A Unified Framework for Bounded and Unbounded Numerical Estimation

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Abstract

Representations of numerical value have been assessed using bounded (e.g., 0-1000) and unbounded (e.g., 0-?) number-line tasks, with considerable debate regarding whether one or both tasks elicit unique cognitive strategies (e.g., addition or subtraction) and require unique cognitive models. To test this, we examined 86 5- to 9-year-olds' addition, subtraction, and estimation skill (bounded and unbounded). Against the measurement-skills hypothesis, estimates were even more logarithmic on unbounded than bounded number lines and were better described by conventional log-linear models than by alternative cognitive models. Moreover, logarithmic index values reliably predicted arithmetic scores, whereas model parameters of alternative models failed to do so. Results suggest that the logarithmic-to-linear shift theory provides a unified framework for numerical estimation with high descriptive adequacy and yields uniquely accurate predictions for children's early math proficiency.

Keywords: cognitive development; numerical cognition; number-line estimation; psychophysical function

Introduction

In this paper, we sought to resolve a debate on what gives rise to developmental changes in numerical estimation and provide a unified framework for understanding seemingly irreconcilable data regarding the psychophysical functions that link numbers to their magnitude estimates (Barth & Paladino, 2011; Cohen & Sarnecka, 2014; Opfer, Thompson, & Kim, 2016; Siegler & Opfer, 2003; Slusser, Santiago, & Barth, 2013).

Conventionally, developmental changes in numerical estimation have been viewed as following a logarithmic-to-linear shift in representations of numeric magnitude (Siegler & Opfer, 2003; Siegler, Thompson, & Opfer, 2009). This shift was first observed on a number-line task, in which a target number was estimated on a line flanked by a number at each end (Fig. 1A). On this task, young children's placement of numbers typically follows an approximately logarithmic function (Siegler & Booth, 2004; Siegler & Opfer, 2003; Opfer & Siegler, 2007; Opfer, Siegler, & Young, 2011; Thompson & Opfer, 2008), but this logarithmic pattern changes to a linear one later with age and experience, with timing depending on the number ranges tested (Siegler, Thompson, & Opfer, 2009). For example, on a 0-100 number line, where estimates of kindergarteners are logarithmic, second graders produce linear estimates (Booth & Siegler, 2006), while they estimate numbers on a log scale on a 0-1000 number line (Siegler & Opfer, 2003). That logarithmic-to-linear transitions appear at different times in development suggests that logarithmic and linear representations can co-exist and compete in the same child. Thus, a simple model of numerical estimation is thought to be a mixed log-linear model (MLLM), in which estimates are predicted as a weighted sum of logarithmic and linear transforms of the number to be estimated (Anobile et al., 2012; Opfer et al., 2016)

Recently, two related challenges to the logarithmic-to-linear shift theory have been raised. The first challenge argued that children's estimates reflect one of three proportional reasoning strategies (Barth & Paladino, 2011; Cohen & Blanc-Goldhammer, 2011; Cohen & Sarnecka, 2014; Slusser, Santiago, & Barth, 2013), which had been modeled in adults using three different cyclic power models (Hollands & Dyre, 2000). By comparing multiple extensions of the cyclic power model (CPM) to a log or linear model, Slusser et al. (2013) showed that one of the power models showed better fits for a majority of 5- to 10-year-olds' estimates than did the logarithmic model.

However, a recent study by Opfer and colleagues (2016) provided evidence against this account, showing that the fit of the cyclic power models was an artifact of an unusual anchoring procedure used by Slusser et al. (2013) that did not characterize estimates using the standard "free" numerical estimation procedure. Further, Opfer et al. (2016) found that both free and anchored estimates were better fit by MLLM than a mixed cyclic power model (MCPM) that included all variants of the CPMs proposed. Within this MLLM, the effect of anchoring could be traced to decreasing the logarithmicity of estimates without any need

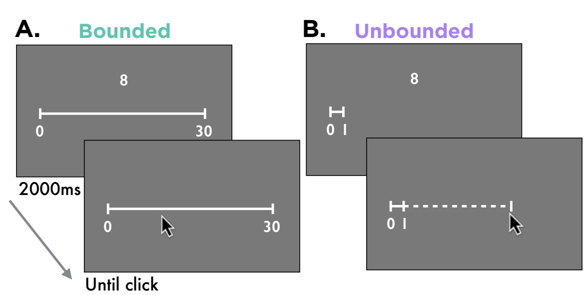

Figure 1. Illustration of the bounded and unbounded number-line task.

for positing unique estimation strategies.

