads to disjoint sectors in the string propagation. States in the twisted sectors have another interesting feature, we will return to later — their quanta of momentum are commonly non-integral.

The orbifold construction results in breaking of the supersymmetries and gauge symmetries that do not commute with G which is a phenomenologically interesting consequence. Also, considering a theory with a tachyon (a state with a negative mass) on an orbifold sometimes results in a tachyon-free spectrum.

Chapter 2

Examples

Let me start by presenting two examples of simple geometric twists and mentioning some of the properties of the theories on these geometries. The examples will introduce concepts and ideas that the rest of my thesis describes. This chapter contains the background needed for understanding these geometric constructions and their importance for goals of this dissertation. The organization of the dissertation content is given in the following Chapter 3.

The first geometry discussed here is called a Melvin twist and the resulting Melvin Universe is an exact solution of Einstein gravity coupled with gauge fields [7]. It describes a consistent gravitational backreaction to a uniform magnetic field. To construct a Melvin twisted theory of interest, on the gravity side, we start by embedding this twisted geometry into string theory and adding D3-branes to the Melvin background. The gauge theory dual to this setup is Puff Field Theory (PFT). In the dual gravity description of PFT in addition to the usual Ramond-Ramond flux along the D3-branes, there is a strong five-form flux. This RR-field is responsible for Lorentz violation and non-locality in PFT. The preserved symmetry in this case includes spacial rotations and thus makes PFT of particular phenomenological interest.

Another twist this dissertation will focus on is S-duality of four-dimensional super Yang-Mills (SYM) gauge theory in a geometric realization which is available at a self-dual value of gauge coupling constant. As S-duality is then a symmetry of the theory we can compactify the four-manifold on a circle and require that the state of our system after making one loop around this circle is S-dual to its initial configuration. Shrinking this circle to zero size leaves us with a three-dimensional theory that has the S-duality operator at every point and enables us to study its effects. Topological Chern-Simons theory arises as the low-energy limit of the gauge theory on this geometry. And if the leftover three-manifold is curved we also find the gravitational Chern-Simons contribution.

2.1 Melvin twist

The simplest example of the Melvin twist is in three dimensions. This geometry can be thought of as the discrete orbifold $(\mathbb{C} \times \mathbb{R})/\mathbb{Z}$ where, for a $(z, \zeta) \in \mathbb{C} \times \mathbb{R}$, the identification generated by element of $\mathbb{Z}$ is given by

$$\begin{aligned} z &\rightarrow e^{i\theta} z, \\ \zeta &\rightarrow \zeta + 2\pi R. \end{aligned} \tag{2.1}$$

The resulting manifold is smooth, with no singularities, as $\mathbb{Z}$ is freely acting.

Although construction (2.1) does not have a fixed point one can think of the origin of the z-plane as the the deformation near which the twisted states of the orbifold are localized. To see this, consider a low-energy state of the w-twisted sector that winds w times as in part b) of Figure 2.1. The endpoints of such a state are separated by an angle $w\theta$ in the z-plane and a classical string placed a distance r from its origin has energy

$$\alpha'^2 M(r)^2 = (Rw)^2 + \left(2rw \sin\left(\frac{\theta}{2}\right)\right)^2 . \tag{2.2}$$

The wave-functions of such winding states falls off exponentially as $r \rightarrow \infty$ and thus strings in this geometry are localized near the origin of the z-plane.

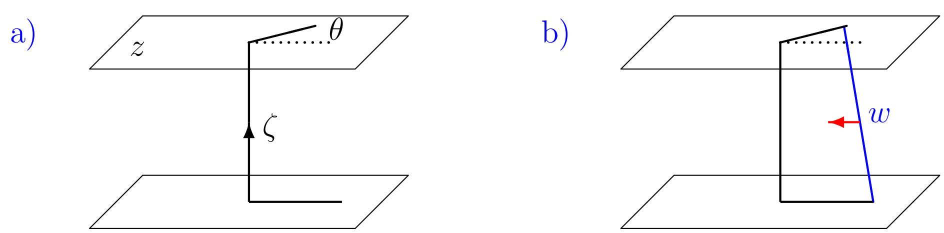


Figure 2.1: Melvin twist in 3 dimensions. a) The construction introduces a rotation in the z-plane by an angle θ for every loop completed in the compact ζ direction. b) A twisted string state in this geometry feels a force toward the origin of the z-plane.

The metric describing this twist in three dimensions can be written in cylindrical coordinates (where z-plane is parametrized by (r, ϕ)) as

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\phi + \frac{\theta}{2\pi R} d\zeta\right)^2 + d\zeta^2 . \tag{2.3}$$

It is obvious that for $\theta = 0$, this twisted space reduces to flat $3+1$ dimensions. As well as that there is a natural string theory realization of this Melvin twist, simply, by embedding metric (2.3) into supergravity. Concretely, let us consider type IIA string theory on this geometry. Then T-dualizing along the ζ-direction results in type IIB theory on a space-time with a background of magnetic fields and scalars

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + \frac{1}{1+\left(\frac{\theta r}{2\pi R}\right)^2}(r^2 d\phi^2 + d\zeta^2) \\ B &= \frac{\frac{\theta r}{2\pi R} r^2}{1+\left(\frac{\theta r}{2\pi R}\right)^2} d\phi \wedge d\zeta \\ e^\Phi &= \sqrt{\frac{1}{1+\left(\frac{\theta r}{2\pi R}\right)^2}}. \end{aligned} \tag{2.4}$$

This is a Melvin Universe supported by the flux of the NS-NS B-field. Its global geometry is that of a teardrop. Resulting background is special from the point of view of string theory, because its world-sheet sigma model is exactly solvable [8]. This is a simple consequence of it being dual to a flat space with periodic identifications.

This simple Malvin solution in particular is not supersymmetric. However, it is possible to repeat this twist in more than one plane, obtaining a more general Melvin configuration, in such a way that some fraction of supersymmetry is preserved. Further details about more general Melvin twists that preserve some supersymmetry will be discussed in Part II of this dissertation.

In general, Melvin twists, also called TsT transformations, are constructions that rely on an a geometric $U(1)\times U(1)$ isometry of a theory [9]. (In the three-dimensional example above these isometries were the $U(1)$ along the compact ζ-direction and a $U(1)$ for rotation in the z-plane.) A TsT transformation consists of a T-duality, followed by an $\mathrm{SL}(2,\mathbb{Z})$ transformation of the complex structure of the dual (a shift) and then another T-duality. The twist could thus be though of as an $\mathrm{SL}(2,\mathbb{Z})$ transformation of the Kähler structure of the original theory.

In string theory, applying Melvin twists to Dp-brane backgrounds gives rise to different field theories depending on the direction of the brane. If both of the $U(1)$ isometries are along the D-brane, one gets a noncommutative field theory [10, 11, 12]. Taking only one $U(1)$ transverse to the brane, the result is a dipole field theory [13, 14, 15]. While if both $U(1)$'s are chosen to be transverse to the brane, we have the Lunin and Maldacena construction [16] also giving the Puff Field Theory [17, 18] which is studied in Part II.

Although these theories were described in terms of metrics on twisted geometries they have another formulation obtained from gauge/gravity duality. On the field theory side, these models can be constructed instead by starting with a Lagrangian and adding a Lorentz violating term which is an IR-irrelevant local operator so that the low-energy behavior is unaffected. The question that then arises is if the UV-completion exists. Without other corrections a term of conformal dimension $\Delta > 4$ would result in a theory that is not UV-complete. However, the above examples are of UV-complete theories with a local Lorentz violating deformations of super Yang-

Mills (SYM) theory. There, UV-completeness is maintained because, in addition to the leading deformation operator, the Lagrangian has an infinite number of nonrenormalizable local terms, which sum up to renormalizable nonlocal interactions.

For SYM on a space of noncommutative geometry the low-energy deformation operator is a two-form of conformal dimension $\Delta = 6$, which breaks the Lorentz group to $SO(2) \times SO(1,1)$. In the example of the dipole theory, this operator is a spacetime vector of dimension $\Delta = 5$. It breaks the Lorentz group to $SO(2,1)$. Puff Field theory, in the IR limit, can be approximated by $\mathcal{N} = 4$ SYM deformed by a dimension $\Delta = 7$ operator. This construction leaves unbroken the spatial subgroup of the Lorentz group $SO(3) \subset SO(1,3)$ and thus is of phenomenological interest for constructing field theory in agreement with the Friedmann-Lematre-Robertson-Walker type cosmology which, on large scale, breaks Lorentz invariance to rotational invariance.

2.2 S-Duality twist

A duality is a mathematical equivalence of two theoretical descriptions of a physical systems which, on the first sight, seem different. Such an equivalence often arises when a single quantum theory has distinct classical limits. Probably the most famous example is that of the particle-wave duality. This is simply the fact that quantum field theory has one limit described by classical field theory and another described by classical particle mechanics. Another well known example is Kramers-Wannier duality of the two dimensional square lattice Ising model at a high-temperature with that at a low-temperature [19].

S-duality is a duality under which the coupling constant of a quantum theory changes nontrivially. Important examples include the $\mathrm{SL}(2,\mathbb{Z})$ self-dualities of IIB string theory and of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory which, in a simplified sense, are just the electric-magnetic duality.

Indeed, S-duality is a generalization of the 19^{th} century electric-magnetic symmetry which can be easily seen from the vacuum Maxwell equations

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \qquad && \vec{\nabla} \cdot \vec{E} = 0, \\ \vec{\nabla} \times \vec{B} &= \frac{\partial \vec{E}}{\partial t}, \qquad && \vec{\nabla} \cdot \vec{B} = 0. \end{aligned} \tag{2.5}$$

These equations have an invariance under $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ while the structure constant $\alpha = e^2/4\pi\hbar c$ transforms as $\alpha \to 1/\alpha$. Alike historic electric-magnetic duality which exchanges electric charges and magnetic monopoles, in a higher dimensional space-time, S-duality automatically exchanges the electric and magnetic eigenbranes.

In 1976, [20] showed that in gauge theories electric charge takes values in the weight lattice of the gauge group, while magnetic charge takes values in the weight lattice of a dual group. This result motivated the Montonen-Olive electric-magnetic duality conjecture [21] according to which a specific gauge theory based on a given gauge group is equivalent to a similar theory with the coupling constant inverted and the gauge group replaced by its dual. The duality is even more natural in a supersymmetric theory [22] and, once θ-angle is included, this $\mathbb{Z}_2$ symmetry can be extended to a full $\mathrm{SL}(2,\mathbb{Z})$ transformation [23, 24, 25] as described bellow.

It is simple to see S-duality arising in abelian gauge theory on a four-manifold M_4 with one-form gauge field A, being the connection on the $U(1)$ bundle, so that the curvature is $F = dA$. This two form can be written in terms of electric and magnetic fields as $F = dt \wedge d\vec{x} \cdot \vec{E} + \frac{1}{2} d\vec{x} \cdot d\vec{x} \times \vec{B}$ whence Maxwell's vacuum equations read $dF = *dF = 0$.

Starting with the Minkowski space action

$$S = -\frac{1}{4g_{\mathrm{YM}}^2} \int_{M_4} F \wedge *F \tag{2.6}$$

we can perform a Legendre transformation with respect to F and implement Bianchi identity $dF = 0$ while introducing a dual connection A_D:

$$S = -\frac{1}{4g_{\mathrm{YM}}^2} \int_{M_4} F \wedge *F + \frac{1}{2} A_D \wedge dF. \tag{2.7}$$

Integrating F out or, equivalently, solving the equation of motion for F gives $dA_D = 1/g_{\mathrm{YM}}^2 * F \equiv F_D$ and the action rewritten in terms of this dual field strength

$$S_D = -\frac{g_{\mathrm{YM}}^2}{4} \int_{M_4} F_D \wedge *F_D \tag{2.8}$$

has inverse coupling constant. This argument shows the $\mathbb{Z}_2$ summery of abelian gauge theory taking $g_{\mathrm{YM}} \to -1/g_{\mathrm{YM}}$.

Now, let us assume M_4 to be a compact and oriented spin manifold, as eventually we will want to include fermions. The third homotopy group

$$\frac{1}{(2\pi)^2} \int_{M_4} F \wedge F \tag{2.9}$$

is an even integer. Therefore, we see a classical symmetry of the four-dimensional abelian gauge theory after addition of θ-term to the action

$$S_\theta = \frac{\theta}{8\pi^2} \int_{M_4} F \wedge F \tag{2.10}$$

as e^{iS_θ} is invariant under $\theta \to \theta + 2\pi$.

The full Yang-Mills gauge theory action including the θ-term is

$$S = -\frac{1}{4g_{\mathrm{YM}}^2} \int_{M_4} F \wedge *F + \frac{i\theta}{8\pi^2} \int_{M_4} F \wedge F. \tag{2.11}$$

Where the θ-angle can be combined with the gauge coupling to form the full complex coupling

$$\tau \equiv \frac{4\pi i}{g_{\mathrm{YM}}^2} + \frac{\theta}{2\pi}\,. \tag{2.12}$$

Since θ-angle is periodic, $\tau \to \tau + 1$ is a symmetry of a combined theory as well as $\tau \to -1/\tau$. A generic element $g \in \mathrm{SL}(2,\mathbb{Z})$ acts on the complex coupling

$$g = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \qquad \tau \to \frac{a\tau + b}{c\tau + d} \tag{2.13}$$

by implementing both of these symmetries as follows

$$\begin{aligned} T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \theta &\to \theta + 2\pi \\ S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & g_{\mathrm{YM}} &\to -1/g_{\mathrm{YM}}\,. \end{aligned} \tag{2.14}$$

The elements S and T are a common choice of generators of $\mathrm{SL}(2,\mathbb{Z})$, where S is the one that inverts the coupling constant.

We can make Yang-Mills gauge theory supersymmetric by coupling it to massless spin-1/2 particles in the adjoin representation of the gauge group. The number of physical fermionic modes described by such a spinor field depends on the number of spacetime dimensions D and the type of spinor it is but it is always a power of two. However, the number of physical modes described by the gauge field A is $D-2$, corresponding to all the possible transverse polarizations. This shows that when $D = 3, 4, 6$ or 10 we can have the same number of physical modes carried by the spinor as by the gauge field which permits an extra symmetry between the gauge bosons and the fermions, called supersymmetry. My dissertation will mostly focus on $D = 4$ case where the largest possible number of supersymmetries is $\mathcal{N} = 4$.

The generators of supersymmetry transformations are fermionic and are called supercharges. For $\mathcal{N} = 4$ supersymmetry there are 16 separate supercharges, $Q_{\alpha a}$ and $\bar{Q}^a_{\dot\alpha}$, where $\alpha, \dot\alpha = 1, 2$ and $a = 1, \ldots,\ 4$ are the spinor and R-symmetry indices, respectively. The R-symmetry in this case is a global $SU(4) \simeq SO(6)$ that transforms supercharges, as well as the fields of this theory, into each other. It commutes with the gauge symmetry.

We can learn new facts about the $\mathrm{SL}(2,\mathbb{Z})$ S-duality of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions by studying a circle compactification of the theory with unconventional boundary conditions as follows. Realizing the circle as the segment $[0, 2\pi R]$ with endpoints 0 and $2\pi R$ identied, we require the configuration at $2\pi R$ to be an S-dual of that at 0. We will refer to this kind of boundary conditions as an S-twist and study the effect it has on the action

$$S = \int_{M_3 \times [0, 2\pi R)} \mathcal{L}_{SYM} d^4x + \int_{M_3} A(0) \wedge dA(2\pi R). \tag{2.15}$$

After taking the $R \to 0$ limit we are left with a three-manifold that has the S-twist at every point. We call the resulting low-energy theory on M_3 Tr-S.

The S-twist is only an option if it is a symmetry of the theory which happens when the coupling, τ, is self-dual. This occurs at the values of τ for which there exists an element $s \in \mathrm{SL}(2,\mathbb{Z})$, other than the identity I, or $-I$, that leaves τ invariant. A self-dual theory can be compactified on an S^1 with an s-twist. The $2+1D$ low-energy limit of this setting has been studied [26, 1] and it was shown that abelian SYM on a flat $M_3 = T^2 \times \mathbb{R}$ gives Chern-Simons theory. The self-dual coupling constant $\tau = i$ is invariant a $\mathbb{Z}_4 \subset \mathrm{SL}(2,\mathbb{Z})$ including $S : \tau \to -1/\tau$ and leads to Chern-Simons at level $k = 2$. While for $\tau = e^{\frac{\pi i}{3}}$ which is invariant under a $\mathbb{Z}_6 \subset \mathrm{SL}(2,\mathbb{Z})$ we can get levels $k = 1,\ 3$.

There is a more geometric realization of the S-twist that I would like to mention now and elaborate on in Part IV of my thesis. Six-dimensional $(2,\ 0)$ theory of the worldvolume of M5-branes is probably the most natural setting for defining this twist. $\mathcal{N} = 4$ super Yang-Mills theory with coupling constant τ is the low-energy limit of $(2,\ 0)$ theory compactified on a two-torus, with τ being the complex structure parameter of the torus, where S-duality can be realized as the $SL(2,\mathbb{Z})$ group of the diffeomorphisms of the T^2.

Now, Tr-S is viewed as the effective low-energy description of the $(2,\ 0)$ theory formulated on $M_3 \times [(\mathbb{C} \times S^1)/\mathbb{Z}_q]$. Here $(\mathbb{C} \times S^1)/\mathbb{Z}_q$ is the discrete orbifold of $\mathbb{C} \times S^1$, parameterized by (z, ζ) with $0 \le \zeta < 2\pi R q$, by $\mathbb{Z}_q$ that is generated by the freely acting $(z, \zeta) \to (e^{2\pi i q} z, \zeta + 2\pi R)$. Where $q = 4$ corresponds to $\tau = i$ while $q = 6$ for $\tau = e^{\frac{\pi i}{3}}$, as above. It is also obvious that we can add an R-symmetry transformation of order q to this $\mathbb{Z}_q$ action. In this description, Tr-S theory is reminiscent of the Melvin twist background described in §2.1.

Chapter 3

Organization

My dissertation will elaborate on these two examples of geometric twists while presenting details of the calculations I worked on. The rest of the document is organized as follows. Part II focuses on the particular Melvin twist that results in Puff Field Theory (PFT) as sketched in §2.1. I will explain further details of this geometry while attempting to explore the Coulomb branch of this field theory. The calculation presented is a project I did with Ori J. Ganor and Shannon McCurdy.

Part III gives a long work on putting static charges into Tr-S theory setup as described in §2.2 which was done with Ori J. Ganor, Yoon Pyo Hong and Hai Siong Tan. This part of my thesis can also be found in [27]. We are interested in the nonabelian Tr-S theory and, in particular, want to show that this theory has no propagating degrees of freedom, and thus is topological, by studying its Witten Index. It is worth note that there are other possible tests of our conjecture that have not been explored here. For example, one could study its Bogomol'nyi-Prasad-Sommerfield (BPS) states — massive representations of the supersymmetry algebra. While momentum is carried by BPS states, if this theory is topological there should not be any propagating degrees of freedom, so one could look for BPS states that carry nonzero momentum along the two-torus the nonabelian SYM is compactified on to disprove our conjecture.

A topological quantum field theory (TQFT) computes topological invariants — properties invariant under homeomorphisms. Correlation functions of a TQFT do not depend on the metric of spacetime. We distinguish two basic kinds of TQFTs. Witten-type, or cohomological, TQFTs have explicit metric dependence in the Lagrangian but the partition function and some correlation functions can me shown to be diffeomorphism invariant. One example of this can be found in the Wess-Zumino-Witten (WZW) model [28, 29].

On the other hand, in a Schwartz-type TQFT the path integral measure as well as all the observables are explicitly independent of the metric. In, particular, TQFT found when studying abelian $\mathcal{N} = 4$ SYM on $M_3 \times S^1$ where the circle has an S-duality twist is Chern-Simons theory on M_3. Chern-Simons theory is a Schwartz-type TQFT and can be specified by a choice of a gauge group and a level k. Although Chern-Simons action is gauge dependent the partition function is not if $k \in \mathbb{Z}$ and gauge field strength vanishes on ∂M_3. The expectation values of Chern-Simons observables are related to knot invariants [30] and can be calculated as Wilson lines at a fixed point on the circle in Tr-S theory.

Even though we are unsure if nonabelian Tr-S theory is topological, its abelian counterpart is understood better [1]. Compactification of $U(1)$ SYM on S^1 with boundary conditions that are twisted by an S-duality (together with an R-symmetry twist to eliminate zero modes and preserve supersymmetry), leads to a $2+1D$ low-energy theory that is pure Chern-Simons theory at a level k determined by the SYM coupling constant τ. However, we know that Chern-Simons theory on a manifold with curvature is not quite invariant but requires a gravitational Chern-Simons counterterm [30]. Also, [31] shows that the partition function of Chern-Simons theory is not a modular invariant but transforms as a modular form with weight quadratic in curvature. These results suggest that an S-duality twist should be accompanied with a gravitational Chern-Simons term.

In Part IV, I compare low-energy gravitational Chern-Simons theory with a one-loop amplitude calculation in $(2,\ 0)$ theory compactified on a two-torus to get $\mathcal{N}=4$ SYM gauge theory, and then again on a circle with an S-duality twist inserted. This calculation was done in collaboration with Ori J. Ganor and Nesty R. Torres-Chicon.

Part II

A Study of the Coulomb branch of Puff Field theory in a Melvin Universe

Chapter 4

Setup

Puff Field Theory (PFT) arises from an embedding of N D-branes in a particular Melvin Universe, and is an example of a non-local, Lorentz-violating field theory in the decoupling limit. PFT is a gauge theory dual to a string theory with D3-branes in the Melvin background with strong Ramond-Ramond fields.

On the gravity side, PFT can be constructed by lifting type IIA supergravity with N D0-branes to M-theory and introducing Melvin twists in three transverse planes before reducing back to type IIA theory. This gives D0-branes in a Melvin universe [7] supported by an Ramond-Ramond two-form flux. Then T-dualizing along three untwisted directions results in a simple supergravity dual to PFT: N D3-branes in type IIB theory supported by an RR five-form field strength background.

Our particular starting configuration has N D0-branes in type IIA string theory compactified on a three-torus T^3 ($S^1 \times S^1 \times S^1$) in directions x, y and z. Let the leftover $\mathbb{R}^6$ directions be parametrized as three (r_i, ϕ_i) planes with $i = 1, \ldots, 3$. We proceed by lifting this setup up to eleven dimensions to get a Kaluza-Klein particle with N units of momentum along the new compact ζ-direction in flat M-theory with $\zeta \sim \zeta + 2\pi R$. The metric is parametrized as follows

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \sum_{i=1}^{3} \left(dr_i^2 + r_i^2 d\phi_i^2 \right) + d\zeta^2.$$

We then twist in each of the three (r_i, ϕ_i) planes with deformation parameters β_i

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \sum_{i=1}^{3} \left(dr_i^2 + r_i^2 (d\phi_i + \beta_i d\zeta)^2 \right) + d\zeta^2.$$

This is equivalent to doing three separate twists as in the simple three-dimensional example in §2.1.

After reducing on the M-theory ζ-circle back to type IIA, we perform a TsT twist. In this case, the twist is T-duality in direction x, followed by an S-duality, followed by

T-dualities in directions y and z, which results in a system of N coincident D3-branes embedded in a Melvin universe with RR four-form potential in the background of type IIB theory.

This IIB supergravity dual solution the has metric

$$ds^2 = \frac{R^2}{r^2}\left(K^{-\frac{1}{2}}\left[dx^2 + dy^2 + dz^2 - \left(dt - \frac{4\pi N}{r^2}\beta\psi\right)^2\right] + K^{\frac{1}{2}} ds^2_{\mathbb{R}^6}\right) \tag{4.1}$$

and the RR four-form potential

$$C_4' = \frac{\pi N}{r^4}K^{-1}\left[dt \wedge dx \wedge dy \wedge dz - \frac{\beta}{r^2}\psi \wedge dx \wedge dy \wedge dz\right] \tag{4.2}$$

with $K = 1 + \left(\frac{4\pi N}{r^2}\right)^2 \frac{\beta^2}{r^2}$ and constant $R^4 = 4\pi g_s N\alpha'^2$, while $g_{\mathrm{IIB}} = 2\pi g^2_{\mathrm{YM}}$. The appropriate decoupling limit was taken for the parameters β, N and R as one sends α' to zero in order to decouple the gauge theory on the D-branes from the rest of the string theory modes.

Here x, y, and z are static coordinates on N coincident D3-branes while six transverse coordinates of $\mathbb{R}^6$ can be chosen to be radial direction r and coordinates on S^5 so that $ds^2_{\mathbb{R}^6} = dr^2 + r^2 d\Omega_5^2$. If we think of the five-sphere as the principal $U(1)$-bundle, or a Hopf fibration of S^1, over the base $\mathbb{C}P^2$, we can write the metric $d\Omega_5^2 = ds^2_{\mathbb{C}P^2} + \psi^2$ where ψ is a one-form such that $d\psi$ is a representative of the second de-Rham cohomology $H^2(\mathbb{C}P^2, \mathbb{R})$. Considering the transverse directions as a Hopf fibration is convenient as in these coordinates the Melvin twist is along the direction of the fiber S^1.

It will also be useful to parametrize the S^5 using six angular coordinates which we will denote $\vec{n}$ with components n_I where $I = 1, \ldots,\ 6$. Locally, in these angular coordinates, one-form ψ can be written as $\beta\psi = \vec{n}^T\varepsilon d\vec{n}$. Rewriting the metric and the RR four-form in the angular coordinates

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2}\left(K^{\frac{1}{2}}\left[dx^2 + dy^2 + dz^2 - \left(dt - \frac{4\pi N}{r^2}\vec{n}^T\varepsilon d\vec{n}\right)^2\right] + K^{-\frac{1}{2}} ds^2_{\mathbb{R}^6}\right) \\ C_4' &= \frac{\pi N}{r^4}K^{-1}\left[dt \wedge dx \wedge dy \wedge dz - \frac{1}{r^2}\vec{n}^T\varepsilon d\vec{n} \wedge dx \wedge dy \wedge dz\right] \end{aligned} \tag{4.3}$$

where $|\vec{n}|^2 = \sum n_I^2 = 1$ and $d\Omega_5^2 = \sum dn_I^2$.

The Melvin twist that we modded out by is a simultaneous translation in the z-direction and a rotation of the S^5, where the rotation is determined by the constant matrix $\varepsilon \in so(6)$. In this calculation we have chosen the background where ε is the following:

$$\varepsilon \;=\; \begin{pmatrix} 0 & \beta_1 & 0 & 0 & 0 & 0 \\ -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 & 0 \\ 0 & 0 & -\beta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_3 \\ 0 & 0 & 0 & 0 & -\beta_3 & 0 \end{pmatrix}; \qquad \beta = \beta_1 = \beta_2 = \beta_3. \tag{4.4}$$

Although described in terms of gravity above, on the gauge theory side this is a deformation of $\mathcal{N} = 4$ SYM which, with our choice $\beta = \beta_1 = \beta_2 = \beta_3$, breaks all supersymmetry. However, it is possible to construct backgrounds that preserve some fraction of supersymmetry by performing different Melvin twists [17]. Choosing $\beta_1 + \beta_2 + \beta_3 = 0$ to stay in a $so(3)$ subgroup preserves $\mathcal{N} = 1$ supersymmetry. Additionally if we set, say, $\beta_3 = 0$ we are in a $so(2)$ invariant subgroup and our gauge theory has $\mathcal{N} = 2$ supersymmetry. Preserving some supersymmetry is important as a generic background can be unstable.

Chapter 5

Result

Puff Field theory, in the IR limit, can be approximated to the leading order by Yang-Mills gauge theory which is not supersymmetric with our choice of the Melvin twist parameters. We study the Coulomb branch of the decoupled field theory which is flat in this limit. It is deformed by an order seven operator [18] whose bosonic part can be deduced from the Dirac-Born-Infeld (DBI) action and the Wess-Zumino-Witten (WZW) term.

Let the coordinates on the full spacetime be indexed with $\mu,\ \nu$ ($\mu, \nu = t,\ \vec{x},\ r,\ \vec{n}$) while we use indices $\alpha,\ \beta$ for coordinates along the brane ($\alpha,\ \beta = t,\ \vec{x}$) and indices $m,\ n$ for the transverse directions. Let $G_{\alpha\beta}$ be the pull-back of the full space-time metric $g_{\mu\nu}$ to the D3-brane. If the gauge fields are A_α, with field strength $F = dA$, on the brane and scalars Φ are describing transverse motion of the brane the DBI action is

$$S_{DBI} = -T_3 \int_{M_4} dt d^3x \sqrt{-\det\left(G_{\alpha\beta} + B_{\alpha\beta} + kF_{\alpha\beta}\right)}. \tag{5.1}$$

While the WZW term is

$$S_{WZW} = \mu_3 \int_{M_4} (C \wedge e^{B+kF}). \tag{5.2}$$

Here C is the pullback of the total RR potential to the D3-brane worldvolume, while μ_3 and T_3 are the D3-brane charge and tension. Constant $k = 2\pi\alpha'$ and we set the NS-NS two-form $B = 0$.

Changing the coordinates on S^5 again, we rescale the six angular denoted $\vec{n}$ (with components n_I where $I = 1, \ldots,\ 6$, where $|\vec{n}|^2 = \sum n_I^2 = 1$ and $d\Omega_5^2 = \sum dn_I^2$). Take our new field $\vec{\phi}$ to be defined as

$$\begin{aligned} \phi^I &= \frac{1}{r} n^I \\ \partial_\alpha \phi^I &= -\frac{1}{r^2}\partial_\alpha r n^I + \frac{1}{r}\partial_\alpha n^I. \end{aligned}$$

Expanding the DBI action with the WZW term to the lowest order in the deformation parameter $\varepsilon \sim \beta$ and keeping up to three derivatives we get the following order seven operator as the leading contribution deforming the Yang-Mills theory with gauge fields and fermions set to zero

$$\frac{N^2\beta}{2\pi}\frac{\varepsilon_{IJ}\varepsilon_{KL}}{\beta^2}\phi^K\epsilon^{0\alpha\beta\gamma}\partial_\alpha\phi^I\partial_\beta\phi^J\partial_\gamma\phi^L.$$

Further details of this expansions can be found in the following Appendix.

Chapter 6

Appendix

6.1 DBI action

The leading order low-energy effective action for the dynamics of the massless excitations on a D-brane (with gauge field A_α supported on the brane and scalars Φ^m describing transverse motion of the brane) is given by the Dirac-Born-Infeld action

$$S_{DBI} = -T_3 \int dt d^3x \sqrt{-\det\left(G_{\alpha\beta} + B_{\alpha\beta} + kF_{\alpha\beta}\right)}$$

where $G_{\alpha\beta}$ is the pull-back of the space-time metric $g_{\mu\nu}$ with coordinates Φ^μ ($\mu,\ \nu = t,\ \vec{x},\ r,\ \vec{n}$) to the brane, $F_{\alpha\beta}$ is the field strength of the gauge field A_α on the brane, and $B_{\alpha\beta} = 0$ is the NS-NS two-form, while constant $k = 2\pi\alpha'$. Let us also set $F_{\alpha\beta} = 0$ for simplicity of current discussion and reintroduce it in §6.1.1. The induced metric on the brane can be written as

$$G_{\alpha\beta} = g_{\mu\nu}\partial_\alpha\Phi^\mu\partial_\beta\Phi^\nu$$

in static coordinates. We use indices ($\alpha,\ \beta = t,\ \vec{x}$) for coordinates on the D3-brane worldvolume, and ($m,\ n = r,\ \vec{n}$) for transverse directions, so that

$$G_{\alpha\beta} = h_{\alpha\beta} + g_{\alpha n}\partial_\beta\Phi^n + g_{m\beta}\partial_\alpha\Phi^m + g_{mn}\partial_\alpha\Phi^m\partial_\beta\Phi^n$$

Where the full space-time metric can be writing in matrix form with coordinates in the order $(t, \vec{x}, r, \vec{n})$, where for convenience we define $c_r = \frac{\sqrt{4\pi N}}{r}$

$$g_{\mu\nu} = \frac{R^2}{\sqrt{K}r^2}\begin{pmatrix} -1 & 0 & 0 & c_r^2\vec{n}^T\varepsilon \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & K & 0 \\ c_r^2\varepsilon^T\vec{n} & 0 & 0 & (Kr^2\mathbb{1} - c_r^4\varepsilon^T\vec{n}\vec{n}^T\varepsilon) \end{pmatrix}$$

and the rotation matrix

$$\varepsilon \;=\; \begin{pmatrix} 0 & \beta & 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & -\beta & 0 \end{pmatrix} \in so(6)$$

with the deformation parameter β.

To calculate the DBI action we need

$$h^\gamma_\beta = h^{\gamma\alpha} G_{\alpha\beta} = \delta^\gamma_\beta + h^{\gamma\alpha} g_{\alpha n}\partial_\beta \Phi^n + h^{\gamma\alpha} g_{m\beta}\partial_\alpha \Phi^m + h^{\gamma\alpha} g_{mn}\partial_\alpha \Phi^m \partial_\beta \Phi^n .$$

The pull-back metric can be written as $h^\gamma_\alpha = \delta^\gamma_\alpha + \mathbb{M}^\gamma_\alpha$, where with static coordinates in the order $(t, \vec{x})$ in the matrix form

$$\mathbb{M} = \begin{pmatrix} h^{tt}(g_{rr}\partial_t r \partial_t r + \partial_t \vec{n}^T g_{\vec{n}\vec{n}} \partial_t \vec{n} + 2\partial_t \vec{n}^T g_{\vec{n}t}) & h^{tt}(g_{t\vec{n}}\partial_i \vec{n} + g_{rr}\partial_t r \partial_i r + \partial_t \vec{n}^T g_{\vec{n}\vec{n}} \partial_i \vec{n}) \\ h^{ik}(\partial_k \vec{n}^T g_{\vec{n}\mathbf{t}} + g_{rr}\partial_k r \partial_t r + \partial_k \vec{n}^T g_{\vec{n}\vec{n}}\partial_t \vec{n}) & h^{ik}(g_{rr}\partial_k r \partial_j r + \partial_k \vec{n}^T g_{\vec{n}\vec{n}} \partial_j \vec{n}) \end{pmatrix}$$

and

$$h_{\alpha\beta} = \frac{R^2}{\sqrt{K} r^2} \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad h^{\alpha\beta} = \frac{\sqrt{K} r^2}{R^2} \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Therefore

$$I_{DBI} = -T_3 \int_{M_4} dt d^3x \sqrt{-\det(h_{\alpha\beta})}\sqrt{det(\mathbb{1} + \mathbb{M})}$$

where

$$\sqrt{-\det(h_{\alpha\beta})} = \frac{R^4}{K r^4}.$$

Since

$$\ln\det(\mathbb{1} + \mathbb{M}) \;=\; \operatorname{tr}\ln(\mathbb{1} + \mathbb{M}) = \operatorname{tr}(\mathbb{M}) - \frac{1}{2}\operatorname{tr}(\mathbb{M}^2) + \frac{1}{3}\operatorname{tr}(\mathbb{M}^3) + \ldots$$

we have

$$\begin{aligned} (\det(\mathbb{1} + \mathbb{M}))^{\frac{1}{2}} \;&=\; (e^{\operatorname{tr}(\mathbb{M}) - \frac{1}{2}\operatorname{tr}(\mathbb{M}^2) + \ldots})^{\frac{1}{2}} \\ &=\; 1 + \frac{1}{2}\operatorname{tr}(\mathbb{M}) - \frac{1}{4}\operatorname{tr}(\mathbb{M}^2) + \frac{1}{6}\operatorname{tr}(\mathbb{M}^3) + \frac{1}{8}(\operatorname{tr}(\mathbb{M}))^2 \\ &\quad -\frac{1}{8}\operatorname{tr}(\mathbb{M})(\operatorname{tr}(\mathbb{M}))^2 + \frac{1}{48}(\operatorname{tr}(\mathbb{M}))^3 + \ldots \end{aligned}$$

The DBI action, without the gauge fields, where as we expected to find an operator of order seven, we kept terms with up to three derivatives

$$\begin{aligned}\mathcal{L}_{DBI} &= -T_3\frac{R^4}{Kr^4}\Big(1 - c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n} \\ &+\frac{1}{2}K(-\partial_t r\partial_t r + \delta^{ij}\partial_i r\partial_j r) + \frac{1}{2}Kr^2(-\partial_t\vec{n}^T(\mathbb{1})\partial_t\vec{n} + \delta^{ij}\partial_i\vec{n}^T(\mathbb{1})\partial_j\vec{n}) \\ &+(-c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})\Big(\frac{1}{2}Kr^2(\partial_t\vec{n}^T)(\mathbb{1})(\partial_t\vec{n}) - \frac{1}{2}Kr^2\delta^{ik}(\partial_k\vec{n}^T)(\mathbb{1})(\partial_i\vec{n})\Big) \\ &+K(c_r^2\vec{n}^T\varepsilon\partial_i\vec{n})\delta^{ik}\partial_k r\partial_t r + \frac{1}{2}K(-c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})(\partial_t r\partial_t r + \delta^{kj}\partial_k r\partial_j r)) + \dots\Big)\end{aligned}$$

where the D3-brane tension, with conventions from [32], is

$$T_3 = \frac{1}{(2\pi)^3 g_s\alpha'^2} = \frac{N}{2\pi^2 R^4}. \tag{6.1}$$

6.1.1 Including Gauge Fields

Recall that the full DBI action is

$$I_{DBI} = -T_3\int dtd^3x\sqrt{-\det(G_{\alpha\beta} + kF_{\alpha\beta})}$$

where we so far have only calculated

$$S_{DBI} = -T_3\int dtd^3x\sqrt{-\det(G_{\alpha\beta})}$$

setting $F_{\alpha\beta} = 0$ in §6.1. Now we need to add

$$S_{DBI} = -T_3\int dtd^3x\sqrt{-\det(G_{\alpha\beta})}\sqrt{\det(\mathbb{1} + kG^{\beta\gamma}F_{\gamma\beta})}.$$

And we know that

$$\det(G_{\alpha\beta} + kF_{\alpha\beta}) = \det(G_{\alpha\beta} + kF_{\alpha\beta})^T = \det(G_{\alpha\beta} - kF_{\alpha\beta})$$

which is an even function of k. Using matrix notation and denoting

$$\mathbb{N} = kG^{-1}F$$

we have

$$\begin{aligned}\sqrt{\det(\mathbb{1} + \mathbb{N})} &= (\det(\mathbb{1} - \mathbb{N}^2))^{\frac{1}{4}} \\ &\approx 1 - \frac{1}{4}\operatorname{tr}(\mathbb{N}^2) - \frac{1}{8}\operatorname{tr}(\mathbb{N}^4) + \frac{1}{32}(\operatorname{tr}(\mathbb{N}^2))^2 + \dots\end{aligned}$$

$$\approx \quad 1+\frac{1}{4}k^2F_{\alpha\beta}G^{\alpha\delta}G^{\beta\gamma}F_{\delta\gamma}-\frac{1}{8}k^2(F_{\alpha\beta}G^{\alpha\delta}G^{\beta\gamma}F_{\delta\gamma})^2+\frac{1}{32}k^4+\dots$$

However, since we are keeping at most three derivatives for the leading order contribution, we actually only need:

$$\sqrt{\det(\mathbb{1}+\mathbb{N})} \quad \approx \quad 1+\frac{1}{4}k^2F_{\alpha\beta}G^{\alpha\delta}G^{\beta\gamma}F_{\delta\gamma}$$

where we only need $G^{\alpha\delta}$ up to single derivatives. This is

$$\begin{aligned}
G_{\alpha\beta} &= h_{\alpha\beta}+\begin{pmatrix} 2\partial_t\vec{n}^T g_{\vec{n}t} & g_{t\vec{n}}\partial_i\vec{n} \\ \partial_i\vec{n}^T g_{\vec{n}\mathbf{t}} & 0\end{pmatrix}+g_{mn}\partial_\alpha\Phi^m\partial_\beta\Phi^n \\
G^{\alpha\delta} &\approx h^{\alpha\beta}+(\frac{\sqrt{K}r^2}{R^2})^2\begin{pmatrix} -2\partial_t\vec{n}^T g_{\vec{n}t} & g_{t\vec{n}}\partial_i\vec{n} \\ \partial_i\vec{n}^T g_{\vec{n}\mathbf{t}} & 0\end{pmatrix}+\dots \\
&\approx h^{\alpha\beta}+(\frac{\sqrt{K}r^2}{R^2})\tilde{h}^{\alpha\beta}+\dots
\end{aligned}$$

and

$$\tilde{h}^{\alpha\beta} \quad = \quad (\frac{\sqrt{K}r^2}{R^2})\begin{pmatrix} -2\partial_t\vec{n}^T g_{\vec{n}t} & g_{t\vec{n}}\partial_i\vec{n} \\ \partial_i\vec{n}^T g_{\vec{n}\mathbf{t}} & 0\end{pmatrix}=\begin{pmatrix} -2c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n} & c_r^2\vec{n}^T\varepsilon\partial_i\vec{n} \\ c_r^2\partial_i\vec{n}^T\varepsilon^T\vec{n} & 0\end{pmatrix}.$$

Therefore

$$\begin{aligned}
\sqrt{\det(\mathbb{1}+\mathbb{N})} &\approx 1+\frac{1}{4}k^2F_{\alpha\beta}h^{\alpha\delta}h^{\beta\gamma}F_{\delta\gamma}+\frac{1}{2}k^2(\frac{\sqrt{K}r^2}{R^2})F_{\alpha\beta}h^{\alpha\delta}\tilde{h}^{\beta\gamma}F_{\delta\gamma} \\
&\approx 1+\frac{1}{4}k^2\frac{Kr^4}{R^4}F_{\alpha\beta}\eta^{\alpha\delta}\eta^{\beta\gamma}F_{\delta\gamma}+\frac{1}{2}k^2\frac{Kr^4}{R^4}F_{\alpha\beta}\eta^{\alpha\delta}\tilde{h}^{\beta\gamma}F_{\delta\gamma}.
\end{aligned}$$

