Title
String Field Equation from the Generalized Sigma Model II

Journal
Nuclear Physics B, B524(3)

Author
Bardakci, K.

Publication Date
1997-11-21

LBNL-40936
Preprint

ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

String Field Equations from Generalized Sigma Model II

Korkut Bardakci

Physics Division

November 1997

Submitted to
Nuclear Physics B

November 21, 1997

LBNL-40936
UCB-PTH-97/53

String Field Equations From Generalized Sigma Model II [1]

Korkut Bardakci[2]

Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

We improve and extend a method introduced in an earlier paper for deriving string field equations. The idea is to impose conformal invariance on a generalized sigma model, using a background field method that ensures covariance under very general non-local coordinate transformations. The method is used to derive the free string equations, as well as the interacting equations for the graviton-dilaton system. The full interacting string field equations derived by this method should be manifestly background independent.

[1] This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-95-14797.

[2] e-mail: kbardakci@lbl.gov

1. Introduction

This paper is the follow up of an earlier paper with the same title [1]. The basic idea of both papers is to derive the dynamical equations satisfied by the string states by requiring a sigma model on the world sheet to be conformally invariant. This idea has a long history, going back to some early papers [2-6], where the equations satisfied by the massless particles in the spectrum of the string were derived by demanding conformal invariance of the effective action in the one loop approximation. These early efforts dealt with only renormalizable sigma models, which restricted the scope of their investigation to the dynamics of the massless states. In order to incorporate the dynamics of the massive levels of the string, one has to start with the most general non-renormalizable sigma model, subject only to some general requirements of invariance. A method of imposing conformal invariance on a a general sigma model was proposed by Banks and Martinec [7], who introduced an explicit cutoff and used the Wilson type renormalization group equations [8,9]. This approach was further developed and was used to derive the tree level closed bosonic string amplitudes by Hughes, Liu and Polchinski[10] and others [11-14]. The idea behind this method is to cancel the conformal anomalies due to the quantum corrections against the classical violation of the conformal symmetry due to the presence of nonrenormalizable terms in the action. Despite its success in reproducing string amplitudes, this approach suffers from some drawbacks, among them lack of a sufficiently powerful gauge invariance to eliminate all the spurious states [10]. Another disadvantage of this approach is the absence of manifest covariance under redefinitions of the target space coordinate $X(\sigma)$. In fact, it will become clear later on that these problems are related; the spurious states are absent in a manifestly covariant treatment.

In the reference cited above [1], we proposed a new method for deriving the string field equations, by combining the advantageous features of both the earlier work on the sigma model [2-6], and of the Wilson renormalization group approach [7,10]. The starting point was the most general nonrenormalizable sigma model on the world sheet, subject only to two dimensional Poincare invariance. The basic idea was again to cancel the quantum conformal anomaly by the terms in the action that violate conformal invariance classically. This was done by first computing the one loop effective action with an explicit cutoff, and then by requiring the effective action to be invariant under conformal transformations. The main goal of the paper was to carry out the calculation of the effective action using a background field method [3,15], which is covariant under field transformations. The transformations in question included not only the local diffeomorphisms of general relativity, but also non-local ones with derivatives with respect to the world sheet coordinates (see eq.(4)). This posed the problem of finding a suitable covariant metric in order to be able to use the tools of differential geometry. Such a metric can easily be extracted from the action when only covariance under the local transformations is required; however, when non-local transformations are also included, the problem becomes difficult. In reference [1], partial progress was made in this direction by using an expansion in the slope parameter; however, only the equations for the first few levels of the string could be derived by this method. Even then, the left-right nonsymmetric string could not be treated .

In the present paper, we show how to overcome all of the difficulties encountered in the earlier work. The main new idea is to forget about the metric and introduce the connection as a completely independent field. It is also necessary to introduce a vector field which generates the conformal transformations (conformal Killing vector), again as an independent quantity. This means that we make no a priori commitment about the metric and the conformal Killing vector, but instead, we let the the equations resulting from conformal invariance (the RG equations) decide the issue. However, one encounters several problems in applying these equations: They are explicitly cutoff dependent and also they do not seem powerful enough to fix the connection and the Killing vector completely. The first problem is not really serious; it turns out that almost all of the cutoff dependendence factorizes, leaving behind cutoff independent equations. The only exception is a set of terms with logarithmic dependence on the cutoff, and these can be eliminated by slope renormalization. This is then the only renormalization needed to render the theory finite. As for the second problem, it is true that the connection and the Killing vector remain mostly undetermined; however, this does not mean that the RG equations contain no useful information. A subset of the equations turn out to be independent of the connection and the Killing vector, and these equations are then the candidates for the string field equations. A major part of this paper is devoted to working out the consequences of this idea to see whether it actually leads to the correct string equations. This comparison is done in two different cases: First, the linearized form of the RG equations are shown to be equivalent to the the well known free string equations. Also, going beyond the linear approximation, the interacting graviton-dilaton equations come out correctly.

The paper is organized as follows: In section 2, we review the version of the RG equations derived in [1], and we rewrite them in a form convenient for future applications. We also discuss in some detail the heat kernel method, the regularization scheme we use in this paper. It has a number of advantages over the explicit cutoff used in [1]. In sections 3 and 4, the linearized RG equations are applied to the massless and the first massive levels of the string. There are several reasons for considering these special cases before embarking on the general problem. The same special cases were considered in [1]; here we show how the present treatment overcomes the difficulties encountered there. Also, many of the important features of the general problem are already present in these special cases, and working them out in detail should be helpful. For example, one can easily verify that the cutoff dependence of the equations causes no problems. Also, it is instructive to see that the covariant treatment helps to eliminate several spurious states and the resulting spectrum is then in agreement with the string spectrum. In section 5, we apply the linearized RG equations to an arbitrary string state, and we show that they can be written in a compact form as a single equation using the standard operator formalism familiar from string theory. Section 6 is devoted to establishing the equivalence of this equation to the standard equations satisfied by the free string. Finally, we go beyond the linear approximation in section 7 by applying the full non-linear RG equations to the dilaton-graviton system, and we show that the resulting equations are the correct ones.

By working out these examples, we hope to have shown that the approach to string field equations proposed here is both correct and useful. As a future project, it seems quite feasible to derive the full set of interacting equations in the operator formalism of section 5. The main motivation for doing this is the realization that these equations should be manifestly background independent. Although initially the calculations are done in the framework of an expansion around the flat background, using the methods of sections 5 and 7, one should be able to sum the series and get rid of the background dependence. Lack of manifest background independence is a problem shared by many different approaches to string field theory, including the BRST formalism [16-19]. In addition to background independence, the field equations derived by the present method will also be invariant under non-local field transformations mentioned earlier. It has been suspected for a long time that string theory has a large class of as yet undiscovered hidden symmetries, and that these symmetries may be important in understanding string dynamics. For example, duality symmetries [20], which have attracted much attention recently, may be the manifestations of a much bigger hidden symmetry. In any case, any new approach to string theory will hopefully deepen our understanding of it.

2. One Loop RG Equations

We start this section with a brief review of the one loop RG equations derived in [1]. The starting point is a two dimensional action S which describes the world sheet structure of an interacting bosonic string theory. The only requirement on this action is two dimensional Lorenz invariance, other than that, it is the most general local non-renormalizable action constructed from the string coordinate $X^{\mu\sigma} \equiv X^{\mu}(\sigma)$. All of the computations of this paper will be carried out in a flat Minkowski background, accordingly, the action is split into free and interacting parts:

$$\begin{aligned} S &= S^{(0)} + S^{(1)}, \\ S^{(0)} &= \int d^2\sigma\, \partial_+ X^{\mu\sigma}\, \partial_- X^{\nu\sigma} \eta_{\mu\nu}, \\ S^{(1)} &= \int d^2\sigma \left(\Phi(X(\sigma)) + \tilde{h}_{\mu\nu}(X(\sigma)) \partial_+ X^{\mu\sigma} \partial_- X^{\nu\sigma} + \cdots \right), \end{aligned} \tag{1}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, Φ is the tachyon field, and $\tilde{h}_{\mu\nu}$ is related to the gravitational metric $g_{\mu\nu}$ and the antisymmetric tensor $B_{\mu\nu}$ through

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad g_{\mu\nu} = \frac{1}{2}(\tilde{g}_{\mu\nu} + \tilde{g}_{\nu\mu}), \quad B_{\mu\nu} = \frac{1}{2}(\tilde{g}_{\mu\nu} - \tilde{g}_{\nu\mu}), \tag{2}$$

and $\partial_+ \equiv \partial_{\sigma_+}$ and $\partial_- \equiv \partial_{\sigma_-}$ are derivatives with respect to the world sheet coordinates

$$\sigma_+ = \frac{1}{2}(\sigma_0 + i\sigma_1), \quad \sigma_- = \frac{1}{2}(\sigma_0 - \sigma_1).$$

The dots represent higher levels which contain more derivatives with respect to σ. Eq.(1) is a quasi-local expansion of the action in the derivatives of the coordinate $X^{\mu}(\sigma)$; the fields are local functions of $X^{\mu}(\sigma)$, as opposed to functionals. Non-locality is introduced gradually through higher powers of $\partial_{\pm} X^{\mu\sigma}$. World sheet Lorentz invariance requires equal numbers of ∂_+ and ∂_-. The presence of higher derivatives makes the model unrenormalizable, and a cutoff is needed to define it. Another way to organize this expansion is to classify the terms according to their classical conformal dimension, which is the naive dimension associated with the scaling of σ: Each derivative with respect to σ adds a unit to the classical conformal dimension. We note that the action is not classically conformal invariant.

The invariance properties of the model will play an important role. The action of eq.(1) is invariant if a total derivative is added to the integrand, setting

$$S = \int d^2\sigma \, I(\sigma).$$

the action is invariant under

$$I \rightarrow I + \partial_+ I_-(\sigma) + \partial_- I_+(\sigma). \tag{3}$$

Later, we will see that in the string language, this corresponds to invariance under adding spurious states generated by the application of the Virasoro operators L_{-1} and $\bar{L}_{-1}$ to the physical states. We will call this a linear gauge transformation. In addition to these invariances, which follow automatically from the definition of the action, we will impose invariance under the infinitesimal coordinate transformations

$$X^{\mu\sigma} \rightarrow X^{\mu\sigma} + f^{\mu}(X(\sigma)) + f^{\mu}_{\nu\lambda}(X(\sigma))\partial_+ X^{\nu\sigma}\partial_- X^{\lambda\sigma} + \cdots \tag{4}$$

where f's are arbitrary local functions of $X(\sigma)$. The first function f^{μ} corresponds to the local diffeomorphisms of general relativity, so it ensures the imbedding of gravity into the model. We shall see later that the transformations with higher derivatives eliminate spurious states.

Finally, we would like the model to be conformally invariant. In the flat world sheet formulation we are using, the two sets of infinitesimal conformal transformations are given by

$$\sigma_+ \rightarrow \sigma_+ + v_+(\sigma_+), \quad \sigma_- \rightarrow \sigma_- + v_-(\sigma_-). \tag{5}$$

The following operators, acting on the coordinates, generate these transformations:

$$\delta_{v_\pm} = \int d^2\sigma \, v_\pm(\sigma_\pm)\, \partial_\pm X^{\mu\sigma} \frac{\delta}{\delta X^{\mu\sigma}}. \tag{6}$$

However, these generators do not transform properly under the coordinate transformations given by eq.(4). To ensure proper transformation properties,

$\partial_\pm X^{\mu\sigma}$ in eq.(6) should be replaced by a vector (Killing vector):

$$\begin{aligned} \delta_{v_\pm} &= \int d^2\sigma F^{\mu\sigma}_{v_\pm}(X)\frac{\delta}{\delta X^{\mu\sigma}}, \\ F^{\mu\sigma}_{v_\pm} &= v_\pm(\sigma_\pm)\partial_\pm X^{\mu\sigma} + \int d^2\sigma' v_\pm(\sigma'_\pm) f^{\mu\sigma}_{\sigma'}(X). \end{aligned} \tag{7}$$

Here, $f^{\mu\sigma}_{\sigma'}$ is introduced so that $F^{\mu\sigma}_{v_\pm}$ will transform like a contravariant vector in the indices $\mu\sigma$ under the transformations of eq.(4). This then guarantees that conformal invariance is coordinate independent. To start with, $f^{\mu\sigma}_{\sigma'}$ will be left arbitrary, and it will eventually be fixed by the string field equations.

