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Inverse Problems on Electrical Networks and in Photoacoustic Tomography

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by

Christina Grace Knox

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This thesis is dedicated to the many excellent mathematics professors that inspired me to pursue a Ph.D.

ABSTRACT OF THE DISSERTATION

Inverse Problems on Electrical Networks and in Photoacoustic Tomography

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2019
Dr. Amir Moradifam, Chairperson

This dissertation investigates two inverse problems, one on electrical networks and another from photo acoustic tomography. First we consider the inverse problem of recovering the conductivities of an electrical network from the knowledge of the magnitude of the current along the edges coupled with either the voltage on the boundary of the network or the current flowing in or out of the network. This problem corresponds to finding the minimizers of a l^1 minimization problem. Additionally, we show that while the conductivities are not determined uniquely the flow of the current is uniquely determined. We will also present a convergent numerical algorithm for solving these problems along with basic numerical simulations. Lastly, we will discuss some applications of this inverse problem. Next we consider the inverse problem of determining both the source of a wave and its speed inside a medium from measurements of the solution of the wave equation on the boundary. This problem arises in photoacoustic and thermoacoustic tomography. We will present a brief overview of previous uniqueness results and then present our two original uniqueness results. If the reciprocal of the wave speed squared is harmonic in a simply connected region and identically one elsewhere then a wave speed satisfying a natural admissibility assumption can be uniquely determined from the solution of the wave equation on the boundary of domain without knowledge of the source. If the wave speed is known and only assumed to be bounded, then, under the same admissibility assumption, the source of the wave can be uniquely determined from boundary measurements.

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Chapter 1

Introduction

What constitutes an inverse problem? Perhaps the best way of understanding what an inverse problem is is to contrast it with that of a forward problem. In a forward problem given an equation with a set of parameters a solution is sought. In contrast, in an inverse problem parts or all of the solution to the equation is known and one seeks to recover the parameters. The part of the solution assumed to be known is commonly referred to as the data for the inverse problem. In practice this would be the measurements that are observed. The first problem usually considered when looking at an inverse problem is that of uniqueness. The question of uniqueness asks, if two sets of data are equal, must the parameters to be recovered also be equal? This dissertation will be focused on the question of uniqueness for two different inverse problems. On a practical level one is also interested in numerical algorithms to recover the parameters and also stability, that is how do errors in the measured data effect the recovery of the parameters. We have developed a numerical algorithm for one of the inverse problems studied in this dissertation.

1.1 An inverse problem on electrical networks

The first inverse problem we consider is on electrical networks. In this problem both exterior and interior data is known with the aim of recovering the conductivities of the graph. This is from a joint work with Amir Moradifam in [42]. The set up of this inverse problem is as follows. Let $G = (V, E)$ be a simple, undirected, weighted graph with n vertices. We can identify G with an electrical network by placing a resistor with resistance R_{ij} between every two vertices i and j, for $0 \leq i, j \leq n$ with $i \neq j$. We assign the weight $\sigma_{ij} = \frac{1}{R_{ij}}$ on each edge E_{ij}, and let $\sigma_{ij} = 0$ if i and j are not connected. Suppose a voltage is applied to a subset of the vertices, denoted by ∂V and called the boundary of V, then a current $J = (J_{ij})_{n\times n}$ will be induced on the edges of the graph, where J_{ij} is the current flowing from vertex i to vertex j. In particular, $J_{ij} = -J_{ji}$ and if the current flows from i to j, then $J_{ij} > 0$. We will also assume that $J_{ij} = 0$ if the vertices i and j are not connected by an edge, and that $J_{ii} = 0$. Note that $V = \partial V \cup \mathrm{int}(V) = \{1, 2, ..., n\}$. We will view the voltage potential on V as a vector $v = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$ where v_i is the voltage potential at vertex i. We will also denote the imposed voltage potential on the boundary nodes by a function $f : \partial V \to \mathbb{R}$. By Kirchhoff's and Ohm's Law

$$\sum_{j=1}^{n} \sigma_{ij}(v_i - v_j) = 0 \quad \text{for all } i \in \mathrm{int}(V), \tag{1.1}$$

where $\mathrm{int}(V) = V \setminus \partial V$ are the interior nodes, and $v = f$ on ∂V is the imposed voltage on the boundary nodes (Dirichlet boundary condition). Assume $((\sigma_{ij})_{n\times n}, f)$ is given on $E \times \partial V$. Then (1.1) can be written as a system of $m = |\mathrm{int}(V)|$ linear equations with m unknowns, i.e.

$$A_D v = b, \tag{1.2}$$

where v is a m dimensional column vector containing the unknown voltage values at the interior nodes, A_D is a $m \times m$ non-singular matrix (see Proposition 1 in chapter 2) depending on the conductivities, and b is a m dimensional column vector depending on the conductivities and the known voltage at the boundary. In particular the forward problem (1.1) always has a unique solution which is indeed the voltage potential associated to the conductivity problem on the network.

On the other hand if a current $0 \neq g \in \mathbb{R}^{|\partial V|}$ is injected to the network on a subset of vertices $\partial V \subset V$ (Neumann boundary condition), then we necessarily have

$$\sum_{i=1}^{|\partial V|} g_i = 0, \tag{1.3}$$

and by Kirchhoff's and Ohm's Law the voltage potential v satisfies

$$\begin{cases} \sum_{j=1}^{n} \sigma_{ij}(v_i - v_j) = 0 \quad \text{for all} \quad i \in \text{int}(V) \\ \\ \sum_{j=1}^{n} \sigma_{ij}(v_i - v_j) = g_i \quad \text{for all} \quad i \in \partial V. \end{cases} \tag{1.4}$$

The above equations can be written as

$$A_N v = b, \tag{1.5}$$

where A_N is an $n \times n$ matrix depending on the conductivity $\sigma = (\sigma_{ij})_{n \times n}$, and b is an n-dimensional column vector depending on the injected current on the boundary ∂V. The matrix A_N also has unique solutions up to adding a constant (see Propositions 17 and 18 in chapter 2) and the solution of (1.5) is the voltage potential on the vertices of the graph. The matrix A_N is in fact the well known graph laplacian of a weighted undirected graph.

As described above, the forward problems always have unique solutions up to a constant and can be easily solved by solving a linear system of equations. We are interested in the inverse problem of determining the conductivity matrix of an electrical network from the knowledge of the induced current along the edges of the network and Dirichlet or Neumann boundary conditions. This problem can also be understood as a design problem where one aims to design an electrical network that induces a prescribed current along its edges when a voltage $f \in \mathbb{R}^{|\partial V|}$ is applied to the boundary nodes ∂V, or when a current $g \in \mathbb{R}^{|\partial V|}$ is injected on ∂V. These inverse problems are in the spirit of Current Density Imaging (CDI) and Current Density Impedance Imaging (CDII) in dimensions $n \geq 2$ which have been actively studied in recent years because of their potential applications in medical imaging, see [29, 37, 31, 35, 39, 40, 46, 49, 50, 52, 51, 54, 55, 56, 53, 57]. In dimension $n = 3$ the induced current inside the conductive body Ω can be measured by Magnetic Resonance Imaging (MRI), see [29, 37].

To the authors' best knowledge the natural inverse problem considered in this paper has not been studied elsewhere. In [13] and [10], the authors investigate the problem of recovering the conductivity of the edges from the measurement of voltages at the boundary vertices, and measurements of the voltage, current, and conductivity on the boundary respectively. In [13] the authors proved injectivity of this inverse problem for critical, circular and planar graphs and provided an explicit reconstruction method. Under the assumption of monotonicity of conductivities, partial uniqueness results are established in [10]. While the general theory of inverse problems on graphs is a rich field of study with applications in various disciplines, the above results are most closely related to this work.

There is a close connection between electrical networks and random walks on graphs (see [14]). Random walks arise in many mathematical and physical models in biology, economics, computer and social networks, epidemiology, and statistical mechanics. The inverse problem we investigate here translates to intriguing questions in various contexts where a random walk model on graphs is utilized. The results could also be useful in the design of effective random walk models for achieving prescribed goals with random steps in a network. For instance, one can think of designing a random walk model with a prescribed high net number of times the walker passes along certain edges of the graph. In Section 2.5 we exploit this connection and apply our results on electrical networks to study the inverse problem of determining transition probabilities of random walk models from the net number of times the walker passes along the edges of the graph. We will also discuss a potential application of our results in public-key encryption, a seemingly unrelated problem.

The first chapter of this dissertation is organized as follows. In Section 2.1 we study the problem of determining the conductivity matrix of an electrical network from the knowledge of the magnitude of the induced current with Dirichlet boundary condition, and in Section 2.2 we study this problem with Neumann boundary data. In Section 2.4 we present a numerical algorithm for finding minimizers of the l^1 minimization problem we obtain in Sections 2.1 and 2.2. In Section 2.5 the connection between random walks and electrical networks is discussed and we apply our results on electrical networks to the inverse problem of determining transition probabilities from the net number of time a random walker passes along the edges of the graph.

1.2 An inverse problem in photoacoustic tomography

The second inverse problem we consider deals with the wave equation. This is from a joint work with Amir Moradifam in [41]. Consider the wave equation

$$\begin{cases} u_{tt} - c^2\Delta u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = f(x), \quad u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The function $c(x)$ is the speed of the wave and the function $f(x)$ represents the source of the wave. We assume that $c(x) - 1$ and $f(x)$ have compact support. We are interested in the inverse problem of recovering f and/or c from the measurements of the solution of the wave equation on $\partial\Omega$ given by the measurement operator

$$\Lambda_{f,c}(x,t) = u(x,t), \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{1.7}$$

We will study this problem in dimension n=3. This problem naturally arises in thermoacoustic (TAT) and photoacoustic (PAT) tomography, both of which have significant potential in clinical applications and biology [28, 43, 44, 44, 69, 70]. We shall refer to $\Lambda_{f,c}(x,t)$ as the PAT data. Photoacoustic tomography aims to combine the high contrast of electromagnetic waves with the high resolution of ultrasound. In photoacoustic tomography a laser probes the medium to be imaged and the absorption of energy generates small levels of heating. This thermoelastic expansion produces an acoustic wave which is measured using ultrasonic transducers outside the medium. This data is then used to recover optical properties of the medium. It is known, for example, that cancerous tissues absorbs more energy than healthy tissues ([69],[70]). There are many mathematical problems that arise in photoacoustic tomography such as uniqueness, stability, partial boundary data, and numerical reconstructions. In this dissertation we will consider only the problem of uniqueness, that is, if $\Lambda_{f_1,c_1}(x,t) = \Lambda_{f_2,c_2}(x,t)$ for two sound speed and source pairs (f_1, c_1) and (f_2, c_2), does it follows that $f_1 = f_2$ and/or $c_1 = c_2$? This inverse problem is called the first step of photoacoustic tomography or the qualitative step of photoacoustic tomography. The case when the sound speed c is previously known and we only look to reconstruct the source f has been extensively studied. The case when the sound speed $c(x)$ is unknown is still a wide open problem. In section 3.1 we provide a brief overview of previous uniqueness results in the literature, in both the case when the sound speed is known and unknown. In section 3.2 we state and prove our new uniqueness results. The first uniqueness result is for the sound speed c when it has a particular form or when a monotonicity condition is satisfied. The second uniqueness result is for the recovery of the source when the sound speed is known.

For completeness we will briefly present a model of the second step of photoacoustic tomography which takes place after the recovery of the source f. The model for this second step is the one considered in [5]. We consider the following model

$$f(x) = \Gamma(x)\sigma(x)w(x)$$

$$\begin{cases} -\nabla \cdot D(x)\nabla w + \sigma(x)u = 0, & x \in \Omega \\ w(x) = g(x) & x \in \partial\Omega \end{cases}$$

where Γ is the Grüneisen coeffcient, σ is the absorption coefficient, w is the intensity of radiation, D is the diffusion coefficient, and g is the initial illumination, that is the incoming source of radiation. The goal of the second step of photoacoustic tomography is to recover σ, Γ, and/or D from knowledge of f and g. It is also assumed that σ, Γ, and D are known on the boundary of Ω. See [5], [6], and [4] for some results in this problem. One result of interest in [5] is that it is not possible to recover σ, Γ, and D no matter the number of illuminations chosen. However if one of the three is known then two well chosen illuminations can uniquely determine the other two. In [4] the authors considered the problem of rotating measurements which necessitated the integration of the two steps.

Chapter 2

Electrical networks with prescribed current

2.1 Dirichlet boundary condition

In this section we study the inverse problem of determining the conductivity matrix $\sigma = (\sigma_{ij})_{n\times n}$ from the knowledge of its induced current $J = (J_{ij})_{n\times n}$ on E and the imposed voltage potential f on ∂V (Dirichlet boundary conditions). Let $G = (V, E)$ be an undirected, simple, connected graph with n vertices, and suppose a voltage is applied to some subset of the vertices inducing the current $J = (J_{ij})_{n\times n}$ on E. Throughout the paper $|J|$ denotes the matrix $|J| := (|J_{ij}|)_{n\times n}$, we will refer to $|J|$ as a measurement matrix.

We first show that the forward problem has a unique solution, i.e. A_D is non-singular. One can find a proof in [13] and we present a brief proof for the sake of completeness.

Proposition 1 *The matrix A_D is non-singular.*

Proof. For every $i \in int(V)$ it follows from (1.1) that v_i is the weighted average of the voltage potential in its neighboring nodes, i.e.

$$v_i = \frac{\sum_{j=1}^{n} \sigma_{ij} v_j}{\sum_{j=1}^{n} \sigma_{ij}}. \tag{2.1}$$

Consequently v satisfies the strong maximum principle in the sense that if v attains its maximum or minimum on an interior node, then v must be constant on V. In particular, v attains its minimum and maximum on the boundary ∂V.

Now suppose $A_D v = A_D \tilde{v} = b$. Then $w = v - \tilde{v}$ satisfies

$$\sum_{j=1}^{n} \sigma_{ij}(w_i - w_j) = 0 \quad \text{for all } i \in \text{int}(V).$$

Since $w = 0$ on ∂V, it follows from the above maximum principle that $w = 0$ on V. Thus the matrix A_D is non-singular. □

An immediate consequence of Proposition 1 is that the forward problem (1.1) always has a unique solution.

Definition 2 *We say that a vertex i is an interior vertex and write $i \in int(V)$ if*

$$J_i := \sum_{j=1}^{n} J_{ij} = 0.$$

Otherwise we say that i is boundary vertex and write $i \in \partial V$. For every $i \in \partial V$, J_i is the current flowing in ($J_i < 0$) or out ($J_j > 0$) of the graph at vertex i. In particular, $V = int(V) \cup \partial V$ and $int(V) \cap \partial V = \emptyset$.

Definition 3 *Given $f : \partial V \to \mathbb{R}$ and a measurement matrix $a = (a_{ij})_{n \times n}$ with $a_{ij} \in [0, \infty)$ for all $1 \leq i, j \leq n$ and $a_{ij} = 0$ when $i = j$ and $E_{i,j} \notin E$, we say that a symmetric matrix*

$\sigma = (\sigma_{ij})_{n\times n}$ with $\sigma_{ij} \in [0,\infty]$ is a conductivity matrix associated to the data (f,a), if there exists a function $v : \{1,2,...,n\} \to \mathbb{R}$ with $v|_{\partial V} = f$, and a matrix $J = (J_{ij})_{n\times n}$ such that

$$J_{ij} = \sigma_{ij}(v_i - v_j) \quad \textit{and} \quad |J_{ij}| = a_{ij} \textit{ for all } i,j \textit{ with } v_i \neq v_j,$$

and

$$\sum_{j=1}^{n} J_{ij} = 0$$

for all $i \in int(V)$. When $a_{ij} \neq 0$ and $v_i = v_j$, then we formally define $\sigma_{ij} = \infty$ and say that the edge between nodes i and j is a perfect conductor. We shall also refer to the function v as a voltage potential and denote the set of all voltage potentials corresponding to the data (f,a) by $\mathcal{V}_{(f,a)}$.

For any measurement matrix $a = (a_{ij})_{n\times n}$, define the function $I : \mathbb{R}^n \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\sum_{i,j} a_{ij}|u_i - u_j|, \tag{2.2}$$

and for $f \in \mathbb{R}^{|\partial V|}$ consider the minimization problem

$$\min\{I(u) : \quad u \in \mathbb{R}^n \ \text{ and } \ u|_{\partial V} = f\}. \tag{2.3}$$

We shall prove that $u \in \mathcal{V}_{(f,a)}$ if and only if it is a minimizer of the least gradient problem. Let us first study the dual of the minimization problem above.

2.1.1 The dual problem

Here we discuss the dual of the least gradient problem (2.3) and study the connection between these two problems.

Let $\mathcal{H}(V)$ be the set of all real valued functions on the vertices. We shall view a function $u \in \mathcal{H}(V)$ as a vector in $\mathbb{R}^n$. Also let $\mathcal{H}(E)$ to be the space of all functions on E, i.e. the space of all $n \times n$ matrices $b = (b_{ij})$, where b_{ij} denotes the value of the function on the edge from vertex i to j, with the additional convention that $b_{ij} = 0$ if the edge from i to j is not in E, and $b_{ii} = 0$.

Definition 4 *Let $u, v \in \mathcal{H}(V)$ and $a, b \in \mathcal{H}(E)$. Then we define the inner products*

$$\langle u, v\rangle_{\mathcal{H}(V)} = \sum_{i=1}^{n} u_i v_i, \qquad \langle b, d\rangle_{\mathcal{H}(E)} = \sum_{i,j} b_{ij} d_{ij} \tag{2.4}$$

on $\mathcal{H}(V) \times \mathcal{H}(V)$ and $\mathcal{H}(E) \times \mathcal{H}(E)$, respectively. The spaces $\mathcal{H}(V)$ and $\mathcal{H}(E)$ equipped with the above inner products are Hilbert spaces.

Next we define two linear operators $D : \mathcal{H}(V) \to \mathcal{H}(E)$ and $\mathrm{div} : \mathcal{H}(E) \to \mathcal{H}(V)$ which play crucial roles in our arguments.

Definition 5 *For $u \in \mathcal{H}(V)$ we define $Du \in \mathcal{H}(E)$ as*

$$(Du)_{ij} = u_i - u_j \tag{2.5}$$

if the edge connecting i to j is in E, and 0 otherwise. Also for $b \in \mathcal{H}(E)$ we define $\mathrm{div}\, b \in \mathcal{H}(V)$ *as follows*

$$(\mathrm{div}\, b)_i = \sum_j b_{ji} - b_{ij}. \tag{2.6}$$

Observe that if $b \in \mathcal{H}(E)$ is anti-symmetric, that is $b_{ij} = -b_{ij}$ for all $1 \leq i, j \leq n$, then the divergence is simply $-2\sum_j b_{ij}$. We shall refer to D and div operators as gradient and divergence, respectively, since they play the role in our setting of the standard gradient and divergence operators on $\mathbb{R}^n$, $n \geq 2$. Note that the definition of the gradient and divergence given here does not depend on the weights (conductivities) of the graph as it would normally when defining these operators on a weighted graph. Since in the inverse problems we consider in this paper, the conductivities are unknown, these definitions are desirable. Let us first show that $-\text{div}$ is the adjoint of D.