And the DBI action, including contributions from gauge fields, is

$$\begin{aligned}
S_{DBI} &= -T_3\int dtd^3x\sqrt{-\det(G_{\alpha\beta})}\sqrt{\det(\mathbb{1}+\mathbb{N})} \\
&= -T_3\int dtd^3x\sqrt{-\det(h_{\alpha\beta})}\sqrt{\det(1+\mathbb{M})}(\det(\mathbb{1}-\mathbb{N}^2))^{\frac{1}{4}} \\
&= -T_3\int dtd^3x\frac{R^4}{r^4}\Big(1-\frac{1}{2}(1+c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})\partial_tr\partial_tr+\frac{1}{2}\delta^{ij}(1-c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})\partial_ir\partial_jr \\
&\quad +\frac{1}{2}r^2(-(1+c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})\partial_t\vec{n}^T(\mathbb{1})\partial_t\vec{n}+(1+c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})\delta^{ij}\partial_i\vec{n}^T(\mathbb{1})\partial_j\vec{n}) \\
&\quad -c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n} \\
&\quad +(c_r^2\vec{n}^T\varepsilon\partial_i\vec{n})\delta^{ik}\partial_kr\partial_tr \\
&\quad +\frac{1}{4}k^2\frac{r^4}{R^4}(1-c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})F_{\alpha\beta}\eta^{\alpha\delta}\eta^{\beta\gamma}F_{\delta\gamma}+\frac{1}{2}k^2\frac{r^4}{R^4}F_{\alpha\beta}\eta^{\alpha\delta}\tilde{h}^{\beta\gamma}F_{\delta\gamma}+\dots\Big).
\end{aligned}$$

6.2 WZW term

We also need to add the terms that come from the Ramond-Ramond field C_4' in the form of Wess-Zumino-Witten action which picks out the form of order four (as it is proportional to the worldvolume of D3-branes) from

$$S_{WZW} \;=\; \mu_3 \int_4 (C \wedge e^{B+kF}).$$

Here C is the total RR potential pulled back to the brane, μ_3 is D3-brane charge, and the NS-NS B-field is set to zero. Ignoring the correction due to the curvature tensor, which is of a higher order in derivatives

$$S_{WZW} \;=\; \mu_3 \int C_4'$$

as we only have a RR four-form potential. For us

$$C_4' = \frac{\pi N}{r^4} K^{-1} dt \wedge dx \wedge dy \wedge dz - \frac{4\pi N}{r^6} K^{-1} \vec{n}^T \varepsilon d\vec{n} \wedge dx \wedge dy \wedge dz$$

where

$$R^4 = 4\pi N g_s \alpha'^2.$$

This term in the Lagrangian is the pull-back of C_4' to the brane in static coordinates

$$\mathcal{L}_{WZW} \;=\; \mu_3 \frac{\pi N}{r^4 K}(1 - c_r^2 \vec{n}^T \varepsilon \partial_t \vec{n})$$

while from above

$$\mathcal{L}_{DBI} \;=\; -\frac{1}{2\pi^2}\frac{N}{K r^4}\Big(1 - c_r^2 \partial_t \vec{n}^T \varepsilon^T \vec{n} + \dots\Big).$$

Remembering that because $\partial_t(\vec{n}^T \varepsilon \vec{n}) = 0$

$$\vec{n}^T \varepsilon \partial_t \vec{n} = \partial_t \vec{n}^T \varepsilon^T \vec{n}$$

we see that these two contributions in DBI and WZW Lagrangians will cancel with D3-brane charge $\mu_3^{-1} = 2\pi^3$:

$$\begin{aligned}
\mathcal{L}_{DBI} + \mathcal{L}_{WZW} \;=\; & -T_3 \frac{R^4}{r^4}\Big(-\frac{1}{2}(1 + c_r^2 \partial_t \vec{n}^T \varepsilon^T \vec{n}) \partial_t r \partial_t r + \frac{1}{2}\delta^{ij}(1 - c_r^2 \partial_t \vec{n}^T \varepsilon^T \vec{n}) \partial_i r \partial_j r \\
& + \frac{1}{2} r^2 (-(1 + c_r^2 \partial_t \vec{n}^T \varepsilon^T \vec{n}) \partial_t \vec{n}^T (\mathbb{1}) \partial_t \vec{n} + (1 + c_r^2 \partial_t \vec{n}^T \varepsilon^T \vec{n}) \delta^{ij} \partial_i \vec{n}^T (\mathbb{1}) \partial_j \vec{n}) \\
& + (c_r^2 \vec{n}^T \varepsilon \partial_i \vec{n}) \delta^{ik} \partial_k r \partial_t r
\end{aligned}$$

$$+\frac{1}{4}k^2\frac{r^4}{R^4}(1-c_r^2\partial_t\vec{n}^T\varepsilon^T\vec{n})F_{\alpha\beta}\eta^{\alpha\delta}\eta^{\beta\gamma}F_{\delta\gamma}+\frac{1}{2}k^2\frac{r^4}{R^4}F_{\alpha\beta}\eta^{\alpha\delta}\tilde{h}^{\beta\gamma}F_{\delta\gamma}+\dots\Big)$$

However, this RR potential C_4' is not complete as it does not have a self-dual field strength $dC_4 = F_5 \neq *F_5$:

$$\begin{aligned}
C_4' &= \frac{N}{2\pi^2r^4K}(dt-c_r^2\vec{n}^T\varepsilon d\vec{n})\wedge dx\wedge dy\wedge dz\\
dC_4' &= \frac{N}{2\pi^2r^4K}\frac{-4}{r}dr\wedge dt\wedge dx\wedge dy\wedge dz\\
&\quad -\frac{N}{2\pi^2r^4K}\frac{-6}{r}c_r^2dr\wedge\vec{n}^T\varepsilon d\vec{n}\wedge dx\wedge dy\wedge dz\\
&\quad +\frac{N}{2\pi^2r^4}\frac{6}{rK}(1-K^{-1})dr\wedge(dt-c_r^2\vec{n}^T\varepsilon d\vec{n})\wedge dx\wedge dy\wedge dz\\
&\quad -\frac{N}{2\pi^2r^4K}c_r^2d\vec{n}^T\wedge\varepsilon d\vec{n}\wedge dx\wedge dy\wedge dz\\
F_5'=dC_4' &= \frac{N}{2\pi^2r^4K}\frac{2}{r}\left(1-\frac{3}{K}\right)dr\wedge dt\wedge dx\wedge dy\wedge dz\\
&\quad +\frac{N}{2\pi^2r^4}\frac{6}{rK^2}dr\wedge c_r^2\vec{n}^T\varepsilon d\vec{n}\wedge dx\wedge dy\wedge dz\\
&\quad -\frac{N}{2\pi^2r^4K}c_r^2d\vec{n}^T\wedge\varepsilon d\vec{n}\wedge dx\wedge dy\wedge dz
\end{aligned}$$

where I used that

$$\begin{aligned}
dc_r^2 &= -2\frac{4\pi N}{r^3}=-\frac{2}{r}c_r^2dr\\
dK &= -\frac{6}{r}\left(\frac{4\pi N\beta}{r^3}\right)^2=-\frac{6}{r}(K-1)dr\\
dK^{-1} &= -\frac{1}{K^2}dK=\frac{1}{K}\frac{6}{r}(1-K^{-1})dr.
\end{aligned}$$

Thus we must add to this calculation a WZW term for another four-form A_4 such that $dA_4 = *F_5'$ so that $d(C_4'+A_4) = F_5' + *F_5' = *d(C_4'+A_4)$.

6.2.1 Writing S^5 As a Hopf Fibration

Due to the symmetries of the problem it is convenient to parametrize the five-sphere in the directions transverse to the N coincident D3-branes as a Hopf fibration instead of using six angular coordinates we denoted $\vec{n}$. As explained in Chapter 4, we can think of the S^5 as the principal $U(1)$-bundle over the base $\mathbb{C}P^2$ and rewrite the metric $d\Omega_5^2 = \sum dn_I^2 = ds^2_{\mathbb{C}P^2} + \psi^2$. Here ψ is a one-form such that $\omega = d\psi$ is a representative of the second de-Rham cohomology $H^2(\mathbb{C}P^2,\mathbb{R})$ and $\beta\psi = \vec{n}^T\varepsilon d\vec{n}$.q

If coordinates on $\mathbb{C}P^2$ are v_2, v_3, φ_2 and φ_3 while ζ is the coordinate along fiber S^1

$$ds^2_{\mathbb{C}P^2} = \frac{dv_2^2+dv_3^2-(v_2dv_3-v_3dv_2)^2}{1-v_2^2-v_3^2}+v_2^2d\varphi_2^2+v_3^2d\varphi_3^2-(v_2^2d\varphi_2+v_3^2d\varphi_3)^2$$

and

$$\psi = d\xi + \Theta = \frac{1}{\beta}\vec{n}^T \varepsilon d\vec{n}$$

where

$$\begin{aligned}
\Theta &= v_2^2 d\varphi_2 + v_3^2 d\varphi_3 \\
\omega &= d\Theta = d\psi = 2v_2 dv_2 \wedge d\varphi_2 + 2v_3 dv_3 \wedge d\varphi_3.
\end{aligned}$$

Then the metric on the S^5 can be rewritten as

$$ds_{S^5}^2 = \frac{dv_2^2 + dv_3^2 - (v_2 dv_3 - v_3 dv_2)^2}{1 - v_2^2 - v_3^2} + v_2^2 d\varphi_2^2 + v_3^2 d\varphi_3^2 + d\xi^2 + 2d\xi(v_2^2 d\varphi_2 + v_3^2 d\varphi_3).$$

The five-form field strength F_5' in these coordinates is

$$\begin{aligned}
F_5' &= \frac{N}{2\pi^2 r^4 K}\frac{2}{r}\Big(1 - \frac{3}{K}\Big) dr \wedge dt \wedge dx \wedge dy \wedge dz \\
&\quad + \frac{N}{2\pi^2 r^4}\frac{6}{rK^2} c_r^2 \beta dr \wedge \psi \wedge dx \wedge dy \wedge dz \\
&\quad - \frac{N}{2\pi^2 r^4 K} c_r^2 \beta \omega \wedge dx \wedge dy \wedge dz \\
&= \frac{N}{2\pi^2 r^4 K}\frac{2}{r}\Big(1 - \frac{3}{K}\Big) dr \wedge dt \wedge dx \wedge dy \wedge dz \\
&\quad + \frac{N}{2\pi^2 r^4}\frac{6}{rK^2} c_r^2 \beta dr \wedge (d\xi + \Theta) \wedge dx \wedge dy \wedge dz \\
&\quad - \frac{N}{2\pi^2 r^4 K} c_r^2 \beta \omega \wedge dx \wedge dy \wedge dz.
\end{aligned}$$

Recalling that the Hodge dual of a m-form η is an n-form $*\eta$, where $m + n = D$ is the number of spacetime dimensions, and the components are

$$*\,\eta_{i_1 \ldots i_n} = \frac{1}{k!}\sqrt{|\det g|}\, g^{j_1 k_1} \ldots g^{j_m k_m} \eta_{j_1 \ldots j_m} \epsilon_{k_1 \ldots k_m i_1 \ldots i_n}$$

for the Hodge dual of our five-form field strength we get

$$*\,F_5' = \frac{N}{4\pi^2}\Big(\omega \wedge \omega \wedge d\xi + \frac{2\pi N\beta}{r^4} dt \wedge \omega \wedge \omega + \frac{8N\pi\beta}{r^5} dr \wedge dt \wedge \omega \wedge \psi\Big)$$

Where we are looking for A_4 such that, locally, $dA_4 = *F_5'$, which is exists because F

is closed[1]. The correction four-form A_4 contains

$$A_4 = \frac{N}{4\pi^2}\Big(\frac{\xi}{a}\omega\wedge\omega + \big(1-\frac{1}{a}\big)\theta\wedge\omega\wedge d\xi + \frac{2N\pi\beta}{r^4}dt\wedge\omega\wedge\psi\Big)$$

for some constant a.

There is only one term in the pull-back of A_4 that contains at most three derivatives of the fields

$$A_4|_{3derivatives} = \frac{N}{4\pi^2}\Big(\frac{2N\pi\beta}{r^4}dt\wedge\omega\wedge\psi\Big).$$

In terms of the $\vec{n}$ coordinates on S^5

$$\begin{aligned}\psi &= \frac{1}{\beta}\vec{n}^T\varepsilon d\vec{n}\\ \omega &= \frac{1}{\beta}d\vec{n}^T\wedge\varepsilon d\vec{n}\end{aligned}$$

we can rewrite it as

$$A_4|_{3derivatives} = \frac{N}{4\pi^2}\Big(\frac{2N\pi\beta}{r^4}\frac{\varepsilon_{IJ}\varepsilon_{KL}}{\beta^2}n^K\epsilon^{0\alpha\beta\gamma}\partial_\alpha n^I\partial_\beta n^J\partial_\gamma n^L dt\wedge dx\wedge dy\wedge dz\Big).$$

Then its pull back to the brane, up to an overall factor, looks like

$$P(A_4|_{3derivatives}) = \frac{N}{4\pi^2}\Big(\frac{2N\pi\beta}{r^4}\frac{\varepsilon_{IJ}\varepsilon_{KL}}{\beta^2}n^K\epsilon^{0\alpha\beta\gamma}\partial_\alpha n^I\partial_\beta n^J\partial_\gamma n^L\Big).$$

6.3 Stress-Energy Tensor

The most instructive way to rewrite all the contributions to the low-energy effective action is in terms of the stress-energy tensor of rescaled transverse fields. Let us define these new fields as

$$\phi^I = \frac{1}{r}n^I$$

[1] If F were not closed ([32] pg. 205)

$$\begin{aligned}F &= F_5' + *F_5'\\ dF &= *J_m\\ d*F &= *J_e(= *J_m = *J).\end{aligned}$$

From this we would have

$$*\,d*F_5' = J.$$

$$\partial_\alpha \phi^I = -\frac{1}{r^2}\partial_\alpha r n^I + \frac{1}{r}\partial_\alpha n^I$$

and remember that

$$\vec{n}^T\vec{n} = 1, \qquad \partial_\alpha(\vec{n}^T\vec{n}) = 0.$$

We have

$$\partial_\alpha \phi^I \delta_{IJ} \partial_\beta \phi^J = \frac{1}{r^4}\partial_\alpha r \partial_\beta r + \frac{1}{r^2}\partial_\alpha \vec{n}^T \partial_\beta \vec{n}$$

and define

$$J_\alpha = \delta\phi = \frac{1}{r^2}\vec{n}^T\varepsilon\partial_\alpha\vec{n}.$$

Then stress-energy tensor contributions from the Yang-Mills gauge field strength $F = dA$ and transverse scalar fields $\vec{\phi}$ can be written as

$$\begin{aligned}
(T_{YM})_{\alpha\beta} &= \eta^{\gamma\nu}F_{\alpha\gamma}F_{\beta\nu} - \frac{1}{4}\eta_{\alpha\beta}F_{\gamma\nu}F^{\gamma\nu} \\
(T_\phi)_{\alpha\beta} &= \partial_\alpha\phi^I\delta_{IJ}\partial_\beta\phi^J - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma\phi^I\delta_{IJ}\partial_\delta\phi^J.
\end{aligned}$$

While the terms

$$\begin{aligned}
\eta^{\delta\alpha}J_\delta(T_\phi)_{\alpha t} &= -J_t(T_\phi)_{tt} + J_i(T_\phi)_{it} \\
&= -J_t(\frac{1}{2}(\partial_t\phi^I\delta_{IJ}\partial_t\phi^J + \partial_i\phi^I\delta_{IJ}\partial_i\phi^J) + J_i(\partial_i\phi^I\delta_{IJ}\partial_t\phi^J) \\
\eta^{\delta\alpha}J_\delta(T_{YM})_{\alpha t} &= -J_t(T_{YM})_{tt} + J_i(T_{YM})_{it} \\
&= -J_t(F_{ti}F_{ti} + \frac{1}{4}F_{\gamma\nu}F^{\gamma\nu}) + J_i(F_{ij}F_{tj}).
\end{aligned}$$

Adding the the contributions of DBI action and WZW term with A_4 correction gives the total Lagrangian density up to three derivatives

$$\begin{aligned}
\mathcal{L}_{DBI} + \mathcal{L}_{WZW} + \mathcal{L}_{A_4} = \;& \frac{N}{2\pi^2}\Big(\mathcal{L}_\phi + \frac{\pi}{g_s N}\mathcal{L}_{YM} \\
& -(4\pi N)\eta^{\delta\alpha}J_\delta(T_\phi + T_{YM})_{\alpha t} \\
& +\Big(N\pi\beta\frac{\varepsilon_{IJ}\varepsilon_{KL}}{\beta^2}\phi^K\epsilon^{0\alpha\beta\gamma}\partial_\alpha\phi^I\partial_\beta\phi^J\partial_\gamma\phi^L\Big)\ldots\Big).
\end{aligned}$$

Part III

Static Charges in the Low-Energy Theory of the S-Duality Twist

Chapter 7

Introduction

For $U(n)$ gauge group, S-duality of $\mathcal{N}=4$ super Yang-Mills theory [21, 22, 33] asserts that the theory at the complex combination of coupling constant and θ angle

$$\tau \equiv \frac{4\pi i}{g_{\mathrm{YM}}^2} + \frac{\theta}{2\pi}$$

is equivalent to the same theory at complex coupling $-1/\tau$. The conjecture has passed many tests (see for instance [34, 35, 36, 37]) and is generally accepted as true, even though no proof exists. Over the years, much insight has been accumulating on the way S-duality works. Some notable breakthroughs include the geometric realization of S-duality in terms of the $(2,0)$-theory [38, 39], the connection with the geometric Langlands program [40], and the discovery [41, 42] of the role of certain strongly-coupled $2+1D$ $\mathcal{N}=4$ theories [43] as intertwiners between a supersymmetric boundary condition and its S-dual.

Another way to explore S-duality was recently examined in [26, 1]. There, an S-duality twist was introduced into a compactification of $\mathcal{N}=4$ super Yang-Mills (SYM) on S^1 in a way that preserves $\mathcal{N}=6$ supersymmetry in $2+1D$. An S-duality twist is an unusual possible boundary condition that is permissible when the complex coupling constant is set to the self-dual value $\tau=i$. The S-duality twist is then achieved by inserting the transformation that realizes $\tau\rightarrow -1/\tau$ at some point along S^1. We can then further compactify the remaining two spatial dimensions on, say, T^2. The Hilbert space of ground states of this compactification, which was studied in [1], provides insight into the operator that realizes S-duality. We refer to the resulting three-dimensional low-energy theory as *Tr-S*, because correlation functions of Wilson loops $\langle W(C_1)\cdots W(C_l)\rangle_{\text{Tr-S}}$ in this theory can be interpreted as, roughly speaking, a regularized version of the trace $\mathrm{tr}((-1)^F SRW(C_1)\cdots W(C_l))$, where the trace is taken over the Hilbert space of $\mathcal{N}=4$ $U(n)$ SYM at the self-dual coupling $\tau=i$, S is the S-duality operator, R is an appropriate $SU(4)$ R-symmetry operator that is inserted in order to preserve $\mathcal{N}=6$ supersymmetry in $2+1D$, F is the fermion number (which is equivalent to a central element of the R-symmetry group), and

$W(C_1), \ldots, W(C_l)$ are Wilson loop operators associated with the loops $C_1, \ldots, C_l$ in $\mathbb{R}^3$.

S-duality is part of a larger $\mathrm{SL}(2,\mathbb{Z})$ group of dualities, and some of them can be used as twists as well. These arise for the special coupling $\tau = e^{\frac{i\pi}{3}}$, which is invariant under $\tau \to (\tau-1)/\tau$ and $\tau \to -1/(\tau-1)$, and the corresponding $\mathrm{SL}(2,\mathbb{Z})$ elements can be used to twist the boundary conditions. Together with $\tau \to -1/\tau$, we thus have three $\mathrm{SL}(2,\mathbb{Z})$ elements to explore as possible twists.[1] We denote a general $\mathrm{SL}(2,\mathbb{Z})$-element by

$$\mathbf{g} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \qquad \tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}, \tag{7.1}$$

and we denote its order in the group by $\mathbf{r}$ (thus $\mathbf{g}^{\mathbf{r}} = 1$). In [1] an integer k and a phase $-\pi < \upsilon < \pi$ were assigned to the three $\mathrm{SL}(2,\mathbb{Z})$ elements $\mathbf{g}$ as follows:

$$e^{i\upsilon} \equiv \mathbf{c}\tau + \mathbf{d}, \qquad k \equiv \frac{2 - \mathbf{a} - \mathbf{d}}{\mathbf{c}}. \tag{7.2}$$

It can easily be checked that in our three cases υ is real and equal to $2\pi/\mathbf{r}$, and k is an integer. Explicitly,

$$\left.\begin{array}{llllll}
k=1, & \mathbf{r}=6, & \upsilon = \frac{\pi}{3} & \text{for} & \tau = e^{\pi i/3}, & \mathbf{g} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}; \\[2ex]
k=2, & \mathbf{r}=4, & \upsilon = \frac{\pi}{2} & \text{for} & \tau = i, & \mathbf{g} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \\[2ex]
k=3, & \mathbf{r}=3, & \upsilon = \frac{2\pi}{3} & \text{for} & \tau = e^{\pi i/3}, & \mathbf{g} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.
\end{array}\right\} \tag{7.3}$$

In [1] the study of the Tr-S theory with gauge group $U(n)$ compactified on T^2 was started, and the Hilbert space of ground states was determined. We use the notation $\mathcal{H}(k,n)$ to refer to the Hilbert space of ground states of the theory with τ and $\mathbf{g}$ that are determined according to the list (7.3). In [1] it was also found convenient to restrict attention to the cases with $n < \mathbf{r}$, since these have no Coulomb branch, as we review in Chapter 8. This restriction arises as in cases where $n \geq \mathbf{r}$ there are elements of the Weyl group $S_n \subset U(n)$ that have order $\mathbf{r}$, and a gauge transformation by such a Weyl group element can cancel the effect of the R-symmetry twist and produce a zero-mode. For $n < \mathbf{r}$ there are no such zero modes. We believe that in

[1] The twists are in $\mathrm{SL}(2,\mathbb{Z})$ and not $\mathrm{PSL}(2,\mathbb{Z})$ because the central element $-1 \in \mathrm{SL}(2,\mathbb{Z})$ acts nontrivially, being equivalent to charge conjugation. For each $\mathbf{g} \in \mathrm{SL}(2,\mathbb{Z})$ in the list (7.3) below, one can also use $-\mathbf{g}$ as a twist, but it is always identical to the inverse of an element that already appears in the list, and the resulting theory with $-\mathbf{g}$ is always the parity transform of another theory from the list.

this case the compactification has a mass-gap, and the $2+1D$ low-energy theory Tr-S is topological.

The purpose of this paper is to continue to explore the Hilbert space of Tr-S on T^2 by introducing supersymmetric static charges corresponding to m pairs of heavy quarks and anti-quarks at fixed locations on T^2 and S^1. We will study the resulting Hilbert space of ground states, and show that the Witten Index of this problem can be calculated by counting the states of a simple quantum mechanical system with action

$$I = \frac{1}{2\pi}\int \sum_{\mathfrak{a},\alpha=1}^{m} \mathcal{M}_{\mathfrak{a}\alpha}\mathfrak{p}^{\mathfrak{a}}\dot{\mathfrak{q}}^{\alpha}\, dt\,, \tag{7.4}$$

where $\mathcal{M}_{\mathfrak{a}\alpha}$ is an $m\times m$ matrix with integer entries, and $\mathfrak{p}^1, \mathfrak{q}^1, \dots, \mathfrak{p}^m, \mathfrak{q}^m$ are periodic coordinates with period 2π. The action (7.4) describes geometric quantization of T^{2m}.

One motivation for introducing static charges into the Tr-S theory — apart from a better understanding of the theory itself — is to clarify the relationship between Tr-S and another (known) topological theory in $2+1D$, namely the Chern-Simons theory. In fact, when the gauge group is abelian, in which case the $\mathcal{N}=4$ SYM is a free theory and we have a complete understanding of its S-duality [31], we have an explicit description for Tr-S: it is simply the abelian Chern-Simons theory at level k given in (7.2) (see §5 of [1]).

This simple picture does not hold true for nonabelian gauge groups, but the result of [1] suggested that there might still exist a close relationship between the two theories. There, we observed that the Hilbert space of the Tr-S theory compactified on T^2 decomposes into different sectors, and for almost all sectors we were able to show that their symmetry operators and behavior under modular transformations of T^2 agree with those of the Hilbert spaces of the Chern-Simons theory with appropriate gauge groups and levels. Introduction of static charges then provides a further test on the identification of the Hilbert spaces of two theories.

Our strategy for extracting the Witten Index of the system with static charges inserted follows closely that of [1]. Since little is known about the S-duality operator itself for nonabelian gauge groups, we will embed our setting into full type IIB string theory and apply a series of string theory dualities, after which the low-energy description of the system is given by the simple quantum mechanical one in (7.4). For abelian gauge group, the result we obtain in this way precisely agrees with what we expect from introducing Wilson line operator in abelian Chern-Simons theory. This is as it must be, because we already know that the Tr-S theory *is* Chern-Simons theory in this case. For nonabelian gauge groups, we show that our result passes a nontrivial consistency check in itself, but does not agree with Chern-Simons theory predictions in general. We will provide more discussion on this discrepancy in the concluding section.

The paper is organized as follows. In Chapter 8 we explain the problem in detail and describe the S-duality and R-symmetry twists, the amount of supersymmetry that is left unbroken, and the absence of the Coulomb branch for $n < \mathbf{r}$. We then construct the type-IIA dual of the theory without the charges. In Chapter 9 we introduce the supersymmetric static charges and derive their type-IIA dual description. We then explain how the geometric quantization systems of the type (7.4) arise. In Chapter 10 we use the known solution of the problem with $U(1)$ gauge group, which reduces to $U(1)$ Chern-Simons theory at level k [defined in (7.3)], to demonstrate how the type-IIA dual reproduces the known results about the Hilbert space of ground states of $U(1)$ Chern-Simons theory with charges. In Chapter 11 we move to the case of $U(n)$ gauge group. The goal of this section is to calculate the Witten Index of this system as a function of n, k, and m. We describe the technical aspects of the calculation in detail and summarize the final results in Table 11.2. Next, using a Wick-rotation we express the Witten index as a supertrace of a combination of spatial Wilson loops over $\mathcal{H}(k, n)$ (the Hilbert space without external charges). This provides us with a consistency check on the final result, and moreover, allows us to calculate the eigenvalues of the spatial Wilson loop operators on $\mathcal{H}(k, n)$. We then compare the results to Chern-Simons theory as conjectured in [1], and show that they do not match. We conclude in Chapter 12 with a discussion and future directions.

Chapter 8

Review of the S-duality twist and its type-IIA dual

In this section, we carry out a brief review of Tr-S theory, how to realize it in string theory, and how to construct the type-IIA dual of the theory on T^2. We refer our readers to [1] for more comprehensive details. At the end of this section we include a small discussion on why we believe Tr-S is topological for $n < \mathbf{r}$.

8.1 Definition of Tr-S

By definition, Tr-S is the $2+1D$ low-energy limit of a compactification of $\mathcal{N}=4$ super Yang-Mills (SYM) theory on S^1 with boundary conditions that include an S-duality twist and an appropriate R-symmetry twist to be discussed below. This compactification was introduced in [26, 1], and similar compactifications have also recently been studied in [44, 45, 46], where the S-duality twist was referred to as a "duality wall."

By itself, S-duality does not commute with supersymmetry [40], and since we want to preserve some supersymmetry we have to supplement the S-twist with an R-symmetry twist. Therefore, in [1] we twisted the boundary conditions on S^1 further by an element γ of the R-symmetry group $SU(4)$, which in a particular basis takes the form:

$$\gamma = \begin{pmatrix} e^{\frac{i}{2}\upsilon} & & & \\ & e^{\frac{i}{2}\upsilon} & & \\ & & e^{\frac{i}{2}\upsilon} & \\ & & & e^{-\frac{3i}{2}\upsilon} \end{pmatrix} \in SU(4)_R\,, \tag{8.1}$$

where υ is given by (7.2). This choice, it turns out, preserves the maximal possible amount of supersymmetry in the presence of S-duality and R-symmetry twists, which is $\mathcal{N}=6$ in $2+1D$. It also preserves a $U(3)\subset SU(4)_R$ R-symmetry. This $U(3)$

can be thought of as the unitary group of rotations that act holomorphically on the transverse $\mathbb{C}^3 \simeq \mathbb{R}^6$.

$3+1D$ $\mathcal{N}=4$ SYM has a Coulomb branch on which the gauge group is broken to $U(1)^n \subset U(n)$. After compactification with the S-twist and R-twist together, the Coulomb branch completely disappears for $n < \mathbf{r}$ [1]. This is because a point on the Coulomb branch $(\mathbb{R}^6)^n/S_n$ of $\mathcal{N}=4$ $U(n)$ SYM is described by an unordered set of noncoincident n points in $\mathbb{R}^6$. S-duality preserves the point in moduli space, but the R-symmetry twist γ acts on it by rotation of $\mathbb{R}^6$. Since γ has order $\mathbf{r} \equiv 2\pi/\upsilon$ when acting on $\mathbb{R}^6$, we see that we need $n \geq \mathbf{r}$ for a point on the Coulomb branch to survive the twist. For $n \geq \mathbf{r}$ the situation is more complicated and some portion of the moduli space of the Coulomb branch survives [1]. In order to avoid these complications we will restrict the discussion that follows to the case $n < \mathbf{r}$.

We can easily realize Tr-S in string theory using D3-branes. Consider the type-IIB background $\mathbb{R}^{9,1}$ with Cartesian coordinates $x_0, \dots, x_9$, and place n D3-branes at $x_4 = x_5 = \cdots = x_9 = 0$. The type-IIB coupling constant is denoted by $\tau = \chi + \frac{i}{g_{\text{IIB}}}$, where g_{IIB} is the string coupling constant, and χ is the RR scalar. The S-duality transformation $\mathbf{g}$ of (7.1) then lifts to an S-duality transformation of the full type-IIB string theory (that we also denote by $\mathbf{g}$), and the R-symmetry element γ lifts to a geometrical rotation in the six directions transverse to the D3-branes. We compactify the x_3-direction on a circle of radius R with boundary conditions given by a simultaneous S-duality twist $\mathbf{g}$ and a $\gamma \in \text{Spin}(6)$ geometrical twist in directions $x_4, \dots, x_9$, where γ is given by (8.1). This means that as we traverse the x_3 circle once, we also apply a $\gamma \in \text{Spin}(6)$ rotation in the transverse directions before gluing $x_3 = 0$ to $x_3 = 2\pi R$.

We now compactify directions x_1, x_2, so that $0 \leq x_1 < 2\pi L_1$ and $0 \leq x_2 < 2\pi L_2$ are periodic. This puts the $2+1D$ field theory on T^2 with area $4\pi^2 L_1 L_2$ and complex structure iL_2/L_1. In the limit

$$L_1, L_2, R \gg \alpha'^{1/2}\,, \tag{8.2}$$

where $\alpha'^{1/2}$ is the type-IIB string scale, we can first reduce the description of the D3-branes to $\mathcal{N}=4$ $U(n)$ SYM at low-energy, and then compactify $\mathcal{N}=4$ SYM with an S-duality and R-symmetry twist.

8.2 Type-IIA dual

We will now transform the type-IIB background, using string dualities, to one where S-duality is realized geometrically. For this we need to consider the opposite limit $L_1, L_2 \to 0$ with $R \to \infty$ (in the order to be specified below). In this limit, the type-IIB description is strongly coupled, but we will perform a U-duality transformation as specified in Table 8.1 to transform the setting to a weakly coupled type-IIA background, enabling us to easily study the ground states of the field theory.

Type	Brane	1	2	3	4	5	6	7	8	9	10	Apply:
IIB	D3	$-$	$-$	$\div$							$\times$	T-duality on 1
IIA	D2		$-$	$\div$							$\times$	Lift to M-theory
M	M2		$-$	$\div$								Reduction to IIA on 2
IIA	F1		$\times$	$\div$								This is the final step!

Table 8.1: The sequence of dualities from n D3-branes in type-IIB to n fundamental strings in type-IIA. A direction that the corresponding brane or string wraps with periodic boundary conditions is represented by $-$, a direction that the object wraps with twisted boundary conditions is represented by $\div$, and a dimension that doesn't exist in the particular string theory is denoted by $\times$. All the branes in the table are at the origin of directions $4, \dots, 9$.

The U-duality transformation proceeds as follows. We first replace type-IIB on a circle of radius L_1 with M-theory on T^2 with complex structure τ and area $\mathcal{A} = (2\pi)^2 \alpha'^2 \tau_2^{-1} L_1^{-2} = (2\pi)^2 M_{\mathrm{p}}^{-3} L_1^{-1}$, where M_{p} is the 11-dimensional Planck scale. We now reduce from M-theory to type-IIA on the circle of radius L_2 to get a theory with string coupling constant

$$g_{\mathrm{IIA}} = (M_{\mathrm{p}} L_2)^{3/2} = \tau_2^{1/2} L_1^{1/2} L_2^{3/2} \alpha'^{-1} ,$$

and new string scale

$$\alpha'_{\mathrm{IIA}} = M_{\mathrm{p}}^{-3} L_2^{-1} = \alpha'^2 \tau_2^{-1} L_1^{-1} L_2^{-1} .$$

After these dualities, the D3-branes become fundamental type-IIA strings with a total winding number n in the x_3 direction. The S-duality twist $\mathbf{g}$ is now a diffeomorphism of the type-IIA torus (in the $x_{10} x_1$ directions), which can be realized as a rotation by an angle υ listed in (7.3). To make this type-IIA background weakly coupled we assume that the limits are taken in such a way that

$$\mathcal{A} \gg \alpha'_{\mathrm{IIA}} , \qquad g_{\mathrm{IIA}} \ll 1 , \qquad R \gg \alpha'^{1/2}_{\mathrm{IIA}} . \tag{8.3}$$

This is a different limit than (8.2), but we can use the weakly coupled type-IIA background to study the Hilbert space of supersymmetric ground states, or more precisely the Witten Index of the Tr-S theory on T^2.

To describe the basic geometry of the dual type-IIA background, it is convenient to divide the 9 directions into three groups and view the spatial manifold as an orbifold of $T^2 \times \mathbb{R} \times \mathbb{C}^3$. We regard the T^2 as the complex plane modded out by a lattice, $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and take

$$z \sim z + 1 \sim z + \tau \tag{8.4}$$

as its coordinate. On $\mathbb{R}$, we take the coordinate

$$-\infty < x_3 < \infty ,$$

and on $\mathbb{C}^3 \simeq \mathbb{R}^6$, we take the coordinates to be

$$(\zeta_1, \zeta_2, \zeta_3)\,, \qquad \zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}\,.$$

The orbifold is then represented by the identification

$$(z, x_3, \zeta_1, \zeta_2, \zeta_3) \sim (e^{iv} z, x_3 + 2\pi R, e^{iv}\zeta_1, e^{iv}\zeta_2, e^{iv}\zeta_3)\,. \tag{8.5}$$

Also, the shift $x_3 \to x_3 + 2\pi R$ ensures that the orbifold has no fixed points, and thus the geometry is flat and free of singularities. In particular, the $\zeta_1 = \zeta_2 = \zeta_3 = 0$ subspace is a T^2-fibration over S^1 with structure group $\mathbb{Z}_{\mathbf{r}}$.

The ground states that are relevant to our problem are those with a total string winding number n along direction x_3. A state with string winding number n is a p-particle (that is, p-string) state consisting of 1-particle states of winding numbers $n_1 \geq n_2 \geq \cdots \geq n_p > 0$ with $n_1 + n_2 + \cdots + n_p = n$. Thus, the Hilbert space of ground states decomposes as a direct sum:

$$\mathcal{H}(k,n) = \bigoplus_{p=1}^{n} \left(\bigoplus_{\substack{n_1 \geq n_2 \geq \cdots \geq n_p > 0 \\ n_1 + n_2 + \cdots + n_p = n}} \mathcal{H}(k; n_1, \ldots, n_p) \right)\,. \tag{8.6}$$

A crucial point is that the problem of finding the ground states can be solved using essentially classical geometry: we simply need to find classical string configurations of minimal length. Consider a superstring with winding number $\tilde{n}$ in direction x_3. It turns out [8] that (for $\tilde{n} \neq 0$) the ground states are bosonic and in the RR sector. (We will independently verify this in Chapter 10.3.) For $\tilde{n}$ that is not divisible by $\mathbf{r}$, there is a basis of ground states that are in one-to-one correspondence with loops of winding number $\tilde{n}$ and minimal length in the geometry (8.5). In the limit $\alpha'_{\mathrm{IIA}} \to 0$, these states reduce to the classical string configurations.

To describe the classical configurations, we can fix an x_3 coordinate and specify the points where the classical string intersects the transverse coordinates $T^2 \times \mathbb{C}^3$ in the geometry (8.5). At winding number $\tilde{n}$, the string intersects $T^2 \times \mathbb{C}^3$ at $\tilde{n}$ (not necessarily distinct) points, and in order to be of minimal length the coordinates of these points should be independent of x_3. The classical configurations are thus characterized by a set of $\tilde{n}$ points in $T^2 \times \mathbb{C}^3$ that is invariant, as a set, under the orbifold operation

$$(z, \zeta_1, \zeta_2, \zeta_3) \sim (e^{iv} z, e^{iv}\zeta_1, e^{iv}\zeta_2, e^{iv}\zeta_3)\,.$$

For $\tilde{n}$ that is not divisible by $\mathbf{r}$, there is a finite number of such sets, and they are all localized at the origin of $\mathbb{C}^3$, i.e., at $\zeta_1 = \zeta_2 = \zeta_3 = 0$. They are therefore entirely described by the z-coordinates of where the string intersects T^2: $z, e^{iv}z, e^{2iv}z, \ldots, e^{i(\tilde{n}-1)v}z$, since as we go once around the x_3 direction the coordinate z switches to $e^{iv}z$. After $\tilde{n}$ loops z becomes $e^{i\tilde{n}v}z$, which should be identified with z (up to a shift in $\mathbb{Z} + \mathbb{Z}\tau$)

in order to close the string. The classical string configurations are then described by solutions $z = \zeta_{M_a,M_b}$ of

$$e^{i\tilde{n}\upsilon}\zeta_{M_a,M_b} = \zeta_{M_a,M_b} + M_a + M_b\tau\,, \qquad (M_a, M_b \in \mathbb{Z}) \tag{8.7}$$

and we consider two solutions ζ_{M_a,M_b} and $\zeta_{M_a{}',M_b{}'}$ as equivalent if they differ by a lattice element, i.e., if $\zeta_{M_a,M_b} - \zeta_{M_a{}',M_b{}'} \in \mathbb{Z}+\mathbb{Z}\tau$. In addition, ζ_{M_a,M_b} and $e^{i\upsilon}\zeta_{M_a,M_b}$ give equivalent solutions, since the intersection points of the string with T^2 are unordered.

There is then only a finite number of inequivalent solutions to (8.7), and we have described them in detail in [1]. The full single-particle string spectrum (including excited states) decomposes into a finite sum of distinct sectors, labeled by M_a, M_b, and the solution ζ_{M_a,M_b}, which is a point on T^2, describes the center of mass of the string in the directions of T^2. Thus, a single-particle ground state with winding number $\tilde{n}$ can be described by the location of the intersection of the classical string configuration with any particular T^2 fiber at a constant x_3:

$$\big|[z, e^{i\upsilon}z, \ldots, e^{(\tilde{n}-1)i\upsilon}z]\big\rangle\,, \tag{8.8}$$

where z coordinates are always taken modulo the lattice $\mathbb{Z} + \mathbb{Z}\tau$. The multi-string states can subsequently be described by

$$\big|\{[z_1, e^{i\upsilon}z_1, \ldots, e^{(n_1-1)i\upsilon}z_1], [z_2, e^{i\upsilon}z_2, \ldots, e^{(n_2-1)i\upsilon}z_2], \ldots, [z_p, e^{i\upsilon}z_p, \ldots, e^{(n_p-1)i\upsilon}z_p]\}\big\rangle\,,$$

where each z_i is a solution $\zeta_{M_{ai},M_{bi}}$ of (8.7) with $\tilde{n} \to n_i$, and $n = \sum_1^p n_i$ is the total winding number. Also, the number of inequivalent solutions of (8.7) for $\tilde{n} = 1$ is equal to k. It is a function of υ alone, as indicated in (7.3).

8.3 $\mathbb{Z}_k$ symmetries

For $k > 1$ there are two useful $\mathbb{Z}_k$ symmetries that can be described geometrically in the type-IIA background as follows [1]:

1. The metric (8.5) has a discrete isometry that is generated by the operator $\mathcal{U}$ defined to act as a translation in the T^2 fiber:

$$\mathcal{U}: \qquad (z, x_3, \zeta_1, \zeta_2, \zeta_3) \mapsto (z + \frac{1}{k}(1+\tau), x_3, \zeta_1, \zeta_2, \zeta_3)\,. \tag{8.9}$$

2. The first homology group H_1 of the space (8.5) is $\mathbb{Z} \oplus \mathbb{Z}_k$ where $\mathbb{Z}_k$ is generated by the homology class of one of the 1-cycles of the T^2 (in the z direction), and $\mathbb{Z}$ is generated by a cycle that wraps around the x_3 direction at $z = 0$. The homology class of a fundamental string is conserved, and the projection onto the $\mathbb{Z}_k$ factor describes a conserved quantum number $q \in \mathbb{Z}_k$. We define the operator $\mathcal{V}$ to have the eigenvalue $e^{\frac{2\pi i}{k}q}$ on a state with quantum number q.