The string field equations can be derived [10] by requiring the conformal invariance of the string theory based on the action S (eq.(1)). This action is not even classically conformal invariant as it stands; the tachyon field and the fields corresponding to massive levels violate classical conformal invariance. Quantum mechanically, there is a further violation (anomaly) coming from higher order graphs. Conformal invariance can be restored by cancelling the classical terms against the quantum anomaly; the resulting conditions are then the string field equations. Below, we write down the version of these equations derived in [1]:

$$E_G + E_M = 0, \tag{8}$$

where,

$$E_G = \left(F^{\mu\sigma}_v \frac{\delta}{\delta X^{\mu\sigma}} + \delta_\Lambda\right)\left(bS - \frac{1}{2}Trlog(G)\right), \tag{9}$$

and

$$\begin{aligned} E_M &= \frac{1}{2}\left(-F^{\lambda\tau}_v \frac{\delta G^{\mu\sigma,\mu'\sigma'}}{\delta X^{\lambda\tau}} + \frac{\delta F^{\mu\sigma}_v}{\delta X^{\lambda\tau}} G^{\lambda\tau,\mu'\sigma'} + \frac{\delta F^{\mu'\sigma'}_v}{\delta X^{\lambda\tau}} G^{\mu\sigma,\lambda\tau} - \delta_\Lambda G^{\mu\sigma,\mu'\sigma'}\right) \\ &\times \; M_{\mu\sigma,\mu'\sigma'}. \end{aligned} \tag{10}$$

Let us define the expressions that appear in the equation above. F and S were discussed earlier, "b" is the slope parameter, and δ_Λ involves the variation of the cutoff and it will be explained when we discuss the cutoff procedure. The "supermetric" G is defined by

$$G_{\mu\sigma,\mu'\sigma'} = \frac{\delta^2 S}{\delta X^{\mu\sigma}\delta X^{\mu'\sigma'}} - \Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'}\frac{\delta S}{\delta X^{\lambda\tau}}, \tag{11}$$

and $G^{\mu\sigma,\mu'\sigma'}$ is the inverse of $G_{\mu\sigma,\mu'\sigma'}$. The connection Γ is introduced in order to preserve covariance under the transformations given by eq.(4), and it will be further specified later on. The term $M_{\mu\sigma,\mu'\sigma'}$ is related to the Jacobian of a change of variables, as explained in [1], and it depends on the connection Γ alone. In this paper, we only need the terms linear in its expansion in terms of Γ:

$$M_{\mu\sigma,\mu'\sigma'} = -\frac{1}{3}\left(\frac{\delta\Gamma^{\lambda\tau}_{\lambda\tau,\mu\sigma}}{\delta X^{\mu'\sigma'}} + \frac{\delta\Gamma^{\lambda\tau}_{\lambda\tau,\mu'\sigma'}}{\delta X^{\lambda\tau}} + \frac{\delta\Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'}}{\delta X^{\lambda\tau}}\right) + \cdots. \tag{12}$$

In the preceding equations, as well as in the rest of the paper, the summation convention is also applied to the world sheet variables; repeated variables are to be integrated over. We also frequently use the matrix(operator) notation for expressions with two sets of indices, for example, $G_{\mu\sigma,\mu'\sigma'}$ is to be thought of as a matrix in the set of indices $\mu\sigma$ and $\mu'\sigma'$, with an obvious definition of the matrix product. Another convention we follow throughout the paper is to write only the set equations corresponding to $v_+(\sigma_+)$, when the set involving $v_-(\sigma_-)$ can be obtained from the first set by the obvious substitution $+ \leftrightarrow -$. Following this convention in the above set of equations, we have not displayed the set corresponding to v_-. Also, the trace in the expression $Trlog(G)$ is over the same set of indices.

The set of eqs.(8,9,10) form the starting point of this paper; they are the analogue of the renormalization group equations of reference [10]. ompared to [10], it has the advantage of being invariant under the transformations of eq.(4), which, as we shall see, is important in eliminating certain spurious states. In contrast, in the non-covariant approach of [10], there does not seem to be enough gauge invariance to decouple all the spurious states.

As they stand, eqs.(8,9,10) are still only formal, since we have not yet specified any cutoff or regularization procedure. We now briefly discuss the heat kernel method, the regularization procedure we are going to use. It differs from the naive cutoff used in [1], and it has several advantages over it: It is simple to implement, and it preserves invariance under coordinate transformations (eq.(4)). There is a further advantantage in using the heat kernel method : Although we have written down eqs.(8,9 and 10) in full generality, we are really interested only in the local terms in these equations. By this, we mean terms that have a local expansion similar to the expansion for $S^{(1)}$ in eq.(1). These terms are the only ones to be considered in a renormalization group analysis such as ours, since only they contribute to the renormalization of the original local action. The heat kernel method provides a very convenient way of extracting these local terms, and it will enable us later on to write a finite and local version of the equations (8,9,10).

There are two divergent terms that need regularization: The $Trlog(G)$ term in eq.(9) and $M_{\mu\sigma,\mu'\sigma'}$ in eq.(12). Let us first consider the $Trlog(G)$. We set

$$2\,G_{\mu\sigma,\mu'\sigma'} = 2\,\Delta_{\mu\sigma,\mu'\sigma'} + H_{\mu\sigma,\mu'\sigma'}, \tag{13}$$

where,

$$\Delta_{\mu\sigma,\mu'\sigma'} = \eta_{\mu\mu'}\Delta_{\sigma,\sigma'}, \quad \Delta_{\sigma,\sigma'} = -\partial_+\partial_-\delta^2(\sigma-\sigma').$$

We shall also need the free propagator $\Delta^{\mu\sigma,\mu'\sigma'}$, which is the inverse of $\Delta_{\mu\sigma,\mu'\sigma'}$. It satisfies

$$\Delta^{\mu\sigma,\mu'\sigma'} = \eta^{\mu\mu'}\Delta^{\sigma,\sigma'}, \quad \partial_+\partial_-\Delta^{\sigma\sigma'} = -\delta^2(\sigma-\sigma'),$$

and its regularized form is given by

$$\Delta^{\mu\sigma,\mu'\sigma'} \to \Delta^{\mu\sigma,\mu'\sigma'}(\epsilon) = \int_\epsilon^\infty dt\, \tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(t), \tag{14}$$

with

$$\begin{aligned}\tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(t) = \theta(t)\left(e^{-t\Delta}\right)_{\mu\sigma,\mu'\sigma'} &= \eta_{\mu\mu'}\,\tilde{G}^{(0)}_{\sigma,\sigma'}(t)\\ &= \eta_{\mu\mu'}\,\frac{\theta(t)}{4\pi t}\exp\left(-\frac{(\sigma-\sigma')^2}{4t}\right).\end{aligned} \tag{15}$$

The term $Trlog(G)$ is regularized by

$$TrlogG \to -\int_\epsilon^\infty \frac{dt}{t}\,Tr(\tilde{G}), \tag{16}$$

where the full heat kernel $\tilde{G}$ is defined by

$$\tilde{G}_{\mu\sigma,\mu'\sigma'}(t) \equiv \theta(t)\left(e^{-tG}\right)_{\mu\sigma,\mu'\sigma'}.$$

We now compute the conformal variation of $Trlog(G)$. Starting with

$$\delta_v(Trlog(G)) = \int_\epsilon^\infty dt\,Tr\left(e^{-tG}\delta_v(G)\right), \tag{17}$$

it is convenient to split it into two terms:

$$\begin{aligned}\delta_{v+}\left(G_{\mu\sigma,\mu'\sigma'}\right) &= \partial_{\sigma+}\left(v(\sigma_+)G_{\mu\sigma,\mu'\sigma'}\right) + \partial_{\sigma'_+}\left(v(\sigma'_+)G_{\mu\sigma,\mu'\sigma'}\right) + \delta^{(2)}_{v+}\left(G_{\mu\sigma,\mu'\sigma'}\right)\\ &= \left(\delta^{(1)}_{v+} + \delta^{(2)}_{v+}\right)G_{\mu\sigma,\mu'\sigma'}.\end{aligned} \tag{18}$$

This split is motivated by the observation that conformal transformations are a special case of the coordinate transformations; the transformation law of a tensor such as $G_{\mu\sigma,\mu'\sigma'}$ under the coordinate transformations contains two types of terms: the first type comes from the transformation of the indices of the tensor and it is represented by $\delta^{(1)}_{v+}$ or the first two terms on the right hand side of eq.(18). The second type of term corresponds to the transformation of the coordinates on which the tensor depends and it is given by $\delta^{(2)}_{v+}$. For example, acting on the first term in eq.(11) for G, $\delta^{(2)}_{v+}$ is given by

$$\delta^{(2)}_{v+}\left(\frac{\delta^2 S}{\delta X^{\mu\sigma}\delta X^{\mu'\sigma'}}\right)=\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\mu'\sigma'}}\left(\int d^2\tau\, v(\tau_+)\,\partial_+ X^{\lambda\tau}\frac{\delta S}{\delta X^{\lambda\tau}}\right). \tag{19}$$

We shall later see that all of the cutoff independent useful information will come from $\delta^{(1)}_{v\pm}$; $\delta^{(2)}_{v\pm}$ will only contribute cutoff dependent terms which will cancel.

We now turn to the evaluation of the right hand side of eq.(17). Using the definition of the heat kernel, and the identity

$$\left(v(\sigma_+)\partial_{\sigma_+}+v(\sigma'_+)\partial_{\sigma'_+}\right)\tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(t)=-\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}\frac{\partial}{\partial t}\left(t\,\tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(t)\right), \tag{20}$$

one can easily establish the following result:

$$\begin{aligned}
&\delta^{(1)}_{v+}\left(Trlog(G)\right)=\\
&=\frac{1}{2}\int_\epsilon^\infty dt\int d^2\sigma\int d^2\sigma'\,\tilde{G}_{\mu\sigma,\mu'\sigma'}\left(\partial_{\sigma_+}(t)\left(v(\sigma_+)H_{\mu\sigma,\mu'\sigma'}\right)+\partial_{\sigma'_+}\left(v(\sigma'_+)H_{\mu\sigma,\mu'\sigma'}\right)\right)\\
&=\frac{1}{2}\int_\epsilon^\infty dt\int d^2\sigma\int d^2\sigma'\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}\left(H_{\mu\sigma,\mu'\sigma'}\frac{\partial}{\partial t}\left(t\tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(t)\right)\right.\\
&\left.-\frac{1}{4}\int_{-\infty}^{+\infty}dt'\left(H\tilde{G}(t')H\right)_{\mu\sigma,\mu'\sigma'}\frac{\partial}{\partial t}\left((t-t')\tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(t-t')\right)\right).
\end{aligned} \tag{21}$$

This equation enables us to make a clean seperation between local and non-local contributions to eqs.(8,9,10). We note that the integrand is a total derivative with respect to the variable t. It can therefore be integrated, with the result that the contribution from the upper limit ∞ is the non-local part of the integral, and the contribution from the lower limit ϵ is the local part. This follows from the well-known properties of the heat kernel, which describes the diffusion of a point source as a function of time t. For small t, $t=\epsilon$, the source can diffuse only a small distance in space, and so in the limit

$\epsilon \to 0$, the contribution from the lower limit is local. On the other hand, the contribution from the upper limit is clearly non-local, since t is very large. We now define δ_Λ (eqs.(9,10)) so as to cancel the unwanted contribution from the upper limit of integration over t:

$$\left(\delta^{(1)}_{v+} + \delta_\Lambda\right) Trlog(G) = -\frac{1}{2}\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\Big(\epsilon\, \tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon) H_{\mu\sigma,\mu'\sigma'} - \frac{1}{2}\int_0^\epsilon dt'(\epsilon - t')\tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon - t')\left(H\tilde{G}H\right)_{\mu\sigma,\mu'\sigma'}\Big). \quad (22)$$

