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THE EQUILIBRIUM LENGTH OF HIGH-CURRENT BUNCHES IN ELECTRON STORAGE RINGS*

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ABSTRACT

An equilibrium theory of the length of intense electron bunches circulating in a storage ring is presented. The consequence of electrical interaction with various resonant structures is expressed in terms of quadratures over the impedance of the structures, and impedance functions for a variety of elements are evaluated. It is shown that elements having resonances at high frequency can, above transition, cause bunches to increase in length with increasing current. The parametric dependence of the bunch lengthening is found to be in good agreement with observations, and numerical estimates, which are in substantial agreement with experiment, are presented.

I. INTRODUCTION

It has been observed that the length of bunches in the electron-positron storage rings at Orsay and Frascati is a function of the stored current, although no such effect has been observed at Stanford or Novosibirsk.[1]

These observations have stimulated considerable theoretical effort.[2-6] Theories based upon coherent synchrotron radiation[2,3] predicted a shortening of bunches with increasing current, in contradiction with the observations. Resonances associated with clearing-field electrodes[4] and resonances associated with radio-frequency cavities[5] have been suggested as the source of the phenomenon.

All the above theories are equilibrium calculations; that is, they are theories in which the effective azimuthal potential well is modified in strength as a result of the high beam current. A recent analysis[6] suggests that the bunch lengthening is due to an instability of the internal coherent synchrotron oscillations. The parametric dependence of the bunch lengthening in this theory is not in good agreement with the observations.

In this paper we present a very general theory of the equilibrium length of high-current bunches. Thus, the theories of Refs. 2 through 5 are included as special cases in our analysis. Because the observations show that the bunch lengthening is an effect independent of the number of bunches and independent of the total beam current, we restrict our theory to include only the interaction of a bunch with itself and, in particular, to include only direct interactions. (Thus we neglect the interaction of a bunch with itself as a result of completing a revolution of the storage ring. Such long-term memory effects can lead to longitudinal instabilities[7] as well as bunch lengthening, but presumably would predict effects which have not been observed.) To simplify the analysis we neglect gas scattering and multiple Coulomb scattering within a bunch; neither effect being important in the regime in which the experimental observations have been made. It is easy to extend our analysis to include these phenomena. We also consider only the above-transition situation; the below-transition behavior follows trivially.

In Section II we obtain an expression for bunch length in terms of the synchrotron oscillation frequency. In Section III a general form for the self-interaction (which is derived in Appendix A) is employed to obtain the synchrotron oscillation frequency as a quadrature either over the self-force Green's function or over the self-force impedance. In Section IV bunch length formulas for beam interaction with structures which resonate at high or at low frequencies are obtained. In Section V we summarize the experimental observations on bunch length and show that they can be fitted by beam interaction with structures which resonate at high frequencies. Numerical examples for some different structures are presented and general scaling laws derived. Electrical properties of smooth chambers, resonant cavities, electrodes, etc. are discussed in Appendices B through E.

The theory presented here is in good agreement with observations, and suggests that the bunch lengthening which has been observed is due to high-frequency resonant elements in the storage rings. In simple terms, bunch lengthening (above transition) requires inductive coupling between the beam and its surroundings.[8] In contrast to a smooth chamber (which is capacitive and hence produces bunch shortening and negative mass instability), many elements--such as a resonant cavity at frequencies below its resonant frequency--are inductive, have a much stronger effect than the smooth chamber, and lead to bunch lengthening.

There is, in general, the possibility of bunch shortening, and this case is included in the general analysis. However, because no observation of shortening of bunches in storage rings has been reported, we limit our examples to those giving bunch lengthening. It should be noted that this lack of observed shortening may be due only to the difficulty associated with such observations.

II. DIFFERENTIAL EQUATION FOR SYNCHROTRON MOTION

In this section we derive the equations which describe the azimuthal motion of electrons under the influence of applied radio-frequency fields, incoherent synchrotron radiation (including the quantum fluctuations), and self-fields. As explained in Section I, we consider only a single bunch of electrons and, furthermore, a constant guide field.

In the absence of self-fields the linearized equation describing synchrotron motion is[9]

$$\frac{d^2\varphi}{dt^2} + 2\alpha_s \frac{d\varphi}{dt} + \Omega_s^2\varphi = - \dot{\tilde{P}}_{\gamma s} , \qquad (2.1)$$

where φ is the phase relative to the synchronous particle, α_a is the radiation damping constant, $\dot{\tilde{P}}_{\gamma s}$ describes the fluctuations, and Ω_s^2--

the small-amplitude synchrotron oscillation frequency--is given by

$$\Omega_s^2 = - \frac{h\eta\omega_s^2 \cos\varphi_s \, eV_{rf}}{2\pi\beta^2 E_s} . \tag{2.2}$$

Note that we have taken a sign convention for phase φ opposite to that in Ref. 9. In the formula for Ω_s^2, h is the rf harmonic number, η is a dispersion coefficient, E_s is the total energy of the synchronous particle, and β its velocity in units of light velocity, ω_s is the revolution frequency of the synchronous particle, V_{rf} is the peak rf voltage, and φ_s the rf phase of the synchronous particle. The coefficient η is given by

$$\eta = \beta^2 \frac{E}{\omega} \frac{d\omega}{dE} \bigg|_{E=E_s} . \tag{2.3}$$