In other words, $\mathcal{U}$ can be viewed as $\mathbb{Z}_k$-momentum, and $\mathcal{V}$ can be viewed as $\mathbb{Z}_k$-winding number. They obey the commutation relations [1]:

$$\mathcal{V}^k = \mathcal{U}^k = 1\,, \qquad \mathcal{V}\mathcal{U}\mathcal{V}^{-1}\mathcal{U}^{-1} = e^{\frac{2\pi i n}{k}}\,. \tag{8.10}$$

This $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry also has a natural interpretation in terms of the original gauge theory. Its conserved charges can be expressed in terms of S-duality invariant combinations of electric and magnetic fluxes in the center $U(1) \subset U(n)$; we refer the reader to [1] for details. For the present paper, however, we only need to know the commutation relations of $\mathcal{U}$ and $\mathcal{V}$ with the Wilson loop operators. Those follow directly from the relation of $\mathcal{U}$ and $\mathcal{V}$ to electric and magnetic fluxes, or can be derived directly in the type-IIA dual. The result will be given later on in the paper, in (11.34).

For completeness, we also note that in addition to this $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry we have an $\mathrm{SL}(2,\mathbb{Z})$ symmetry that acts as the mapping class group of T^2 on which the Tr-S theory is defined. In the type-IIA dual picture, the complex structure parameter of this T^2 becomes the complexified area modulus of the type-IIA T^2 (in $x_{10}x_1$ directions):

$$\rho = \frac{i}{\alpha'_{\mathrm{IIA}}}\mathrm{Area}(T^2) + \frac{1}{2\pi}\int_{T^2} B\,. \tag{8.11}$$

Here B is the NS-NS two-form potential. The $\mathrm{SL}(2,\mathbb{Z})$ group acts by T-duality, and is generated by

$$\mathcal{S} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \qquad \mathcal{S}: \qquad \rho \to -\frac{1}{\rho}\,,$$

and

$$\mathcal{T} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \qquad \mathcal{T}: \qquad \rho \to \rho + 1\,.$$

In [1], this correspondence was used to read off the modular transformation properties of the ground states of Tr-S theory.

8.4 Is Tr-S a topological theory?

Before we proceed to study static charges for Tr-S on T^2, let us explain in more detail why we believe Tr-S is a topological theory. The setting we described above has $\mathcal{N} = 6$ supersymmetry in $2+1D$ and a $U(3)$ R-symmetry, which is the subgroup of $SU(4)$ that commutes with γ in (8.1). If Tr-S is not topological and has propagating degrees of freedom, it must be an interacting $\mathcal{N} = 6$ superconformal field theory. Let us then consider the low-energy limit of Tr-S on S^1. This compactification has $\mathcal{N} = (6,6)$ supersymmetry in $1+1D$. To gain more insight about this $1+1D$ theory, let us look at the list of dualities in Table 8.1, but instead of performing all the dualities all the way to type-IIA at the last line, let us stop one line before the last, at the point where we have M-theory and n M2-branes. At this point direction x_2 is not yet compact, but directions x_1, x_{10} are compact and form a torus with complex structure τ. The directions transverse to the M2-branes are parameterized by the complex coordinates $(z, \zeta_1, \zeta_2, \zeta_3)$. The M2-branes wrap direction x_3, and the boundary conditions along x_3 are twisted by a geometrical twist, which is a rotation in the directions transverse to the M2-branes. This twist acts as

$$(z, \zeta_1, \zeta_2, \zeta_3) \to (e^{iv} z, e^{iv} \zeta_1, e^{iv} \zeta_2, e^{iv} \zeta_3) \tag{8.12}$$

and corresponds to a rotation by an angle v in 4 transverse planes.

For $n = 1$, the twisted boundary conditions create a mass gap of $1/\mathbf{r}R$, where $\mathbf{r} = 2\pi/v$ [see the discussion below (7.2)], and the $1+1D$ low-energy theory has no propagating degrees of freedom. This is consistent with the identification of Tr-S at $n = 1$ with abelian Chern-Simons theory, as will be discussed in Chapter 10 in more detail, and indeed Chern-Simons theory has no propagating degrees of freedom.

What about the nonabelian case, say $n = 2$? In this case we need to understand the low-energy limit describing two M2-branes compactified on S^1 with transverse directions $T^2 \times \mathbb{C}^3$ parameterized by $(z, \zeta_1, \zeta_2, \zeta_3)$ and with boundary conditions twisted by (8.12) along S^1. The $1+1D$ low-energy theory corresponds to configurations where $(z, \zeta_1, \zeta_2, \zeta_3)$ are independent of x_3. Because of the twist, this implies that $\zeta_1 = \zeta_2 = \zeta_3 = 0$ and z is a fixed point of the twist. For given v, the twist has k fixed points on T^2, as we explained at the end of §8.2. It is easy to check that these fixed points are at

$$z_j = \frac{j}{k}(1+\tau)\,, \qquad j = 0, \ldots, k-1. \tag{8.13}$$

The $1+1D$ low-energy theory therefore has $k(k+1)/2$ sectors. In $k(k-1)/2$ of the sectors the two M2-branes sit at different fixed points $z_j \neq z_{j'}$. In this case it is clear that no massless excitations survive the low-energy limit and the low-energy $1+1D$ theory has no propagating degrees of freedom. The remaining k sectors have two M2-branes at the same z_j. Clearly, all these sectors are equivalent and we can concentrate on one of them, say at $z_0 = 0$. Since the M2-branes are pinned to the origin, we can safely replace T^2 with $\mathbb{C}$ and set $z \to \zeta_0$, with $\zeta_0 \in \mathbb{C}$. We have now reduced the problem to understanding the compactification of two M2-branes on S^1 with transverse directions $\mathbb{C}^4$ and a twist along S^1 given by

$$(\zeta_0, \zeta_1, \zeta_2, \zeta_3) \to (e^{iv}\zeta_0, e^{iv}\zeta_1, e^{iv}\zeta_2, e^{iv}\zeta_3). \tag{8.14}$$

Up until recently we would have had to proceed indirectly from here, but the recent progress in the low-energy description of M2-branes [47]-[48], culminating in the discovery of the ABJM action [49], allows us, in principle, to explore this problem directly. We need to take the ABJM action and compactify all fields on S^1 with boundary conditions twisted by (8.14). This corresponds to an element of the $SO(8)$ R-symmetry group. However, proceeding to compactify the ABJM theory in this manner involves subtleties that require a separate treatment, which we hope to present elsewhere. Instead, for now we will settle for an indirect approach, modifying the problem a little bit. Instead of taking the discrete value $\upsilon = 2\pi/\mathbf{r}$ in (8.14), let us consider the case that $|\upsilon| \ll 1$. More precisely, consider the double-scaling limit

$$\upsilon \to 0\,, \qquad R \to 0\,, \qquad \beta \equiv \frac{\upsilon}{R} \to \text{const.}$$

Using a standard technique, we change variables to

$$z'_j \equiv e^{-\frac{i\upsilon x_3}{2\pi R}}\zeta_j\,, \qquad j = 0, 1, 2, 3,$$

and write the metric as

$$\begin{aligned} ds^2 &= -dx_0^2 + dx_2^2 + dx_3^2 + \sum_{j=0}^{3} |d\zeta_j|^2 \\ &= -dx_0^2 + dx_2^2 + \bigl(1 + \frac{\beta^2}{4\pi^2}\sum_{0}^{3}|z'_j|^2\bigr)dx_3^2 + \sum_{j=0}^{3}|dz'_j|^2 - \frac{\beta}{\pi}dx_3\,\mathrm{Im}\sum_{j=0}^{3} z'_j d\overline{z}'_j\,. \end{aligned}$$

We can now reduce to type-IIA along direction x_3 to obtain a "Melvin background" [7]-[50]. This background has a Ramond-Ramond field strength

$$F^{RR} = dA^{\mathrm{RR}} = -\frac{\beta}{2\pi}\mathrm{Im}\sum_{j=0}^{3} dz'_j \wedge d\overline{z}'_j + O(\beta^3)\,,$$

and a dilaton

$$e^{\Phi} = (M_p R)^{3/2}\bigl(1 + \frac{\beta^2}{4\pi^2}\sum_{0}^{3}|z'_j|^2\bigr)^{3/2}\,,$$

where M_p is the 10+1D Planck scale. This background creates mass terms for all low-energy fields that propagate on a long string in direction x_2. Such a string is pinned to the origin $z'_0 = z'_1 = z'_2 = z'_3$, as is obvious from the M-theory description. In the type-IIA world-sheet description, the fermion mass terms are generated from the coupling of the string modes to the RR field strength, while the bosonic mass terms are generated from the string-frame metric. For small β we can trust the approximate perturbative string analysis, and we see that all propagating modes along the remaining noncompact direction x_2 have mass of order $\beta = \upsilon/R$. It is not immediately clear that we can extrapolate this analysis to $\upsilon = 2\pi/\mathbf{r}$, but we know that for a single string in this background no complications should arise, and in the limit $R \to 0$ string interactions are small.

Furthermore, if Tr-S is a nontrivial SCFT, and if it has a moduli space, then this moduli space must be compact because we have eliminated all noncompact modes along the Coulomb branch via the twist, and because the type-IIA picture shows no trace of noncompact flat directions. But $\mathcal{N} = 6$ supersymmetry in $2+1D$ is very restrictive and requires the moduli space to be locally flat. It must therefore be of the form T^{8d}/Γ, where Γ is a discrete isometry group, and d is an integer. On the other hand, the unbroken R-symmetry group must act nontrivially on the moduli space (by supersymmetry), but the maximal continuous isometry group of T^{8d}/Γ is abelian and cannot contain $SU(3)$, which is a contradiction.

We will proceed under the assumption that Tr-S is topological, but we note that even if this assumption is incorrect, the results of this paper are still meaningful, but they then correspond to the Witten Index of an interacting SCFT, rather than a TQFT. We now proceed to the calculation of the Witten Index.

Chapter 9

Static charges and their type-IIA dual description

In this paper we study what happens when we insert static charges into the Tr-S theory defined in Chapter 8. We add $2m$ static external sources to the system at S^1 coordinate $x_3 = 0$. Specifically, we insert m heavy (non dynamical) quarks at the fixed T^2 coordinates $(a_1^{(\mathfrak{a})}, a_2^{(\mathfrak{a})})$ (where $\mathfrak{a} = 1, \ldots, m$), and to cancel the net $U(1)$ charge[1] we insert an equal number m of anti-quarks, which we take to be at fixed T^2 coordinates $(a_1^{(\mathfrak{a}+m)}, a_2^{(\mathfrak{a}+m)})$. (Here $0 \leq a_i^{(\mathfrak{a})} < 2\pi L_i$ $(i = 1, 2)$ are periodic coordinates in type-IIB directions x_1, x_2.)

The $\mathfrak{a}^{th}$ static charge can be incorporated by introducing a matrix element of a time-like Wilson line

$$\operatorname{tr} P \exp\left(i \int_{-\infty}^{\infty} A_0(t, a_1^{(\mathfrak{a})}, a_2^{(\mathfrak{a})}, 0) dt \right) \tag{9.1}$$

into the path integral. This prescription, however, breaks all the supersymmetry. To preserve some supersymmetry we follow [51, 52] and add one of the adjoint scalar fields of $\mathcal{N} = 4$ SYM to A_0 in (9.1). For concreteness, we take

$$\operatorname{tr} P \exp\left(i \int_{-\infty}^{\infty} [A_0(t, a_1^{(\mathfrak{a})}, a_2^{(\mathfrak{a})}, 0) + \Phi^9(t, a_1^{(\mathfrak{a})}, a_2^{(\mathfrak{a})}, 0)] dt \right), \tag{9.2}$$

where Φ^9 is the scalar field that corresponds to D3-brane fluctuations in the x_9 direction. In §9.1 we will show that inserting charges that interact with Tr-S as the low-energy limit of (9.2) preserves 4 real supercharges.

[1] Actually, it is not necessary for the net charge to be zero, thanks to the S-duality twisted boundary conditions in the x_3 direction. But the system with nonzero net charge is more complicated and will not be studied here.

9.1 Charges as endpoints of type-IIB strings

Our main task now is to identify the type-IIA dual realization of the charges. We start in type-IIB and follow a standard technique to introduce static charges with interactions (9.2) into the type-IIB construction described at the beginning of Chapter 8.

Following [51, 52] we introduce fundamental strings with one endpoint on the D3-branes and extending indefinitely in direction x_9. We label the strings by $\mathfrak{c} = 1, \ldots, 2m$ and let the strings labeled by $\mathfrak{c} = 1, \ldots, m$ extend along $0 \leq x_9 < \infty$ and the strings labeled by $\mathfrak{c} = m+1, \ldots, 2m$ extend along $-\infty < x_9 \leq 0$. The low-energy description of this system holds the information about the ground states of Tr-S with static charges, and the (x_1, x_2) coordinates of the endpoints of the strings can be set to $(a_1^{(\mathfrak{c})}, a_2^{(\mathfrak{c})})$. We are only interested in the low-energy excitations of the system at energies well below the string scale, as well as the compactification scales:

$$E \ll M_{\mathrm{st}}, \frac{1}{L_1}, \frac{1}{L_2}, \frac{1}{R}.$$

The semi-infinite strings, in this limit, are very heavy and can be treated semi-classically.

The long-wavelength excitations of each string are described by 8 free scalars $X^{\mu}_{\mathfrak{c}}(x_9, t)$ $(\mu = 1, \ldots, 8)$ and a free Majorana-Weyl fermion $\psi_{\mathfrak{c}}$ satisfying the chirality condition

$$\Gamma^{0123456789}\psi_{\mathfrak{c}} = \psi_{\mathfrak{c}}$$

and the free massless Dirac equation along the string:

$$(\Gamma^0\partial_t + \Gamma^9\partial_9)\psi_{\mathfrak{c}} = 0\,.$$

Their low-energy effective action is of the form

$$I = I^{(\mathrm{int})} + \sum_{\mathfrak{c}=1}^{2m} I^{(F1)}_{\mathfrak{c}}\,, \tag{9.3}$$

where $I_{\mathfrak{c}}$ is the bulk $1+1D$ action of the free fields $X^{\mu}_{\mathfrak{c}}, \psi_{\mathfrak{c}}$, and $I^{(\mathrm{int})}$ is the 0+1D action that couples the fields $X^{\mu}_{\mathfrak{c}}, \psi_{\mathfrak{c}}$ at $x_9 = 0$ to the low-energy degrees of freedom of Tr-S theory. In addition to the boundary values of $X^{\mu}_{\mathfrak{c}}, \psi_{\mathfrak{c}}$, the interaction term $I^{(\mathrm{int})}$ depends on additional local 0+1D degrees of freedom, whose form we seek to find. (See Figure 9.1.)

We will now provide a preview of what $I^{(\mathrm{int})}$ looks like, and we will explain the derivation at length in the following subsections. The term $I^{(\mathrm{int})}$ is formulated in terms of additional variables that are localized at the interaction point $x_9 = 0$. These variables include a discrete variable that specifies the "sector," with a finite number of sectors altogether. Each sector is then described by m periodic variables $\mathfrak{p}^a$ $(a =$

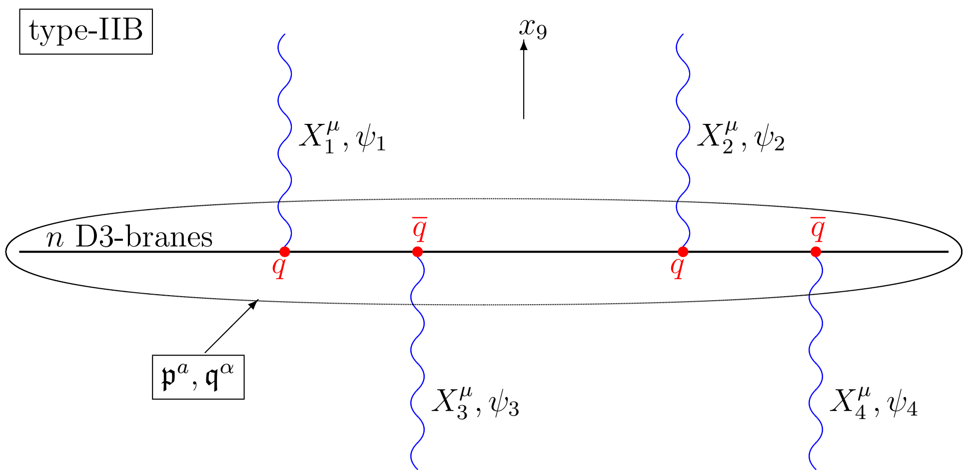


Figure 9.1: External quark and anti-quark sources are realized as endpoints of fundamental strings. At low-energy, the strings are described by free $1+1D$ fields $X_{\mathfrak{c}}^\mu(x_9,t), \psi_{\mathfrak{c}}(x_9,t)$ and the low-energy modes of the compact interacting system of D3-branes and charges are described by periodic variables $\mathfrak{p}^a(t), \mathfrak{q}^\alpha(t)$.

$1,\dots,m$) and additional m' periodic variables $\mathfrak{q}^\alpha$ ($\alpha = 1,\dots,m'$), both of which take values in the range $[0, 2\pi)$. The number m' of $\mathfrak{q}^\alpha$'s depends on the sector, but generally $m' \geq m$. The action $I^{(\text{int})}$ is then a sum of two terms, which we write schematically as:

$$I^{(\text{int})} = I_0 + I_1\,, \qquad I_0 \equiv I_0(\{\mathfrak{p}^a, \mathfrak{q}^\alpha\})\,, \qquad I_1 \equiv I_1(\{\mathfrak{p}^a, \mathfrak{q}^\alpha\}, \{X_{\mathfrak{c}}^1(0), X_{\mathfrak{c}}^2(0)\}).$$

The first term I_0 describes the local system at $x_9 = 0$, while I_1 is the interaction term that couples the system to the boundary values of the scalar fields on the $2m$ fundamental strings. (The fermions will be discussed later, but are suppressed at the moment.)

For the sectors for which $m' = m$ the configuration space of $\{\mathfrak{p}^{\mathfrak{a}}, \mathfrak{q}^\alpha\}$ is T^{2m}, and the system described by I_0 is equivalent to geometric quantization on T^{2m} with the following action:

$$I_0^{(\text{g.q.})} = \frac{1}{2\pi} \int \sum_{a,\alpha=1}^{m} \mathcal{M}_{\mathfrak{a}\alpha} \mathfrak{p}^{\mathfrak{a}} \dot{\mathfrak{q}}^\alpha dt\,, \tag{9.4}$$

where $\mathcal{M}_{\mathfrak{a}\alpha}$ is a nonsingular matrix of integers that we will describe below. As mentioned in the Introduction, the sectors of most interest to us will be of this form. Any additional kinetic terms that are quadratic in $\dot{\mathfrak{q}}^\alpha, \dot{\mathfrak{p}}^{\mathfrak{a}}$ are irrelevant at low-energy. The other sectors, with $m' > m$, also have a piece of the form $I_0^{(\text{g.q.})}$ in their action, but it is necessary to include additional kinetic terms.

The coupling term I_1 is linear in $X^1_{\mathfrak{c}}, X^2_{\mathfrak{c}}$ and the derivatives $\dot{\mathfrak{q}}^\alpha, \dot{\mathfrak{p}}^{\mathfrak{a}}$. It is of the form:

$$I_1 = \frac{1}{2\pi}\int\sum_{\mathfrak{c}\alpha}\mathcal{N}_{\mathfrak{c}\alpha}X^1_{\mathfrak{c}}(0)\dot{\mathfrak{q}}^\alpha dt + \frac{1}{2\pi}\int\sum_{\mathfrak{c}\mathfrak{a}}\mathcal{K}_{\mathfrak{c}\mathfrak{a}}X^2_j(0)\dot{\mathfrak{p}}^{\mathfrak{a}}dt\,, \tag{9.5}$$

where $\mathcal{N}_{\mathfrak{c}\alpha}, \mathcal{K}_{\mathfrak{c}\mathfrak{a}}$ are matrices of integers to be specified later. The remaining fields $X^\mu_{\mathfrak{c}}$ with $\mu = 4, \dots, 8$ have Dirichlet boundary conditions $X^\mu_{\mathfrak{c}}(0) = 0$, while $X^3_{\mathfrak{c}}$ has Neumann boundary conditions. These fields are however irrelevant for our discussion. In §9.3 we will explain how the interactions (9.3)-(9.5) are derived from the type-IIA dual. But first, let us discuss how much supersymmetry is left.

9.2 Supersymmetry

To see how much supersymmetry is preserved we consider once again the realization of (9.2) in type-IIB. We have fundamental strings that stretch along direction x_9 and end on the n D3-branes. Let Γ^A $(A = 0, \dots, 9)$ be 9+1D Dirac gamma matrices, which we take to be real. Let $\epsilon = \epsilon_1 + i\epsilon_2$ be a complex 9+1D Weyl spinor, where ϵ_1, ϵ_2 are Majorana-Weyl, and $\epsilon^* = \epsilon_1 - i\epsilon_2$ its complex conjugate.

The supersymmetry preserved by the n D3-branes is parameterized by those combinations of the supercharges with coefficients ϵ that satisfy:

$$\Gamma^{0123}\epsilon = -i\epsilon\,. \tag{9.6}$$

The S-R-twist preserves

$$\epsilon = e^{-\frac{iv}{2}}e^{\frac{v}{2}(\Gamma^{45}+\Gamma^{67}+\Gamma^{89})}\epsilon\,, \tag{9.7}$$

where the first factor is from the S-twist, and the second from the R-twist, and the interaction (9.2) preserves the same combinations of supersymmetry generators that a fundamental string in directions $0, 9$ preserves, which is given by

$$\Gamma^{09}\epsilon = \epsilon^*\,. \tag{9.8}$$

Combining (9.8) and (9.7) we find

$$\epsilon^* = e^{\frac{iv}{2}}e^{\frac{v}{2}(\Gamma^{45}+\Gamma^{67}-\Gamma^{89})}\epsilon^*\,,$$

while taking complex conjugate of (9.7) yields (keeping in mind that the gamma matrices are all real)

$$\epsilon^* = e^{-\frac{iv}{2}}e^{\frac{v}{2}(\Gamma^{45}+\Gamma^{67}+\Gamma^{89})}\epsilon^*\,.$$

Together, these two equations imply

$$\Gamma^{89}\epsilon = i\epsilon\,.$$

Brane	1	3	4	5	6	7	8	9	10
F1		$\div$							
D2								$\vdash\!\dashv$	$-$

Table 9.1: Open D2-branes are the U-duals of the type-IIB strings. Here $-$ denotes a direction that the brane wraps, $\div$ denotes a direction that the brane/string wraps with a twist, and $\vdash\!\dashv$ denote a direction in which the brane extends but with endpoints (see §9.9).

Then, from (9.7), we obtain

$$\Gamma^{45}\epsilon = -\Gamma^{67}\epsilon\,, \tag{9.9}$$

and together with (9.6) this leaves four linearly independent complex supersymmetry parameters. But (9.8) then puts a reality condition on these parameters, and leaves four real supercharges unbroken.

Out of the $U(3)$ R-symmetry that is preserved by the R-twist (8.1), the static charges only preserve $SU(2) \times U(1) \subset U(3)$. This is the double-cover of the unitary group $U(2) \simeq [SU(2)\times U(1)]/\mathbb{Z}_2$ that acts as unitary rotations of the variables x_4+ix_5, $x_6 + ix_7$, and preserves $x_8 + ix_9$. The surviving supercharges transform as a doublet of $SU(2)$ and are neutral under $U(1)$ (which is generated by $\Gamma^{45} + \Gamma^{67}$).

9.3 Constructing the type-IIA dual of charges

Now we transform the system of D3-branes and fundamental strings to type-IIA by applying the U-duality transformation described in Table 8.1. The $2m$ fundamental strings turn into D2-branes, and the n D3-branes turn into fundamental strings, as listed in Table 9.1. In the type-IIB picture the strings end on the n D3-branes, but in the type-IIA picture the D2-branes are too big to end on the n strings. The system must therefore rearrange itself, and we have to pair up each D2-brane that corresponds to a quark (extending in the positive x_9 direction) with a D2-brane that corresponds to an anti-quark (extending in the negative x_9 direction), and glue them into a single smooth D2-brane. We thus get m D2-branes whose worldvolume is equivalent to an infinite cylinder.

Some of the type-IIA closed strings that we had in Chapter 8 must now be allowed to break into open strings and end on the D2-branes. Every D2-brane must have at least one such open string attached to it, because otherwise the corresponding type-IIB string would not be bound to any of the n D3-branes. For ease of discussion it will be convenient to slightly separate the D2-branes in the x_3 direction. The resulting configuration is depicted in Figure 9.2.

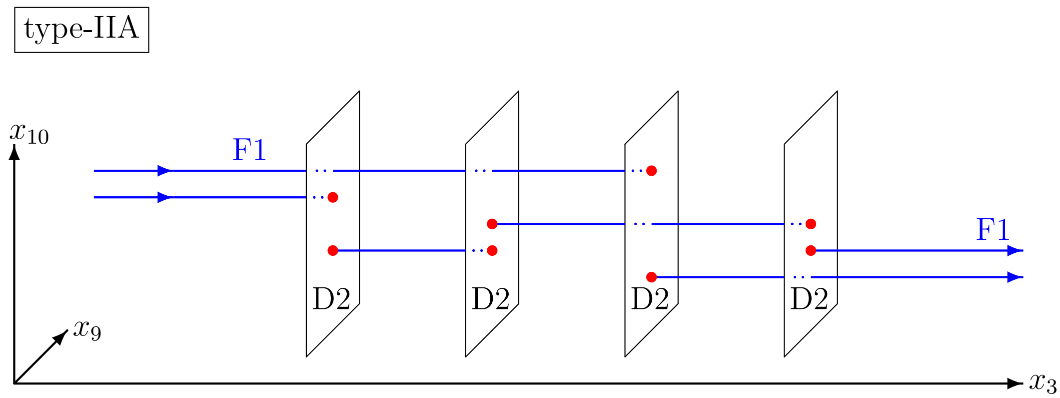


Figure 9.2: The type-IIA configuration following the U-duality transformation of Table 8.1. The m pairs of type-IIB open strings become m continuous D2-branes. The n D3-branes become n fundamental strings, which in the presence of the D2-branes can break up into open strings. At least one pair of open strings must be attached to each D2-brane. In this example $m = 4$ and $n = 2$.

9.4 Local $\mathfrak{p}^{\mathfrak{a}}, \mathfrak{q}^{\alpha}$ variables and their action I_0

To understand how the $\mathfrak{p}^{\mathfrak{a}}, \mathfrak{q}^{\alpha}$ that appear in (9.4) arise, it is instructive to start with a simple example: one D2-brane ($m = 1$) and one string ($n = 1$) for the case $k = 2$ with $v = \frac{\pi}{2}$. We have one fundamental string wrapping direction x_3 and bound to the D2-brane, which wraps direction x_{10} and extends in direction x_9. The system is therefore dual to $U(1)$ Tr-S theory with an external quark and anti-quark pair.

To bind to a D2-brane, the closed string must break to become an open string that starts and ends on the D2-brane located at $x_3 = 0$. Let the string start at T^2 coordinate $z = x_{10} + ix_1 \equiv \mathfrak{q} + i\mathfrak{u}$ on the D2-brane, where $\mathfrak{u}$ and $\mathfrak{q}$ are functions of time. To make $\mathfrak{u}$ and $\mathfrak{q}$ compact variables with period 2π, we will rescale the x_{10} and x_1 coordinates so that from now on they take values in $[0, 2\pi)$. To be of minimal length the string must remain at $x_9 = 0$ and at constant z, as x_3 varies from 0 to $2\pi R$. The point with coordinates $x_3 = 2\pi R$ and $z = \mathfrak{q} + i\mathfrak{u}$ is equivalent in the geometry of (8.5) to the point with coordinates $x_3 = 0$ and $z = e^{iv}(\mathfrak{q} + i\mathfrak{u}) = -\mathfrak{u} + i\mathfrak{q}$. This point must also be on the D2-brane, which wraps direction x_{10} but is at a fixed x_1 coordinate. We thus find that $\mathfrak{u} = \mathfrak{q}$. In other words, the bound fundamental string starts at $z' = (1 + i)\mathfrak{q}$ on the D2-brane, and ends at $z'' = (-1 + i)\mathfrak{q}$ on the same D2-brane (see Figure 9.3).

The starting point and endpoint of the string are oppositely charged under the $U(1)$ gauge field that resides on the D2-brane. Let A_{10} be the component of this

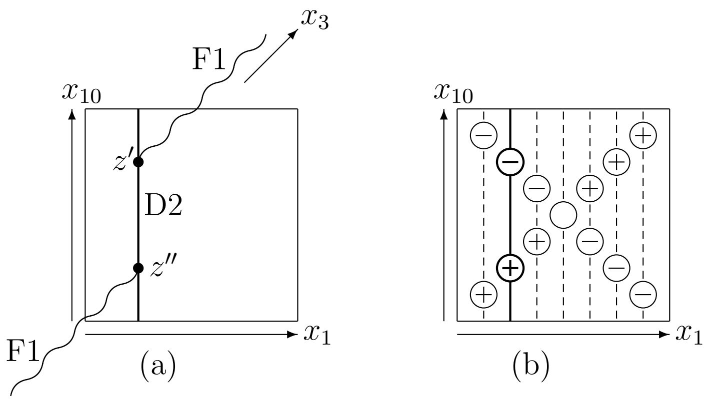


Figure 9.3: (a) A fundamental string (F1) bound to the D2-brane. The D2-brane wraps the compact direction x_{10} (the vertical direction) and extends indefinitely in direction x_9 (not shown in the picture). The fundamental string is at $x_9 = 0$ and extends in direction x_3 (perpendicular to the plane of the drawing). Because of the S-duality twist, which in the type-IIA picture translates to a rotation, the fundamental string's endpoint z'' can be different from its starting point z'. (b) Configurations of the D2-brane with the two endpoints z', z'' of the string marked as oppositely charged points. As the D2-brane changes its x_1-position, the positions of the charges change accordingly.

gauge field in the x_{10} direction. We can fix the gauge so that A_{10} is independent of x_{10}, and set $\mathfrak{p} \equiv 2\pi A_{10}$. The starting point and endpoint of the string are separated along the x_{10} direction by $\Delta x_{10} = 2\mathfrak{q}$, so the action of the system includes the term:

$$I_0 \equiv 2 \int A_{10} d\mathfrak{q} = \frac{1}{2\pi} \int 2\mathfrak{p} d\mathfrak{q} \,. \tag{9.10}$$

We claim that I_0 is the only relevant term at low-energy. For example, the kinetic energies of the fundamental string and of the D2-brane are irrelevant at low-energy, because they are proportional to $(\partial_0 \mathfrak{q})^2$. Indeed, setting the mass dimensions of $\mathfrak{p}, \mathfrak{q}$ to zero, we see that the kinetic term has dimension 2 and is irrelevant in a 0+1D theory.

The action I_0 describes the geometric quantization of a torus with the symplectic form $2d\mathfrak{p} \wedge d\mathfrak{q}$. The Hilbert space has two states, and this is indeed the number of states we expect, since the $U(1)$ Tr-S theory is known to be equivalent to Chern-Simons theory at level k. (See Chapter 10 for more details.)

We can now turn to the general case. The 0+1D variables $\mathfrak{p}^a, \mathfrak{q}^\alpha$ that appear in (9.4) arise out of the type-IIA picture as follows. The low-energy description of each of the m D2-branes includes a gauge field $A^{(\mathfrak{a})}$ ($\mathfrak{a} = 1, \ldots, m$). The periodic variable

$\mathfrak{p}^{\mathfrak{a}}$ is identified with the holonomy of $A^{(\mathfrak{a})}$ around the x_{10} circle at $x_9 = 0$:

$$\mathfrak{p}^{\mathfrak{a}}(t) = \int_0^{2\pi} A_{10}^{(\mathfrak{a})} dx_{10}\Big|_{x_9=0}. \tag{9.11}$$

In the presence of the D2-branes, fundamental strings can break into open strings. A string doesn't have to break at every D2-brane, but for ease of notation let us first assume that each of the n fundamental strings does break at every D2-brane. We thus have $n \times (m+1)$ open string segments. Each segment must be located at a constant x_{10} in order for its length to be minimal. We denote the x_{10} coordinates of the open string segments by variables $\mathfrak{q}^{\mathfrak{i}\mathfrak{a}'}$ (with $\mathfrak{i} = 1, \dots, n$ and $\mathfrak{a}' = 0, \dots, m$). The string segments are ordered via the index $\mathfrak{a}'$ in the direction of increasing x_3. (See Figure 9.4 for an illustration.) In addition to the x_{10} coordinates we also need to know the x_1 coordinates of the strings. We denote them by $\mathfrak{u}^{\mathfrak{i}\mathfrak{a}'}$:

$$(x_1, x_{10}) \to (\mathfrak{u}^{\mathfrak{i}\mathfrak{a}'}, \mathfrak{q}^{\mathfrak{i}\mathfrak{a}'}).$$

However, as noted above, an open string doesn't have to break at every D2-brane. For a given state, we define the *binding matrix* $\mathfrak{B}$ to be the following $n \times m$ matrix of 0 and 1's that encodes which strings break and which do not:

$$\mathfrak{B}_{\mathfrak{i}\mathfrak{a}} = \begin{cases} 1 & \text{if the } \mathfrak{i}^{th} \text{ string is bound to the } \mathfrak{a}^{th} \text{ D2-brane,} \\ 0 & \text{otherwise,} \end{cases} \tag{9.12}$$

for $\mathfrak{i} = 1, \dots, n$ and $\mathfrak{a} = 1, \dots, m$. We therefore have

$$\mathfrak{u}^{\mathfrak{i}\mathfrak{a}} = \mathfrak{u}^{\mathfrak{i}(\mathfrak{a}-1)}, \quad \mathfrak{q}^{\mathfrak{i}\mathfrak{a}} = \mathfrak{q}^{\mathfrak{i}(\mathfrak{a}-1)}, \qquad \text{if } \mathfrak{B}_{\mathfrak{i}\mathfrak{a}} = 0,$$

which indicates that the $\mathfrak{i}^{th}$ string is continuous at the $\mathfrak{a}^{th}$ brane. On the other hand, at every break point $(\mathfrak{i}, \mathfrak{a})$ on the $\mathfrak{a}^{th}$ D2-brane the $\mathfrak{i}^{th}$ string is charged under the gauge field on the $\mathfrak{a}^{th}$ D2-brane $A^{(\mathfrak{a})}$, and the $A_{10}^{(\mathfrak{a})} dx_{10}$ interaction of the gauge field with the charged particle produces a term proportional to $\mathfrak{p}^{\mathfrak{a}} d\mathfrak{q}^{\mathfrak{i}\mathfrak{a}}$ in the effective action. The sign of this term is positive for one end of the string and negative for the other, as the charges of the two ends are opposite. Thus an open segment of the $\mathfrak{i}^{th}$ string that starts on the $\mathfrak{a}^{th}$ D2-brane and ends on the $\mathfrak{b}^{th}$ D2-brane contributes $\mathfrak{p}^{\mathfrak{a}} d\mathfrak{q}^{\mathfrak{i}\mathfrak{a}} - \mathfrak{p}^{\mathfrak{b}} d\mathfrak{q}^{\mathfrak{i}(\mathfrak{b}-1)}$ to the action.

On top of this, direction x_3 is compact with $x_3 \in [0, 2\pi R]$ and the S-duality and R-symmetry operators act at $x_3 = 0$. The S-R-twisted boundary conditions induce linear relations between the x_1 and x_{10} coordinates of the strings they connect, where we are free to add a permutation $\sigma \in S_n$ among the n strings before applying the S-R-twist. A particular sector of the Hilbert space is thus described by the $\mathfrak{B}_{\mathfrak{i}\mathfrak{a}}$ matrix, as well as the permutation $\sigma \in S_n$. We collect this information in a matrix and denote

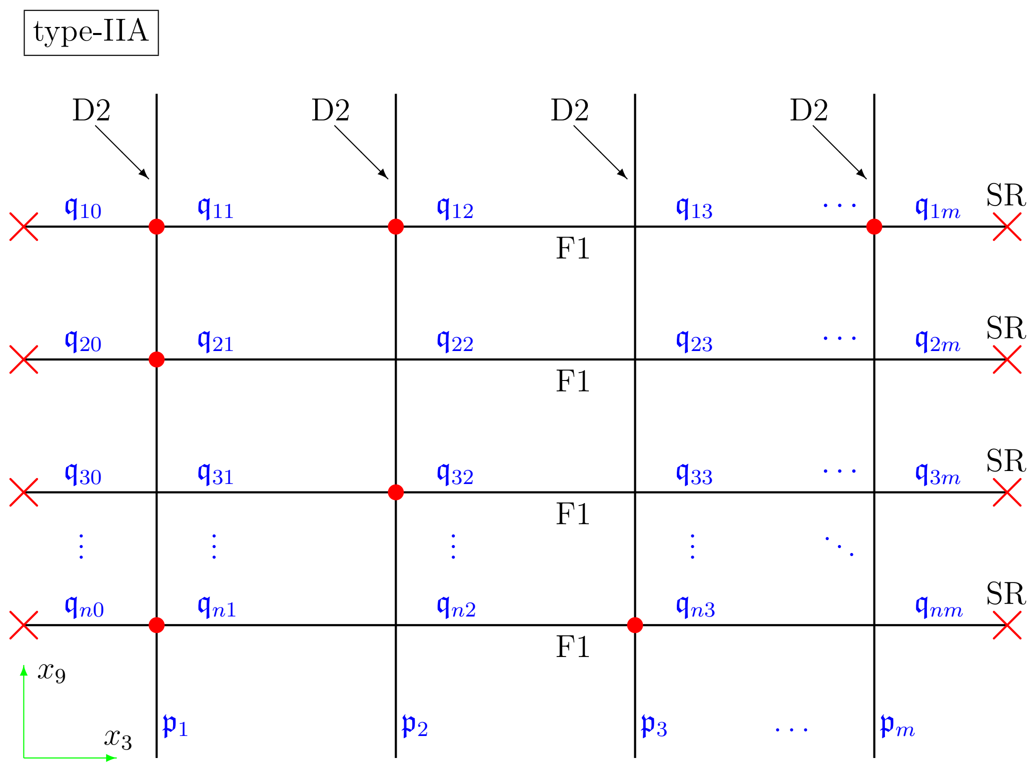


Figure 9.4: A variable $\mathfrak{p}^{\mathfrak{a}}$ ($\mathfrak{a} = 1, \ldots, m$) is associated with each D2-brane, ordered in the direction of increasing x_3. Let every string break at every brane into a total of $m+1$ open segments. The constant x_{10} coordinate of each of these segments is denoted by $\mathfrak{q}^{\mathfrak{i}\mathfrak{a}'}$ with $\mathfrak{i} = 1, \ldots, n$ and $\mathfrak{a}' = 0, \ldots, m$ in the direction of increasing x_3. However, the strings actually break only at positions marked with a $\bullet$ and the main contribution to the action comes from these break points. The strings are coincident, but we separated them in the picture for clarity.

a given sector as

$$\left[\begin{array}{cccc|c} \mathfrak{B}_{11} & \mathfrak{B}_{12} & \cdots & \mathfrak{B}_{1m} & \sigma(1) \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} & \cdots & \mathfrak{B}_{2m} & \sigma(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{B}_{n1} & \mathfrak{B}_{n2} & \cdots & \mathfrak{B}_{nm} & \sigma(n) \end{array}\right].$$

Obviously, two sectors $[\mathfrak{B}_{\mathfrak{i}\mathfrak{a}}|\sigma]$ and $[\mathfrak{B}_{\sigma'(\mathfrak{i})\mathfrak{a}}|\sigma'\circ\sigma\circ\sigma'^{-1}]$ are equivalent (with $\sigma'\in S_n$). The boundary conditions can now be written as

$$e^{iv}(\mathfrak{q}^{\mathfrak{i}m}+\tau\mathfrak{u}^{\mathfrak{i}m}) = \mathfrak{q}^{\sigma(\mathfrak{i})0}+\tau\mathfrak{u}^{\sigma(\mathfrak{i})0} \pmod{2\pi(\mathbb{Z}+\tau\mathbb{Z})}. \tag{9.13}$$

Using (7.2) and $\tau = \frac{\mathbf{a}\tau+\mathbf{b}}{\mathbf{c}\tau+\mathbf{d}}$ we can rewrite (9.13) as an equation with integer coefficients:

$$\mathfrak{q}^{\sigma(\mathfrak{i})0} = \mathbf{d}\mathfrak{q}^{\mathfrak{i}m}+\mathbf{b}\mathfrak{u}^{\mathfrak{i}m}\,, \qquad \mathfrak{u}^{\sigma(\mathfrak{i})0} = \mathbf{c}\mathfrak{q}^{\mathfrak{i}m}+\mathbf{a}\mathfrak{u}^{\mathfrak{i}m}\,. \tag{9.14}$$

For example, for $v=\frac{\pi}{2}$ ($k=2$), this becomes

$$\mathfrak{u}^{\mathfrak{i}m} = -\mathfrak{q}^{\sigma(\mathfrak{i})0}\,, \qquad \mathfrak{q}^{\mathfrak{i}m} = \mathfrak{u}^{\sigma(\mathfrak{i})0} \pmod{2\pi\mathbb{Z}}.$$

Finally, let $\mathfrak{u}^{\mathfrak{a}}$ ($\mathfrak{a}=1,\dots,m$) be the x_1 coordinate of the $\mathfrak{a}^{th}$ D2-brane. Then we have the equations

$$\mathfrak{u}^{\mathfrak{a}} = \mathfrak{u}^{\mathfrak{i}\mathfrak{a}} = \mathfrak{u}^{\mathfrak{i}(\mathfrak{a}-1)}\,, \qquad \text{whenever } \mathfrak{B}_{\mathfrak{i}\mathfrak{a}}=1, \tag{9.15}$$

since the $\mathfrak{i}^{th}$ string connects with the $\mathfrak{a}^{th}$ D2-brane. The equations (9.13)-(9.15) reduce the total number of independent $\mathfrak{q}^{\mathfrak{i}\mathfrak{a}'}$ variables. A linearly independent basis can be chosen, and these furnish the $\mathfrak{q}^{\alpha}$ variables in (9.4). In Chapter 10-Chapter 11 we will present explicit detailed examples.