Proposition 6 *Let $u \in \mathcal{H}(V)$ and $b \in \mathcal{H}(E)$. Then*

$$\langle u, -\text{div} b\rangle_{\mathcal{H}(V)} = \langle Du, b\rangle_{\mathcal{H}(E)}.$$

Proof. Let $u \in \mathcal{H}(V)$ and $b \in \mathcal{H}(E)$. Then

$$\begin{aligned}
\langle u, -\text{div} b\rangle_{\mathcal{H}(V)} &= \sum_i u_i(-(\text{div} b)_i) \\
&= \sum_i u_i \sum_j (b_{ij} - b_{ji}) \\
&= \sum_i \sum_j u_i b_{ij} - \sum_j \sum_i u_j b_{ij} \\
&= \sum_{i,j} (u_i - u_j) b_{ij} \\
&= \sum_{i,j} (Du)_{ij} b_{ij} \\
&= \langle Du, b\rangle_{\mathcal{H}(E)}.
\end{aligned}$$

□

Let $f \in \mathbb{R}^{|\partial V|}$ and define

$$\mathcal{H}_f = \{u \in \mathcal{H}(V) : u|_{\partial V} = f\}.$$

For $a \in \mathcal{H}(E)$ we take $a \geq 0$ to mean that every entry is non-negative. Then for $0 \leq a \in \mathcal{H}(E)$ and $f \in \mathbb{R}^{|\partial V|}$, the least gradient problem (2.3) can be written as

$$\min_{u \in \mathcal{H}_f} \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j| = \min_{u \in \mathcal{H}_f} \frac{1}{2} \langle a, |Du|\rangle_{\mathcal{H}(E)}, \tag{2.7}$$

where we have used the notation $|Du|_{ij} = |(Du)_{ij}|$. We point out at this point that $\mathcal{H}_f$ is not a Hilbert space. We will redefine the problem so that we can work with a Hilbert space.

Choose $u_f \in \mathcal{H}_f$. Define $\mathcal{H}_0(V) \subset \mathcal{H}(V)$ to be the space of functions on V which are equal to zero on ∂V. Then we can equivalently write the primal problem (2.7) as

$$\min_{u \in \mathcal{H}_0(V)} \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j + (u_f)_i - (u_f)_j| = \min_{u \in \mathcal{H}_0(V)} \frac{1}{2} \langle a, |Du + Du_f| \rangle_{\mathcal{H}(E)}. \tag{2.8}$$

Define $F : \mathcal{H}(E) \to \mathbb{R}$ and $G : \mathcal{H}_0(V) \to \mathbb{R}$ as follows

$$F(d) = \frac{1}{2} \langle a, |d + Du_f| \rangle_{\mathcal{H}(E)} \quad \text{and} \quad G(u) \equiv 0. \tag{2.9}$$

Then (2.8) can be written as

$$(P) \qquad \alpha_P := \min_{u \in \mathcal{H}_0(V)} F(Du) + G(u).$$

Before discussing the dual of this problem we will provide some background. The following contains information from [60]. We first introduce the convex conjugate function.

Definition 7 *Let V be a vector space and V^* its dual. For $F : V \to \bar{\mathbb{R}}$ define a function $F^* : V^* \to \bar{\mathbb{R}}$ by*

$$F^*(u^*) = \sup_{u \in V} \{ < u, u^* > - F(u) \}.$$

We call F^* the convex conjugate of F.

We then define Rockafellar-Fenchel duality.

Definition 8 *Given the paired space V and V^*, Y and Y^* assume that Λ is a continuous linear operator from V into Y with adjoint Λ^* from Y^* into V^*. Further assume that V is a reflexive Banach space. Consider the primal problem*

$$\inf_{u \in V} \{ F(u) + G(\Lambda u) \}.$$

Then the corresponding dual problem is

$$\sup_{p\in Y^*}\{-F^*(\Lambda^* p^*) - G(-p^*)\}.$$

The primal problem 2.8 then admits a dual problem which can be expressed as

$$\max_{b\in\mathcal{H}(E)} \; -G^*(-\text{div} b) - F^*(-b). \tag{2.10}$$

We will look at computing the convex conjugate of our F and G. We first have that

$$\begin{aligned} G^*(u) &= \sup_{v\in\mathcal{H}_0(V)} \sum_i u_i v_i \\ &= \begin{cases} 0 & \text{if } u \equiv 0 \text{ on } int(V) \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Next we compute the convex conjugate of F.

Lemma 9 *Let $a = (a_{ij}) \in \mathcal{H}(E)$ with $a_{ij} \geq 0$ and $u_f \in \mathcal{H}_f(V)$. Then*

$$F^*(b) = \begin{cases} -\langle b, Du_f\rangle_{\mathcal{H}(E)} & \text{if} \quad |b| \leq \frac{1}{2}a \\ \infty & \textit{otherwise.} \end{cases} \tag{2.11}$$

Proof. Suppose $|b| \leq \frac{1}{2}a$, that is $|b_{ij}| \leq \frac{1}{2}a_{ij}$ for all i, j. Then

$$\begin{aligned} F^*(b) &= \sup_{d\in\mathcal{H}(E)} (\langle d, b\rangle_{\mathcal{H}(E)} - \frac{1}{2}\langle a, |d + Du_f|\rangle_{\mathcal{H}(E)}) \\ &= -\langle b, Du_f\rangle_{\mathcal{H}(E)} + \sup_{d\in\mathcal{H}_a(E)} (\langle d, b\rangle_{\mathcal{H}(E)} - \frac{1}{2}\langle a, |d|\rangle_{\mathcal{H}(E)}) \\ &= -\langle b, Du_f\rangle_{\mathcal{H}(E)} + \sup_{d\in\mathcal{H}_a(E)} (\sum_{i,j} d_{ij}b_{ij} - \frac{1}{2}a_{ij}|d_{ij}|) \\ &\leq -\langle b, Du_f\rangle_{\mathcal{H}(E)} + \sup_{d\in\mathcal{H}_a(E)} \sum_{i,j} |d_{ij}|(|b_{ij}| - \frac{1}{2}a_{ij}) \\ &\leq -\langle b, Du_f\rangle_{\mathcal{H}(E)}. \end{aligned}$$

Taking $d = 0$ we also get $F^*(b) \geq -\langle b, Du_f \rangle_{\mathcal{H}(E)}$.

Now suppose that there exists $1 \leq i_0, j_0 \leq n$ such that $|b_{i_0 j_0}| > \frac{1}{2} a_{i_0 j_0}$. Let $d_{i_0 j_0} = \lambda b_{i_0 j_0}$, and $d_{ij} = 0$ otherwise, where $\lambda \in \mathbb{R}$. Then we have

$$\begin{aligned}
F^*(b) &= -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{d \in \mathcal{H}_a(E)} (\sum_{i,j} d_{ij} b_{ij} - \frac{1}{2} a_{ij} |d_{ij}|) \\
&\geq -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{\lambda > 0} \lambda (b_{i_0 j_0}^2 - \frac{1}{2} a_{i_0 j_0} |b_{i_0 j_0}|) \\
&= -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{\lambda > 0} \lambda |b_{i_0 j_0}| (|b_{i_0 j_0}| - \frac{1}{2} a_{i_0 j_0}) \\
&= \infty.
\end{aligned}$$

□

Thus the dual problem (2.10) can be written as

$$(D) \qquad \alpha_D := \sup\{-\langle b, Du_f \rangle_{\mathcal{H}(E)} : \quad b \in \mathcal{H}(E), \quad |b| \leq \frac{1}{2} a, \\ \text{and } \operatorname{div}(b) \equiv 0 \text{ on } int(V)\}.$$

Given that $u_i = 0$ for at least one $i \in V$ one can show that any minimizing sequence of the the primal problem is uniformly bounded. Hence a convergent subsequence exists and a minimizer of the primal problem (P) always exists. We now consider whether a solution to the dual problem (D) always exists. We will apply the following theorem from [60] (Theorem III.4.1).

Theorem 10 *Given the paired spaces V and V^*, Y and Y^* assume that Λ is a continuous linear operator from V into Y with adjoint Λ^* from Y^* into V^*. Suppose the primal problem, (P) takes the form*

$$\inf_{u \in V} J(u, \Lambda u)$$

where J is a convex function from $V \times Y$ into $\bar{\mathbb{R}}$. The corresponding dual problem (D) is

$$\sup_{p^* \in Y^*} -J^*(\Lambda^* p^*, -p^*).$$

Assume that $\inf(P)$ *is finite and that there exists $u_0 \in V$ such that $J(u_0, \Lambda u_0) < \infty$ and the function $p \mapsto J(u_0, p)$ is continuous at Λu_0. Then*

$$\inf(P) = \sup(D)$$

and the dual problem (D) has at least one solution $\bar{p}^*$.

Since $I(u) = \frac{1}{2}\langle a, |Du + Du_f|\rangle_{\mathcal{H}(E)}$ is convex and $J : \mathcal{H}(E) \to \mathbb{R}$ with $J(p) = \frac{1}{2}\langle a, |p|\rangle_{\mathcal{H}(E)}$ is continuous at $p = 0$, the conditions in the statement of the Theorem 10 are satisfied. Thus a solution to the dual problem (D) always exists. The weighted l^1 minimization problem (2.3) does not have an unique minimizer and thus the conductivity inducing the current J on E is not unique. However we can characterize the non-uniqueness.

Theorem 11 *The infimum of the primal problem (P) is equal to the supremum of the dual problem (D). Moreover, the dual problem has an optimal solution b, and $J = -2b$ satisfies*

$$|J_{ij}| = a_{ij} \text{ for every } i, j \text{ with } v_i \neq v_j \tag{2.12}$$

and

$$J_{ij}(v_i - v_j) \geq 0 \text{ for all } 1 \leq i, j \leq n, \tag{2.13}$$

for every minimizer v of (2.3)*. Conversely, if $u \in \mathcal{H}_f$ and the above equation holds then then u is a minimizer of* (2.3)*.*

Proof. A solution b to the dual problem always exists and the infimum of the primal problem (P) is equal to the supremum of the dual problem by Theorem III.4.1 in [17] as discussed above. Let v be a minimizer of (2.3). Then

$$\alpha_P = I(v) = \frac{1}{2}\sum_{i,j} a_{ij}|v_i - v_j| \geq \sum_{i,j} |b_{ij}||v_i - v_j| \geq \sum_{i,j} -b_{ij}(v_i - v_j) \tag{2.14}$$

$$=\langle -b, Dv\rangle_{\mathcal{H}(E)} = \langle \text{div}b, v\rangle_{\mathcal{H}(V)}$$

$$= \sum_{i\in\partial V} (\text{div}b)_i v_i = \sum_{i\in\partial V} (\text{div}b)_i f_i = \alpha_D = \alpha_P.$$

Hence the inequalities in 2.14 are indeed equalities and thus

$$|b_{ij}| = \frac{1}{2}a_{ij} \text{ for every } i,j \text{ with } v_i \neq v_j$$

and

$$b_{ij}(v_i - v_j) \leq 0 \text{ for all } 1 \leq i,j \leq n.$$

Therefore if we let $J = -2b$ we we see that (2.12) and (2.13) hold. We can also see that the converse also holds from the above computations. □

Corollary 12 *If u and v are two arbitrary minimizers of (2.3), then*

$$(u_i - u_j)(v_i - v_j) \geq 0 \text{ for all } 1 \leq i,j \leq n.$$

2.1.2 Voltage potentials have minimum energy

We are now ready to prove the following theorem.

Theorem 13 *Let f be a function on ∂V and a be a measurement matrix. Then $v \in \mathcal{V}_{(f,a)}$ if and only if it is a minimizer of the least gradient problem* (2.3)*.*

Proof. Suppose $v \in \mathcal{V}_{(f,a)}$ and let J be the corresponding current on E. Then

$$\begin{aligned} I(v) = \frac{1}{2}\sum_{i,j} a_{ij}|v_i - v_j| &= \frac{1}{2}\sum_{i,j}|J_{ij}||v_i - v_j| \geq \frac{1}{2}\sum_{i,j} J_{ij}(v_i - v_j) \\ &= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} J_{ij} = \sum_{i \in int(V)} v_i J_i + \sum_{i \in \partial V} v_i J_i \\ &= \sum_{i \in \partial V} v_i J_i = \sum_{i \in \partial V} f_i J_i. \end{aligned} \tag{2.15}$$

Therefore the minimum of the least gradient problem (2.3) is equal to $\sum_{i \in \partial V} f_i J_i$. Moreover the minimum is achieved for every $v \in \mathcal{V}_{(f,|J|)}$.

Now suppose v is a minimizer of the problem (2.3) and let b be a solution of the dual problem (D) and let $J = -2b$. Then by Theorem 11

$$|J_{ij}| = a_{ij} \text{ for all } i,j \text{ with } v_i \neq v_j$$

and since $\text{div}J = 0$ on $int(V)$

$$\sum_{j=1}^{n} J_{ij} = 0 \text{ for all } i \in int(V).$$

For $v_i \neq v_j$ define $\sigma_{ij} = \frac{J_{ij}}{v_i - v_j} \geq 0$. Then

$$J_{ij} = \sigma_{ij}(v_i - v_j) \quad \text{for all } \ i,j \ \text{ with } \ v_i \neq v_j.$$

Thus $v \in \mathcal{V}_{(f,a)}$ and the proof is complete. □

Remark 14 *Note that every minimizer v of* (2.3) *uniquely determines a conductivity matrix σ. Corollary 12 indicates that the directions of the flow of the current along the edges is unique, despite multiplicity of the minimizer of* (2.3). *Indeed if two conductivity matrices σ^1 and σ^2 with $0 \leq \sigma^1_{ij}, \sigma^2_{ij} < \infty$ induce the currents J^1 and J^2 on a network when the* *voltage f is imposed on ∂V, and $|J^1| = |J^2|$, then $J^1 = J^2$. This is a counter-intuitive result.*

2.1.3 Multiple measurements

Suppose we have two data sets (f^1, a^1) and (f^2, a^2), and would like to find a conductivity matrix σ inducing the currents with magnitudes a^1 and a^2, when the voltage potentials f^1 and f^2 are imposed on the boundary vertices ∂V^1 and ∂V^2, respectively.

Let I^1 and I^2 be defined by Equation (2.2) for a^1 and a^2 respectively and for $u = (u^1, u^2) \in \mathbb{R}^n \times \mathbb{R}^n$ define

$$\Phi(u^1, u^2) = \sum_{\mathcal{C}^2} \left| \frac{u_i^1 - u_j^1}{|J_{ij}^1|} - \frac{u_i^2 - u_j^2}{|J_{ij}^2|} \right|^2, \tag{2.16}$$

where

$$\mathcal{C}^2 = \{(i,j) : \quad 1 \le i, j \le n \;\text{ and }\; J_{ij}^1, J_{ij}^2 \neq 0\}.$$

Define

$$\mathcal{F}(u^1, u^2) = I^1(u^1) + I^2(u^2) + \Phi(u^1, u^2) \tag{2.17}$$

and

$$\mathcal{A} := \{(u^1, u^2) \in \mathbb{R}^n \times \mathbb{R}^n : \quad u^1 = f^1 \;\text{ on } \partial V^1 \;\text{ and }\; u^2 = f^2 \;\text{ on } \partial V^2\}.$$

Now consider

$$\inf_{(u^1,u^2)\in\mathcal{A}} \mathcal{F}(u^1, u^2). \tag{2.18}$$

It is easy to see that (2.18) always has a minimizer.

Theorem 15 *Let (u^1, u^2) be a minimizer of (2.18).*

1. *If there exists a conductivity matrix* σ *which induces the current* J^i *with* $|J^i| = a^i$ *when the voltage potential* f^i *is imposed on the boundary, denoted* $\partial^i V$, $i = 1, 2$, *then* $\Phi(u^1, u^2) = 0$. *Moreover,*

$$\sigma_{ij} = \frac{a^1_{ij}}{|u^1_i - u^1_j|} \quad \textit{for all} \quad i, j \quad \textit{with} \quad u^1_i \neq u^1_j,$$

and

$$\sigma_{ij} = \frac{a^2_{ij}}{|u^2_i - u^2_j|} \quad \textit{for all} \quad i, j \quad \textit{with} \quad u^2_i \neq u^2_j.$$

2. *If there doesn't exist a conductivity matrix* σ *inducing the current* J^i *with* $|J^i| = a^i$ *when the voltage potential* f^i *is imposed on the boundary noted* ∂V^i, $i = 1, 2$, *then* $\Phi(u^1, u^2) > 0$.

Proof. (1) Suppose there exists a conductivity matrix σ producing the data (f^1, a^1) and (f^2, a^2). It follows directly from Theorem 13 that the set of minimizers of (2.18) is equal to $\mathcal{V}_{(f^1,a^1)} \times \mathcal{V}_{(f^2,a^2)}$. So the first statement follows.
(2) Suppose $\Phi(u^1, u^2) = 0$. Then u^1 and u^2 minimize I^1 and I^2 over the appropriate spaces and so by Theorem 13, $u^1 \in \mathcal{V}_{(f^1,a^1)}$ and $u^2 \in \mathcal{V}_{(f^2,a^2)}$ and thus they each have corresponding conductivity matrices σ^1 and σ^2 that generate currents J^1 and J^2 respectively. However $\Phi(u^1, u^2) = 0$ implies that these conductivities are in fact equal. □

Now suppose a finite data set of measurements is given:

$$(f^1, a^1), (f^2, a^2), ..., (f^k, a^k), \quad k \geq 2.$$

Define

$$I^l = \frac{1}{2} \sum_{ij} a^l_{ij} |u_i - u_j|, \quad 1 \leq l \leq k,$$

and

$$\Phi^k(u^1, u^2, ..., u^k) = \sum_{l=2}^{k} \sum_{\mathcal{B}^l} \left| \frac{u_i^1 - u_j^1}{|J_{ij}^1|} - \frac{u_i^l - u_j^l}{|J_{ij}^l|} \right|^2,$$

where

$$\mathcal{C}^l = \{(i,j) : \quad 1 \le i,j \le n \ \text{ and } \ J_{ij}^1, J_{ij}^l \ne 0\}.$$

Consider the weighted l^1 minimization problem

$$\inf_{(u^1,u^2,...,u^k)\in\mathcal{A}^k} \sum_{l=1}^{k} I^l(v^l) + \Phi^k(u^1, u^2, ..., u^k), \tag{2.19}$$

where

$$\mathcal{A}^k := \{(u^1, u^2, ..., u^k) : \quad u^l \in \mathbb{R}^n \ \text{ and } \ u^l = f^l \ \text{ on } \ \partial V^l, \quad i = 1, 2, ..., k\}.$$

One can similarly prove the following theorem.

