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Theory and simulations of principle of minimum dissipation rate

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We perform a self-consistent, time-dependent numerical simulations of dissipative turbulent plasmas at a higher Lundquist number, typically up to $\mathcal{O}(10^6)$, using full three-dimensional compressible magnetohydrodynamics code with a numerical resolution of 128^3. Our simulations follow the time variation of global helicity, magnetic energy, and the dissipation rate and show that the global helicity remains approximately constant, while magnetic energy is decaying faster and the dissipation rate is decaying even faster than the magnetic energy. This establishes that the principle of minimum dissipation rate under the constraint of (approximate) conservation of global helicity is a viable approach for plasma relaxation. © *2008 American Institute of Physics*.
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I. INTRODUCTION

In a seminal work, Taylor[1] proposed that for a slightly dissipative turbulent magnetized plasma, the global magnetic helicity $\int_V \mathbf{A}\cdot\mathbf{B}dV$ (where $\mathbf{B}$ is the magnetic field and $\mathbf{A}$ is the vector potential) remains relatively invariant (so-called "rugged-invariant") while the magnetic energy $\int_V B^2/8\pi dV$ tends to a minimum value. This leads to a variational principle,

$$\delta\int_v\left(\frac{B^2}{8\pi}-\lambda\mathbf{A}\cdot\mathbf{B}\right)dV=0, \tag{1}$$

where λ is the Lagrange multiplier. The variation yields the Euler–Lagrange equation

$$\nabla\times\mathbf{B}=\lambda\mathbf{B}. \tag{2}$$

Obviously, the state denoted by Eq. (2) as proposed by Taylor, is a force-free (FF), or zero pressure gradient, state since the current $\mathbf{J}\|\mathbf{B}$, leading to $\mathbf{J}\times\mathbf{B}=0$. Taylor's theory successfully explained the field reversal of reversed field pinches (RFPs).

Since then, Taylor's theory force-free model has been almost universally adopted to explain various long-lived structures observed in fusion physics and astrophysics. A vast literature illustrates this (see Refs. 2–6 and the references cited therein).

However, despite the wide acceptance of the force-free relaxed state proposed by Taylor, it is nonetheless subject to many intrinsic and severe limitations. Taylor's model does not produce a pressure balanced structure, and is not applicable to long-lived plasma states in other fusion machines, such as the tokamak, the formation of field reversed configuration by merging two counterhelicity spheromaks, etc. Several numerical works, particularly those by Sato and his collaborators,[7,8] have established the existence of self-organized states with finite pressure gradient; i.e., these states are governed by the magnetohydrodynamic force balance relation $\mathbf{J}\times\mathbf{B}=\nabla p$ rather than $\mathbf{J}\times\mathbf{B}=0$.

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In this paper, we describe our theoretical and simulation results, based on the magnetohydrodynamics (MHD) model, to investigate the principle of minimum dissipation rates through a turbulent relaxation of an initial nonequilibrium plasma. In Sec. II, we describe theoretical overview of the problem. Section III describes our simulation results which are consistent with the theory. Section IV contains conclusions.

II. THEORETICAL BACKGROUND

A different approach to self-organization and relaxation in plasma can be obtained from a law of irreversible thermodynamics; namely, the principle of minimum entropy production. First formulated by Prigogine,[9] this law states that a steady irreversible process is characterized by a minimum rate of entropy production. Prigogine[10] envisaged that an open dissipative system far from equilibrium is able to transform small-scale irregularities into large-scale patterns, which he described as "Dissipative Structures." An equivalent and more accessible version of the above principle, which could be applied to laboratory and astrophysical plasma, is the principle of minimum dissipation rate (MDR) of energy. It should be mentioned that Chandrasekhar and Woltjer[11] did consider minimum dissipation states with constant energy and obtained $\nabla\times\nabla\times\mathbf{B}=\alpha\mathbf{B}$, commenting that solutions to this equation are "much wider" that those corresponding to a force-free field.

