Title
Spontaneous Breaking of Scale Invariance and the Ultraviolet Fixed Point in O (N)-Symmetric (φ36) Theory

Journal
Physical Review Letters, 52(14)

ISSN
0031-9007

Authors
Bardeen, William A
Moshe, Moshe
Bander, Myron

Publication Date
1984-04-02

DOI
10.1103/physrevlett.52.1188

Spontaneous Breaking of Scale Invariance and the Ultraviolet Fixed Point in O(N)-Symmetric (ϕ_3^6) Theory

William A. Bardeen and Moshe Moshe[a]
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

and

Myron Bander
Department of Physics, University of California, Irvine, California 92717

(Received 28 June 1983)

At large N, the $\eta\vec{\phi}^6$ theory is shown to possess a nontrivial ultraviolet fixed point. A new phase is found where asymptotic scale invariance is spontaneously broken and a dynamical mass is generated through dimensional transmutation. At the tricritical limit, the spontaneous breaking of an exact scale invariance at leading N results in the formation of a massless composite Goldstone mode, the dilaton. We compare these results to standard $1/N$ expansion and emphasize the nonperturbative nature of these phenomena.

PACS numbers: 11.10.-z, 11.30.Qc

The O(N) vector model, $\eta(\vec{\phi}^2)^3$, presents a unique laboratory for the study of several interesting aspects of quantum field theory. In three dimensions the theory is just renormalizable and, indeed, the coupling constant, η, is not renormalized in the leading-N approximation. However, recent analyses[1,2] of the $1/N$ corrections have shown the existence of an ultraviolet fixed point at a finite coupling, $\eta=\eta^*$. Hence the theory represents an example of a nonasymptotically free field theory with a nontrivial ultraviolet limit governed by the fixed point. In contrast, for the $\lambda_0\vec{\phi}^4$ theory in four dimensions, there is an accumulation of evidence that the renormalized theory[3-6] is either inconsistent ($\lambda_0<0$) or "trivial" ($\lambda_0>0$). The $\eta\vec{\phi}^6$ theory has also been used as a laboratory for studying interesting effects of critical and tricritical behavior in field theory.

In this paper, variational methods will be exploited to reveal novel, nonperturbative aspects of $\eta(\vec{\phi}^2)^3$ theory which fundamentally alter our understanding of the ground-state structure, the spectrum, and the ultraviolet behavior of this theory. We will show that the ultraviolet fixed point $\eta=\eta^*$ found through a perturbative analysis[1,2] lies in the region of instability where nonperturbative effects dominate the physics. The essential structure of this theory is, however, governed by a different ultraviolet fixed point, $\eta=\eta_c<\eta^*$. The true ground state is characterized by dynamical mass generation through dimensional transmutation. This massive phase reflects the nontrivial-fixed-point structure of the β function which occurs even at leading order in the $1/N$ expansion where the perturbative β function vanishes. At the tricritical point, the theory is scale invariant at leading-N order. This scale invariance is spontaneously broken by the dynamical mass generation, and a massless Goldstone boson, the dilaton, appears as a dynamical bound state. The stress tensor remains traceless but now contains a direct, induced coupling to the dilaton. The $1/N$ corrections will break the exact scale invariance, even at the tricritical point, and the dilaton will gain a mass of order $1/N$.

We wish to consider the O(N) vector model described in three dimensions by the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\vec{\phi})^2-\tfrac{1}{2}\mu_0^2(\vec{\phi}^2)-\tfrac{1}{4}(\lambda_0/N)(\vec{\phi}^2)^2-\tfrac{1}{6}(\eta_0/N^2)(\vec{\phi}^2)^3, \quad (1)$$

where $\{\vec{\phi}\}$ is an N-component scalar field. We have scaled the coupling constants consistent with the large-N limit where μ_0^2, λ_0, and η_0 are held fixed as $N\to\infty$. In this limit, only the cactus diagrams will contribute. The theory may be analyzed either through Euclidean functional integrals or through direct Hamiltonian methods.