Theorem 16 *Let $(u^1, u^2, ..., u^k)$ be a minimizer of* (2.19).

1. *If there exists a conductivity matrix σ which induces the current J^l with $|J^l| = a^l$ when the voltage potential f^l is imposed on the boundary noted ∂V^l, $l = 1, 2, ..., k$, then $\Phi(u^1, u^2, ..., u^k) = 0$. Moreover,*

$$\sigma_{ij} = \frac{a_{ij}^l}{|u_i^l - u_j^l|} \quad \textit{for all } \ i,j \ \textit{ with } \ u_i^l \ne u_j^l, \quad l = 1, 2, ..., k.$$

2. *If there doesn't exist a conductivity matrix σ inducing the current J^l with $|J^l| = a^l$ when the voltage potential f^l is imposed on the boundary noted ∂V^l, $l = 1, 2, 3, ..., k$, then $\Phi(u^1, u^2, ..., u^k) > 0$.*

2.2 Neumann boundary condition

Let $G = (V, E)$ be an undirected simple connected graph with n vertices, and suppose the current $0 \neq g \in \mathbb{R}^{|\partial V|}$ is injected to a subset ∂V of V, regarded as boundary of V, inducing the current $J = (J_{ij})$ on E. Then g should satisfy the compatibility assumption

$$\sum_{i=1}^{|\partial V|} g_i = 0. \tag{2.20}$$

We will again denote $|J| := (|J_{ij}|)_{n\times n}$ and refer to $|J|$ as a measurement matrix. The following proposition characterizes solutions of the forward problem (1.4).

Proposition 17 *Let A_N be the matrix defined in* (1.5). *Then*

$$Ker(A_N) = \{(c, c, ..., c) \in \mathbb{R}^n : \quad c \in \mathbb{R}\}.$$

Proof. Suppose $A_N w = 0$ for some $w \in \mathbb{R}^n$. Then it follows from (1.4) that

$$\begin{aligned} \frac{1}{2}\sum_{i,j} \sigma_{ij}(w_i - w_j)^2 &= \frac{1}{2}\sum_{i=1}^n w_i \sum_{j=1}^n \sigma_{ij}(w_i - w_j) - \frac{1}{2}\sum_{j=1}^n w_j \sum_{i=1}^n \sigma_{ij}(w_i - w_j) \\ &= \sum_{i=1}^n w_i \sum_{j=1}^n \sigma_{ij}(w_i - w_j) \\ &= 0. \end{aligned}$$

Hence $w_i = w_j$ for all i and j connected by an edge. Since G is connected the proof is complete. □

Proposition 18 *The equation $A_N v = b$ has a solution if and only if $\sum_{i=1}^n b_i = 0$.*

Proof. By the Fredholm Alternative from linear algebra, $A_N v = b$ has a solution if and only if $b \in Ker({A_N}^T)^\perp$. By the previous proposition and the fact that A_N is symmetric we have

$$Ker({A_N}^T)^\perp = Ker(A_N)^\perp = \{b \in \mathbb{R}^N : \quad \sum_{i=1}^n b_i = 0\}.$$

□

Therefore if $\sum_{i=1}^n b_i = 0$, up to adding a constant the equation (1.4) has a unique solution. The following is the analog to Definition 3.

Definition 19 *Given* $0 \neq g : \partial V \to \mathbb{R}$ *satisfying* $\sum_{i=1}^{|\partial V|} g_i = 0$ *and a measurement matrix* $a = (a_{ij})_{n\times n}$ *with* $a_{ij} \in [0,\infty)$ *for all* $1 \leq i, j \leq n$ *and* $a_{ij} = 0$ *when* $i = j$ *and* $E_{ij} \notin E$, *we say that a symmetric matrix* $\sigma = (\sigma_{ij})_{n\times n}$ *with* $\sigma_{ij} \in [0,\infty]$ *is a conductivity matrix associated to the data* (g, a), *if there exists a function* $v : \{1, 2, ..., n\} \to \mathbb{R}$ *with and a matrix* $J = (J_{ij})_{n\times n}$ *such that*

$$J_{ij} = \sigma_{ij}(v_i - v_j) \quad \textit{and} \quad |J_{ij}| = a_{ij} \textit{ for all } i, j \textit{ with } v_i \neq v_j,$$

$$\sum_{j=1}^{n} J_{ij} = g_i \textit{ for all } i \in \partial V$$

and

$$\sum_{j=1}^{n} J_{ij} = 0 \textit{ for all } i \in int(V).$$

When $a_{ij} \neq 0$ *and* $v_i = v_j$, *then we formally define* $\sigma_{ij} = \infty$ *and say that the edge between nodes* i *and* j *is a perfect conductor. We shall also refer to the function* v *as a voltage potential and denote the set of all voltage potentials corresponding to the data* (g, a) *by* $\mathcal{V}_{(g,a)}$.

For a measurement matrix $a = (a_{ij})_{n\times n}$, define the function $I : \mathbb{R}^n \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\sum_{i,j} a_{ij}|u_i - u_j|. \tag{2.21}$$

Also for $g \in \mathbb{R}^{|\partial V|}$ satisfying (2.20) define

$$\mathcal{M}_g := \{u \in \mathbb{R}^n : \sum_{i\in\partial V} u_i g_i = 1\}.$$

We shall prove that the voltage potential is a minimizer of the l^1 minimization problem

$$\min_{u\in\mathcal{M}_g}\frac{1}{2}\sum_{i,j}a_{ij}|u_i-u_j|. \tag{2.22}$$

Let us first study the dual of this problem.

The dual problem

In this section we discuss the dual of the least gradient problem (2.22) and study its connection to the primal problem. Let $0\neq g\in\mathbb{R}^{|\partial V|}$ satisfying (2.20). Choose $u_g\in\mathcal{H}(V)$ such that

$$\sum_{i\in\partial V}(u_g)_i g_i=1.$$

Define

$$\mathcal{M}_0:=\{u\in\mathcal{H}(V):\ \sum_{i\in\partial V}u_i g_i=0\}.$$

Then we can equivalently write the primal problem (2.22) as

$$\min_{u\in\mathcal{M}_0}\frac{1}{2}\sum_{i,j}a_{ij}|u_i-u_j+(u_g)_i-(u_g)_j|=\min_{u\in\mathcal{M}_0}\frac{1}{2}\langle a,|Du+Du_g|\rangle_{\mathcal{H}(E)}. \tag{2.23}$$

Define $F:\mathcal{H}(E)\to\mathbb{R}$ and $G:\mathcal{M}_0\to\mathbb{R}$ as follows

$$F(d)=\frac{1}{2}\langle a,|Du+Du_g|\rangle_{\mathcal{H}(E)}\ \text{ and }\ G(u)\equiv 0. \tag{2.24}$$

Then (2.23) can be written as

$$(P_N)\qquad \alpha_{P_N}:=\min_{u\in\mathcal{M}_0}F(Du)+G(u).$$

As before this problem admits a dual problem which can be expressed as

$$\max_{b\in\mathcal{H}(E)}-G^*(-\mathrm{div}b)-F^*(-b). \tag{2.25}$$

From Lemma 9 we have

$$F^*(b) = \begin{cases} -\langle b, Du_g \rangle_{\mathcal{H}(E)} & \text{if} \quad |b| \leq \frac{1}{2}a \\ \\ \infty & \text{otherwise.} \end{cases}$$

Next we compute G^*.

Lemma 20 *Let $G : \mathcal{M}_0 \to \mathbb{R}$ be defined as $G \equiv 0$. Then for $G^* : (\mathcal{M}_0)^* \to \mathbb{R}$ we have*

$$G^*(D^*b) = \begin{cases} 0 & \textit{if} \;\; b \in \mathcal{B} \\ \\ \infty & \textit{otherwise,} \end{cases} \tag{2.26}$$

where

$$\mathcal{B} := \{b \in \mathcal{H}(E) : \quad \text{div} b \equiv 0 \textit{ on } int(V) \textit{ and } (\text{div} b)_i = \lambda g_i \textit{ for all } i \in \partial V,$$
$$\textit{for some } \lambda \in \mathbb{R}\}.$$

Proof. First note that

$$\begin{aligned} G^*(D^*b) &= \sup_{u \in \mathcal{M}_0} \langle D^*b, u \rangle_{\mathcal{H}(V)} = \sup_{u \in \mathcal{M}_0} \langle b, Du \rangle_{\mathcal{H}(E)} = \sup_{u \in \mathcal{M}_0} -\langle \text{div} b, u \rangle_{\mathcal{H}(V)} \\ &= \begin{cases} 0 & \text{if div} b \in {\mathcal{M}_0}^{\perp} \\ \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Let $h \in \mathcal{H}(V)$ with $h_i = 0$ if $i \in int(V)$ and $h_i = g_i$ if $i \in \partial V$, and

$$N = \{\lambda h : \lambda \in \mathbb{R}\} \subset \mathcal{H}(V).$$

Observe that $\mathcal{M}_0 = \{u \in \mathcal{H}(V) : \langle h, u \rangle_{\mathcal{H}(V)} = 0\}$. Hence $\mathcal{M}_0 = N^{\perp}$. Since $N^{\perp\perp} = N$ (see [27]),

$$M_0^{\perp} = N,$$

and the result follows. □

Therefore the dual problem (2.25) can be written as

$$(D_N) \qquad \alpha_{D_N} := \sup_{b \in \mathcal{D}} \{-\langle b, Du_g \rangle_{\mathcal{H}(E)}\},$$

where $\mathcal{D} = \{b \in \mathcal{B} : |b| \leq \frac{1}{2}a\}$.

Similar to before one can show that (2.22) has a minimizer. Similar to the Dirichlet boundary condition case, it follows from Theorem III.4.1 in [17] that the dual problem (D_N) also has a solution and characterizes the non-uniqueness of solutions of the primal problem (2.22).

Theorem 21 *The infimum of the primal problem (P_N) is equal to the supremum of the dual problem (D_N). Moreover, the dual problem has an optimal solution b, and $J = -2b$ satisfies*

$$|J_{ij}| = a_{ij} \textit{ for every } i, j \textit{ with } u_i \neq u_j \tag{2.27}$$

and

$$J_{ij}(u_i - u_j) \geq 0 \textit{ for all } 1 \leq i, j \leq n, \tag{2.28}$$

for every minimizer u of (2.22)*. Conversely, if* (2.27) *and* (2.28) *hold for some $\mathcal{M}_g$, then then u is a minimizer of* (2.22)*.*

Proof. Let b be a solution to the dual problem with corresponding $\lambda \in \mathbb{R}$. Suppose u is a minimizer of 2.22. Then

$$\alpha_{P_N} = I(u) = \frac{1}{2}\sum_{i,j} a_{ij}|u_i - u_j| \geq \sum_{i,j} |b_{ij}||u_i - u_j| \geq \sum_{i,j} -b_{ij}(u_i - u_j) \tag{2.29}$$
$$= \langle -b, Du\rangle_{\mathcal{H}(E)} = \langle \text{div} b, u\rangle_{\mathcal{H}(V)}$$
$$= \lambda \sum_{i\in\partial V} g_i u_i = \lambda = \alpha_{D_N} = \alpha_{P_N}.$$

Thus the inequalities in (2.29) are indeed equalities and taking $J = -2b$ we we see that (2.27) and (2.28) hold. We can also see from the above compuations that the converse also holds. □

Corollary 22 *If u and v are two arbitrary minimizers of (2.22), then*

$$(u_i - u_j)(v_i - v_j) \geq 0 \textit{ for all } 1 \leq i, j \leq n.$$

2.2.1 Voltage potentials have minimum energy

We can now prove the analog to Theorem 13.

Theorem 23 *Let $g \neq 0$ be a function on ∂V satisfying 2.20 and a be a measurement matrix. If $v \in \mathcal{V}_{(g,a)}$, then v is a minimizer of the least gradient problem* (2.22)*. Conversely, given any $a = (a_{i,j})$ with $a_{i,j} \geq 0$ and $g \in \mathbb{R}^{|\partial V|}$ satisfying* (2.20)*, if v is a minimizer of the least gradient problem* (2.22)*, then $v \in \mathcal{V}_{(\lambda g,a)}$ for some $\lambda > 0$.*

Proof. Suppose $v \in \mathcal{V}_{(g,a)}$ and let J be the corresponding current on E. Following similar computations as in the proof of Theorem 13 we have

$$I(v) = \frac{1}{2}\sum_{i,j} a_{ij}|v_i - v_j| = \frac{1}{2}\sum_{i,j} |J_{ij}||v_i - v_j| \geq \frac{1}{2}\sum_{i,j} J_{ij}(v_i - v_j) \tag{2.30}$$
$$= \sum_{i\in\partial V} v_i g_i = 1.$$

Therefore the minimum of the least gradient problem (2.22) is equal to 1. Moreover the minimum is achieved for every $v \in \mathcal{V}_{(g,|J|)}$.

Now suppose v is a minimizer of the problem (2.22) and let b be a solution of the dual problem (D_N) with the corresponding $\lambda \in \mathbb{R}$. Let $J = -2b$. Then by Theorem 21 we see that $v \in \mathcal{V}_{(\lambda g, a)}$. $\square$

Remark 24 *Note that Corollary 22 indicates that the direction of the flow of the current along the edges is unique, despite multiplicity of the minimizers of* (2.3) *(see also Remark 14).*

2.2.2 Multiple measurements

Suppose we have two data sets (g^1, a^1) and (g^2, a^2), and would like to find a conductivity matrix σ inducing the currents with magnitudes $|J^1|$ and $|J^2|$, when the currents g^1 and g^2 are injected on the boundary vertices $\partial^1 V$ and $\partial^2 V$, respectively. We can consider the minimization problem

$$\inf_{(v^1,v^2)\in\mathcal{K}} F(v^1, v^2). \tag{2.31}$$

where F is defined by (2.17) and

$$\mathcal{K} := \{(v^1, v^2) \in \mathbb{R}^n \times \mathbb{R}^n : \ \sum_{j=1}^{n} v_i^1 g_i^1 = 1 \ \text{ on } \ \text{ and } \ \sum_{j=1}^{n} v_i^2 g_i^2 = 1\}.$$

The analog to Theorem 15 can be formulated and proved in this setting and we can also similarly extend to a finite number of measurements.

2.3 Algorithms for finding minimizers

In this section we present numerical algorithms for finding minimizers of the l^1 minimization problems discussed in Sections 3 and 4, yielding voltage potentials for Dirichlet or Neumann boundary conditions. The primal problem (P_D) and (P_N) can be written as

$$\min_{\{u\in H, d\in\mathcal{H}(E)\}} F(d) \quad \text{subject to} \quad Du = d, \tag{2.32}$$

where $H = \mathcal{H}_0(V)$ for the Dirichlet case and $H = \mathcal{M}_0$ for the Neumann boundary problem. This leads to the unconstrained problem

$$\min_{\{u\in H, d\in\mathcal{H}(E)\}} F(d) + \frac{\alpha}{2}\|Du - d\|^2. \tag{2.33}$$

To solve the above minimization problem, we use and develop an algorithm in the spirit of the alternating Split Bregman method which was first introduced by Goldstein and Osher [26]. The Split Bregman algorithm suggests initiating the vectors b^0 and d^0, and producing the sequences u^k, b^k, and d^k as follows

$$\begin{aligned} (u^{k+1}, d^{k+1}) &= \text{argmin}_{u\in H, d\in H(E)}\{F(d) + \frac{\alpha}{2} \parallel b^k + Du - d \parallel_2^2\}, \\ b^{k+1} &= b^k + Du^{k+1} - d^{k+1}, \end{aligned} \tag{2.34}$$

where $\alpha > 0$. Since the joint minimization problem (2.34) in both u and d is in general expensive to solve exactly, Goldstein and Osher [26] proposed the following Alternating Split Bregman algorithm for solving problems of type (2.32)

$$u^{k+1} = \text{argmin}_{u\in H} \parallel b^k + Du - d^k \parallel_2^2, \tag{2.35}$$

$$d^{k+1} = \text{argmin}_{d\in\mathcal{H}(E)}\{F(d) + \frac{\alpha}{2} \parallel b^k + Du^{k+1} - d \parallel_2^2\}, \tag{2.36}$$

$$b^{k+1} = b^k + Du^{k+1} - d^{k+1}. \tag{2.37}$$

See [18, 8, 26, 24, 62, 63] for more details. It is pointed out by Esser [18] and Setzer [63] that the above idea to minimize alternatingly was first presented for the augmented Lagrangian algorithm by Gabay and Mercier [24] and Glowinski and Marroco [25]. The resulting algorithm is called the alternating direction method of multipliers (ADMM) [23] and is equivalent to the alternating split Bregman algorithm. The convergence of ADMM in finite dimensional Hilbert spaces was established by Eckstein and Bertsekas [16]. This in particular implies convergence of the alternating split Bregman algorithm in finite dimensional Hilbert spaces. Cai, Osher, and Shen [8] and Setzer [62, 63] also independently presented convergence results for the alternating split Bregman in finite dimensional Hilbert spaces. In [50] and [52] the authors proved the convergence of the alternating split Bregman algorithm in infinite dimensional Hilbert spaces by showing that the alternating split bregman algorithim corresponds to the Douglas-Rachford splitting algorithm for the dual problem. Indeed the dual problems (2.10) and (2.25) can be written in the form

$$0 \in A(-b) + B(-b), \tag{2.38}$$

where $A := \partial G^* o(-\text{div})$ and $B = \partial F^*$ are maximal monotone operators on H. For a set valued operator $P : H \to 2^H$, let J_P denote its resolvent, i.e. $J_P = (Id + P)^{-1}$. Douglas-Rachford splitting algorithm states that for any initial elements x_0 and p_0 and any $\alpha > 0$, the sequences p_k and x_k generated by the following algorithm

$$\begin{aligned} x_{k+1} &= J_{\alpha A}(2p_k - x_k) + x_k - p_k \\ p_{k+1} &= J_{\alpha B}(x_{k+1}), \end{aligned} \tag{2.39}$$

converges to some x and p respectively. Furthermore $p = J_{\alpha B}(x)$ and p satisfies