A second challenge to the logarithmic-to-linear shift account was the measurement-skills account proposed by Cohen and colleagues (2011, 2014). Here, log-like patterns in estimates are proposed to arise because conventional number-line tasks are "bounded" by two numbers, thereby requiring sophisticated arithmetic skills, like subtraction, which young children may not have acquired (Fig. 1A). In this account, when a target number such as 70 is given on a 0-100 number line, a participant may need to consider both the number 70 and the difference 30 after subtracting 70 from 100. In contrast, estimation of numbers on a number line "unbounded" by the larger endpoint is thought to require only addition (Fig. 1B), is thought to more accurately reveal numeric magnitude judgments, and is thought to be best modeled using three more variants of power models called "scallop power models" (SPM) (Cohen & Blanc-Goldhammer, 2011).

To test this second challenge, Cohen and Sarnecka (2014) tested the effects of boundedness in number-line tasks by giving both bounded and unbounded tasks to 3- to 8-year-olds. Model fits of a log or linear function were compared to those of multiple variations of the CPM for the bounded estimates and to those of multiple SPMs for the unbounded estimates. In the bounded condition, the extensions of cyclic models provided better fits to estimates, and the response bias parameters (βs) changed with age. For the unbounded number-line estimates, in contrast, one of three scallop power models predicted the estimates better, but the estimation bias parameters (βs) stayed constant across age groups. These findings led the researchers to conclude that there were no such things as log-to-linear shifts in numerical representation, but these changes stemmed from the poor use of arithmetic strategies. Also, they suggested that numerical estimation be made on an unbounded number line for a better assessment of number representations.

The Current Study

In this paper, we tested three rival hypotheses regarding bounded and unbounded numerical estimation. The primary hypothesis is that the cognitive process of estimation in the two tasks is essentially similar and does not require positing six estimation strategies associated with six unique psychophysical functions. From this perspective, bounded and unbounded numerical estimates -- like free and anchored numerical estimates -- are best viewed as reflecting children's representations of numeric magnitude and best modeled using the same mixed log-linear model. The second hypothesis is that both tasks are equally well suited for characterizing numerical magnitude estimates, with *the parameters of the MLLM derived from each task providing better predictors of children's use of numbers in other contexts (such as addition and subtraction) than even the models that allegedly track addition and subtraction skill*. Finally, because number-line estimation tasks do assess children's representations of numeric magnitude, providing children with more numbers against which to anchor their estimates will result in improved performance, leading to bounded numerical estimates being more accurate and more linear than unbounded estimates.

To test these hypotheses, we nearly replicated the procedure used by Cohen and Sarnecka (2014) with the only exception being to use a fully-balanced design that could detect order effects and the administration of a battery of math tests (addition and subtraction). Then, we pit the MLLM tested by Opfer et al. (2016) against the MCPMs for bounded number-line tasks and against the mixed scallop power model (MSPM) for unbounded number-line tasks.

The MLLM consists of logarithmic and linear components and is defined as:

$$y = a\left(\lambda \frac{U}{\ln(U)} \ln(x) + (1-\lambda)x\right), \quad (1)$$

in which y indicates an estimate of number x on a 0-U number line. Also, a denotes a scaling parameter, and λ is a logarithmicity index that measures the degrees of logarithmic compression in estimates. If estimation is perfectly linear, a λ value converges to 0, whereas the value of the logarithmicity index gets close to 1 as estimation shows more logarithmic compression.