Congested and decongested matrices

We say that a binding matrix $\mathfrak{B}_{\mathfrak{i}\mathfrak{a}}$ is *congested* if there is at least one $\mathfrak{a}$ for which there are two distinct $\mathfrak{i}\neq\mathfrak{j}$ such that $\mathfrak{B}_{\mathfrak{i}\mathfrak{a}}=\mathfrak{B}_{\mathfrak{j}\mathfrak{a}}=1$. This means that there is at least one D2-brane from which at least two strings emanate. If every D2-brane has exactly one string emanating from it, we say that the binding matrix is *decongested.* In this case, for every $\mathfrak{a}=1,\dots,m$ there is exactly one $\mathfrak{i}$ for which $\mathfrak{B}_{\mathfrak{i}\mathfrak{a}}=1$. The difference between congested and decongested binding matrices will become relevant when we discuss fermionic zero modes in §11.2.

9.5 The interaction term I_1

We will now derive the interaction of the 0+1D variables $\mathfrak{p}^{\mathfrak{a}},\mathfrak{q}^{\alpha},\mathfrak{u}^{\mathfrak{a}}$ with the $1+1D$ fields. The low-energy $1+1D$ fields that are relevant for the present discussion can be described either in type-IIA or in type-IIB. In type-IIB, these fields are the two scalars $X^1(x_9,t), X^2(x_9,t)$ (with the index $\mathfrak{c}$ of the string suppressed). In type-IIA, the two relevant low-energy fields on the D2-brane are the gauge field component A_{10} and the x_1 coordinate of the D2-brane, which we denote by Φ^1. Note that direction 1 in type-IIA is related to direction 1 in type-IIB via T-duality. Following the U-duality of Table 8.1, it is then easy to see that X^1, X^2 are the duals (as $1+1D$ free compact scalar fields) of Φ^1, A_{10}, respectively:

$$\partial_9 X^1 = \partial_t \Phi^1\,, \quad \partial_t X^1 = -\partial_9 \Phi^1\,, \quad \partial_9 X^2 = \partial_t A_{10}\,, \quad \partial_t X^2 = \partial_9 A_{10}\,. \tag{9.16}$$

From this simple observation, it is easy to derive the requisite interaction between X^1, X^2 and $\mathfrak{p}^{\mathfrak{a}}, \mathfrak{q}^{\alpha}, \mathfrak{u}^{\mathfrak{a}}$. In the type-IIA picture $\{\mathfrak{p}^{\mathfrak{a}}, \mathfrak{u}^{\mathfrak{a}}\}$ determine the boundary conditions at $x_9 = 0$ of Φ^1, A_{10}. If $\mathfrak{a}$ is the index of the brane, then by definition, we have the Dirichlet boundary conditions:

$$\begin{aligned} 2\pi A_{10}(x_9 = 0, t) &= \mathfrak{p}^{\mathfrak{a}}(t)\,, \\ \Phi^1(x_9 = 0, t) &= \mathfrak{u}^{\mathfrak{a}}(t)\,, \end{aligned}$$

where we suppressed brane indices on the left-hand sides. The duality (9.16) then converts the Dirichlet boundary conditions of type-IIA to Neumann boundary conditions of type-IIB. The latter can be incorporated into the action with the addition of the term

$$\int (\mathfrak{u}^{\mathfrak{a}} dX^1 + \mathfrak{p}^{\mathfrak{a}} dX^2)\,. \tag{9.17}$$

For most sectors the $\mathfrak{u}^{\mathfrak{a}}$'s can be written as linear combinations of the $\mathfrak{q}^{\alpha}$'s by using the various constraints discussed at the end of §9.4. In these cases the sum of the terms (9.17) for all the D2-branes takes the form of I_1 in (9.5).

In the present paper we will not have much use for the interaction term I_1, since we keep the charge coordinates X^1, X^2 constant. We have nevertheless presented it here for completeness. In a future work we hope to explore the dependence of the Hilbert space on the position of the charges, and the term I_1 will then play a central role. We conclude the discussion of the interaction term by presenting a more geometrical interpretation of I_1.

D2-F1 intersections

The interaction term $\mathfrak{p}^{\mathfrak{a}} dX^2$ in (9.17) has a simple interpretation in terms of the geometry of the lift of the D2-branes and F1-strings to M-theory. To see this, let us focus on a single F1-string and a single D2-brane. We start by recalling some facts about this system, following the techniques developed in [53]. There, a configuration of D4-branes and NS5-branes was analyzed by lifting it to M-theory. Here, we need to analyze a similar configuration of fundamental strings and D2-branes, and we are also going to lift it to M-theory.

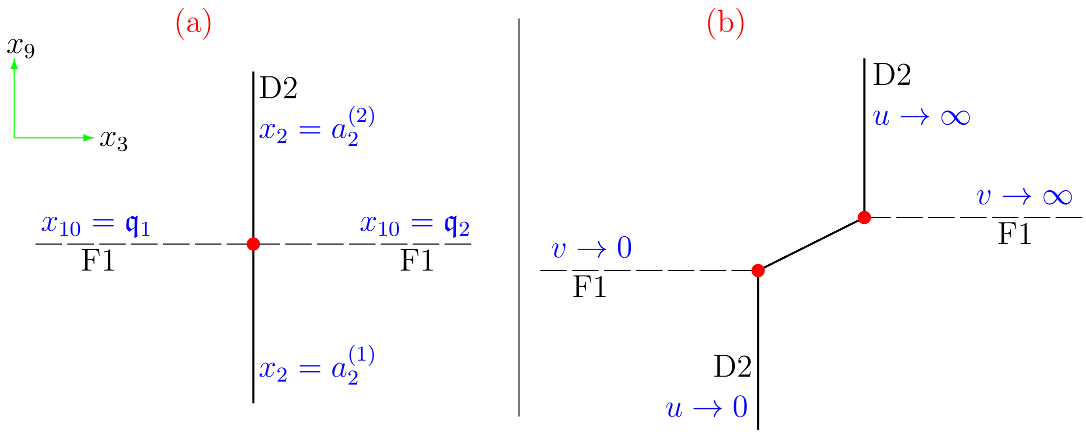


Figure 9.5: (a) A junction of two open fundamental strings, one starting and one ending on a D2-brane. (b) The D2-brane wraps direction $10 \equiv \natural$ and the configuration can be deformed so that it lifts to a smooth holomorphic curve in M-theory.

The lift essentially brings us back to the second-from-last row of Table 8.1, and the M-theory direction in the present discussion is therefore denoted by x_2. We assume that x_2, x_{10} are periodic with period 2π. Let us also assume for simplicity that the length of the x_3 direction is very large, so that $-\infty < x_3 < \infty$. (That is, we will suspend the effect of S-R-twist for the moment.) Following [53], we define complex variables

$$v = e^{x_9 + ix_{10}}, \qquad u = e^{x_3 + ix_2}.$$

The configuration of an F1-string intersecting a D2-brane then lifts to a single M2-brane which extends along the locus of the complex equation

$$v = \frac{u - e^{i\alpha}}{u - e^{-i\alpha}}, \tag{9.18}$$

where α is a real constant. Equation (9.18) is designed so that as $x_9 \to \infty$ (where v has a pole) $x_2 = -\alpha$, and as $x_9 \to -\infty$ (where v has a zero) $x_2 = \alpha$ (see Figure 9.5b). Equation (9.18) also tells us that as $x_3 \to \infty$ we have $v = 1$, while as $x_3 \to -\infty$ we have $v = e^{2i\alpha}$. If we now let $\mathfrak{q}^1$, $\mathfrak{q}^2$ be the x_{10} coordinates of the string between $x_3 = \infty$ and $x_3 = -\infty$, then we have

$$\Delta x_{10} \equiv \mathfrak{q}^2 - \mathfrak{q}^1 = 2\alpha = -X^2(x_9 = +\infty) + X^2(x_9 = -\infty) \mod 2\pi. \tag{9.19}$$

Going to the low-energy limit, we can interpret the boundary condition $X^2(x_9 = +\infty)$ as the boundary value at $x_9 = 0$ of the X^2 field on the $x_9 > 0$ portion of the string (which we denoted by $X^2_{\mathfrak{a}}$), and similarly the boundary condition $X^2(x_9 = -\infty)$ is the low-energy boundary value at $x_9 = 0$ of the X^2 field on the $x_9 < 0$ portion of the string (which we denoted by $X^2_{\mathfrak{a}+m}$). The geometrical equation (9.19) is therefore consistent with the equations of motion derived from varying $\mathfrak{p}$ in the action

$$\int \mathfrak{p}(\dot{\mathfrak{q}}^2 - \dot{\mathfrak{q}}^1 + \dot{X}^2_{\mathfrak{a}} - \dot{X}^2_{\mathfrak{a}+m}).$$

After taking into account the S-R-twist, $\mathfrak{q}^1$ and $\mathfrak{q}^2$ are not independent anymore, and the above expression then becomes the contribution of $\mathfrak{p}$ to the action (9.3).

9.6 Bosonic zero modes

In §9.4 we explained how to determine the bosonic part of the action in terms of the variables $\mathfrak{p}^{\mathfrak{a}}, \mathfrak{u}^{i\mathfrak{a}'}$, and $\mathfrak{q}^{i\mathfrak{a}'}$. Generally, this action can be further simplified because equations (9.13)-(9.15) reduce the total number of independent $\mathfrak{q}^{i\mathfrak{a}'}$ variables. So in general, as we shall see in concrete examples in Chapter 10 and Chapter 11, some of the $\mathfrak{u}^{i\mathfrak{a}'}$ variables can be expressed as linear combinations of the $\mathfrak{q}^{i\mathfrak{a}'}$'s, while the remaining $\mathfrak{u}^{i\mathfrak{a}'}$'s are reduced to discrete values. After eliminating the $\mathfrak{u}^{i\mathfrak{a}'}$'s we get an action of the form (9.4). If the corresponding constants $\mathcal{M}_{\mathfrak{a}\alpha}$ that appear in this action form a nonsingular square matrix, quantization of (9.4) gives rise to a finite dimensional Hilbert space.

However, for some sectors we end up with more than m independent $\mathfrak{q}^{i\mathfrak{a}'}$'s and then the procedure of §9.4 yields an expression similar to (9.4) but where $\mathcal{M}_{\mathfrak{a}\alpha}$ is not a square matrix. The simplest sector for which this happens is for $n = m = 2$ with

$$[\mathfrak{B}|\sigma] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] . \tag{9.20}$$

In general for $n = 2$ there are more than m independent $\mathfrak{q}^{i\mathfrak{a}'}$'s when $\mathfrak{B}$ has at least two columns of the form $(11)^\top$ [where $(\cdots)^\top$ denotes the transposed matrix]. As will be explained in more detail in one of the examples of §11.1, for $k = 2$ the action for the sector (9.20) is

$$I_0 = \frac{1}{2\pi} \int \left\{ \mathfrak{p}^1 \left[d\left(\mathfrak{q}^{11} + \mathfrak{q}^{21}\right) + 2d\mathfrak{q}^{12} \right] + \mathfrak{p}^2 \left[-d\left(\mathfrak{q}^{11} + \mathfrak{q}^{21}\right) + 2d\mathfrak{q}^{12} \right] \right\} . \tag{9.21}$$

In this sector there are two strings connecting the two D2-branes, and the action only depends on $\mathfrak{q}^{11}$ and $\mathfrak{q}^{21}$ through the center of mass $Q_{com} \equiv \mathfrak{q}^{11} + \mathfrak{q}^{21}$, and is independent of the relative coordinate $Q_{rel} \equiv \mathfrak{q}^{11} - \mathfrak{q}^{21}$. Therefore, in order to proceed we need to add a kinetic term, proportional to $\dot{Q}_{rel}^2$. However, if we are only interested in ground states we may simply rewrite (9.21) in terms of Q_{com},

$$I_0 = \frac{1}{2\pi} \int \left\{ \mathfrak{p}^1 (dQ_{com} + 2d\mathfrak{q}^{12}) + \mathfrak{p}^2 (-dQ_{com} + 2d\mathfrak{q}^{12}) \right\} , \tag{9.22}$$

which is of the form (9.4) with a nonsingular square matrix $\mathcal{M} = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$. From (9.22) we can determine the number of ground states, which happens to be 4.

We will show later on that sectors with bosonic zero modes invariably also possess fermionic zero modes and therefore do not contribute to the Witten Index. In fact, the sectors with bosonic zero modes form a proper subset of the set of congested sectors, and all congested sectors have fermionic zero modes. In this paper we are only concerned with the Witten Index, and we therefore do not need to consider sectors with bosonic zero modes anymore.

9.7 Fermionic zero modes

So far in this section, we have mainly focused on the bosonic degrees of freedom of the system. While we will not present the explicit form of the fermionic part of the action, it will turn out to be important to understand the fermionic zero modes of the system in each sector described by the binding matrix $\mathfrak{B}_{\mathfrak{i}\mathfrak{a}}$ and permutation $\sigma \in S_n$. Specifically, they will be crucial to our argument that only decongested sectors contribute to the Witten Index.[2] Therefore, in this subsection, we will discuss various chirality and boundary conditions that these fermionic zero modes have to satisfy.

Our conventions for fermions are as follows. In describing the fermionic modes of type-IIA theory we find it more convenient to consider its M-theory lift. Therefore, we denote a fermion by a real 32-component 10+1D spinor on which the Dirac matrices $\Gamma^0, \dots, \Gamma^9, \Gamma^\natural$ act. (We use the notation $\natural \equiv 10$ to avoid confusion between Γ^{10} and $\Gamma^1\Gamma^0$.) They satisfy the identity

$$\Gamma^{0123456789\natural} = 1. \tag{9.23}$$

The low-energy fermionic modes of the system can then be described in terms of $1+1D$ fermionic fields that are supported on the type-IIA open strings and on the dimensional reduction (on x_{10} direction) of the D2-branes. (See Figure 9.6 for illustration.) The fermionic field along the $\mathfrak{i}^{th}$ open F1-string between the $\mathfrak{a}'^{th}$ and $(\mathfrak{a}'+1)^{st}$ D2-branes is denoted by $\psi_{\mathfrak{i}\mathfrak{a}'}$ (with $\mathfrak{i} = 1, \dots, n$ and $\mathfrak{a}' = 1, \dots, m-1$). For every $\mathfrak{i} = 1, \dots, n$, there is another piece of string starting on the m^{th} D2-brane, going through the S-R-twist, and ending on the 1^{st} D2-brane. To capture the fields on this string using the same notation, we extend the range of $\mathfrak{a}'$ to $0, \dots, m$ and postulate that the fields $\psi_{\mathfrak{i}0}$ and $\psi_{\mathfrak{i}m}$ are identified up to the S-R-twist

$$\psi_{\sigma(\mathfrak{i})0} = e^{\frac{v}{2}(\Gamma^{1\natural}+\Gamma^{45}+\Gamma^{67}+\Gamma^{89})}\psi_{\mathfrak{i}m}\,, \tag{9.24}$$

where we have also allowed the possibility of the action of the permutation $\sigma \in S_n$, as discussed in §9.4. We set

$$\mathcal{P} \equiv e^{-\frac{v}{2}(\Gamma^{1\natural}+\Gamma^{45}+\Gamma^{67}+\Gamma^{89})}\,. \tag{9.25}$$

[2] This will be our only use for the fermionic degrees of freedom, so a reader who doesn't wish to go into the detailed proof of this statement can skip the present section and §10.3,§11.2, as well as some portions of §9.9.

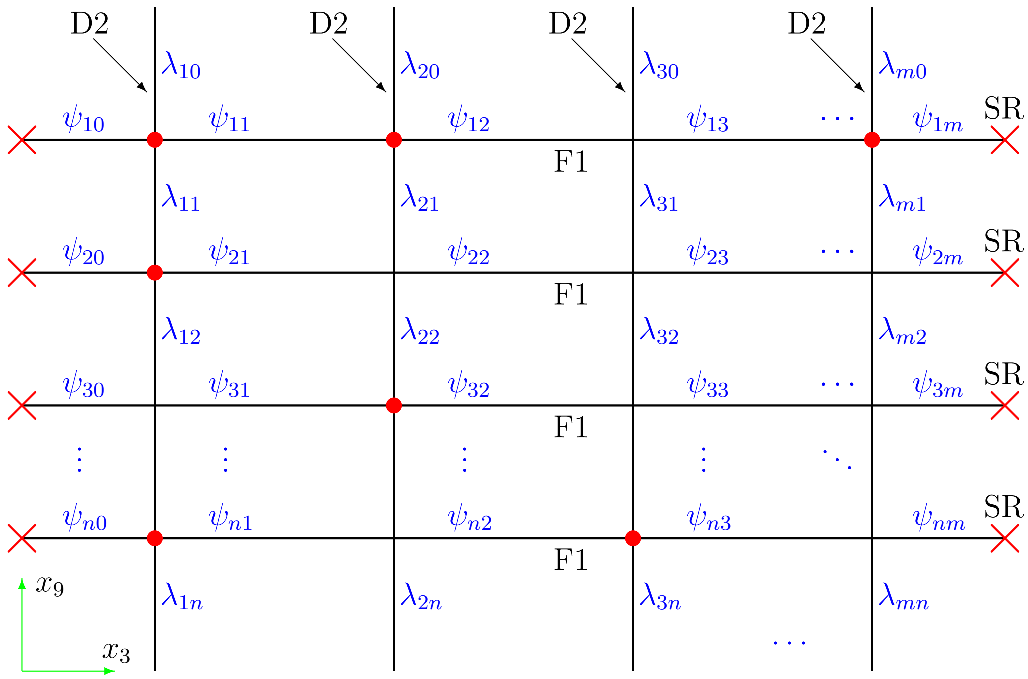


Figure 9.6: The fermionic zero modes are constructed from solutions of the linear equations for the boundary conditions of the gluino fields $\lambda_{\mathfrak{a}\mathfrak{i}'}$ on the D2-brane sections (viewed as 1-dimensional segments below the x_{10} compactification scale) and the fermionic modes $\psi_{\mathfrak{i}\mathfrak{a}}$ on string sections. The S-R-twist is denoted by a $\times$.

Generally, the $\psi_{\mathfrak{i}\mathfrak{a}'}$ fields are functions of (x_3, t) (with the appropriate finite range for x_3), but at low-energy only the zero modes are important, so we can assume that $\psi_{\mathfrak{i}\mathfrak{a}'}$ is independent of x_3. The $\psi_{\mathfrak{i}\mathfrak{a}'}$ fields also satisfy the obvious boundary conditions that if the $\mathfrak{i}^{th}$ string is not bound to the $\mathfrak{a}^{th}$ D2-brane then it continuously connects with $\psi_{\mathfrak{i}(\mathfrak{a}-1)}$:

$$\psi_{\mathfrak{i}\mathfrak{a}} = \psi_{\mathfrak{i}(\mathfrak{a}-1)} \qquad \text{if } \mathfrak{B}_{\mathfrak{i}\mathfrak{a}} = 0.$$

All the $\psi_{\mathfrak{i}\mathfrak{a}}$ fields satisfy the chirality condition

$$0 = (1 + \Gamma^{023})\psi_{\mathfrak{i}\mathfrak{a}} \tag{9.26}$$

corresponding to the low-energy fields on an M2-brane with x_2 being the M-theory direction.

Next, we define the fields along the D2-branes. Dimensionally reducing along the compact direction of x_{10}, the D2-branes become $1+1D$ objects, and we denote by

$\lambda_{\mathfrak{a}\mathfrak{i}'}$ (with $\mathfrak{a} = 1, \dots, m$ and $\mathfrak{i}' = 0, \dots, n$) the low-energy fermionic fields supported on the segment of $\mathfrak{a}^{th}$ D2-brane between the $\mathfrak{i}'^{th}$ and $(\mathfrak{i}'+1)^{st}$ F1-string, where we use the convention that $\mathfrak{i}' = 0$ corresponds to the segment that continues from the first string to $x_9 > 0$, and $\mathfrak{i}' = n$ continues from the n^{th} string to $x_9 < 0$ (see Figure 9.6). These fields are generally functions of (x_9, t), but at low-energy they can again be assumed to be constant. Formally, the range of x_9 for $1 \leq \mathfrak{i}' < n$ is zero, but because the fields are constant this does not matter. A more elaborate treatment starting with 2D fields that are harmonic functions of $x_9 + ix_{10}$, with poles at the intersections with the F1-string, will lead to a similar result.

Similarly as for $\psi_{\mathfrak{i}\mathfrak{a}'}$, we have for $\lambda_{\mathfrak{a}\mathfrak{i}'}$ the continuity conditions

$$\lambda_{\mathfrak{a}\mathfrak{i}} = \lambda_{\mathfrak{a}(\mathfrak{i}-1)} \qquad \text{if } \mathfrak{B}_{\mathfrak{i}\mathfrak{a}} = 0$$

and we also have the chirality conditions:

$$0 = (1 + \Gamma^{09\natural})\lambda_{\mathfrak{a}\mathfrak{i}'}. \tag{9.27}$$

At the D2-F1 junctions where the $\mathfrak{a}^{th}$ D2-brane and $\mathfrak{i}^{th}$ F1-string intersect (so that $\mathfrak{B}_{\mathfrak{i}\mathfrak{a}} = 1$), the zero modes have to satisfy the following boundary conditions:

$$(1 - \Gamma^{239\natural})\psi_{\mathfrak{i}\mathfrak{a}} = (1 - \Gamma^{239\natural})\psi_{\mathfrak{i}(\mathfrak{a}-1)} = (1 - \Gamma^{239\natural})\lambda_{\mathfrak{a}\mathfrak{i}} = (1 - \Gamma^{239\natural})\lambda_{\mathfrak{a}(\mathfrak{i}-1)}\,, \tag{9.28}$$

and

$$0 = (1 + \Gamma^{239\natural})(\psi_{\mathfrak{i}(\mathfrak{a}-1)} - \psi_{\mathfrak{i}\mathfrak{a}} + \Gamma^{39}\lambda_{\mathfrak{a}\mathfrak{i}} - \Gamma^{39}\lambda_{\mathfrak{a}(\mathfrak{i}-1)})\,. \tag{9.29}$$

These equations are derived by first going to the M-theory picture as in Figure 9.5, where the D2-F1 junction is described by a single M2-brane, and then deforming the M2-brane worldvolume. Details of the derivation of (9.28)–(9.29) are provided in Appendix 13.1.

9.8 The eigenvalues of $\mathcal{P}$

The operator $\mathcal{P}$, defined in (9.25), realizes the S-R-twist on the fermionic modes of the fundamental strings in the type-IIA picture. In this subsection we calculate its eigenvalues. This will be important in §10.3 and §11.2 where we prove the absence of zero modes for certain sectors.

Consider a spinor ψ that satisfies $\mathcal{P}\psi = \varepsilon\psi$ for some eigenvalue ε. Since $\mathcal{P}$ commutes with Γ^{023} we may assume that ψ has a specific Γ^{023} chirality. We first assume that

$$\Gamma^{023}\psi = \psi.$$

Then, $\Gamma^{0123456789\natural} = 1$ implies

$$0 = (1 - \Gamma^{023})\psi = (1 - \Gamma^{1456789\natural})\psi\,,$$

and hence

$$\Gamma^{451\natural}\psi = \Gamma^{451\natural}\Gamma^{1456789\natural}\psi = \Gamma^{6789}\psi\,.$$

We can therefore rewrite $\mathcal{P}\psi$ as

$$\mathcal{P}\psi = e^{\frac{v}{2}\Gamma^{45}(1-\Gamma^{451\natural})+\frac{v}{2}\Gamma^{67}(1-\Gamma^{6789})}\psi = e^{\frac{v}{2}\Gamma^{45}(1-\Gamma^{6789})+\frac{v}{2}\Gamma^{67}(1-\Gamma^{6789})}\psi\,.$$

Since Γ^{6789} has eigenvalues ± 1, and Γ^{ij} has eigenvalues $\pm i$ for all spatial indices i, j ($i \neq j$), we deduce that $\mathcal{P}$ has eigenvalues 1 and $e^{\pm 2iv}$ on the subspace with Γ^{023}-chirality 1.

We can similarly analyze the eigenvalues of $\mathcal{P}$ on the subspace of ψ's with the opposite Γ^{023}-chirality, i.e., $\Gamma^{023}\psi = -\psi$. On that subspace we find

$$\mathcal{P}\psi = e^{\frac{v}{2}\Gamma^{45}(1-\Gamma^{451\natural})+\frac{v}{2}\Gamma^{67}(1-\Gamma^{6789})}\psi = e^{\frac{v}{2}\Gamma^{45}(1+\Gamma^{6789})+\frac{v}{2}\Gamma^{67}(1-\Gamma^{6789})}\psi\,, \tag{9.30}$$

and hence deduce that $\mathcal{P}$ has eigenvalues $e^{\pm iv}$. Note that $e^{\pm iv} \neq 1$ and so there is no nontrivial solution to $\psi = \mathcal{P}\psi$ on the subspace with Γ^{023}-chirality -1. This fact will come in handy later on.

9.9 Constructing a Witten Index

The states of the system discussed in §9.1, while they contain all the information about the Hilbert space of Tr-S theory with external charges, also contain superfluous excitations in the form of long wavelength modes of $X^\mu_{\mathfrak{c}}, \psi_{\mathfrak{c}}$ along the semi-infinite strings. We can eliminate these excitations by imposing appropriate boundary conditions on the modes $X^\mu_{\mathfrak{c}}, \psi_{\mathfrak{c}}$ at some finite distance from the origin, say at $x_9 = \pm\Delta$ (where the $+$ sign is for $\mathfrak{c} \leq m$ and the $-$ sign is for $\mathfrak{c} > m$, for some positive constant Δ). The following boundary conditions preserve the four real supersymmetries left unbroken by (9.6)-(9.8).

In the type-IIB picture, we pick Neumann boundary conditions for fluctuations in directions $4, \dots, 8$:

$$\partial_9 X^4_{\mathfrak{c}}(\pm\Delta) = \partial_9 X^5_{\mathfrak{c}}(\pm\Delta) = \partial_9 X^6_{\mathfrak{c}}(\pm\Delta) = \partial_9 X^7_{\mathfrak{c}}(\pm\Delta) = \partial_9 X^8_{\mathfrak{c}}(\pm\Delta) = 0\,, \tag{9.31}$$

Dirichlet boundary conditions in directions $1, 2, 3$:

$$X^1_{\mathfrak{c}}(\pm\Delta) = a_1^{(\mathfrak{c})}\,, \quad X^2_{\mathfrak{c}}(\pm\Delta) = a_2^{(\mathfrak{c})}\,, \quad X^3_{\mathfrak{c}}(\pm\Delta) = 0\,, \tag{9.32}$$

and supersymmetric boundary conditions for the fermions:

$$\Gamma^{045678}\psi_{\mathfrak{c}}(\pm\Delta) = \psi_{\mathfrak{c}}(\pm\Delta)\,. \tag{9.33}$$

These boundary conditions are formally what we would get if we let the string end on a D5-brane that extends in directions $4, \dots, 8$ and is fixed in directions $1, 2, 3, 9$.

However, we must note that because directions $1, 2, 3$ are compact, such a D5-brane will back-react strongly on the metric and will require additional orientifolds or other objects to make a complete string-theory solution. (This is similar to the situation with D8-branes as developed in [54], or with D7-branes in [55].) Here, we will simply regard (9.31)—(9.33) as formal boundary conditions that we impose to get rid of unwanted zero modes.

To write down the boundary conditions equivalent to (9.33) in the type IIA picture, we first decompose the fermionic zero mode $\psi_{\mathfrak{c}}$ on the type-IIB fundamental string into left-moving and right-moving components as $\psi_{\mathfrak{c}} = \psi_{\mathfrak{c}+} + \psi_{\mathfrak{c}-}$, where

$$\psi_{\mathfrak{c}\pm} = \pm\Gamma^{09}\psi_{\mathfrak{c}\pm} = \pm\Gamma^{12345678}\psi_{\mathfrak{c}\pm}\,. \tag{9.34}$$

The boundary condition (9.33) can now be written as a relation between the left-moving and right-moving modes at the end of the string ($x_9 = \pm\Delta$):

$$\psi_{\mathfrak{c}+} = \Gamma^{045678}\psi_{\mathfrak{c}-}\,. \tag{9.35}$$

We now follow the dualities of Table 8.1, each time transforming the fermionic field to a dual field on a dual brane. The type-IIA boundary conditions then become:

$$0 = (1 - \Gamma^{\natural})(1 + \Gamma^{0145678})\lambda_{\mathfrak{a}\mathfrak{i}'} \tag{9.36}$$

for $\mathfrak{i}' = 0, n$. We can now define the Witten index $I = \mathrm{Tr}(-1)^F$ of the quantum mechanical system of D3-branes and open strings in the type IIB picture, and we can calculate it using the system of D2-branes and F1-strings in the dual type-IIA picture. With boundary conditions (9.31)—(9.33), the semi-infinite open strings can be regarded as external charges with no internal dynamics, so the Witten index I will simply count the number of ground states of the theory with external charges inserted.

On the other hand, in the type IIA picture, we have various configurations of D2-branes and F1-strings, which can be classified into sectors described by the binding matrix $\mathfrak{B}$ and permutation σ. With the extra conditions (9.36) we will show (see §10.3 and §11.2) that only decongested sectors do not have fermionic zero modes. Such sectors will make a nonzero contribution to the Witten Index.

We also remark that it is possible to create open strings in a supersymmetric configuration by using D3-branes instead of D5-branes. This will avoid the problem of incompleteness of the background, but will instead create additional fermionic zero modes to make the Witten Index identically zero. We briefly discuss this construction in Appendix 13.3.

9.10 Summary of the rules

A sector of Tr-S theory with $U(n)$ gauge group in the presence of m external quark and anti-quark pairs is described by a permutation $\sigma \in S_n$ and an $n \times m$ binding matrix $\mathfrak{B}_{\mathfrak{ia}}$ of 0's and 1's. We build the action in terms of periodic variables $\mathfrak{u}^{\mathfrak{ia}'}, \mathfrak{q}^{\mathfrak{ia}'}$, $\mathfrak{u}^{\mathfrak{a}}$, and $\mathfrak{p}^{\mathfrak{a}}$, with $\mathfrak{i} = 1, \dots, n$, $\mathfrak{a} = 1, \dots, m$, and $\mathfrak{a}' = 0, \dots, m$. All variables have 2π periodicity. The variables are further restricted by linear relations with integer coefficients:

$$\mathfrak{u}^{\mathfrak{ia}} = \mathfrak{u}^{\mathfrak{i}(\mathfrak{a}-1)}, \quad \mathfrak{q}^{\mathfrak{ia}} = \mathfrak{q}^{\mathfrak{i}(\mathfrak{a}-1)}, \qquad \text{whenever } \mathfrak{B}_{\mathfrak{ia}} = 0, \tag{9.37}$$

$$\mathfrak{u}^{\mathfrak{a}} = \mathfrak{u}^{\mathfrak{ia}} = \mathfrak{u}^{\mathfrak{i}(\mathfrak{a}-1)}\,, \qquad \text{whenever } \mathfrak{B}_{\mathfrak{ia}} = 1, \tag{9.38}$$

and

$$\mathfrak{q}^{\sigma(\mathfrak{i})0} = \mathbf{d}\mathfrak{q}^{\mathfrak{i}m} + \mathbf{b}\mathfrak{u}^{\mathfrak{i}m}\,, \qquad \mathfrak{u}^{\sigma(\mathfrak{i})0} = \mathbf{c}\mathfrak{q}^{\mathfrak{i}m} + \mathbf{a}\mathfrak{u}^{\mathfrak{i}m}\,. \tag{9.39}$$

The action is

$$I_0 = \frac{1}{2\pi}\int dt \sum_{\mathfrak{i}=1}^{n}\sum_{\mathfrak{a}=1}^{m} \mathfrak{B}_{\mathfrak{ia}}\mathfrak{p}^{\mathfrak{a}}(\dot{\mathfrak{q}}^{\mathfrak{ia}} - \dot{\mathfrak{q}}^{\mathfrak{i}(\mathfrak{a}-1)})\,. \tag{9.40}$$

Furthermore, the coordinates $(X^1_{\mathfrak{a}}, X^2_{\mathfrak{a}})$ of the m quarks, as well as the, the coordinates $(X^1_{\mathfrak{a}+m}, X^2_{\mathfrak{a}+m})$ of the m anti-quarks are encoded in the action via an extra term

$$I_1 = \frac{1}{2\pi}\int dt \sum_{\mathfrak{a}=1}^{m}[(\dot{X}^1_{\mathfrak{a}} - \dot{X}^1_{\mathfrak{a}+m})\mathfrak{u}^{\mathfrak{a}} + (\dot{X}^2_{\mathfrak{a}} - \dot{X}^2_{\mathfrak{a}+m})\mathfrak{p}^{\mathfrak{a}}]\,. \tag{9.41}$$

Finally, we only keep those sectors for which $\sum_{\mathfrak{i}} \mathfrak{B}_{\mathfrak{ia}} = 1$ for all $\mathfrak{a} = 1, \dots, m$ (we called these *decongested* sectors), because sectors for which $\sum_{\mathfrak{i}} \mathfrak{B}_{\mathfrak{ia}} > 1$ for some $\mathfrak{a}$ have zero modes and therefore do not contribute to the Witten Index, while sectors for which $\sum_{\mathfrak{i}} \mathfrak{B}_{\mathfrak{ia}} = 0$ for some $\mathfrak{a}$ have a D2-brane disconnected from the rest of the system.

Chapter 10

$U(1)$ gauge group

We will now apply the rules of §9.10 to study our setup with $U(1)$ gauge group. We already know that abelian Tr-S is equivalent to $U(1)$ Chern-Simons theory at level k [1]. This model will therefore provide us with a good example of how the rules of §9.10 work. In §10.1-§10.3 we apply the rules of §9.10 to $U(1)$ Tr-S with m charge pairs, and in §10.4 we show how they reproduce the predictions from Chern-Simons theory.

In solving the problem, a central role is played by the action (7.4). The general actions that we will consider are equivalent to quantum mechanical systems that are obtained by geometric quantization of T^{2m}. They are of the form:

$$I = \frac{1}{2\pi}\int \mathcal{M}_{\mathfrak{a}\alpha}\mathfrak{p}^{\mathfrak{a}}d\mathfrak{q}^{\alpha}\,, \qquad \mathfrak{a},\alpha = 1,\ldots,m. \tag{10.1}$$

Here $\mathfrak{p}^{\mathfrak{a}},\mathfrak{q}^{\alpha}$ are periodic coordinates parameterizing T^{2m}, in the range $[0,2\pi)$, and $\mathcal{M}_{\mathfrak{a}\alpha}$ are the components of a nonsingular $m\times m$ matrix of integers. We denote its determinant by

$$\Delta \equiv \det_{m\times m}(\mathcal{M}_{\mathfrak{a}\alpha}) \neq 0.$$

The dimension of the Hilbert space is then $|\Delta|$. We denote the inverse matrix by $(\mathcal{M}^{-1})^{\alpha\mathfrak{a}}$, i.e., $\mathcal{M}_{\mathfrak{a}\alpha}(\mathcal{M}^{-1})^{\alpha\mathfrak{b}} = \delta^{\mathfrak{b}}_{\mathfrak{a}}$. Suppose that $(k_1,\ldots,k_m)$ and $(l_1,\ldots,l_m)$ are vectors of integers. Then the operators $\exp(i\sum_{\mathfrak{a}} k_{\mathfrak{a}}\mathfrak{p}^{\mathfrak{a}})$ and $\exp(i\sum_{\alpha} l_{\alpha}\mathfrak{q}^{\alpha})$ are well-defined, and we have the commutation relation

$$e^{i\sum_{\mathfrak{a}} k_{\mathfrak{a}}\mathfrak{p}^{\mathfrak{a}}}e^{i\sum_{\alpha} l_{\alpha}\mathfrak{q}^{\alpha}}e^{-i\sum_{\mathfrak{a}} k_{\mathfrak{a}}\mathfrak{p}^{\mathfrak{a}}}e^{-i\sum_{\alpha} l_{\alpha}\mathfrak{q}^{\alpha}} = e^{2\pi i\sum_{\alpha,\mathfrak{a}}(\mathcal{M}^{-1})^{\alpha\mathfrak{a}}k_{\mathfrak{a}}l_{\alpha}}\,. \tag{10.2}$$

10.1 A quark and anti-quark pair $(m = 1)$

Using the rules summarized in §9.10, we can now write down explicitly the low-energy effective action for the type-IIA configuration that corresponds to inserting a quark and anti-quark pair in Tr-S theory. As we have explained in the previous section, in such a case we have one fundamental string wrapping the x_3 circle and attached to one D2-brane, forming a bound state with it. The closed string that we would have in the absence of external charges breaks to become an open string starting and ending on the D2-brane. In the notations explained in §9.10, for fixed charges we have the action

$$I_0 = \frac{1}{2\pi}\int \mathfrak{p}^1\left(d\mathfrak{q}^{11} - d\mathfrak{q}^{10}\right), \tag{10.3}$$

and if we wish to allow the charges to move around we need to add the term

$$I_1 = -\frac{1}{2\pi}\int [\left(X_1^1 - X_2^1\right) d\mathfrak{u}^1 + \left(X_1^2 - X_2^2\right) d\mathfrak{p}^1]. \tag{10.4}$$

To proceed, we recall that the S-R-twisted boundary conditions induce linear relations among the variables, following (9.39). For $k = 2$ this yields $\mathfrak{u}^1 = \mathfrak{q}^{11} = -\mathfrak{q}^{10} \equiv \mathfrak{q}^1$, and thus (10.3)-(10.4) can be simplified to:

$$\begin{aligned} I &\equiv I_0 + I_1 && (10.5)\\ I_0 &= \frac{1}{2\pi}\int 2\mathfrak{p}^1 d\mathfrak{q}^1, && (10.6)\\ I_1 &= -\frac{1}{2\pi}\int\left\{\left[a_1^{(1)}(t) - a_1^{(2)}(t)\right]d\mathfrak{q}^1 + \left[a_2^{(1)}(t) - a_2^{(2)}(t)\right]d\mathfrak{p}^1\right\}, && (10.7)\end{aligned}$$

where, as defined in §7, $\left(a_1^{(j)}, a_2^{(j)}\right) = (X_j^1, X_j^2)$ refer to the T^2 coordinates of the quark's and anti-quark's world-lines.

After taking into account similarly the constraints in (9.14) and (9.15) for the cases $k = 1, 3$, the action for static charges (10.3) can be written collectively for all three values of k as[1]

$$I_0 = \frac{k}{2\pi}\int \mathfrak{p} d\mathfrak{q}. \tag{10.8}$$

This action is of the form (10.1) and describes geometric quantization of T^2. It gives rise to a k-dimensional Hilbert space.

We also note that the action (10.5) is of the same form as the action for the ground states of the well-known Landau problem describing the low-energy spectrum of a two-dimensional charged particle moving on a torus with k units of magnetic flux. In this context, the velocity of the quark relative to the anti-quark, which is given by

$$(\dot{a}_1^{(1)}(t) - \dot{a}_1^{(2)}(t),\ \dot{a}_2^{(1)}(t) - \dot{a}_2^{(2)}(t))$$

is interpreted as the electric field in the Landau problem. This is consistent with the interpretation of the quark and anti-quark as charges in $U(1)$ Chern-Simons theory (see §10.4 below). To simplify matters, in the following, we consider static charges and let the quarks and anti-quarks be located at the origin.

[1] For $k = 1$, $\mathfrak{q}^{11} = 0$ and $\mathfrak{u}^1 = -\mathfrak{q}^{10}$, whereas for $k = 3$, $-\mathfrak{q}^{10} = 2\mathfrak{q}^{11} = 2\mathfrak{u}^1$.

10.2 Multiple quark and anti-quark pairs $(m > 1)$

Generalization of the results of §10.1 to $m > 1$ cases is straightforward. There is only one sector: the permutation $\sigma \in S_1$ is the identity, and the $1 \times m$ binding matrix $\mathfrak{B}$ is

$$\mathfrak{B} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}.$$

The relations (9.38) imply that all $\mathfrak{u}^{\mathfrak{i}\mathfrak{a}}$ variables are equal:

$$\mathfrak{u}^{10} = \mathfrak{u}^{11} = \mathfrak{u}^{12} = \cdots = \mathfrak{u}^{1m} = \mathfrak{u}^{1} = \cdots = \mathfrak{u}^{m} \equiv \mathfrak{u}.$$

Just like what we have done in §10.1, we can write down the low-energy effective action as

$$\begin{aligned} I = \frac{1}{2\pi} \int \sum_{\mathfrak{a}=1}^{m} \mathfrak{p}^{\mathfrak{a}} \left(d\mathfrak{q}^{1\mathfrak{a}} - \mathfrak{q}^{1(\mathfrak{a}-1)} \right) \\ - \frac{1}{2\pi} \int \sum_{\mathfrak{a}=1}^{m} \left[\left(a_1^{(\mathfrak{a})}(t) - a_1^{(\mathfrak{a}+m)}(t) \right) d\mathfrak{u} - \left(a_2^{(\mathfrak{a})}(t) - a_2^{(\mathfrak{a}+m)}(t) \right) d\mathfrak{p}^{\mathfrak{a}} \right] . \end{aligned} \tag{10.9}$$

Furthermore, (9.39) gives linear constraints among $\mathfrak{u}, \mathfrak{q}^{10}, \mathfrak{q}^{1m}$ which depend on k:

$$\mathfrak{q}^{10} = \mathbf{d}\mathfrak{q}^{1m} + \mathbf{b}\mathfrak{u}\,, \qquad \mathfrak{u} = \mathbf{c}\mathfrak{q}^{1m} + \mathbf{a}\mathfrak{u}\,. \tag{10.10}$$

This can be solved using the explicit expressions for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in terms of k, given in (7.3), and we get

$$\mathfrak{q}^{10} = -\mathfrak{u}\,, \quad \mathfrak{q}^{1m} = 0\,, \qquad \text{for } k = 1, \tag{10.11}$$

and

$$\mathfrak{q}^{10} = (1-k)\mathfrak{u}\,, \quad \mathfrak{q}^{1m} = \mathfrak{u}\,, \qquad \text{for } k = 2, 3. \tag{10.12}$$

After taking into account these linear constraints, the action for fixed external charges becomes

$$I_0 = \frac{1}{2\pi} \int \left\{ \mathfrak{p}^1 (d\mathfrak{q}^1 + d\mathfrak{q}^m) + \sum_{\mathfrak{a}=2}^{m-1} \mathfrak{p}^{\mathfrak{a}} (d\mathfrak{q}^{\mathfrak{a}} - d\mathfrak{q}^{\mathfrak{a}-1}) - \mathfrak{p}^m d\mathfrak{q}^{m-1} \right\} , \qquad \text{for } k = 1, \tag{10.13}$$

where we set $\mathfrak{q}^{\mathfrak{a}} = \mathfrak{q}^{1\mathfrak{a}}$ for $\mathfrak{a} = 1, \ldots, m-1$ and $\mathfrak{q}^m = \mathfrak{u}$, and

$$I_0 = \frac{1}{2\pi} \int \left\{ \mathfrak{p}^1 [d\mathfrak{q}^1 + (k-1) d\mathfrak{q}^m] + \sum_{\mathfrak{a}=2}^{m} \mathfrak{p}^{\mathfrak{a}} (d\mathfrak{q}^{\mathfrak{a}} - d\mathfrak{q}^{\mathfrak{a}-1}) \right\} , \qquad \text{for } k = 2, 3, \tag{10.14}$$

where we set $\mathfrak{q}^{\mathfrak{a}} \equiv \mathfrak{q}^{1\mathfrak{a}}$. The action (10.14) is of the form (10.1) with

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & k-1 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}, \tag{10.15}$$

and it is also straightforward to write down the corresponding matrix $\mathcal{M}$ for $k = 1$ case. Following the discussion below (10.1), we can immediately compute the dimension of each Hilbert space as $|\Delta| = k$.