This alternate principle has been used to derive the relaxed state of a magnetized plasma, in an effort to eliminate the deficiencies of Taylor's model; Hameiri and Bhattacharjee[12] derived the relaxed plasma state using the minimization of entropy production rate. Montgomery and Phillips[13] first applied the principle of minimum dissipation to plasma relaxation for an open system with constant helicity injection. Farengo and Sobehart[14,15] obtained the magnetic field and current density profile of a steady-state tokamak plasma assuming minimum rate of Ohmic dissipation. Dasgupta *et al.*[16] considered relaxed states of a magnetized plasma with minimum dissipation to obtain a pressure balanced structure using a variational method,

$$\delta \int_v (\eta j^2 - \Lambda \mathbf{A} \cdot \mathbf{B}) dV = 0, \tag{3}$$

where η is the plasma resistivity and Λ is the Lagrange multiplier, which yielded the Euler–Lagrange equation

$$\nabla \times \nabla \times \nabla \times \mathbf{B} = \Lambda \mathbf{B}. \tag{4}$$

The solution of the above equation (4) is obtained as a superposition of Chandrasekhar and Kendall eigenfunctions. Dasgupta *et al.* showed that this can produce a pressure profile for the RFP and also the field reversal. Furthermore this model was applied to produce the observed pressure and q-profile of a tokamak,[17] a field-reversed configuration,[18] and a spheromak configuration.[19] Bhattacharyya and Janaki[20] extended these ideas to address dissipative relaxed states in a two-fluid plasma with external drive.

Astrophysical systems are invariably characterized by dissipation and often a high plasma β. The Taylor model of force-free states is inadequate to describe such systems and it is expected that minimum dissipative states are more relevant. Bhattacharyya *et al.*[22] considered solar arcades as minimum dissipative relaxed states and showed that the generation of different types of arcades is possible by varying a single parameter characterizing the relaxed state. Based on the principle of MDR, Hu and Dasgupta[21] developed an approach to derive three-dimensional non-force-free coronal magnetic fields from vector magnetograms. In this approach, those authors took advantage of the governing equation for the magnetic field, resulted from MDR, whose solution can be exactly expressed as linear superposition of three linear FF fields. They developed a fast and easy, yet more general approach to derive coronal magnetic field in a finite volume in a non-force-free state. It has potential significance and application to addressing the problem of solar magnetic activity central to solar physics research, especially given increasing amount of high-quality solar vector magnetograph measurements in present times.

A small amount of resistivity, present in any realistic plasma, is essential to allow reconnection processes that leads to relaxation. Dissipation, along with nonlinearity, is universal in systems evolving towards self-organized states and it is natural to assume that dissipation plays a decisive role in the self-organization of a system. The prime objective of this work is therefore to corroborate our claim of using MDR as viable constraint for plasma relaxation. For this purpose, we use full three-dimensional (3D) MHD simulations in the context of (freely) decaying turbulence to measure decay rates associated with the MHD invariant as described in the subsequent section.

III. NONLINEAR SIMULATION RESULTS

The turbulent magnetofluid flow in our simulations is described by the magnetohydrodynamic equations, in which statistically homogeneous, isotropic, and isothermal plasma fluctuations are cast in terms of single fluid magnetic $\mathbf{B}(\mathbf{r},t)$ and velocity $\mathbf{U}(\mathbf{r},t)$ fields and pressure $p(\mathbf{r},t)$ as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{5}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{6}$$

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right)\mathbf{U} = -\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{U} + \hat{\eta} \nabla (\nabla \cdot \mathbf{U}). \tag{7}$$

The equations are closed with a nondivergent magnetic field $\nabla \cdot \mathbf{B} = 0$ and an equation of state relating the perturbed density to the pressure variables. The last term in Eq. (7) is a result of compressibility, which leads to a finite solenoidal component of the turbulent velocity that corresponds to dissipation. Here, $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ is a three-dimensional vector, and η and ν are, respectively, magnetic and kinetic viscosities. The above equations can be normalized using a typical length scale (ℓ_0), density (ρ_0), pressure (p_0), magnetic field (B_0), and the velocity (U_0). With respect to these normalizing ambient quantities, one may define a constant sound speed $C_{s_\circ} = \sqrt{\gamma p_0 / \rho_0}$, sonic Mach number $\mathrm{Ma}_{s_\circ} = U_0 / C_{s_\circ}$, Alfvén speed $V_{A_\circ} = B_0 / \sqrt{4\pi\rho_0}$, and Alfvénic Mach number $M_{A_\circ} = U_0 / V_{A_\circ}$. The magnetic and mechanical Reynolds numbers are $R_{m_\circ} \approx U_0 \ell_0 / \eta$ and $R_{e_\circ} \approx U_0 \ell_0 / \nu$, and the plasma beta $\beta_0 = 8\pi p_0 / B_0^2$.