In the presence of self-fields, which we treat in linear approximation, the only modification to the above formulas is to replace $\cos\varphi_s \, eV_{rf}$, in (2.2), by

$$\cos\varphi_s \, eV_{rf} + e \frac{dU}{d\varphi} \bigg|_{\varphi=\varphi_s} , \tag{2.4}$$

where eU is the energy change per revolution of a particle due to self-fields and the subscript s indicates evaluation for the synchronous particle. Thus Ω_s^2 is changed to Ω^2, where

$$\Omega^2 = \Omega_s^2 - \frac{h\eta\omega_s^2 e \frac{dU}{d\varphi}\big|_s}{2\pi\beta^2 E_s} . \tag{2.5}$$

If the fluctuation term is assumed to be dominated by quantum effects in the synchrotron radiation, the mean square equilibrium bunch length is given by[10]

$$\Delta^2 = \left(\frac{\alpha c}{\Omega E_s}\right)^2 \left(\frac{\rho_s}{R_s}\right)\left[\frac{55\ e^2 c^2 \hbar \gamma^7}{2^3 3^{3/2} \rho_s{}^3 4\alpha_s}\right], \tag{2.6}$$

where ρ_s is the radius of curvature in the bending magnets, R_s is the average radius, $\hbar$ is Planck's constant, γ is E_s/mc^2, and α--the momentum compaction factor--is given by

$$\alpha = \beta^2 \frac{E}{R} \frac{dR}{dE}\bigg|_s . \tag{2.7}$$

We write (2.6) in the convenient form

$$\Delta^2 = \Delta_0{}^2 \frac{1}{\left[1 + \frac{\Omega^2 - \Omega_s{}^2}{\Omega_s{}^2}\right]}, \tag{2.8}$$

where $\Delta_0{}^2$ is the mean square equilibrium bunch length in the limit of zero bunch current. From (2.6), and using well-known expressions for the various quantities therein, we have--in the relativistic limit ($\gamma^2 \gg 1/\alpha$) in which one is above transition--that

$$\Delta_0{}^2 = \frac{55(3)^{1/2} \pi \alpha R_s \lambda\!\!\!^{-}_c mc^2 \gamma^3}{96\ J_s heV_{rf} \cos\varphi_s}, \tag{2.9}$$

where $\lambda\!\!\!^{-}_c$ is the electron Compton wavelength, and J_s is a dimensionless parameter which relates the energy radiated per electron per unit time, W, to the damping constant α_s,

$$\alpha_s = \frac{W}{E_s} J_s . \tag{2.10}$$

In the remainder of this paper we consider only the above-transition case and hence, as can be seen from (2.2), $\cos\varphi_s$ is positive.

III. SELF-FIELDS

The electromagnetic self-fields acting on the bunch are assumed to arise from the interaction between the bunch itself and the material structures which are near the bunch trajectory, such as, for instance, rf cavities or monitoring electrodes.

It is shown in Appendix A that the force on one electron in the bunch due to the self-fields can be written, in the ultrarelativistic limit, as

$$eE_z(\sigma, t) = e\int_\sigma^\infty d\sigma'\ \lambda(\sigma')\ G(\sigma - \sigma', t), \tag{3.1}$$

where σ is the distance from the synchronous particle measured along the bunch $\sigma = R_s(\varphi - \varphi_s)/h$, $\lambda(\sigma)$ is the linear charge density of the bunch, and G is a function which characterizes the particular structure interacting with the beam. The energy change of a particle per revolution, eU, is proportional to the integral of (3.1) over one revolution period,

$$eU(\sigma) = e\beta c\int_0^T dt\int_\sigma^\infty d\sigma'\ \lambda(\sigma')\ G(\sigma' - \sigma, t). \tag{3.2}$$

Assuming the synchrotron oscillation period to be much larger than T, and defining a new function

$$\mathcal{G}(\sigma' - \sigma) = \beta c\int_0^T dt\ G(\sigma' - \sigma), \tag{3.3}$$

we can write the energy change $eU(\sigma)$ approximately as

$$eU(\sigma) = e\int_{\sigma}^{\infty} d\sigma' \, \lambda(\sigma') \mathcal{E}'(\sigma' - \sigma). \tag{3.4}$$

To evaluate the change in synchrotron oscillation frequency we need, according to (2.5), the quantity

$$e \left.\frac{dU}{d\varphi}\right|_s = \frac{Re}{h}\int_0^{\infty} d\sigma \, \mathcal{E}(\sigma) \frac{d\lambda(\sigma)}{d\sigma} , \tag{3.5}$$

where we have used $\sigma = R\varphi/h$ (which is consistent with our sign convention for φ). Using (3.5), (2.5), and (2.2), we have

$$\frac{\Omega^2 - \Omega_s^2}{\Omega_s^2} = \frac{e\beta c}{h\omega_s \cos\varphi_s \, eV_{rf}} \int_0^{\infty} d\sigma \, \mathcal{G}(\sigma) \frac{d\lambda(\sigma)}{d\sigma} . \tag{3.6}$$

Consistent with the linear approximation, already made, the charge density $\lambda(\sigma)$ has the form

$$\lambda(\sigma) = \frac{Ne}{(2\pi)^{1/2}\Delta} e^{-\sigma^2/2\Delta^2}, \tag{3.7}$$

where N is the total number of electrons in the bunch, and Δ must be determined as a solution of (2.8), (3.6), and (3.7). Inserting (3.7) into (3.6), we obtain

$$\frac{\Omega^2 - \Omega_s^2}{\Omega_s^2} = \frac{-Ne^2\beta c}{h\omega_s \cos\varphi_s \, eV_{rf}(2\pi)^{1/2}\Delta^3} \int_0^{\infty} \sigma \, d\sigma \, \mathcal{G}(\sigma) e^{-\sigma^2/2\Delta^2} . \tag{3.8}$$