In the following, we use the variational methods of Ref. 6 to find the best plane-wave ground state, which can be represented in the Schrödinger picture[7] by the wave functional

$$\psi\{\vec{\phi}\}=\exp\{-\int d^3x\, d^3y\,[\vec{\phi}(x)-\vec{\phi}_c]\cdot[\vec{\phi}(y)-\vec{\phi}_c]G_m(\vec{x}-\vec{y})\}, \quad (2)$$

where the constant background field, $\vec{\phi}_c$, and the correlation function, $G_m(\vec{x}-\vec{y})$, can be determined by minimizing the ground-state energy. To leading order in N, the vacuum energy can be expressed in terms of the kinetic and potential energies,

$$W(m^2,\vec{\phi}_c)=K(m^2)+\langle V(\vec{\phi})\rangle, \tag{3}$$

where

$$K(M^2)=-\tfrac{1}{2}\int_0^{m^2} dm^2\, m^2\, \partial\langle\vec{\phi}^2\rangle/\partial m^2, \tag{4}$$

$$\langle V(\vec{\phi})\rangle=\tfrac{1}{2}\mu_0^2(\vec{\phi}_c^2+\langle\vec{\phi}^2\rangle)+\tfrac{1}{4}(\lambda_0/N)(\vec{\phi}_c^2+\langle\vec{\phi}^2\rangle)^2+\tfrac{1}{6}(\eta_0/N^2)(\vec{\phi}_c^2+\langle\vec{\phi}^2\rangle)^3. \tag{5}$$

It is sufficient to parametrize the ground state by a common mass, m. The vacuum value of $\langle\vec{\phi}^2\rangle$ depends on m and in three dimensions is given by

$$\langle\vec{\phi}^2\rangle=\frac{N}{(2\pi)^3}\int_0^\Lambda \frac{d^3k}{k^2+m^2}=\frac{N}{2\pi^2}\left(\Lambda-\frac{\pi}{2}m\right), \tag{6}$$

where we have used a symmetric Euclidean cutoff, Λ, to define the divergent integrals.

The complete vacuum energy is expressed in terms of the variational parameters, m and $\vec{\phi}_c$, by

$$W(m,\vec{\phi}_c)=\frac{N}{24\pi}m^3+\frac{1}{2}\mu^2\left(\vec{\phi}_c^2-\frac{N}{4\pi}m\right)+\frac{1}{4}\frac{\lambda}{N}\left(\vec{\phi}_c^2-\frac{N}{4\pi}m\right)^2+\frac{1}{6}\frac{\eta}{N^2}\left(\vec{\phi}_c^2-\frac{N}{4\pi}m\right)^3, \tag{7}$$

where the first term is the kinetic energy and we have introduced renormalized potential parameters, $\eta=\eta_0$, $\lambda=\lambda_0+\eta_0\Lambda/\pi^2$, and $\mu^2=\mu_0^2+\lambda_0\Lambda/2\pi^2+\eta_0\Lambda^2/4\pi^4$. We observe that the $\vec{\phi}^6$ coupling, η, requires no divergent renormalization.

Our subsequent analysis will focus on the coupling η as it governs the ultimate stability of this theory in three dimensions. The renormalized parameters μ^2 and λ are both dimensional constants and their effects may be ignored in the ultraviolet limit. We remark that when μ^2 is zero the theory is said to be at the critical point and when both μ^2 and λ are zero the theory is tricritical. At the tricritical point, the leading-N theory is scale invariant since the remaining coupling, η, is unrenormalized. As an immediate consequence, the perturbative β function can be seen to vanish in this limit.

