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A Characterization of Strategic Complementarities [*]

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Abstract

I characterize games for which there is an order on strategies such that the game has strategic complementarities. I prove that, with some qualifications, games with a unique equilibrium have complementarities if and only if Cournot best-response dynamics has no cycles; and that all games with multiple equilibria have complementarities. This is a negative result because it implies that the predictive power of complementarities alone is very weak. As an application of my results I show that generic $2X2$ games either have no pure-strategy equilibria, or are GSC.

JEL Classification: C62, C71.
Keywords: Strategic complementarities, supermodular games, non-standard analysis, Cournot best-response dynamics.

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1   Introduction

A game has strategic complementarities (Topkis 1979, Vives 1990) if, given an order on players' strategies, an increase in one player's strategy makes the other players want to increase their strategies. For example, the players could be firms in price competition; if each firm's optimal price is an increasing function of the prices set by their opponents, the game has strategic complementarities. In GSC, Nash equilibria have a certain order structure; in particular, there is a smallest and largest equilibrium (Topkis, Vives, Zhou (1994)). Further, the set of all rationalizable strategies, and the set of limits of adaptive learning, is bounded below by the smallest equilibrium and above by the largest equilibrium (Vives 1990, Milgrom and Roberts 1990, Milgrom and Shannon 1994). Games of strategic complementarities (GSC) are a well-behaved class of games, and a useful tool for economists. In this paper I basically characterize the full set of GSC, and argue that complementarities is a useful assumption only when coupled with additional structure, not by itself.

Strategic complementarities are commonly introduced as an assumption on player's payoffs, but the crucial feature for most results on GSC is that the game's best-response correspondence is monotone increasing. The assumption of supermodular—or quasisupermodular—payoffs in GSC is sufficient for monotone increasing best-responses, but I shall argue that it is enough, for the purpose of this paper, to define a GSC as a game for which there is a partial order on strategies so that best-responses are monotone increasing (and such that strategies have a lattice structure).

Consider the coordination game in Figure 1, is it a GSC? That is, is there an order on player's strategies so that best responses are monotone increasing? Yes, let α be smaller than β. Then, player 2's best response to α is α and to β is β. So, when 1 increases her strategy from α to β, 2's best response increases from α to β. Similarly for 1. So, with this simple order the coordination game is a GSC.

The order on strategies that makes the coordination game a GSC is artificial. This is always the case: how strategies are ordered is never part of the description of a game. We as analysts use the order as a tool, therefore we are justified in choosing the order to conform to our theory. In fact, typically results about GSC are used in applications by introducing an order on strategies that was not present in the problem

that motivated the model.

	α	β
α	$2,2$	$0,0$
β	$0,0$	$1,1$

Coordination

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Matching Pennies

Figure 1: 2x2 games, GSC?

Now consider the game "Matching Pennies" in Figure 1, is this game a GSC? Let H be smaller than T, then 2's best response to H, T, is larger than 2's best response to T, H. The same problem appears if we set T smaller than H. So, we need to order the player's strategies in different ways. Say that for player 1 H is smaller than T, but that for player 2 T is smaller than H. Now 2's best responses are increasing, but 1's best response is not—when 2 increases her strategy from T to H, 1's best response decreases from T to H. So matching pennies is not a GSC.

What does this depend on? Why is the simple coordination game a GSC but not matching pennies? My results are that

- a game with a unique pure-strategy Nash equilibrium is a GSC if and only if Cournot best-response dynamics have no cycles except for the equilibrium;

- a game with two or more pure-strategy Nash equilibria is always a GSC;

- generically, $2X2$ games are either GSC or have no pure-strategy equilibria.