By considering the turbulent-relaxation of the fluid in three dimensions, we can investigate the nonlinear mode coupling interaction of decaying compressible MHD turbulence [Eqs. (5)–(7)]. For this purpose, we have developed a fully three-dimensional compressible magnetohydrodynamic code.[24] All the fluctuations are initialized isotropically with random phases and amplitudes in Fourier space and evolved further by integration of Eqs. (5)–(7) using a fully de-aliased pseudospectral numerical scheme. Fourier spectral methods are remarkably successful in describing turbulent flows in a variety of plasma and hydrodynamic (i.e., nonmagnetized) fluids. Not only do they provide an accurate representation of the fluid fluctuations in Fourier space, but they are also nondissipative. Because of the latter, nonlinear mode coupling interactions preserve ideal rugged invariants of fluid flows, unlike finite difference or finite volume methods. The conservation of the ideal invariants (energy, enstrophy, magnetic potential, helicity, etc.) in turbulence is an extremely important feature in general, and *particularly* in our simulations, because these quantities describe the cascade of energy in the inertial regime, where turbulence is, in principle, independent of both large-scale forcing and small scale dissipation. The precise measurement of the decay rates associated with the MHD invariants is therefore of particular concern and importance in understanding MDR. Dissipation is nonetheless added physically in our simulations to extend the spectral cascades to the smallest scales and also to allow minimal dissipation. The evolution variables are discretized in Fourier space and we use periodic boundary conditions. The initial isotropic turbulent spectrum was chosen to be close to k^{-2} with random phases in all three directions. The choice of such (or even a flatter than -2) spectrum does not influence the dynamical evolution of the turbulent fluctuations as final the state in all our simulations leads to identical results that are consistent with the proposed analytic theory. The equa-

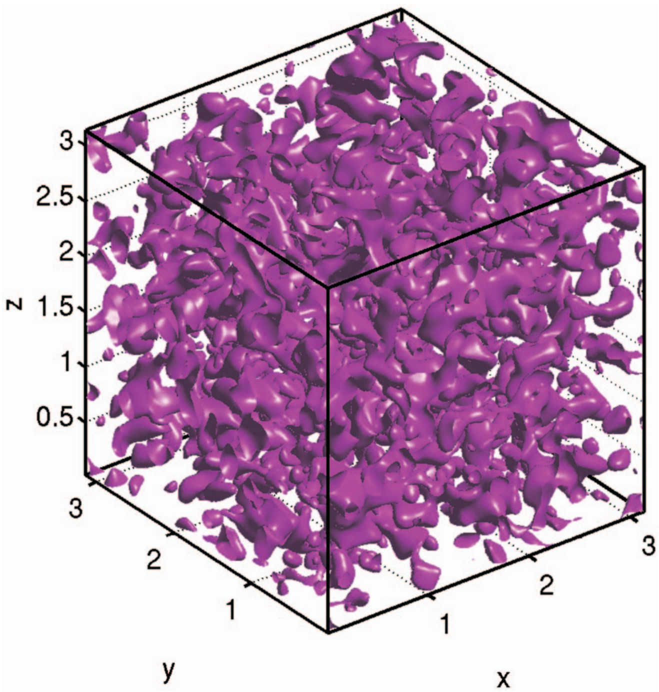

FIG. 1. (Color) Evolution of random initial turbulent fluctuations lead to the formation of relatively small-scale isotropic structures in 3D compressible MHD simulations. The numerical resolution is 128^3 in a cubic box of volume π^3. The dissipation parameter $\eta=\nu=10^{-4}$. Shown are the isosurfaces of the x component of the magnetic field.