$$0 \in A(p) + B(p).$$

Let us introduce the sequences b^k and d^k with

$$x_k = \alpha(b^k + d^k) \quad \text{and} \quad p_k = \alpha b_k.$$

Notice that both sequences b^k and d^k converge. The resolvents $J_{\alpha A}(2p_k - x_k)$ and $J_{\alpha B}(x_{k+1})$ can be computed as follows

$$J_{\alpha A}(2p_k - x_k) = \alpha(b^k + Du^{k+1} - d^k) \tag{2.40}$$

and

$$J_{\alpha B}(x_{k+1}) = \alpha(b^k + Du^{k+1} - d^{k+1}), \tag{2.41}$$

where u^{k+1} and d^{k+1} are minimizers of

$$I_1(u) = \sum_{i,j} |b^k_{ij} + (Du)_{ij} - d^k_{ij}|^2 \tag{2.42}$$

and

$$I_2(d) = \frac{1}{2}\sum_{i,j} a_{ij}|d_{ij} + (Du_f)_{ij}| + \frac{\alpha}{2}\sum_{i,j} |b^k_{ij} + (Du^{k+1})_{ij} - d_{ij}|^2 \tag{2.43}$$

over $u \in \mathcal{H}_0(V)$ for the Dirichlet problem and over $u \in \mathcal{M}_0$ for the Neumann problem, and over $d \in \mathcal{H}(E)$. We will first consider the case of finding minimizers of I_1. Let $u, v \in \mathbb{R}^n$. For any $t \in \mathbb{R}$ define

$$i(t) = I_1(u + tv).$$

Then

$$i'(t) = 2\sum_{i,j}(b_{ij} + (Du + tv)_{ij} - d_{ij})(Dv)_{ij}$$

and so

$$\begin{aligned} i'(0) &= 2\sum_{i,j}(b_{ij} + (Du)_{ij} - d_{ij})(Dv)_{ij} \\ &= 2\sum_{i,j} b_{ij}(Dv)_{ij} + (Du)_{ij}(Dv)_{ij} - d_{ij}(Dv)_{ij} \\ &= 2\sum_{i} -(\text{div}b)_i v_i - (\text{div}(Du))_i v_i + (\text{div}d)_i v_i \\ &= \sum_{i}(2(\text{div}d)_i - 2(\text{div}b)_i + 4\sum_{j}(Du)_i)v_i \end{aligned}$$

where we have used Proposition 6 and the fact that $(\text{div}(Du))_i = -2\sum_j (Du)_i$ to simplify. In the case of the Dirichlet boundary problem since I_1 is convex if $i'(0) = 0$ for a fixed $u \in \mathcal{H}_0(V)$ and all $v \in \mathcal{H}_0(V)$ then u is a minimizer of I_1 over $\mathcal{H}_0(V)$. Thus if u satisfies the Euler-Lagrange equation

$$\begin{cases} \sum_{j=1}^{n}(Du)_{ij} = \frac{1}{2}[(\text{div}b^k)_i - (\text{div}d^k)_i], \quad \forall i \in \text{int}(V) \\ u_i = 0 \quad \text{for all} \quad i \in \partial V. \end{cases} \tag{2.44}$$

then u is a minimizer of I_1 over $\mathcal{H}_0(V)$. It follows from Proposition 1 that the above system is uniquely solvable.

In the case of Neumann boundary condition, I_1 also has a unique minimizer in $\mathcal{M}_0$ up to adding a constant, but identifying the solutions is more subtle. First note that if u satisfies the Euler-Lagrange equation

$$\begin{cases} \sum_{j=1}^{n}(Du)_{ij} = \frac{1}{2}[(\text{div}b^k)_i - (\text{div}d^k)_i], \quad \forall i \in \text{int}(V) \\ \sum_{j=1}^{n}(Du)_{ij} = \beta g_i + [\frac{1}{2}(\text{div}b^k)_i - (\text{div}d^k)_i], \quad \text{for all} \quad i \in \partial V \end{cases} \tag{2.45}$$

for $\beta \in \mathbb{R}$ and $u \in \mathcal{M}_0$ then u is minimizer of I_1 over $\mathcal{M}_0$. Since $\sum_{i\in\partial V} g_i = 0$ and $\sum_{i=1}^{n}(\text{div}c)_i = 0$ for any $c \in \mathcal{H}(E)$, by Propositions 17 and 18 the system (2.45) has a unique solution in $\mathcal{H}(V)$ for every $\beta \in \mathbb{R}$, up to adding a constant. To identify β and find a solution of (2.45) in $\mathcal{M}_0$, let z be a solution of

$$\begin{cases} \sum_{j=1}^{n}(Dz)_{ij} = 0, & \forall i \in \text{int}(V) \\ \sum_{j=1}^{n}(Dz)_{ij} = g_i & \text{for all} \quad i \in \partial V. \end{cases} \tag{2.46}$$

Then

$$\begin{aligned} 0 < \frac{1}{2}\sum_{i,j}(D_z)_{ij} &= \frac{1}{2}\sum_{i=1}^{n} w_i \sum_{j=1}^{n}(Dz)_{ij} - \frac{1}{2}\sum_{j=1}^{n} z_j \sum_{i=1}^{n}(Dz)_{ij} \\ &= \sum_{i=1}^{n} z_i \sum_{j=1}^{n}(Dz)_{i,j} \\ &= \sum_{i\in\partial V} z_i g_i. \end{aligned}$$

Hence

$$\sum_{i\in\partial V} z_i g_i > 0.$$

Now let u be a solution of

$$\sum_{j=1}^{n}(Du)_{ij} = \frac{1}{2}[(\text{div}b^k)_i - (\text{div}d^k)_i], \qquad \forall i \in V. \tag{2.47}$$

Define

$$\beta := -\frac{\sum_{i\in\partial V} u_i g_i}{\sum_{i\in\partial V} z_i g_i}.$$

Then $v = u + \beta z$ belongs to $\mathcal{M}_0$ and satisfies the equation (2.45), and hence v is the unique minimizer of I_1 over $\mathcal{M}_0$, up to adding a constant.

The minimizer of I_2 for the Dirichlet problem can be directly computed as

$$d_{ij}^{k+1} = \begin{cases} \max\{|w_{ij}| - \frac{a_{ij}}{2\alpha}, 0\}\frac{w_{ij}}{|w_{ij}|} - (Du_f)_{ij} & \text{if } w_{ij} \neq 0 \\ -(Du_f)_{ij} & \text{if } w_{ij} = 0, \end{cases} \tag{2.48}$$

where $w_{ij} = (Du^{k+1})_{ij} + (Du_f)_{ij} + b^k_{ij}$. For the Neumann problem u_f is replaced by v_g.

Therefore Douglas-Rachford splitting leads to the following convergent algorithms for the Dirichlet and Neumann problems.

Algorithm 1 (Finding a minimizer of the Dirichlet Problem)

Let $\alpha > 0$, $u_f \in \mathcal{H}(V)$ with $u = f$ on ∂V and initialize $b^0, d^0 \in \mathcal{H}(E)$. For $k \geq 0$:

1. Solve

$$\begin{cases} \sum_j (Du^{k+1})_{ij} = \frac{1}{2}[(\mathrm{div} b^k)_i - (\mathrm{div} d^k)_i], & \forall i \in \mathrm{int}(V) \\ u_i^{k+1} = 0 \quad \text{for all} \quad i \in \partial V. \end{cases} \tag{2.49}$$

2. Compute d^{k+1}

$$d^{k+1}_{ij} = \begin{cases} \max\{|w_{ij}| - \frac{a_{ij}}{2\alpha}, 0\} \frac{w_{ij}}{|w_{ij}|} - (Du_f)_{ij} & \text{if } w_{ij} \neq 0 \\ -(Du_f)_{ij} \quad \text{if } w_{ij} = 0, \end{cases} \tag{2.50}$$

   where $w_{ij} = (Du^{k+1})_{ij} + (Du_f)_{ij} + b^k_{ij}$.

3. Set

$$b^{k+1}_{ij} = b^k_{ij} + (Du^{k+1})_{ij} - d^{k+1}_{ij}.$$

The following proposition follows directly from the convergence of Douglas-Rachford splitting algorithm and Theorem 1.2 in [50]. See also [8, 62, 63].

Proposition 25 *Let u^k b^k, and d^k be the sequences produced by the Algorithm 1. Then $u^k \to u$ and $b^k \to \frac{1}{2\alpha} J$, where u and J are solutions of the* (2.8) *and it's dual problem (D), respectively. In addition $d^k \to Du$. In particular u is a voltage potential corresponding to the data (f, a) and J is the induced current with $|J| = a$.*

Algorithm 2 (Finding a minimizer of the Neumann Problem)

Let $\alpha > 0$, $v_g \in \mathcal{H}(V)$ with $\sum\limits_{i \in \partial V} v_i g_i = 1$ and initialize $b^0, d^0 \in \mathcal{H}(E)$. Also let $z \in \mathbb{R}^n$ be a solution of (2.46) with $z_1 = 0$. For $k \geq 0$:

1. (a) Solve

$$\left\{ \sum_j (Du^{k+1})_{ij} = \frac{1}{2}[(\text{div}b^k)_i - (\text{div}d^k)_i], \qquad \forall i \in V \right. \tag{2.51}$$

   with $u_1^{k+1} = 0$.

   (b) Compute

$$\beta^{k+1} = -\frac{\sum_{i \in \partial V} u_i^{k+1} g_i}{\sum_{i \in \partial V} z_i g_i}$$

   and set $v^{k+1} = u^{k+1} + \beta^{k+1} z$.

2. Compute d^{k+1}

$$d_{ij}^{k+1} = \begin{cases} \max\{|w_{ij}| - \frac{a_{ij}}{2\alpha}, 0\} \frac{w_{ij}}{|w_{ij}|} - (Dv_g)_{ij} & \text{if } w_{ij} \neq 0 \\ -(Dv_g)_{ij} \quad \text{if } w_{ij} = 0, \end{cases} \tag{2.52}$$

   where $w_{ij} = (Dv^{k+1})_{ij} + (Dv_g)_{ij} + b_{ij}^k$.

3. Set

$$b_{ij}^{k+1} = b_{ij}^k + (Dv^{k+1})_{ij} - d_{ij}^{k+1}.$$

Convergence of Douglas-Rachford splitting algorithm implies the following convergence result, see Theorem 1.2 in [50] and [8, 62, 63].

Proposition 26 *Let v^k b^k, and d^k be the sequences produced by the Algorithm 2. Then $v^k \to v$ and $b^k \to \frac{1}{2\alpha} J$, where v and J are solutions of the* (2.23) *and it's dual problem (D_N),* *respectively. In addition* $d^k \to Dv$*. In particular* v *is a voltage potential corresponding to the data* $(\lambda g, a)$ *for some* $\lambda \in \mathbb{R}$ *and* J *is the induced current with* $|J| = a$*. Moreover* λ *is the optimal values of the primal and dual problems* (P_N) *and* (D_N)*, i.e.* $\lambda = \alpha_{P_N} = \alpha_{D_N}$.

2.3.1 Numerical simulations

We performed a set of numerical simulations in MATLAB to demonstrate convergence of Algorithm 1 and 2. A simple graph with 100 vertices was generated and edges were randomly assigned between nodes with a approximate density of 0.125. Random numbers uniformly distributed between 0 and 1 were then assigned to each edge as their conductivity. We then selected 5 boundary nodes and randomly assigned values between 0 and 1 as boundary data. For the Dirichlet boundary data, the forward problem was solved to determine the current J, generating the data $a = |J|$. To generate the boundary data for the Neumann problem we found the current entering/leaving the system at each boundary vertex. The simulations for both the Dirichlet and Neumann boundary data were done on the same graph structure with the same current data $|J|$. The nonsingular linear systems in algorithm 1 were solved using the MATLAB mldivide function and the singular linear systems in algorithm were solved using the pinv function. The vector u_f was chosen to be zero on $int(V)$ and f on the ∂V. The vector v_g in Algorithm 2 was chosen using the MATLAB mldivide function. Tables 1 and 2 show the numerical errors for algorithms 1 and 2 on the same graph for different levels of tolerance. Simulations were run on a late 2013 MacBook Pro with a 2.4 GHZ Intel Core i5 processor. We used the L^2 matrix norm for error computations.

Table 2.1: Numerical errors for algorithm 1 on 100 node graph with 1121 edges

Tolerance	**Relative L2 Error**	**Number of Iterations**	**Elapsed Time(s)**
10^{-3}	1.2171×10^{-3}	16	0.069309
10^{-4}	1.3160×10^{-4}	22	0.102846
10^{-5}	1.4494×10^{-5}	92	0.358250
10^{-6}	1.3615×10^{-6}	133	0.405979

Table 2.2: Numerical errors for algorithm 2 on 100 node graph with 1121 edges

Tolerance	**Relative L2 Error**	**Number of Iterations**	**Elapsed Time(s)**
10^{-2}	1.3069×10^{-3}	7	0.055400
10^{-3}	1.3908×10^{-4}	9	0.071342
10^{-4}	1.0235×10^{-5}	12	0.086956
10^{-5}	1.1987×10^{-6}	24	0.147310

While running our simulations we observed that the speed of convergence of Algorithm 1 varied quite wildly depending on the choice of boundary data. We also observed that the speed of convergence of Algorithm 2 was always the same or faster than that of Algorithm 1. To test this observation, we ran algorithms 1 and 2 on the same graph used in Tables 1 and 2 for 1000 different choices of Dirichlet boundary. The average number of iterations for each algorithm is shown in Table 3. We also remark that changing the structure of the graph also effects the speed of convergence. It is not clear to the authors that why Algorithm 2 converges faster than Algorithm 1, and an in depth analysis of the

Table 2.3: Average number of iterations for algorithm 1 versus algorithm 2

Tolerance	**Algorithm 1**	**Algorithm 2**
10^{-3}	21.175	15.918
10^{-4}	46.097	18.905
10^{-5}	111.847	23.486
10^{-6}	227.624	32.846

speed of convergences of algorithms 1 and 2 remain open.

2.4 Applications

In this section we discuss potential applications of our results on electrical networks on random walks on graphs and Cryptography.

2.4.1 Random walks on graphs

We will now investigate how the inverse problem here can be related to random walks. Random walk models have been used to model infection on graphs such as spread of epidemics and rumours with mobile agents, see [9, 15], voting patterns [71, 12], and stock market prices [20]. Random walk models have also been proven to be a simple yet powerful method for extracting information from computer and social networks such as identification of reputable entities in a network. For instance Google's PageRank algorithm uses random walks to rank websites in their search engine results, see [59, 30], and the survey papers [48] and [61] for applications of random walks on graph in computer networks. Also see [64] for a wide variety of applications of random walks on graphs in statistical mechanics.

Let $G = (V, E')$ be a connected, directed, and simple graph with n nodes and consider a random walk on G. Suppose a random walker begins at node a and walks until they reach node b and if they return to a before reaching b they keep walking. Let $P = (P_{ij}) \in \mathcal{H}(E)$ be the matrix of transition probabilities, i.e. $0 \leq P_{ij} \leq 1$ is the probability of the random walker walking from node i to node j. In particular $\sum_j P_{ij} = 1$ for all $1 \leq i \leq n$. Let W_{ij} be the expected number of times the walker walks from node i

to node j before exiting the graph at node b. Note that $W_{ij} = -W_{ji}$. Can one determine transition probabilities $P = (P_{ij})$ from the knowledge of the boundary vertices $\{a, b\}$ and $W = (W_{i,j})$? In this section, among other results, we show that the answer is yes, and describe an algorithm for determining such P.

There is a close connection between electrical networks and random walks on graphs [14]. Let $G = (V, E)$ be an electrical network with conductivity matrix $\sigma = (\sigma_{ij})$, $\sigma_{i,j} \in [0, \infty)$, and let $\partial V = \{a, b\}$. Suppose a current g with $g(a) = 1$ and $g(b) = -1$ is injected to the network inducing a current J along the edges. Define

$$\sigma_i := \sum_{j=1}^{n} \sigma_{ij} \quad \text{and} \quad P_{ij} = \frac{\sigma_{ij}}{\sigma_i} \tag{2.53}$$

and assign the transition probability matrix P to the graph $G = (V, E')$. Then the net number of times the walker taking an step from node i to node j is indeed J_{ij}, i.e.

$$J = W.$$

Therefore if the boundary nodes $\partial V = \{a, b\}$ and the magnitude of expected net number of times the walker should walk along the edges of the graph is prescribed, by the method presented in Section 5, one can first find a conductivity matrix σ inducing the current $J = W$ on network and compute transition probability matrix P by (2.53).

The connection between random walks on graphs and electrical networks with Neumann boundary condition can be generalized to the case when $\partial V = \Gamma_a \cup \Gamma_b$ with $\Gamma_a \cap \Gamma_b = \emptyset$ and $\Gamma_a, \Gamma_b \neq \emptyset$. Let $g \in \mathbb{R}^{|\partial V|}$ with $g|_{\Gamma_a} \geq 0$ and $g|_{\Gamma_b} \leq 0$ and

$$\sum_{i \in \Gamma_1} g_i = 1 \quad \text{and} \quad \sum_{i \in \Gamma_2} g_i = -1.$$

Suppose we would like to determine a transition matrix P such that if a random walker enters the network from a vertex k in Γ_a with probability g_k, then

- they exit the network at a node $l \in \Gamma_b$ with probability $|g_l|$
- the expected net number of times they pass from vertex i to node j before exiting the network is W_{ij}, $1 \leq i, j \leq n$.

As explained above, to determine the transition matrix P it suffices to find a conductivity matrix σ inducing the current $J = W$ with Neumann data g on ∂V. Then P can be computed from (2.53).

Suppose $\partial V = \{a, b\}$ and consider the inverse problems of determining the transition probabilities from the relative net number of times the walker walks between the edges of the graphs, i.e. $\alpha W = (\alpha W_{i,j})$ where α is a unknown constant. Then one can determine a transition probability P by finding a conductivity matrix σ by minimizing the l^1 minimization problem (2.2) with $a = \alpha W$, $f(a) = 1$ and $f(b) = 0$. A transition matrix can also be obtained by minimizing (2.22) with the Neumann boundary condition $g(a) = 1$ and $g(b) = -1$.