In games that have a unique Nash equilibrium, strategic complementarities is equivalent to the absence of cycles—this result is subject to some qualifications that I discuss below. If b is the game's best-response function (the product of the players' best-response functions) then Cournot best-response dynamics starting at x is defined by $x_0 = x$, $x_n = b(x_{n-1})$, $n = 1, 2, \ldots$. Absence of cycles means that, if x is not an equilibrium, Cournot best-response dynamics starting at x never returns to x, i.e. $x_n \neq x$ for all $n \geq 1$, or, equivalently, that $x \neq b^n(x)$ for all $n \geq 1$. In finite games this condition is equivalent to global stability, that is that $b^n(x)$ converges to the equilibrium for all x. In infinite games, absence of cycles is a weaker condition than global stability—a game with a unique, globally stable, equilibrium is a GSC,

but the converse is not true. When best responses are not unique, so that b is a correspondence, this result holds with some qualifications.

A game with two or more equilibria is a GSC, so the vast majority of games that we encounter in applied work are GSC. The order in the coordination game of Figure 1 that makes its best-response function increasing involves making one Nash equilibrium the smallest point in the joint strategy space, and the other equilibrium the largest point in the strategy space. I show that, if a game has at least two equilibria, the same trick always works; we can order the strategies such that one equilibrium is the largest strategy profile and the other equilibrium is the smallest, and such that best-responses are monotone increasing.

The literature on GSC has developed a set-valued prediction concept: the set of strategies that are larger than the smallest Nash equilibrium and smaller than the largest Nash equilibrium. This "interval prediction" contains all rationalizable strategies, and all strategies that are limits of adaptive learning. Is the interval prediction in general a sharp prediction? Milgrom and Roberts suggest that the answer may be negative:

> Indeed, for some games, these bounds are so wide that our result is of little help: it is even possible that these bounds are so wide that the minimum and maximum elements of the strategy space are equilibria. (Milgrom and Roberts 1990, p. 1258)

Milgrom and Roberts go on to argue that, in some models, "the bounds are quite narrow." They present as examples an arms-race game, and a class of Bertrand oligopoly models, where there is a unique equilibrium.

My results imply that this is *generally* the case: if a game does not have a unique equilibrium the interval prediction is essentially vacuous, as all games with multiple equilibria are GSC where the smallest and largest equilibria are the smallest and largest strategy profiles. The explanation for this is that games with multiple equilibria always involve a kind of coordination problem, namely the problem of coordinating on one equilibrium. This coordination problem can be formalized through an order on strategies that make the game a GSC.

I wish to emphasize that GSC are still a very useful tool. In a given game,

complementarities jointly with the rest of the structure of the game can provide sharp results. In a sense, complementarities is like compactness—most spaces we work with can be compactified, but compactness is still a very powerful tool. The point of this paper is that complementarities alone do not possess important predictive power, but that we can use complementarities to understand properties of a particular game. For example, Topkis's (1979) algorithm for finding the smallest and largest Nash equilibria in GSC is very useful on a given game with complementarities; and once Topkis's algorithm has delivered an interval prediction we can judge if it is sharp or not *in that particular game.* Other examples are Amir's (1996) elegant methods for analyzing Cournot oligopoly models using complementarities, and the equilibrium uniqueness results in the literature on global games (Morris and Shin 2000). [1]

Lippman, Mamer, and McCardle (1987), Sobel (1988), Milgrom and Roberts (1990), Milgrom and Shannon (1994), Milgrom and Roberts (1994), and Echenique (2001) present comparative statics results for a parameterized GSC. They prove that, if a parameter is complementary to players' choices, some selections of equilibria are monotone increasing in the parameter. My results prompt the question: can any systematic comparative statics conclusion be rationalized as coming from a parameterized GSC? I show that this is not the case: there are parameterized models (with multiple equilibria) such that no order delivers the comparative statics as an application of results for GSC. In this sense, comparative statics acts as an "identifying restriction." Comparative statics imply that we do not have enough degrees of freedom in selecting the order on strategies so as to make the game a GSC and, at the same time, preserve monotone comparative statics.

