Title
LINEAR PROGRAM APPROACH TO LINEAR DIFFERENTIAL PROBLEMS

Author
Young, Jonathan D.

Publication Date
1963-11-11

UCRL-9498 Rev.

University of California

Ernest O. Lawrence Radiation Laboratory

LINEAR PROGRAM APPROACH TO LINEAR DIFFERENTIAL PROBLEMS

DISCLAIMER

Rept. submitted for pub. in the
International Journal of Engineering Science.

UCRL-9498 Rev.

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

LINEAR PROGRAM APPROACH TO LINEAR DIFFERENTIAL PROBLEMS

Jonathan D. Young

November 11, 1963

LINEAR PROGRAM APPROACH TO LINEAR DIFFERENTIAL PROBLEMS*

Jonathan D. Young

Lawrence Radiation Laboratory
University of California
Berkeley, California

November 11, 1963

ABSTRACT

The numerical solution of a linear differential problem is formulated as an overdetermined linear model with the objective of minimizing approximation errors. The dual of this model is a linear program.

I. INTRODUCTION

In recent years linear programming has become a useful approach to many decision problems in various fields of business, industry, and science. Riley and Gass in [1] present some thousand such applications. With respect to problems phrased in mathematical language per se, Goldstein [2] has applied the technique to overdetermined linear systems and Mangasarian [3] has used it for error evaluation in connection with the numerical solution of the biharmonic problem.

The approximation of linear differential problems by linear systems has been extensively treated. For example, Fox [4] uses this method for the linear ordinary second-order differential equation with boundary conditions. Likewise, Forsythe and Wasow [5] employ it for second-order partial differential equations. In all cases the systems are determinate, the unknowns are approximate solution values on mesh points, and no estimate of the error is explicit in the process.

The linear model by which we represent the differential problem is more extensive: first, the unknowns include derivative as well as solution values at mesh points; second, sufficient mesh points and approximation formulas are used so that the system is overdetermined; third, the maximum absolute value of approximation remainders is introduced as an unknown to be minimized; and fourth, the model is the dual of a linear program.

II. LINEAR DIFFERENTIAL PROBLEM

A linear differential problem consists of linear differential equations and prescribed conditions involving an unknown function u and some of its derivatives with respect to an independent variable s. Either u or s or both may be vectors, as in the cases, respectively, of an ordinary system, a partial differential equation, or a partial system.

The differential equations are assumed to hold on a simply connected domain, D, of points, s. The prescribed conditions apply to a set, T, of points, s. We consider only the case where D is closed and T is a subset of D. The method can be extended to D open and T in the closure of D (for example, when T is the boundary of D).

The differential equation(s) can be written as

$$Lu = c \quad \text{for} \quad s \in D \tag{1}$$

and the prescribed conditions by

$$L_p u = c_p \quad \text{for} \quad s \in T < D . \tag{2}$$

The problem is said to be well-posed if there exists a unique solution, u, on D, which has continuous derivatives of orders appearing in L. The "coefficients" involved in L and L_p, together with c_p, are assumed to be continuous functions of s on D, and we require that the highest-ordered terms of L do not vanish on D.

III. DISCRETIZATION AND THE LINEAR MODEL

We construct a suitable mesh over D and concern ourselves with numerical values at the mesh points for the solution u and its derivatives up to the orders which appear in L and L_p. These values appear as the unknown variables in a linear system obtained by applying (1) to all the mesh points by applying (2) to mesh points in T and by using truncated Taylor expansions as approximations to relate solution and derivative values at adjacent points. The last involve approximation errors. If we considered each of these errors as a distinct unknown variable, our linear system would have more unknowns than equations. Hence, we introduce a single error variable which is the maximum absolute value for all the error terms. Each approximation equation of the form

$$e + av = 0,$$

where e is an approximation error, v is a vector of solution and derivative values, and a is a vector of coefficients, is replaced by two inequalities of the form

$$u_0 + av \geq 0,$$

$$u_0 - av \geq 0,$$

where $u_0 = \max |e|$.

The other equations (obtained from (1) and (2)) have the form

$$av = c,$$

where c is a numerical value.

They are replaced by the equivalent limiting inequalities of the form

$$av \geq c,$$

$$-av \geq -c.$$

Thereby, all our restraints will all be inequalities of the direction $\geq$. The resulting complete linear model has the form

$$Av \geq c$$

(where v now includes u_0 and c is a vector of all the constant terms) and we wish to minimize the linear function,

$$bv = u_0,$$

where b is a coefficient vector whose components are all zero except the one corresponding to u_0 which is 1.

IV. THE LINEAR PROGRAM

According to reference [6] a linear model of the form above has as its dual

$$\begin{aligned} &\text{Maximize} && z = cx \\ &\text{subject to} && A'x = b \\ &\text{with} && x \geq 0, \end{aligned}$$

which is a linear program in primal form. The Simplex algorithm [7] applied to it yields an optimal solution for both the linear program and the original linear model. Thus we are provided with numerical values for the solution and derivatives at the mesh points together with a value for the maximum magnitude of approximation errors. Linear programming codes [8] exist which perform the necessary computation.