It is convenient to introduce an impedance $Z(\omega)$ by

$$Z(\omega) = \int_{-\infty}^{\infty} \mathcal{G}(\beta c\tau) e^{i\omega\tau} d\tau, \tag{3.9}$$

so that (3.8) becomes, after we employ the inversion of (3.9) and after we interchange the order of integration,

$$\frac{\Omega^2 - \Omega_s^2}{\Omega_s^2} = \frac{-Ne^2\beta c}{(2\pi)^{3/2} h\omega_s \cos\varphi_s \, eV_{rf} \, \Delta^3} \int_{-\infty}^{\infty} d\omega \, Z(\omega) \int_0^{\infty} \sigma \, d\sigma \, e^{-\frac{\sigma^2}{2\Delta^2} - \frac{i\omega\sigma}{\beta c}} . \tag{3.10}$$

Performing the integration over σ, we obtain

$$\frac{\Omega^2 - \Omega_s^2}{\Omega_s^2} = \frac{-2Ne^2\beta c}{(2\pi)^{3/2} h\omega_s \cos\varphi_s \, eV_{rf} \, \Delta} \int_{-\infty}^{\infty} d\omega \, Z(\omega)$$

$$\times \left[{}_1F_1\left(1; \frac{1}{2}; -\frac{\Delta^2\omega^2}{2\beta^2c^2}\right) + \frac{i\pi^{1/2}}{2^{3/2}} \frac{\Delta\omega}{\beta c} e^{-\Delta^2\omega^2/2\beta^2c^2} \right], \tag{3.11}$$

where ${}_1F_1$ is a confluent hypergeometric function. In performing the ω integration in (3.11) the contour should be closed in the lower half-plane.

In general, to obtain the bunch length Δ, one must first evaluate the impedance function $Z(\omega)$, or equivalently, the Green's function $\mathcal{G}(\sigma)$, of the storage ring. Then one evaluates Ω^2, using (3.8) or (3.11), and then the bunch length from (2.8). In the Appendices B through E we present $\mathcal{G}(\sigma)$ or $Z(\omega)$ or both for a variety of elements which can--and do--appear in storage rings.

For situations where $Z(\omega)$ is a complicated function (coming, for example, from a variety of resonant elements) and known either by computation or from measurement, it is convenient to employ (3.11). For simple models it is often easier to employ (3.8) directly.

IV. HIGH- AND LOW-FREQUENCY RESONANCES

In the previous section we have obtained a general expression for Ω^2 in terms of the self-force impedance $Z(\omega)$. It is illuminating to consider two limiting cases of this expression. Suppose, firstly, that $Z(\omega)$ is nonzero only for low frequencies, i.e., for frequencies such that $|\omega| \ll \omega_{crit} \equiv \beta c/\Delta$. In this case (3.11) becomes

$$\frac{\Omega^2 - \Omega_s^{\;2}}{\Omega_s^{\;2}} \approx \frac{-2Ne^2\beta c\,\mathcal{F}_{LF}}{(2\pi)^{3/2}\hbar\omega_s \cos\varphi_s \; eV_{rf}\,\Delta}, \tag{4.1}$$

with

$$\mathcal{F}_{LF} = \int_{-\omega_{crit}}^{\omega_{crit}} Z(\omega)\,d\omega. \tag{4.2}$$

If, on the other hand, $Z(\omega)$ is nonzero only for high frequencies, i.e., for frequencies with $|\omega| \gg \omega_{crit}$, then

$$\frac{\Omega^2 - \Omega_s^{\;2}}{\Omega_s^{\;2}} \approx \frac{2Ne^2\beta^3c^3 \;{}_{FH}}{(2\pi)^{3/2}\hbar\omega_s \cos\varphi_s \; eV_{rf}\,\Delta^3}, \tag{4.3}$$

with

$$\mathcal{F}_{HF} = \int_{\omega_{crit}}^{\infty} \frac{Z(\omega)\,d\omega}{\omega^2} + \int_{-\infty}^{-\omega_{crit}} \frac{Z(\omega)\,d\omega}{\omega^2}. \tag{4.4}$$

In deriving (4.2) and (4.4) we have used small-argument and large-argument expansions for the confluent hypergeometric function.[11] Note that $\mathcal{F}_{LF}$ and $\mathcal{F}_{HF}$ have different dimensions.

With the aid of (4.1) through (4.4) we may readily solve (2.8) for the bunch length Δ. In the low frequency case

$$\frac{\Delta}{\Delta_0} = \frac{-\mu_{LF}\mathcal{F}_{LF}}{2} + \left[\frac{\mu^2_{LF}\mathcal{F}^2_{LF}}{4} + 1\right]^{1/2}, \quad \text{(LF)} \tag{4.5}$$

where

$$\mu_{LF} = \frac{-2Ne^2\beta c}{(2\pi)^{3/2} h\omega_s \cos\varphi_s \, eV_{rf} \, \Delta_0} . \tag{4.6}$$

In the high-frequency case

$$\Delta^2 = \Delta_0^2\left[1 + \mu_{HF}\mathcal{F}_{HF}\frac{\Delta_0}{\Delta}\right], \quad \text{(HF)} \tag{4.7}$$

where

$$\mu_{HF} = \frac{-2Ne^2\beta^3 c^3}{(2\pi)^{3/2} h\omega_s \cos\varphi_s \, eV_{rf} \, \Delta^3} . \tag{4.8}$$