It can easily be shown that this definition ensures that the long distance behavior of the free propagator is unchanged under conformal transformations. For this reason, in the case of free propagator, the regularization we are using agrees with the cutoff used in reference [10]. Having extracted the local part of eq.(9), we can rewrite eqs.(9,10) in the following form:

$$b\int d^2\sigma \int d^2\sigma'\, v(\sigma'_+) F^{\mu\sigma}_{\sigma'}(X)\,\frac{\delta S}{\delta X^{\mu\sigma}} + \frac{1}{2}\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}$$
$$\times\left(\frac{1}{2}\epsilon\, \tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon) H_{\mu\sigma,\mu'\sigma'} - \frac{1}{4}\int dt'(\epsilon - t')\tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon - t')\left(H\tilde{G}(t')H\right)_{\mu\sigma,\mu'\sigma'}\right)$$
$$-\frac{1}{2}\int_\epsilon^\infty dt\, \tilde{G}_{\mu'\sigma',\mu\sigma}(t)\Big(\int d^2\tau \int d^2\tau'\, v(\tau'_+) f^{\lambda\tau}_{\tau'}(X)\,\frac{\delta G_{\mu\sigma,\mu'\sigma'}}{\delta X^{\lambda\tau}}$$
$$+\frac{1}{2}\,\delta^{(2)}_v(H_{\mu\sigma,\mu'\sigma'})\Big) + E_M = 0. \quad (23)$$

We shall often need the part of the above equation linear in fields. It is quite straightforward to linearize various terms except perhaps E_M. Since M already starts at the linear order(eq.(12)), in the factor in front of this term, we can replace G by the zeroth order term in its expansion:

$$G^{\mu\sigma,\mu'\sigma'} \to \int_\epsilon^\infty dt\, \tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(t),$$

and arrive at the result

$$-F^{\lambda\tau}_v \frac{\delta G^{\mu\sigma,\mu'\sigma'}}{\delta X^{\lambda\tau}} + \frac{\delta F^{\mu\sigma}_v}{\delta X^{\lambda\tau}} G^{\lambda\tau,\mu'\sigma'} + \frac{\delta F^{\mu'\sigma'}_v}{\delta X^{\lambda\tau}} G^{\mu\sigma,\lambda\tau} - \delta_\Lambda(G^{\mu\sigma,\mu'\sigma'})$$
$$\to \int_\epsilon^\infty dt\left(v(\sigma_+)\partial_{\sigma_+} + v(\sigma'_+)\partial_{\sigma'_+} - \delta_\Lambda\right)\tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(t)$$
$$= \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\,\epsilon\, \tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(\epsilon), \quad (24)$$

Next, we turn our attention to eq.(12) for $M_{\mu\sigma,\mu'\sigma'}$. This expression needs regularization, since the integration over the variable τ will lead to divergences. Again, we use the heat kernel method to regulate it. Making use of a basic property of the heat kernel,namely, as $\epsilon \to 0$,

$$lim \left(\tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(\epsilon)\right) \to \eta_{\mu\mu'}\, \delta^2(\sigma-\sigma'),$$

we can regularize the integration over τ by setting, for example

$$\int d^2\tau\, \Gamma^{\lambda\tau}_{\lambda\tau,\mu\sigma} \to \int d^2\tau \int d^2\tau'\, \Gamma^{\lambda\tau}_{\lambda'\tau',\mu\sigma} \tilde{G}^{(0)}_{\lambda\tau,\lambda'\tau'}(\epsilon).$$

Combining this with eq.(24) yields the following regulated expression for E_M:

$$\begin{aligned} E_M &= -\frac{1}{3}\int d^2\sigma \int d^2\sigma' \int d^2\tau \int d^2\tau' \frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+ - \sigma'_+}\epsilon\, \tilde{G}^{(0)}_{\mu\sigma,\mu'\sigma'}(\epsilon)\tilde{G}^{(0)}_{\lambda\tau,\lambda'\tau'}(\epsilon) \\ &\times \left(2\,\frac{\delta\Gamma^{\lambda\tau}_{\lambda'\tau',\mu\sigma}}{\delta X^{\mu'\sigma'}} + \frac{\delta\Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'}}{\delta X^{\lambda'\tau'}}\right) + \cdots, \end{aligned} \tag{25}$$

where the dots represent higher order terms in the fields that we have not written down. Finally, putting everything together, we have the following linear version of eq.(23):

$$\begin{aligned} & b\int d^2\sigma\, v(\sigma_+)\, \partial_+ X^{\mu\sigma}\frac{\delta S^{(1)}}{\delta X^{\mu\sigma}} - 2b\int d^2\sigma \int d^2\sigma' v(\sigma'_+)\, f^{\mu\sigma}_{\sigma'}(X)\, \partial_+\partial_-(X^{\mu\sigma}) \\ & +\frac{1}{4}\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+ - \sigma'_+}\, \epsilon\, \tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon) H_{\mu\sigma,\mu'\sigma'} \\ & -\frac{1}{4}\int_\epsilon^\infty dt\, \tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(t)\, \delta^{(2)}_v(H_{\mu\sigma,\mu'\sigma'}) + E_M = 0, \end{aligned} \tag{26}$$

where,

$$\begin{aligned} H_{\mu\sigma,\mu'\sigma'} &= \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\mu'\sigma'}} - \Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'}\frac{\delta S}{\delta X^{\lambda\tau}} \\ &\simeq \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\mu'\sigma'}} + 2\,\Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'}\partial_+\partial_- X^{\lambda\tau}. \end{aligned} \tag{27}$$

In closing this section, let us comment on what has been accomplished so far. Using the heat kernel method, we have both regularized the basic renormalization group equations (8,9,10) and also extracted their local component. The result is eq.(23) and its linearized version, eq.(26). At first sight, it is not clear that these equations are powerful enough to give useful information. For example, the connection Γ and the field f (eq.(7)) were introduced as independent fields in our equations. On the other hand, since the string field equations should ultimately be expressible only in terms of the string fields that appear in the basic action of eq.(1), Γ and f should somehow be eliminated in favor of these fundamental fields. In the following sections, we shall see that there is no need of an a priori determination of the auxilliary fields Γ and F ; the equations themselves will do this job for us. The situation is somewhat similar to the first order formulation of general relativity, when the connection is introduced as an independent field and then determined from the equations of motion. We have somewhat oversimplified the situation here; the equations we have, unlike those in general relativity, are not quite powerful enough to determine all the components of these auxiliary fields. However, the undetermined components are also unneeded; they do not appear in the equations for the string fields. As a consequence, the auxiliary fields can be completely eliminated from the final string field equations.

Another question concerns the cutoff dependence of the equations. We shall see that the equations neatly seperate into cutoff independent and cutoff dependent parts. The cutoff dependent pieces have a different structure than the cutoff independent ones, and as a result, they have to cancel among themselves. The resulting equations partially fix Γ and F, but they do not lead to any relations between the string fields. As we shall see, the useful equations come exclusively from the cutoff independent pieces in eq.(26).

In the next two sections, eq.(26) will be applied to the massless and the first massive levels of the string, neglecting all the rest of the levels. In section 5, we generalize our treatment to include all of the levels. The reasons for specializing to these two levels are the following: In [1], we considered the same two levels of the string, with somewhat unsatisfactory results for the first massive level. We feel that it is instructive to compare the improved treatment given here to the treatment given in [1], and to show that all the dificulties encountered in the earlier paper are easily overcome. In addition, since the general treatment of all the levels given in section 5 is somewhat formal, we felt that working out two simple examples in some detail might be useful.

3. Linearized Equations For the Zero Mass States

In this section, we apply eq.(26) to the massless states of the string, the graviton, the dilaton and the antisymmetric tensor, suppressing for the time being all the other states of the string. Also, we confine ourselves to a linearized treatment, which serves as an introduction to the full non-linear treatment of section 6. The action that describes the graviton and the antisymmetric tensor is the second term in eq.(1):

$$S^{(1)} = \int d^2\sigma\, \tilde{h}_{\mu\nu}\left(X(\sigma)\right) \partial_+ X^{\mu\sigma} \partial_- X^{\nu\sigma}. \tag{28}$$

The dilaton is at the moment missing, and it will make its appearence later as part of the connection. As explained earlier, the connection will not be specified yet, and only the following general conditions will be imposed on it:

a) The connection should be a local function of $X(\sigma)$.

b) Its classical conformal dimension should be determined by requiring that the two terms in eq.(11) have the same dimension. This requirement guarantees that $G_{\mu\sigma,\mu'\sigma'}$ will have a well defined classical conformal dimension.

These requirements fix the form of Γ to be

$$\Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'} = \Gamma^{\lambda}_{\mu\mu'}\left(X(\sigma)\right) \delta^2(\tau-\sigma)\delta^2(\sigma-\sigma'). \tag{29}$$

We should make it clear that although we are using for it the same symbol as the usual metric derived connection of general relativity, $\Gamma^{\lambda}_{\mu\mu'}$ is as yet an undetermined function of $X(\sigma)$. In fact, in the end, it will turn out to be different from the standard result.

We now discuss the expected invariances of the model. Since all the higher levels in the action are neglected, invariance under coordinate transformations, eq.(4), is restricted to the coordinate transformations of general relativity,

$$X^{\mu\sigma} \to X^{\mu\sigma} + f^{\mu}\left(X(\sigma)\right). \tag{30}$$

In addition to these coordinate transformations, there is invariance under the gauge transformations given eq.(3). Taking

$$I_+ = -\partial_+ X^{\mu\sigma} \Lambda_\mu, \quad I_- = \partial_- X^{\mu\sigma} \Lambda_\mu,$$

we have the well-known gauge transformations of the antisymmetric tensor:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{31}$$

Finally, the action given by eq.(28) is conformally invariant in the classical limit.

As a consequence of these invariances, in computing the contribution of various terms to eq.(26), a number of simplifications occur:
a) Two of the terms in (26) vanish as a result of the conformal invariance of the action,

$$\begin{aligned} \int d^2\sigma\, v(\sigma_+)\, \partial_+ X^{\mu\sigma} \frac{\delta S^{(1)}}{\delta X^{\mu\sigma}} &= 0, \\ \delta_v^{(2)}(H_{\mu\sigma,\mu'\sigma'}) &= 0. \end{aligned} \tag{32}$$

b) The second term in the same equation also vanishes, since $f^{\mu\sigma}_{\sigma'} = 0$. This is because the first term for F in eq.(7) already transforms as a vector under (30).
The remaining of the terms are given by,

$$\begin{aligned} &\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \epsilon\, \tilde{G}_{\mu'\sigma',\mu\sigma}(\epsilon) H_{\mu\sigma,\mu'\sigma'} \\ &= \frac{1}{4\pi} \int d^2\sigma\, v'(\sigma_+) \partial_+ X^{\mu\sigma} \partial_- X^{\mu'\sigma} \Big(\Box \tilde{h}_{\mu\mu'} - \partial_\mu \partial_\lambda \tilde{h}_{\lambda\mu'} \\ &+ \partial_{\mu'} \partial_\lambda \tilde{h}_{\lambda\mu} - 2\, \partial_{\mu'} \Gamma^{\mu}_{\lambda\lambda} \Big) + \frac{1}{2\pi\epsilon} \int d^2\sigma\, v'(\sigma_+) \tilde{h}_{\lambda\lambda}, \end{aligned} \tag{33}$$

$$E_M = -\frac{1}{3(4\pi)^2 \epsilon} \int d^2\sigma\, v'(\sigma_+) \Big(2\partial_\mu \Gamma^{\nu}_{\nu\mu}(X(\sigma)) + \partial_\nu \Gamma^{\nu}_{\mu\mu}(X(\sigma)) \Big). \tag{34}$$

Substituting these results in eq.(26), we note that terms propotional to the factor $\partial_+ X^{\mu\sigma} \partial_- X^{\mu'\sigma}$ and terms that do not have this factor must cancel seperately among themselves. Since the terms without this factor are proportional to $1/\epsilon$, it follows that cutoff dependent terms cancel among themselves, and the cutoff factor does not appear in the resulting equations:

$$\Box \tilde{h}_{\mu\mu'} - \partial_\lambda \partial_\mu \tilde{h}_{\lambda\mu'} + \partial_{\mu'} \partial_\lambda \tilde{h}_{\lambda\mu} - 2\, \partial_{\mu'} \Gamma^{\mu}_{\lambda\lambda} = 0, \tag{35}$$

$$\tilde{h}_{\lambda\lambda} - \frac{1}{6\pi} (2\, \partial_\mu \Gamma^{\lambda}_{\lambda\mu} + \partial_\lambda \Gamma^{\lambda}_{\mu\mu}) = 0. \tag{36}$$