Remark 27 *Note that in this section we assume that the conductivity matrix* $\sigma = (\sigma_{ij})$ *satisfies* $\sigma_{i,j} \in [0, \infty)$. *Indeed we do not allow perfect conductors as otherwise the probability matrix* P *in* (2.53) *will not be well-defined. As described in the introduction, if for a minimizer* v *of* (2.3) *or* (2.22) *we have* $v_i = v_j$ *and* $|J_{i,j}| \neq 0$ *for some* $1 \leq i, j \leq n$, *then the edge* (i, j) *is a perfect conductor, i.e.* $\sigma_{i,j} = \infty$. *If* v *is minimizer of* (2.3) *or* (2.22) *leading to perfect conductance on an edge, then one may look for an increasing function*

$F : \mathbb{R} \to \mathbb{R}$ such that $u = (u_1, u_2, ..., u_n) := (F(v_1), F(v_2), ..., F(v_n))$ satisfies $u_i \neq u_j$ for $i \neq j$. Note that such u will also be a minimizer of (2.3) *or* (2.22) *and would provide a conductivity matrix σ with $\sigma_{ij} \in [0, \infty)$, and hence the transition probabilities can be computed from* (2.53)*. If such increasing function F does not exists, then there exists no transition probability matrix P for which the expected number of times the walker passes along the edges is W.*

2.4.2 Applications in cryptography

In this section we discuss a potential application of our results on electrical networks in public-key encryption. As stated in Remark 24, Theorem 13 implies that a mass preserving flow $J = (J_{ij})$ along the edges of a graph $G = (V, E)$ can be recovered from the knowledge of $|J| = (|J_{ij}|)$ and its net flux on the boundary nodes ∂V. More precisely, suppose $J_{i,j}$ is the current from node i to node j ($J_{ij} = -J_{ji}$ for $(i, j) \in E$), and suppose

$$\sum_{j=1}^{n} J_{ij} = 0 \quad \text{for every interior node } \ i \notin \partial V$$

and

$$\sum_{j=1}^{n} J_{ij} = f_i \quad \text{for every boundary node } \ i \in \partial V.$$

Then J can be reconstructed from the knowledge of $(|J|, f, \partial V)$. This counter-intuitive result has a potential application in cryptography. To see the connection, let us translate a special case of this result to the language of matrices.

Let I_n be a subset of $\{1, 2, ..., 2n + 1\}$ with n elements and $\mathcal{A}_{I_n}$ be the space of $(2n + 1) \times (2n + 1)$ anti-symmetric matrices $A = (a_{ij})$ satisfying the following properties:

Note that $f \in \mathbb{R}^n$. Suppose a pair of communicators have agreed on a set of indices

$I_n \subset \{1, 2, ..., 2n+1\}$ with n elements, both are aware of I_n, and would like to securely communicate a matrix $A \in \mathcal{A}_{I_n}$. Then the first party can just send the key $(|A|, f)$ where $f \in R^n$ is the sum of the entries of the rows of A that belong to I_n. The second party can decrypt the message and find A from the knowledge of $(|A|, f, I_n)$, using the algorithm we developed in Section 4. Since a_{ij} only takes integer values in $\{-1, 0, +1\}$, a few iterations of the algorithm should be enough to determine A. On the other hand, finding A from the knowledge of $(|A|, f)$ would be extremely difficult for an adversary who is not aware of I_n. Indeed since all rows of $|A|$ have an even number entries equal to 1, the adversary could not determine the boundary nodes I_n from $|A|$. To decrypt the message, the adversary faces the problem of guessing I_n among $\binom{2n+1}{n}$ subsets of $\{1, ..., 2n+1\}$ with n elements and matching it with f. The number of different possibilities are

$$n!\binom{2n+1}{n} \simeq \frac{2^{2n+1}}{\sqrt{\pi n}} n!,$$

which grows very fast and makes the decryption for adversaries extremely difficult for large n.

Chapter 3

Determining both the source of a wave and its speed in a medium from boundary measurements

3.1 Prior uniqueness results

Unique determination of the source function f and the wave speed c has been studied by many authors and several interesting results have been obtained. However, most of the results in the literature have been concerned with determination of f from the knowledge of $\Lambda_{f,c}$ under the assumption that the sound speed is known. When the sound speed is known, smooth, and non-trapping (see Definition 29), then the source f can be uniquely recovered ([3, 2, 22, 32, 33, 45, 65, 68]). We will highlight some of these results in more detail in the rest of this section. The case when the sound speed is known and constant was first studied. In [3], Agranovksy and Quinto proved the following uniqueness theorem.

Theorem 28 *If the known speed c is constant then the PAT data $\Lambda_{f,c}(x,t)$ determines $f \in L^2_c(\mathbb{R}^n)$ uniquely.*

When the sound speed is constant the solution is given by Kirchhoff Poisson formulas. This relates the PAT problem to a problem involving the mean spherical operator, and thus becomes a problem in integral geometry. This result was refined by Finch, Patch, and Rakesh in [22], in which they proved uniqueness given PAT data of a finite time, specifically for for $0 \leq t \leq \frac{D}{2}$ where D is the diameter of Ω. Next the case when the sound speed is variable was considered. Before stating these uniqueness results we need a definition.

Definition 29 *For $(x,\xi) \in \mathbb{R}^{2n}_{x,\xi}$ and the Hamiltonian $H = \frac{1}{2}c^2(x)|\xi|^2$, consider the Hamiltonian system*

$$\begin{cases} x'_t = \frac{\partial H}{\partial \xi} = c^2\xi \\ \xi'_t = -\frac{\partial H}{\partial x} = -c(x)\nabla(c(x))|\xi|^2, \\ x|_{t=0} = x_0, \xi|_{t=0} = \xi_0. \end{cases} \tag{3.1}$$

The solution $(x(t),\xi(t))$ is called a bicharacteristic and $x(t)$ is called a ray. We say that the sound speed c is non-trapping if all rays with $\xi_0 \neq 0$ tend to infinity as $t \to \infty$.

It was proved by Agranovsky and Kuchment in [1] that a smooth non-trapping sound speed can be recovered uniquely by the PAT data.

Theorem 30 *Assume that the sound speed is strictly positive and both $c - 1$ and f have* *compact support. If the known speed $c(x)$ is smooth and non-trapping then the PAT data $\Lambda_{f,c}(x,t)$ determines $f(x)$ uniquely.*

Similar to the constant speed case uniqueness also holds under finite time. Stefanov and Uhlmann proved the following theorem in [68].

Theorem 31 *Assume that the sound speed is strictly positive and both $c-1$ and f have compact support in Ω. If the known speed $c(x)$ is smooth and non-trapping then the data $\Lambda_{f,c}(x,t)$ measured till any time $T > T(\Omega)$ is sufficient for unique recovery of f where $T(\Omega)$ is the supremum of the time it takes for a ray to reach $\partial\Omega$ over all rays originating in Ω.*

We also remark at this point that the inverse problem is satisfactorily stable when the sound speed is known and non-trapping, see [45] for an brief overview of stability results.

In practice the sound speed inside the medium is often unknown [36]. It has been observed that even replacing a sound speed with small variation by its average value can significantly distort the reconstruction of f [33]. One suggested solution is to additionally perform an ultrasonic transmission tomography (UTT) to recover the sound speed [36]. Thus from both a theoretical and practical point of view it would be advantageous to know whether both the sound speed and the source term can be uniquely recovered from $\Lambda_{f,c}$. This is currently an open problem. We will now outline the work that has been done is this area. The first result for the recovery of a unknown sound speed was proved by Hristova, Kuchment, and Nguyen in [33] where they proved that a constant sound speed can be uniquely recovered using support conditions. They were able to prove this result for all odd dimensions $n > 1$. In particular they proved the following explicit formula for c. (Theorem 5 in [33]).

Theorem 32 *Let* $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ *where* $n > 1$ *is odd. Let* f *be supported inside* Ω. *Define*

$$t_0 = \inf\{t > 0 : \text{there exists} \;\; y \in \partial\Omega \;\; \text{such that} \;\; \Lambda(y,t) \neq 0\}$$

and

$$T_0 = \sup\{t > 0 : \text{there exists} \;\; y \in \partial\Omega \;\; \text{such that} \;\; \Lambda(y,t) \neq 0\}.$$

Then the sound speed c *satisfies the equality*

$$c = \frac{2}{t_0 + T_0}$$

and thus is uniquely determined by the PAT data Λ.

The existence of t_0 and T_0 is due to the finiteness of the speed of propagation.

In [21], Finch and Hickman proved uniqueness of the sound speed under a monotonicity condition and that if the sound speed is radial then both f and c can be recovered uniquely. To understand their results we first state one of their definitions.

Definition 33 *An acoustic profile on a domain* Ω *is a smooth function* $c(x) \in C^\infty(\mathbb{R}^n)$ *with* $0 < \sigma < c(x) < \infty$ *for all* $x \in \Omega$ *for some* $\sigma > 0$ *and* $supp(1 - c(x)) \subset \Omega$.

We can now state Finch and Hickman's main results.

Theorem 34 *Let* Ω *be a domain in* $\mathbb{R}^n$. *Suppose photoacoustic data* $\Lambda(x,t)$ *is generated by an acoustic profile in some* $\mathcal{D}$. *Assume also that for every pair* $c(x), b(x) \in \mathcal{D}$, $c(x) - b(x) \geq 0$ *or* $c(x) - b(x) \leq 0$ *on* Ω. *Then the acoustic profile generating the data* $\Lambda(x,t)$ *is determined uniquely in* $\mathcal{D}$.

Theorem 35 *Let $\Omega = B_1(0)$ and $n \geq 3$ be odd. Suppose that the photoacoustic data $\Lambda(x,t)$ on $\partial\Omega \times (0, \infty)$ is generated by a radially symmetric, non-trapping acoustic profile. Then the acoustic profile generating data $\Lambda(x,t)$ is uniquely determined among the set of radially symmetric, non-trapping acoustic profiles.*

These results are proved by connecting the PAT problem to the transmission eigenvalue problem.

In [47], Liu and Uhlmann showed that under additional assumptions on the wave speed and the source term both can be uniquely recovered simultaneously. It is the techniques used in this paper that inspired our work. In their paper and our results soon to be stated we restrict the dimension $n = 3$. Before stating their result we need to include some background.

Let $u(x,t)$ be the solution of the wave equation (1.8) and for $(x,k) \in \mathbb{R}^3 \times \mathbb{R}_+$ define the temporal Fourier transform of the function $u(x,t)$ by

$$\hat{u}(x,k) := \frac{1}{2\pi} \int_0^\infty u(x,t) e^{ikt} dt. \tag{3.2}$$

We then define an admissible pair for a source and wave speed.

Definition 36 *Let $0 < c_0 < c \in L^\infty(\Omega)$, $f \in L^\infty(\Omega)$, $supp(c-1) \subset \Omega$, and $supp(f) \subset \Omega$. We say that the pair (f,c) is admissible if there exists $\epsilon > 0$ such that $\hat{u}(x,k) \in H^1_{loc}(\mathbb{R}^3)$ for all $k \in (0, \epsilon)$.*

Indeed if $u(x,t)$ decays fast enough in time such that $\hat{u}(x,k) \in H^1_{loc}(\mathbb{R}^3)$, then (f,c) is admissible. Note that the admissibility assumption above is a weak form of the non-trapping assumption on the sound speed. As pointed out in [47] by Liu and Uhlmann, if the sound speed c is smooth and non-trapping then (f, c) is admissible, but not vice versa. See [21, 68] and the references cited therein for more details.

We can now state Liu and Uhlmann's result.

Theorem 37 *Let (f, c) be admissible and suppose that c is constant, f^{-2} is harmonic in a simply connected region $\omega \subset\subset \Omega$ and identically zero on ω^c. Furthermore, assume that $f(x) \geq 0$ for a.e. $x \in \Omega$ and $\int_\Omega f(x)dx > 0$. Then both f and c are uniquely determined by $\Lambda_{f,c}(x, t)$.*

We quickly remark that this result is a consequence of the results as stated in their paper. We will conclude this section with two more remarks. The case when c is unknown but f is known was considered by Stefanov and Uhlmann in [68]. They were able to prove uniqueness of the sound speed from the knowledge of the source f under the assumption that the domain Ω is foliated by strictly convex hypersurfaces with respect to the Riemannian metric $g = c^{-2}dx$. It was proved by Stefanov and Uhlmann in [67] that the linearized problem of recovery of both f and c is unstable. This suggests that the recovery of both f and c may be unstable as well. This instability result is as follows.

Theorem 38 *Let $\mathcal{K} = supp(f) \subset \Omega$. There is no stability estimate of the type*

$$||\delta f||_{H^{s_1}(\Omega)} + ||\delta c^2||_{H^{s_1}(\mathcal{K})} \leq C||\delta\Lambda\{\delta f, \delta c^2\}||_{H^{s_2}(\Omega)},$$

$s_1 \geq 0, s_2 \geq 0$, regardless of s_1, s_2.

3.2 New results

We prove that if c^{-2} is harmonic in $\omega \subset \mathbb{R}^3$ and identically 1 on ω^c, where ω is a simply connected region, then a non-trapping wave speed c can be uniquely determined from the solution of the wave equation on boundary of $\Omega \supset\supset \omega$ without the knowledge of the source. We also show that if the wave speed c is known and only assumed to be bounded then, under the admissibility assumption in definition 36, the source of the wave can be uniquely determined from boundary measurements. Indeed we prove Theorem 39 and Theorem 40 below.

Theorem 39 *Let (f_1, c_1) and (f_2, c_2) be two admissible pairs such that*

$$\Lambda_{f_1,c_1}(x,t) = \Lambda_{f_2,c_2}(x,t) \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}_+.$$

Then

$$C^* := \int_\Omega c_1^{-2}(x) f_1(x) dx = \int_\Omega c_2^{-2}(x) f_2(x) dx.$$

If $C^* \neq 0$, then

$$\int_\Omega (c_2^{-2} - c_1^{-2}) \varphi dy = 0, \quad \textit{for all harmonic functions } \varphi. \tag{3.3}$$

In particular, if $(c_2^{-2} - c_1^{-2})$ is harmonic in a simply connected region $\omega \subset\subset \Omega$ and identically zero on ω^c, then $c_1 \equiv c_2$ in Ω.

We are also able to prove Corollary 45 regarding unique recovery of the sound speed under monotonicity conditions.

Theorem 40 *Let* (f_1, c) *and* (f_2, c) *be admissible pairs. If*

$$\Lambda_{f_1,c}(x,t) = \Lambda_{f_2,c}(x,t) \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

then $f_1 = f_2$.

Theorem 40 should be compared to the uniqueness results in [2] and [65] where the authors assume that the sound speed is smooth. Models with discontinuous sound speed arise in thermoacoustic and photoacoustic tomography in order to understand the effect of sudden change of the sound speed in the skull in imaging of the human brain [66]. The results in [66] assume that the sound speed is smooth but allow for jumps across smooth surfaces. The rest of this thesis is devoted to proving the theorems in this section.

3.2.1 Uniqueness of the wave speed

In this section we present the proof of Theorem 39. Let us first develop a few basic facts about solutions of the wave equation (1.8) and gather some known results which will be used in our proofs.

The temporal Fourier transform $\hat{u}$, defined in (3.2), satisfies the elliptic partial differential equation

$$\Delta \hat{u}(x,k) + \frac{k^2}{c^2(x)} \hat{u}(x,k) = \frac{ik}{2\pi} \frac{f(x)}{c^2(x)}, \quad (x,k) \in \mathbb{R}^3 \times (0,\epsilon), \tag{3.4}$$

which is well-posed under the classical Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x| \left(\frac{\partial \hat{u}(x,k)}{\partial |x|} - ik\hat{u}(x,k) \right) = 0. \tag{3.5}$$

A good reference for more information on this Helmholtz type equation is [11]. Note also that under the temporal Fourier transform the measurement operator becomes

$$\hat{\Lambda}_{f,c}(x,k) = \hat{u}(x,k), \qquad (x,k) \in \mathbb{R}^3 \times (0,\epsilon). \tag{3.6}$$

Remark 41 *Suppose that for some admissible pairs (f_1, c_1) and (f_2, c_2) we have that $\Lambda_{f_1,c_1}(x,t) = \Lambda_{f_1,c_1}(x,t)$ for all $(x,t) \in \partial\Omega \times (0,\infty)$. Since $c_1 = c_2 \equiv 1$ and $f_1 = f_2 \equiv 0$ outside Ω, u_1 and u_2 both satisfy the same equation outside Ω. It then follows that $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \partial\Omega^c \times (0,\infty)$ since they both solve the same exterior boundary value problem. Thus*

$$\hat{u_1}(x,k) = \hat{u_2}(x,k), \qquad (x,k) \in \Omega^c \times (0,\epsilon)$$

and it follows that

$$\frac{\partial \hat{u}_1(x,k)}{\partial \nu} = \frac{\partial \hat{u}_2(x,k)}{\partial \nu}, \qquad (x,k) \in \partial\Omega \times (0,\epsilon).$$

In the rest of this chapter we will frequently use integration by parts (sometimes also referred to as Green's formula). We can assume without loss of generality that the boundary of Ω is sufficiently regular for all such computations. If the boundary of Ω were to lack the necessary regularity the previous remark tells us that we can simply assume the PAT data is equal on the boundary of some sphere containing Ω and use this sphere in place of Ω moving forward. We shall need the following lemma proved by Liu and Uhlamm in [47] (they also use results from [11])

Lemma 42 ([47]) *Let $\hat{u}(x,k) \in H^1_{loc}(\mathbb{R}^3)$ be the solution to (3.4)-(3.5). Then $\hat{u}(x,k)$ is uniquely given by the following integral equation*

$$\hat{u}(x,k) = k^2 \int_{\mathbb{R}^3} \left(c^{-2}(y) - 1\right) \hat{u}(y,k)\Phi(x-y)dy - \frac{ik}{2\pi} \int_{\mathbb{R}^3} \frac{f(y)}{c^2(y)} \Phi(x-y)dy, \quad x \in \mathbb{R}^3. \tag{3.7}$$

Moreover, as $k \to 0$, we have

$$\hat{u}(x,k) = -\frac{ik}{2\pi}\int_{\Omega}\frac{f(y)}{c^2(y)}\Phi_0(x-y)dy + \frac{k^2}{8\pi^2}\int_{\Omega}\frac{f(y)}{c^2(y)}dy + \mathcal{O}(k^3). \tag{3.8}$$

Here

$$\Phi(x) := \frac{e^{ik|x|}}{4\pi|x|} \quad \text{for} \quad |x| \neq 0$$

is the fundamental solution of $-\Delta - k^2$ and Φ_0 is the fundamental solution of $-\Delta$.

Define the space

$$\mathcal{A} := \{v \in L^2(\Omega) : \int_{\Omega} g\varphi dx = 0 \;\text{ for all harmonic functions } \varphi\}.$$

We shall frequently use the following two lemmas.

Lemma 43 *Let $g \in L^2(\Omega)$ and suppose $w \in H^1(\Omega)$ satisfies*

$$\begin{cases} -\Delta w = g & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Then $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ if and only if $g \in \mathcal{A}$.