The parametric dependence of Δ may be explicitly exhibited in the two limiting cases. The functions $\mathcal{F}_{LF}$ and $\mathcal{F}_{HF}$ are, in these limits, independent of ω_{crit} [since the $Z(\omega)$ is assumed to cut off the integrations]. Thus, the $\mathcal{F}$ functions characterize the structure, but are independent of beam energy, beam current, and bunch length. For example, in the high-frequency case we obtain, from (4.7), (4.8), and (2.9),

$$\Delta^2 = \Delta_0^2\left[1 + k\frac{I}{E_s^3\Delta}\right], \quad \text{(HF)} \tag{4.9}$$

where I is the beam current, and the constant k, which depends only upon machine parameters, is given by

$$k = \frac{-192\, J_s(mc^2)^3}{55\,\pi(6\pi)^{1/2}\alpha}\left(\frac{R_s}{\lambda_c}\right)\left[\mathcal{F}_{HF}\omega_s c\right]\frac{R_s}{I_0} , \tag{4.10}$$

with $I_0 = mc^3/e = 17\ 000$ A. Note that applied voltage appears only through the dependence of Δ_0 upon V_{rf}. The functional form (4.9) has been previously obtained by Le Duff and Robinson.[4,5]

V. DISCUSSION AND COMPARISON WITH OBSERVATIONS

In this section we first summarize the experimental observations on bunch lengthening in ACO and ADONE. The experimental observations have been empirically fitted with the phenomenological forumlas[1]

$$\Delta^2 = \Delta_0{}^2 \left[1 + 2 \times 10^{-3} \frac{I(\mathrm{mA})}{E^4(\mathrm{GeV})\ \Delta(\mathrm{ns})} \right], \quad (\mathrm{ACO}) \qquad (5.1)$$

$$\frac{\Delta(\mathrm{ns})}{\Delta_0^{2/3}(\mathrm{ns})} = \frac{0.46\ I(\mathrm{mA})^{1/3}}{E(\mathrm{GeV})^{7/6}} \left[\frac{30}{V_{rf}(\mathrm{keV})} \right]^{1/6}, \text{ for } \Delta \gg \Delta_0. \quad (\mathrm{ADONE}) \qquad (5.2)$$

For ACO the functional dependence (5.1) is quite close to (4.9), with only the E^{-4} factor replaced by E^{-3}; however, a recent analysis (private communication from J. Le Duff) indicates that the data may be equally well fitted with E^{-3}. For ADONE the agreement is again quite good, but now $E^{-7/6}V^{-1/6}$ is replaced by E^{-1}. Actually the ADONE dependence upon V could be weaker than that given in (5.2).[1]

It is rather remarkable that the simple model of only high-frequency resonances yields a formula for the bunch length (4.9) in such good accord with the phenomenological formulas (5.1) and (5.2). Such would not be the case for low-frequency resonances, and in the remainder of this section we consider only high-frequency resonances (although either bunch lengthening or shortening through low-frequency resonances is, in principal, possible).

Also with high frequency resonances, where the functional form (4.9) is assured, there is the possibility of bunch lengthening or shortening corresponding to positive or negative values of k. For the interaction of a beam with a smooth chamber one finds, in fact, bunch shortening as was discussed in Ref. 3 and as can be readily seen from $Z(\omega)$, which is given, for this case, in Appendix B.

To the contrary, there are a variety of structures such as rf cavities, pickup electrodes, and clearing electrodes, which are always present in storage rings and which usually will cause bunch lengthening.

We leave the task of detailed computations of bunch length, for any particular machine, to the interested reader. Here, we limit ourselves to a few illustrative examples.

<u>Example No. 1: Single-Resonance Model</u>

As a special case of the general discussion in Appendix C, we are led to consider an impedance $Z(\omega)$ of the form

$$Z(\omega) = \frac{\omega_R Z_R}{2i}\left[\frac{1}{\omega - \omega_R + i\Gamma_R} + \frac{1}{\omega + \omega_R + i\Gamma_R}\right], \qquad (5.3)$$

so that ω_R is the real frequency of the resonant structure, Γ_R is the damping constant, and Z_R is the structure impedance. The sign of Z_R is taken positive, corresponding to resonant cavity. We evaluate $\mathcal{F}_{HF}$, where

$$\mathcal{F}_{HF} = \int_{\omega_{crit}}^{\infty} \frac{Z(\omega)d\omega}{\omega^2} + \int_{-\infty}^{-\omega_{crit}} \frac{Z(\omega)d\omega}{\omega^2}, \qquad (5.4)$$

by extending the range of integration from $-\infty$ to ∞ and excluding the pole at $\omega = 0$. The contour must be closed in the lower half-plane, and thus

$$\mathcal{F}_{HF} = -\, 2\pi Z_R \omega_R \frac{\omega_R^2 - \Gamma_R^2}{(\omega_R^2 + \Gamma_R^2)^2} \, . \tag{5.5}$$

We assume the damping constant is such as to reduce the induced fields to a negligible value in a time of the order of one particle revolution (and hence there will be negligible bunch-bunch coupling), while--at the same time--the decay of fields during the passage of one bunch is negligible. Thus we take $\Gamma\Delta \ll 1$ and, since we have assumed (high-frequency resonance) $\omega_R\Delta \gg 1$, we obtain $\omega_R \gg \Gamma_R$, so that

$$\mathcal{F}_{HF} \approx \frac{-2\pi Z_R}{\omega_R} \, . \tag{5.6}$$

The machine parameter k becomes, from (4.10),

$$k = \frac{384\ J_s}{55(6\pi)^{1/2}} \frac{1}{\alpha} \left(\frac{\omega_s}{\omega_R} \right) (Z_R c) \left(\frac{R_s}{\lambda_c} \frac{R_s}{I_0} \right) (mc^2)^3 \, . \tag{5.7}$$