10.3 Absence of fermionic zero modes

In this subsection, we show that there is no fermionic zero mode in the sector discussed in §10.2. This is necessary for the consistency of our result, because otherwise the contribution of the sector to the Witten index would be zero, and since this is the only sector for $n = 1$ cases, this would mean that the Witten index of $U(1)$ Tr-S theory would be zero too, regardless of the value of k. On the other hand, we know that $U(1)$ Tr-S theory is simply $U(1)$ Chern-Simons theory, which contains k bosonic ground states only.

As explained in §9.7, the low-energy fermionic modes of our system can be understood via $1+1D$ fermionic fields $\psi_{\mathfrak{i}\mathfrak{a}'}$ and $\lambda_{\mathfrak{a}\mathfrak{i}'}$ supported on the open strings and D2-branes. Each $\psi_{\mathfrak{i}\mathfrak{a}'}$ and $\lambda_{\mathfrak{a}\mathfrak{i}'}$ satisfies the chirality conditions (9.26) and (9.27), respectively. In addition, the $\lambda_{\mathfrak{a}\mathfrak{i}'}$'s at the two far ends of the D2-branes (that is, those with $\mathfrak{i}' = 0, n$) satisfy the constraint (9.36) which derives from dualizing the boundary conditions for open strings ending on D5-branes in the type-IIB setting. Each intersection of D2-F1 satisfies the two junction conditions (9.28) and (9.29). Finally, the S-R-twist on the F1-string gives rise to the boundary condition (9.24). We now proceed to prove that in the $n = 1$ abelian case, these various boundary conditions dictate that we have no fermionic zero modes. (See Figure 10.1 for notation.)

As a warm-up, we start with the $m = 0$ case. In this case there is only one variable ψ_{10} which satisfies the boundary conditions (9.24) and the chirality condition (9.26):

$$\psi_{10} = \mathcal{P}\psi_{10}\,, \qquad (\Gamma^{023}+1)\psi_{10} = 0. \tag{10.16}$$

But in §9.8 we saw that all the eigenvectors of $\mathcal{P}$ with eigenvalue 1 have the opposite Γ^{023} chirality from that of ψ_{10}. It follows that(10.16) does not have any non-trivial solutions and the $m = 0$ states have no zero modes.

Now, let us study the $m > 0$ case. We first consider the continuity of $\lambda_{\mathfrak{a}\mathfrak{i}'}$ at each D2-F1 junction. Define

$$\zeta_{\mathfrak{a}} \equiv \lambda_{\mathfrak{a}1} - \lambda_{\mathfrak{a}0}$$

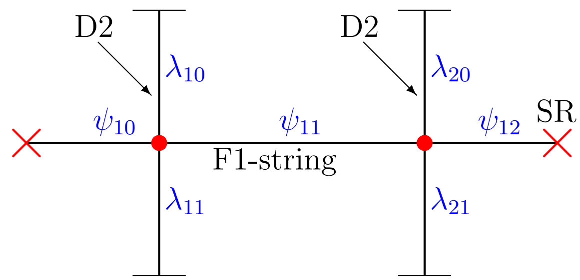


Figure 10.1: Illustration of fermionic zero modes for $n = 1$ and $m = 2$ case. The fundamental string breaks at every D2-brane intersection.

for $\mathfrak{a} = 1, \ldots, m$. Since in the abelian case ($n = 1$) both $\lambda_{\mathfrak{a}0}$ and $\lambda_{\mathfrak{a}1}$ satisfy the chirality condition (9.27) and the boundary condition (9.36), $\zeta_{\mathfrak{a}}$ must also satisfy the equations

$$\begin{aligned} 0 &= (1 + \Gamma^{09\natural})\zeta_{\mathfrak{a}}\,, && (10.17)\\ 0 &= (1 - \Gamma^{\natural})(1 + \Gamma^{0145678})\zeta_{\mathfrak{a}}\,, && (10.18)\\ 0 &= (1 - \Gamma^{239\natural})\zeta_{\mathfrak{a}}\,, && (10.19)\end{aligned}$$

where the last equation comes from (9.28). Equations (10.17) and (10.19) together imply

$$\zeta_{\mathfrak{a}} = \Gamma^{023}\zeta_{\mathfrak{a}}\,. \tag{10.20}$$

Since in our convention $\Gamma^{0123456789\natural} = 1$ (see Appendix 13.1), this in turn implies

$$\Gamma^{0145678}\zeta_{\mathfrak{a}} = \zeta_{\mathfrak{a}}\,, \tag{10.21}$$

and (10.18) becomes

$$(1 - \Gamma^{\natural})\zeta_{\mathfrak{a}} = 0\,.$$

Then (10.19) now reads

$$\Gamma^{239}\zeta_{\mathfrak{a}} = \zeta_{\mathfrak{a}}\,, \tag{10.22}$$

which has no non-trivial solution, because Γ^{239} has no real eigenvalues (it squares to -1). Therefore, we obtain $\zeta_{\mathfrak{a}} = 0$, or $\lambda_{\mathfrak{a}0} = \lambda_{\mathfrak{a}1}$ for all $\mathfrak{a} = 1, \ldots, m$.

Next, we consider the $\psi_{1\mathfrak{a}'}$ fields. Since $\zeta_{\mathfrak{a}} = 0$, equations (9.28)—(9.29) now become

$$(1 \pm \Gamma^{239\natural})(\psi_{1,\mathfrak{a}-1} - \psi_{1\mathfrak{a}}) = 0\,,$$

or simply

$$\psi_{10} = \psi_{11} = \cdots = \psi_{1m}\,. \tag{10.23}$$

On the other hand, we have from the S-R-twist condition $\psi_{1m} = \mathcal{P}\psi_{10}$, where the operator $\mathcal{P}$ is defined in (9.25), and together with the above equalities,

$$\psi_{10} = \mathcal{P}\psi_{10}\,. \tag{10.24}$$

At this point, we can again use the fact that all the eigenvectors of $\mathcal{P}$ with eigenvalue 1 have the opposite Γ^{023} chirality from that of ψ_{10}. Therefore, (10.24) does not have a non-trivial solution, and hence

$$\psi_{10} = \psi_{11} = \cdots = \psi_{1m} = 0\,. \tag{10.25}$$

Finally, let us define

$$\xi_{\mathfrak{a}} = \lambda_{\mathfrak{a}0} + \lambda_{\mathfrak{a}1}\,.$$

With $\zeta_{\mathfrak{a}} = \psi_{1\mathfrak{a}'} = 0$, we find that $\xi_{\mathfrak{a}}$ satisfies the same set of equations (10.17)-(10.19) as $\zeta_{\mathfrak{a}}$:

$$0 = (1+\Gamma^{09\natural})\xi_{\mathfrak{a}}\,, \tag{10.26}$$

$$0 = (1-\Gamma^{\natural})(1+\Gamma^{0145678})\xi_{\mathfrak{a}}\,, \tag{10.27}$$

$$0 = (1-\Gamma^{239\natural})\xi_{\mathfrak{a}}\,. \tag{10.28}$$

The first two equations follow from the chirality and boundary conditions for $\lambda_{\mathfrak{a}\mathfrak{i}'}$, while the third follows from (9.28) and the previous result $\psi_{1\mathfrak{a}'} = 0$ for all $\mathfrak{a}'$. Therefore, we find $\xi_{\mathfrak{a}} = 0$ as before. Together with $\zeta_{\mathfrak{a}} = 0$, this implies $\lambda_{\mathfrak{a}\mathfrak{i}'} = 0$ for all $\mathfrak{a} = 1, \ldots, m$ and $\mathfrak{i}' = 0, 1$. To summarize, we conclude that there is no fermionic zero mode for the abelian $n = 1$ case with an arbitrary number m of D2-branes.

10.4 Comparison with Chern-Simons theory results

At this point, it is pertinent to discuss consistency with abelian Chern-Simons theory, which we had explained in [1] to be the low-energy limit of abelian Tr-S theory. For $U(1)$ Chern-Simons theory the dimension of the Hilbert space with m external charge pairs is always equal to the level k, independently of m. This is indeed what we found from the type-IIA dual picture in §10.2. There, the dimension $\dim \mathcal{H}(k, n = 1, m)$ can be calculated as the determinant of the matrix $\mathcal{M}$ that appears in (10.15) and its $k = 1$ counterpart, and we indeed find the result $\dim \mathcal{H}(k, n = 1, m) = k$ independently of m. The underlying reason for this coincidence is the fact that there are no fermionic zero modes, as we have proved in §10.3. If follows that the dimension we calculated is in fact the Witten Index of the system.

To go beyond the mere equality of dimensions of the Hilbert spaces, we can consider for $k > 1$ the action of the $\mathbb{Z}_k$ symmetry operators $\mathcal{U}, \mathcal{V}$ discussed in §8.3. At the classical level, the discrete translation $\mathcal{U}$ acts as

$$\mathcal{U}: \qquad \mathfrak{u}^{\mathfrak{i}\mathfrak{a}'} \to \mathfrak{u}^{\mathfrak{i}\mathfrak{a}'} + \frac{2\pi}{k}\,, \quad \mathfrak{q}^{\mathfrak{i}\mathfrak{a}'} \to \mathfrak{q}^{\mathfrak{i}\mathfrak{a}'} + \frac{2\pi}{k}\,, \quad \mathfrak{u}^{\mathfrak{a}} \to \mathfrak{u}^{\mathfrak{a}} + \frac{2\pi}{k}\,, \quad \mathfrak{p}^{\mathfrak{a}} \to \mathfrak{p}^{\mathfrak{a}}\,, \tag{10.29}$$

while $\mathcal{V}$, which is related to the homology class of the fundamental string, can be interpreted as electric flux on the D2-branes and acts as

$$\mathcal{V}: \qquad \mathfrak{u}^{\mathfrak{i}\mathfrak{a}'} \to \mathfrak{u}^{\mathfrak{i}\mathfrak{a}'}\,, \quad \mathfrak{q}^{\mathfrak{i}\mathfrak{a}'} \to \mathfrak{q}^{\mathfrak{i}\mathfrak{a}'}\,, \quad \mathfrak{u}^{\mathfrak{a}} \to \mathfrak{u}^{\mathfrak{a}}\,, \quad \mathfrak{p}^{\mathfrak{a}} \to \mathfrak{p}^{\mathfrak{a}} + \frac{2\pi}{k}\,. \tag{10.30}$$

After geometric quantization, the actions (10.29)-(10.30) translate to actions on quantum operators of the system. The action is by conjugation; for example the rightmost expression of (10.30) is to be read as $\mathcal{V}^{-1}\mathfrak{p}^{\mathfrak{a}}\mathcal{V} = \mathfrak{p}^{\mathfrak{a}} + \frac{2\pi}{k}$.

For $n = 1$ we find in terms of the variables of (10.14)

$$\mathcal{V}^{-1}\mathfrak{p}^{\mathfrak{a}}\mathcal{V} = \mathfrak{p}^{\mathfrak{a}} + \frac{2\pi}{k}\,, \qquad \mathcal{V}^{-1}\mathfrak{q}^{\mathfrak{a}}\mathcal{V} = \mathfrak{q}^{\mathfrak{a}}\,, \qquad \mathcal{U}^{-1}\mathfrak{p}^{\mathfrak{a}}\mathcal{U} = \mathfrak{p}^{\mathfrak{a}}\,, \qquad \mathcal{U}^{-1}\mathfrak{q}^{\mathfrak{a}}\mathcal{U} = \mathfrak{q}^{\mathfrak{a}} + \frac{2\pi}{k}\,, \tag{10.31}$$

for $\mathfrak{a} = 1, \ldots, m$. Using the commutation relation (10.2), together with the inverse of (10.15):

$$\mathcal{M}^{-1} = \begin{pmatrix} \frac{1}{k} & \frac{1}{k}-1 & \frac{1}{k}-1 & \frac{1}{k}-1 & \cdots & \frac{1}{k}-1 & \frac{1}{k}-1 \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k}-1 & \frac{1}{k}-1 & \cdots & \frac{1}{k}-1 & \frac{1}{k}-1 \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \frac{1}{k}-1 & \cdots & \frac{1}{k}-1 & \frac{1}{k}-1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & \frac{1}{k} \end{pmatrix}\,, \tag{10.32}$$

we find that, up to a possible constant phase, we can identify

$$\mathcal{U} = e^{-i\mathfrak{p}^1}\,, \qquad \mathcal{V} = e^{i\mathfrak{q}^m}\,. \tag{10.33}$$

Now, let us consider the effect of the interaction I_1 of (9.5). For charge positions $a_1^{(\mathfrak{c})}, a_2^{(\mathfrak{c})}$ that are independent of time, I_1 contributes a total derivative term in (10.9):

$$-\frac{1}{2\pi}\sum_{\mathfrak{a}=1}^{m}\left(a_1^{(\mathfrak{a})} - a_1^{(\mathfrak{a}+m)}\right)\int d\mathfrak{q}^m - \frac{1}{2\pi}\int\sum_{\mathfrak{a}=1}^{m}\left(a_2^{(\mathfrak{a})} - a_2^{(\mathfrak{a}+m)}\right)d\mathfrak{p}^{\mathfrak{a}}\,,$$

where we used (10.12). We now claim that the effect of the interaction term I_1 is to modify (10.33) to

$$\mathcal{U} = e^{-i\mathfrak{p}^1 + \frac{i}{k}\sum_{\mathfrak{a}=1}^{m}\left(a_1^{(\mathfrak{a})} - a_1^{(m+\mathfrak{a})}\right)}\,, \qquad \mathcal{V} = e^{i\mathfrak{q}^m + \frac{i}{k}\sum_{\mathfrak{a}=1}^{m}\left(a_2^{(\mathfrak{a})} - a_2^{(m+\mathfrak{a})}\right)}\,. \tag{10.34}$$

This is not so obvious for static charge positions, and to see it we actually need to let the position, say $a_1^{(\mathfrak{a})}$, vary as a function of time. It can then be checked that an initial state $|i\rangle$ at time $t = -\infty$ evolves into

$$|f\rangle = e^{\frac{i}{2\pi}[a_1^{(\mathfrak{a})}(\infty) - a_1^{(\mathfrak{a})}(-\infty)]\mathfrak{q}^m}|i\rangle$$

at $t = \infty$. But in order for $\mathcal{U}|i\rangle$ to evolve into $\mathcal{U}|f\rangle$, the operator $\mathcal{U}$ must add a phase of $\frac{1}{k}a_1^{(\mathfrak{a})}(-\infty)$ to $|i\rangle$ and a similar phase of $\frac{1}{k}a_1^{(\mathfrak{a})}(\infty)$ to $|f\rangle$, since $\mathcal{U}^{-1}\mathfrak{q}^m\mathcal{U} = \mathfrak{q}^m + \frac{2\pi}{k}$. A similar method can be used to derive the extra phase $\frac{1}{k}\sum_{\mathfrak{a}=1}^{m}\left(a_2^{(\mathfrak{a})} - a_2^{(m+\mathfrak{a})}\right)$ in the action of $\mathcal{V}$.

Now we can compare the above discussion with Chern-Simons theory. It was argued in [1] that in terms of $U(1)$ Tr-S theory (namely Chern-Simons theory) defined on T^2 in the x_1x_2 directions, $\mathcal{U}$ and $\mathcal{V}$ can be understood as gauge transformations with discontinuous gauge parameters:

$$\Lambda_{\mathcal{U}} = e^{\frac{i}{k}x_1}\,, \qquad \Lambda_{\mathcal{V}} = e^{\frac{i}{k}x_2}\,, \tag{10.35}$$

where the coordinates x_1, x_2 take values in $[0, 2\pi)$. (This was argued by relating $\mathcal{U}, \mathcal{V}$ to momentum and winding number in type-IIA and then mapping these quantum numbers to electric fluxes on the D3-brane in type-IIB.) Equation (10.35) in conjunction with (10.34) allows us to directly map states of Chern-Simons theory to states of the system we got from geometric quantization of T^{2m} by, for example, mapping eigenstates of $\mathcal{U}$ in (10.35) to eigenstates of $\mathcal{U}$ in (10.34). Moreover, the extra $a_i^{(\mathfrak{a})}$-dependent phase that we got in (10.34) has a natural interpretation in Chern-Simons theory. This is precisely the phase that we would expect to pick up when acting with a gauge transformation (10.35) on a system that contains m positive charges at positions $(a_1^{(\mathfrak{a})}, a_2^{(\mathfrak{a})})$ and m negative charges at $(a_1^{(\mathfrak{a}+m)}, a_2^{(\mathfrak{a}+m)})$, for $\mathfrak{a} = 1, \ldots, m$. This concludes our map from the Hilbert space of the geometric quantization system to the Hilbert space of $U(1)$ Chern-Simons theory.

Chapter 11

$U(n)$ gauge group

We now turn to the nonabelian case with a $U(n)$ gauge group. Our goal is to calculate the Witten Index of Tr-S theory on T^2 as a function of k, n and the number of charge pairs m. We will begin in §11.1 with a few examples of sectors for $n = 2$, including a brief description of all its sectors with low m. We then show in §11.2 that only decongested sectors contribute to the Witten Index. This greatly simplifies the computation, since decongested sectors are equivalent to a product of decoupled $U(1)$ Hilbert spaces. We describe the results in §11.3. A reader who wishes to skip the details is advised to jump directly to §11.3. In §11.4 we test our results by rewriting the Witten Index as a trace of products of Wilson loop operators in Tr-S theory without charges ($m = 0$). This provides us with a consistency check, and also allows us to calculate the eigenvalues of Wilson loop operators acting on the $m = 0$ Hilbert space. Appendix 13.2 includes some additional details of the combinatorics involved, and for curiosity, we included in Appendix 13.2.2 a combinatorical derivation of the total number of sectors. Interestingly, it is described by a Fibonacci sequence.

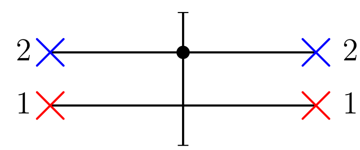
11.1 Examples of $U(2)$ sectors and states

As explained in §9.3, each sector corresponds to a different choice of the binding matrix $\mathfrak{B}$ and a permutation $\sigma \in S_n$ that accompanies the action of the S-R-twist on the n strings. Since for $n = 2$ the permutation group is $S_n \simeq \mathbb{Z}_2$, in the following we shall simply express the permutation as $\sigma \in \{1, -1\}$, and write it as a subscript of $\mathfrak{B}$, so a sector will be denoted as $\mathfrak{B}_{\pm 1}$. Physically, $\sigma = 1$ implies that each string's endpoint is connected to its own starting point so that we have two strings, each with winding number 1 along the x_3 circle, whereas $\sigma = -1$ means that the string's endpoint is connected to the other string's starting point so that we end up with one string with winding number 2. If the permutation results in an equivalent configuration, the subscript is omitted. This happens when at least two strings start on the same D2-brane (i.e., the binding matrix is congested), and relabeling the string indices results in equivalent sectors with different σ's (see for example the third configuration in §11.1). It is also useful to recall that our conventions are such that all strings begin at $x_3 = 0$, and the S-R-twist is located at $x_3 = 2\pi R$.

In this subsection, we will count the states of each sector following the rules of §9.10, and show that in each case the low-energy action can be reduced to the form (7.4).

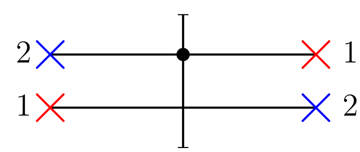
$m = 1$

We have one D2-brane and a string winding number of $n = 2$. This yields three sectors described below [where $(\cdots)^\top$ denotes the transposed matrix]. The diagram for each sector is a miniature of Figure 9.4, with the S-R-twists colored to reflect σ,[1] while the black circles depict junctions where open strings attach to D2-branes.



1. $\mathfrak{B} = (1\,0)^\top_1$

   One open string of winding number 1 bound to the D2-brane and one closed string of winding number 1. We saw in §10.1 that the open string plus the D2-brane system yields k states, while the closed string, being dual to the abelian Chern-Simons theory without charges, also gives rise to k states. In total, we get $k \times k = k^2$ states.



2. $\mathfrak{B} = (1\,0)^\top_{-1}$

   One open string of winding number 2. Let the string start at $z = \mathfrak{q} + i\mathfrak{u}$ and end at $e^{2i\upsilon}(\mathfrak{q} + i\mathfrak{u})$ (the phase is 2υ because the string passes through the S-R-twist twice before ending on the D2-brane). For $k = 2$, for which $\upsilon = \frac{\pi}{2}$, this means $\mathfrak{u} = -\mathfrak{u} \mod 2\pi$, or $\mathfrak{u} = 0$ or π. For each choice of $\mathfrak{u}$, the string starts at $x_{10} = \mathfrak{q}$ on the D2-brane and end at $x_{10} = -\mathfrak{q}$, so the effective action is

   $$I_{k=2} = \frac{2}{2\pi}\int \mathfrak{p}d\mathfrak{q}, \text{ with } \mathfrak{u} = 0, \text{ or } \pi$$

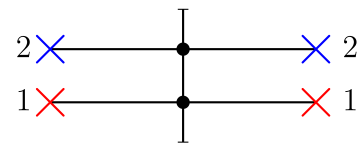
   This gives us $2+2 = 4$ states. Note that in this case the parameter $\mathfrak{u}$ is discrete and decoupled from the geometrically quantized T^2, which in turn is described only by $(\mathfrak{p}, \mathfrak{q})$.

[1]For configurations equivalent under σ, we chose $\sigma = 1$ for our analysis.

For both $k=1$ and $k=3$, the phase e^{2iv} is effectively what we had for $k=3$ case in §10.1. Therefore, from similar analysis, the effective action is

$$I_{k=1,3} = \frac{3}{2\pi}\int \mathfrak{p} d\mathfrak{q}\,,$$

which gives rise to 3 states each.



3. $\mathfrak{B} = (1\,1)^{\top}$

   Here we have two open strings of winding number 1 both of which bound to the D2-brane. For the D2-brane worldvolume gauge field this means that there are twice as many charged particles as in the sector $(1\,0)_1^{\top}$. Therefore, the bosonic part of the action is also twice that of the $(1\,0)_1^{\top}$ sector, i.e.,

   $$\frac{1}{2\pi}\int 2k\mathfrak{p} d\mathfrak{q}.$$

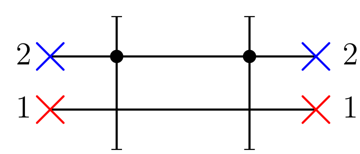
   This is a *congested* sector that also has 4 real fermionic zero modes, as will be shown in §11.2. The bosonic part of the action has 4 states, but the actual Hilbert space is more complicated because of the fermionic zero modes and because of possible interactions between the bosonic and fermionic modes.

$m=2$

We have seven sectors in the $m=2$ case, as briefly described below. For each sector, the effective action is given by

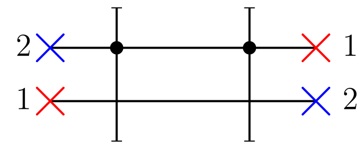
$$I = \frac{1}{2\pi}\int \mathfrak{B}_{\mathfrak{i}\mathfrak{a}}\mathfrak{p}^{\mathfrak{a}}(d\mathfrak{q}^{\mathfrak{i}\mathfrak{a}} - d\mathfrak{q}^{\mathfrak{i}(\mathfrak{a}-1)})\,,$$

but the $\mathfrak{q}^{\mathfrak{i}\mathfrak{a}}$ variables are constrained by the relations (9.37)-(9.39). We derive the dimension of the Hilbert space of each sector in detail for $k=2$, but simply state the results for $k=1,3$.



1. $\mathfrak{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}_1$

   One closed string of winding number 1 breaks on each of the D2-branes to form two open strings, one of which passes through the S-R-twist once. In addition, there is also a closed string. The open string states were analyzed in §10.2 to give rise to k states, while the closed string gives rise to k states as well (as mentioned in the first sector of §11.1). Thus, there are a total of k^2 states in this sector.

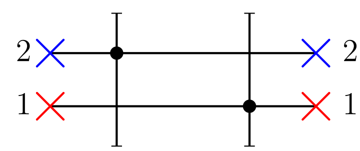


2. $\mathfrak{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}_{-1}$

One string connects the two D2-branes [and hence $\mathfrak{u}^{11} = \mathfrak{u}^{12} = \mathfrak{u}^1 = \mathfrak{u}^2 \equiv \mathfrak{u}$ from (9.38)], while the other starts at the second D2-brane, winds around the x_3 circle and passes through the S-R-twist twice before ending on the first D2-brane. For $k = 2$, the string that starts on the second D2-brane at $\mathfrak{q}^{12} + i\mathfrak{u}$ ends on the first brane at $-\mathfrak{q}^{12} - i\mathfrak{u}$, giving us the constraint $2\mathfrak{u} = 0$ (modulo 2π), which implies $\mathfrak{u} = 0$, or π. The effective action becomes

$$I_{k=2} = \frac{1}{2\pi} \int \left[\mathfrak{p}^1 \left(d\mathfrak{q}^{11} + d\mathfrak{q}^{12}\right) + \mathfrak{p}^2 \left(d\mathfrak{q}^{12} - d\mathfrak{q}^{11}\right)\right],$$

which gives us 2 states for each of the two possible values of $\mathfrak{u}$. In total, we get 4 states. For $k = 1, 3$, the computation is similar to that of the second sector of §11.1, and we get 3 states in both cases.

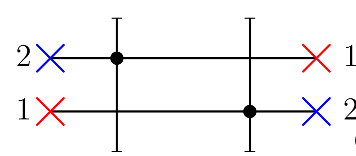


3. $\mathfrak{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{1}$

One open string starts and ends on each D2-brane, and each string passes through the S-R-twist once. It is easy to see that the two strings are completely decoupled from each other, each being bound to a separate D2-brane. We therefore get two decoupled $n = 1$ sectors, each with $m = 1$. The action is a sum of two terms:

$$I_{k=2} = \frac{2}{2\pi} \int \left[\mathfrak{p}^1 d\mathfrak{q}^{11} + \mathfrak{p}^2 d\mathfrak{q}^{12}\right],$$

which gives us 4 states. A similar computation gives us 1 and 9 states for $k = 1, 3$, respectively.

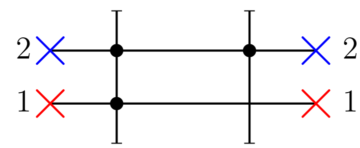


4. $\mathfrak{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{-1}$ One open string starts on the first D2-brane, winds around the x_3 circle once, passing through the S-R-twist, before ending on the second D2-brane. For $k = 2$, it starts at $\mathfrak{q}^{11} + i\mathfrak{u}^{11}$ and ends at $\mathfrak{q}^{21} + i\mathfrak{u}^{21} = -\mathfrak{u}^{11} + i\mathfrak{q}^{11}$. The other open string starts on the second D2-brane at $\mathfrak{q}^{22} + i\mathfrak{u}^{22}$, passes the S-R-twist once before ending on the first D2-brane at $\mathfrak{q}^{10} + i\mathfrak{u}^{10} = -\mathfrak{u}^{22} + i\mathfrak{q}^{22}$. Since $-\mathfrak{u}^{11} + i\mathfrak{q}^{11}$ and $\mathfrak{q}^{22} + i\mathfrak{u}^{22}$ are on the same D2-brane, it follows that $\mathfrak{q}^{11} = \mathfrak{u}^{22} = -\mathfrak{q}^{10}$, and since $\mathfrak{q}^{11} + i\mathfrak{u}^{11}$

and $-\mathfrak{u}^{22}+i\mathfrak{q}^{22}$ are on the same D2-brane it follows that $\mathfrak{u}^{11}=\mathfrak{q}^{22}=-\mathfrak{q}^{21}$. We therefore get the effective action

$$\frac{2}{2\pi}\int[\mathfrak{p}^1 d\mathfrak{q}^{11}+\mathfrak{p}^2 d\mathfrak{q}^{22}]\,,$$

which has the same form as that in Sector 3, and thus there are 4 states. A similar computation gives 3 states for both $k=1,3$ respectively.

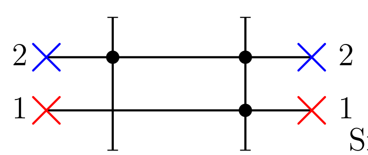


5. $\mathfrak{B}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$

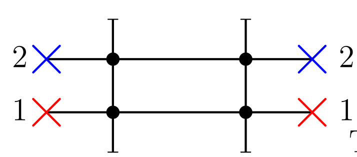
One open string stretches between the two D2-branes and is located at $\mathfrak{q}^{11}+i\mathfrak{u}^1$ for $k=2$. Another string starts on the second D2-brane, passes through the S-R-twist and winds around the x_3 circle once before ending on the first D2-brane. It starts at $\mathfrak{q}^{12}+i\mathfrak{u}^2$ and ends at $-\mathfrak{u}^2+i\mathfrak{q}^{12}$. A third string starts on the first D2-brane at $\mathfrak{q}^{21}+i\mathfrak{u}^1$, passes the S-R-twist once before ending on the same D2-brane at $-\mathfrak{u}^1+i\mathfrak{q}^{21}$. Taking into account the constraints (9.37)-(9.39), we get $\mathfrak{q}^{12}=\mathfrak{q}^{21}=-\mathfrak{q}^{20}=-\mathfrak{q}^{10}=\mathfrak{u}^1=\mathfrak{u}^2$, and the bosonic part of the effective action simplifies to

$$I_{k=2}=\frac{1}{2\pi}\int\left[\mathfrak{p}^1\left(d\mathfrak{q}^{11}+3d\mathfrak{q}^{12}\right)+\mathfrak{p}^2\left(d\mathfrak{q}^{12}-d\mathfrak{q}^{11}\right)\right]\,.$$

This is a congested sector which, as will be shown in §11.2, has 4 real fermionic zero modes.



6. $\mathfrak{B}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ Similar to Sector 5, but with the two D2-branes exchanged. All results mentioned in Sector 5 are identical to those in this sector.



7. $\mathfrak{B}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ This is the sector given as an example for bosonic zero modes in §9.6. Each closed string of winding number 1 breaks on each of the D2-branes to form two open strings, one of which passes through the S-R-twist once. One string starts on the second D2-brane at $\mathfrak{q}^{\mathrm{i}2}+i\mathfrak{u}^1$, passes the twist once before ending on the first D2-brane at $-\mathfrak{u}^1+i\mathfrak{q}^{\mathrm{i}2}$. The remaining string stretches between the two D2-branes and is located at $\mathfrak{q}^{\mathrm{i}1}+i\mathfrak{u}$, where

$\mathfrak{u} = \mathfrak{u}^1 = \mathfrak{u}^2$. We have two sets of such strings, for $\mathfrak{i} = 1, 2$. Taking into account the constraints, we get $\mathfrak{q}^{\mathfrak{i}0} = -\mathfrak{q}^{\mathfrak{i}2} = -\mathfrak{u}$ for $\mathfrak{i} = 1, 2$, and the bosonic part of the effective action simplifies to the expression given in (9.21). This is again a congested sector which, as will be shown in §11.2, has 12 real fermionic zero modes.

We thus see that the binding matrix $\mathfrak{B}$ and the permutation σ can be used to help us visualize the string and brane configurations and determine the effective action rather easily.

$m = 3$

As a last explicit example, let us enumerate the sectors for $m = 3$. It turns out that there are 18 sectors described by the following set of binding matrices:

$$\mathfrak{B} \in \left\{ \begin{array}{lll} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}_{\{1,-1\}}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}_{\{1,-1\}}, & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}_{\{1,-1\}}, \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\{1,-1\}}, & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{1}, & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}_{1}, & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. & \end{array} \right\} \tag{11.1}$$

In (11.1), there are 4 decongested $\mathfrak{B}$'s of which each permutation σ gives rise to a distinct sector, and thus these binding matrices generate 8 sectors in total. Apart from the following pair

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{\{1,-1\}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}_{\{-1,1\}},$$

which are equivalent after relabeling of the strings (note that σ has to be changed as well), the rest of the $\mathfrak{B}$'s remain invariant under relabeling. There are thus a total of 18 different sectors for $m = 3$.

11.2 Counting fermionic zero modes

In order to properly compute the Witten index of our system in the type-IIA picture, we will now count the number of fermionic zero modes in each sector characterized by the binding matrix $\mathfrak{B}$ and permutation σ. If a sector does not support a fermionic zero mode, then its contribution to the Witten index is just the number of ground states of its Hilbert space; if on the other hand a sector does support fermionic zero modes, then after quantization, its Hilbert space will contain an equal number of bosonic and fermionic ground states, thereby making the net contribution to the Witten index zero. It is therefore crucial in the computation of the Witten index to determine which sectors support fermionic zero modes and which sectors do not.

In §11.2, we will address this question for the three sectors that arise in the case $n = 2$ with $m = 1$, as discussed in §11.1. This simplest example will serve to illustrate the salient points of the discussion. We will then tackle the cases with general m in §11.2. For ease of discussion, we will explicitly treat $k = 2$ and $n = 2$ cases only; generalization to other values of k and n however is straightforward, and leads to the same conclusion.

$m = 1$

Of the three sectors described in §11.1, the first two sectors do not support fermionic zero modes. This essentially follows from our abelian result in §10.3. Sector 1 consists of the abelian $n = 1$, $m = 1$ sector plus a closed string, neither of which supports a fermionic zero mode. The fermionic zero modes of Sector 2 must satisfy the same set of equations as those of the abelian $n = 1$, $m = 1$ sector, except for those coming from the S-R-twist. In other words, in the notation of Figure 11.1, the $\psi_{1\mathfrak{a}'}$ for $\mathfrak{a}' = 0, 1$ and $\lambda_{1\mathfrak{i}'}$ for $\mathfrak{i}' = 0, 1$ (note that $\lambda_{11} = \lambda_{12}$ and $\psi_{20} = \psi_{21}$ in this sector) will satisfy all the equations of §10.3 with $\mathcal{P}$ replaced by $\mathcal{P}^2$, because the open string starting on the D2-brane passes through the S-R-twist twice before ending on the same D2-brane. Therefore, the boundary condition from the S-R-twist now reads

$$\psi_{11} = \mathcal{P}\psi_{20} \equiv \mathcal{P}\psi_{21} = \mathcal{P}^2\psi_{10}\,,$$

due to the permutation $\sigma = -1$. But the argument otherwise does not change, because the only property of $\mathcal{P}$ that we used there was the fact that it does not have an eigenvalue $+1$, and neither does $\mathcal{P}^2$. We conclude that Sector 2 does not support fermionic zero modes.

It remains to consider Sector 3. The full set of equations that we need to solve is as follows (see Figure 11.1). First, we have the chirality conditions:

$$0 = (1+\Gamma^{023})\psi_{10} = (1+\Gamma^{023})\psi_{11} = (1+\Gamma^{023})\psi_{20} = (1+\Gamma^{023})\psi_{21}\,, \quad (11.2)$$
$$0 = (1+\Gamma^{09\natural})\lambda_{10} = (1+\Gamma^{09\natural})\lambda_{11} = (1+\Gamma^{09\natural})\lambda_{12}\,. \quad (11.3)$$

Then, we have the boundary conditions at the end of the D2-branes (9.36):

$$0 = (1-\Gamma^{\natural})(1+\Gamma^{0145678})\lambda_{10} = (1-\Gamma^{\natural})(1+\Gamma^{0145678})\lambda_{12}\,. \quad (11.4)$$

Next, we have two junctions with boundary conditions (9.28)-(9.29), which read:

$$(1-\Gamma^{239\natural})\psi_{10} = (1-\Gamma^{239\natural})\psi_{11} = (1-\Gamma^{239\natural})\lambda_{10} = (1-\Gamma^{239\natural})\lambda_{11}\,, \quad (11.5)$$
$$0 = (1+\Gamma^{239\natural})(\psi_{10} - \psi_{11} + \Gamma^{39}\lambda_{11} - \Gamma^{39}\lambda_{10})\,. \quad (11.6)$$

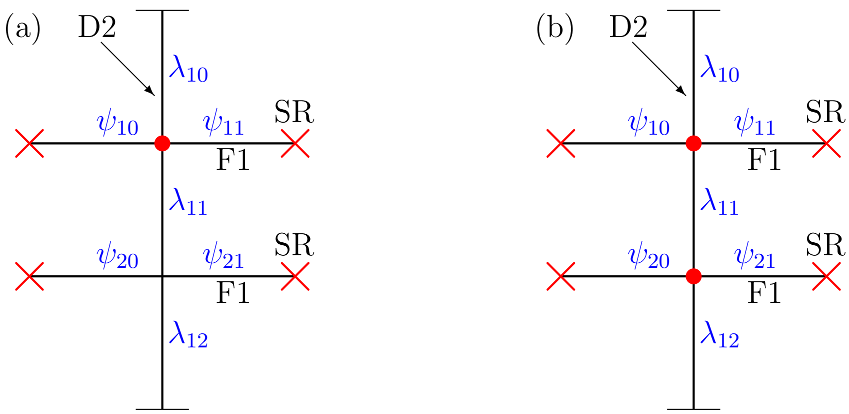


Figure 11.1: Fermionic zero modes on the D2-brane and F1-strings for (a) Sectors 1 and 2, and (b) Sector 3 of the $m = 1$ cases listed in §11.1. In (a), we have trivial identifications $\psi_{20} = \psi_{21}$ and $\lambda_{10} = \lambda_{11}$.

for the first junction, and

$$(1 - \Gamma^{239\natural})\psi_{20} \;=\; (1 - \Gamma^{239\natural})\psi_{21} = (1 - \Gamma^{239\natural})\lambda_{11} = (1 - \Gamma^{239\natural})\lambda_{12}\,, \tag{11.7}$$

$$0 \;=\; (1 + \Gamma^{239\natural})(\psi_{20} - \psi_{21} + \Gamma^{39}\lambda_{12} - \Gamma^{39}\lambda_{11})\,. \tag{11.8}$$

for the second junction. And finally we have the S-R-twist condition (9.24)-(9.25):

$$\psi_{11} = \mathcal{P}\psi_{10}\,, \qquad \psi_{21} = \mathcal{P}\psi_{20}\,. \tag{11.9}$$

To solve these equations, we first eliminate the fermionic mode λ_{11} which lives on the middle section of the D2-brane. To do this, we note that the junction conditions (11.5)-(11.6) together imply

$$\lambda_{11} = \tfrac{1}{2}(1 + \Gamma^{239\natural})\lambda_{11} + \tfrac{1}{2}(1 - \Gamma^{239\natural}\lambda_{11} = \lambda_{10} + \tfrac{1}{2}(1 + \Gamma^{239\natural})\Gamma^{39}(\psi_{10} - \psi_{11})\,. \tag{11.10}$$

It is not hard to check that if we set λ_{11} to the RHS of (11.10), and if we assume that $\lambda_{10}, \psi_{10}, \psi_{11}$ satisfy the chirality conditions that are required of them in (11.2)-(11.3), then λ_{11} will automatically satisfy the chirality condition that is required of it in (11.3). It follows that we can safely eliminate λ_{11} from the equations using (11.10). But if we choose to eliminate λ_{11} from the second junction (11.7)-(11.8), we get, instead of (11.10),

$$\lambda_{11} = \lambda_{12} + \tfrac{1}{2}(1 + \Gamma^{239\natural})\Gamma^{39}(\psi_{21} - \psi_{20})\,. \tag{11.11}$$

Comparing (11.10) and (11.11), we obtain

$$\lambda_{10} + \tfrac{1}{2}(1 + \Gamma^{239\natural})\Gamma^{39}(\psi_{10} - \psi_{11}) = \lambda_{12} + \tfrac{1}{2}(1 + \Gamma^{239\natural})\Gamma^{39}(\psi_{21} - \psi_{20})\,. \tag{11.12}$$

Now, we need to solve (11.4)-(11.9) together with (11.12) for

$$\lambda_{10}, \lambda_{12}, \psi_{10}, \psi_{11}, \psi_{20}, \psi_{21}\,,$$

that are subject to the chirality conditions specified in (11.2)-(11.3).