The MCPMs were formulated as proposed by Hollands and Dyre (2000) (also see Opfer, Thompson, & Kim, 2016, for details). The first MCPM (MCPM1) is formalized based on Slusser et al. (2013)'s study that hypothesized that the number of reference points used for estimation changes in development: children with poor proportion skills would only use a single reference point, i.e., the lower bound, (0 cyclic power model), and then learn to use the lower and upper bounds (1 cycle power model) and the middle point with the two endpoints (2 cycle power model) as they become more familiar with the number range. The MCPM1 is defined as:

$$y = w_1 \cdot 0\mathrm{CPM} + w_2 \cdot 1\mathrm{CPM} + w_3 \cdot 2\mathrm{CPM}, \quad (2)$$

where each of w_1, w_2, and w_3 denotes a weight for each variant of the CPM respectively. Each weight and the sum of weights are constrained to be between 0 and 1, so that contribution of three models in a response can be assessed individually. The MCPM2 is identical to the MCPM1 except that 0CPM is replaced with the subtraction bias cyclic model (SBCM) in the MCPM2 as proposed by Cohen & Sarnecka (2014). The SBCM was similar to 1CPM, but includes an additional parameter (s) that is associated with the subtraction bias.

The MSPM for unbounded number-line tasks is also formed in the same manner based on Cohen and Blanc-Goldhammer (2011)'s assumption that the unbounded tasks are solved using an addition strategy. The following is the formalization of the mixed model:

$$y = w_1 \cdot 1\mathrm{SPM} + w_2 \cdot 2\mathrm{SPM} + w_3 \cdot \mathrm{MSPM}. \quad (3)$$

In the model, 1SPM indicates the single scallop model, 2SPM the dual scallop model, and MSPM the multiple scallop model. The same constraints set on weights in the mixed CPM are set in this model.

After obtaining model fits and parameter estimates from each of the models, we next compared fits of the MLLM with the MCPMs for the bounded condition and with the MSPM for the unbounded condition to examine the best-fitting model for each number-line task. Whereas the alternative functions are task-specific, the MLLM is theoretically applicable to both bounded and unbounded conditions. Therefore, if the MLLM is an unifying, generalizable function that captures numerical representations regardless of number-line boundedness, the MLLM should not only describe the bounded estimates better (Opfer et al., 2016), but also predict data from the unbounded condition better. Also, we examined whether model parameters, such as subtraction bias (s) in the MCPM2 and logarithmicity (λ) in the MLLM, actually predict addition and subtraction accuracy. If compressive estimates appear due to a lack of subtraction skills in the bounded number-line tasks, significant correlation between subtraction bias (s) and actual subtraction performance should be observed. On the other hand, if compression in estimates reflects logarithmic representations of numbers, the logarithmicity index (λ) should predict both addition and subtraction achievement better than other arithmetic-strategy or estimation bias parameters.

Experiment

Methods

Subject Thirty 5- to 6-year-old kindergarten, 30 first grade, and 26 second grade students were recruited in Columbus, OH (kindergarteners: 17 female children, *M* = 5.90 years, *SD* = .32 years; first graders: 19 female children, *M* = 6.74 years, *SD* = .39 years; second graders: 21 female children, *M* = 7.91 years, *SD* = .46 years).

Materials and Procedure Participants completed both bounded and unbounded number-line tasks given in a counterbalanced order. In the bounded condition, a number was shown for 2,000 ms above a number line flanked by 0 and 30/100/1000 (Fig. 1A). On every trial, the mouse cursor was reset to be located at the 0 point and moved only horizontally on a number line. Participants were instructed to estimate a given number on a number line with the following instruction:

> Now we're going to play a game with numbers. This is a number line. In this game, each number line will have a 0 at one end and 30/100/1000 at the other end. There will be a number up here. Your job is to show me where that number goes on a number line like this one. When you decide where the number goes, you have to drag this little mark to where the number should go. When you're ready to go again, press the green bar (spacebar with a green sticker on) on the keyboard.