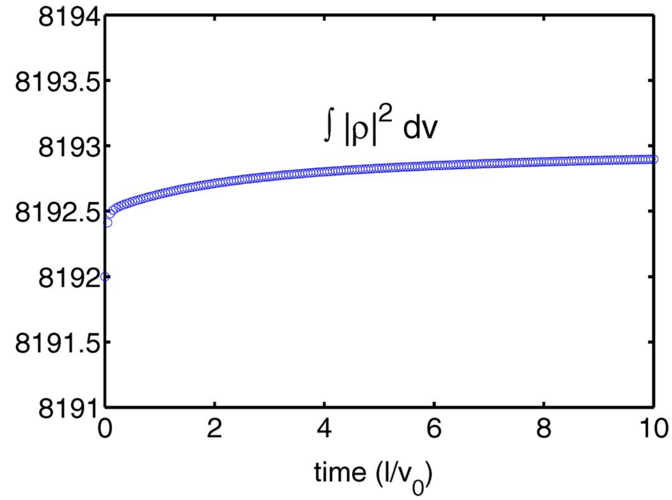

FIG. 2. (Color online) Evolution of the energy associated with the density fluctuations in 3D MHD simulations; i.e., $\int \rho^2 d\mathbf{v}$. In the absence of external sources or sinks, density is conserved. This quantity is used to check the validity of our numerical results while computing the decay rates of several MHD invariants.

tions are advanced in time using a second-order predictor-corrector scheme. The code is stabilized by a proper dealiasing of spurious Fourier modes and choosing a relatively small time step in the simulations. Additionally, the code preserves the $\nabla \cdot \mathbf{B}=0$ condition at each time step. Our code is massively parallelized using Message Passing Interface (MPI) libraries to facilitate higher resolution in a 3D volume. Kinetic and magnetic energies are also equipartitioned between the initial velocity and the magnetic fields. The latter helps treat the transverse or shear Alfvén and the fast/slow magnetosonic waves on an equal footing, at least during the early phase of the simulations.

Magnetized, fully compressible MHD turbulence evolves under the action of nonlinear interactions in which random initial fluctuations, containing sources of free energy, lead to the excitation of unstable modes. These modes often displace the initial evolutionary system substantially from its equilibrium state. When the instability begins to saturate, unstable modes lead to fully developed turbulence in which larger eddies transfer their energy to smaller ones through a forward cascade until the process is terminated by the small-scale dissipation. During this process, MHD turbulent fluctuations are dissipated gradually due to the finite Reynolds number, thereby damping small-scale motion as well. The energy in the smaller Fourier modes migrates towards the higher Fourier modes following essentially the vector triad interactions $\mathbf{k}+\mathbf{p}=\mathbf{q}$. These interactions involve the neighboring Fourier components $(\mathbf{k},\mathbf{p},\mathbf{q})$ that are excited in the local inertial range turbulence. A typical case of turbulent relaxation is shown in Fig. 1 in which the x component of the magnetic field generates relatively small-scale fully developed isotropic and homogeneous structures. Only isosurfaces of constant $|B_x|$ contours in three dimensions are shown in Fig. 1. Furthermore, we can follow the evolution of density fluctuations to monitor the validity of our simulations. For instance, in the absence of external sources or sinks, density is conserved. The volume integration of density in Eq. (5) is proportional to a surface integral that corresponds to the flux of plasma flow distributed across the volume. The latter, nevertheless, diminishes for a periodic system, thereby leading to $\int \rho^2 d\mathbf{v}=\text{const}$. This quantity is evaluated at each time step of the numerical integration to check the validity of our simulation results while computing the decay rates of several MHD invariants. The evolution of $\int \rho^2 d\mathbf{v}$, shown in Fig. 2, is almost unchanged.

The decay rates associated with the turbulent relaxation of the rugged ideal invariants of MHD, viz, magnetic helicity $(K=\int \mathbf{A}\cdot\mathbf{B}d\mathbf{v})$, magnetic energy $[E=(1/2)\int B^2 d\mathbf{v}]$ and energy dissipation rate $(R=\eta\int J^2 d\mathbf{v})$ are shown together in Fig. 3. Clearly, the magnetic energy E decays faster than the magnetic helicity K, and the energy dissipation (R) decays even faster than the two invariants. This state corresponds to minimum dissipation in which selective decay processes lead to a faster decay rate for the magnetic energy (when compared with the magnetic helicity decay rate). The selective decay process (in addition to dissipations) depends critically on the cascade properties associated with the rugged MHD invariants that eventually govern spectral transfer in the inertial range. This can be elucidated as follows.[23] The magnetic vector potential in 3D MHD dominates the magnetic field fluctuations at the smaller Fourier modes, which in turn leads to a domination of the magnetic helicity invariant over the magnetic energy. On the other hand, dissipation occurs predominantly at the higher Fourier modes, which introduces a rapid damping of the energy dissipative quantity R. A heuristic argument for this process can be formulated in the following way. The decay rates of helicity K and the dissipation