V. SIMPLE SPECIFIC EXAMPLE

$$\begin{aligned} &u'' + u = s \\ &\text{on} \quad D: 0 \leq s \leq 0.2; \end{aligned} \tag{3}$$

$$u(0) + u'(0) = 2.0, \tag{4}$$

$$u(0.2) = 1.18007. \tag{5}$$

Full details of the construction of the linear model for the mesh

$$\begin{aligned} &s_1 = 0, && \\ &s_2 = 0.08, && h_1 = s_2 - s_1 = 0.08, \\ &s_3 = 0.2, && h_2 = s_3 - s_2 = 0.12 \end{aligned}$$

are given [9]. In brief, we have:

from (3),

$$u_k + u_k'' \geq s_k , \qquad k = 1(1)3 ,$$

$$-u_k - u_k'' \geq -s_k ;$$

from (4),

$$u_1 + u_1' \geq 2 ,$$

$$-u_1 - u_1' \geq -2 ;$$

from (5),

$$u_1 \geq 1.18007 ,$$

$$-u_1 \geq -1.18007 ;$$

from Taylor approximations,

$$\left.\begin{array}{l} u_0 \pm (u_k + h_k u_k' + 0.5\, h_k^2 u_k'' - u_{k+1}) \geq 0 \\ u_0 \pm (h_k u_k' + h_k^2 u_k'' - h_k u_{k+1}') \geq 0 \end{array}\right\} k = 1, 2 ,$$

$$\left.\begin{array}{l} u_0 \pm (u_k - h_k u_k' + 0.5\, h_k^2 u_k'' - u_{k-1}) \geq 0 \\ u_0 \pm (h_k u_k' - h_k^2 u_k'' - h_k u_{k-1}) \geq 0 \end{array}\right\} k = 2, 3 .$$

Thus the linear model consists of 26 inequalities in 10 unknowns, u_0, u_k, u_k', u_k'', and we wish to minimize u_0. The dual of this model is a linear program with 10 equations in 26 unknowns, a linear function of which is to be minimized.

Computation of the program gave $u(0.08) = 1.07682$, with the maximum error $u_0 = 0.00004$. The problem has an analytic solution, $u = s + \cos s$, for which $u(0.08) = 1.07680$. The finite-difference method described by Fox [4] gives $u(0.08) = 1.07681$.

This latter method requires a uniform mesh and provides only solution values, whereas approximations for the derivative values are obtained in the linear program approach.

Many other examples are given in full detail in [9].

VI. CONCLUSION

The linear program approach to the numerical solution of linear differential problems has these advantages:

(A) It is global, that is, all mesh points are considered simultaneously.

(B) It applies to problems with initial boundary or mixed conditions.

(C) An overlay of approximation formulas can be used.

(D) It applies to single equations or systems.

(E) It applies to ordinary or partial problems.

(F) The mesh need not be uniform.

(G) It provides values for derivatives as well as for the solution at mesh points.

(H) It provides an estimate of the error.

Disadvantages of the approach are:

(A) It applies only to linear differential problems.

(B) Construction of the linear model is tedious.

(C) Extensive computation is involved.

The linear program approach appears most likely to be useful as follows:

(A) For problems where derivative values are desired.

(B) When a uniform mesh is not suitable to the problem.

(C) For problems with unusual conditions.

(D) As a starting process for initial condition problems.

We should note that Stiefel [10] presents an algorithm for error minimization in overdeterminate linear systems which may be applied to some linear models approximating differential problems. With his algorithm it is not possible to attain the versatility and generality afforded by dualizing to a linear program.

The author wishes to thank Dr. L. A. Zadeh for his advice and assistance in preparing this article for publication.

FOOTNOTES AND REFERENCES

* Work done under the auspices of the U. S. Atomic Energy Commission.

[1] V. Riley and S. I. Gass, Linear Programming and Associated Techniques, Johns Hopkins Press, Baltimore (1958).

[2] A. A. Goldstein, On the Method of Descent in Convex Domains and Its Application to the Minimal Approximation of Overdetermined Systems of Linear Equations, Math. Preprint No. 1, Convair Astronautics, San Diego, Calif. (1956).

[3] O. L. Mangasarian, Numerical Solution of the First Biharmonic Problem by Linear Programming, Intern. J. Eng. Sci. 1, 231-240 (1963).

[4] L. Fox, The Numerical Solution of Two-Point Boundary Problems in Ordinary Differential Equations, Oxford University Press, London (1957).

[5] G. E. Forsythe and W. R. Wasow, Finite Difference for Partial Differential Equations, John Wiley & Sons, Inc., New York (1960).

[6] S. I. Gass, Linear Programming, McGraw-Hill Book Co., Inc., New York (1958).

[7] G. B. Dantzig, The Simplex Method, RAND Report P-891, The Rand Corporation, Santa Monica, Calif. (1956).

[8] W. Orchard-Hays, SCROL, A Comprehensive Operating System for Linear Programming on the IBM-704, CEIR, Inc., Washington (1959).

[9] J. D. Young, Application of Linear Programming to the Numerical Solution of Linear Differential Equations, Lawrence Radiation Laboratory Report UCRL-10110, 1962 (unpublished).

[10] E. L. Stiefel, Note on Jordan Elimination, Linear Programming, and Tchebycheff Approximation, Numerische Mathematik 2, 1960.