The unbounded number-line task was identical to the bounded one except that a single-unit line (0-1) was presented instead of the full range number line (Fig. 1B). An experimenter introduced the task with the following instruction that was created based on Cohen and Sarnecka (2014):

> Now we're going to play a game with numbers. This is a number line. In this game, each number line will have a 0 here and 1 over here. All the other numbers go after 1. There will be a number up here. Your job is to show me where that number goes on a number line like this one. When you decide where the number goes, you have to drag this little mark to where the number should go. When you're ready to go again, press the green bar on the keyboard.

The length between 0 and 1 was adjusted based on the ranges of number lines. In other words, a displayed length for 0-1 intervals was the shortest for a 0-1000 number line, whereas it was the longest in the 0-30 number line task.

Based on Slusser et al. (2013), different number ranges that would elicit compressive estimates were used depending on children's grades. 5- and 6-year-old kindergarteners were given 0-30 number-line tasks with to-be-estimated numbers sampled evenly from a 0-30 range: 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 27, 28, 29, 30. For first graders, a 0-100 number line was used with 25 numbers between 0 and 100: 3, 4, 6, 8, 12, 17, 21, 23, 25, 29, 33, 39, 43, 48, 52, 57, 61, 64, 72, 29, 81, 84, 90, 96, 100. Second graders were asked to estimate numbers chosen between 0 and 1000 on a 0-1000 number line: 2, 5, 18, 34, 56, 78, 100, 122, 147, 150, 163, 179, 246, 366, 486, 606, 722, 725, 738, 754, 818, 938, 1000. In both bounded and unbounded conditions, the same target numbers were randomly presented. To keep children's attention on the tasks, a neutral sound was produced once a stimulus was displayed or a response was made by mouse click. The tasks started after an instruction without any practice, and there was no feedback provided over trials.

Upon completion of the two number-line tasks, arithmetic performance was assessed with paper-and-pencil addition and subtraction tests. The addition test consisted of 50 one-digit addend problems, such as 1+1 and 5+3. The subtraction items were generated by rearranging the addition. To be specific, the first addends in the addition test were used as subtrahends, while the second addends became the answers. For the questions like 1+1 = 2 and 5+3 = 8, they were rearranged to be 2-1 and 8-5 in the subtraction. Participants were given the tests in a random order and asked to solve as many problems as they could within 1 minute for each test.

Results

Whether cognitive process of estimation is qualitatively different between bounded and unbounded tasks was examined using psychophysical functions. For the bounded condition, individual data were fit by the MLLM, MCPM1, and MCPM2, whereas estimates on unbounded number lines were fit with the MLLM and MSPM. The model fits were then assessed by comparing AICc values.

Against the measurement-skills account, none of the MCPMs was the best fitting model for bounded number-line estimation. For 0-30 and 0-100 number-line estimates, most children's performance was better explained by the MLLM (Table 1). Estimates of 50% of participants on 0-1000 number lines were also best fit by the MLLM. In the unbounded condition, across all the three number ranges, 100% of children produced estimates that were better described by the MLLM than the MSPM. The results suggest that logarithmically distorted estimates in a number-line task do not stem from a lack of proportion, subtraction, or addition skills, but come from representations of numerical magnitude that are well predicted with the mixed log-linear model.

Would variants of the power models based on arithmetic strategies predict children's actual arithmetic performance? To address this question, we next correlated each child's addition and subtraction performance with the best-fitting parameter values from the models. If response patterns in the bounded condition came from poor subtraction skills, parameters, such as βs of the MCPM1 & MCPM2 and *s* of MCPM2, would be expected to correlate with arithmetic scores. As presented in Table 2, however, none of the βs from the MCPMs was a reliable predictor for either addition or subtraction scores: only β_{1CPM} of MCPM1 correlated with subtraction scores, but such correlation was not found in the subtraction bias parameter *s* of MCPM2. This correlation also remained insignificant if adjusted parameter values (absolute values of β-1 or s-1) were used for analyses. In contrast, the logarithmicity parameter λ of the MLLM reliably predicted both addition and subtraction performance, $r(84) = -.42$, $p < .001$ for addition; $r(84) = -.36$, $p < .01$ for subtraction. The finding suggests that the more logarithmic their estimates were in bounded tasks, the worse performance in arithmetic tests was observed.