In the usual definition of GSC—supermodular games and games with ordinal complementarities—there is a link between the order and the topology on the strategy spaces that, among other things, ensures the existence of equilibria (by Tarski's Theorem), and that if equilibrium is unique it must be globally stable. Here I construct orders that make best-responses monotone increasing, but that do not have any

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[1]Amir's analysis does not proceed by introducing a convenient order on strategy spaces like I do (see also Amir and Lambson (2000)). My results—which, unlike Amir's, require prior knowledge of some equilibria of the game—imply that there are more Cournot oligopoly models that can be made into a GSC than the ones identified by Amir. But the structure, in addition to complementarities, that Amir uses in his papers enables him to obtain substantive results about Cournot oligopoly.

relation to the topology on the strategy spaces. In finite games this does not matter: a finite game without equilibria cannot be a GSC, and a game with a unique equilibrium is a GSC if and only if this equilibrium is globally stable. In infinite games, though, there may be GSC without equilibria, and GSC with a unique equilibrium that is not globally stable. I wish to emphasize that, despite this technical problem, for games with multiple equilibria, all of Vives's, Topkis's and Milgrom and Roberts's results hold trivially with the order that I construct.

2 Preliminaries

A detailed discussion of the concepts defined in this subsection can be found in Topkis (1998). A pair $(X, \leq)$, where X is a set and $\leq$ is a transitive, reflexive, antisymmetric binary relation, is a **partially ordered** set; $(X, \leq)$ is **totally ordered** if, for all $x, y \in X$, $x \leq y$ or $y \leq x$ ($\leq$ is then a total order on X); $(X, \leq)$ is a **lattice** if whenever $x, y \in X$, both $x \wedge y = \inf \{x, y\}$ and $x \vee y = \sup \{x, y\}$ exist in X. Note that a totally ordered set is a lattice.

A lattice $(X, \leq)$ is **complete** if for every nonempty subset A of X, $\inf A, \sup A$ exist in X. A nonempty subset A of X is a **sublattice** if for all $x, y \in A$, $x \wedge_X y, x \vee_X y \in A$, where $x \wedge_X y$ and $x \vee_X y$ are obtained taking the infimum and supremum as elements of X (as opposed to using the relative order on A). A nonempty subset $A \subseteq X$ is **subcomplete** if $B \subseteq A$, $B \neq \emptyset$ implies $\inf_X B, \sup_X B \in A$, again taking inf and sup of B as a subset of X. For two subsets A, B of X, say that A is smaller than B in the **strong set order** if $a \in A$, $b \in B$ implies $a \wedge b \in A, a \vee b \in B$.

Let $(X, \leq)$ be a lattice. Say that a correspondence $\phi : X \twoheadrightarrow X$ is **weakly increasing** over $A \subseteq X$ if $x, y \in A$ and $x \leq y$ implies that there is $z \in \phi(x)$ and $z' \in \phi(y)$ with $z \leq z'$. Also, say that ϕ is **increasing in the strong set order** if $x < y$ implies that $\phi(x)$ is smaller in the strong set order than $\phi(y)$. Note that when ϕ is a function, i.e. single valued, both concepts coincide with the usual notion of "monotone weakly increasing."

A correspondence $\phi : X \twoheadrightarrow X$ takes **finite values** if $\phi(x)$ is a finite set for all $x \in X$.

3   Unique Equilibrium

I shall present all my results as results about functions, or correspondences, on a set X. The reader should think of the best-response function, or correspondence, of a game as the main application of my results. Let $\Gamma = \left\{ I, \{u_i\}_{i \in I} \, \{S_i\}_{i \in I} \right\}$ be a normal-form game. That is, I is a set of players, and each player $i \in I$ is endowed with a strategy space S_i and a payoff function $u_i : S = \times_{j \in I} S_j \to \mathbf{R}$. If $\beta_i : S \twoheadrightarrow S_i$ is player i's best-response correspondence, $\beta_i(s) = \operatorname{argmax}_{\tilde{s}_i} u_i(\tilde{s}_i, s_{-i})$, then $X = S$ and $\phi : X \twoheadrightarrow X$ defined by $\phi = \times_{j \in I} \beta_i$ is the game's best-response correspondence. The set of fixed points of ϕ coincides with the set of pure-strategy Nash equilibria of Γ. A game $\Gamma = \left\{ I, \{u_i\}_{i \in I} \, \{S_i\}_{i \in I} \right\}$ is a **_game of strategic complementarities_** if there is an order $\le$ on S such that $(S, \le)$ is a lattice and ϕ is weakly increasing. If best-responses are always unique, then ϕ is a function and ϕ is weakly increasing if and only if it is monotone increasing.