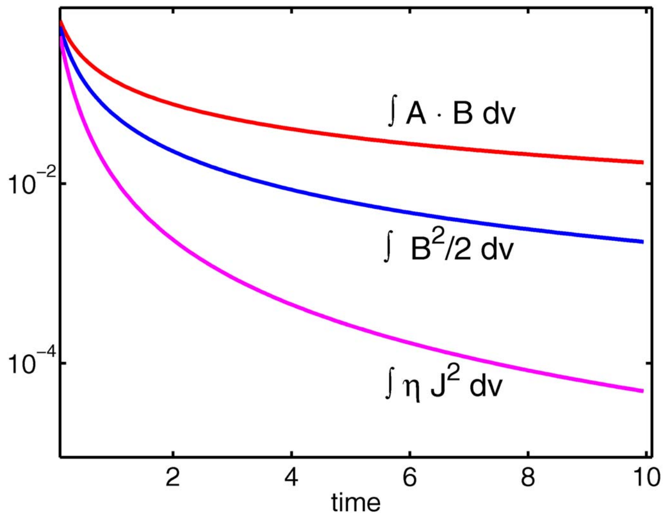

FIG. 3. (Color online) Evolution of decay rates associated with turbulent relaxation of the rugged ideal invariant of MHD, viz., magnetic helicity ($K=\int \mathbf{A}\cdot\mathbf{B}d\mathbf{v}$), magnetic energy ($E=1/2\int B^2 d\mathbf{v}$), and dissipative current ($R=\eta\int J^2 d\mathbf{v}$) are shown simultaneously.

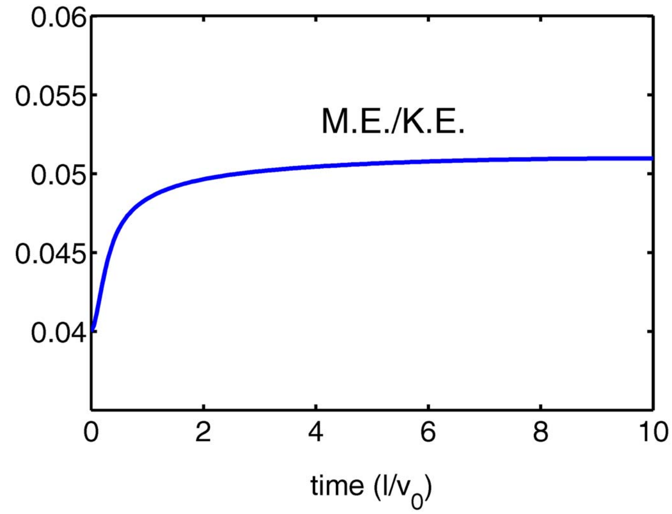

FIG. 4. (Color online) The decay rates of kinetic energy of turbulent fluctuations are initially higher than the magnetic energy. Hence the ratio of magnetic to kinetic energies shows a sharp rise in the initial evolution. The two decay rates eventually become identical, thereby leading to a constant value of the ratio.

rate for $R=\int_V \eta j^2 dV$, in dimensionless form, with the magnetic field Fourier decomposed as $\mathbf{B}(\mathbf{k},t)=\Sigma_k \mathbf{b}_\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r})$, can be expressed as

$$\frac{dK}{dt}=-\frac{2\eta}{S}\Sigma_k k\mathbf{b}_\mathbf{k}^2, \quad \frac{dR}{dt}=-\frac{2\eta^2}{S^2}\Sigma_k k^4\mathbf{b}_\mathbf{k}^2, \tag{8}$$