To proceed, we note that we can combine (11.5) with (11.7) to get

$$\begin{aligned}(1-\Gamma^{239\natural})\psi_{10} = (1-\Gamma^{239\natural})\psi_{11} = (1-\Gamma^{239\natural})\psi_{20} = (1-\Gamma^{239\natural})\psi_{21} \\ = (1-\Gamma^{239\natural})\lambda_{10} = (1-\Gamma^{239\natural})\lambda_{12}\,.\end{aligned} \tag{11.13}$$

Let

$$\zeta_1 \equiv \lambda_{10} - \lambda_{12}\,.$$

It satisfies the same set of equations (10.17)—(10.19) of the abelian case: (10.17) because of the chirality condition (11.3) on λ_{ab}, (10.18) because both λ_{10} and λ_{12} have the boundary conditions (11.4) that is dual to type-IIB strings ending on (formal) NS5-branes, and (10.19) because of (11.13). Hence, the same argument we used before, next to (10.20)-(10.22), implies $\zeta_1 = 0$ again, and so

$$\lambda_{10} = \lambda_{12}\,. \tag{11.14}$$

Next, substitute (11.14) into (11.12) to obtain

$$0 = (1+\Gamma^{239\natural})(\psi_{10} - \psi_{11} + \psi_{20} - \psi_{21})\,. \tag{11.15}$$

On the other hand, from (11.13) we have

$$0 = (1-\Gamma^{239\natural})(\psi_{10} - \psi_{11} + \psi_{20} - \psi_{21})\,. \tag{11.16}$$

The two equations (11.15) and (11.16) together imply

$$\psi_{10} - \psi_{11} + \psi_{20} - \psi_{21} = 0\,,$$

or

$$\psi_{10} + \psi_{20} = \psi_{11} + \psi_{21}\,. \tag{11.17}$$

But from the boundary condition (11.9) that describes the S-R-twist we can write (11.17) as:

$$\psi_{11} + \psi_{21} = \mathcal{P}(\psi_{10} + \psi_{20}),$$

and since $\mathcal{P}$ does not have an eigenvalue $+1$, we get

$$\psi_{10} + \psi_{20} = \psi_{11} + \psi_{21} = 0\,. \tag{11.18}$$

Now, from (11.13) we have

$$(1-\Gamma^{239\natural})(\psi_{10} - \psi_{20}) = 0\,,$$

and together with (11.18), we deduce that

$$(1-\Gamma^{239\natural})\psi_{10} = (1-\Gamma^{239\natural})\psi_{20} = 0\,, \tag{11.19}$$

and hence all the other expressions appearing in (11.13) also vanish.

If we now define

$$\xi_1 \equiv \lambda_{10} + \lambda_{12}\,,$$

the result of the last paragraph implies that ξ_1 satisfies the same equations that ζ_1 satisfies in (10.17)-(10.19), and hence $\xi_1 = 0$. Therefore, we conclude that

$$\lambda_{10} = \lambda_{12} = 0\,. \tag{11.20}$$

At this point, there is essentially only one unknown variable, say ψ_{11}. The other variables ψ_{ij} and λ_{11} are determined in terms of it by (11.11), (11.18), and (11.9). The equations it should satisfy are

$$0 = (1-\Gamma^{239\natural})\mathcal{P}^{-1}\psi_{11} = (1-\Gamma^{239\natural})\psi_{11} = (1+\Gamma^{023})\mathcal{P}^{-1}\psi_{11} = (1+\Gamma^{023})\psi_{11}\,. \tag{11.21}$$

where we substituted $\psi_{10} = \mathcal{P}^{-1}\psi_{11}$ from (11.9). Next, we recall that for $k=2$ the operator $\mathcal{P}$ realizes a rotation by $\frac{\pi}{2}$ in four transverse two-planes, and so

$$\mathcal{P}\Gamma^{239\natural}\mathcal{P}^{-1} = -\Gamma^{1238}\,, \qquad \mathcal{P}\Gamma^{023}\mathcal{P}^{-1} = \Gamma^{023}.$$

Using these relations, we can write (11.21) as

$$\psi_{11} = \Gamma^{239\natural}\psi_{11} = -\Gamma^{023}\psi_{11} = -\Gamma^{1238}\psi_{11}. \tag{11.22}$$

We can now work in a basis for which $\Gamma^{23}, \Gamma^{18}, \Gamma^{9\natural}$ and Γ^0 are simultaneously diagonal. It is then easy to see that (11.22) has 4 linearly independent solutions. This corresponds to 4 zero modes of our system. These four real zero modes transform as singlets under the $SU(2)$ factor of the $SU(2)\times U(1)$ symmetry group that was mentioned at the end of §9.2 and they have ± 1 charges under the $U(1)$ factor, which is generated by $\frac{i}{2}(\Gamma^{45}+\Gamma^{67})$. These statements are easy to derive from (11.22), which together with (9.23) implies that $\Gamma^{4567}\psi_{11} = -\psi_{11}$. (The other fermionic fields of the problem are determined in terms of ψ_{11} and are easily seen to satisfy the same chirality condition.) In this subsection we have restricted for simplicity to the $k=2$ case, but the same result of 4 zero modes is also true for the other cases $k=1,3$.

$m > 1$

Having considered the fermionic zero modes for the $n=2$, $m=1$ case, we now move on to consider the cases with general m. In this subsection, we prove the following criterion for the existence of fermionic zero modes: *the fermionic zero modes exist precisely in those sectors for which $\mathfrak{B}_{1\mathfrak{a}} = \mathfrak{B}_{2\mathfrak{a}} = 1$ for some $\mathfrak{a} = 1,\ldots,m$.* In

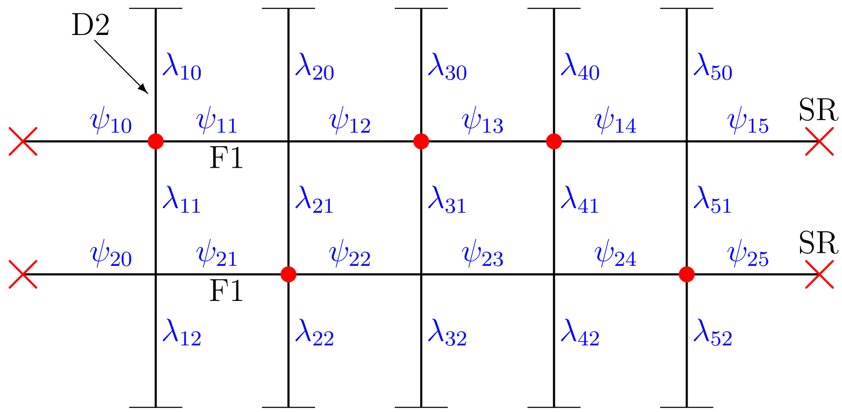


Figure 11.2: An example of sectors without fermionic zero modes. Only one string is attached to each D2-brane.

other words, they exist if and only if there is at least one D2-brane to which both open strings attach. These are what we called *congested* sectors.

It is easy to see that there is no fermionic zero mode if for each of the m D2-branes there is only one string attached to it (see Figure 11.2 for an example). For those sectors for which the permutation σ that accompanies the S-R-twist is the identity, we can divide the D2-branes into two groups: those connecting to the F1-string that is labeled by $\mathfrak{i} = 1$, and those connecting to the F1-string labeled by $\mathfrak{i} = 2$. Each group together with the respective F1-string then forms an abelian system discussed in §10.3, and hence fermionic zero modes are absent.

For sectors with $\sigma = -1$, we can divide the D2-brane indices $\mathfrak{a} = 1, \ldots, m$ into two groups so that those with $\mathfrak{a} = \mathfrak{a}_1, \ldots, \mathfrak{a}_k$ attach to the F1-string that is labeled by $\mathfrak{i} = 1$, and those with $\mathfrak{a} = \mathfrak{a}_{k+1}, \ldots, \mathfrak{a}_m$ to the F1-string labeled by $\mathfrak{i} = 2$. Then the system can again be regarded as an abelian case, the D2-branes now being arranged in the new order $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$, except for the S-R-twist condition, which should now read

$$\psi_{1\mathfrak{a}_k} = \mathcal{P}\psi_{2\mathfrak{a}_{k+1}}\,, \quad \psi_{2\mathfrak{a}_m} = \mathcal{P}\psi_{1\mathfrak{a}_1}\,.$$

The effect of this new boundary condition is that instead of (10.23), we get

$$\psi_{1\mathfrak{a}_1} = \psi_{1\mathfrak{a}_2} = \cdots = \psi_{1\mathfrak{a}_k} = \mathcal{P}\psi_{2\mathfrak{a}_{k+1}} = \cdots = \mathcal{P}\psi_{2\mathfrak{a}_m} = \mathcal{P}^2\psi_{1\mathfrak{a}_1}\,.$$

But since $\mathcal{P}^2$ does not have an eigenvalue $+1$, the conclusion (10.25) remains the same, and hence there is no zero mode.

So let us now establish the fact that if there is at least one D2-brane to which both strings attach, there are fermionic zero modes (see Figure 11.3 for an example). Let us first note that if only one string is attached to the $\mathfrak{a}^{th}$ brane, then we have

$$\psi_{1,\mathfrak{a}-1} = \psi_{1\mathfrak{a}}\,, \quad \psi_{2,\mathfrak{a}-1} = \psi_{2\mathfrak{a}}\,. \tag{11.23}$$

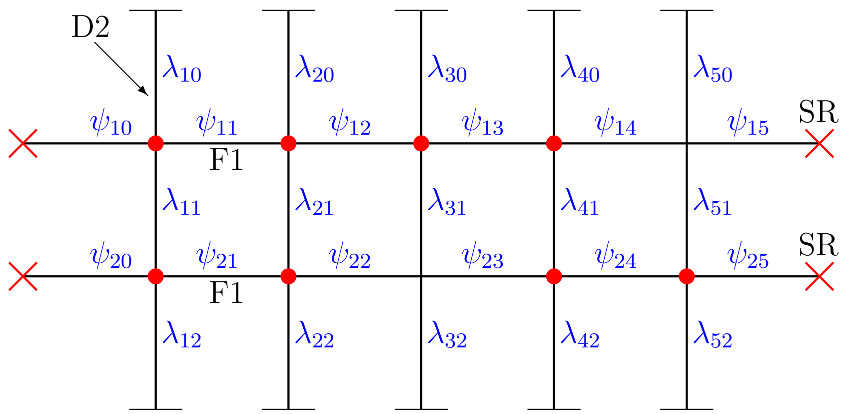


Figure 11.3: An example of sectors *with* fermionic zero modes. Both strings attach to the first, second, and fourth D2-branes.

This follows from the abelian result of §10.3: for example, if it is the $\mathfrak{i} = 1$ F1-string that attaches to the $\mathfrak{a}^{th}$ D2-brane, then we have a trivial identification $\psi_{2,\mathfrak{a}-1} = \psi_{2\mathfrak{a}}$ (and $\lambda_{\mathfrak{a}1} = \lambda_{\mathfrak{a}2}$), while the equality $\psi_{1,\mathfrak{a}-1} = \psi_{1\mathfrak{a}}$ follows from the same reasoning that leads to (10.23) in §10.3. If, on the other hand, both strings attach to the $\mathfrak{a}^{th}$ D2-brane, then we have a weaker identity

$$\psi_{1,\mathfrak{a}-1} + \psi_{2,\mathfrak{a}-1} = \psi_{1\mathfrak{a}} + \psi_{2\mathfrak{a}} . \tag{11.24}$$

This follows from the same reasoning that leads to (11.17) in §11.2.

From (11.23) and (11.24), we now get

$$\psi_{10} + \psi_{20} = \psi_{11} + \psi_{21} = \cdots = \psi_{1m} + \psi_{2m} , \tag{11.25}$$

and from the S-R-twist,

$$\psi_{1m} + \psi_{2m} = \mathcal{P}(\psi_{10} + \psi_{20}) , \tag{11.26}$$

regardless of the choice of permutation σ. These two relations together imply that

$$\psi_{10} + \psi_{20} = \psi_{11} + \psi_{21} = \cdots = \psi_{1m} + \psi_{2m} = 0 . \tag{11.27}$$

Let us now divide the D2-brane indices $\mathfrak{a} = 1, \ldots, m$ so that the subset $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_l\}$ refer to those D2-branes to which both strings attach. Then we can eliminate the fermionic modes $\lambda_{\mathfrak{a}_1 1}, \ldots, \lambda_{\mathfrak{a}_l 1}$ that reside on the middle D2-brane segments, similarly to (11.10). Then the reasoning leading to (11.20) gives us

$$\lambda_{\mathfrak{a}_1 0} = \lambda_{\mathfrak{a}_1 2} = \lambda_{\mathfrak{a}_2 0} = \lambda_{\mathfrak{a}_2 2} = \cdots = \lambda_{\mathfrak{a}_l 0} = \lambda_{\mathfrak{a}_l 2} = 0 .$$

For $\mathfrak{a} \notin \{\mathfrak{a}_1, \ldots, \mathfrak{a}_l\}$, only one string attaches to the $\mathfrak{a}^{th}$ D2-brane, so we have trivial identification $\lambda_{\mathfrak{a}1} = \lambda_{\mathfrak{a}0}$ or $\lambda_{\mathfrak{a}1} = \lambda_{\mathfrak{a}2}$. But since $\lambda_{0\mathfrak{a}} = \lambda_{2\mathfrak{a}} = 0$ from the same argument as in the abelian result, we have $\lambda_{1\mathfrak{a}} = 0$ in either case. Also for $\mathfrak{a} \notin \{\mathfrak{a}_1, \dots, \mathfrak{a}_l\}$, we have $\psi_{\mathfrak{i}\mathfrak{a}} = \psi_{\mathfrak{i},\mathfrak{a}-1}$ for $\mathfrak{i} = 1, 2$ from (11.23). It follows therefore that the only independent variables that we have at this point are $\psi_{1\mathfrak{a}_1}, \dots, \psi_{1\mathfrak{a}_l}$, since $\psi_{2\mathfrak{a}_1}, \dots, \psi_{2\mathfrak{a}_l}$ can be recovered from (11.27), and $\psi_{\sigma(\mathfrak{i})0} = \mathcal{P}^{-1}\psi_{\mathfrak{i}m}$. The remaining equations are

$$0 = (1 - \Gamma^{239\natural})\psi_{1\mathfrak{a}} = (1 + \Gamma^{023})\psi_{1\mathfrak{a}}\,, \qquad \mathfrak{a} \in \{\mathfrak{a}_1, \dots, \mathfrak{a}_l\}. \tag{11.28}$$

Similarly to the discussion following (11.21), there are 4 zero modes for ψ_{1a_l} and 8 zero modes for each of $\psi_{1\mathfrak{a}_1}, \dots, \psi_{1\mathfrak{a}_{l-1}}$. We get $8l - 4$ zero modes in total.

11.3 The Witten Index — results

According to the discussion of §11.2 (and its extension to $n > 2$), sectors with a congested binding matrix have a nonzero number of zero modes. They therefore do not contribute to the Witten Index. Only sectors with no zero modes contribute to the Witten Index, and these are precisely the decongested sectors.

By our definition at the end of §9.4, a decongested sector is a sector whose binding matrix has exactly one '1' in each column. It can be alternatively described as follows. Start with p closed strings of winding numbers $n_1, n_2, \dots, n_p$, such that $n = n_1 + \cdots + n_p$, and attach each of the m D2-branes to one string. The point of attachment along the string is also important, and since the x_3 coordinate of the D2-brane is fixed, there are n_j choices for the j^{th} string, since the string passes through the x_3 coordinate n_j times. The partition $n = n_1 + \cdots + n_p$ determines the conjugacy class $[\sigma]$ of the permutation $\sigma \in S_n$ (where σ is represented as a product of cycles and $n_1, \dots, n_p$ are the lengths of the cycles), and the points of attachment determine the binding matrix. Thus, there are initially n^m choices for the attachment points, but choices that are equivalent up to relabeling of the strings should be counted only once.

Let m_j be the number of D2-branes that end up being attached to the j^{th} string. Then $m = m_1 + \cdots + m_p$, and it is not difficult to see that the Hilbert space of the corresponding sector is equivalent to a tensor product $\otimes_{j=1}^{p}\mathcal{H}_j(n_j, m_j)$ of decoupled Hilbert spaces. The dimension of $\mathcal{H}_j(n_j, m_j)$ can be determined explicitly from the *effective twist phase* $e^{in_j v}$. Since we have the constraint $n_j \le n < \mathbf{r}$, the effective phase is never trivial. In fact, in the cases relevant to us, the effective twist phase takes one of the following seven values:

$$e^{in_j v} \in \{\pm i, e^{\pm\frac{\pi i}{3}}, e^{\pm\frac{2\pi i}{3}}, -1\}.$$

The first six values are the same twist we got in §10 for $U(1)$ theory at levels $k = \pm 2, \pm 1, \pm 3$. (It is necessary to keep track of the sign if relative parity is important.) The dimensions in these cases are given by $\dim \mathcal{H}_j(n_j, m_j) = |k|$, regardless of m_j. The last phase $e^{in_j v} = -1$ is a new case that hasn't been discussed in §10, but appeared for example in the second sectors of §11.1 and §11.1 (for $k = 2$). As we saw there, it is not hard to check that in this case $\dim \mathcal{H}_j(n_j, m_j) = 4$, independently of m_j, n_j.

We can now summarize the results for the dimensions in Table 11.1. The contribution to the Witten Index is calculated as

$$I_m(n_1, \ldots, n_j) = \sum_{\{m_j\}: \sum m_j = m} \left(\prod_{j=1}^{p} \dim \mathcal{H}_j(n_j, m_j) \right).$$

As an example for how the entries in Table 11.1 were derived, take the case $n = 3$ and $k = 2$ with partition $3 = 1 + 1 + 1$. The number of decongested binding matrices is 3^m. (This is the total number of ways to attach m D2-branes to the strings.) But this over-counts binding matrices that are related by a permutation of the string labels $1, 2, 3$. There are three binding matrices in which all m D2-branes are attached to the same string (so that we have one row of all 1's and two other rows of all 0's). They are of course equivalent to one another after relabeling the string indices. Excluding these 3 configurations, to which we shall return later, we are left with $3^m - 3$ configurations, and accounting for the relabeling redundancy $3! = 6$, we get $\frac{1}{6}(3^m - 3)$ inequivalent configurations. For each of this type of configurations, let m_1, m_2, m_3 denote the number of D2-branes that are attached to the $1^{st}, 2^{nd}$, and 3^{rd} string, respectively (so that $m = m_1 + m_2 + m_3$). For each configuration, the Hilbert space is a product of three Hilbert spaces of the $U(1)$ theory with $2m_1, 2m_2$, or $2m_3$ charges, respectively. These Hilbert spaces were analyzed in §10 and have $k = 2$ states each. So we get a total of 8 states for each configuration of this form. On the other hand, the remaining 3 configurations, for which all m D2-branes are attached to the same string, are equivalent to each other up to relabeling. The string with its m attached D2-branes has a Hilbert space that is equivalent to that of the $U(1)$ theory with $2m$ quarks, and hence possesses 2 states, while the remaining 2 unattached strings form a Hilbert space that corresponds to the $\sigma = 1$ sector of the $U(2)$ Tr-S theory with no charges, and has 3 states [1]. The total number of states for this configuration is therefore $2 \times 3 = 6$ and altogether we get the total number of states for $k = 2$ and $n = 3 = 1 + 1 + 1$:

$$8 \times \tfrac{1}{6}(3^m - 3) + 6 = \tfrac{4}{3}3^m + 2.$$

As another example, take $k = 1$ and $n = 4 = 2 + 2$. The binding matrix has four rows, labeled by string index $i = 1, \ldots, 4$, and for definiteness we take the permutation $\sigma = (12)(34)$. For this discussion it is convenient to pretend that this sector has two closed strings, one formed by connecting $i = 1$ and $i = 2$ strings, the other by connecting $i = 3$ and $i = 4$. Each closed string has winding number 2, to which some D2-branes are possibly attached. We start by considering all 4^m possible binding matrices, and note that there are 2×2^m binding matrices for which all D2-branes are attached to the same (pretend closed) string. The other $(4^m - 2 \times 2^m)$ binding matrices have a nonzero number of D2-branes attached to each of the two (pretend closed) strings. For this latter type of configurations, the over-counting factor is 8 because we can exchange the two strings (namely exchanging $\{1,2\}$ and $\{3,4\}$, contributing an over-counting factor of 2) and within each string we can exchange the two string indices ($i=1$ and $i=2$ for the first string, and similarly $i=3$ and $i=4$ for the second). The total number of inequivalent configurations for which neither of the strings is unattached is therefore $\frac{1}{8}(4^m - 2^{m+1})$. Let $m_1 > 0$, $m_2 > 0$ be the number of D2-branes attached to each string, with $m = m_1 + m_2$. Each string has an effective twist phase of $e^{2iv} = e^{\frac{2\pi i}{3}}$, and therefore its Hilbert space corresponds to the Hilbert space of a $U(1)$ theory with $k' = 3$ (which is the value of k for which the phase is $e^{iv'} = e^{\frac{2\pi i}{3}}$), and with $2m_1$ or $2m_2$ charges. Each configuration therefore has $3 \times 3 = 9$ states. The remaining configurations have one unattached (pretend closed) string, and all m D2-branes are attached to the other (pretend closed) string. There are 2×2^m such binding matrices, with an over-counting factor of $4 = 8/2$ (where 2 is the symmetry factor which corresponds to switching the two string indices of the unattached string), so we get 2^{m-1} configurations. Each configuration has a Hilbert space that corresponds to $U(1)$ theory with $k' = 3$ and $2m$ charges inserted times the Hilbert space of $U(2)$ theory with $k = 1$ in the sector $\sigma = (12)$, and no external charges. The latter Hilbert space is two-dimensional [1]. The Hilbert space of the configuration thus has a total dimension of $3 \times 2 = 6$. Altogether we find the total number of states of the $n = 4 = 2 + 2$ sector of the $k = 1$ theory to be:

$$9 \times \tfrac{1}{8}(4^m - 2^{m+1}) + 6 \times 2^{m-1} = \tfrac{9}{8}4^m + \tfrac{3}{4}2^m.$$

In addition to the sector-by-sector analysis described above, we can also write down closed formulas for certain types of sectors. In general, it is useful to have a formula for the number $f(n,m)$ of non-equivalent decongested binding matrices that have at least one nonzero entry in each of the n rows. This corresponds to the number of configurations for which each of the n strings is attached to at least one D2-brane. In Appendix 13.2.1 we show that:

$$f(n,m) = \sum_{j=1}^{n} \frac{(-1)^{n-j}}{j!(n-j)!} j^m \,. \tag{11.29}$$

Using this result we can write a general expression for the Witten Index in the sectors with partition $n = 1 + 1 + \cdots + 1$. Such a sector can have $1 \leq l \leq n$ unattached strings. The dimension of the Hilbert space of the unattached strings is the same as the number of states that a system of l identical bosons each occupying one of k states has, which is $\begin{pmatrix} k+l-1 \\ k-1 \end{pmatrix}$, while the dimension of the Hilbert space of all $(n-l)$ attached strings is k^{n-l}. So, we get a total of

$$\sum_{l=0}^{n-1} \begin{pmatrix} k+l-1 \\ k-1 \end{pmatrix} k^{n-l} f(n-l,m) = \sum_{j=1}^{n} \left(\sum_{l=0}^{n-j} \frac{(-1)^{n-l-j} k^{n-l} (k+l-1)!}{(k-1)! l! j! (n-l-j)!} \right) j^m$$

states. Using this and similar techniques we get the results listed in Table 11.1. We can now add the contribution of the various sectors to the Witten Index for each k and n, listed in Table 11.1, and obtain the results of Table 11.2.

11.4 Wilson loop operators and their eigenvalues

From the expressions for the dimensions of Hilbert spaces with static charges we can get information about basic properties of Wilson loop operators along the two (spatial) cycles of T^2. Let us define the two basic 1-cycles of T^2, one along the x_1 direction, and the other along the x_2 direction. We denote the low-energy limits of the two supersymmetric Wilson loop operators that correspond to these cycles, and in the fundamental representation $\square$ of $U(n)$, by $\mathcal{W}_1$ and $\mathcal{W}_2$:

$$\mathrm{tr}_\square\left(Pe^{i\int_0^{2\pi L_1}(A_1(t,x_1,x_2,x_3)+\Phi^9(t,x_1,x_2,x_3))dx_1}\right) \xrightarrow{\text{low-energy limit}} \mathcal{W}_1\,, \qquad (11.30)$$

$$\mathrm{tr}_\square\left(Pe^{i\int_0^{2\pi L_2}(A_2(t,x_1,x_2,x_3)+\Phi^9(t,x_1,x_2,x_3))dx_2}\right) \xrightarrow{\text{low-energy limit}} \mathcal{W}_2\,. \qquad (11.31)$$

Here Φ^9 is the adjoint scalar from the $\mathcal{N}=4$ multiplet that corresponds to fluctuations of the D3-branes in direction x_9 as in (9.2), and $\mathcal{W}_1, \mathcal{W}_2$ are operators on the Hilbert space $\mathcal{H}(k,n,m=0)$ (namely, Hilbert space without external charges). Assuming that the low-energy theory is topological we expect $\mathcal{W}_1, \mathcal{W}_2$ to be independent of t, x_1, x_2 altogether.

A simple Wick rotation now allows us to derive the eigenvalues of $\mathcal{W}_i^\dagger\mathcal{W}_i$ from the dimensions of the Hilbert spaces $\mathcal{H}(k,n,m)$ as a function of m. Obviously, if the theory is topological the eigenvalues are the same for $i=1,2$. Let us compactify time on a circle with (supersymmetric) periodic boundary conditions so that $0 \le x_0 < 2\pi T$. Tr-S theory is now formulated on T^3 in directions x_0, x_1, x_2. Now insert the m quark and anti-quark pairs. At this point, if we let x_2, for example, play the role of Euclidean time then every quark corresponds to a Wilson loop operator for a loop around direction x_0. In the microscopic $3+1D$ theory, let $\mathcal{W}'$ be such a supersymmetric Wilson loop around direction x_0 and at a fixed x_3. In the Hilbert space of Tr-S theory on T^2 (in directions x_0, x_1), let $\mathcal{W}$ be the operator that is the low-energy limit of $\mathcal{W}'$. It is, of course, independent of x_3. Now we can write

$$\dim\mathcal{H}(k,n,m) = \mathrm{tr}[(\mathcal{W}^\dagger\mathcal{W})^m]\,. \qquad (11.32)$$

Thus, if we calculate $\dim\mathcal{H}(k,n,m)$ for all m, we will be able to read off the eigenvalues of $\mathcal{W}^\dagger\mathcal{W}$. Since $\mathcal{W}^\dagger\mathcal{W}$ is a matrix of dimension $\dim\mathcal{H}(k,n,0)$, it follows that $\dim\mathcal{H}(k,n,m)$ has to be a sum of at most $\dim\mathcal{H}(k,n,0)$ m-powers. We therefore expect

$$\dim\mathcal{H}(k,n,m) = \sum_{j=1}^{\dim\mathcal{H}(k,n,0)} \lambda_j(k,n)^m\,, \qquad (11.33)$$

where the λ_j's are eigenvalues of $\mathcal{W}^\dagger\mathcal{W}$ and thus are independent of m. If indeed we can write $\dim\mathcal{H}(k,n,m)$ in the form (11.33), that will provide us with a nice test of our construction, and in particular the conjecture that Tr-S is topological. Moreover, we will be able to find the eigenvalues of $\mathcal{W}^\dagger\mathcal{W}$.

Also, note that for $k=1,2$ we have $\mathcal{W}=\mathcal{W}^\dagger$ for the following reason. For $k=1,2$ we find that the order $\mathbf{r}=6,4$ of the S-duality twist is even. Thus, the cyclic group $\{1,\mathbf{g},\mathbf{g}^2,\dots,\mathbf{g}^{\mathbf{r}-1}\}$ that is generated by the S-duality twist $\mathbf{g}\in\mathrm{SL}(2,\mathbb{Z})$ contains $\mathbf{g}^{\mathbf{r}/2}$ which is equal to the central element $-I\in\mathrm{SL}(2,\mathbb{Z})$. This is physically equivalent to the charge-conjugation operator $\mathcal{C}$ [see (7.3)]. Therefore, when we continuously change the x_3 position of the Wilson loop $\mathcal{W}'$ until it completes $\frac{\mathbf{r}}{2}$ cycles along the x_3 circle, it becomes the charge conjugate $(\mathcal{W}')^\dagger$. Since we assumed that the low-energy limit of $\mathcal{W}'$ is independent of x_3, we find that $\mathcal{W}$ is hermitian for those values of k. Thus, for $k=1,2$ the eigenvalues of $\mathcal{W}$ are simply the square-roots of the eigenvalues of $\mathcal{W}^\dagger\mathcal{W}$, and are therefore known up to an overall sign.

11.5 Consistency checks

Now let us check the consistency of our results. Comparing the first three columns of Table 11.2 with the corresponding columns of Table 11.1 we observe an interesting phenomenon — whereas individual sectors in Table 11.1 do not generally conform to the required form (11.33), their total contribution in Table 11.2 does! We believe this result is a nontrivial test of our construction and derivation, and we will discuss its meaning further in §11.6. Moreover, from the behavior of the Witten Index as a function of m in Table 11.2 we can read off the eigenvalues of the Wilson loop operator combination $\mathcal{W}^\dagger\mathcal{W}$ on the Hilbert space of Tr-S without charges ($m=0$). The results are listed in the last column of Table 11.2. In deriving the eigenvalues of $\mathcal{W}^\dagger\mathcal{W}$ we matched the expressions for the Witten Index with (11.33). In (11.33) the total number of eigenvalues, taking multiplicities into account, has to be equal to the dimension of the Hilbert space without charges. These dimensions are listed in the 4^{th} column of Table 11.2 as $\dim\mathcal{H}(k,n,0)$, and in cases where the number of powers appearing in the expression for the Witten Index in the 3^{rd} column falls short of $\dim\mathcal{H}(k,n,0)$ we have to add zero eigenvalues. (That the number of powers is always smaller than $\dim\mathcal{H}(k,n,0)$ constitutes another consistency check.) There is, however, an independent check on these results and the number of zero eigenvalues as follows.

For $k>1$, there are symmetry operators $\mathcal{U},\mathcal{V}$ that act on the Hilbert spaces $\mathcal{H}(k,n,0)$. They were introduced in [1] and reviewed in §8.3. These operators have a geometrical interpretation in the type-IIA description, but in the original gauge theory description they are understood as large gauge transformations in the $U(1)\subset U(n)$ center. This latter interpretation allows us to immediately write their commutation relations with $\mathcal{W}$. For concreteness, let's assume that $\mathcal{W}\equiv\mathcal{W}_1$ is a Wilson loop around the x_1 direction of the (type-IIB) T^2. Then,

$$\mathcal{V}^{-1}\mathcal{W}\mathcal{V} = \mathcal{W}\,, \qquad \mathcal{U}^{-1}\mathcal{W}\mathcal{U} = e^{\frac{2\pi i}{k}}\mathcal{W}\,. \tag{11.34}$$

For $k = 2$ we argued above that $\mathcal{W} = \mathcal{W}^\dagger$, and $\mathcal{W}$ is therefore diagonalizable. The second equation of (11.34) then shows that the nonzero eigenvalues of $\mathcal{W}$ must come in pairs $(\lambda, -\lambda)$, and so the multiplicities of the nonzero eigenvalues $|\lambda|^2$ of $\mathcal{W}^\dagger\mathcal{W}$ are all even. For $k = 3$ we don't have a similar argument to show that $\mathcal{W}$ is diagonalizable, but assuming that it is, the nonzero eigenvalues of $\mathcal{W}$ must come in triplets $(\lambda, e^{\frac{2\pi i}{3}}\lambda, e^{-\frac{2\pi i}{3}}\lambda)$ and therefore the multiplicities of the nonzero eigenvalues $|\lambda|^2$ of $\mathcal{W}^\dagger\mathcal{W}$ must all be divisible by 3. This is indeed the case, as we can see from Table 11.2.

For $k = n = 2$ we can say more. In this case $\mathcal{U}$ and $\mathcal{V}$ commute and we can write $\mathcal{H}(k = 2, n = 2, 0)$ as a direct sum $\bigoplus \mathcal{H}_{(u,v)}(2, 2, 0)$ of simultaneous eigenstates of $(\mathcal{U}, \mathcal{V})$, with eigenvalues $(u = \pm 1, v = \pm 1)$. It is easy to check that

$$\dim \mathcal{H}_{(+1,+1)} = \dim \mathcal{H}_{(+1,-1)} = \dim \mathcal{H}_{(-1,+1)} = 2\,, \qquad \dim \mathcal{H}_{(-1,-1)} = 0.$$

Now take a state $|\psi\rangle \in \mathcal{H}_{(+1,-1)}$ and consider the $(\mathcal{U}, \mathcal{V})$ eigenvalues of $\mathcal{W}|\psi\rangle$. By (11.34) they must be $(-1, -1)$, and since $\mathcal{H}_{(-1,-1)}$ is trivial it follows that $\mathcal{W}|\psi\rangle = 0$. Therefore, $\mathcal{W}$ is identically zero on the two-dimensional subspace $\mathcal{H}_{(+1,-1)}$. It follows that $\mathcal{W}$ has at least two eigenvalues that are identically zero, and so does $\mathcal{W}^\dagger\mathcal{W}$. From Table 11.2 we see that this is indeed the case, and that the multiplicity of the zero eigenvalue of $\mathcal{W}^\dagger\mathcal{W}$ is exactly 2.

11.6 Comparison with Chern-Simons theory

So far we have found the Witten Indices of Tr-S theory on T^2 in individual sectors, listed in Table 11.1, and their sum over all sectors, listed in Table 11.2. We have also seen that the results pass some nontrivial consistency checks in §11.5. These results are however supposed to provide some clues about what Tr-S theory is. Is it a known theory, or is it an entirely new theory? How should we interpret the results from Table 11.1?

As a first step, we have to know whether different "sectors" correspond to different theories, or whether they are part of the same theory. Following the results in [1] regarding the Hilbert spaces $\mathcal{H}(k, n, m = 0)$ and their decomposition as representations of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of (the type-IIB) T^2, it was proposed there that a sector $[\sigma]$ corresponds to a superselection sector of Tr-S theory on $\mathbb{R}^{2,1}$ — perhaps a discrete remnant of an expectation value of a Wilson loop along the compact x_3 direction. Furthermore, it was observed in [1] that strictly as representations of $\mathrm{SL}(2, \mathbb{Z})$ and operators $\mathcal{U}, \mathcal{V}$, the Hilbert spaces of most of the sectors are equivalent to the Hilbert spaces of (pure) Chern-Simons theories at various levels and with various gauge groups that are in general subgroups of $U(n)$. We have reproduced the general results of [1] in Table 11.3. For $n = 2$, for example, the breakdown into individual sectors is reproduced in Table 11.4. (Note that, as explained in [1], the Chern-Simons theory level of the $U(1)$ part of the gauge group is given by $k' = nk$ in all cases.) Naturally, it was then conjectured that Tr-S in each of these sectors is equivalent to the Chern-Simons theory at the corresponding level and with the corresponding gauge group. But given the results of Table 11.1, we can now take a critical look at some of these conjectures.

To extract useful information out of Table 11.1 we need to know how to match a sector of Tr-S with $m > 0$ charge pairs to a sector of Tr-S with no charges ($m = 0$). A sector with $m = 0$ is described entirely by the conjugacy class $[\sigma]$ of the permutation $\sigma \in S_n$, or alternatively, by the partition $n = n_1 + n_2 + \cdots + n_p$. A sector with $m > 0$, on the other hand, is described by $[\sigma]$ together with a binding matrix $\mathfrak{B}_{ia}$, up to relabeling of string indices i, and for general sectors, combinations $(\mathfrak{B}, [\sigma])$ and $(\mathfrak{B}', [\sigma'])$ with different conjugacy classes ($[\sigma] \neq [\sigma']$) may be equivalent. In general, therefore, we cannot unambiguously assign a sector of $m = 0$ theory to a given sector with $m > 0$. This is also clear because the $m > 0$ sectors have open strings while the $m = 0$ sectors only have closed strings.

However, if we restrict ourselves to *decongested* sectors we can overcome this problem. Since a decongested sector has exactly one pair of open strings ending on each D2-brane, we can formally align and recombine without ambiguity these two ends to form a configuration of closed strings, thereby creating a unique $m = 0$ sector out of a decongested $m > 0$ sector. In fact, the "pretend closed" terminology of §11.3 and the partitions $n = n_1 + \cdots + n_p$ appearing in Table 11.1 took advantage of this fact.

But now we face a serious obstacle. It was argued in §11.5 that any sector whose entry in Table 11.1 does not conform to (11.33) — one for which the coefficient of any m^{th} power in its contribution to the Witten Index is not an integer — cannot possibly be a stand-alone theory. For consistency we have to, at the very least, combine sectors so that their total contribution to the Witten Index will be of the form (11.33). Thus, for example, both of the $k = n = 2$ sectors might be individual theories corresponding to different "superselection" sectors. But the $k = 2$ and $n = 3$ sectors corresponding to the partitions $3 = 1 + 1 + 1$ and $3 = 3$ cannot be separate theories. Similarly, the $k = 1$ and $k = 3$ sectors with partitions $2 = 1 + 1$ and $2 = 2$ cannot be separate theories either. This, we have to admit, is evidence against at least some of the conjectures that are implicit in Table 11.3.

So, still focusing on the $U(2)$ case, let us assume that we need to combine both $2 = 1 + 1$ and $2 = 2$ sectors for $k = 1, 3$, and let us remain agnostic about whether we need to combine or not the two sectors for the $k = 2$ case. Let us proceed and ask whether in this way Tr-S theory can still be a pure Chern-Simons theory in these cases. What can we learn from Table 11.2? We are going to make the assumption that if indeed Tr-S is identified with pure Chern-Simons theory then the Wilson loop operator $\mathcal{W}$ is identified with a Wilson loop in Chern-Simons theory (wound around one of the nontrivial 1-cycles of T^2). We will now compare the information from Table 11.2 about the eigenvalues of Wilson loops with what we know about $U(2)$ Chern-Simons theory.

Wilson loop operators in $U(2)$ Chern-Simons theory

Let us begin by reviewing the known $U(2)$ Chern-Simons results. The Hilbert space of $SU(2)$ Chern-Simons theory on a torus T^2 at level k'' is $(k''+1)$-dimensional, and as explained in [30, 56], there exists a canonical basis in which basis states are labeled by $SU(2)$ spin $j = 0, \frac{1}{2}, \ldots, \frac{k''}{2}$, once we choose a basis of 1-cycles a and b for the first homology group $H_1(T^2; \mathbb{Z})$ of the torus. When we think of T^2 as the boundary of a solid torus, the a-cycle is the one that becomes contractible inside the solid torus, while the b-cycle remains non-trivial. The state labeled by spin j is then defined in terms of the wave-function whose value is given by the path integral of Chern-Simons theory on the solid torus with a Wilson loop in the spin j representation inserted along the b-cycle. We will denote such basis states of the Hilbert space by $|\mathfrak{m}\rangle$, with $\mathfrak{m} \equiv 2j = 0, \ldots, k''$.

The action of a Wilson loop operator $W^{(n_a,n_b)}$ in any representation of $SU(2)$ that winds around the torus n_a times along the a-cycle and n_b times along the b-cycle was given in [57]. For our present purpose, we need the result for the Wilson loop in the fundamental representation with $n_a = 1$, $n_b = 0$:

$$W \equiv W^{(1,0)} = \sum_{\mathfrak{m}} 2\cos\frac{\pi(\mathfrak{m}+1)}{k''+2} |\mathfrak{m}\rangle\langle\mathfrak{m}| \,. \tag{11.35}$$

On the other hand, the Hilbert space of $U(1)$ Chern-Simons theory at level k' is k'-dimensional, and the Wilson loops act as

$$W^{(1,0)} = \sum_{p=0}^{k'-1} e^{\frac{2\pi i}{k'}p} |p\rangle\langle p| \,, \qquad W^{(0,1)} = \sum_{p=0}^{k'-1} |p+1\rangle\langle p| \,, \tag{11.36}$$

where $|p\rangle$ for $p = 0, \ldots, k'-1$ are the basis states.

We can now combine the results for the $U(1)$ and $SU(2)$ theories to construct the Hilbert space for the $U(2)$ theory. The Hilbert space, denoted by $U(2)_{k',k''}$, can be obtained by first taking the tensor product of the Hilbert space of $U(1)$ theory at level $k' = 2k$ and that of $SU(2)$ at level k'', and then restricting to the subspace where a certain "large" gauge transformation acts trivially. This is because the group $U(2)$ is not simply the product of $U(1)$ and $SU(2)$, but rather $U(2) = [U(1) \times SU(2)]/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the center of $SU(2)$.

Specifically, let us first consider the following "illegal" gauge transformations of the $U(1)$ gauge theory on T^2:

$$\Lambda'_1(x_1, x_2) = e^{ix_1/2} \,, \quad \Lambda'_2(x_1, x_2) = e^{ix_2/2} \,. \tag{11.37}$$

Here, x_1, x_2 are periodic coordinates on the torus with $0 \le x_i \le 2\pi$, $i = 1, 2$. Since Λ'_i ($i = 1, 2$) changes its value from $+1$ to $-1 \in \mathbb{Z}_2$ as x_i changes from 0 to 2π, it is not a genuine gauge transformation, and hence acts nontrivially on the physical Hilbert space. If we let Ω'_1, Ω'_2 be the corresponding operators on the Hilbert space of $U(1)$ Chern-Simons theory at level $k' = 2k$, then their action on the basis states $|p\rangle$ defined in (11.36) is given by

$$\Omega'_1|p\rangle = |p+k\rangle\,, \quad \Omega'_2|p\rangle = (-1)^p|p\rangle\,. \tag{11.38}$$

We can similarly define the "illegal" gauge transformations for the $SU(2)$ theories:

$$\Lambda''_i(x_1, x_2) = \mathrm{diag}(e^{ix_i/2}, e^{-ix_i/2})\,, \quad i = 1, 2\,. \tag{11.39}$$

They also change their values from the identity to $-1 \in \mathbb{Z}_2$ as x_i change from 0 to 2π. The corresponding operators Ω''_i ($i = 1, 2$) act on the Hilbert space of $SU(2)$ Chern-Simons theory at level $k'' > 0$ by

$$\Omega''_1|\mathfrak{m}\rangle = |k'' - \mathfrak{m}\rangle\,, \quad \Omega''_2|\mathfrak{m}\rangle = (-1)^{\mathfrak{m}}|\mathfrak{m}\rangle\,. \tag{11.40}$$

In both $U(1)$ and $SU(2)$ theories, the action of Ω'_2 and Ω''_2 is easy to understand from the definition of the basis states $|p\rangle$ and $|\mathfrak{m}\rangle$, and then the action of Ω'_1 and Ω''_1 can be inferred from the modular transformation properties of the basis states.