We also examined relations between the parameter βs of the MSPM and arithmetic achievement in the unbounded tasks. According to the measurement-skills account, these parameters in the scallop models should reflect representation of numeric magnitude most accurately of all because the task is supposedly pure of the sins of boundedness and the model is correct. Against this idea, however, none of the parameter βs were significantly correlated with arithmetic performance, whereas the MLLM parameter λ again showed a significant correlation with addition and subtraction, $r(84) = -.33$, $p < .01$ for addition; $r(84) = -.34$, $p < .01$ for subtraction (Table 2). These findings again support the idea that logarithmic estimates on a number line should be viewed as a reflection of logarithmic representation of numbers, which interferes with use of numbers in other contexts.

Table 1. Percent of participants best fit by each model.

	Bounded			Unbounded	
	MLLM	MCPM1	MCPM2	MLLM	MSPM
0-30	96.67	.00	3.33	100	.00
0-100	80	10	10	100	.00
0-1000	50	3.85	46.15	100	.00

The final issue we examined was to quantify the effect of boundedness on numerical estimation. If the bounded number-line tasks required more challenging measurement skill, the bounded tasks would be expected to elicit more erroneous and compressive estimates. On the other hand, if the two endpoints provide better anchors against which to judge large numbers, estimates in the bounded condition would be expected to yield less logarithmic and more accurate estimates. Consistent with the latter account, on all the 0-30/100/1000 number-line tasks, percent absolute errors (PAE) in median estimates were always greater in the unbounded than bounded conditions (bounded $M = 12\%$, $SD = 5\%$, unbounded $M = 17\%$, $SD = 7\%$ for 0-30 range; bounded $M = 12\%$, $SD = 4\%$, unbounded $M = 14\%$, $SD = 7\%$, for 0-100 range; bounded $M = 20\%$, $SD = 12\%$, unbounded $M = 24\%$, $SD = 14\%$, for 0-1000 range).

Additionally, as shown in Figure 2A, in all the three number ranges, the values of the logarithmicity parameter (λ) were greater for median estimates in the unbounded condition than those in the bounded condition (bounded $\lambda = .30$, unbounded $\lambda = .76$ for 0-30 range; bounded $\lambda = .42$, unbounded $\lambda = .55$ for 0-100 range; bounded $\lambda = .73$, unbounded $\lambda = .75$ for 0-1000 range). These results do not support the measurement-skills prediction that unbounded number-line tasks are easier and require less advanced mensuration skills than conventional bounded tasks.

Finally, the same analyses were repeated on individual children's data. For the 0-30 number-lines, greater PAEs were again observed in the unbounded than bounded condition ($M = 17\%$, $SD = 7\%$ for bounded; $M = 21\%$, $SD = 7\%$ for unbounded), $t(29) = 3.82$, $p < .001$. For the 0-100 number-lines, greater PAEs were also observed in the unbounded than bounded condition ($M = 15\%$, $SD = 7\%$ for

Table 2. Correlation between parameter values and arithmetic performance after number-line ranges were controlled for.

			Addition	Subtraction
Bounded	MLLM	λ	-.42***	-.36**
	MCPM1	β_{0CPM}	-.01	-.10
		β_{1CPM}	.17	.27*
		β_{2CPM}	-.08	.10
	MCPM2	s	.13	.06
		β_{SBCM}	.02	-.09
		β_{1CPM}	.17	.14
		β_{2CPM}	-.06	-.13
Unbounded	MLLM	λ	-.33**	-.34**
	MSPM	β_{1SPM}	-.05	.00
		β_{2SPM}	.19	.01
		β_{MSPM}	.16	.15