In addition to eq.(34), which came from conformal transformations on the variable σ_+, there is a σ_- counterpart, obtained by letting

$$\tilde{h}_{\mu\mu'} \to \tilde{h}_{\mu'\mu}$$

in that equation:

$$\square\tilde{h}_{\mu'\mu} - \partial_\mu\partial_\lambda\tilde{h}_{\mu'\lambda} + \partial_{\mu'}\partial_\lambda\tilde{h}_{\mu\lambda} - 2\,\partial_{\mu'}\Gamma^{\mu}_{\lambda\lambda} = 0. \tag{37}$$

Combining eqs.(34) and (36) yields the following result:

$$\Gamma^{\lambda}_{\mu\mu} = \partial_\mu h_{\mu\lambda} - \frac{1}{2}\partial_\lambda h_{\mu\mu} + \partial_\lambda\phi, \tag{38}$$

$$\square h_{\mu\mu'} - \partial_\mu\partial_\lambda h_{\mu'\lambda} - \partial_{\mu'}\partial_\lambda h_{\mu\lambda} + \partial_\mu\partial_{\mu'}h_{\lambda\lambda} - 2\,\partial_\mu\partial_{\mu'}\phi = 0, \tag{39}$$

$$\square B_{\mu\mu'} - \partial_\mu\partial_\lambda B_{\lambda\mu'} + \partial_{\mu'}\partial_\lambda B_{\lambda\mu} = 0. \tag{40}$$

We now make a few observations:
a) Eq.(38) determines only $\Gamma^{\lambda}_{\mu\mu}$, the contracted part of the connection, up to a total derivative of a new field. We identify this field ϕ with the dilaton field.
b) Eq.(39) is the correct linearized equation for the gravitational field (symmetric part of $\tilde{h}$), coupled to the dilaton field.
c) Eq.(40) is the correct linearized equation for the antisymmetric tensor.
d) Those components of Γ not determined by eqs.(35) and (36) play no role in the equations for the fundamental string fields. In fact, Γ is completely absent from the equations for h and B. One can think of this as some kind of gauge invariance operating on Γ, although we will not stress this point of view in this paper.
e) The linearized equation for the dilaton

$$\square\phi = 0,$$

is still missing. It can in fact be derived from the eq.(38) for gravity as follows: This equation is invariant under

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu\kappa_\nu + \partial_\nu\kappa_\mu, \tag{41}$$

the standard linearized gauge transformations of gravity. If the d'Alembertian acting on $h_{\mu\mu'}$ in (39) is invertible, then h is a pure gauge. Therefore, the only physical part of h comes from the non-invertible part of the d'Alembertian. One can then fix the gauge so that

$$\square h_{\mu\nu} = 0.$$

In this gauge, applying the d'Alembertian on both sides of (38), we find that ϕ satisfies the massles free field equation.

To summarize, we have shown in this section that eq.(26) correctly reproduces the coupled gravity-dilaton equations in the linear approximation. We stress that no a priori choice of metric or connection was made; in fact, the metric played no role at all in our derivation. This is in contrast to the standard treatment [2-6], where an initial choice of the metric is made. The problem is that when the higher levels of the string are present, they will in general contribute to the metric, it is no longer easy to guess the form of this contribution. An incorrect initial choice would in general conflict with the renormalization group equations. We avoid this problem by letting the equations determine as much of the connection as possible; the components of the connection that are left undetermined are spurious and do not appear in the equation for the physical fields. The dilaton field, which was not present in the original action(eq.(1)), emerges from as part of the connection. Again, this differs from the standard approach [5], which introduces the dilaton field in the original action.

4. Linearized Equations For The First Massive Level

In this section, we apply eq.(26) to the first massive level of the string. In general, the action for the first massive level contains 8 terms; however, using gauge transformations of the form given by eq.(3), it was shown in [1] that five of those terms can be eliminated, resulting in the following completely gauge fixed form:

$$\begin{aligned} S^{(1)} &= \int d^2\sigma \Big(e_{\mu_1\mu_2,\nu_1\nu_2} \partial_+ X^{\mu_1} \partial_+ X^{\mu_2} \partial_- X^{\nu_1} \partial_- X^{\nu_2} \\ &+ e_{\mu_1\mu_2\mu_3} \partial_+\partial_- X^{\mu_1} \partial_+ X^{\mu_2} \partial_- X^{\mu_3} + e_{\mu_1\mu_2} \partial_+\partial_- X^{\mu_1} \partial_+\partial_- X^{\mu_2} \Big). \quad (42) \end{aligned}$$

In this formula, the fields, as usual, are assumed to be local functions of $X(\sigma)$. The full action is again sum of free and interacting terms:

$$S = S^{(0)} + S^{(1)},$$

with $S^{(0)}$ given by eq.(1). The coordinate transformations relevant for this action are

$$X^{\mu\sigma} \to X^{\mu\sigma} + f^{\mu}_{\nu\lambda}(X(\sigma))\partial_+ X^{\nu\sigma}\partial_- X^{\lambda\sigma} + f^{\mu}_{\nu}(X(\sigma))\partial_+\partial_- X^{\nu\sigma}. \quad (43)$$

These transformations, acting on $S^{(0)}$, generate terms of the same form as the terms proportional to $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ in $S^{(1)}$. In fact, these terms can be eliminated by choosing

$$2\, f^{\mu_1}_{\mu_2\mu_3} = e_{\mu_1\mu_2\mu_3}, \quad f^{\mu_1}_{\mu_2} + f^{\mu_2}_{\mu_1} = e_{\mu_1\mu_2}, \tag{44}$$

which shows that the corresponding states are spurious, so long as the theory is invariant under (43). The only set of states which cannot be decoupled are the states represented by the first term in eq.(42); these are therefore the only physical states.

At this point, we would like to make the following observations:

a) The coordinate transformations , acting $S(1)$, generate additional terms. Since we are investigating only the linear portion of the theory, these terms can be neglected.

b) The terms eliminated by coordinate invariance are the same terms which would vanish, if the free equations of motion for X,

$$\partial_+\partial_- X^{\mu\sigma} = 0, \tag{45}$$

were imposed. Of course, in the linearized theory, the free field equations are what remain from the full set of interacting classical equations that follow from the action (1). Covariantizing the theory with respect to coordinate transformations therefore enables one to use the classical equations of motion in conjunction with the renormalization group equations. We should stress that, since the renormalization group equations deal with off mass shell quantities, the use of the classical equations of motion is in general not permissible in a non-covariant approach. This is clearly a good feature of the covariant approach, since the states eliminated by the equations of motion are also absent in the standard treatment of the string theory. In contrast, in a non-covariant treatment of the renormalization group equations, it is not clear how to eliminate these unwanted states [10]. It is also interesting to know whether what we are doing here is related to the Batalin-Vilkovisky program [21,22], which also makes it possible to use the equations of motion in an off-shell formulation. In this context, Henneaux [23] discussed the connection between field redefinitions, equations of motion and Batalin-Vilkovisky method.

c) Let us compare the coordinate transformations of general relativity(eq.(30)), which are completely local on the world sheet, with the transformations of eq.(43), which, in contrast, contain derivatives with respect to the world sheet coordinates. Invariance under either set of transformations serves to eliminate spurious states. There is, however, a difference: Invariance under diffeomorphisms of general relativity, in contrast to invariance under (43), does not lead to on mass shell constraints .

The next step is to secure invariance under (43) by introducing a suitable connection. The conditions that the connection must satisfy are the same ones that lead to eq.(29); namely, locality and the correct classical conformal dimension, plus world sheet Lorentz invariance. The expansion in terms of delta functions in world sheet coordinates and their derivatives is rather lengthy; it contains ten terms. To give the reader an idea, we exhibit a few typical terms below:

$$\begin{aligned}\Gamma^{\lambda\tau}_{\mu\sigma,\mu'\sigma'} &= \delta^2(\tau-\sigma)\delta^2(\sigma-\sigma')\Big(\Gamma^{(1)\lambda}_{\mu\mu',\alpha\beta}(X(\sigma))\,\partial_+X^{\alpha\sigma}\partial_-X^{\beta\sigma} \\ &+ \Gamma^{(2)\lambda}_{\mu\mu',\alpha}\,\partial_+\partial_-X^{\alpha\sigma}\Big)+\delta^2(\tau-\sigma)\partial_{\sigma_+}\delta^2(\sigma-\sigma')\Gamma^{(3)\lambda}_{\mu\mu',\alpha}\,\partial_-X^{\alpha\sigma} \\ &+ \delta^2(\tau-\sigma)\partial_{\sigma_-}\delta^2(\sigma-\sigma')\Gamma^{(4)\lambda}_{\mu\mu',\alpha}\,\partial_+X^{\alpha\sigma}+\cdots \end{aligned} \tag{46}$$

The reader should have no trouble in constructing the remaining terms acording to the following rules: There are always two delta functions setting the worldsheet variables σ,σ' and τ equal to each other, and there is one derivative with respect to σ_+ and one derivative with respect σ_-, acting on the delta functions or on X's. Each term contains also a local function of $X(\sigma)$, denoted by Γ's with superscripts. In a similar fashion, using locality and dimensional analysis, the unknown function in the definition of the conformal Killing vector (eq.(7)) can be written as

$$\int d^2\tau\, v(\tau_+)f^{\mu\sigma}_{\tau} = v'(\sigma_+)\left(f^{(1)\mu}_{\nu\lambda}\,\partial_+X^{\nu\sigma}\partial_-X^{\lambda\sigma}+f^{(2)\mu}_{\nu}\,\partial_+\partial_-X^{\nu\sigma}\right). \tag{47}$$

We now substitute eqs.(42),(46) and (47) into (26); the resulting equations are the linear part of the string field equations satisfied by the states at the first massive level. Since these equations are rather lengthy and a knowledge of their detailed form is not particularly important, in what follows some their important general features will be described, and a few of them that are really needed will be written down. First of all, it is useful to exhibit the cutoff dependence of the equations by writing them in the following form:

$$\begin{aligned}&\int d^2\sigma\, v'(\sigma_+)\Big(\partial_+X^{\mu_1\sigma}\partial_+X^{\mu_2\sigma}\partial_-X^{\nu_1\sigma}\partial_-X^{\nu_2\sigma}A^{(1)}_{\mu_1\mu_2,\nu_1\nu_2}(X(\sigma)) \\ &+\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\mu_3}A^{(2)}_{\mu_1\mu_2\mu_3}+\partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2}A^{(3)}_{\mu_1\mu_2} \\ &+\partial_+^2X^{\mu_1}\partial_-X^{\mu_2}\partial_-X^{\mu_3}A^{(4)}_{\mu_1\mu_2\mu_3}+\partial_+^2X^{\mu_1}\partial_-X^{\mu_2}A^{(5)}_{\mu_1\mu_2}\Big)\end{aligned}$$

$$
\begin{aligned}
&+log(\epsilon)\int d^2\sigma\, v'(\sigma_+)\left(\partial_+X^{\mu_1\sigma}\partial_+X^{\mu_2\sigma}\partial_-X^{\nu_1\sigma}\partial_-X^{\nu_2\sigma}B^{(1)}_{\mu_1\mu_2,\nu_1\nu_2}+\cdots\right)\\
&+\frac{1}{\epsilon}\int d^2\sigma\, v'(\sigma_+)\partial_+X^{\mu\sigma}\partial_+X^{\nu\sigma}C_{\mu\nu}(X(\sigma))+\frac{1}{\epsilon^2}\int d^2\sigma\, v'(\sigma_+)D(X(\sigma))\\
&=0.
\end{aligned}
\tag{48}
$$