We can now consider the $U(2)$ gauge theory on T^2 and perform the transformations Λ'_i and Λ''_i simultaneously. The point is that while they are "illegal" gauge transformations when applied separately, they together become a genuine $U(2)$ gauge transformation, as can be seen explicitly from the above expressions. Therefore, the Hilbert space $U(2)_{k',k''}$ is the subspace of the tensor product of the Hilbert spaces of $U(1)_{k'}$ and $SU(2)_{k''}$ theories on which the operators $\Omega'_i \otimes \Omega''_i$ act trivially. We can then read off the action of the Wilson loop operators on this subspace from those of the $U(1)_{k'}$ and $SU(2)_{k''}$ theories. The results for the cases listed in Table 11.4 are as follows. (In the following, we consider only the action of $W = W^{(1,0)}$, the Wilson loop going around the a-cycle once, but $W^{(0,1)}$ is related to W by modular transformation.)

- $k = 1$: For $U(2)_{2,1}$, the invariant subspace is one-dimensional, spanned by

  $$|0\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |1\rangle_{U(1)} \otimes |1\rangle_{SU(2)}\,.$$

  The Wilson loop operator is just the identity: $\mathcal{W} = 1$.

  For $U(2)_{2,3}$, the invariant subspace is two-dimensional, spanned by

  $$|0\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |1\rangle_{U(1)} \otimes |3\rangle_{SU(2)}\,,$$
  $$|0\rangle_{U(1)} \otimes |2\rangle_{SU(2)} + |1\rangle_{U(1)} \otimes |1\rangle_{SU(2)}\,.$$

  The Wilson loop operator is given in this basis by

  $$\mathcal{W} = \mathrm{diag}(\phi, \phi - 1),$$

where $\phi = \frac{1}{2}(1+\sqrt{5})$ is the "golden ratio." On the other hand, according to Table 11.2 the Tr-S results are:

$$\mathcal{W} = \mathrm{diag}(\pm\sqrt{2}, \pm\sqrt{2}, 0),$$

and they clearly don't agree with Chern-Simons results for $U(n)_{2,k''}$ for any k''. (We have only explicitly written down the cases $k'' = 1, 3$ above, since they appear in Table 11.4, but it can be easily checked that other values don't give the right answer either.)

- $k = 2$: For $U(2)_{4,2}$, the invariant subspace is three-dimensional, spanned by

$$|0\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |2\rangle_{U(1)} \otimes |2\rangle_{SU(2)} \,,$$
$$|1\rangle_{U(1)} \otimes |1\rangle_{SU(2)} + |3\rangle_{U(1)} \otimes |1\rangle_{SU(2)} \,,$$
$$|0\rangle_{U(1)} \otimes |2\rangle_{SU(2)} + |2\rangle_{U(1)} \otimes |0\rangle_{SU(2)} \,.$$

The Wilson loop operator in this basis is given by

$$\mathcal{W} = \mathrm{diag}(\sqrt{2}, 0, -\sqrt{2}) \,,$$

which is also the result for $U(2)_{4,-2}$. Thus, the conjectures from Table 11.4 of $U(2)_{4,2}$ and $U(2)_{4,-2}$ for the sectors $2 = 1+1$ and $2 = 2$, respectively, are in precise agreement with the eigenvalues of $\mathcal{W}$ that we calculated and summarized in Table 11.2.

- $k = 3$: For $U(2)_{6,1}$ the invariant subspace is three-dimensional, spanned by

$$|0\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |3\rangle_{U(1)} \otimes |1\rangle_{SU(2)} \,,$$
$$|1\rangle_{U(1)} \otimes |1\rangle_{SU(2)} + |4\rangle_{U(1)} \otimes |0\rangle_{SU(2)} \,,$$
$$|2\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |5\rangle_{U(1)} \otimes |1\rangle_{SU(2)} \,.$$

The Wilson loop operator in this basis is given by

$$\mathcal{W} = \mathrm{diag}(1, -\omega, \omega^2) \,,$$

where $\omega = e^{\pi i/3}$.

For $U(2)_{6,3}$ the invariant subspace is six-dimensional, spanned by

$$|0\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |3\rangle_{U(1)} \otimes |3\rangle_{SU(2)} \,,$$
$$|0\rangle_{U(1)} \otimes |2\rangle_{SU(2)} + |3\rangle_{U(1)} \otimes |1\rangle_{SU(2)} \,,$$
$$|1\rangle_{U(1)} \otimes |1\rangle_{SU(2)} + |4\rangle_{U(1)} \otimes |2\rangle_{SU(2)} \,,$$
$$|1\rangle_{U(1)} \otimes |3\rangle_{SU(2)} + |4\rangle_{U(1)} \otimes |0\rangle_{SU(2)} \,,$$
$$|2\rangle_{U(1)} \otimes |0\rangle_{SU(2)} + |5\rangle_{U(1)} \otimes |3\rangle_{SU(2)} \,,$$
$$|2\rangle_{U(1)} \otimes |2\rangle_{SU(2)} + |5\rangle_{U(1)} \otimes |1\rangle_{SU(2)} \,.$$

The Wilson loop operator in this basis is given by

$$\mathcal{W} = \mathrm{diag}(\phi, 1-\phi, -\omega(1-\phi), -\omega\phi, \omega^2\phi, \omega^2(1-\phi))\,,$$

where ϕ is the golden ratio as before. The Tr-S eigenvalues that we expect have to have an absolute value of $\sqrt{2}$ or 0, and so we don't find an agreement in this case either.

In the above, we explicitly compared the eigenvalues of Wilson loop operators only for the gauge group $U(2)$, but we can do similar computations for other gauge groups as well using the formula of [57]. We find that in general the results of Table 11.2 do not agree with the eigenvalues of Wilson loop operators in Chern-Simons theories with gauge group listed in Table 11.3, except for the $k = 2$, $n = 2$ case discussed above.

k	$n = n_1 + \cdots + n_p$	Contribution to Witten Index
3	$1 = 1$	3
3	$2 = 1 + 1$	$\frac{9}{2}2^m$
3	$2 = 2$	$\frac{3}{2}2^m$
2	$1 = 1$	2
2	$2 = 1 + 1$	$2 \cdot 2^m$
2	$2 = 2$	$2 \cdot 2^m$
2	$3 = 1 + 1 + 1$	$\frac{4}{3} \cdot 3^m + 2$
2	$3 = 2 + 1$	$4 \cdot 3^m + 2$
2	$3 = 3$	$\frac{2}{3}3^m$
1	$1 = 1$	1
1	$2 = 1 + 1$	$\frac{1}{2}2^m$
1	$2 = 2$	$\frac{3}{2}2^m$
1	$3 = 1 + 1 + 1$	$\frac{1}{6}3^m + \frac{1}{2}$
1	$3 = 2 + 1$	$\frac{3}{2}3^m + \frac{1}{2}$
1	$3 = 3$	$\frac{4}{3}3^m$
1	$4 = 1 + 1 + 1 + 1$	$\frac{1}{24}4^m + \frac{1}{4}2^m + \frac{1}{3}$
1	$4 = 2 + 1 + 1$	$\frac{3}{4}4^m + 2^m$
1	$4 = 2 + 2$	$\frac{9}{8}4^m + \frac{3}{4}2^m$
1	$4 = 3 + 1$	$\frac{4}{3}4^m + \frac{2}{3}$
1	$4 = 4$	$\frac{3}{4}4^m$
1	$5 = 1 + 1 + 1 + 1 + 1$	$\frac{1}{120}5^m + \frac{1}{12}3^m + \frac{1}{6}2^m + \frac{3}{8}$
1	$5 = 2 + 1 + 1 + 1$	$\frac{1}{4}5^m + \frac{5}{6}3^m + \frac{1}{2}2^m + \frac{1}{4}$
1	$5 = 2 + 2 + 1$	$\frac{9}{8}5^m + \frac{3}{4}3^m + \frac{9}{8}$
1	$5 = 3 + 1 + 1$	$\frac{2}{3}5^m + \frac{2}{3}3^m + \frac{1}{3}2^m$
1	$5 = 3 + 2$	$2 \cdot 5^m + \frac{2}{3}3^m + 2^m$
1	$5 = 4 + 1$	$\frac{3}{4}5^m + \frac{5}{4}$
1	$5 = 5$	$\frac{1}{5}5^m$

Table 11.1: The contribution to the Witten Index of the sector with given k and n, and a permutation σ in the conjugacy class that corresponds to the partition $n = n_1 + \cdots + n_p$.

k	n	Witten Index	$\dim \mathcal{H}(k,n,m=0)$	$\mathcal{W}^\dagger\mathcal{W}$ eigenvalues
3	1	3	3	$1_{(3)}$
3	2	$6 \cdot 2^m$	9	$2_{(6)}, 0_{(3)}$
2	1	2	2	$1_{(2)}$
2	2	$4 \cdot 2^m$	6	$2_{(4)}, 0_{(2)}$
2	3	$6 \cdot 3^m + 4$	12	$3_{(6)}, 1_{(4)}, 0_{(2)}$
1	1	1	1	$1_{(1)}$
1	2	$2 \cdot 2^m$	3	$2_{(2)}, 0_{(1)}$
1	3	$3 \cdot 3^m + 1$	5	$3_{(3)}, 1_{(1)}, 0_{(1)}$
1	4	$4 \cdot 4^m + 2 \cdot 2^m + 1$	10	$4_{(4)}, 2_{(2)}, 1_{(1)}, 0_{(3)}$
1	5	$5 \cdot 5^m + 3 \cdot 3^m + 2 \cdot 2^m + 3$	15	$5_{(5)}, 3_{(3)}, 2_{(2)}, 1_{(3)}, 0_{(2)}$

Table 11.2: The Witten Index as a function of k, n, and m. The behavior of the Witten Index as a function of the number of quark and anti-quark pairs m allows us to calculate the eigenvalues λ_l of the operator $\mathcal{W}^\dagger\mathcal{W}$ and their multiplicities N_l. They are listed in the last column as $\lambda_{1(N_1)}, \lambda_{2(N_2)}, \ldots$.

$\upsilon = \frac{\pi}{3}$ $(k=1)$	$n=1$	$U(1)_1$
	$n=2$	$U(2)_{2,1} \oplus U(2)_{2,-3}$
	$n=3$	$U(3)_{3,1} \oplus [U(1)_1 \times U(2)_{2,-3}] \oplus U(3)_{3,-2}$
	$n=4$	$U(4)_{4,1} \oplus 2[U(2)_{2,1} \times U(2)_{2,-3}] \oplus [U(1)_1 \times U(3)_{3,-2}] \oplus \mathcal{H}_{(2,2)}$
	$n=5$	$U(5)_{5,1} \oplus U(5)_{5,1} \oplus 2[U(3)_{3,1} \times U(2)_{2,-3}] \oplus [U(1)_1 \times \mathcal{H}_{(2,2)}] \oplus$ $[U(2)_{2,1} \times U(3)_{3,-2}] \oplus [U(2)_{2,-3} \times U(3)_{3,-2}]$
$\upsilon = \frac{\pi}{2}$ $(k=2)$	$n=1$	$U(1)_2$
	$n=2$	$U(2)_{4,2} \oplus U(2)_{4,-2}$
	$n=3$	$U(3)_{6,2} \oplus [U(1)_2 \times U(2)_{4,-2}] \oplus U(3)_{6,-1}$
$\upsilon = \frac{2\pi}{3}$ $(k=3)$	$n=1$	$U(1)_3$
	$n=2$	$U(2)_{6,3} \oplus U(2)_{6,-1}$

Table 11.3: The results of [1] regarding the equivalence of the Hilbert spaces of Tr-S and Chern-Simons theory on T^2 as representations of the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ together with $\mathcal{U}, \mathcal{V}$. The notation $U(n)_{k',k''}$ corresponds to a Chern-Simons theory where the $U(1)$ part is at level $k'$and the $SU(n)$ part is at level k''. One of the sectors $(4 = 2+2)$ for $n=4$ and $k=1$ could not be matched with a Chern-Simons theory and is therefore written explicitly as $\mathcal{H}_{(2,2)}$. It also appears in the $5 = 2+2+1$ decomposition of the $n=5$ theory.

k	$n = n_1 + \cdots + n_p$	Hilbert space
1	$2 = 1 + 1$	$U(2)_{2,1}$
1	$2 = 2$	$U(2)_{2,-3}$
2	$2 = 1 + 1$	$U(2)_{4,2}$
2	$2 = 2$	$U(2)_{4,-2}$
3	$2 = 1 + 1$	$U(2)_{6,3}$
3	$2 = 2$	$U(2)_{6,-1}$

Table 11.4: The $n = 2$ results of [1], sector by sector. Each Hilbert space of a Tr-S sector is equivalent, as a representation of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ and $\mathcal{U}, \mathcal{V}$, to a corresponding Hilbert space of Chern-Simons theory. The notation $U(2)_{k',k''}$ corresponds to a Chern-Simons theory where the $U(1)$ part is at level k' and the $SU(2)$ part is at level k''.

Chapter 12

Discussion

We have computed the Witten Index of Tr-S theory on T^2 with charges, and we have used the results to calculate the eigenvalues of simple Wilson loop operators in the theory. The Witten Index of the $U(n)$ theory with parameter k and $2m$ charges is listed in Table 11.2. We found that for gauge group $U(2)$ and for the $k = 2$ case (the basic S-duality twist corresponding to $\tau \to -1/\tau$) the results are consistent with a conjecture put forward in [1] relating the theory to two $U(2)$ Chern-Simons theories with $U(1) \subset U(2)$ at level 4 and the $SU(2)$ at levels ± 2. This would imply that the low-energy theory has two "superselection" sectors. On the other hand, we saw that in most other cases of n and k, the simple decomposition into superselection sectors labeled by a conjugacy class in the permutation group S_n (as conjectured in [1]) is inconsistent with the form of the Witten Index results, and several sectors have to be combined together to yield a consistent theory. What this theory is we do not know, but we were able to show that it is inconsistent with pure Chern-Simons theory, at least at low levels.

A physical perspective for understanding the discrepancy is plausibly as follows. Well-defined Wilson loops in the four-dimensional $\mathcal{N} = 4$ SYM theory with twisted boundary conditions flow, in the low-energy limit, to a S-duality invariant operator $\mathcal{W}_{\text{inv.}}$ that is a linear combination of Wilson loops and dual monopole operators. For example, as first briefly discussed in Section 6.6 of [1], for $k = 2$, the relevant operator supported on any curve C flows as[1]

$$\mathcal{W}_{\text{inv.}}(C, x_3) \longrightarrow \mathcal{W}(C, x_3) + \mathcal{M}(C, x_3) + \mathcal{W}(C, x_3)^\dagger + \mathcal{M}(C, x_3)^\dagger$$

where $\mathcal{M}$ is the dual 't Hooft operator. It is thus possible that Wilson loops in Tr-S theory correspond to an appropriate dimensional reduction of $\mathcal{W}_{\text{inv.}}$. As a simple check, we note that for the abelian case, computing $\text{tr}[(\mathcal{W}_{\text{inv.}}^\dagger \mathcal{W}_{\text{inv.}})^m]$ yields also the index to be k. Moreover, for the only nonabelian case which agrees with Chern-Simons theory, namely $k = n = 2$, the expectation values of monopole operators

[1] For other values of k, $\mathcal{W}_{\text{inv.}}$ involves mixed Wilson-'t Hooft operators. See, for example, [58] for an illuminating discussion.

and Wilson loops are identical as first explained in [59]. To phrase it simply, the discrepancy between Tr-S and Chern-Simons theory may be understood physically as coming from non-trivial electromagnetic boundary conditions that descend from the four-dimensional twisted theory.

We conjecture that Tr-S is a topological theory for $n < \mathbf{r}$, and we presented some arguments in favor of this in §8.4. Another possible test of this could be to look for BPS states that carry nonzero momentum along T^2. If the low-energy theory is topological we would expect to find only states with energies of the order of $1/R$. In the type-IIA dual the momentum quantum numbers become D0-brane and D2-brane charge (where the D2 branes wrap directions $1, 10$). It would be interesting to study the bound states of D0-branes with the n fundamental strings. In the limit $R \to 0$ this system can be mapped to a sector of a $U(1)$ dipole-theory [13, 14]. We hope to explore this further in a separate work.

To calculate the Witten Index we divided the Hilbert space into "sectors" according to the pattern of closed and open strings of the dual type-IIA system. We saw that only a subset of sectors contributes to the Witten Index — the "decongested" sectors. The remaining (congested) sectors have fermionic zero modes, and they do not contribute to the Index. It would be interesting to explore these sectors further. For example, we noted that the supersymmetric system of charges has a global $U(1)$ symmetry that is generated by the element $\mathcal{J}$ of rotations in transverse directions that acts on spinors as $\frac{i}{2}(\Gamma^{45} + \Gamma^{67})$. Since $\mathcal{J}$ commutes with all the surviving supersymmetry generators it is possible to generalize the Index to

$$I(u) \equiv \operatorname{tr}\left[(-)^F e^{iu\mathcal{J}}\right].$$

This modified index receives contributions only from ground states, but can get contributions from some congested sectors as well. At the end of §11.2 we gave an example of a congested sector with 2 complex fermionic zero modes that are all charged under $\mathcal{J}$. Quantizing these gives a Hilbert space with 4 states with $\mathcal{J}$ charges $-1, 0, 0, +1$ and which contributes a term proportional to $(2 - 2\cos u)$ to the index. It is possible, however, that the fermionic zero modes interact with the bosonic modes and calculating $I(u)$ therefore requires a separate treatment and will not be pursued here.

Taking a different approach, it would be interesting to construct Tr-S directly in terms of the duality-generating theories $T(U(n))$ defined in [41]. For example, Gaiotto and Witten argued that S-duality for $SU(2)$ is generated by starting with $T(SU(2))$, which they identified with the strongly-coupled low-energy limit of the $2+1D$ $\mathcal{N} = 4$ theory of two equally charged hypermultiplets coupled to a $U(1)$ vector-multiplet. This theory has a manifest $SU(2) \times U(1)$ global symmetry, but as conjectured in [43] and further explained in [60], the low-energy limit has an enhanced $SU(2) \times SU(2)$ symmetry. The S-duality twist, according to Gaiotto and Witten, is then realized by gauging one $SU(2)$ with the original gauge field (at $x_3 = 0$ in our context) and the other $SU(2)$ with the dual gauge field (the one at $x_3 = 2\pi R$). It would be interesting to derive our results for the Witten Index directly from this construction.

The computation is not so trivial, of course, because the $T(SU(2))$ theory is strongly coupled.

Recently, Terashima and Yamazaki [44] studied a related compactification with an S-duality twist but only $\mathcal{N} = 2$ supersymmetry in $2+1D$. They computed the partition function of the theory on S^3 and related it to $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory. It would be interesting to understand if this construction can be modified to provide information on the $\mathcal{N} = 6$ setting that we studied in this paper.

In [1] another way to reproduce Tr-S from the $T(SU(n))$ theories was also offered. This made use of the low-energy limit of a D3-brane boundary on a (p,q) 5-brane, as constructed by Gaiotto and Witten using $T(SU(n))$ [41]. The starting point for [1] in this context was the (2, 0)-theory wrapping the three-dimensional submanifold of the space (8.5) that is defined by $\zeta_1 = \zeta_2 = \zeta_3 = 0$. Recently, a beautiful picture of the low-energy limit of the (2, 0)-theory compactified on a general three-dimensional manifold has emerged [45]-[61]. It would be interesting to analyze Tr-S from that perspective as well.

If Tr-S is topological then correlation functions of the low-energy limits of Wilson loops, discussed in §11.4, construct knot and link invariants. The general question, to which this paper provides only partial answers in special cases, is what are these invariants. Recently, there have been exciting new developments in the realization of knot invariants in terms of field theories and string theory (see [62] -[63] for a sample of the recent literature). A better understanding of Tr-S might provide new physical constructions of knot invariants. We hope to explore more general Wilson loops in future papers.

Chapter 13

Appendices

13.1 Supersymmetry and fermionic zero modes — details

In this appendix we expand on various statements made in §9.2, §9.9, and §10.3 about the amount of supersymmetry preserved by intersections of strings and branes and the fermionic modes that describe these systems at low-energy.

Our conventions are as follows. 10+1D directions are denoted by

$$I, J, K, \dots = 0, \dots, 10 \equiv \natural\,,$$

and we use $\natural \equiv 10$ in indices of Dirac matrices to avoid confusion with $1, 0$. We work in Minkowski signature

$$\eta_{IJ} dx^I dx^J = -dx_0^2 + dx_1^2 + \dots + dx_\natural^2\,.$$

All our spinors, whether in 10+1D M-theory on 9+1D type-IIA/B are 32-component Majorana spinors on which the 11-dimensional Dirac matrices Γ^I can act. When we need type-IIA spinors, we will specify which direction is eliminated (as the "M-theory direction"). For example, the 2^{nd} row of Table 8.1 is obtained from the 3^{rd} by eliminating direction 10, so the resulting type-IIA spinors ϵ are still 32-component Majorana spinors, but they can be decomposed into left-chirality and right-chirality spinors:

$$\epsilon = \epsilon_+ + \epsilon_-\,, \qquad \epsilon_\pm \equiv \frac{1}{2}(1 \pm \Gamma^\natural)\epsilon.$$

We will construct type-IIB spinors by performing T-duality on another direction. For example, the 1^{st} row of Table 8.1 is obtained from the 2^{rd} by T-duality on direction 1, so we can define the complex Weyl type-IIB SUSY parameters as

$$\epsilon_{\text{IIB}} \equiv \epsilon_+ + i\Gamma^1\epsilon_-\,.$$

The Dirac matrices Γ^I are real and satisfy

$$\{\Gamma^I, \Gamma^J\} = 2\eta^{IJ}\,, \qquad \Gamma^{0123456789\natural} = 1\,.$$

Now take an M2-brane in directions $0, 9, \natural$. We denote

$$\mu, \nu, \cdots = 0, 9, \natural; \qquad a, b, c, \cdots = 1, \ldots, 8\,.$$

The M2-brane low-energy fields are the scalars Φ^a ($a = 1, \ldots, 8$) and the spinors λ, which satisfy the chirality condition

$$\Gamma^{09\natural}\lambda = -\lambda\,.$$

Let ϵ be the 10+1D SUSY parameter and set

$$\epsilon_l \equiv \tfrac{1}{2}(1 - \Gamma^{09\natural})\epsilon\,, \qquad \epsilon_r \equiv \tfrac{1}{2}(1 + \Gamma^{09\natural})\epsilon$$

The SUSY transformations are

$$\delta\lambda = \epsilon_l + \partial_\mu\Phi^a\Gamma^\mu\Gamma_a\epsilon_r + (\overline{\lambda}\Gamma^\mu\epsilon_r)\partial_\mu\lambda\,, \qquad \delta\Phi^a = \overline{\lambda}\Gamma^a\epsilon_r + \partial_\mu\Phi^a\overline{\lambda}\Gamma^\mu\epsilon_r\,.$$

The spinors are real and $\overline{\lambda} \equiv \lambda^t\Gamma^0$. From the point of view of the $2 + 1D$ worldvolume theory, the ϵ_r parameters generate worldvolume supersymmetry transformations, while ϵ_l generate the κ-symmetry [64]. If we compactify this M2-brane on T^2 by making directions $1, 2$ periodic, the spinors λ will have 16 linearly independent zero modes, which generate a multiplet of 256 states. These states are invariant under all supersymmetries with $\epsilon_l = 0$, but not invariant under supersymmetries with ϵ_r. As is customary, we refer to the supersymmetry transformation with $\epsilon_l = 0$ as the "unbroken supersymmetries."

Now consider an M2-brane stretched along directions $2, 3$, which upon reduction to type-IIA on direction 2 will become an F1 in direction 3. At low-energy there are 8 scalar fields in the vector representation 8_v of the group $SO(8)$ of rotations in transverse directions $1, 4, 5, 6, 7, 8, 9, 10$, as well as their superpartners which are spinors in $(2, 8_s)$ of $SO(2, 1) \times SO(8)$. These spinors ψ satisfy

$$\psi = -\Gamma^{023}\psi = -\Gamma^{1456789\natural}\psi\,.$$

Upon reduction to type-IIA we write

$$\psi = \psi_L + \psi_R$$

where

$$\psi_L = \frac{1}{2}(1 + \Gamma^2)\psi\,, \qquad \psi_R = \frac{1}{2}(1 - \Gamma^2)\psi\,,$$

satisfy

$$\Gamma^{03}\psi_L = \psi_L\,, \quad \Gamma^{03}\psi_R = -\psi_R\,.$$

which become left- and right-moving massless fields along the string. Note that in type-IIA both ψ_L and ψ_R are in 8_s of $SO(8)$. Now, consider an M2-brane along directions $9, 10$, which becomes a D2-brane in type-IIA, and compactify direction 10 as well. This M2-brane has low-energy fermions χ satisfying

$$\chi = -\Gamma^{09\natural}\chi = -\Gamma^{12345678}\chi\,.$$

Now compactify direction x_{10}. At low-energy, below the x_{10} compactification scale, the wrapped D2-brane looks like a string which has left- and right-moving massless fields $\chi_{L,R}$ along it. The are defined by:

$$\chi = \chi_R + \chi_L\,,$$

where

$$\chi_L = \tfrac{1}{2}(1+\Gamma^{\natural})\chi\,,\qquad \chi_R = \tfrac{1}{2}(1-\Gamma^{\natural})\chi\,,\qquad \Gamma^{09}\chi_L = \chi_L\,,\qquad \Gamma^{09}\chi_R = -\chi_R\,.$$

Next, we consider a configuration where such a D2-brane has two F1 strings (with the same orientation) emanating from a point on it: one string in the positive x_3 direction and one string in the negative x_3 direction. Note that the total charge at the point of origin is zero, since the charge of the endpoint of one string cancels the charge of the endpoint of the other string. Denote the low-energy fields on the string in the positive x_3-direction by $\psi^{(>)}$, and denote the low-energy fields on the string in the negative x_3-direction by $\psi^{(<)}$. Similarly, the wrapped D2-brane, at energies below the x_{10} compactification scale, has low-energy fields $\chi^{(<)}$ for the $x_9 < 0$ side and $\chi^{(>)}$ for the $x_9 > 0$ side. We are interested in the boundary conditions that connect the values of the 4 fields $\psi^{(<,>)}, \chi^{(<,>)}$ at the intersection point.

If we lift this system back to M-theory we get two M2-branes that intersect at a point. Perhaps the easiest way to derive the requisite boundary conditions is to deform this system to a smooth M2-brane that extends along a surface that, in appropriate complex coordinates described below, is a holomorphic curve. The low-energy reduction, below the x_2, x_{10} compactification scales, looks like a (p, q)-web as in figure Figure 9.5(b) (see [65, 66] for some examples). The smooth geometry can be described by techniques similar to those developed in [53]. We define two complex coordinates

$$u \equiv e^{\frac{ix_2+x_3}{L_2}}\,,\qquad v \equiv e^{\frac{ix_{10}+x_9}{L_{10}}}$$

where L_2, L_{10} are the radii of directions 2 and 10. The smooth holomorphic curve is given by

$$(u-1)(v-1) = C \tag{13.1}$$

where $C \neq 0$ is a constant. Note that this is a deformation of the singular curve $(u-1)(v-1) = 0$. An M2-brane that wraps this holomorphic curve will have a low-energy fermionic field λ on it. We are looking for zero modes of this field which have a finite limit at the 4 semi-infinite directions $x_3 \to \pm\infty$ and $x_9 \to \pm\infty$. Below, we explain how to make the connections:

$$\begin{aligned} &\lambda(x_9 = -\infty) \to \chi^{(<)}\,, \qquad \lambda(x_9 = \infty) \to \chi^{(>)}\,, \\ &\lambda(x_3 = -\infty) \to \psi^{(<)}\,, \qquad \lambda(x_3 = \infty) \to \psi^{(>)}\,. \end{aligned} \tag{13.2}$$

The linear algebraic relations among these four limit values will constitute the requisite boundary conditions.

Consider a part of the M2-brane near $x_3 \to \infty$. It is approximately at constant x_9, x_{10} and stretches in directions x_2, x_3. The spinor can be decomposed according to the eigenvalue of Γ^{23} as

$$\lambda = \eta_+ + \eta_-\,, \qquad \eta_\pm \equiv \tfrac{1}{2}(1 \pm i\Gamma^{23})\lambda\,, \qquad \Gamma^{23}\eta_\pm = \mp i\eta_\pm\,, \qquad (\text{near } x_3 \to \infty)\,. \tag{13.3}$$

We then calculate the zero-mode equation

$$0 = (\Gamma^2\partial_2 + \Gamma^3\partial_3)\eta_\pm = \Gamma^2(\partial_2 \mp i\partial_3)\eta_\pm$$

and so η_+ is holomorphic in $x_3 + ix_2$, and hence in u, while η_- is anti-holomorphic. When extending to the bulk of the holomorphic curve, we have to keep track of the tangent and normal bundles of the M2-brane surface given by (13.1). At an arbitrary point p on this surface the tangent plane T_p can be thought of as a subspace of $\mathbb{R}^4$ that is the constant tangent space in the x_2, x_3, x_9, x_{10} directions. As p varies the embedding $T_p \subset \mathbb{R}^4$ varies. Locally, we can pick a rotation $\Omega_p \in U(2) \subset \mathrm{Spin}(4)$ that maps the tangent plane T_p to a common plane, which we choose to be the $x_2 - x_3$ plane, and also varies smoothly with p. At any fixed point p on the surface, this rotation Ω_p is unique up to $SO(2) \times SO(2)$ (rotations in the $x_2 - x_3$ and $x_9 - x_{10}$ planes separately). Near $x_9 \to \infty$, for example, T_p is the $x_9 - x_{10}$ plane and we can take the rotation in spinor representation to be

$$\Omega = e^{\frac{\pi}{4}(\Gamma^{2\natural} + \Gamma^{93})} = e^{\frac{\pi}{4}(1 + \Gamma^{239\natural})\Gamma^{93}} = \tfrac{1}{2}(1 + \Gamma^{2\natural})(1 + \Gamma^{93}). \tag{13.4}$$

If we decompose the fermionic field near $x_9 \to \infty$ as

$$\chi^{(>)} = \chi_R^{(>)} + \chi_L^{(>)}\,, \qquad \chi_R^{(>)} = \tfrac{1}{2}(1 + \Gamma^{239\natural})\chi^{(>)}\,, \qquad \chi_L^{(>)} = \tfrac{1}{2}(1 - \Gamma^{239\natural})\chi^{(>)} \tag{13.5}$$

the components $\chi_R^{(>)}$ and $\chi_L^{(>)}$, after rotation of the $x_9 - x_{10}$ plane into the $x_2 - x_3$ plane, are

$$\Omega\chi_R^{(>)} = \chi_R^{(>)}\,, \qquad \Omega\chi_L^{(>)} = \Gamma^{93}\chi_L^{(>)} = e^{\frac{\pi}{2}\Gamma^{93}}\chi_L^{(>)} = \Gamma^{2\natural}\chi_L^{(>)} = e^{\frac{\pi}{2}\Gamma^{2\natural}}\chi_L^{(>)}. \tag{13.6}$$

Thus, using Ω we can map chiral spinors at any point on the surface to a common space, and thus extend (13.3) by setting

$$\eta_\pm = \frac{1}{2}(1 \pm i\Gamma^{23})\Omega\lambda. \tag{13.7}$$

Let $\mathcal{K}$ be the canonical bundle (i.e., the bundle whose sections are holomorphic $(1,0)$-forms on the curve), and let $\mathcal{N} = \mathcal{K}^{-1}$ be the normal bundle (where we embed the curve in $\mathbb{C}^2$ in directions $2,3,9,10$). The modes η_+ transform as sections of $\mathcal{K}^{1/2} \otimes (\mathcal{N}^{1/2} \oplus \mathcal{N}^{-1/2}) = \mathcal{O} \oplus \mathcal{K}$, where $\mathcal{O}$ is the trivial bundle. The relation (13.7) thus maps a spinor λ to a section of $\mathcal{O} \oplus \mathcal{K}$ (times a trivial spinor bundle in the transverse directions). So, altogether, we can decompose zero modes into

$$\lambda = \lambda_R + \lambda_L\,, \qquad \lambda_R \equiv \tfrac{1}{2}(1+\Gamma^{239\natural})\lambda\,, \qquad \lambda_L \equiv \tfrac{1}{2}(1-\Gamma^{239\natural})\lambda\,.$$

Then, zero modes λ_L are sections of $\mathcal{K}^{1/2}\otimes\mathcal{N}^{1/2} = \mathcal{O}$ which is the trivial bundle, while zero modes λ_R are sections of $\mathcal{K}^{1/2} \otimes \mathcal{N}^{-1/2} = \mathcal{K}$. Thus, λ_L is simply a holomorphic function of u, with finite limits at the 4 ends, while λ_R is a holomorphic 1-form with finite limits at the 4 ends.

In terms of the coordinate u, the curve (13.1) is mapped to the complex u-plane with 4 singular points: $u = 0, \infty$ correspond to the two ends of the F1-string, while $u = 1 - C, 1$ correspond to $v = 0, \infty$, which are the two ends of the D2-brane. Equation (13.2) becomes

$$\begin{aligned} \lambda(u = 1-C) &\to \chi^{(<)}\,, \qquad & \lambda(u=1) &\to \chi^{(>)}\,, \\ \lambda(u=0) &\to \psi^{(<)}\,, \qquad & \lambda(u=\infty) &\to \psi^{(>)}\,. \end{aligned} \tag{13.8}$$

We denote

$$\chi_R^{(<,>)} \equiv \tfrac{1}{2}(1+\Gamma^{239\natural})\chi^{(<,>)}\,, \qquad \chi_L^{(<,>)} \equiv \tfrac{1}{2}(1-\Gamma^{239\natural})\chi^{(<,>)}\,.$$

and

$$\psi_R^{(<,>)} \equiv \tfrac{1}{2}(1+\Gamma^{239\natural})\psi^{(<,>)}\,, \qquad \psi_L^{(<,>)} \equiv \tfrac{1}{2}(1-\Gamma^{239\natural})\psi^{(<,>)}\,.$$

As λ_L modes are sections of the trivial line-bundle, and are therefore constant functions, their boundary conditions must be:

$$\chi_L^{(<)} = \chi_L^{(>)} = \psi_L^{(<)} = \psi_L^{(>)}\,. \tag{13.9}$$

On the other hand, the λ_R modes are sections of the canonical bundle. They correspond to holomorphic 1-forms which we denote by $\omega(u)du$. Being constant near $x_3 \to \infty$ means that ωdu is proportional at $u = \infty$ to $d\log u = du/u$, and so has a first-order zero there. Similar analysis of the behavior near the other three singular points $u = 0, 1-C, 1$, shows that the 1-form needs to have at most a simple pole, and since it vanishes at $u = \infty$, the 1-form is of the form

$$\omega(u) = \left(\frac{\alpha}{u-(1-C)} + \frac{\beta}{u-1} + \frac{\gamma}{u}\right) du\,.$$

Here $\chi_R^{(<)}$ is proportional to the constant α, $\chi_R^{(>)}$ is proportional to the constant β, $\psi_L^{(<)}$ is proportional to the constant γ, and $\psi_L^{(>)}$ is proportional to the constant $-(\alpha+\beta+\gamma)$. Converting back to spinors using (13.7), we find:

$$0 = \Gamma^{39}\chi_R^{(>)} - \Gamma^{39}\chi_R^{(<)} + \psi_R^{(>)} - \psi_R^{(<)}\,. \tag{13.10}$$

Equations (13.9)-(13.10) are the requisite boundary conditions!

13.2 Additional Combinatorics

13.2.1 The number of decongested binding matrices

In (11.29) we quoted the number $f(n,m)$ of non-equivalent decongested binding matrices that have at least one nonzero entry in each of the n rows. We will now derive this expression. We can easily find a recursion formula for $f(n,m)$ by noting that we can uniquely relabel the strings so that the m^{th} D2-brane is attached to the n^{th} string. Suppose there are $0 \le l \le m-n$ additional D2-branes attached to the n^{th} string, then the remaining $(n-1)$ strings have $f(n-1, m-l-1)$ configurations, and therefore

$$f(n,m) = \sum_{l=0}^{m-n} \binom{m-1}{l} f(n-1, m-l-1) = \sum_{j=n-1}^{m-1} \binom{m-1}{j} f(n-1, j). \quad (13.11)$$

Define the generating function

$$f_n(u) \equiv \sum_{m=n}^{\infty} f(n,m) u^{-m} .$$

Then, (13.11) implies

$$\begin{aligned} f_n(u) &= \sum_{m=n}^{\infty} \sum_{j=n-1}^{m-1} \binom{m-1}{j} f(n-1,j) u^{-m} \\ &= \sum_{j=n-1}^{\infty} \sum_{m=j+1}^{\infty} \binom{m-1}{j} f(n-1,j) u^{-m} \\ &= \sum_{j=n-1}^{\infty} (u-1)^{-(j+1)} f(n-1,j) \\ &= \frac{1}{u-1} f_{n-1}(u-1) . \end{aligned}$$

It follows that

$$f_n(u) = \frac{1}{\prod_{j=1}^{n} (u-j)} = \sum_{j=1}^{n} \frac{(-1)^{n-j}}{(j-1)!(n-j)!(u-j)} ,$$

and therefore

$$f(n,m) = \sum_{j=1}^{n} \frac{(-1)^{n-j}}{j!(n-j)!} j^m .$$

13.2.2 A generating function for the number of sectors and Fibonacci numbers

In the following, we will count the number of sectors for all $m \geq 1$. Remarkably, it turns out that the number of sectors for consecutive m's follows a generalized Fibonacci sequence $= \{3, 7, 18, 47, 123 \ldots\}$. To solve this combinatorial problem, we begin by introducing another set of notations to describe the binding matrices $\mathfrak{B}$. We denote any continuous stretch of rows of $\mathfrak{B}$ using letters $[j]$ defined as:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv [1], \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \equiv [2], \ldots \quad \underbrace{\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}}_{j \text{ rows}} \equiv [j] \tag{13.12}$$

and similarly so when the zeroes are located in the first row. Each letter represents two possible configurations: [1] represents both $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; [2] represents both $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; and so on. Thus, for example,

$$\mathfrak{B} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \text{ is translated to the word } [4][2][1][1][2][1]. \tag{13.13}$$

In general we cannot recover $\mathfrak{B}$ from the word, but note that different $\mathfrak{B}$'s can be made equivalent after relabeling of the open strings. In particular, whenever there is a column with two 1's, we can relabel the open strings that are to the right of that column. This will change the matrix $\mathfrak{B}$, and potentially also the accompanying permutation σ, but will give an equivalent physical sector. For example, if we relabel all strings after the 5^{th} column of (13.13) we get [in the notation of (11.1) of §11.1]:

$$\begin{aligned} \mathfrak{B}_\sigma &= \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}_{+1} \sim \\ \sim \mathfrak{B}'_{\sigma'} &= \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}_{-1}. \end{aligned} \tag{13.14}$$

This means that up to changing σ we can always assume that after a string of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ columns there appears $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The letters $[2], [3], \ldots$ thus translate back to a unique sequence of columns in $\mathfrak{B}$. But the letter [1] can translate back to either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We denote the number of [1] letters in a given word by p. Then, there are 2^p ways to translate the word back to $\mathfrak{B}$. There are also two possibilities for σ, which gives 2^{p+1} possibilities, but now each sector is counted twice because we can exchange the entire two rows of $\mathfrak{B}$ to get equivalent sectors. Altogether, we find that a word with p letters [1] corresponds to 2^p sectors.

We can now count the number of sectors for a generic $m > 1$ as follows. Let us consider words that have p letters [1] and $r \geq 0$ other letters, which makes $(r+p)$ letters in total. The total number of different ways to fill in the [1]'s is then $\frac{(r+p)!}{r!p!}$. We need to compute also the total number of ways to write $(m-p)$ as a sum of r numbers from $2, 3, 4, \ldots$. This is calculated by subtracting 2 from each letter and then computing the number of ways to write $(m-p-2r)$ in this form as a sum of r non-negative integers, which is simply equal to $\frac{(m-p-r-1)!}{(m-p-2r)!(r-1)!}$. Putting these facts together, the total number of different configurations d_m represented by this class of words is then

$$d_m = \sum_{r,p} \frac{(r+p)!(m-p-r-1)!}{(m-p-2r)!(r-1)!r!p!} 2^p \tag{13.15}$$

Finally, we note that the set of $\mathfrak{B}$'s that can be represented by a word lacks those which end with a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ column, i.e., $\mathfrak{B}_{1m} = \mathfrak{B}_{2m} = 1$. Including such configurations of which there are d_{m-1}, the total number of different sectors D_m is then found to be

$$D_m = d_m + d_{m-1}\,. \tag{13.16}$$

To find a closed form for D_m, we can sum over m, r, p to write down a rational generating function whose Taylor coefficients will yield D_m. It is convenient to consider first

$$\begin{aligned} G(t) &= \sum_{m=0}^{\infty} d_m t^m = \sum_{r,p,m} \frac{(r+p)!(m-p-r-1)!}{(m-p-2r)!(r-1)!r!p!} 2^p t^m \\ &= \sum_{r,p,m} \frac{(r+p)!(m-p-r-1)!}{(m-p-2r)!(r-1)!r!p!} 2^p t^{m-p-2r} t^{p+2r} = \sum_{r,p} \frac{(r+p)!}{r!p!} 2^p (1-t)^{-r} t^{p+2r} \\ &= \sum_p (2t)^p \big(1 - \frac{t^2}{1-t}\big)^{-p-1} = \frac{1-t}{1-3t+t^2}\,. \end{aligned} \tag{13.17}$$

Note that $d_0 = 1$. Now, (13.16) and (13.17) allow us to construct the full generating function for D_m, defined for $m \geq 1$. This gives us

$$\begin{aligned} F(t) &= \sum_{m=1}^{\infty} D_m t^m = \sum_{m=1}^{\infty} (d_m + d_{m-1}) t^m = \sum_{m=1}^{\infty} (d_m t^m) + t \sum_{m=0}^{\infty} d_m t^m \\ &= (1+t)G(t) - 1 = \frac{3t - 2t^2}{1-3t+t^2}\,. \end{aligned} \tag{13.18}$$

We can thus compute D_m for all $m \geq 1$ easily from (13.18), and we get:

$$D_m = 3, 7, 18, 47, 123, \ldots, \tag{13.19}$$

which is a sequence that is constructed by taking the even-numbered terms of a Fibonacci sequence L_n that starts with the first two seed values $L_0 = 2, L_1 = 1$. This sequence (Lucas numbers) is related to the standard Fibonacci sequence F_n, and we can write down an exact expression[1] for D_m:

$$D_m = L_{2m-1} + L_{2m-2} = F_{2m+1} + F_{2m-1} = \phi^{2m} + (1-\phi)^{2m} \tag{13.20}$$

where $\phi = (1+\sqrt{5})/2$ is the Golden ratio.