Proof. Let φ be harmonic in Ω and $w \in H^2(\Omega)$ be the solution of (3.9). We have that

$$-\int_{\Omega}\Delta w(y)\varphi(y)dy = \int_{\Omega} g(y)\varphi(y)dy.$$

Then integration by parts two times gives that

$$-\int_{\Omega} w(y)\Delta\varphi(y)dy - \int_{\partial\Omega}\varphi(y)\frac{\partial w}{\partial\nu}dS + \int_{\partial\Omega} w(y)\frac{\partial\varphi}{\partial\nu}dS = \int_{\Omega} g(y)\varphi(y)dy.$$

Since φ is harmonic and $w = 0$ on $\partial\Omega$ we have that

$$-\int_{\partial\Omega}\varphi(y)\frac{\partial w}{\partial\nu}dS = \int_{\Omega} g(y)\varphi(y)dy.$$

Hence $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ if and only if $g \in \mathcal{A}$. $\square$

Lemma 44 *For $g \in L^\infty(\mathbb{R}^3)$ with compact support in Ω define*

$$w(x) := \int_{\mathbb{R}^3} g(y)\Phi_0(x-y)dy = \int_\Omega g(y)\Phi_0(x-y)dy. \tag{3.10}$$

Then

$$-\Delta w = g \quad in \quad \mathbb{R}^3$$

in the weak sense. Moreover if $w = 0$ on Ω^c, then for any harmonic function φ on $\mathbb{R}^3$ we have

$$\int_{\mathbb{R}^3} g(y)\varphi(y)dy = \int_\Omega g(y)\varphi(y)dy = 0.$$

Proof. The function w is often referred to as the Newton potential of g. For details on the Newton potential we refer the reader to chapters 2 and 13 of [38]. One thing of note is that if $g \in L^\infty(\Omega)$ then $w \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0,1)$. Since $w = 0$ and $\frac{\partial w}{\partial \nu} = 0$, the second part of this lemma follows from Lemma 43. □

For $g \in L^2(\Omega)$ we will denote the solution of (3.9) by $\Delta^{-1}(g)$. Note that if $g \in L^\infty(\mathbb{R}^3)$ has compact support in Ω and w defined by (3.10) vanishes on Ω^c, then $w = \Delta^{-1}(g)$.

Proof of Theorem 39. By Lemma 42 we have

$$\hat{u}(x,t) = -\frac{ik}{2\pi}\int_\Omega \frac{f(y)}{c^2(y)}dy\Phi_0(x-y)dy + \frac{k^2}{8\pi^2}\int_\Omega \frac{f(y)}{c^2(y)}dy + \mathcal{O}(k^3). \tag{3.11}$$

For $i = 1, 2$ define

$$w_i(x) := \lim_{k\to 0}\frac{\hat{u}_i(x,k)}{k} = -\frac{i}{2\pi}\int_\Omega \frac{f_i(y)}{c_i^2(y)}\Phi_0(x-y) = \frac{i}{2\pi}\Delta^{-1}(\frac{f_i(y)}{c_i^2(y)}).$$

Then $w := w_2 - w_1$ satisfies

$$\Delta w = \frac{i}{2\pi}\left(\frac{f_2(y)}{c_2^2(y)} - \frac{f_1(y)}{c_1^2(y)}\right),$$

and $w = \frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ via the argument in remark 41. Therefore it follows from Lemma 43 that

$$\int_\Omega \left(\frac{f_2(y)}{c_2^2(y)} - \frac{f_1(y)}{c_1^2(y)}\right)\varphi(y)dy = 0, \tag{3.12}$$

for every harmonic function φ. On the other hand we have

$$\Delta(\hat{u}_2(x,k) - \hat{u}_1(x,k)) + \frac{k^2}{c_2^2(x)}\hat{u}_2(x,k) - \frac{k^2}{c_1^2(x)}\hat{u}_1(x,k) = -\frac{ik}{2\pi}\left(\frac{f_2(x)}{c_2^2(x)} - \frac{f_1(x)}{c_1^2(x)}\right), \tag{3.13}$$

for $(x,k) \in \mathbb{R}^3 \times (0,\epsilon)$. Multiplying both sides of the above equation by a harmonic function φ, using (3.12) and the fact that $\hat{u}_2 - \hat{u}_1 \equiv 0$ on Ω^c, and integrating by parts we get

$$\frac{1}{k}\int_\Omega \left(\frac{\hat{u}_2(y,k)}{c_2^2(y)} - \frac{\hat{u}_1(y,k)}{c_1^2(y)}\right)\varphi dy = 0, \qquad \forall k \in (0,\epsilon). \tag{3.14}$$

Since $\hat{u}_2(x,k) = \hat{u}_1(x,k)$ for all $(x,k) \in \partial\Omega \times (0,\epsilon)$, it follows from (3.11) that

$$\int_\Omega c_1^{-2} f_1 dy = \int_\Omega c_2^{-2} f_2 dy. \tag{3.15}$$

Combining this with (3.14) and Lemma 42 we have

$$\begin{aligned} &\frac{i}{2\pi}\int_\Omega \left(\frac{(\int_\Omega c_2^{-2}(z) f_2(z)\Phi_0(y-z)dz}{c_2^2(y)} - \frac{\int_\Omega c_1^{-2}(z) f_1(z)\Phi_0(y-z)dz}{c_2^2(y)}\right)\varphi(y)dy \\ &-\frac{k}{8\pi^2}\int_\Omega c_1^{-2}(y) f_1(y)dy \int_\Omega (c_2^{-2}(y) - c_1^{-2}(y))\varphi(y)dy + \mathcal{O}(k^2) = 0, \end{aligned} \tag{3.16}$$

for all $k \in (0,\epsilon)$. Thus

$$\int_\Omega \left(\frac{\int_\Omega c_2^{-2}(z) f_2(z)\Phi_0(y-z)dz}{c_2^2(y)} - \frac{\int_\Omega c_1^{-2}(z) f_1(z)\Phi_0(y-z)dz}{c_2^2(y)}\right)\varphi(y)dy \tag{3.17}$$

and

$$\int_\Omega (c_2^{-2} - c_1^{-2})\varphi dy = 0 \tag{3.18}$$

for any harmonic function φ, provided $C^* \neq 0$. If $(c_2^{-2} - c_1^{-2})$ is harmonic in a simply connected region $\omega \subset\subset \Omega$ and identically zero on ω^c, then taking $\varphi = (c_2^{-2} - c_1^{-2})$ in (3.2.1) implies $c_1 \equiv c_2$. □

Corollary 45 *If $c_1 \geq c_2$ or $c_1 \leq c_2$ in Ω, then $c_1 \equiv c_2$.*

Proof. Let $\varphi \equiv 1$ in Ω. Then by Theorem 39 we have

$$\int_\Omega c_2^{-2}(x)dx = \int_\Omega c_1^{-2}(x)dx,$$

and the result follows immediately. □

We can compare this to Finch and Hickman's result in Theorem 34. The result does not require the sound speed to be smooth, we instead require the admissibility condition.

At this point we would like to provide some of the inspiration behind the proof of Theorem 39. As previously mentioned, [47] was the inspiration for our work. In this paper they used the series expansion of $e^{ik|x-y|}$ to expand $\Phi(x-y)$ as

$$\Phi(x-y) = \Phi_0(x-y) + \frac{1}{4\pi}\sum_{n=1}^{\infty} i^n k^n |x-y|^{n-1}.$$

Using this expansion and Lemma 42 they get that

$$\begin{aligned}\hat{u}(x,k) = &-\frac{-ik}{2\pi}\int_\Omega c^{-2}(y)f(y)\Phi_0(x-y)dy + \frac{k^2}{8\pi^2}\int_\Omega c^{-2}(y)f(y)dy \\ &+ k^3\left(\frac{i}{2\pi}\int_\Omega (1-c^{-2}(y))\Delta^{-1}(c^{-2}f)(y)\Phi_0(x-y)dy + \frac{i}{16\pi^2}\int_\Omega c^{-2}(y)f(y)|x-y|dy\right) \\ &+ O(k^4).\end{aligned}$$

When $\hat{u_1} = \hat{u_2}$ on $\partial\Omega$ they use the corresponding three equations (when k=1,2,3) to prove their results. In particular they prove that since

$$\int_\Omega c_1^{-2}(y) f_1(y) \Phi_0(x-y) dy = \int_\Omega c_2^{-2}(y) f_2(y) \Phi_0(x-y) dy \tag{3.19}$$

for all $x \in \Omega^c$ (this follows from the k=1 terms) then

$$\int_\Omega c_1^{-2}(y) f_1(y) \varphi(y) dy = \int_\Omega c_2^{-2}(y) f_2(y) \varphi(y) dy \tag{3.20}$$

for all harmonic functions φ. Since $\Phi_0(x-y)$ is itself harmonic this gives that equation 3.19 in fact holds for all $x \in \Omega$ as well. We can use Lemmas 43 and 44 to verify this despite the fact that $\Phi_0(x-y)$ blows up when $x = y$. In fact the computations in 43 will hold with φ replaced by $\Phi_0(x-y)$ by isolating the singularity.

We then asked the natural question as to what will happen if we continuing expanding. Furthering this expansion we can obtain the k^4 term in the expansion of $\hat{u}$ to be

$$\frac{-1}{8\pi^2} \int_\Omega (1 - c^{-2}(y)) \Phi_0(x-y) dy \int_\Omega c^{-2}(y) f(y) dy - \frac{1}{48\pi^2} \int_\Omega c^{-2}(y) f(y) |x-y|^2.$$

If $\hat{u_1} = \hat{u_2}$ on Ω^c we already know (from the k=2 terms) that

$$\int_\Omega c_1^{-2}(y) f_1(y) dy = \int_\Omega c_2^{-2}(y) f_1(y) dy.$$

Suppose $\hat{u_1} = \hat{u_2}$ on Ω^c. We know from equation 3.19 that

$$C^* = \Delta^{-1}(c_1^{-2} f_1 - c_2^{-2} f_2) = \frac{\partial \Delta^{-1}(c_1^{-2} f_1 - c_2^{-2} f_2)}{\partial \nu} = 0, \qquad \partial\Omega$$

an we can obtain by integration by parts that (see 46 for more details)

$$\int_\Omega (c_1^{-2}(y) f_1(y) - c_2^{-2}(y) f_2(y)) |x-y|^2 = -6 \int_\Omega (\Delta^{-1}(c_1^{-2} f_1 - c_2^{-2} f_2)(y)) dy = 0$$

for any $x \in \Omega^c$. Then the equation obtained from the fourth order terms gives that

$$\int_\Omega (1 - c_1^{-2}(y))\Phi_0(x-y)dy \int_\Omega c_1^{-2}(y) f_1(y)dy = \int_\Omega (1 - c_2^{-2}(y))\Phi_0(x-y)dy \int_\Omega c_2^{-2}(y) f_2(y)dy$$

for all $x \in \Omega^c$. So if $C* \neq 0$ we have for all $x \in \Omega^c$

$$\int_\Omega (c_1^{-2}(y) - c_2^{-2}(y))\Phi_0(x-y)dy = 0.$$

As mentioned previously this is equivalent to saying that

$$\int_\Omega (c_2^{-2} - c_1^{-2})\varphi dy = 0$$

for any harmonic function φ. Note that we have just provided an alternate proof of Theorem 39, although the underlying proof mechanism is the same. After obtaining the new result using the fourth order terms we thought that perhaps we could compute higher order terms and obtain further results regarding the sound speed. However these terms become complicated very quickly and we were unable to make sense of them. We did however notice that if we took $c_1 = c_2$ that these higher order terms simplified significantly and we were able to use them to prove Theorem 40. This proof is provided in the next section.

3.2.2 Uniqueness of the source

In this section we prove that if $c_1 = c_2 = c \in L^\infty(\mathbb{R}^3)$, then the source function f can be uniquely recovered from the knowledge of $\Lambda_{f,c}(x,t)$ on $\partial\Omega \times \mathbb{R}_+$. Throughout this section we shall assume that $c_1 = c_2 = c \in L^\infty(\mathbb{R}^3)$.

By Lemma 42, $\hat{u_2} - \hat{u_1}$ satisfies the following integral equation

$$\begin{aligned}(\hat{u_2} - \hat{u_1})(x, k) &= k^2 \int_\Omega (c^{-2}(y) - 1)(\hat{u_2} - \hat{u_1})(y, k)\Phi(x - y)dy \\ &\quad - \frac{ik}{2\pi} \int_\Omega c^{-2}(y)(f_2 - f_1)(y)\Phi(x - y)dy. \qquad (3.21)\end{aligned}$$

We shall need the following lemma.

Lemma 46 *Let g have compact support in Ω and $g \in L^\infty(\mathbb{R}^3)$. Suppose that w defined by* (3.10) *vanishes on Ω^c. Then for $n \geq 1$*

$$\int_\Omega g(y)|x - y|^n dy = -n(n+1) \int_\Omega \Delta^{-1}(g)(y)|x - y|^{n-2} dy \qquad (3.22)$$

for all $x \in \mathbb{R}^3$.

Proof. Lemma 44, integration by parts twice, and $w = \frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ give us that

$$\begin{aligned}\int_\Omega g(y)|x - y|^n dy &= -\int_\Omega \Delta w(y)|x - y|^n dy \\ &= -\int_\Omega w(y)\Delta|x - y|^n dy + \int_{\partial\Omega} w(y)\frac{\partial|x - y|^n}{\partial\nu} - |x - y|^n(y)\frac{\partial w}{\partial\nu} dS \\ &= -n(n+1) \int_\Omega \Delta^{-1}(g)(y)|x - y|^{n-2} dy.\end{aligned}$$

We remark that some care must be taken when $n = 1$ as the derivatives of $|x - y|$ blow up when $x = y$. However it can be shown though removing the singularity that the above equations still hold. □

For every $g \in L^\infty(\mathbb{R}^3)$ with compact support in Ω define

$$Lg(x) := \Delta^{-1}(c^{-2}g)(x), \quad x \in \mathbb{R}^3.$$

Proposition 47 *For every $n \in \mathbb{N}$ there exists functions $p_m(x)$, $m = 1, 2, ..., n$, such that*

$$(\hat{u}_2 - \hat{u}_1)(x,k) = \sum_{m=1}^{n} p_m(x) k^j + \mathcal{O}(k^{n+1}), \tag{3.23}$$

as $k \to 0$. Moreover if $u_2(x,k) - u_1(x,k) = 0$ for all $x \in \Omega^c$, then

$$p_n(x) = \begin{cases} \frac{-i}{2\pi} L^{\frac{n+1}{2}} (f_2 - f_1)(x) & \text{if } n \text{ is odd} \\ \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{3.24}$$

Proof. By Lemma 42, (3.23) holds for $n = 1, 2$. Suppose it holds for all $j \leq n$. Then there exists functions $p_m(x)$, $m = 1, 2, ..., n$, such that

$$(\hat{u}_2 - \hat{u}_1)(x) = \sum_{m=1}^{n} p_m(x) k^j + \mathcal{O}(k^{n+1}) \quad \text{as} \quad k \to 0.$$

Plugging this expression for $\hat{u}_2 - \hat{u}_1$ into equation (3.21) and expanding Φ we find that

$$(\hat{u}_2 - \hat{u}_1)(x) = \sum_{m=1}^{n} p_m(x) k^j + p_{n+2}(x) k^{n+2} + \mathcal{O}(k^{n+3}) \quad \text{as} \quad k \to 0,$$

where

$$\begin{aligned} p_{n+2}(x) = & \sum_{m=0}^{n} \frac{i^m}{4\pi m!} \int_{\Omega} (c^{-2} - 1)(y) p_{n-m}(y) |x - y|^{m-1} dy \\ & - \frac{i^{n+2}}{8\pi^2 (n+1)} \int_{\Omega} c^{-2}(y) (f_2 - f_1)(y) |x - y|^n dy. \end{aligned}$$

To prove (3.24) we proceed by strong induction. First notice that $p_m \equiv 0$ on Ω^c for all $m \in \mathbb{N}$. By Lemma 42, (3.24) holds when $n = 0, 1, 2$. Suppose (3.24) holds for all $j \leq n+1$. First assume that n is odd. Using the integral equation (3.21) and the induction hypothesis we compute that

$$\begin{aligned} p_{n+2}(x) = & \sum_{\substack{m=0 \\ \text{m even}}}^{n-1} \frac{i^{m+1}}{8\pi^2 m!} \int_{\Omega} (1 - c^{-2})(y) L^{\frac{n-m+1}{2}} (f_2 - f_1)(y) |x - y|^{m-1} dy \\ & - \frac{i^{n+2}}{8\pi^2 (n+1)} \int_{\Omega} c^{-2}(y) (f_2 - f_1)(y) |x - y|^n dy. \end{aligned}$$

For even m with $m \le n-1$ define

$$q_m(x) := \frac{i^{m+1}}{8\pi^2 m!} \int_\Omega (1-c^{-2})(y) L^{\frac{n-m+1}{2}}(f_2-f_1)(y)|x-y|^{m-1}dy,$$

and

$$r(x) := -\frac{i^{n+2}}{8\pi^2(n+1)!} \int_\Omega c^{-2}(y)(f_2-f_1)(y)|x-y|^n dy.$$

Then

$$p_{n+2}(x) = \sum_{\substack{m=0 \\ \text{m even}}}^{n-1} q_m(x) + r(x).$$

It follows from the induction hypothesis and Lemma 46 that

$$\begin{aligned}
q_2(x) &= \frac{-i}{8\pi^2 2!} \int_\Omega (1-c^{-2})(y) L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy \\
&= \frac{-i}{8\pi^2 2!} \int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy \\
&\quad + \frac{i}{8\pi^2 2!} \int_\Omega c^{-2}(y) L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy \\
&= \frac{-i}{8\pi^2 2!} \int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy \\
&\quad - \frac{i}{8\pi^2} \int_\Omega \Delta^{-1}(c^{-2}(y) L^{\frac{n-1}{2}}(f_2-f_1))(y)|x-y|^{-1}dy \\
&= \frac{-i}{8\pi^2 2!} \int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy - \frac{i}{8\pi^2} \int_\Omega L^{\frac{n+1}{2}}(f_2-f_1)(y)|x-y|^{-1}dy.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(q_0+q_2)(x) &= \frac{-i}{8\pi^2} \int_\Omega c^{-2}(y) L^{\frac{n+1}{2}}(f_2-f_1)(y)|x-y|^{-1}dy \\
&\quad - \frac{i}{8\pi^2 2!} \int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy \\
&= \frac{-i}{2\pi} \int_\Omega c^{-2}(y) L^{\frac{n+1}{2}}(f_2-f_1)(y)\Phi_0(x-y)dy \\
&\quad - \frac{i}{8\pi^2 2!} \int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy \\
&= -\frac{i}{2\pi} L^{\frac{n+3}{2}}(f_2-f_1)(x) - \frac{i}{8\pi^2 2!} \int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy.
\end{aligned}$$