Without doing any calculation, the general form given above follows again from locality and naive dimensional analysis. The dots stand for terms not written down, which can be obtained by replacing $A^{(i)}$'s by $B^{(i)}$'s in the line above them. We now examine each of these terms in turn:
a) The last two terms have a dependence on derivatives of X quite different from the first two terms and also from each other. As a consequence, it follows that C and D must vanish seperately,

$$C_{\mu\nu}=0,\quad D=0.$$

These equations constrain various pieces of the connection; since they are rather lengthy and they do not contribute to string field equations, which are our main interest, we are not going to write them out explicitly. The important point is that cutoff dependences of the form $1/\epsilon$ and $1/(\epsilon)^2$ have completely dissappeared from the equations.
b) There is still a cutoff dependence of the form $\log(\epsilon)$ in the second term, and since this term has exactly the same structure as the first (cutoff independent) term, we cannot demand that it vanishes seperately. However, after some manipulation of the equations, it is not difficult to show that

$$\int d^2\sigma\left(\partial_+X^{\mu_1\sigma}\partial_+X^{\mu_2\sigma}\partial_-X^{\nu_1\sigma}\partial_-X^{\nu_2\sigma}B^{(1)}_{\mu_1\mu_2,\nu_1\nu_2}+\cdots\right)=\frac{1}{16\pi}S^{(1)}, \tag{49}$$

where $S^{(1)}$ is given by eq.(42). Since $S^{(1)}$ is multiplied by the slope parameter b (see eq.(9)), we can get rid of the logarithmic cutoff dependence by redefining b:

$$b'=b+\frac{\log(\epsilon)}{16\pi} \tag{50}$$

Exactly the same redefinition also eliminates the $\log(\epsilon)$ dependence from the equations for the tachyon and for the higher massive levels. Therefore, the renormalization of the slope parameter, which gets rid of terms proportional to $\log(\epsilon)$ is the only renormalization needed ; all the cutoff dependence drops out of the equations automatically.

c) Having disposed of all the cutoff dependent terms in (48), we are left with five cutoff independent equations

$$A^{(i)}_{\mu_1\mu_2,\nu_1\nu_2} = 0, \quad i = 1, 2, .., 5.$$

Three of these equations, those involving $A^{(5)}$, $A^{(2)}$ and $A^{(3)}$, provide further constraints on the connection and the conformal Killing vector, whereas the remaining two, involving $A^{(1)}$ and $A^{(4)}$, can be written exclusively in terms of the string field $e_{\mu_1\mu_2,\nu_1\nu_2}$. As the discussion leading to eq. (44) shows, among the fields of the first massive level given by eq.(42), this field is the only physical one, so we expect that the string field equations should finally be expressible in terms of only this field. Taking into account the σ_- counterpart of (48) and simplifying, we have,

$$\Box e_{\mu_1\mu_2,\nu_1\nu_2} + 16\pi b' \, e_{\mu_1\mu_2,\nu_1\nu_2} = 0, \tag{51}$$

and,

$$\begin{aligned} \partial_\nu e_{\mu_1\mu_2,\nu\nu_1} - \frac{1}{6}\,\partial_{\nu_1} e_{\mu_1\mu_2,\nu\nu} &= 0, \\ \partial_\mu e_{\mu\mu_1,\nu_1\nu_2} - \frac{1}{6}\,\partial_{\mu_1} e_{\mu\mu,\nu_1\nu_2} &= 0. \end{aligned} \tag{52}$$

The above are indeed the correct equations for the first massive level of the closed string. In case of eq.(49), this is obvious; on the other hand, eqs.(52) may not look familiar. This is because, in writing down eq.(42), we have made use of linear gauge transformations of the form of eq.(3) to eliminate some spurious states. In the next section, we will show that, in the string language, these correspond to gauges generated by the operators L_{-1} and $\bar{L}_{-1}$. What we have done amounts to explicitly solving the string equations

$$L_1|s>= \bar{L}_1|s>= 0.$$

Eqs.(50) are then equivalent to the remaining string equations

$$L_2|s>= \bar{L}_2|s>= 0.$$

In the standard string approach, one starts with the redundant set of fields $E_{\mu_1\mu_2,\nu_1\nu_2}$, $E_{\mu,\nu_1\nu_2}$, $E_{\mu_1\mu_2,\nu}$ and $E_{\mu\nu}$ (see Appendix B of [1]) without initially imposing any string equations. The connection between our $e_{\mu_1\mu_2,\nu_1\nu_2}$ and the string field $E_{\mu_1\mu_2,\nu_1\nu_2}$ is

$$e_{\mu_1\mu_2,\nu_1\nu_2} = E_{\mu_1\mu_2,\nu_1\nu_2} + \frac{5}{12}\left(\eta_{\mu_1\mu_2}E_{\mu\mu,\nu_1\nu_2} + \eta_{\nu_1\nu_2}E_{\mu_1\mu_2,\nu\nu}\right) + \frac{5}{48}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}E_{\mu\mu,\nu\nu}. \tag{53}$$

It is then not difficult to show that the free string equations for the E's are equivalent to eqs.(52).

At this point, we would like to compare the results obtained here for the first massive level to the results of [1]. The main difference is that in [1], only left-right symmetric models could be treated, whereas here there is no such restriction. Also, the restriction that the coordinate transformations of eq.(4) should have unit determinant has been removed. These restrictions were due to an improper choice of the connection in [1]; they dissappear when the connection is freed from any a priori constraint.

Finally, in closing this section, let us try to understand how starting with a non-renormalizable action (eq.(42)) and a largely arbitrary connection (eq.(46)), we were able to derive unique and cutoff independent equations. This result follows from.the structure of eqs.(48). Since the theory is non-renormalizable, there is a singular dependendence on the cutoff, in the form of terms proportional to $1/\epsilon$ and $1/\epsilon^2$. Because of their different structure, however, these terms satisfy seperate equations and never mix with finite terms or terms proportional to $log(\epsilon)$. Dimensional analysis dictates the structure of these terms; each additional power of ϵ must go with an additional derivative with respect to the world sheet coordinate. Moreover, the equations proportional to $1/\epsilon$ and $1/\epsilon^2$ determine partially only the connection and the conformal Killing vector; they impose no constraints on the string fields. The remaining equations are similar those coming from a renormalizable theory; the $log(\epsilon)$ is absorbed into slope renormalization, and at the end, one is left with the cutoff independent equations of the form

$$A^{(i)}_{\mu_1\mu_2,\nu_1\nu_2} = 0. \tag{54}$$

Furthermore, these equations can be neatly seperated into two sets: Those that receive contribution from the connection and the Killing vector and those that do not. The structures of these two sets are different; the first set of terms can be written in the form

$$\sum_{m,n}\int d^2\tau\, Q^{m,n}_{\lambda\tau}\, \partial_+^m\partial_-^n X^{\lambda\tau}, \tag{55}$$

where both m and n are integers ≥ 1. The second set consists of terms that cannot be written in this form. Or, stated otherwise, the first set of terms vanish upon imposing the free field equations (45), whereas the second set of terms do not. Glancing at eq.(48), we see that $A^{(2)}$, $A^{(3)}$ and $A^{(5)}$ belong to the first set and therefore only they receive contributions from the connection and the conformal Killing vector. On the other hand, $A^{(1)}$ and $A^{(4)}$ belong to the second set, and setting them equal to zero yields the string field equations (51, 52). It is not too difficult to see why the connection and the Killing vector contribute only to the first set: Both contributions include a factor

$$\frac{\delta S^{(0)}}{\delta X^{\mu\sigma}} = -2\,\partial_+\partial_- X^{\mu\sigma},$$

which vanishes when the free field equations (45) are used. In the next section, we shall see that the same seperation into two sets also works in the case of all the higher levels.

5. Linear Equations For All Levels

In this section, we shall show that, the standard free string equations

$$\begin{aligned}(L_0 - 1)|s> &= 0, \quad (\bar{L}_0 - 1)|s>= 0,\\ L_n|s> &= 0, \quad \bar{L}_n|s>= 0,\end{aligned} \tag{56}$$

can be derived from eq.(26). In the previous two sections, we have already shown this for the massless and the first massive levels. Here, we present a general proof that applies to all the levels. In constructing the proof, we will make use of the following results of the last section:

a) Only the cutoff independent part of eq.(26) gives useful information about the string states; the cutoff dependent equations proportional to inverse powers of ϵ provide only constraints on the connection and the Killing vector. This result can be established for the higher levels without much trouble by appropriately generalizing eqs.(46),(47) and (48) to these states. Just as in the case of the first massive level, the number of constraints on the connection, for example, are far fewer than the number of allowed components of the connection, and therefore, the connection is only partially fixed.

b) The terms that depend logarithmically on ϵ turn out to be proportional to the action, with the same constant of proportionality as in eq.(49). They are eliminated by slope renormalization.

c) The cutoff independent equations can be split into the two sets discussed at the end of the last section (see eq.(55)). Only the second set of equations, which do not vanish upon imposing the free field equations (45), are free of the connection and the Killing vector and hence lead to useful string field equations. This is established by the same argument given at the end of the last section.

Let us now extract these "useful" equations from (26), by first eliminating all of the cutoff dependent terms and terms belonging to the first set. The third and the last terms on the left hand side of this equation are purely cutoff dependent and can be dropped. One can also drop the terms proportional to the connection and the Killing vector, since they belong to the first set. The only remaining term with ϵ dependence is the second term, which, in addition to singular pieces, contains a cutoff independent subterm. To extract it, we note that, because of locality, H will turn out to be the sum of terms proportional to $\delta^2(\sigma-\sigma')$ or some order derivative of it with respect to σ_+ and σ_-. The terms in H that are proportional to the delta function without the derivatives have cutoff independendent contributions. This follows from eq.(15) for $\tilde{G}^{(0)}$:

$$\epsilon\, \tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon)\delta^2(\sigma-\sigma') = \frac{1}{4\pi}\,\eta_{\mu\mu'}\,\delta^2(\sigma-\sigma').$$

For this term, we can replace $\tilde{G}^{(0)}$ by $1/4\pi\eta_{\mu\mu'}$. On the other hand, in the case of delta function with derivatives, we can integrate by parts with respect to either σ or σ' to transfer all the derivatives on the prefactor in front of H. If any of the derivatives act on $\tilde{G}^{(0)}$, it again follows from eq.(15) that the result either vanishes or is proportional to an inverse power of ϵ. These are the cutoff dependent contributions and they can therefore safely be dropped. The only term that survives is the one where all the derivatives act on $\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}$, and $\tilde{G}^{(0)}$ is multiplied by a delta function without derivatives. In this term, $\tilde{G}^{(0)}$ can again be replaced by $1/4\pi$, and the derivatives acting on $\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}$ can then be shifted back on H. All this amounts to simply replacing $\tilde{G}^{(0)}$ by $1/4\pi\,\eta_{\mu\mu'}$. As a result, we arrive at a cutoff independent relation:

$$b'\int d^2\sigma\, v(\sigma_+)\partial_+ X^{\mu\sigma}\frac{\delta S^{(1)}}{\delta X^{\mu\sigma}} + \frac{1}{16\pi}\int d^2\sigma \int d^2\sigma'\,\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}\,\frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}} \cong 0. \tag{57}$$

Although cutoff dependent terms have disappeared, there are still terms belonging to the first set that have to be eliminated. This explains the need for the $\cong$ sign; the above equation is in reality an equivalence relation modulo terms of the first set, namely, terms which vanish on the free field equations. The source of these unwanted terms is the structure of $S^{(1)}$, to see this, we split it into two pieces:

$$S^{(1)} = S_f + S_s,$$

where S_f can be written in the same form as (55),

$$S_f = \sum_{m,n} \int d^2\sigma \, N^{(m,n)}_{\mu\sigma} \, \partial_+^m \partial_-^n X^{\mu\sigma}, \tag{58}$$

with m and n ≥ 1, and S_s is the rest. It is natural to expect that S_f will contribute terms of the first set to eq.(57). We will now show that this is indeed the case by making use of the identity

$$\frac{\delta^2}{\delta X^{\mu\sigma} \delta X^{\mu\sigma'}} \left(\sum_{m,n} \int d^2\tau \, N^{(m,n)}_{\lambda\tau} \, \partial_+^m \partial_-^n X^{\lambda\tau} \right)$$
$$\sum_{m,n} \int d^2\tau \, \frac{\delta^2 N^{(m,n)}_{\lambda\tau}}{\delta X^{\mu\sigma} \delta X^{\mu\sigma'}} \, \partial_+^m \partial_-^n X^{\lambda\tau} + \sum_{m,n} \partial_{\sigma_+}^m \partial_{\sigma_-}^n (\cdots)$$
$$+ \partial_{\sigma'_+}^m \partial_{\sigma'_-}^n (\cdots). \tag{59}$$

We have not written out the explicitly the terms represented by dots, since they do not contribute to eq.(57). To see this, consider the term with derivatives with respect to $\sigma_\pm$. There is at least one derivative each with respect to σ_+ and σ_-. Upon integration by parts, this will kill the factor $\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}$. A similar argument goes through for terms derivatives with respect to $\sigma'_\pm$, and also for the equation which is the σ_- counterpart of (57). The remaining term clearly belongs to the first set. It is therefore justified to drop the contribution of S_f altogether, and replace $S^{(1)}$ by S_s in (57). There is, however, an ambiguity in the seperation of $S^{(1)}$ into S_f and S_s that we have just outlined. This seperation depends on the form of the integrand, as in eq.(58), and partial integration may convert an integrand that appears to belong to the second set into one of the first set. One way to eliminate this ambiguity is to completely fix the linear gauges generated by integration by parts (eq.(3)), as we have done in writing eq.(42). This is not a practical procedure in the case of higher levels, so, we shall leave this gauge ambiguity unfixed for the time being. Later, we shall see that, in the string language, it corresponds to the gauge transformations generated by L_{-1} and $\bar{L}_{-1}$.