For numerical evaluation we adopt the parameters given in Table I, and--arbitrarily--take $\omega_R = 10^{10}$ sec^{-1} and $Z_R c = 33$ (corresponding to an impedance of 1 kΩ). We find, for ADONE, $k = 5.1 \times 10^{-3}$ (GeV$^3\cdot$ns/mA) so that $k^{1/3} = 0.17$, which is to be compared with the coefficient 0.46 in (5.2). For ACO, $k \approx 10^{-3}$ to 10^{-4} (GeV^3 ns/mA) (corresponding to the range in α), which is to be compared with the coefficient 2×10^{-3} in (5.1). Clearly if there were more than one resonant structure (having $Z_R = 1\ k\Omega$) the coefficient k would be increased.

Table I. Machine parameters employed in the numerical examples.

	ADONE	ACO
J_s	1	1
α	$6. \times 10^{-2}$	0.06 to 0.6
ω_s	$2. \times 10^{7}\ \mathrm{sec}^{-1}$	$9 \times 10^{7}\ \mathrm{sec}^{-1}$
R_s	1.7×10^{3} cm	3.41×10^{2} cm

Example No. 2: Resonant Cavity Model

The closed-cavity model of Appendix D may be employed to evaluate $Z(\omega)$. In the approximation of a bunch long compared with the cavity dimensions, $Z(\omega)$ is given by (D.8), which is precisely of the form (5.3). Thus we obtain from the cavity model, in contrast to the above evaluation, explicit formulas for the quantities Z_R and ω_R,

$$\omega_R = 2.41c/b \tag{5.8}$$

and

$$Z_R c = 8.5\ \frac{b}{L}\left[1 - \cos\left(\frac{2.41L}{b}\right)\right] , \tag{5.9}$$

where we have taken $\beta = 1$, b is the cavity radius, and L the cavity length. Note that sign of Z_R produces bunch lengthening.

The bracket in (5.9) varies between 0 and 2, so we take it as 1. If we choose b = 7.2 cm, then $\omega_R = 10^{10}\ \mathrm{sec}^{-1}$, and if we

Take L = 1.85 cm then $Z_R c = 33$, as in the preceding example.

Example No. 3: Electrodes

The interaction of a beam with an electrode is considered in Appendix E, where it is shown that for an electrode of length ℓ, terminated at both ends by its characteristic impedance Z_0, the Green's function is

$$\mathcal{G}(\sigma) = -Z_0 \left[\delta\left(\frac{\sigma}{\beta c}\right) - \delta\left(\frac{\sigma}{\beta c} - \frac{2\ell}{c}\right)\right]. \tag{5.10}$$

From (3.8),

$$\frac{\Omega^2 - \Omega_s^2}{\Omega_s^2} = - \frac{N e^2 \beta^3 c^2 Z_0 2\ell \, e^{-2\ell^2\beta^2/\Delta^2}}{(2\pi)^{1/2} h\omega_s \cos\varphi_s \, eV_{rf} \, \Delta^3} . \tag{5.11}$$

This yields bunch lengthening which, when $\ell >> \Delta$, is a small effect. This appears to be in agreement with the observation on ADONE that removal of the long clearing-field electrodes had no effect on bunch length.[1] If $\ell < \Delta$ we have, from (2.8), a formula of the form of (4.9),

$$k = \frac{192}{55}\left(\frac{2}{3\pi}\right)^{1/2} \frac{(Z_0 c)(m_0 c^2)^3 J_s \ell R_s}{\alpha \lambda_c I_0} . \tag{5.12}$$

Taking the parameters of Table I, ℓ = 10 cm and $(Z_0 c) = 0.33$ (corresponding to 10 Ω), we find for ADONE $k = 9.2 \times 10^{-4}$ (GeV^3 ns/mA). A hundred of such elements--or ten elements of impedance 100 Ω--would explain the observations. For ACO, taking $\alpha = 0.06$, ten 10-Ω elements would give the observed effect, whereas for $\alpha = 0.6$, 100 are required. Pickup electrodes, in both these machines, could well be described by this model.

Finally, we turn from examples to some general remarks. In practice, it is extreme bunch lengthening which is a serious effect, so it is interesting to examine scaling laws in this case.

In the limit of bunch length, Δ, much larger than the natural length, Δ_0, we have, from (4.9), the relation

$$\Delta^3 = \frac{\Delta_0{}^2 kI}{E_s{}^3} \, . \tag{5.13}$$

In order to see the general dependence upon parameters we can evaluate k for the single-resonance model, as in (5.7), and also use (2.9) to obtain

$$\frac{\Delta}{\Delta_0} = \left[\frac{(16)^{1/6}(384)^{1/3}}{(55)^{1/2}3^{1/12}\pi^{1/3}}\right]\left(\frac{J_s}{\alpha}\right)^{1/2}\left(\frac{R_s}{\lambda_c}\right)^{1/2}\left(\frac{heV_{rf}\cos\varphi_s}{mc^2}\right)^{1/6}$$
$$\times \left(\frac{\omega_s}{\omega_R}\right)^{1/3} \frac{(Z_R c)^{1/3}}{\gamma^{3/2}}\left(\frac{I}{I_0}\right)^{1/3} . \tag{5.14}$$

The dependence of Δ/Δ_0 upon rf parameters is weak ($h^{1/6}$ and $V_{rf}{}^{1/6}$), as is the dependence upon beam current ($I^{1/3}$). The strongest readily controllable dependence is upon energy ($\gamma^{-3/2}$), and upon momentum compaction ($\alpha^{-1/2} \approx \nu_r$). The last dependence could explain why bunch lengthening was not observed at the weak-focusing ($\nu_r < 1$) Stanford and Novosibirsk storage rings.