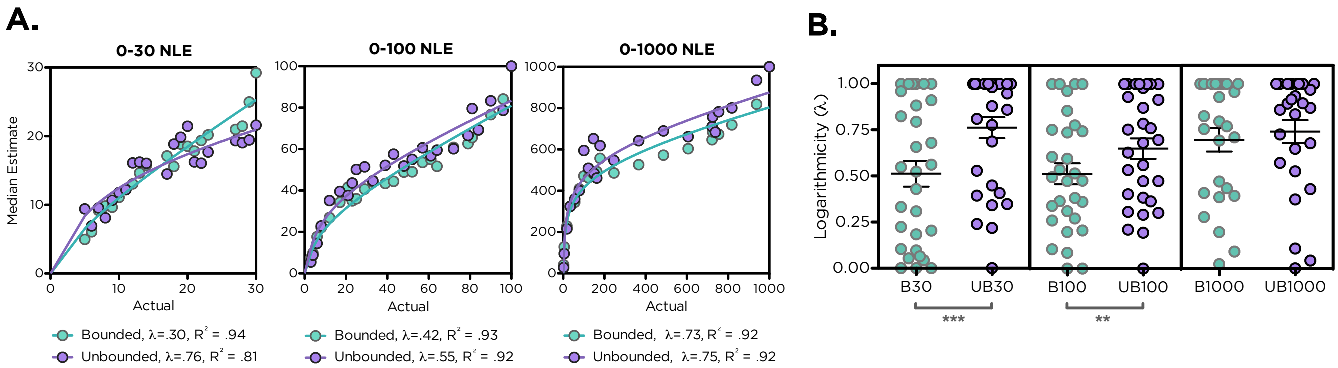

Figure 2. A: Median estimates in 0-30, 0-100, and 0-1000 number lines with and without bounds. B: Individual logarithmicity (λ) across the three number-line tasks.

bounded; M = 19%, SD = 9% for unbounded), $t(29) = 3.51$, $p < .01$. And for 0-1000 number-lines, greater PAEs were again observed in the unbounded than bounded condition (M = 20%, SD = 8% for bounded; M = 27%, SD = 11% for unbounded), $t(25) = 5.21$, $p < .001$. Therefore, for all the three number ranges, conventional number lines with two bounds elicited more accurate responses than unbounded number lines. These results are not at all consistent with the measurement-skills account.

Although number-line tasks without an upper bound produced more erroneous estimates, it is not clear whether the unboundedness increased random noise or logarithmic compression in estimates. To address this issue, individual children's degrees of logarithmicity were obtained with the MLLM fitting.

As shown in Figure 2B, in all the three number ranges, the values of the logarithmicity measure were greater for a number line without bounds. In 0-30 and 0-100 number-line tasks, estimates in the unbounded condition showed significantly greater logarithmic compression in the 0-30 range (M = .51, SD = .38 for bounded; M = .76, SD = .31 for unbounded), $t(29) = 4.33$, $p < .001$, and in the 0-100 range (M = .51, SD = .31 for bounded; M = .65, SD = .30 for unbounded), $t(29) = 3.37$, $p < .01$. Although logarithmicity was slightly greater for the unbounded 0-1000 number-line task than for the bounded one, the difference was not statistically significant (M = .70, SD = .32 for bounded; M = .74, SD = .32 for unbounded), $p > .05$. Against the arithmetic strategy hypothesis, negatively accelerating patterns in estimates were observed in both bounded and unbounded number-line tasks, and these patterns were stronger when the number line did not have a right-end bound. Even if the number-line task without the right endpoint revealed more accurate representation of numbers as the measurement-skills account claimed, what it revealed in the unbounded task was more logarithmic compression.