Equation (13.20) can also be derived more directly. For this purpose, consider the family of sectors for $(m-1)$ D2-branes. To enumerate the sectors for m D2-branes, we can add another column to the right[2] of $\mathfrak{B}$, i.e. any of $(1\ 0)^\top$, $(0\ 1)^\top$ and $(1\ 1)^\top$. This gives a new set of $\mathfrak{B}$'s which includes all the sectors for m as a subset. Thus, we can write

$$D_m = 3D_{m-1} - \mathcal{O}_m \tag{13.21}$$

where $\mathcal{O}_m$ counts the sectors that have been over-counted. It turns out that $\mathcal{O}_m$ is exactly D_{m-2}.

To prove this, we observe that the family of sectors for $m-1$ D2-branes can always be divided into two classes: (i)those which are invariant under σ and (ii) those which are not. Class (i) matrices are bounded at both ends by at least one $(1\ 1)^\top$, whereas for class (ii), no $(1\ 1)^\top$ appears at either the left or right end. Now, for class (ii), when we add $(1\ 1)^\top$ to the right end, the resulting $\mathfrak{B}$ is now invariant under σ, and thus $3D_{m-1}$ over-counts by one for each distinct case. By removing the 1^{st} column, each such over-counted matrix can be mapped to a matrix of class (i) with $(m-2)$ D2-branes which end with either $(1\ 0)^\top$ or $(0\ 1)^\top$. Similarly, for class (i), consider each pair of terms generated by adding $(1\ 0)^\top$ or $(0\ 1)^\top$ at the right end. They can be easily seen to be equivalent, and thus each such matrix can be mapped to a matrix of class (i) with $(m-2)$ D2-branes and which end with $(1\ 1)^\top$. Taking into account the over-counting for both classes, we see that the total number of over-counts can be mapped to precisely D_{m-2}.

We conclude therefore that $\mathcal{O}_m = D_{m-2}$. This means that we have, from (13.21),

$$D_m = 3D_{m-1} - D_{m-2}. \tag{13.22}$$

This is precisely the recurrence relation for the generalized Fibonacci sequence we have found in (13.20). Given D_1 and D_2, we can generate the rest of the sequence.

Finally, we note that in the case of $k = 2$, the number of states in each sector is always 4. (This is not the case for $k = 1, 3$ as can be seen from our earlier computations.) For any m, we can verify this straightforwardly using the methods discussed in this section. Below, we present a short inductive derivation for $\sigma = 1$ sectors.

---

[1] One can also use a binomial-Fibonacci identity to write $D_m = \sum_{k=0}^{m} {}^{2m-k}C_k + \sum_{k=0}^{m-1} {}^{2(m-1)-k}C_k$. Invoking (13.16), this gives us a simple closed form for (13.15).

[2] We can also add another column to the left, but to avoid over-counting, one can choose to add in only one direction.

To be definite, let us consider an arbitrary $k = 2$ sector in $m + 1$ which begins with $\mathfrak{B}_{i1} = (1\,1)^\top$. Such a sector can be thought of as an $(m - 1)$ sector augmented by two more columns of $\mathfrak{B}$ as represented in Figure 13.1.

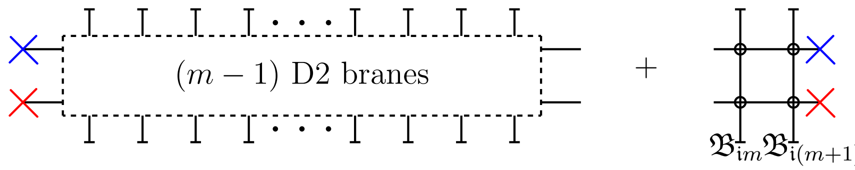


Figure 13.1: The above decomposes the $\sigma = 1$ class of sectors (with $(m + 1)$ D2-branes) in a particular way. The circles on the last two D2-branes indicate various possibilities for the last two columns of $\mathfrak{B}$, giving possibly different sectors.

There are five different possibilities for the last two columns of $\mathfrak{B}$ that we need to consider, the rest being related by symmetries. Below, we assume that $\mathfrak{B}_{\mathfrak{i}1} = 1$ for $\mathfrak{i} = 1, 2$. Also, we denote the action that corresponds to the configuration before adding the last column by I_m, and the resulting action to be I_{m+1}. After some algebra, we simplify the various actions to be, in each case,

1. $\mathfrak{B}_{\mathfrak{i}m} = (1\,1)^\top, \mathfrak{B}_{\mathfrak{i}(m+1)} = (1\,1)^\top$:
   $I_{m+1} = I_m + \int \left(\mathfrak{p}^{m+1} - \mathfrak{p}^m\right)\left(2d\mathfrak{q}^{1m} - d\mathfrak{q}^{1(m+1)} - dq^{2m}\right)$;

2. $\mathfrak{B}_{\mathfrak{i}m} = (1\,1)^\top, \mathfrak{B}_{\mathfrak{i}(m+1)} = (1\,0)^\top$:
   $I_{m+1} = I_m + \int \left(\mathfrak{p}^{m+1} - \mathfrak{p}^m\right)\left(d\mathfrak{q}^{1m} - d\mathfrak{q}^{1(m+1)}\right)$;

3. $\mathfrak{B}_{\mathfrak{i}m} = (1\,0)^\top, \mathfrak{B}_{\mathfrak{i}(m+1)} = (1\,0)^\top$:
   $I_{m+1} = I_m + \int \left(\mathfrak{p}^{m+1} - \mathfrak{p}^m\right)\left(d\mathfrak{q}^{1m} - d\mathfrak{q}^{1(m+1)}\right)$;

4. $\mathfrak{B}_{\mathfrak{i}m} = (1\,0)^\top, \mathfrak{B}_{\mathfrak{i}(m+1)} = (0\,1)^\top$:
   $I_{m+1} = I_m + \int \left(\mathfrak{p}^{m+1} - \mathfrak{p}^l\right)\left(d\mathfrak{q}^{1m} - d\mathfrak{q}^{1(m+1)}\right)$, where $\mathfrak{B}_{2l}$ is the second rightmost column of $\mathfrak{B}_{2\mathfrak{a}}$ that is unity;

5. $\mathfrak{B}_{\mathfrak{i}m} = (1\,0)^\top, \mathfrak{B}_{\mathfrak{i}(m+1)} = (1\,1)^\top$:
   $I_{m+1} = I_m + \int \left(\mathfrak{p}^l - \mathfrak{p}^{m+1}\right)\left(d\mathfrak{q}^{2m} - d\mathfrak{q}^{1m}\right) + \left(\mathfrak{p}^m - \mathfrak{p}^{m+1}\right)\left(d\mathfrak{q}^{1(m+1)} - d\mathfrak{q}^{1m}\right)$.

Thus, we see that in all of these cases, the new action (for $m + 1$ branes) differs from the previous one (for m branes) by a term dependent on new conjugate pairs of $(\mathfrak{p}, \mathfrak{q})$. This implies that the determinant, and hence the number of states remains the same. Assuming other choices of $\mathfrak{B}_{\mathfrak{i}1}$ and $\sigma = -1$ leads to a similar conclusion too, but we will leave details to the interested reader. Essentially, only in the $k = 2$ case, the number of states $= 4$ for all sectors with $m = 2$ branes. The calculation above, together with similar ones for other choices of $\mathfrak{B}_{\mathfrak{i}1}$ and $\sigma = -1$, then implies that the number of states $= 4 \forall\, m$ by induction.

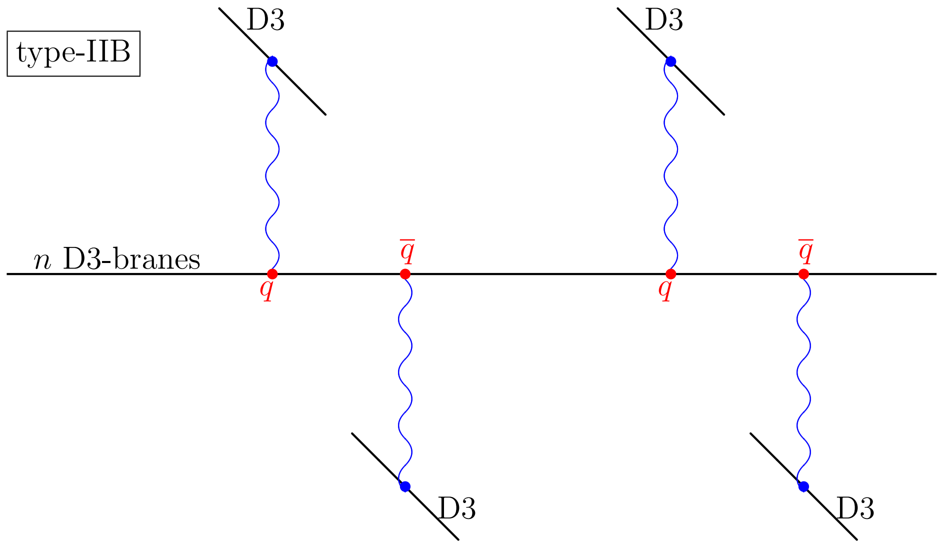


Figure 13.2: External quark and anti-quark sources are realized as endpoints of fundamental strings. $2m$ D3-branes ($m = 2$ in the picture) control the (x_1, x_2) coordinates of the sources. (x_1, x_2) are along the direction of the n D3-branes.

13.3 An alternative set of boundary conditions using tilted D3-branes

At the end of §9.9 we mentioned an alternative possibly useful set of boundary conditions for the fields $X^\mu_{\mathfrak{c}}, \psi_{\mathfrak{c}}$ ($\mathfrak{c} = 1, \dots, 2m$) at the $x_9 = \Delta$ or $x_9 = -\Delta$ end of the $2m$ open strings, which we will now describe. The boundary conditions that we used in the main text are formally realized by D5-branes. Here we will instead realize the boundary conditions by D3-branes. By tilting the D3-branes, these boundary conditions can be made to preserve one real supercharge. They also have the advantage that they can be realized more comfortably in string theory, avoiding the complications mentioned below (9.33). However, they suffer from additional fermionic zero modes which render the Witten Index identically zero. We discuss this construction below.

In this alternative set-up, we realize the $2m$ sources as endpoints of $2m$ strings that end on the n D3-branes. The coordinates of one end of the j^{th} string are thus given by

$$(x_0 = t, x_1 = a_1^{(j)}, x_2 = a_2^{(j)}, x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = 0).$$

We control the coordinates (x_1, x_2) of the endpoint of the j^{th} string by letting its other endpoint lie on another D3-brane whose (x_1, x_2) position is fixed. Thus, for each $j =$

Brane	type	(number)	$1,4$	$2,7$	3	$5,6$	8	9
Original	D3	(n)	$\longrightarrow$	$\longrightarrow$	$\div$			
String	F1	$(2m)$						$\vdash\!\dashv$
additional	D3	$(2m)$	$\nearrow$	$\nearrow$			$-$	

Table 13.1: Open strings end on the original n D3 branes and additional $2m$ D3-branes. $\nearrow$ denotes a brane that extends along the diagonal of the corresponding plane (such as $x_1 - x_4 =$ const.) and $\longrightarrow$ denotes a brane that extends along the first direction ($x_4 =$ const.).

$1, \dots, 2m$ we introduce a D3-brane which controls the j^{th} source (see Figure 13.2).[3]. We take the $2m$ D3-branes to be parallel to each other, and let the $\mathfrak{c}^{th}$ one occupy the locus

$$x_1 - x_4 = a_1^{(j)}\,, \quad x_2 - x_7 = a_2^{(j)}\,, \quad x_3 = 0\,, \quad x_9 = \Delta_j\,, \quad x_5 = x_6 = 0\,. \qquad (13.23)$$

We will assume that

$$\Delta_{j+m} = -\Delta_j < 0\,, \qquad j = 1, \dots, m,$$

so that a D3-brane that controls a quark ($j \leq m$) is at a positive x_9 and a D3-brane that controls an anti-quark ($j > m$) is at a negative x_9, and all strings have a nonzero mass. We will also set $\Delta_1 = \cdots = \Delta_m = \Delta$ for simplicity. As we argued below (9.10), Δ will not affect the low-energy description, and in fact the mass of the string is an irrelevant operator in the IR.

Equation (13.23) describes a D3-brane that extends along the x_8 direction and along the diagonals of the $x_1 - x_4$ and $x_2 - x_7$ planes. The directions of the D-branes are summarized in Table 13.1. They are designed so that the combined system of original n D3-branes and the additional $2m$ D3-branes preserves some amount of supersymmetry. More precisely, we find 4 unbroken supersymmetries that are preserved by this combined system [68, 69, 70, 71]. Including the fundamental string and the S-R-twist we find that there is only one unbroken real supercharge left. Another way of saying this is that out of the original 12 supercharges that are preserved by the n D3-branes and the twist, 11 are broken by the $2m$ branes and the fundamental strings.

Recall that the original n D3-branes extend in directions x_1, x_2, x_3 and occupy the locus

$$x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = 0\,. \qquad (13.24)$$

[3]This is reminiscent of the way Wilson loops were calculated in the topological string realization of Chern-Simons theory by Ooguri and Vafa [67]. We are grateful to Kevin Schaeffer for pointing out to us the connection with that work.

Brane	1	3	4	5	6	7	8	9	10
F1		$\div$							
D2								$\vdash\!\dashv$	$-$
NS5	$-$		$-$			$-$	$-$		$-$

Table 13.2: In the type IIA dual of Table 13.1, the D2-branes end on NS5-branes. Appropriate low-energy background fields on the NS5-brane worldvolumes control the position of the Wilson loop in the type-IIB picture.

Therefore, an open string with one endpoint on the original D3-branes and the other endpoint on one of the $2m$ D3-branes will have minimal length (of Δ) if and only if all its coordinates except x_9 are constant:

$$x_1 = a_1^{(\mathfrak{c})}\,, \quad x_2 = a_2^{(\mathfrak{c})}\,, \quad x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\,.$$

The positions of the $2m$ D3-branes therefore control the positions of the $2m$ quarks and anti-quarks.

Now we transform the system to type-IIA by applying the U-duality transformation described in Table 8.1. After the series of dualities of Table 8.1 the $2m$ D3-branes turn into type-IIA NS5-branes that wrap directions $x_1, x_4, x_7, x_8, x_{10}$. The parameters $(a_1^{(\mathfrak{c})}, a_2^{(\mathfrak{c})})$ that enter into the conditions (13.23) are encoded in the compact scalar Φ and 2-form B that are part of the low-energy tensor multiplet of the NS5-brane. We have

$$B = (x_4 + a_1^{(j)})dx_1 \wedge dx_{10} - x_7 dx_0 \wedge dx_8\,, \qquad \Phi = x_7 + a_2^{(j)}. \tag{13.25}$$

The type-IIA system is described in Table 13.2.

Similarly to the set-up in the main text, we have to connect each D2-brane that corresponds to a quark with a D2-brane that corresponds to an anti-quark and glue them into a smooth D2-brane that ends on one NS5-brane at $x_9 = \Delta$ ($j = 1, \ldots, m$) and another NS5-brane at $x_9 = -\Delta$.

For the specific purpose of computing the Witten Index however, this configuration is not so useful because, in an analogous computation as was done in §10.3, we found that there are fermionic zero modes that will make the contribution to the Witten Index vanish. Nonetheless, we also found that this configuration preserves one real supercharge, and thus it may turn out to be useful in understanding other aspects of the problem.

Part IV

Abelian Self-duality and Gravitational Chern-Simons Theory

Chapter 14

Motivation

There exists an intriguing connection between three-dimensional Chern-Simons theory and S-duality of four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) gauge theory. For abelian gauge groups the connection is simple and can be stated as follows [1]. Compactification of SYM on a circle with boundary conditions that are twisted by an S-duality accompanied with an R-symmetry twist in order to eliminate zero modes and preserve supersymmetry, leads to a $2+1D$ low-energy theory that is pure Chern-Simons theory at a level k determined by the complex coupling of SYM given by

$$\tau \equiv \frac{4\pi i}{g_{\rm YM}^2} + \frac{\theta}{2\pi}\,,$$

with θ and $g_{\rm YM}$ being a theta angle and gauge coupling constant.

In particular, the self-dual coupling constant $\tau = i$ is invariant under $S : \tau \to -1/\tau$ and leads to Chern-Simons level $k = 2$. In [1] the compactification of SYM was on a flat three-dimensional manifold ($T^2 \times \mathbb{R}$). Since the resulting low-energy theory is topological, it is independent of the metric. But this assertion is not completely obvious from the outset. In particular, the symmetries of the problem appear to allow a gravitational Chern-Simons term, which is defined by analogy with the Yang-Mills Chern-Simons term in the familiar form

$$S_{GCS} = \frac{1}{4\pi}\int_{M_3} (\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega)$$

where ω is the Levi-Civita connection one-form on the spin bundle of M_3. This term can be rewritten on a four-manifold M_4 with boundary $\partial M_4 = M_3$ like

$$S_{GCS} = \frac{1}{2\pi}\int_{M_4} \mathrm{tr}(R \wedge R)$$

as in §15.1 and must be studied on M_4 with non-vanishing curvature $R = d\omega + \omega \wedge \omega$.

In fact, we can expect that adding the gravitational Chern-Simons term is required for regularizing this theory [30]. Chern-Simons partition function, at least for large

k, can be defined as a topological invariant of the oriented, framed three manifold M_3 (a framed three manifold being one that is endowed with a homotopy class of trivializations of the tangent bundle). This can be done only after adding the gravitational Chern-Simons term which is necessary for anomaly cancellation. However, the Lorentz Chern-Simons term is of a higher-order in derivatives, so only the Yang-Mills Chern-Simons term is a part of the low-energy effective supergravity theory, as was found in the study of the ground states [1].

The $U(1)$ gauge theory on a four-manifold M_4 has S-duality symmetry which is an electric-magnetic duality that inverts the gauge coupling constant and extends to an action of full $\mathrm{SL}(2,\mathbb{Z})$ on the complex coupling. However, the partition function of abelian SYM theory is not a modular-invariant function under this transformation. It instead transforms as a modular form, with weights that depend on the Euler characteristic, χ, and the signature, σ, of the four-manifold which can be written as

$$\chi = \frac{1}{2\pi}\int_{M_4} \mathrm{tr}(R\wedge *R)\,, \qquad \sigma = \frac{1}{2\pi}\int_{M_4} \mathrm{tr}(R\wedge R)$$

in terms of the curvature two-form [31]. Thus the partition function includes the gravitational Chern-Simons term contribution.

Indeed, we do find such a contribution to be, at least to the lowest order in the deviation from flat four-dimensional space, in agreement with the S-duality of the abelian gauge theory at the self-dual point. We realize the S-duality geometrically starting with the six-dimensional description of the worldvolume of an M5-brane in M-theory, called (2, 0) theory, which gives $\mathcal{N}=4$ SYM when compactified on a two-torus [38], and introduce a twist on this T^2. The one-loop perturbative contribution in this theory is in accord with low-energy approximation of the gravitational Chern-Simons theory.

14.1 A Review of (2, 0) Theory

The wonderfully thing about supersymmetry is that it makes supersymmetric theories amenable to exact treatment allowing holomorphic quantities to be exactly computed. Like with any other symmetry, the more supersymmetry a theory has, the more constrained the field content and interactions are. The largest number of supercharges possible in free field theory is sixteen. With a greater number of supercharges than that, the free multiplet would include fields with spin larger than one which result in inconsistent theories without gravity.

The most symmetric classical field theory with sixteen supercharges is supersymmetric Yang-Mills (SYM) theory in ten dimensions. And the simplest such theory is an Abelian gauge theory. Its supermultiplet includes only a massless photon and a massless fermion. The nonabelian extension of this theory exists as a classical field theory, but its quantum version is anomalous and therefore inconsistent.

We can construct theories with sixteen supercharges in less than ten dimensions by taking a toroidal compactification and retaining only the states of zero compact momentum. The theory in d dimensions, which is obtained by dimensional reduction of the abelian classical theory in ten dimensions, is anomaly free. Its Lorentz symmetry is $\mathrm{Spin}(d-1,1)$ while its R-symmetry group, $\mathrm{Spin}(10-d)$, originates from the rest of ten dimensional Lorentz group.

The most widely known theory with sixteen supercharges is the $\mathcal{N}=4$ SYM theory in four dimensions. But in six dimensions, there is also a supersymmetry algebra with sixteen supercharges [72]. It includes four spinors of the same chirality and it is usually called the (2, 0) algebra and thus the corresponding class of theories are (2, 0) theories. Upon compactification on a two-torus down to four-dimensions these theories become $\mathcal{N}=4$ SYM [38]. The complex coupling parameter τ of the four-dimensional $U(1)$ gauge theory is simply the complex structure τ of the T^2.

These theories appeared first in the study of K3 compactications of the Type IIB theory at an ADE singularity [38] and later in the context of coincidental 5-branes in M-theory [73, 74]. They are expected to be non-trivial fixed points of the renormalization group in six dimensions and thus not to have a dimensionful parameter. Furthermore, since these fixed points are isolated, they have no dimensionless parameter.

In six dimensions, (2, 0) supersymmetry is maximal for a non-gravitational theory, and there is a unique matter multiplet with this supersymmetry. The (2, 0) supersymmetry constrains the metric on the moduli space to be locally flat. Along this moduli space there is a tensor multiplets of (2, 0) supersymmetry which includes five scalars and a two-form B, whose field strength three form is selfdual

$$H \equiv dB = *dB. \tag{14.1}$$

Even without supersymmetry or fermions the theory needs a spin structure for its definition. With supersymmetry, the mater multiplet also contains four chiral fermions. This tensor multiplet reduces to a vector multiplet of four-dimensional $\mathcal{N}=4$ supersymmetry when compactified on a two-torus.

The theory with a single matter multiplet can be realized as the low energy limit of the worldvolume theory of a 5-brane in M-theory. Since the only parameter of the 5-brane theory is the eleven-dimensional Planck scale, we again see that, in this limit, there are no adjustable parameters, dimensionful or dimensionless. The presence of a self-dual field makes a covariant Lagrangian description tricky [75], but a noncovariant Lagrangian can be written [76] and it is unique, with no free parameters. However, known results about the low-energy description of the theory will be sufficient for understanding our result.

This six-dimensional theory has string-like excitations of an unusual kind. From the point of view of the construction via type IIB theory at an ADE singularity the lightest scale in the theory is set by the tension of the strings. However, (2, 0) theories are scale-invariant just as are the four-dimensional $\mathcal{N}=4$ gauge theories.

And thus the strings in a (2, 0) theories are entirely different from other strings we have studied: they live in six dimensions, they are not associated with gravity, and they have no adjustable coupling constant — their interactions in fact are of order one. This is why (2, 0) theories are also called tensionless string theories. Of all the new phases of gauge and string theories that have been discovered this is perhaps the most mysterious, and may be a key to understanding many other things.

14.2 (2, 0) Theory on a Twisted Geometry

We are interested in studying S-duality of the abelian $\mathcal{N} = 4$ SYM theory in four dimensions, which can be realized as a geometrical twist of the six-dimensional (2, 0) theory [38]. This section describes the construction of this theory by twisting of the worldvolume of the coincident M5-branes while applying appropriate boundary conditions to the associated fields. This may well be the proper setting for understanding the S-duality of $\mathcal{N} = 4$ gauge theory.

Let the coordinates on the six-dimensional Minkowski manifold of the (2, 0) theory be x_μ with $\mu = 0, \ldots, 5$ while we will use x_a with $a = 0, \ldots, 3$ for coordinates on M_4. Let us compactify this six-dimensional manifold to $M_4 \times T^2$ with the coordinates on the two-torus, x_4 and x_5, given in terms of complex $z = x_4 + ix_5 \in \mathbb{C}$. We regard the T^2 with structure constant τ as this complex plane $\mathbb{C}$ modded out by the lattice, $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, so that $z \sim z + 1 \sim z + \tau$. After shrinking this torus to zero size the leftover gauge theory on M_4 has manifest S-duality coming from the diffeomorphisms of the T^2. This is similar to type IIB $SL(2, \mathbb{Z})$ strong-weak coupling duality viewed as a lower dimensional manifestation of M-theory diffeomorphism invariance [77].

Indeed, $\mathcal{N} = 4$ $U(1)$ super Yang-Mills theory with coupling constant τ is the low-energy limit of six-dimensional (2, 0) theory compactified on a T^2, with τ being the complex structure parameter of the torus, where S-duality can be realized as the $SL(2, \mathbb{Z})$ transformation. If we denote a general $\mathrm{SL}(2, \mathbb{Z})$-element by

$$\mathbf{g} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \qquad \mathbf{g} : \tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}} \tag{14.2}$$

is the action of the S-duality transformation on the coupling τ where $\mathbf{ad} - \mathbf{bc} = 1$.

We set this complex coupling to the self-dual value $\tau = i$ and introduce S-duality twist into the compactification of the $\mathcal{N} = 4$ SYM on a circle by introducing an unusual boundary conditions on the fields along this S^1, which we choose to be in the direction x_3. This is achieved by inserting a strong-weak coupling transformation that realizes $\tau \to -1/\tau$ and corresponds to

$$\mathbf{g} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{14.3}$$

This choice of $\mathbf{g}$ is permissible as it leaves the coupling $\tau = i$ invariant. Thinking of the circle as the segment $x_3 \in [0, 2\pi R]$ with endpoints 0 and $2\pi R$ identified, we require the field configuration at $2\pi R$ to be an S-dual of that at 0. This kind of boundary condition is what I have been referring to as an S-twist.

In particular, on the gauge field two-form B of the (2, 0) theory, the boundary conditions that results from this S-twist are

$$B_{az}(x_3 + 2\pi R) = iB_{az}(x_3), \qquad B_{a\bar{z}}(x_3 + 2\pi R) = -iB_{a\bar{z}}(x_3)\,, \tag{14.4}$$

while other components are not effected. This is exactly the action of the S-duality on the electric and magnetic fields $[\vec{E} \pm i\vec{B}](x_3 + 2\pi R) = \pm i[\vec{E} \pm i\vec{B}](x_3) = [-\vec{B} \pm i\vec{E}](x_3)$.

We recall [40] that $\mathbf{g} \in \mathrm{SL}(2, \mathbb{Z})$ acts nontrivially on the supercharges

$$\mathbf{g} : Q_{a\alpha} \to \left(\frac{c\tau + d}{|c\tau + d|}\right)^{-\frac{1}{2}} Q_{a\alpha} = e^{-\frac{i\pi}{4}} Q_{a\alpha}\,. \tag{14.5}$$

In order to get a supersymmetric theory, we therefore need to supplement the S-twist with an R-symmetry twist so that the phase in (14.5) is canceled and supersymmetry restored. However, only the S-twist affects the gauge field, and only the R-twist affects the scalars and fermions. So we will not need to consider details of the R-twist for our calculation.

This setup has been studied in detail on a flat space (with $M_3 = T^2 \times \mathbb{R}$) in [1] as well as Part III of this dissertation and it was found to correspond at low-energy to the pure Chern-Simons theory at level $k = 2$. Here, I extend that study of the ground states to include the first order quantum effect arising as a consequence of curvature on M_4. This effect was calculated to one-loop and presented in §15.2. It turns out that this contribution agrees with the lowest order effect of the gravitational Chern-Simons term as in §15.1.

Chapter 15

Calculation

In this Chapter I present details of the calculations that compare the low-energy effective gravitational Chern-Simons theory with the one-loop correlation function in (2, 0) theory twisted as described above. A reader that does not wish to get bogged down in these can find a summary of the results in Chapter 16.

15.1 Low-Energy Expansion of Gravitational Chern-Simons Theory

Although the Chern-Simons term was first introduced in three-dimensional gauge field and gravitational models [78], it can also deform physical theories in four-dimensional space-time, where it modifies conventional kinematics and dynamics in a Lorentz violating fashion. This possibility has been investigated for Maxwell electrodynamics [79] and in the present calculation we study a similar deformation of Einstein's general relativistic gravity theory.

On a four-manifold M_4 gravitational Chern-Simons Lagrangian density is given by the Pontryagin density which is the divergence of topological Chern-Simons current in analogy with the Yang-Mills case

$$S_{GSC} = \frac{1}{2\pi} \int_{M_4} \mathrm{tr}(R \wedge R) \tag{15.1}$$

where R is the curvature two-form. We want to expand this Lagrangian in terms of a small deviation h from the flat metric on a Minkowski manifold M_4

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,. \tag{15.2}$$

For this low-energy approximation we will be keeping just the lowest order in h so that, for example, Christoffel symbol to the first order in h looks like

$$\Gamma^{\nu}{}_{\mu\lambda} = \frac{1}{2}\delta^{\nu\sigma}(\partial_\mu h_{\lambda\sigma} + \partial_\lambda h_{\sigma\mu} - \partial_\sigma h_{\lambda\mu}) \,. \tag{15.3}$$

To complete this calculation we are switching to the non-coordinate basis

$$e_a = e_a{}^{\mu}\partial_{\mu} \tag{15.4}$$

where $e_a{}^{\mu}$ are valued in $GL(4,\mathbb{R})$ and ∂_{μ} are the coordinate basis. We start by writing the matrix valued connection one-form as

$$\omega^a{}_b = e^a{}_{\nu}(\partial_{\mu}e_b{}^{\nu} + e_b{}^{\lambda}\Gamma^{\nu}{}_{\mu\lambda})dx^{\mu}\,. \tag{15.5}$$

When expanded to the first order in h the connection looks like

$$\omega^a{}_b = \frac{1}{2}\delta^{a\sigma}\delta_b{}^{\lambda}\partial_{[\lambda}h_{\sigma]\mu}dx^{\mu}\,. \tag{15.6}$$

And the curvature two-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{15.7}$$

to the first order in h is given by

$$R^a{}_b = \frac{1}{2}\delta^{a\sigma}\delta_b{}^{\lambda}\partial_{\nu}\partial_{[\lambda}h_{\sigma]\mu}dx^{\nu} \wedge dx^{\mu}. \tag{15.8}$$

The gravitational Chern-Simons Lagrangian density to the lowest order in the deviation from flat metric h is

$$\mathcal{L}_{GSC} = \frac{1}{2\pi}R^a{}_b \wedge R^b{}_a = \frac{1}{4\pi}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}\partial_{[\sigma}h_{\tau]\beta}\partial_{\nu}\partial^{\sigma}h^{\tau}_{\mu}dV \tag{15.9}$$

which is in agreement with [80].

For a plane monochromatic gravitational waves with wave-vector p_{α} satisfying $p_{\alpha}p^{\alpha} = 0$ we can rewrite $\partial_{\alpha}\partial_{\sigma}h_{\tau\beta} = -p_{\alpha}p_{\sigma}h_{\tau\beta}$ and the above approximation becomes

$$\mathcal{L}_{GSC} = \frac{1}{2\pi}R^a{}_b \wedge R^b{}_a = \frac{1}{4\pi}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}p_{[\sigma}h_{\tau]\beta}p_{\nu}p^{\sigma}h^{\tau}_{\mu}. \tag{15.10}$$

This is the result we are comparing with the one-loop amplitude calculation of the twisted abelian gauge theory at the self-dual coupling constant $\tau = i$, presented in the next section.

15.2 Field theory loop calculation

As described in Chapter 14, we start with the six-dimensional (2, 0) theory compactified first on a two-torus down to four dimensional $\mathcal{N} = 4$ SYM and then on a circle with an S-duality twist at self dual SYM coupling $\tau = i$ so that the twist is a symmetry of the theory. We are shrinking the T^2 in directions x_4 and x_5 to zero size and retaining only the states of zero compact momentum by setting $p_4 = p_5 = 0$.

However, momentum along the S^1 direction is not vanishing, but quantized in an unusual way. As a result of the geometrical twist introduced along the circle the Kaluza-Klein modes along S^1 are shifted by $\pm 1/4$.

This effect of the S-twist can be seen as follows. Recall the action of the twist on the B-field two-form (14.4) and let us first focus on B_{az} (with $a = 0, \ldots, 3$, while $z = x_4 + ix_5$ and $x_3 \in [0, 2\pi R]$) which S-twist multiplies by i. Starting at $x_3 = 0$ and making a full circle in the x_3-direction shifts momentum eigenstates $e^{ip_3 x_3/R}$ of B_{az} by $e^{2\pi i p_3}$. But they must be the same at $x_3 = 0$ after the S-twist by $i = e^{i\pi/2}$. It follows that $p_3 \in \mathbb{Z} - 1/4$ for the modes of B_{az}. Equivalently, the Kaluza-Klein modes of $B_{a\bar{z}}$ are $p_3 \in \mathbb{Z} + 1/4$ instead as S-twist multiplies $B_{a\bar{z}}$ by $-i$.

The bosonic field content of the $(2,\ 0)$ consists of five scalars, corresponding to five broken translations, and a self dual two-form potential, B, with self-dual field strength $H = dB$. There are also the Fermionic superpartners, symplectic Majorana Spinors, all together making up the tensor multiplet. Due to the self-duality constraint on H, it is not possible to write a simple action for the tensor multiplet even at the linearized level [77]. This may be achieved, however, by introducing an auxiliary scalar that effectively allows one to gauge away the anti-self-dual degrees of freedom. This is often referred to as the PST approach [75].

While it is difficult to write the full action for $(2,\ 0)$ theory [76], as we are looking for the highest order effect of spacetime curvature on this theory, using an incomplete low-energy approximation of the action will suffice

$$S_0 = \int \mathcal{L}_0 = -\frac{1}{12g^2} \int d^6x H_{\mu\nu\lambda} H^{\mu\nu\lambda} \tag{15.11}$$

where the three-form field strength $H_{\mu\nu\lambda}$, in terms of two-form gauge field $B_{\nu\lambda}$, is given by

$$H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]}.$$

Only the self-dual part $H^{(+)}_{\mu\nu\lambda}$ contributes to the $(2,\ 0)$ theory action, where

$$H^{(\pm)}_{\mu\nu\lambda} = \frac{1}{2}\left(H_{\mu\nu\lambda} \pm \frac{1}{6}\epsilon_{\mu\nu\lambda\alpha\beta\gamma} H^{\alpha\beta\gamma} \right). \tag{15.12}$$

After gauge-fixing, the correlation function of the three-form can be written in momentum space as

$$\left\langle H_{\mu\nu\sigma}(\mathbf{p}) H^{\alpha\beta\gamma}(\mathbf{q}) \right\rangle = \frac{(2\pi)^6 (2g^2)\delta(\mathbf{p}+\mathbf{q})}{p^2} p_{[\mu} p^{[\alpha} \delta^{\beta}_{\nu} \delta^{\gamma]}_{\sigma]}. \tag{15.13}$$

Separating the T^2 coordinates and using that $g_{\sigma z} = 0$ except for $g_{z\bar{z}} = 1$ as well as that $p_z = p_{\bar{z}} = 0$ in low-energy limit gives correlation

$$\left\langle H_{mnz}(\mathbf{p}) H^{abz}(\mathbf{q}) \right\rangle = \left(\frac{2}{9}\right) \frac{(2\pi)^6 (2g^2)\delta(\mathbf{p}+\mathbf{q})}{p^2}\ p_{[m} p^{[a} \delta^{b]}_{n]} \tag{15.14}$$

and a similar result for $\left\langle H_{mn\bar{z}}(\mathbf{p})H^{ab\bar{z}}(\mathbf{q})\right\rangle$ where Greek indices run over six-dimensions ($\alpha,\ \beta = 0,\ldots,5$) while Roman indices are reserved for M_4 ($a,\ b = 0,\ldots,3$) and z is the complex coordinate on the T^2.

To calculate first order correction to this theory coming from curvature, let the metric be given by its deviation from flat $g_{\phi\chi} = \eta_{\phi\chi} + h_{\phi\chi}$, as before, and expand the Lagrangian in powers of $h_{\phi\chi}$ as

$$\mathcal{L} = \mathcal{L}_0 + h_{\phi\chi}T^{\phi\chi} + h_{\phi\chi}h_{\dot{\phi}\dot{\chi}}T^{\phi\chi}T^{\dot{\phi}\dot{\chi}} + \mathcal{O}(h^3)$$

in order to get a low-energy effective description. We will calculate one-loop amplitude

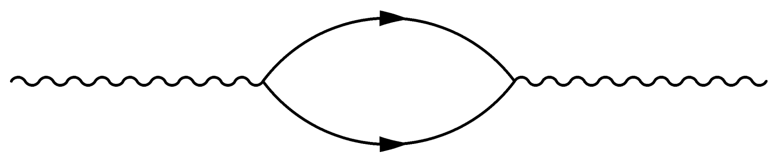


Figure 15.1: One-loop diagram computed in twisted (2, 0) theory with vertices $h_{\phi\chi}T^{\phi\chi}$ and $h_{\dot{\phi}\dot{\chi}}T^{\dot{\phi}\dot{\chi}}$ where $h_{\phi\chi}$ are deviations from flat metric and $T^{\phi\chi}$ is the low-energy-momentum tensor.

contribution with interactions $h_{\phi\chi}T^{\phi\chi}$ where $T^{\phi\chi}$ is the energy-momentum tensor of the incomplete Lagrangian $\mathcal{L}_0$

$$T^{\phi\chi} = -\frac{1}{2g^2}H^{\chi}{}_{\nu\sigma}H^{\phi\nu\sigma} + \frac{1}{12g^2}\eta^{\phi\chi}H_{\mu\nu\sigma}H^{\mu\nu\sigma}. \tag{15.15}$$

We also must separate self-dual and anti-self-dual contributions to energy-momentum tensor $T^{\phi\chi} = T^{(+)\phi\chi} + T^{(-)\phi\chi}$. Here $T^{(+)\phi\chi}$ comes from the self-dual part of the field strength, $H^{(+)}_{\mu\nu\lambda}$, only while $T^{(-)\phi\chi}$ has only $H^{(-)}_{\mu\nu\lambda}$ components. Rewritten in terms of full $H_{\mu\nu\lambda} = H^{(+)}_{\mu\nu\lambda} + H^{(-)}_{\mu\nu\lambda}$ this gives

$$\begin{aligned} T^{(\pm)\phi\chi} = &\mp\frac{1}{48g^2}\epsilon^{\phi\nu\sigma\alpha\beta\gamma}H^{\chi}{}_{\nu\sigma}H_{\alpha\beta\gamma} \mp \frac{1}{48g^2}\epsilon^{\chi\nu\sigma\alpha\beta\gamma}H^{\phi}{}_{\nu\sigma}H_{\alpha\beta\gamma} \\ &- \frac{1}{4g^2}H^{\chi}{}_{\beta\gamma}H^{\phi\beta\gamma} + \frac{1}{24g^2}\eta^{\phi\chi}H^{\alpha\beta\gamma}H_{\alpha\beta\gamma} \end{aligned} \tag{15.16}$$

where just $T^{(+)\phi\chi}$ shows up in our calculation as (2, 0) theory has only a self-dual three-form field strength.

The correlation function of the energy-momentum tensor vanishes in the flat space-time. But we expect even the one-loop amplitude contribution

$$\int \frac{d^4\mathbf{p}}{(2\pi)^4}\frac{d^4\mathbf{q}}{(2\pi)^4}h_{\phi\chi}h_{\dot{\phi}\dot{\chi}}\left\langle T^{(+)\phi\chi}(\mathbf{p})T^{(+)\dot{\phi}\dot{\chi}}(\mathbf{q})\right\rangle \tag{15.17}$$

to be non-zero in our setup. We find that, after the S-twist, a non-vanishing effect of $h_{\phi\chi}$ is a consequence of the modification in the treatment of loop momentum. In particular, the momentum component in the S^1 direction x_3 only takes discrete values which, instead of being simply integers, are shifted by the S-twist as $p_3 \in \mathbb{Z} \pm 1/4$.

The non vanishing term comes from replacing

$$\int \frac{d^4\mathbf{p}}{(2\pi)^4} = \frac{1}{R} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n \pm \frac{1}{4}}$$

and after regularizing this sum. This indeed is the effect of the S-twist itself which results in the $\pm 1/4$ shifts. In fact, this contribution is in exact agreement with the gravitational Chern-Simons term (15.10).

Chapter 16

Interpretation

It was shown in [1] that the low-energy limit of $\mathcal{N} = 4$ SYM theory on a flat four-dimensional manifold compactified on a shrinking circle with an S-duality twist gives a topological field theory, whose correlation functions do not depend on the metric of the spacetime. However, we know that Chern-Simons theory on a manifold with curvature is not quite invariant but requires a gravitational Chern-Simons counter-term [30]. Also, [31] shows that the partition function of Chern-Simons theory is not a modular invariant but transforms as a modular form with weight quadratic in curvature. These results are suggesting that an S-duality twist should be accompanied with a gravitational Chern-Simons term.

To check this, we geometrically realize S-duality in SYM as a twist of $(2,\ 0)$ theory of coincident M5-branes compactified on a two-torus. We find that the effective low-energy contribution is in agreement with the $\mathcal{O}(h^2)$ contribution of the gravitational Chern-Simons term

$$\mathcal{L}_{GSC} = \frac{1}{2\pi} R^a{}_b \wedge R^b{}_a = \frac{1}{4\pi} \epsilon^{\mu\nu\alpha\beta} p_\alpha p_{[\sigma} h_{\tau]\beta} p_\nu p^\sigma h^\tau_\mu. \tag{16.1}$$

This is the first step toward proving the conjecture that adding S-duality twist to $\mathcal{N} = 4$ super Yang-Mills gauge theory gives Chern-Simons theory with gravitational Chern-Simons contributions.

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