The ultimate stability of the ground state depends on the balance between the kinetic energy and the leading terms of the potential energy. By examining Eq. (7), the existence of a stable ground state requires a constraint on the fundamental coupling constant, $0<\eta\le\eta_c=(4\pi)^2$, where the lower bound comes from the large $\vec{\phi}_c$ dependence and the upper bound from the large m dependence. The apparent instability for $\eta>\eta_c$ is a surprising new feature of this analysis. The fixed point η^* discussed by Pisarski[1] lies in this region. He computes the $1/N$ corrections to the perturbative β function and finds $\beta(\eta)=0+(3/2\pi^2)N^{-1}(1-\eta/\eta^*)$, where $\eta^*=192>\eta_c=(4\pi)^2$. Our combined results seem to have disastrous implications for the theory. The Pisarski β function causes the coupling to run slowly to values larger than η_c in the ultraviolet limit. Hence there seems to be no stable domain for the existence of the theory without an explicit ultraviolet cutoff to keep the coupling less than η_c. We will see that this disaster is avoided because of the existence of a different, nonperturbative fixed point associated with the critical coupling, η_c.

The instability that we have discovered at large coupling is, indeed, a strange result. Namely, the $(\vec{\phi}^2)^3$ operator associated with the coupling η would seem to be positive definite and incapable of generating a vacuum instability. It is only the large negative renormalization of the $\vec{\phi}^2$ operator which makes the potential energy unbounded from below. However, it should also be clear that any physically regulated theory should have a stable ground state as the positivity properties will be maintained. The observed instability will be reflected by the generation of a large mass, m, typically of the order of the physical cutoff Λ, when the coupling is larger, $\eta>\eta_c$, in the regulated theory. One may then question the existence of a "continuum" limit if all masses are of order the cutoff. The emergence of a stable ground state with nontrivial infrared structure is possible, however, if the theory has an ultraviolet fixed point at η_c. Indeed, we can establish the existence of this nonperturbative fixed point, and a finite dynamical mass is generated through the mechanism of dimensional transmutation.

The physically regularized theory may be defined in many ways. We choose to introduce a consistent momentum cutoff but we have obtained the same results using lattice, Pauli-Villars,[8] and dimensional regularization methods. The essential modification of the regularization is to keep the nonleading terms in the evaluation of Eq. (6), $\langle\vec{\phi}^2\rangle$

$=(N/2\pi^2)(\Lambda-\frac{1}{2}\pi m+m^2/\Lambda+\ldots)$, where we have assumed $m \ll \Lambda$. The kinetic energy may be computed through Eq. (4) and used to obtain a regularized expression for the vacuum energy at the tricritical point,

$$W=\frac{N}{24\pi}\left(m^3-\frac{3}{\pi}\frac{m^4}{\Lambda}+\ldots\right)+\frac{1}{6}\frac{\eta}{N^2}\left(\vec{\phi}_c^{\,2}-\frac{N}{4\pi}m+\frac{N}{2\pi^2}\frac{m^2}{\Lambda}+\ldots\right)^3. \tag{8}$$

A stable minimum now exists for the theory as the potential energy no longer dominates the kinetic energy as m increases. Since there will be no symmetry breaking, we take $\vec{\phi}_c=0$ and compute the gap equation

$$0=\frac{\partial W}{\partial m}=\frac{N}{8\pi}\left(1-\frac{4}{\pi}\frac{m}{\Lambda}\right)\left[m^2-\frac{\eta}{(4\pi)^2}\left(m-\frac{2}{\pi}\frac{m^2}{\Lambda}\right)^2\right]. \tag{9}$$

The relevant solution is

$$m=\Lambda f(\eta)=\Lambda(\pi/2)[1-(\eta_c/\eta)^{1/2}]. \tag{10}$$

Indeed we find that a finite mass is generated for the ϕ particles if the coupling is allowed to approach the fixed point, $\eta=\eta_c+\tilde{\mu}/\Lambda$, as we approach the continuum limit, $\Lambda\to\infty$. Equation (10) has precisely the structure expected for a nonperturbative fixed point and dimensional transmutation. We can use Eq. (10) to derive the β function for this massive phase,

$$\beta(\eta)=\Lambda\partial_\Lambda(\eta)|_m\to\eta_c-\eta,\quad \eta\to\eta_c^+. \tag{11}$$

Near the fixed point, the β function is expected to be universal (however, it appears to be difficult to extract directly the correct β function with use of dimensional regularization), and we can verify Eq. (11) by a direct examination of the physical six-point vertex as seen below.