Definition 1 _Let X be a set and $\phi : X \twoheadrightarrow X$ be a correspondence. A point $x \in X$ is a **cycle** of ϕ if there is $n \in \mathbf{N}$ such that $x \in \phi^n(x)$. Let X be a topological space and $\phi : X \twoheadrightarrow X$ a correspondence. A fixed point e is **globally stable** for ϕ if, for every $x \in X$ and every sequence $\{x_k\}$ with $x = x_0$ and $x_k \in \phi(x_{k-1})$, $x_k \to e$._

If players meet to play Γ over and over again, and if they, in each round of play n, choose a best response to their opponents' play in $n - 1$, we say that they follow Cournot best-response dynamics. If ϕ is Γ's best-response correspondence, all Cournot best-response dynamics are generated by $x_n \in \phi(x_{n-1})$ (for some initial x_0). Then, absence of cycles means that no Cournot best-response dynamics will get caught in a cycle. And global stability means that players that engage in Cournot best-response dynamics will eventually approach Nash equilibrium play. [2]

The existence of an order such that a function, or a correspondence, is increasing is a non-topological statement, so it cannot depend on global stability. The absence of cycles is a non-topological condition that is weaker than global stability: global

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[2]Cournot best-response dynamics is a very naive learning model, my results do not depend of any virtues of Cournot dynamics as a learning model, the absence of cycles should be viewed as a technical condition. It is true, though, that Cournot dynamics is easy to implement on a computer, and therefore the condition of absence of cycles is easy to verify computationally (in a finite game).

stability implies the absence of cycles other than e, but absence of cycles does not imply global stability. For example, let X be the unit disk in $\mathbf{R}^2$ and f be a rotation by an irrational number, then f has a unique fixed point, $(0,0)$, and no cycles, but $(0,0)$ is clearly not globally stable.

Theorem 2 *Let X be a set and $f : X \to X$ be a function on X that has a unique fixed point $e \in X$. There is a total order $\leq$ on X such that f is monotone increasing if and only if f has no cycles besides e.*

Theorem 2 may have some mathematical interest, independently of the application to game theory that I emphasize. The proof of Theorem 2 follows from Theorem 3 below.

Let $\Gamma = \left\{ I, \{u_i\}_{i \in I} \{S_i\}_{i \in I} \right\}$ be a normal-form game with a unique Nash equilibrium. If all players have unique best-responses—which, as these are pure-strategy best-responses, is generically the case when Γ is a finite game—then, by Theorem 2, Γ is a GSC if and only if Cournot best-response dynamics has no cycles except for e. [3] In particular, if e is globally stable under Cournot best-response dynamics, then Γ is a GSC. One implication of these results is that a game that satisfies the dominant diagonal condition in Gabay and Moulin (1980), and that has thus a unique, globally stable, equilibrium, is a GSC.

As an illustration of Theorem 2, consider Γ_1, the $3X3$ game on the left in Figure 2. This game has a unique pure-strategy equilibrium, and best-response dynamics has the cycles $(\alpha, \alpha) \to (\alpha, \alpha)$, $(\beta, \alpha) \to (\alpha, \gamma) \to (\gamma, \alpha) \to (\alpha, \beta) \to (\beta, \alpha)$, and $(\beta, \beta) \to (\beta, \gamma) \to (\gamma, \gamma) \to (\gamma, \beta) \to (\beta, \beta)$. Therefore there is no order on strategies such that the game in Figure 2 is a GSC.

Consider Γ_2, the zero-sum game on the right in Figure 2 as a second illustration of Theorem 2. This is a GSC because it has a unique equilibrium and no cycles. The intuitive idea that zero-sum games and GSC are different is therefore false.