where $S=\tau_R/\tau_A$ is the Lundquist number and $\tau_R=4\pi\ell_0^2/c^2\eta_0$, $\tau_A=\ell_0/V_{A_0}$ are the resistive and Alfvén time scales, respectively. The Lundquist number in our simulations varies up to 10^6, corresponding to the parameter range $\beta_0=0.01$, $\eta_0\sim 10^{-4}$, $V_{A_0}\sim 0.1-1$, $\ell_0\sim 0.01-0.1$. We find that at scale lengths for which $k\approx S^{1/2}$, the decay rate of energy dissipation is $\sim O(1)1$. But at these scale lengths, helicity dissipation is only $\sim O(S^{-1/2})\ll 1$. This physical scenario is further consistent with our 3D simulations (Fig. 3). Interestingly, the decay rate of kinetic energy of turbulent fluctuations is initially higher than that of the magnetic energy. Thus, the ratio of magnetic to kinetic energy shows a sharp rise in the initial evolution, as shown in Fig. 4. However, as the evolution progresses, the two decay rates become identical and the ratio (magnetic to kinetic energies) is asymptote to a constant value (see Fig. 4). Although not shown in the figure, we have seen that $J_\perp=J-(\mathbf{J}\cdot\mathbf{B}/|\mathbf{B}|)\hat{b}$—where $\hat{b}=\mathbf{B}/|\mathbf{B}|$ is the unit vector along $\mathbf{B}$—is nonzero, thus proving conclusively that the resultant state is non-force-free. For the force-free state, $J_\perp$ is zero.

We further confirm in our simulations that a slight variation in the initial value of plasma beta (from $\beta_0=0.01$) does not alter the final MDR state. By contrast, a significant change in the initial plasma beta can altogether lead to a different final plasma state that may not necessarily be a representative of the MDR characteristic property within the realm of MHD. This can be elucidated as follows. For instance, an initial large value of plasma beta essentially characterizes the predominance of hydrodynamiclike effects in MHD,[24,25] where the nonlinear processes are governed predominantly by the eddy interactions, i.e., hydrodynamic convective force $\mathbf{V}\cdot\nabla\mathbf{V}$, instead of magnetic field forces ($\mathbf{J}\times\mathbf{B}$). In such a case, the final state is dominated possibly by the hydrodynamic invariants.

IV. CONCLUSION

In conclusion, we have performed self-consistent three-dimensional simulations of isothermal compressible dissipative magnetized plasmas to investigate the principle of minimum dissipation rates in a regime that corresponds to a high Lundquist number ($S\sim 10^6$), relevant typically to the fusion plasmas. Starting from a nonequilibrium state, the evolution of the MHD invariants, viz. magnetic helicity and magnetic energy, proceeds towards a minimum dissipation state in which magnetic helicity ($K=\int\mathbf{A}\cdot\mathbf{B}d\mathbf{v}$) dominates magnetic energy [$E=(1/2)\int B^2 d\mathbf{v}$] evolution, whereas the energy dissipation rates ($R=\eta\int J^2 d\mathbf{v}$) are predominantly minimal. The most important outcome to emerge from our investigations is that a state corresponding to the minimum dissipation rates, is more plausible in a dissipative plasma. Thus, we may conclude, notwithstanding the Taylor hypothesis of a *force-free state*, that our simulations clearly demonstrate that nonlinear selective decay processes will cause plasma fluctuations to relax towards a *non-force-free state* in a natural and self-consistent manner. Note that we have not yet included forcing effects, such as those due to solar flares, coronal mass ejecta, supernovae blasts, etc. Our work thus assumes that the magnetized compressible plasma is evolving freely without experiencing external forcing.

Finally, we reiterate that predictions from a relaxation model based on minimum dissipation rate yield the profiles for magnetic field components and pressure profile that are very distinct from those obtained for a force-free state described in terms of Chandrasekhar–Kendall functions. These various profiles for some of the laboratory plasma devices are shown in our earlier works.[16–19] The magnetic field components for reversed field pinch and spheromak obtained from our model correspond to the experimental observations and are similar to those obtained from a force-free model. For the tokamak (where the force-free model is not applicable), our model predicts magnetic field profile, q-profile that are very close to the experimentally observed profiles. The most important aspect of our model is the prediction of a nonzero pressure gradient, our model produces a pressure profile for all the above devices, whereas a force-free model always produces a zero pressure gradient.

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