Similarly by Lemma 46 we get

$$\begin{aligned} q_4(x) &= \frac{i}{8\pi^2 4!}\int_\Omega (1-c^{-2})(y)L^{\frac{n-3}{2}}(f_2-f_1)(y)|x-y|^3 dy \\ &= \frac{i}{8\pi^2 4!}\int_\Omega L^{\frac{n-3}{2}}(f_2-f_1)(y)|x-y|^3 dy + \frac{i}{8\pi^2 2!}\int_\Omega L^{\frac{n-1}{2}}(f_2-f_1)(y)|x-y|dy. \end{aligned}$$

Hence

$$(q_0+q_2+q_4)(x) = -\frac{i}{2\pi}L^{\frac{n+3}{2}}(f_2-f_1)(x) + \frac{i}{8\pi^2 4!}\int_\Omega L^{\frac{n-3}{2}}(f_2-f_1)(y)|x-y|^3 dy.$$

We can continue this process in general. Let m be even with $m \leq n-3$ and suppose

$$(q_0+q_2+...+q_m)(x) = -\frac{i}{2\pi}L^{\frac{n+3}{2}}(f_2-f_1)(x) + \frac{i^{m+1}}{8\pi^2 m!}L^{\frac{n-m+1}{2}}(f_2-f_1)(y)|x-y|^{m-1}dy.$$

Then by Lemma 46

$$\begin{aligned} q_{m+2}(x) &= \frac{i^{m+3}}{8\pi^2(m+2)!}\int_\Omega (1-c^{-2})(y)L^{\frac{n-m-1}{2}}(f_2-f_1)(y)|x-y|^{m+1}dy \\ &= \frac{i^{m+3}}{8\pi^2(m+2)!}\int_\Omega L^{\frac{n-m-1}{2}}(f_2-f_1)(y)|x-y|^{m+1}dy \\ &\quad + \frac{i^{m+3}}{8\pi^2 m!}\int_\Omega L^{\frac{n-m+1}{2}}(f_2-f_1)(y)|x-y|^{m-1}dy. \end{aligned}$$

Noting that $i^{m+3} = -i^{m+1}$ we get that

$$\begin{aligned} (q_0+q_2+...+q_m+q_{m+2})(x) &= -\frac{i}{2\pi}L^{\frac{n+3}{2}}(f_2-f_1)(x) \\ &\quad + \frac{i^{m+3}}{8\pi^2(m+2)!}\int_\Omega L^{\frac{n-m-1}{2}}(f_2-f_1)(y)|x-y|^{m+1}dy. \end{aligned}$$

Repeating the above process until $m = n-1$ we obtain

$$\begin{aligned} (q_0+q_2+...+q_{n-3}+q_{n-1})(x) &= -\frac{i}{2\pi}L^{\frac{n+3}{2}}(f_2-f_1)(x) \\ &\quad + \frac{i^n}{8\pi^2(n-1)!}\int_\Omega L(f_2-f_1)(y)|x-y|^{n-2}dy. \end{aligned}$$

In addition by Lemma 46

$$r(x) = \frac{i^{n+2}}{8\pi^2(n-1)!}\int_\Omega L(f_2 - f_1)(y)|x-y|^{n-2}dy.$$

Hence

$$p_{n+2}(x) = -\frac{i}{2\pi}L^{\frac{n+3}{2}}(f_2 - f_1)(x).$$

This finishes the proof for the case that n is odd.

Now suppose n is even. Using the integral equation (3.21) and the induction hypothesis we compute that

$$p_{n+2}(x) = \sum_{\substack{m=1 \\ \text{m odd}}}^{n-1} \frac{i^{m+1}}{8\pi^2 m!}\int_\Omega (1-c^{-2})(y)L^{\frac{n-m+1}{2}}(f_2 - f_1)(y)|x-y|^{m-1}dy$$
$$- \frac{i^{n+2}}{8\pi^2(n+1)}\int_\Omega c^{-2}(y)(f_2 - f_1)(y)|x-y|^n dy.$$

As before for odd m with $m \leq n-1$ define

$$q_m(x) := \frac{i^{m+1}}{8\pi^2 m!}\int_\Omega (1-c^{-2})(y)L^{\frac{n-m+1}{2}}(f_2 - f_1)(y)|x-y|^{m-1}dy,$$

and

$$r(x) := -\frac{i^{n+2}}{8\pi^2(n+1)!}\int_\Omega c^{-2}(y)(f_2 - f_1)(y)|x-y|^n dy.$$

Then

$$p_{n+2}(x) = \sum_{\substack{m=0 \\ \text{m even}}}^{n-1} q_m(x) + r(x).$$

It follows from the induction hypothesis and Lemma 46 that

$$
\begin{aligned}
q_3(x) &= \frac{1}{8\pi^2 3!}\int_\Omega (1-c^{-2})(y)L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy \\
&= \frac{1}{8\pi^2 3!}\int_\Omega L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy \\
&\quad - \frac{1}{8\pi^2 3!}\int_\Omega c^{-2}(y)L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy \\
&= \frac{1}{8\pi^2 3!}\int_\Omega L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy \\
&\quad + \frac{1}{8\pi^2}\int_\Omega \Delta^{-1}(c^{-2}(y)L^{\frac{n-2}{2}}(f_2-f_1))(y) dy \\
&= \frac{1}{8\pi^2 3!}\int_\Omega L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy + \frac{1}{8\pi^2}\int_\Omega L^{\frac{n}{2}}(f_2-f_1)(y) dy.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(q_1+q_3)(x) = \frac{-1}{8\pi^2}&\int_\Omega (1-c^{-2})(y)L^{\frac{n}{2}}(f_2-f_1)(y) dy \\
&+ \frac{1}{8\pi^2 3!}\int_\Omega L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy + \frac{1}{8\pi^2}\int_\Omega L^{\frac{n}{2}}(f_2-f_1)(y) dy \\
= &\frac{1}{8\pi^2}\int_\Omega c^{-2}(y)L^{\frac{n}{2}}(f_2-f_1)(y) dy + \frac{1}{8\pi^2 3!}\int_\Omega L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy
\end{aligned}
$$

Similarly by Lemma 46 we get

$$
\begin{aligned}
q_5(x) &= -\frac{1}{8\pi^2 5!}\int_\Omega (1-c^{-2})(y)L^{\frac{n-4}{2}}(f_2-f_1)(y)|x-y|^4 dy \\
&= -\frac{1}{8\pi^2 5!}\int_\Omega L^{\frac{n-4}{2}}(f_2-f_1)(y)|x-y|^4 dy - \frac{1}{8\pi^2 3!}\int_\Omega L^{\frac{n-2}{2}}(f_2-f_1)(y)|x-y|^2 dy.
\end{aligned}
$$

Hence

$$
(q_1+q_3+q_5)(x) = \frac{1}{8\pi^2}\int_\Omega c^{-2}(y)L^{\frac{n}{2}}(f_2-f_1)(y) dy - \frac{1}{8\pi^2 5!}\int_\Omega L^{\frac{n-4}{2}}(f_2-f_1)(y)|x-y|^4 dy.
$$

We can continue this process in general. Let m be odd with $1 \le m \le n-3$ and suppose

$$(q_1 + q_3 + ... + q_m)(x) = \frac{1}{8\pi^2} \int_\Omega c^{-2}(y) L^{\frac{n}{2}} (f_2 - f_1)(y) dy + \frac{i^{m+1}}{8\pi^2 m!} \int_\Omega L^{\frac{n-m+1}{2}} (f_2 - f_1)(y) |x-y|^{m-1} dy.$$

Then as in the odd case by Lemma 46

$$\begin{aligned} q_{m+2}(x) &= \frac{i^{m+3}}{8\pi^2 (m+2)!} \int_\Omega (1 - c^{-2})(y) L^{\frac{n-m-1}{2}} (f_2 - f_1)(y) |x-y|^{m+1} dy \\ &= \frac{i^{m+3}}{8\pi^2 (m+2)!} \int_\Omega L^{\frac{n-m-1}{2}} (f_2 - f_1)(y) |x-y|^{m+1} dy \\ &\quad + \frac{i^{m+3}}{8\pi^2 m!} \int_\Omega L^{\frac{n-m+1}{2}} (f_2 - f_1)(y) |x-y|^{m-1} dy. \end{aligned}$$

Noting that $i^{m+3} = -i^{m+1}$ we get that

$$\begin{aligned} (q_1 + q_3 + ... + q_m + q_{m+2})(x) &= \frac{1}{8\pi^2} \int_\Omega c^{-2}(y) L^{\frac{n}{2}} (f_2 - f_1)(y) dy \\ &\quad + \frac{i^{m+3}}{8\pi^2 (m+2)!} \int_\Omega L^{\frac{n-m+1}{2}} (f_2 - f_1)(y) |x-y|^{m-1} dy. \end{aligned}$$

Repeating the above process until $m = n-1$ we obtain

$$\begin{aligned} (q_1 + q_3 + ... + q_{n-3} + q_{n-1})(x) &= \frac{1}{8\pi^2} \int_\Omega c^{-2}(y) L^{\frac{n}{2}} (f_2 - f_1)(y) dy \\ &\quad + \frac{i^n}{8\pi^2 (n-1)!} \int_\Omega L (f_2 - f_1)(y) |x-y|^{n-2} dy. \end{aligned}$$

In addition by Lemma 46

$$r(x) = \frac{i^{n+2}}{8\pi^2 (n-1)!} \int_\Omega L (f_2 - f_1)(y) |x-y|^{n-2} dy.$$

Hence

$$p_{n+2}(x) = \frac{1}{8\pi^2} \int_\Omega c^{-2}(y) L^{\frac{n}{2}} (f_2 - f_1)(y) dy.$$

By the induction hypothesis for $j = n+1$ we have

$$p_{n+1}(x) = L^{\frac{n+2}{2}} (f_2 - f_1)(x) = 0, \quad \forall x \in \Omega^c.$$

Hence

$$\Delta^{-1}(c^{-2}L^{\frac{n}{2}}(f_2 - f_1)) = L^{\frac{n+2}{2}}(f_2 - f_1)(x) = 0 \quad \forall x \in \Omega^c,$$

and it follows from Lemma 44 that

$$\int_\Omega c^{-2}(y)L^{\frac{n}{2}}(f_2 - f_1)(y)\varphi dy = 0,$$

for every harmonic function φ. Letting $\phi \equiv 1$ we get

$$p_{n+2}(x) = \int_\Omega c^{-2}(y)L^{\frac{n}{2}}(f_2 - f_1)(y)dy = 0.$$

Hence $p_{n+2}(x) = 0$. □

Theorem 48 *Suppose $c_1 = c_2 = c$ and that u_1, u_2 be solutions of the wave equation* (1.8) *with $u_1(x,0) = f_1$ and $u_2(x,0) = f_2$. If*

$$u_2(x,t) = u_1(x,t) \text{ for all } (x,t) \in \Omega^c \times \mathbb{R}_+,$$

then

$$\int_\Omega F_n \varphi c^{-2} dx = 0, \tag{3.25}$$

for all $n \geq 0$ and all harmonic functions φ, where $F_0 = f_2 - f_1$ and

$$F_n = \Delta^{-1}(c^{-2}F_{n-1}), \quad n \geq 1. \tag{3.26}$$

Proof. The proof follows directly from Proposition 47, Lemma 43, and the observation that $F_n = p_n$. □

The proof of the next lemma will use Theorem 6.3 in [34]. It is provided below for reference.

Theorem 49 *Let Ω be an open connected subset of $\mathbb{R}^n$ with $n > 2$ and $V \in L^{\frac{n}{2}}_{loc}(\Omega)$. Let $q = \frac{2n}{n+2}$. If $u \in H^{2,q}_{loc}(\Omega)$ satisfies*

$$|\Delta u(x)| \leq |V(x)||u(x)|$$

for all $x \in \Omega$ and if u vanishes in a non-empty open subset of Ω then u is identically zero in Ω.

We can now prove the following lemma.

Lemma 50 *Let $g \in L^2(\Omega)$ satisfy*

$$\int_\Omega c^{-2} g \varphi dx = 0, \tag{3.27}$$

for all φ harmonic in Ω. Suppose $c_0 < c \in L^\infty(\Omega)$ for some $c_0 > 0$. If

$$\lambda g = \Delta^{-1}(c^{-2} g) \quad in \quad \Omega \tag{3.28}$$

for some $\lambda > 0$, then $g \equiv 0$.

Proof. It follows from Lemma 43 that $g|_\Omega = \frac{\partial g}{\partial \nu} = 0$. Note that

$$-\Delta g = \frac{c^{-2}}{\lambda} g \;\text{ in }\; \Omega.$$

Since $c \in L^\infty(\Omega)$ and $g \in L^2(\Omega)$, by elliptic regularity $g \in H^2(\Omega)$. Thus $g \in W^{1,2}(\Omega)$ and so by the $k < \frac{n}{p}$ case of the general Sobolev inequality (see Theorem 5.6 in [19]) $g \in L^6(\Omega)$. Since $c \in L^\infty(\Omega)$ again by elliptic regularity we have that $g \in W^{2,6}(\Omega)$. Then by the $k > \frac{n}{p}$ case of the general Sobolev inequality we have that $g \in C^{1,\frac{1}{2}}(\bar{\Omega})$. Hence we can extend g to a function $\tilde{g} \in C^{0,\frac{1}{2}}(\mathbb{R}^3)$ by defining $\tilde{g} = 0$ on Ω^c. Let

$$w(x) := \frac{1}{\lambda} \int_{\mathbb{R}^3} c^{-2}(y) \tilde{g}(y) \Phi_0(x-y) dy = \frac{1}{\lambda} \int_\Omega c^{-2}(y) g(y) \Phi_0(x-y) dy.$$

Since $c^{-2}\tilde{g} \in L^{\infty}(\mathbb{R}^3)$, it follows from elliptic regularity that $w \in C^{1,\alpha}(\alpha)$ and it satisfies

$$-\Delta w = \frac{c^{-2}}{\lambda}\tilde{g}.$$

Furthermore since $g \in \mathcal{A}$, $w = 0$ on Ω^c. Thus $w = \tilde{g}$ and hence $\tilde{g} \in C^{1,\alpha}(\mathbb{R}^3)$ solves

$$-\Delta \tilde{g} = \frac{c^{-2}}{\lambda}\tilde{g}, \quad \text{in} \quad \mathbb{R}^3$$

and $\tilde{g} = 0$ on Ω^c. We will then apply the unique continuation result in [34] (see Theorem 49. Let Ω' be a bounded domain in $\mathbb{R}^3$ containing Ω. Since $c \in L^{\infty}(\mathbb{R}^3)$ we have that $\frac{c^{-2}}{\lambda} \in L^{\frac{3}{2}}(\Omega')$. Since $\tilde{g} \in H^2(\Omega')$ (by elliptic regularity) and Ω' is bounded we have that $\tilde{g} \in W^{2,\frac{6}{5}}(\Omega')$. Since

$$|\Delta \tilde{g}| \leq \frac{c^{-2}}{\lambda}|\tilde{g}|$$

for all $x \in \Omega'$ and $\tilde{g}$ vanishes in $\Omega' - \Omega$ we can conclude that $\tilde{g} = 0$ in Ω'. Thus $g = 0$ in Ω.

□

Lemma 51 *Let $\mathcal{H}$ be defined by*

$$\mathcal{H} := \{v \in L^2(\Omega, c^{-2}dx) : c^{-2}v \in \mathcal{A} \quad and \quad c^{-2}L^n(v) \in \mathcal{A} \quad \forall n \in \mathbb{N}\}.$$

Then $\mathcal{H}$ is a Hilbert space equipped with the inner product

$$\langle v, w\rangle|_{\mathcal{H}} = \int_{\Omega} vwc^{-2}dx.$$

Proof. The linearity of L gives that $\mathcal{H}$ is a subspace. It remains to verify that $\mathcal{H}$ is closed. Suppose v_n converges to v in $L^2(\Omega, c^{-2}dx)$ and $v_n \in \mathcal{H}$. We can show $c^{-2}v \in \mathcal{A}$ simply. Let

φ be harmonic. Then by Holder's inequality

$$\begin{aligned}
|\int_\Omega c^{-2} v\phi| &= |\int_\Omega c^{-2}(v_n - v)\phi| \\
&\leq \int_\Omega |c^{-1}(v_n - v)||c^{-1}\phi| \\
&\leq (\int_\Omega c^{-2}|v_n - v|^2)^{\frac{1}{2}} (\int_\Omega c^{-2}|\phi|^2)^{\frac{1}{2}} \\
&= ||v_n - v||_{L^2(\Omega, c^{-2}dx)} ||c^{-2}\phi||_{L^2(\Omega)}.
\end{aligned}$$

Since $c^{-2}\phi \in L^2(\Omega)$ and $||v_n - v||_{L^2(\Omega,c^{-2}dx)} \to 0$ as $n \to \infty$ we have that $\int_\Omega c^{-2} v\phi = 0$.

We claim that $w_n := Lv_n$ converges to $w := Lv$ in $L^2(\Omega, c^{-2}dx)$. Since

$$-\Delta(w_n - w) = c^{-2}(v_n - v) \ \text{ in } \ \Omega, \quad w_n = 0 \ \text{ on } \ \partial\Omega,$$

we have

$$\| w_n - w \|_{H^2(\Omega)} \leq C \| c^{-2}(v_n - v) \|_{L^2(\Omega)} \to 0,$$

(See the Remark after Theorem 4 in section 6.3 of [19]) and hence w_n converges to w in $L^2(\Omega, c^{-2}dx)$. Moreover, $c^{-2}L^n(w) = c^{-2}L^{n+1}(v) \in \mathcal{A}$ for all $n \geq 0$. Thus $w \in \mathcal{H}$ and the proof is complete. □

Lemma 52 *The linear operator $L : \mathcal{H} \to \mathcal{H}$ is a non-negative, self-adjoint, and compact operator.*

Proof. It follows from integration by parts that

$$\langle Lv, v\rangle_{\mathcal{H}} = \int_\Omega \Delta^{-1}(c^{-2}v)c^{-2}v = \int_\Omega |\nabla\Delta^{-1}(c^{-2}v)|^2 \geq 0,$$

and hence L is non-negative. Similarly,

$$\langle Lv, w\rangle_{\mathcal{H}} = \int_\Omega \Delta^{-1}(c^{-2}v) w c^{-2} dx = \int_\Omega c^{-2} v \Delta^{-1}(c^{-2}w) dx = \langle v, Lw\rangle_{\mathcal{H}},$$

and hence L is self-adjoint. To show that L is compact we need to to prove that $L(B_{\mathcal{H}})$ has compact closure in the strong topology, where $B_{\mathcal{H}}$ is the unit ball in $\mathcal{H}$ (see [7]). Let $v_n \in B_{\mathcal{H}}$. We need to show that $\{w_n\} := \{L(v_n)\}$ has a subsequence that converges in $L^2(\Omega, c^{-2}dx)$. Since

$$-\Delta w_n = c^{-2} v_n, \quad w_n = 0 \ \text{ on } \ \partial\Omega,$$

we have

$$\| w_n \|_{H^2(\Omega)} \leq C \| c^{-2} v_n \|_{L^2(\Omega)} \leq C.$$

Thus w_n is bounded in $H^2(\Omega)$ and hence w_n has a subsequence, denoted by w_n again, that converges weakly in $H^2(\Omega)$. Therefore w_n converges strongly in $L^2(\Omega, c^{-2}dx)$ to some $w \in L^2(\Omega, c^{-2}dx)$ and thus L is compact. □

Proposition 53 *Let F_n be defined by* (3.26) *and suppose*

$$\int_\Omega F_n \varphi c^{-2} dx = 0, \tag{3.29}$$

for all $n \geq 0$ and all harmonic functions φ. Then $F_0 \equiv 0$ in Ω.