It is interesting, in this same limit of a strong effect, to examine the parametric dependence of the bunch length itself,

$$\Delta = \left[2^{3/2}\pi^{1/2}\left(\frac{mc^2}{heV_{rf}\cos\varphi_s}\right)\left(\frac{\omega_s}{\omega_R}\right)(Z_R c)\left(\frac{I}{I_0}\right)\right]^{1/3} R_s. \qquad (5.15)$$

It is seen that Δ is independent of energy and momentum compaction and depends--only weakly-- upon rf parameters and beam current.

In the limit of a strong effect, the bunch length is set by the balance between the potential-well-reducing self-forces (which are strongly dependent upon bunch length) and the applied rf voltage. Thus the bunch length is independent of radiation damping and quantum-fluctuation undamping.

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APPENDICES

A. Formal Expression for the Self-Force

We need the axial electric field, E_z, associated with a rigid collection of charges moving axially at a constant speed v. Because the current, j, and charge, ρ, are trivially related, we may write

$$E_z(z,t) = e \sum_{k=1}^{N} \int dt' \int dz' \, \delta(z' - vt' - \sigma_k) \, \bar{\bar{G}}(z - z', t - t', t), \quad \text{(A.1)}$$

where σ_k is the position of particle k at time t = 0, and $\bar{\bar{G}}$ is a Green's function in which the integration over transverse coordinates has already been performed. In writing (A.1) we have assumed a passive system (no feedback loops). From (A.1)

$$E_z(z,t) = e \sum_{k=1}^{N} \int dt' \, \bar{\bar{G}}(z - vt' - \sigma_k, \, t - t', t), \quad \text{(A.2)}$$

and writing

$$z = vt + \sigma_i, \quad \text{(A.3)}$$

we obtain the field at the position of the ith particle,

$$E_{zi}(t) = e \sum_{k=1}^{N} \int dt' \, \bar{G}(\sigma_i - \sigma_k, \, t - t', t). \quad \text{(A.4)}$$

In (A.4) we have introduced $\bar{G}$, where

$$\bar{G}(\sigma,\tau,t) = \bar{\bar{G}}(\sigma + v\tau, \tau, t). \quad \text{(A.5)}$$

Causality implies that $\bar{\bar{G}}$ is nonzero only in the past light cone, namely where

$$(z - z')^2 - c^2(t - t')^2 < 0, \qquad (A.6)$$

and $$t - t' > 0. \qquad (A.7)$$

When we employ (A.3), (A.6) implies

$$t - t' > \frac{\sigma_i - \sigma_k}{c - v}, \qquad \text{for } \sigma_i > \sigma_k , \qquad (A.8)$$

$$t - t' > \frac{\sigma_k - \sigma_i}{c + v}, \qquad \text{for } \sigma_i < \sigma_k . \qquad (A.9)$$

Hence (A.4) becomes

$$E_{zi}(t) = e \sum_{k=1}^{N} \left\{ \int_{\frac{\sigma_k - \sigma_i}{c+v}}^{\infty} \bar{G}(\sigma_i - \sigma_k, \tau, t) d\tau + \int_{\frac{\sigma_i - \sigma_k}{c-v}}^{\infty} \bar{G}(\sigma_i - \sigma_k, \tau, t) d\tau \right\}. \qquad (A.10)$$

The second term in (A.10) should be small, in the relativistic limit, as the lower limit of the integral is large. If we neglect this term, (A.10) is of the form

$$E_{zi}(t) = e \sum_{k=1}^{N} G(\sigma_k - \sigma_i, t) \quad (\text{for } \sigma_k - \sigma_i > 0) \qquad (A.11)$$

where (A.7) has been employed in writing the restriction on the summation of k, and G is defined by the τ-integration of $\bar{G}$. Expressing (A.11) as an integral, we finally obtain

$$E_z(\sigma_i, t) = \int_{\sigma_i}^{\infty} d\sigma \, \lambda(\sigma) \, G(\sigma - \sigma_i, t), \qquad (A.12)$$

where $\lambda(\sigma)$ is the line charge density.

B. Impedance Function for a Smooth Chamber

The interaction of a circulating bunch with a smooth perfectly conducting vacuum chamber has been studied in Ref. 3, under the approximation of replacing the vacuum chamber with two parallel perfectly conducting infinite planes. A pillbox model has been studied by A. Entis and L. Smith.[12]

For the infinite plate geometry, and in the approximation that the bunch length is much larger than the distance, H, between the planes

$$E_z(\sigma) = -2\left[\frac{1}{\gamma^2}\left(1 + 2\ \ln\frac{2H}{\lambda a}\right) + \left(\frac{\beta H}{\pi R}\right)^2\right]\frac{\partial\lambda(\sigma)}{\partial\sigma}\ , \qquad \text{(B.1)}$$

where R is the orbit radius, a is the bunch transverse radius, $\beta = v/c$, and $\gamma = (1 - \beta^2)^{-1/2}$.