Discussion

In this paper, we attempted to address the nature of developmental changes in numerical estimation and to provide a unified framework for seemingly irreconcilable data regarding the psychophysical functions that link numbers to their magnitude estimates (Barth & Paladino, 2011; Cohen & Sarnecka, 2014; Opfer, Thompson, & Kim, 2016; Siegler & Opfer, 2003; Slusser, Santiago, & Barth, 2013). Specifically, we wished to see whether the logarithmic-to-linear shift existed on both bounded and unbounded number lines. As Cohen and Sarnecka (2014) stated, "For those children whose data are best fit by the logarithmic function, we are skeptical that this reflects a logarithmically organized quantity representation. If it did, we should expect to see the same pattern on the unbounded task…." (p. 1650).

In the teeth of this skepticism, we found robust evidence for a logarithmically organized quantity representation on the unbounded task, as well as the bounded task. Evidence came from multiple sources. First, the MLLM—which had previously unified discrepant data from free and anchored numerical estimation (Opfer et al., 2016)—predicted data from bounded and unbounded tasks better than rival psychophysical models. Additionally, for both tasks, the logarithmicity parameter was significantly greater than zero in each of the three age groups tested. This success of a single simple model over many competing complex models suggests there may be a simple cognitive process lurking behind many varieties of numerical estimation.

Second, individual differences were stable across the two tasks: children whose estimates were most logarithmic in the bounded task were also most logarithmic patterns in the unbounded task, $r(84) = .73$, $p < .001$. This wouldn't be expected if the two tasks elicited radically different estimation strategies or if only one of the tasks provided an accurate picture of children's numeric magnitude judgments.

Third, although the two tasks differed in overall accuracy, it was not in the direction predicted by the measurement-skills account: PAEs and logarithmicity values of both median and individuals' estimates were always greater for the unbounded condition across all three number ranges. Therefore, even if the more difficult unbounded tasks provide a better picture of the internal mental number line, what they show is *stronger* evidence for logarithmic representation of numbers.

Finally, the parameters expected by the measurement-skills account to track arithmetic biases unique to each task never predicted actual arithmetic performance, calling into question the psychological meaning of these parameters (Table 2). In contrast, the logarithmicity component of the MLLM reliably predicted children's arithmetic proficiency regardless of whether bounded or unbounded number lines were used, which is wholly inconsistent with the idea that bounded number lines require greater arithmetic skills than unbounded ones.

Why might our results have differed so strongly from those of Cohen and colleagues (2011, 2014)? Two reasons—methodological and analytical—seem likely. First, unlike Cohen and Sarnecka (2014), we counterbalanced the order of bounded and unbounded tasks so that any effects of task order would not influence our overall results. Counterbalancing also made it possible to test the presence of order effects. And we found that the order of tasks sometimes influenced estimates. For example, on the 0-1000 task, more erroneous estimates were produced in *both* tasks if the unbounded task was first presented, $F(1, 24) = 9.79$, $p < .01$, $\eta_p^2 = .29$, suggesting that the greater difficulty of the unbounded task resulted in fatigue or confusion for the second task. This is critical because it suggests that unbounded tasks are not intrinsically easier for children – quite the opposite.

Another important difference came from the analytic strategies employed. Cohen and Sarnecka (2014) tested the log *or* linear model against the 0CPM *or* 1CPM *or* 2CMP model. Because the number of *or*'s corresponds to greater overall model complexity (i.e., greater degrees of freedom), a poorer fit for one of *two* family A models than one of *three* family B models is ambiguous. Our own analytic strategy, in contrast, was to explicitly include all of the same family of models in the same equation (e.g., MLLM, MCPM1&2, MSPM), and thereby include the correct number of degrees of freedom. This strategy allowed us to improve the validity of our model comparison. Additionally, it yielded two unique insights: that individual differences in the logarithmicity of estimates were stable across tasks and that individual differences in logarithmicity of estimates predicted math skills better than alternative models. These findings directly contradicted the predictions of the measurement-skills account and could not be tested using the analytic strategy of Cohen and Sarnecka (2014).

In conclusion, the present study shows that the log-to-linear shift theory provides a framework for numerical estimation that is high in descriptive adequacy for both bounded and unbounded number lines and is reliably predictive of children's mathematics performance. An interesting question for future studies is whether the same framework provides a good account of magnitude estimation more broadly.

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