Proof. It follows from Lemma 51 and Lemma 52 that L has an orthonormal basis of eigenfunctions $e_n \in \mathcal{H}$ with corresponding eigenvalues $\lambda_n \geq 0$, where $\lambda_n \geq \lambda_{n+1}$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose $F_0 \not\equiv 0$. Then there exists constants $\alpha_j \in \mathbb{R}$ such that

$$F_0 = \sum_{j=1}^{\infty} \alpha_j e_j. \tag{3.30}$$

Let φ be harmonic in Ω. Then (3.29) implies

$$\int_\Omega \left(\sum_{j=1}^{\infty} \lambda_j^n \alpha_j e_j(x) \right) \varphi c^{-2} dx = 0, \quad \forall n \geq 0. \tag{3.31}$$

Now let

$$\lambda_* = \max\{\lambda_j : \quad \alpha_j \neq 0\}.$$

Dividing equality (3.31) by λ_*^n yields

$$\int_\Omega \left(\sum_{j=1}^{m} \alpha_j e_j \right) \varphi c^{-2} dx + \int_\Omega \left(\sum_{j=m+1}^{\infty} \left(\frac{\lambda_j}{\lambda_*} \right)^n \alpha_j e_j \right) \varphi c^{-2} dx = 0, \quad \forall n \geq 0,$$

where $L(e_j) = \lambda_* e_j$, $j = 1, 2, ...m$. We observe that

$$\begin{aligned} \left\| \sum_{j=m+1}^{\infty} \left(\frac{\lambda_j}{\lambda_*} \right)^n \alpha_j e_j \right\|^2_{L^2(\Omega, c^{-2}dx)} &= \sum_{j=m+1}^{\infty} \left\| \left(\frac{\lambda_j}{\lambda_*} \right)^n \alpha_j e_j \right\|^2_{L^2(\Omega, c^{-2}dx)} \\ &\leq \left(\frac{\lambda_{m+1}}{\lambda_*} \right)^n \sum_{j=m+1}^{\infty} \alpha_j \to 0, \end{aligned}$$

as $n \to \infty$. Thus

$$\int_\Omega g c_1^{-2} \varphi dx = 0, \quad \text{for every harmonic functions} \quad \varphi,$$

where g satisfies $L(g) = \Delta^{-1}(c_1^{-2} g) = \lambda_* g$. By Lemma 50 we have $g = \sum_{j=1}^{m} \alpha_j e_j \equiv 0$ in Ω, which is a contradiction. Thus $F_0 \equiv 0$ and the proof is complete. □

Note that Proposition 53 also implies $\mathcal{H} = \{0\}$, where $\mathcal{H}$ is the Hilbert space defined in the statement of Lemma 51.

Proof of Theorem 40. The proof follows directly from Theorem 48 and Proposition 53. □

Chapter 4

Conclusion

We shall conclude with a summary of the main results of this paper. We considered the inverse problem of recovering the conductivities of an electrical network from the data of the magnitude of the current on the interior of the graph and from Dirichlet or Neumann boundary conditions. We discovered through relating this problem to a minimization problem that the recovery is not unique. This contrasts with the continuous case in $\mathbb{R}^n$ for $n \geq 2$. However we were able to characterize the non-uniqueness. Additionally through the use of the dual problem we were able to show that the direction of the current on the electrical network can be uniquely recovered from this data. We then developed a numerical algorithm that generates sequences that converge to the solution of the dual problem and one solution to the primal problem. Lastly, we were able to connect our inverse problem on electrical networks to random walks. In fact, our motivation for considering the Neumann boundary conditions came from thinking about the problem in terms of random walks. There are still some open questions with regards to this inverse problem, especially with regards to the algorithms. Analysis of the speed of convergence of the algorithms would be something of interest. Particularly why algorithm 2 seems to converge faster than algorithm 1 could be investigated. Another interesting problem would be to see if there is a way to adapt the algorithms to somehow take into account multiple measurements.

The inverse problem of recovering the sound speed from PAT is an open problem in which not much research has been done. We were able to prove that if the sound speed c satisfies the properties that c^{-2} is harmonic in a simply connected region in Ω, is equal to 1 elsewhere, and (c, f) is admissible that it can be recovered uniquely. Although this may seem like a small result it still represents a step towards solving a wide open problem. Another advantage of the technique we used is that we do not require the sound speed to be smooth, only that is satisfy an admissibility condition. We also obtained the bonus result that if two sound speeds satisfy a monotonicity condition they can be recovered uniquely. This result was previously known but we have improved upon if for the sound speed is no longer required to be smooth. Next we focused on the more studied case when only the source is being recovered. We were able to duplicate previous results with our technique but with a weakening of the regularity required on the sound speed. Specifically we were able to prove uniqueness for the source as long as the sound speed is $L^\infty(\mathbb{R}^3)$ and the admissibility condition is met. This result is significant because in practice the sound speed may be discontinuous. Recovery of the sound speed is an open problem and other strategies will be needed to continue to make progress.

One problem of interest would be to see if the techniques used in the proof of our theorems could be adapted for dimensions other than 3. Another problem of interest would be to address the issue of stability in the cases when the sound speed is able to be uniquely recovered.

Bibliography

[1] Mark Agranovsky and Peter Kuchment. Uniqueness of reconstruction and an inversion procedure for thermoacoustic and photoacoustic tomography with variable sound speed. *Inverse Problems*, 23(5):20892102, 2007.

[2] Mark Agranovsky, Peter Kuchment, and Leonid Kunyansky. On reconstruction formulas and algorithms for the thermoacoustic tomography. *Photoacoustic Imaging and Spectroscopy, CRC Press*, pages 89–101, 2009.

[3] Mark L. Agranovsky and Eric Todd Quinto. Injectivity sets for the radon transform over circles and complete systems of radial functions. *Journal of Functional Analysis*, 139(2):383414, 1996.

[4] Guillaume Bal and Amir Moradifam. Photo-acoustic tomography in a rotating measurement setting. *Inverse Problems*, 32(10):105012, 2016.

[5] Guillaume Bal and Kui Ren. Multi-source quantitative photoacoustic tomography in a diffusive regime. *Inverse Problems*, 27(7):075003, 2011.

[6] Guillaume Bal and Gunther Uhlmann. Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions. *Communications on Pure and Applied Mathematics*, 66(10):16291652, 2013.

[7] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Springer, 2011.

[8] Jian-Feng Cai, Stanley Osher, and Zuowei Shen. Split Bregman methods and frame based image restoration. *Multiscale modeling & simulation*, 8(2):337–369, 2009.

[9] Robert M Christley, GL Pinchbeck, RG Bowers, D Clancy, NP French, R Bennett, and J Turner. Infection in social networks: using network analysis to identify high-risk individuals. *American journal of epidemiology*, 162(10):1024–1031, 2005.

[10] Soon-Yeong Chung and Carlos A Berenstein. ω-harmonic functions and inverse conductivity problems on networks. *SIAM Journal on Applied Mathematics*, 65(4):1200–1226, 2005.

[11] David Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*. Springer, 2013.

[12] Colin Cooper, Robert Elsasser, Hirotaka Ono, and Tomasz Radzik. Coalescing random walks and voting on connected graphs. *SIAM Journal on Discrete Mathematics*, 27(4):1748–1758, 2013.

[13] Edward B Curtis and James A Morrow. *Inverse problems for electrical networks*, volume 13. World Scientific, 2000.

[14] Peter G Doyle and J Laurie Snell. *Random walks and electric networks*. Mathematical Association of America,, 1984.

[15] Moez Draief and Ayalvadi Ganesh. A random walk model for infection on graphs: spread of epidemics & rumours with mobile agents. *Discrete Event Dynamic Systems*, 21(1):41–61, 2011.

[16] Jonathan Eckstein and Dimitri P Bertsekas. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Mathematical Programming*, 55(1):293–318, 1992.

[17] Ivar Ekeland and Roger Temam. *Convex analysis and variational problems*. SIAM, 1999.

[18] Ernie Esser. Applications of Lagrangian-based alternating direction methods and connections to split bregman. *CAM report*, 9:31, 2009.

[19] Lawrence C. Evans. *Partial differential equations*. American Mathematical Society, 2nd edition, 2010.

[20] Eugene F Fama. Random walks in stock market prices. *Financial analysts journal*, 51(1):75–80, 1995.

[21] David Finch and Kyle S. Hickmann. Transmission eigenvalues and thermoacoustic tomography. *Inverse Problems*, 29(10):104016, 11, 2013.

[22] David Finch and Rakesh. Recovering a function from its spherical mean values in two and three dimensions. 144, 03 2009.

[23] D. Gabay. Applications of the method of multipliers to variational inequalities. In Fortin M. and Glowinski R., editors, *Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary Value Problems, Studies in Mathematics and its Applications*, volume 15, chapter 9, pages 299–331. 1983.

[24] Daniel Gabay and Bertrand Mercier. A dual algorithm for the solution of nonlinear variational problems via finite element approximation. *Computers & Mathematics with Applications*, 2(1):17–40, 1976.

[25] Roland Glowinski and A Marroco. Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité d'une classe de problèmes de Dirichlet non linéaires. *Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique*, 9(2):41–76, 1975.

[26] Tom Goldstein and Stanley Osher. The split Bregman method for L1-regularized problems. *SIAM journal on imaging sciences*, 2(2):323–343, 2009.

[27] Paul Richard Halmos. *Finite-dimensional vector spaces.* Springer Science & Business Media, 2012.

[28] M Haltmeier, O. Scherzer, P. Burgholzer, and G. Paltauf. Thermoacoustic computed tomography with large planar receivers. *Inverse Problems*, 20:1663–1673, 10 2004.

[29] Karshi F Hasanov, Angela W Ma, Richard S Yoon, Adrian I Nachman, and ML Joy. A new approach to current density impedance imaging. In *Engineering in Medicine and Biology Society, 2004. IEMBS'04. 26th Annual International Conference of the IEEE*, volume 1, pages 1321–1324. IEEE, 2004.

[30] Monika R Henzinger, Allan Heydon, Michael Mitzenmacher, and Marc Najork. Measuring index quality using random walks on the web. *Computer Networks*, 31(11):1291–1303, 1999.

[31] Nicholas Hoell, Amir Moradifam, and Adrian Nachman. Current density impedance imaging of an anisotropic conductivity in a known conformal class. *SIAM Journal on Mathematical Analysis*, 46(3):1820–1842, 2014.

[32] Yulia Hristova. Time reversal in thermoacoustic tomography - an error estimate. *Inverse Problems*, 25, 01 2009.

[33] Yulia Hristova, Peter Kuchment, and Linh Nguyen. Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media. *Inverse Problems*, 24(5):055006, 2008.

[34] David Jerison and Carlos E. Kenig. Unique continuation and absence of positive eigenvalues for schrodinger operators. *Annals of Mathematics*, 121:463–488, 5 1985.

[35] Robert L Jerrard, Amir Moradifam, and Adrian I Nachman. Existence and uniqueness of minimizers of general least gradient problems. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013.

[36] Xing Jin and Lihong V. Wang. Thermoacoustic tomography with correction for acoustic speed variations. *Physics in Medicine & Biology*, 51(24):6437, 2006.

[37] M Joy, G Scott, and M Henkelman. In vivo detection of applied electric currents by magnetic resonance imaging. *Magnetic resonance imaging*, 7(1):89–94, 1989.

[38] Jost Jurgen. *Partial differential equations.* Springer, 2013.

[39] Sungwhan Kim, Ohin Kwon, Jin Keun Seo, and Jeong-Rock Yoon. On a nonlinear partial differential equation arising in magnetic resonance electrical impedance tomography. *SIAM journal on mathematical analysis*, 34(3):511–526, 2002.

[40] Yong Jung Kim, Ohin Kwon, Jin Keun Seo, and Eung Je Woo. Uniqueness and convergence of conductivity image reconstruction in magnetic resonance electrical impedance tomography. *Inverse Problems*, 19(5):1213, 2003.

[41] Christina Knox and Amir Moradifam. Determining both the source of a wave and its speed in a medium from boundary measurements. 2018. sumitted.

[42] Christina Knox and Amir Moradifam. Electrical networks with prescribed current and applications to random walks on graphs. *Inverse Problems and Imaging*, 13(2):353375, 2019.

[43] R. A. Kruger, W. L. Kiser, D. R. Reinecke, and G. A. Kruger. Thermoacoustic computed tomography using a conventional linear transducer array. *Med Phys*, 30:856–860, May 2003.

[44] Robert A. Kruger, Daniel R. Reinecke, and Gabe A. Kruger. Thermoacoustic computed tomographytechnical considerations. *Medical Physics*, 26(9):1832–1837, 1999.

[45] Peter Kuchment and Leonid Kunyansky. Mathematics of thermoacoustic tomography. *European J. Appl. Math.*, 19(2):191–224, 2008.

[46] Ohin Kwon, June-Yub Lee, and Jeong-Rock Yoon. Equipotential line method for magnetic resonance electrical impedance tomography. *Inverse Problems*, 18(4):1089, 2002.

[47] Hongyu Liu and Gunther Uhlmann. Determining both sound speed and internal source in thermo- and photo-acoustic tomography. *Inverse Problems*, 31(10):105005, 2015.

[48] László Lovász. Random walks on graphs. *Combinatorics, Paul erdos is eighty*, 2:1–46, 1993.

[49] Amir Moradifam. Existence and structure of minimizers of least gradient problems. *Indiana University Mathematics Journal*, to appear.

[50] Amir Moradifam and Adrian Nachman. Convergence of the alternating split Bregman algorithm in infinite-dimensional Hilbert spaces. submitted.

[51] Amir Moradifam, Adrian Nachman, and Alexandru Tamasan. Conductivity imaging from one interior measurement in the presence of perfectly conducting and insulating inclusions. *SIAM Journal on Mathematical Analysis*, 44(6):3969–3990, 2012.

[52] Amir Moradifam, Adrian Nachman, and Alexandre Timonov. A convergent algorithm for the hybrid problem of reconstructing conductivity from minimal interior data. *Inverse Problems*, 28(8):084003, 2012.

[53] Adrian Nachman, Alexandru Tamasan, and Alexander Timonov. Current density impedance imaging. In *Tomography and inverse transport theory*, volume 559, pages 135–149. Amer. Math. Soc. Providence, RI, 2011.

[54] Adrian Nachman, Alexandru Tamasan, and Alexandre Timonov. Conductivity imaging with a single measurement of boundary and interior data. *Inverse Problems*, 23(6):2551, 2007.

[55] Adrian Nachman, Alexandru Tamasan, and Alexandre Timonov. Recovering the conductivity from a single measurement of interior data. *Inverse Problems*, 25(3):035014, 2009.

[56] Adrian Nachman, Alexandru Tamasan, and Alexandre Timonov. Reconstruction of planar conductivities in subdomains from incomplete data. *SIAM Journal on Applied Mathematics*, 70(8):3342–3362, 2010.

[57] Adrian Nachman, Alexandru Tamasan, and Johann Veras. A weighted minimum gradient problem with complete electrode model boundary conditions for conductivity imaging. *SIAM Journal on Applied Mathematics*, 76(4):1321–1343, 2016.

[58] Lauri Oksanen and Gunther Uhlmann. Photoacoustic and thermoacoustic tomography with an uncertain wave speed. *Mathematical Research Letters*, 21(5):1199–1214, 2014.

[59] Sabir Ribas, Berthier Ribeiro-Neto, Rodrygo LT Santos, Edmundo de Souza e Silva, Alberto Ueda, and Nivio Ziviani. Random walks on the reputation graph. In *Proceedings of the 2015 International Conference on The Theory of Information Retrieval*, pages 181–190. ACM, 2015.

[60] R Tyrrell Rockafellar. Convex analysis, 1997.

[61] Purnamrita Sarkar and Andrew W Moore. Random walks in social networks and their applications: a survey. In *Social Network Data Analytics*, pages 43–77. Springer, 2011.

[62] Simon Setzer. Split Bregman algorithm, Douglas-Rachford splitting and frame shrinkage. In *International Conference on Scale Space and Variational Methods in Computer Vision*, pages 464–476. Springer, 2009.

[63] Simon Setzer. Operator splittings, Bregman methods and frame shrinkage in image processing. *International Journal of Computer Vision*, 92(3):265–280, 2011.

[64] Abram Skogseid and Vicente Fasano. *Statistical mechanics and random walks: principles, processes, and applications*. Nova Science Publishers, 2012.

[65] Plamen Stefanov and Gunther Uhlmann. Thermoacoustic tomography with variable sound speed. *Inverse Problems*, 25(7):075011, 2009.

[66] Plamen Stefanov and Gunther Uhlmann. Thermoacoustic tomography arising in brain imaging. *Inverse Problems*, 27(4):045004, 2011.

[67] Plamen Stefanov and Gunther Uhlmann. Instability of the linearized problem in multiwave tomography of recovery both the source and the speed. *Inverse Probl. Imaging*, 7(4):1367–1377, 2013.

[68] Plamen Stefanov and Gunther Uhlmann. Recovery of a source term or a speed with one measurement and applications. *Trans. Amer. Math. Soc.*, 365(11):5737–5758, 2013.

[69] Lihong V. Wang. *Photoacoustic imaging and spectroscopy.* CRC, 2009.

[70] Minghua Xu and Lihong V. Wang. Photoacoustic imaging in biomedicine. *Review of Scientific Instruments*, 77(4):041101, 2006.

[71] Mehmet E Yildiz, Roberto Pagliari, Asuman Ozdaglar, and Anna Scaglione. Voting models in random networks. In *Information Theory and Applications Workshop (ITA), 2010*, pages 1–7. Institute of Electrical and Electronics Engineers, 2010.