Eq.(57) has exactly the same form as the linear part of renormalization group equation derived in reference [10]. The only difference, but an important one, is that, $S^{(1)}$ has to be replaced by S_s. This replacement gets rid of spurious terms which vanish upon imposing the free field equations (45). However, the non-linear terms in the equation derived in this paper, eq.(23), appear to be different from the quadratic interaction term given in [10].

The next step is to translate eq.(57), with $S^{(1)}$ replaced by S_s, into the string language, in order to compare with (56). We shall write the analogue of eq.(1) for S_s in the form

$$S_s = \int d^2\sigma \, |s, \sigma>, \tag{60}$$

where the integrand is represented by a state labeled by s, which can be built from the "vacuum" by applying creation operators. These operators stand for the derivatives of X with respect to σ_+ and σ_-:

$$\partial_+^m X^{\mu\sigma} \leftrightarrow \alpha_m^{\dagger\mu}, \quad \partial_-^n X^{\mu\sigma} \leftrightarrow \bar{\alpha}_n^{\dagger\mu}. \tag{61}$$

The sigma dependence of the operators α and $\bar{\alpha}$ has been suppressed to simplify writing. For example, the state corresponding to the graviton (eq.(28)) is

$$|s, \sigma> = \tilde{h}_{\mu\nu}(X(\sigma)) \, \alpha_1^{\dagger\mu} \bar{\alpha}_1^{\dagger\nu} |0> . \tag{62}$$

We note that, since by definition, no mixed derivatives of X, such as $\partial_+ \partial_- X^{\mu\sigma}$, appear in S_s, one can write the most general S_s in terms of the operators defined above. The situation here closely parallels the standard quantization of the modes of the free string. There is, however, a difference in the way the integers m and n in eq.(61) are assigned; in the standard string quantization, these would stand for the Fourier modes of X. Here, they represent the number of derivatives acting on X, more in parallel with the representation of the vertex operator.

We will now rewrite the linear gauge transformations in the operator language, by noticing that the derivatives with respect to σ_+ and σ_-, acting on a state, can be represented by

$$\begin{aligned} \partial_{\sigma+} \to L_{-1} &= \alpha_1^{\dagger\mu} \partial_\mu + \sum_{m=1}^{\infty} \alpha_{m+1}^{\dagger\mu} \alpha_{m,\mu}, \\ \partial_{\sigma_-} \to \bar{L}_{-1} &= \bar{\alpha}_1^{\dagger\mu} \partial_\mu + \sum_{n=1}^{\infty} \bar{\alpha}_{n+1}^{\dagger\mu} \bar{\alpha}_{n,\mu}, \end{aligned} \tag{63}$$

and so (3) can be rewritten as

$$|s,\sigma>\to|s,\sigma>+L_{-1}|s_+,\sigma>+\bar{L}_{-1}|s_-,\sigma>. \tag{64}$$

In the above equations, ∂_μ acts on the argument $X(\sigma)$ of a wavefunction such as $\tilde{h}$ in eq.(62). We have also introduced annihilation operators with the standard commutation relations

$$[\alpha^\mu_m,\alpha^{\dagger\nu}_n]=\eta^{\mu\nu}\,\delta_{m,n},\quad [\bar{\alpha}^\mu_m,\bar{\alpha}^{\dagger\nu}_n]=\eta^{\mu\nu}\,\delta_{m,n}.$$

Eq.(63) for L_{-1} and $\bar{L}_{-1}$ is not the familiar one given in string theory, but one can recover the standard form by the following scaling which preserves the commutation relations:

$$\alpha^\mu_m=\frac{1}{\sqrt{2m}\,(m-1)!}\,a^\mu_m,\quad \alpha^{\dagger\mu}_m=\sqrt{2m}\,(m-1)!\,a^{\dagger\mu}_m, \tag{65}$$

and similarly for the barred operators. Here, we are guilty of an abuse of notation; a and $a^\dagger$ are Hermitian conjugates, as the notation indicates, whereas α and $\alpha^\dagger$ are not. In what follows, we will nevertheless continue using the α's, since the resulting formulas look somewhat simpler.

With these preliminaries over, we will first recast the first term in (57) into the operator language. A straightforward calculation gives

$$\begin{aligned}&\int d^2\sigma\, v(\sigma_+)\partial_+X^{\mu\sigma}\frac{\delta S_s}{\delta X^{\mu\sigma}}\\ &=\int d^2\sigma\Big(\sum_{m=1}^{\infty}\sum_{k=1}^{m}v^{(k)}(\sigma_+)\frac{m!}{k!(m-k)!}\alpha^{\dagger\mu}_{m-k+1}\alpha_{m,\mu}-v'(\sigma_+)\Big)|s,\sigma>\\ &=\int d^2\sigma\, v'(\sigma_+)\sum_{m=1}^{\infty}\sum_{k=1}^{m}\Big((-1)^{k-1}\frac{m!}{k!(m-k)!}L^{k-1}_{-1}\alpha^{\dagger\mu}_{m-k+1}\alpha_{m,\mu}\\ &-1\Big)|s,\sigma>\end{aligned} \tag{66}$$

where $v^{(m)}$ stands for the m'th derivative of v with respect to its argument. The last step follows upon integration by parts and by replacing the derivatives with respect to σ_+ by L_{-1}, as in eq.(63).

We now turn our attention to the second term in (57), and convert the integrand of this term into an operator expression:

$$\frac{\delta^2 S_s}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}}=\left(\delta^2(\sigma-\sigma')\Box+\sum_m(\partial^m_{\sigma_+}\delta^2(\sigma-\sigma'))\alpha^\mu_m\,\partial_\mu\right)|s,\sigma>$$

$$+ \quad \sum_m (-1)^m \partial^m_{\sigma_+} \left(\left(\delta^2(\sigma - \sigma')\alpha^\mu_m \partial_\mu + \sum_k \partial^k_{\sigma_+} (\delta^2(\sigma - \sigma'))\alpha^\mu_m \alpha_{k,\mu} \right) |s, \sigma > \right)$$
$$+\partial_{\sigma_-}(\cdots) + \partial_{\sigma'_-}(\cdots) + I_f. \qquad (67)$$

The terms represented by dots have not been written out, since they will drop out after muliplication by $\frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}$ and integration over σ and σ'. Also, the terms represented by I_f belong to the first set (eq.(55)), and as explained earlier, they do not contribute to the equations for the physical fields. Using this result and also the identity

$$\partial^m_{\sigma_+} \partial^k_{\sigma'_+} \left(\frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \right)_{\sigma = \sigma'} = \frac{m!k!}{(m+k+1)!}\, v^{(m+k+1)}(\sigma_+),$$

we have,

$$\int d^2\sigma \int d^2\sigma' \,\frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \frac{\delta^2 S_s}{\delta X^{\mu\sigma} \delta X^{\mu\sigma'}}$$
$$= \int d^2\sigma \Big(v'(\sigma_+)\, \Box + \sum_{m=1}^{\infty} v^{(m+1)}(\sigma_+)\, \frac{2}{m+1}\, \alpha^\mu_m\, \partial_\mu$$
$$+ \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} v^{(m+k+1)}(\sigma_+)\, \frac{m!k!}{(m+k+1)!}\, \alpha^\mu_m \alpha_{k,\mu} \Big) |s, \sigma >$$
$$= \int d^2\sigma\, v'(\sigma_+) \Big(\Box + 2 \sum_m \frac{(-1)^m}{m+1}\, L^m_{-1}\, \alpha^\mu_m\, \partial_\mu$$
$$+ \sum_m \sum_k \frac{(-1)^{m+k} m!k!}{(m+k+1)!}\, L^{m+k}_{-1}\, \alpha^\mu_m \alpha_{k,\mu} \Big) |s, \sigma > . \qquad (68)$$

In arriving at this result, again integration by parts and eq.(63) has been used. Finally, combining eqs.(66) and (68) gives us the following operator version of (57):

$$T|s >= 0, \qquad (69)$$

where

$$T = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!}\, L^m_{-1} L_m - 1 \right).$$

For convenience, we have set $16\pi b' = 1$, which differs from the conventional slope normalization by a factor of two. We have also introduced the conformal operators L_0 and L_m, with $m > 1$:

$$
\begin{aligned}
L_0 &= \Box + \sum_{k=1}^{\infty} k\alpha_k^{\dagger\mu}\alpha_{k,\mu} - 1, \\
L_m &= \frac{2}{m!}\,\alpha_m^{\mu}\,\partial_\mu + \sum_{k=1}^{m-1}\frac{k!(m-k)!}{m+1}\,\alpha_{m-k}^{\mu}\alpha_{k,\mu} \\
&+ \sum_{k=1}^{\infty}\frac{(k+m)!}{(k-1)!}\,\alpha_k^{\dagger\mu}\alpha_{k+m,\mu}.
\end{aligned} \tag{70}
$$

Converting the α's into the a's through eq.(65), the L's defined above are readily identified with the usual Virasoro operators of string theory, with the standard commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{71}$$

Eq.(69) is the main result of this section. It is to be supplemented by its left moving counterpart, where α's are replaced by $\bar{\alpha}$'s and L's by $\bar{L}$'s. These two equations are then the linearised form of the string field equations. It is easy to check directly that they are invariant under the gauge transformations of eq.(64); this follows from the identity

$$TL_{-1} = 0, \tag{72}$$

where T is the operator defined in eq.(69). In their present form, these equations ((69) and its left moving counterpart) look quite different from the standard string equations (56); for one thing, there are only two equations instead of an infinite set. In the next section, we will show that, by a suitable gauge fixing, the standard string equations follow from (69).

6. Derivation Of The Standard String Equations

The goal of this section is to show that the equations with L's in (56) follow from eq.(69). Since the derivation of the equations with $\bar{L}$'s is exactly the same, we will not consider them any further in this section. Let us first write (69) in the form

$$(L_0 - 1)|s> = L_{-1}|s'> .$$

Since the right hand side of this equation is pure gauge, the left hand side will also be pure gauge, except for states satisfying

$$(L_0 - 1)|s> = 0, \tag{73}$$

for which $L_0 - 1$ is not invertible. This is the mass shell condition for physical states, which amounts to a partial choice of gauge. We note that a rectricted set of gauge transformations, which preserve the mass shell condition, are still allowed. Combining (69) with (73) gives

$$U|s>=\sum_{m=1}^{\infty}\frac{(-1)^m}{(m+1)!}L_{-1}^{m-1}L_m|s>=0. \tag{74}$$

In arriving at this equation, we have used the fact that, if

$$L_{-1}|>=0,$$

acting on a state $|>$, then that state must vanish. Let us now grade the states by the eigenvalues n of the number operator

$$N=\sum_{m=1}^{\infty}m\,\alpha_m^{\dagger\mu}\alpha_{m,\mu}.$$

The eigenvalues are then the level numbers. The advantage of labeling the states by the level number follows from the fact that N commutes with T, and so, the states with different level numbers satisfy seperate equations of the form (69). We will now work out the consequences of this equation, or, equivalently, of eqs.(73) and (74) for a few small values of n, starting with $n = 0$. The state $|0>$ represents the tachyon, it is annihilated by all the α's, and it corresponds to $p^2 = -1$. For the next state, at $n = 1$ and $p^2 = 0$, eq.(74) gives the constraint

$$L_1|1>=0.$$

When combined with their left moving (barred) counterparts, these are then the full set of string equations for the massless states.