At the tricritical point, both the mass and four-point vertex are expected to vanish. We have seen that a dynamical mass is generated in the strong-coupling phase. The normal ordering of the $\vec{\phi}^6$ interaction in the massive phase also generates a correction to the four-point vertex, $\bar{\lambda}=\lambda_0+2(\eta/N)\langle\vec{\phi}^2\rangle=\lambda-\eta m/2\pi$. This induced coupling must now be included in computing the amplitudes of the theory.

At leading N, the four-point function involves the sum of bubble diagrams. One finds $\Gamma_4(p)=(2/N)\bar{\lambda}[1-\bar{\lambda}B(p)]^{-1}$, where the standard bubble integral, in three dimensions, is given by

$$B(p)=-(1/8\pi)\int_0^1 d\alpha[m^2-\alpha(1-\alpha)p^2]^{-1/2}. \tag{12}$$

At the tricritical point, $\lambda=0$, and the renormalized four-point amplitude becomes

$$\Gamma_4(p)=-(16\pi/N)m\{1-\int_0^1 d\alpha[1-\alpha(1-\alpha)p^2/m^2]^{-1/2}\}^{-1}, \tag{13}$$

where we must use $\eta=\eta_c$ in the continuum as implied by the discussion following Eq. (10).

A remarkable feature of the amplitude in Eq. (13) is the existence of a pole at p^2-0,

$$\Gamma_4(p)\to\frac{192\pi m^2}{N}\frac{1}{p^2}\quad\text{as}\quad p^2\to 0. \tag{14}$$

This bound-state pole in the O(N) singlet amplitude can be interpreted as the massless Goldstone mode, the dilaton, expected because scale invariance is spontaneously broken in the massive phase. The infrared coupling between the dilaton and two massive ϕ particles can be read off from Eq. (14).

The six-point function also has induced corrections and can be written in the form

$$\Gamma_6(p_i)=8\eta_c[1-V(p_1p_2p_3)]\prod_{k=1}^{3}[1-\lambda B(p_k)]^{-1}, \tag{15}$$

where V is an induced vertex and $B(p)$ is given by Eq. (12). The bubble sums generate dilaton poles in each of the singlet channels. We can use the six-point amplitude to define a physical running coupling constant, $\eta(p)=\Gamma_6(p)/8$. In the ultraviolet limit, B and V vanish and $\eta(p)$ approaches η_c. In the infrared limit, $1-V$ is of order p_k^2, but the coupling $\eta(p)$ still blows up as p^{-4} because of the dilaton poles. The β function can be computed directly for this coupling constant: $\beta(\eta)\to\eta_c-\eta$ as $\eta\to\eta_c^+$ and $\beta(\eta)\to-4\eta$ as $\eta\to\infty$.

We can examine the scale invariance and its spontaneous breaking by evaluating matrix elements of the stress tensor. We note that the scale current is usually defined as $S_\mu = x^\nu \theta_{\mu\nu}$ and is conserved if the stress tensor is traceless. The two-point matrix element is simply computed, including the induced terms, and is given by

$$\langle P'|\theta_{\mu\nu}|P\rangle = (P'_\mu P_\nu + P_\mu P'_\nu - g_{\mu\nu}P'\cdot P + g_{\mu\nu}m^2) + (q_\mu q_\nu - q^2 g_{\mu\nu})\tfrac{1}{4}\{1 - 8\int_0^1 d\alpha\,\alpha(1-\alpha)[1-\alpha(1-\alpha)q^2/m^2]^{-1/2}\} \times \{1 - \int_0^1 d\alpha[1-\alpha(1-\alpha)q^2/m^2]^{-1/2}\}^{-1}, \qquad (16)$$

where the first term is the standard tree amplitude for a particle of mass, m, and the second is the induced term with $q = p' - p$. The induced term contains the dilaton pole and can be simplified with use of an identity. We find

$$\langle P'|\theta_{\mu\nu}|P\rangle = (P'_\mu P_\nu + P_\mu P'_\nu - g_{\mu\nu}P'\cdot P + g_{\mu\nu}m^2) + (q_\mu q_\nu - q^2 g_{\mu\nu})(m^2/q^2 + \tfrac{1}{4}). \qquad (17)$$