Comparing (B.1) and (A.12), we may write

$$G(\sigma) = 4\left[\frac{1}{\gamma^2}\left(1 + 2\ \ln\frac{2H}{\pi a}\right) + \left(\frac{\beta H}{\pi R}\right)^2\right]\delta'(\sigma)\ , \qquad \text{(B.2)}$$

where $\delta'(\sigma)$ is the first derivative of the Dirac delta function, and the extra factor of 2 arises from the δ function of the limit of the integral. Thus, from (3.3), (3.9), and (B.2),

$$Z(\omega) = -\frac{8\pi i}{\beta C\omega_s}\left[\frac{1}{\gamma^2}\left(1 + 2\ \ln\frac{2H}{\pi a}\right) + \left(\frac{\beta H}{\pi R}\right)^2\right]\omega\ , \qquad \text{(B.3)}$$

with $\omega_s = \beta c/R$, and with this formula valid only for frequencies $\omega < \pi\beta c/H$.

In Ref. 3 it was shown that this (nonresonant) $Z(\omega)$ leads, above transition, to bunch shortening. In this simple $G(\sigma)$ it is more convenient to employ (3.8) directly rather than the formalism of (3.11).

C. Impedance Function for General Resonant Structures

In Appendix A it was shown that the average axial self-field is of the form

$$E_z(\sigma_0, t) = \int_{\sigma_0}^{\infty} d\sigma\, \lambda(\sigma)\, G(\sigma - \sigma_0, t). \tag{C.1}$$

For a general resonant structure, there will be characteristic frequencies ω_n, where n characterizes the various modes. These frequencies are complex, and

$$\omega_n = \bar{\omega}_n - i\Gamma_n, \tag{C.2}$$

with $\bar{\omega}_n$, Γ_n real and $\Gamma_n > 0$. The impedance function, $Z(\omega)$, as defined by (3.3) and (3.9) is simply

$$Z(\omega) = \sum_n \frac{\omega_n Z_n}{2i} \left[\frac{1}{\omega - (\bar{\omega}_n - i\Gamma_n)} + \frac{1}{\omega + (\bar{\omega}_n + i\Gamma_n)} \right], \tag{C.3}$$

where the real constants Z_n are characteristic impedances of the structure. For a cavity model the constants Z_n are positive, corresponding to the inductive impedance of a cavity below resonance. It might be noted that the Γ_n ensure that $\mathcal{G}(\sigma)$ is zero for $\sigma < 0$, and it is also easy to check that $Z^*(-\omega) = Z(\omega)$, so that $\mathcal{G}(\sigma)$ is real.

D. Impedance Function for a Closed Cavity

The model considered in this appendix is a right circular perfectly conducting cavity with radius b and length L. A bunch having negligible radial extent and longitudinal charge density described by

$\lambda(z - vt)$ is assumed to move at constant speed v along the axis of the cavity.

The interaction of the bunch with the cavity and hence, also, of the bunch with itself--via the cavity--may be easily calculated by employing the method of Condon.[13] The analysis is sufficiently closely similar to calculations in the literature[14,15] that we merely sketch the method and present the result.

Taking

$$\rho(r, \varphi, z, t) = \frac{\delta(r)}{2\pi r} \lambda\,(z - vt), \tag{D.1}$$

it is easy to compute the ω component of the bunch current (axially directed):

$$J_{z\omega} = \frac{\delta(r)}{2\pi r}\, e^{(-i\omega z/v)}\, \tilde{\lambda}\left(\frac{\omega}{v}\right), \tag{D.2}$$

where

$$\tilde{\lambda}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\sigma} \lambda(\sigma)\, d\sigma\ . \tag{D.3}$$

Computation of the cavity fields excited by the Fourier component $J_{z\omega}$ follows the method presented in the references cited. The axial electric field may then be evaluated at the position $z = \sigma_0 + vt$ and the average field experienced (in crossing the cavity) computed as a function of σ_0:

$$\langle E_z(\sigma_0)\rangle = \frac{16i}{b^2 L^2 v^2} \sum_{s=1}^{\infty} \sum_{p=0}^{\infty} (1 + \delta_{p0}) \int_{-\infty}^{\infty} \frac{\omega\, d\omega \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{p\pi}{L}\right)^2\right]}{\left[\left(\frac{p\pi}{L}\right)^2 - \left(\frac{\omega}{v}\right)^2\right]^2 J_1{}^2(\mu_s b)}$$

$$\times \quad \frac{e^{i\omega\sigma_0}\, \tilde{\lambda}\left(\frac{\omega}{v}\right)\left[1 - (-1)^p \cos\frac{\omega L}{v}\right]}{\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{p\pi}{L}\right)^2 - \mu_s{}^2\right]}\ , \tag{D.4}$$

where δ is a Kroneker delta, s and p are integers, and the quantities μ_s are determined from

$$J_0(\mu_s b) = 0, \text{ for } s = 1, 2, \cdots. \tag{D.5}$$

The only poles with nonzero residue in (D.4) are

$$\omega_{ps} = c\left[\mu_s^2 + \left(\frac{p\pi}{L}\right)^2\right]^{1/2}, \tag{D.6}$$

which are just the resonant frequencies of the cavity. These poles really have a negative imaginary part corresponding to the resistive decay of cavity-mode fields.