Before going on to the next level, we observe that, since L_1 and L_{-1} are mutually adjoint operators, any state can be decomposed as

$$|s>=|s'>+L_{-1}|s''>,\quad L_1|s'>=0. \tag{75}$$

Using the gauge freedom (eq.(64)), it is therefore always possible to impose the condition

$$L_1|s>=0,$$

on any state. If we impose this condition on the first massive level at $n = 2$, then eq.(74), which in this case is

$$(L_1-\frac{1}{3}L_{-1}L_2)|2>=0,$$

gives

$$L_2|2>=0.$$

We have thus recovered the full set of string equations for the level at $n = 2$.

Up to this point, the situation has been relatively simple, but at the next level at $n = 3$, new technical problems arise. Using the string language, what we need is the decomposition of an arbitrary state into a "physical" state, plus a number of "spurious" states [24,25]. This decomposition, a generalization of (75), reads

$$|s>=|p>+|sp>. \tag{76}$$

The physical state $|p>$ satisfies the subsidiary conditions of eq.(56); it is annihilated by all L_n's with $n \geq 1$. On the other hand, the spurious states $|sp>$ are formed by applying the product of various powers of L_{-n}'s, again with $n \geq 1$, on a physical state. Since the L's do not commute, it is convenient to order this product according to increasing values of n. A general spurious state can be written as

$$|sp>=L_{-1}^{n_1}L_{-2}^{n_2}L_{-3}^{n_3}\cdots|>,$$

where the state $|>$ on the right is annihilated by all L_n's with $n \geq 1$. In our case, we can set $n_1 = 0$, since the terms with $n_1 \geq 1$ can be eliminated by the gauge transformation generated by L_{-1} (eq.(64)). After this gauge fixing, the decomposition (76) can be written as

$$\begin{aligned}|s,n> &= |p,n>+\Big(c_1 L_{-2}^{n_2}L_{-3}^{n_3}L_{-4}^{n_4}\cdots+c_2 L_{-3}^{n'_3}L_{-4}^{n'_4}L_{-5}^{n_5}\cdots \\ &+ c_3 L_{-4}^{n''_4}L_{-5}^{n''_5}\cdots+\cdots\Big)|m>+(\tilde{c}_1 L_{-2}^{\tilde{n}_2}\cdots+\cdots)|\tilde{m}>+\cdots\end{aligned} \tag{77}$$

In this equation, the c's are constants, and the integers n,m, etc. are the level numbers of the states. The right hand side is a sum over all the possible spurious states with level number n. We are now going to apply eq.(74) to the state $|s,n>$. It is easily verified that the physical state $|p,n>$ satisfies this equation, and if we could show that all the c's must vanish, then we would have reached our goal of establishing the string equations (56). Before tackling the general problem, as a simple example, let us now consider the case $n = 3$,

$$|s,3>=|p,3>+c_1 L_{-2}|1>+c_2 L_{-3}|0>. \tag{78}$$

The state $|p, 3>$ satisfies eq.(74), and because $|1>$ and $|2>$ satisfy the physical state conditions and they have different level numbers, each one must satisfy a seperate equation:

$$\begin{aligned} UL_{-2}|1> &= \left(-\frac{1}{2}L_1 + \frac{1}{6}L_{-1}L_2\right) L_{-2}|1>= 0, \\ UL_{-3}|0> &= \left(-\frac{1}{2}L_1 + \frac{1}{6}L_{-1}L_2 - \frac{1}{24}L_{-1}^2 L_3\right) L_{-3}|0>= 0. \end{aligned} \tag{79}$$

Using the algebra the L's satisfy (eq.(71)), the second equation can be simplified to

$$\left(4\, L_{-2} + (\frac{c}{6} - \frac{8}{3})L_{-1}^2\right) |0>= 0.$$

This equation is clearly impossible, since the two states are linearly independent and they cannot add up to zero. Hence, we must set $c_2 = 0$ in (78). Similarly, the first equation in (79) gives

$$(c - 26)L_{-1}|1>= 0.$$

Away from the critical dimension $c = 26$, we can conclude that $c_1 = 0$, reaching our goal. However, at $c = 26$, there is an ambiguity in the definition of the physical state; it is possible to add to it a multiple of the state $L_{-2}|1>$. We note that this state is gauge equivalent to the zero norm state

$$|z>= \left(L_{-2} + \frac{3}{2}L_{-1}^2\right) |1>,$$

and that $|z>$ satisfies the string equations (56). The conclusion is that, even though the physical state is not unique, nevertheless eq.(69) and gauge invariance under (64) still imply the standard string equations. The possibility of adding zero norm states to a physical state is well known from the theory of the critical string [24,25].

We would like now to apply the experience gained by working out these special cases to the general expansion (77). Let us first sketch our strategy. As we have noticed in working out the examples, eq.(74) applied to (77) gives rise to several seperate equations; one for each different state $|m>$, $|\tilde{m}>$, etc. The idea is to pick a generic equation, and try to isolate a term from it which has a different structure from the rest of the terms. Such a term has to vanish all by itself. The next step is to iterate this procedure and construct an inductive argument. What follows is an outline of the various steps of the argument: We first apply the operator U (eq.(74)) to the right hand side of (77), and rearrange the products over powers of L_{-n}'s so that n increases from left to right, just as in (77). This is done using the commutation relations of L's to move L_n's for positive n to the right till they hit the state $|m>$ and annihilate it. The result is a complicated sum with many terms; however, one term among all others is easy to isolate; it comes only from the application of the first term in U, $-\frac{1}{2}L_1$, to the first term in (77):

$$\begin{aligned} U|s,n> &\simeq -\frac{1}{2}c_1\, L_1 L_{-2}^{n_2} L_{-3}^{n_3}\cdots|m> \\ &= \Big(-2\, n_3\, c_1\, L_{-2}^{n_2+1} L_{-3}^{n_3-1}\cdots+\cdots\Big)\,|m>\,. \end{aligned} \tag{80}$$

Since this contribution has to vanish all by itself, we conclude that $n_3 = 0$. After setting $n_3 = 0$ in (77), we next isolate a term of the form

$$\begin{aligned} U|s,n> &\simeq \frac{1}{6}c_1\, L_{-1}L_2 L_{-2}^{n_2} L_{-4}^{n_4}\cdots|m> \\ &= \Big(n_4\, c_1\, L_{-1} L_{-2}^{n_2+1} L_{-4}^{n_4-1}\cdots+\cdots\Big)\,|m>\,. \end{aligned} \tag{81}$$

This term, which is again unique, is generated by the application of the second term in U to the first term in the expansion of $|s,n>$. Since it cannot be cancelled by any other term, it must again vanish all by itself, leading to the result that $n_4 = 0$. Continuing this line of reasoning, it is easy to show that all the n's except for n_2 must vanish. We can therefore rewrite eq.(77) in the following form:

$$|s,n>= \Big(c_1\, L_{-2}^{n_2} + c_3\, L_{-2}^{n_2-1}L_{-4} + \cdots\Big)\,|m> + \cdots. \tag{82}$$

We note that the form of the non-leading terms are severely restricted, since their grading with respect to the number operator must match the grading of the leading term, which is $2\,n_2$. Again applying U to $|s,n>$ given above, we identify two terms which must vanish individually:

$$\begin{aligned} U|s,n> &\simeq \Big(-\frac{1}{2}L_1 + \frac{1}{6}L_{-1}L_2\Big)|s,n> \\ &\simeq \Big(\Big(-\frac{1}{2}(3\,n_2 + \frac{4}{3}n_2^2 - \frac{1}{6}c\,n_2)c_1 + c_3\Big)L_{-1}L_{-2}^{n_2-1} \\ &+ \Big(\frac{3}{4}n_2(n_2-1)c_1 - \frac{5}{2}\,c_3\Big)L_{-2}^{n_2-2}L_{-3}\Big)|m>\,. \end{aligned} \tag{83}$$

The resulting equations

$$3\,n_2(n_2-1)c_1 - 10\,c_3 = 0,$$

and

$$(3\,n_2 + \frac{4}{3}n_2^2 - \frac{1}{6}c\,n_2)c_1 - 2\,c_3 = 0,$$

have the only non-trivial solution $n_2 = 1$, $c_3 = 0$, in the critical dimension $c = 26$. This solution, which can be absorbed into into the physical state by a redefinition, was discussed following eq.(78). We can therefore conclude that

$$c_1 = 0, \quad c_3 = 0.$$

Incorporating this result into the expansion (77) gives

$$|s,n>= \left(c_4\,L_{-3}^{n_3}L_{-4}^{n_4}\cdots + c_5\,L_{-3}^{n_3-1}L^{n_4'}\cdots + \cdots\right)|m> . \tag{84}$$

The absence of factors which contain L_{-2} simplies greatly the next step in the argument. We again isolate a unique term from eq.(74):

$$\begin{aligned} U|s,n> &\simeq -\frac{1}{2}L_1\left(c_4\,L_{-3}^{n_3}L_{-4}^{n_4}\cdots + \cdots\right)|m> \\ &\simeq \left(-2\,n_3\,c_4\,L_{-2}L_{-3}^{n_3-1}\cdots + \cdots\right)|m> . \end{aligned} \tag{85}$$

For this term to vanish, n_3 must be equal to zero. This line of resoning can be continued inductively to show that all the spurious states on the right hand side of eq.(77) are absent, leaving behind only the physical state. Since the physical states by definition satisfy the string field equations of (56), we have therefore succeded in deducing these equations from eq.(69).

7. Gravity To Higher Orders

So far, we have only studied the linearized form of the string field equations. In this section, we will consider higher order interaction terms of the massless fields. To keep the discussion simple, we restrict ourselves to a symmetric metric, with $\tilde{h}_{\mu\nu} = h_{\mu\nu} = h_{\nu\mu}$ in eq. (28), and drop the antisymmetric tensor B, although there is no real difficulty in treating the general case. We have already shown in section 3 that the linear terms in the equations of motion are those of gravity coupled to a dilaton, and in view of the postulated invariance under coordinate transformations (30), one would expect that the full non-linear set of equations will also turn out to be Einstein's equations for the dilaton-graviton system. We think, nevertheless, that it is worthwhile to verify that the field equations come out correctly for several reasons. For one thing, it is important to demonstrate that the scheme of regularization we are using respects the covariance under coordinate transformations. Also, it is nice to have a check on the method of using an initially undetermined connection. In the linear approximation, it turned out that the connection could be completely eliminated from the final field equations, leaving behind only the contribution of the dilaton field. We would like to verify that the same thing continues to hold true when the higher order terms are taken into account. Finally, the computation carried out in this section is a necessary preliminary to an explicit construction of a full set cutoff independent non-linear equations for all the levels of the string. Although this latter problem is not considered in this paper, we hope to return to it in a future publication.

The starting point is eq.(23), with the action given by eq.(28) and a symmetric $\tilde{h}_{\mu\nu} = h_{\mu\nu}$. We can simplify this equation simplify using eq.(32) and setting $f^{\mu\sigma}_{\sigma'} = 0$. With these simplifications, eq.(23) becomes

$$W^{(1)} + W^{(2)} + E_M = 0, \tag{86}$$

where,

$$\begin{aligned} W^{(1)} &= \frac{1}{4}\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\, \epsilon\, \tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon) H_{\mu\sigma,\mu'\sigma'}, \\ W^{(2)} &= -\frac{1}{8}\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \\ &\times \int_0^\epsilon dt'(\epsilon - t')\tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon - t') \left(H\tilde{G}(t')H\right)_{\mu\sigma,\mu'\sigma'}. \end{aligned} \tag{87}$$

$W^{(1)}$, the linear term, was already computed in section 3, so we turn our attention to the second term, $W^{(2)}$, which contains all the non-linear contribution . In order to keep the exposition simple, we will present here only the details of the computation of the quadratic terms in the fields in $W^{(2)}$, although it not too difficult to treat higher order terms by the same method we are using. We will also not carry out the calculation of the determinental term E_M to higher orders. Just as in the linear case (eq.(34)), E_M turns out to be purely cutoff dependent also in the higher orders, and it is cancelled by the cutoff dependent parts of $W^{(1)}$ and $W^{(2)}$. This is the same as computing the higher order corrections to eq.(36), and since we are only interested in computing the higher order contributions to eq.(35), we will not consider E_M any further.