The amplitude is clearly traceless and the direct coupling of the dilaton to the stress tensor is made explicit. Of course, a general analysis of the leading-N amplitudes can be made to verify the traceless condition as an operator statement. The beauty of an O(N) theory is that all amplitudes are simply computed in the large-N limit.

We have established an interesting new phase structure for the $\eta(\vec{\phi}^2)^3$ theory in three dimensions. This phase structure is directly related to the existence of a nonperturbative fixed point found by our analysis. At the tricritical point, the theory is scale invariant at leading-N approximation. At strong coupling, the scale invariance is spontaneously broken with mass being generated for the ϕ particles and the dilaton established as the Goldstone boson of broken scale invariance. At leading N, all amplitudes can be computed in closed form. While we have examined the tricritical limit, the nonleading terms related to λ and μ^2 can be included as explicit symmetry breaking. The $O(1/N)$ corrections will also break scale invariance and generate mass for the dilaton. However, the general picture of the phase structure should remain intact. It would be interesting to verify the results of our large-N analysis by other methods, such as through direct lattice calculations.[4] We also raise the question as to whether the new phase structure we have observed here may actually correspond to phenomena observed in real physical systems in three dimensions.[9,10]

We would like to thank A. Duncan for a discussion. Two of us (M. M. and M. B.) would like to thank the Fermilab Theory Group for its support and warm hospitality. This work was supported in part by the National Science Foundation under Contract No. PHY 79-10262.

(a)On leave from Physics Department, Technion, Haifa, Israel.

[1]R. D. Pisarski, Phys. Rev. Lett. **48**, 574 (1982).

[2]T. Appelquist and U. Heinz, Phys. Rev. D **24**, 2169 (1981), and **25**, 2620 (1982). See also earlier works by P. K. Townsend, Phys. Rev. D **12**, 2269 (1975), and **14**, 1715 (1976), and Nucl. Phys. **B118**, 199 (1977).

[3]K. Wilson, Phys. Rev. D **6**, 419 (1972); K. Symanzik, in *Mathematical Problems in Theoretical Physics,* edited by R. Schrader, Lecture Notes in Physics, Vol. 153 (Springer-Verlag, New York, 1982), and references therein.

[4]J. Kogut and K. Wilson, Phys. Rep. **12C**, 75 (1975); G. A. Baker, Jr., in *Bifurcation Phenomena in Mathematical Physics,* edited by C. Bardos and D. Bessis (Reidel, Higham, Mass., 1980), and references therein; C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. D **23**, 2976, 2999 (1981), and **24**, 2772(E) (1982); B. Freedman, P. Smolensky, and D. Weingarten, Phys. Lett. **113B**, 481 (1982).

[5]M. Aizenman, Phys. Rev. Lett. **47**, 1 (1981), and Commun. Math. Phys. **86**, 1 (1982); J. Fröhlich, Nucl. Phys. **B200**, 281 (1982).

[6]William A. Bardeen and Moshe Moshe, Fermilab Report No. 83/23-Thy (to be published).

[7]R. Jackiw and A. Strominger, Phys. Lett. **B99**, 133 (1981), and references therein to early works.

[8]C. Bernard and A. Duncan, Phys. Rev. D **11**, 848 (1975).

[9]R. D. Pisarski, Phys. Rev. D **26**, 3543 (1982).

[10]The model has been considered in a different context by S. Sarbach and M. E. Fisher, Phys. Rev. B **20**, 2797 (1979).