3.1 Non-unique best responses.

Theorem 2 for functions can be generalized to correspondences with finite values, but the characterization is not completely tight. Absence of cycles implies the existence of

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[3]See section 3.2 for the difference between lattice and totally ordered strategy spaces.

	α	β	γ
α	$2,2$	$0,0$	$0,0$
β	$0,0$	$1,-1$	$-1,1$
γ	$0,0$	$-1,1$	$1,-1$

Γ_1

	α	β
α	$1,-1$	$2,-2$
β	$0,0$	$1,-1$

Γ_2

Figure 2: A non-GSC, and a GSC that is zero-sum.

a total order that makes the correspondence weakly monotone, but a weakly monotone correspondence may have cycles, as weak monotonicity does not control all selections from the correspondence.

Theorem 3 *Let $\phi : X \twoheadrightarrow X$ be a correspondence with a unique fixed point $e \in X$. If ϕ takes finite values and has no cycles besides e, there is a total order $\leq$ on X such that ϕ is weakly increasing. Further, if $\{e\} = \phi(e)$ and there is a total order $\leq$ on X such that ϕ is increasing in the strong set order, then ϕ has no cycles besides e.*

The proof of Theorem 3 is in section 7.

I need the hypothesis of a correspondence with finite values to use non-standard analysis the way I do. If Γ is a game with unique best responses, of course the best-responses have finite values, and both weakly increasing and increasing in the strong set order coincide with monotone increasing, so Theorem 3 implies Theorem 2. Also, if Γ is a game with finite strategy spaces (but, possibly, infinitely many players) then best responses take finite values. I have not been able to extend the characterization in Theorem 3 to games with infinite strategy spaces and non-unique best responses.

Global stability implies absence of cycles, thus we obtain

Corollary 4 *Let X be a topological space and $\phi : X \twoheadrightarrow X$ a correspondence with finite values and a unique fixed point $e \in X$. If e is globally stable then there is a total order $\leq$ on X such that ϕ is weakly increasing.*

3.2   Totally ordered strategy space vs. a lattice strategy space

Theorems 2 and 3 give necessary and sufficient conditions for a game to be a GSC with a totally ordered strategy space. We defined GSC as games with lattice strategy
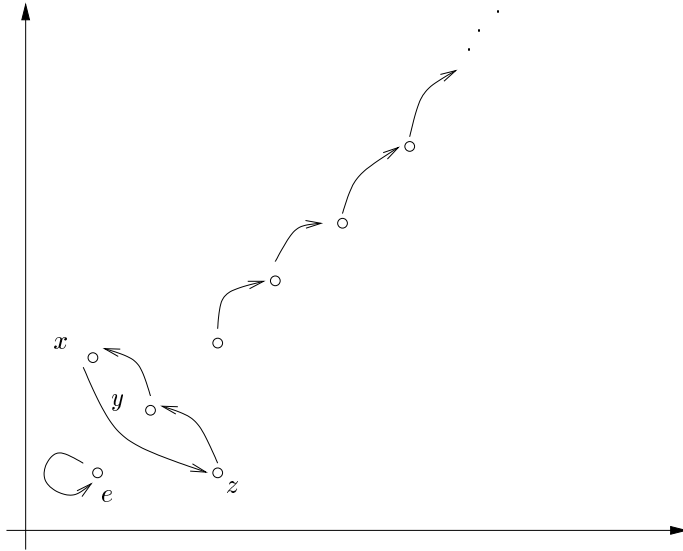
Figure 3: Arrows show the action of f

spaces, is there some loss in focusing on totally ordered strategies? Yes, but it is essentially a technical problem.

Consider Theorem 2. If a game's best-response function does not have cycles, then the game is a GSC because there is a total order on strategies such that best responses are monotone increasing, and a totally ordered set is a lattice. On the other hand, if Γ is a GSC where each S_i is a complete lattice, and best responses f are continuous, then a unique Nash equilibrium is globally stable (Vives 1990, Milgrom and Roberts 1990) so f has no cycles and there is a total order on strategies such that f is increasing. In particular this implies that, in finite games, a game with a unique equilibrium is a GSC if and only if best responses have no cycles.