The part of $W^{(2)}$ quadratic in the fields is given by

$$\begin{aligned} W^{(2)} &\simeq -\frac{1}{8}\int d^2\sigma \int d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \\ &\times \int_0^\epsilon dt'(\epsilon - t')\tilde{G}^{(0)}_{\mu'\sigma',\mu\sigma}(\epsilon - t')\left(H\tilde{G}^{(0)}(t')H\right)_{\mu\sigma,\mu'\sigma'}. \end{aligned} \tag{88}$$

From its definition (first line of eq.(27)), H can be expressed in terms of the metric and the connection:

$$\begin{aligned} H_{\mu\sigma,\mu'\sigma'} &= \partial_{\sigma_+}\partial_{\sigma_-}\left(\delta^2(\sigma - \sigma')A_{\mu\mu'}\right) \\ &+ \partial_{\sigma_+}\left(\delta^2(\sigma - \sigma')\partial_- X^{\lambda\sigma}A_{\mu\mu',\lambda}\right) + \partial_{\sigma_-}\left(\delta^2(\sigma - \sigma')\partial_+ X^{\lambda\sigma}A_{\mu\mu',\lambda}\right) \\ &+ \delta^2(\sigma - \sigma')\left(\partial_+ X^{\lambda\sigma}\partial_- X^{\lambda'\sigma}\, B_{\mu\mu',\lambda\lambda'} + \partial_+\partial_- X^{\lambda\sigma}\, B_{\mu\mu',\lambda}\right), \end{aligned} \tag{89}$$

where the A's and the B's are given by

$$\begin{aligned} A_{\mu\nu} &= -2\, h_{\mu\nu}, \\ A_{\mu\nu,\lambda} &= \partial_\lambda h_{\mu\nu} - \partial_\nu h_{\mu\lambda} + \partial_\mu h_{\nu\lambda}, \\ B_{\mu\nu,\lambda\lambda'} &= \partial_\mu\partial_\nu h_{\lambda\lambda'} - \partial_\mu\partial_\lambda h_{\nu\lambda'} - \partial_\mu\partial_{\lambda'} h_{\nu\lambda} \\ &+ \Gamma^\eta_{\mu\nu}(\partial_\lambda h_{\eta\lambda'} + \partial_{\lambda'} h_{\eta\lambda} - \partial_\eta h_{\lambda\lambda'}), \\ B_{\mu\nu,\lambda} &= -2\,\partial_\mu h_{\nu\lambda} + 2\,\Gamma^\eta_{\mu\nu}\, g_{\eta\lambda}. \end{aligned} \tag{90}$$

Substituting this in eq.(85) expresses the quadratic contributions in terms of the metric and the connection. This is not the end of the story, however, since we still have to extract the finite and cutoff dependent terms from this expression in the limit of $\epsilon \to 0$. We observe that the cutoff dependendence comes from a factor of the form

$$D^{(m,n)}_{\sigma,\sigma'}(\epsilon) = \int_0^\epsilon dt'(\epsilon - t')\tilde{G}^{(0)}_{\sigma',\sigma}(\epsilon - t')\partial^m_{\sigma_+}\partial^n_{\sigma_-}\left(\tilde{G}^{(0)}_{\sigma,\sigma'}(t')\right). \tag{91}$$

The derivatives acting on the second $\tilde{G}$ come from the derivatives of delta functions in the expression for H (eq.(86)), and so $0 \le m \le 2$, $0 \le n \le 2$. From eq.(15), it can easily be shown that, in the limit of $\epsilon \to 0$, the D's either vanish or tend to various derivatives of delta functions. Below is a list of the only non-vanishing limits of the D's for both m and n less than or equal to two:

$$
\begin{aligned}
D^{(1,1)}_{\sigma,\sigma'}(\epsilon) &\to -\frac{1}{8\pi}\delta^2(\sigma-\sigma'),\\
D^{(2,1)}_{\sigma,\sigma'}(\epsilon) &\to -\frac{1}{6\pi}\partial_{\sigma+}\delta^2(\sigma-\sigma'),\\
D^{(1,2)}_{\sigma,\sigma'}(\epsilon) &\to -\frac{1}{6\pi}\partial_{\sigma-}\delta^2(\sigma-\sigma'),\\
D^{(2,2)}_{\sigma,\sigma'}(\epsilon) &\to -\frac{5}{24\pi}\partial_{\sigma+}\partial_{\sigma-}\delta^2(\sigma-\sigma')+\frac{1}{4\pi\epsilon}\delta^2(\sigma-\sigma').
\end{aligned}
\tag{92}
$$

Putting everything together, in the limit of $\epsilon \to 0$, $W^{(2)}$ can be written as

$$
W^{(2)} = \int d^2\sigma v'(\sigma_+)\partial_+ X^{\lambda\sigma}\partial_- X^{\lambda'\sigma} Z^{(2)}_{\lambda\lambda'} - \frac{1}{8\pi\epsilon}\int d^2\sigma v'(\sigma_+)(h_{\mu\nu}h_{\mu\nu}), \tag{93}
$$

where,

$$
\begin{aligned}
Z^{(2)} &= -\frac{1}{8\pi}\Big(\frac{1}{2}\partial_\mu h_{\nu\lambda}\partial_\mu h_{\nu\lambda'} - \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial_{\lambda'}h_{\mu\nu} - \frac{1}{2}\partial_\mu h_{\nu\lambda}\partial_\nu h_{\mu\lambda'}\\
&+ \frac{1}{2}h_{\mu\nu}(\partial_\mu\partial_\nu h_{\lambda\lambda'} - \partial_\mu\partial_\lambda h_{\nu\lambda'} - \partial_\mu\partial_{\lambda'}h_{\nu\lambda}) + \frac{1}{2}\Gamma^{(1)\eta}_{\mu\mu}(\partial_\eta h_{\lambda\lambda'} - \partial_\lambda h_{\eta\lambda'} - \partial_{\lambda'}h_{\eta\lambda})\\
&+ \partial_{\lambda'}\Big(h_{\mu\nu}(\partial_\mu h_{\nu\lambda} - \Gamma^{(1)\lambda}_{\mu\nu}) + \Gamma^{(1)\eta}_{\mu\mu}h_{\eta\lambda} + \Gamma^{(2)\lambda}_{\mu\mu}\Big)\Big).
\end{aligned}
\tag{94}
$$

The first order contribution to Z was calculated in section 3:

$$
Z^{(1)}_{\lambda\lambda'} = \frac{1}{16\pi}\left(\Box h_{\lambda\lambda'} - \partial_\lambda\partial_\mu h_{\mu\lambda'} + \partial_{\lambda'}\partial_\mu h_{\mu\lambda} - 2\,\partial_{\lambda'}\Gamma^{(1)\lambda}_{\mu\mu}\right), \tag{95}
$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ stand for the linear and quadratic parts of the connection. The generalization of eq.(35) to include quadratic terms is then given by

$$
Z^{(1)}_{\lambda\lambda'} + Z^{(2)}_{\lambda\lambda'} = 0. \tag{96}
$$

From this equation, using the symmetry of $h_{\mu\nu}$ in μ and ν, it is easy to show that

$$
\partial_{\lambda'}\Big((g_{\mu\nu}g_{\eta\lambda}\Gamma^\eta_{\mu\nu})_{(2)} - \partial_\mu h_{\mu\lambda} + h_{\mu\nu}\partial_\mu h_{\nu\lambda}\Big) - (\lambda \leftrightarrow \lambda') = 0,
$$

where the subscript (2) means that only up to second order contributions are included. The solution to this equation can be written as

$$
(g^{\mu\nu}g_{\eta\lambda}\Gamma^\eta_{\mu\nu})_{(2)} = \partial_\mu h_{\mu\lambda} - h_{\mu\nu}\partial_\mu h_{\nu\lambda} - \frac{1}{2}\partial_\lambda(h_{\mu\mu} - h_{\mu\nu}h_{\mu\nu}) + \partial_\lambda\phi, \tag{97}
$$

where ϕ is identified with the dilaton field. There is always some ambiguity in the definition of the dilaton field; for example, we could have defined a different dilaton field by

$$\bar{\phi} = \phi - \frac{1}{2} h_{\mu\mu} + \frac{1}{2} h_{\mu\nu} h_{\mu\nu},$$

and thereby simplified eq.(93). This ambiguity arises from the well known possibility of mixing the dilaton with the determinant of the metric. In the definition we have chosen, the dilaton transforms as a scalar under coordinate transformations.

Finally, the terms involving the connection in eq.(92) can be eliminated using eq.(93). As promised earlier, to second order in h, the resulting field equations coincide with the equations

$$2\, R_{\mu\nu} + D_\mu D_\nu \phi = 0, \tag{98}$$

of the gravity-dilaton system.

8. Conclusions

In this paper, we developed further and extended the method for deriving string field equations proposed in an earlier paper [1]. As a check on the method, we derived the linearized equations for all string states and the full non-linear equations for the dilaton-graviton system and compared them with the well known results. There seems to be no obstacle to obtaining a full set of interacting equations for all levels. These equations would then enjoy the desirable properties of background independence and covariance under general non-local coordinate transformations.

Acknowledgements

I would like to thank Luis Bernardo for discussions.

References

1. K.Bardakci and L.M. Bernardo,*String Field Equations from Generalized Sigma Model*, hep-th/9701171.
2. D.Friedan, Phys.Rev.Lett.**45**(1080) 1057.
3. L.Alvarez-Gaumé, D.Z.Freedman and S.Mukhi, Ann. of Phys.**134** (1981) 85.
4. C.Lovelace, Phys. Lett. **135B**(1984) 75.
5. E.S.Fradkin and A.A.Tseytlin, Nucl. Phys. **B261** (1985) 1.
6. C.G.Callan, D.Friedan, E.Martinec and M.J.Perry, Nucl.Phys. **B262** (1985) 593.
7. T.Banks and E.Martinek, Nucl.Phys.**B294** (1987) 733.
8. K.G.Wilson and J.G.Kogut, Phys. Rep. **12** (1974) 75.
9. J.Polchinski, Nucl. Phys. **B231** (1984) 413.
10. J.Hughes, J.Liu and J.Polchinski, Nucl. Phys. **B316** (1989) 15.
11. A.N.Redlich, Phys. Lett. **213B** (1988) 285.
12. R.Brustein and K.Roland, Nucl.Phys. **B372** (1992) 201.
13. A.A.Tseytlin, Int. J. Mod. Phys. **A16** (1989) 4249.
14. U.Ellwanger and J.Fuchs, Nucl. Phys. **B312** (1989) 95.
15. E.Braaten, T.L.Curtright and C.K.Zachos, Nucl. Phys **B260** (1985) 630.
16. C.B.Thorn, Phys. Rep. **175** (1989) 1.
17. M.Bochicchio, Phys. Lett. **193B** (1987) 31.
18. E.Witten, Phys. Rev. **D46** (1992) 5467.
19. B.Zwiebach, Nucl. Phys. **B480** (1996) 541, A.Sen and B.Zwiebach, Nucl. Phys. **B423** (1994) 580.

20. For a review, see C.Vafa, *Lectures on Strings and Dualities*, hep-th/9702201

21. I.A.Batalin and G.A.Vilkovisky, Phys. Rev. **D28** (1983) 2567.

22. R.Brustein and S.de Alwis, Nucl. Phys. **B352** (1991) 451.

23. M.Henneaux, in *Quantum Mechanics of Fundamental Systems 3*, edited by C.Teitelboim and J.Zanelli. Plenum Press, New York, 1992.

24. M.B.Green, J.H.Schwarz and E.Witten, *Superstring Theory*, Cambridge University Press, 1987.

25. P.Goddard, C.Rebbi and C.Thorn, Nuovo Cim. **12A** (1972) 425.

Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098