When (3.3) and (3.9) are employed, the impedance function $Z(\omega)$ becomes

$$Z(\omega) = \frac{16i}{b^2 L v^2} \sum_{s=1}^{\infty} \sum_{p=0}^{\infty} \frac{(1 + \delta_{p0})\omega\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi p}{L}\right)^2\right]\left[1 - (-1)^p \cos\frac{\omega L}{v}\right]}{J_1^{\,2}(\mu_s b)\left[\left(\frac{p\pi}{L}\right)^2 - \left(\frac{\omega}{v}\right)^2\right]^2\left[\left(\frac{\omega}{c}\right)^2 - \omega_{sp}^2\right]} \tag{D.7}$$

When L and b are much smaller than the length of a bunch, then the terms s = 1, p = 0 dominates in the sum and

$$Z(\omega) \approx \frac{16iv^2}{b^2 L} \frac{\left[1 - \cos\frac{\omega L}{v}\right]}{J_1^{\,2}(\mu_1 b)\omega^2}\left[\frac{1}{\omega + c\mu_1} + \frac{1}{\omega - c\mu_1}\right] \tag{D.8}$$

where $\mu_1 b = 2.41$ and $J_1(\mu_1 b) = 0.52$. In the case in which the range of interesting frequency is such that $\omega \ll c\mu_1$, $Z(\omega)$ becomes

$$Z(\omega) \approx -\frac{16iL\omega}{c^2\left[\mu_1 b J_1(\mu_1 b)\right]^2} \tag{D.9}$$

Robinson has given an approximate analysis of a cavity-like structure[5] and obtained an impedance function

$$Z(\omega) \simeq \frac{Li}{\pi c^2} \left(\ln \frac{r_2}{r_1} \right) \omega, \qquad \text{(D.10)}$$

which can be seen to be of the same functional form as (D.9).

E. Impedance Function for Electrodes

As long ago as 1967, L. J. Laslett produced an electrodynamical analysis for clearing electrodes, and thus supplied the first model which could explain bunch lengthening.

Subsequent to his unpublished work, computations for general electrodes have appeared in the literature.[16] We summarize these formulas, in this Appendix, as a convenience for the reader.

From Ref. 16, a beam of radius a moving at speed v, down the axis of a vacuum chamber of radius b, interacts with an electrode which is a cylindrical segment of angular extent $2\varphi_0$, length ℓ, and radius approximately b. The electrode has characteristic impedance Z_0 and is terminated, at the ends, with impedances $Z_1(\omega)$ and $Z_2(\omega)$. The space and time Fourier transform of the longitudinal electric field on the chamber axis, $\tilde{E}_z(k,\omega)$, is given in terms of the double Fourier transform of the beam charge density, $\tilde{\lambda}(k,\omega)$, by

$$\tilde{E}_z(k,\omega) = \frac{8iZ_0c}{2\pi R} (K^{long})^2 g^{*long}(k)\ P^{long}(k,\omega)\ \tilde{\lambda}(k,\omega), \qquad \text{(E.1)}$$

where $$K^{long} = \frac{\varphi_0}{\pi}$$

and $$g^{*long} = \frac{2\gamma}{ka} \frac{I_1(ka/\gamma)}{\left[I_0(kb/\gamma)\right]^2} ,$$

and $$P^{long}(k,\omega) = \frac{\beta}{2}\left\{\frac{2ir_p(\omega)\left[\cos \frac{\ell\omega}{c} - \cos k\ell\right] + \sin \frac{\ell\omega}{c}}{2 \cos \frac{\ell\omega}{c} - 2iW(\omega) \sin \frac{\ell\omega}{c}}\right\},$$

with $$r_p(\omega) = \frac{Z_1 Z_2}{Z_0(Z_1 + Z_2)}$$

and $$W(\omega) = \frac{Z_0^2 + Z_1 Z_2}{Z_0(Z_1 + Z_2)} . \qquad (E.2)$$

Note that our definition of Fourier transform differs from that of Ref. 16 by the interchange $k \to -k$, $\omega \to -\omega$.

From (3.3), (3.9), and (E.1) we obtain

$$Z(\omega) = \frac{8iZ_0}{\beta} (K^{long})^2 \, g^{*long}(\omega/\beta c) \, P^{long}(\omega/\beta c,\omega). \qquad (E.3)$$

Of particular interest is a pickup electrode which extends around the full chamber and is terminated in its characteristic impedance. In the limit of a relativistic beam $\left[(\omega a/\beta c\gamma) \ll 1\right.$, for all ω of $\left.\text{importance}\right]$

$$Z(\omega) \approx 8iZ_0 P^{long}, \qquad (E.4)$$

with $$P^{long} \approx \frac{1}{4}\left[\frac{i(\cos \frac{\ell\omega}{c} - \cos \frac{\ell\omega}{\beta c}) + \sin \frac{\ell\omega}{c}}{(\cos \frac{\ell\omega}{c} - i \sin \frac{\ell\omega}{c})}\right]. \qquad (E.5)$$

Neglecting terms of order $1/\gamma^2$, the impedance function can be written, using (E.4), (E.5), as

$$Z(\omega) \approx - Z_0(1 - e^{2i\omega\ell/c}). \tag{E.6}$$

It is convenient in this case to evaluate the function $\mathcal{G}(\sigma)$; using (3.9) and (E.6) one has

$$\mathcal{G}(\sigma) = - Z_0 \left[\delta\left(\frac{\sigma}{\beta c}\right) - \delta\left(\frac{\sigma}{\beta c} - \frac{2\ell}{c}\right)\right]. \tag{E.7}$$

In the limit of electrodes short compared with the bunch length, (E.7) becomes

$$\mathcal{G}(\sigma) \approx - \frac{2Z_0\ell}{c} \delta'\left(\frac{\sigma}{\beta c}\right) . \tag{E.8}$$

FOOTNOTES AND REFERENCES

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