In GSC where strategy spaces are non-complete lattices, best-responses may have cycles. Consider the example in Figure 3. The function f has a unique fixed point, e, a cycle $z \to y \to x \to z$, and it is monotone increasing. The example works because the monotone increasing sequence obtained by iterating f on points larger than x, y and x does not converge to a fixed point different from e. So the characterization in Theorem 2 is not completely tight. As I remarked earlier, complementarities is a non-topological condition, so the characterization must be independent of completeness

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on strategy spaces. In a sense, then, it is impossible to avoid problems like the one in Figure 3.

4   Multiple Equilibria

A game with two or more Nash equilibria is easily transformed into a GSC. It is enough to set one equilibrium as the smallest strategy profile, and the other as the largest. I argue below that it is in general not possible to improve on this result.

Theorem 5 *Let $\phi : X \twoheadrightarrow X$ be a correspondence. If ϕ has at least two different fixed points, then there is an order $\leq$ on X such that $(X, \leq)$ is a complete lattice, ϕ is weakly increasing, and the set of fixed points of ϕ is a complete sublattice of X.*

Proof: Let $\underline{e}$, $\overline{e}$ be different fixed points of ϕ. Define $\leq$ on X by $x \leq y$ if and only if any of the following is true: $x = y$, $x = \underline{e}$, or $y = \overline{e}$. Then, for any non-singleton $A \subseteq X$, $\underline{e}$ is the unique lower bound on A, hence $\inf A = \underline{e}$; similarly $\sup A = \overline{e}$ as $\overline{e}$ is the unique upper bound on A. Thus, for all $x, y \in X$, $x \vee y, x \wedge y \in X$, and for all non-empty $A \subseteq X$, $\inf A$, $\sup A \in X$, so X is a complete lattice.

Now, let $x \leq y$ and $x \neq y$; if $x = \underline{e}$ then $\underline{e} \in \phi(x)$, so $\underline{e} \leq z$ for all $z \in \phi(y)$, and similarly if $y = \overline{e}$. If $x = y$ then any $z = z' \in \phi(x) = \phi(y)$ satisfies $z \leq z'$. Thus if $x \leq y$ there is $z \in \phi(x)$ and $z' \in \phi(y)$ with $z \leq z'$, so ϕ is weakly increasing.

Finally, if A is a non-singleton set of fixed points, then $\underline{e}$ is the unique lower bound, and $\overline{e}$ is the unique upper bound, on A. So, $\inf A = \underline{e}$ and $\sup A = \overline{e}$. But $\underline{e}$ and $\overline{e}$ are fixed points, so the set of fixed points is a complete sublattice of X. ∎

Remarks:

1. The usual definitions of GSC ensure that best-responses are increasing in the strong set order, and that ϕ takes subcomplete- sublattice-values. The order that I construct in Theorem 5 does not guarantee that ϕ has these properties (unless, of course, ϕ is a function), but Topkis's, Vives's, Milgrom and Roberts's, and Milgrom and Shannon's results are (trivially) true with the constructed order.

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	α	β	γ	δ
α	$2,2$	$0,0$	$0,0$	$0,0$
β	$0,0$	$0,1$	$1,0$	$0,0$
γ	$0,0$	$1,0$	$0,1$	$0,0$
δ	$0,0$	$0,0$	$0,0$	$1,1$

Figure 4: A $4X4$ Coordination Game.

2. By Zhou's version of Tarski's Fixed Point Theorem the equilibrium set of a GSC is a complete lattice. Topkis, Vives and Zhou present examples where it is not a sublattice. By Theorem 5 there is a partial order on X such that this equilibrium set is a complete sublattice, without eliminating complementarities.

The order constructed implies that the interval prediction concept in, among others, Milgrom and Roberts (1990) is trivial. Is there some order that also makes games with multiple equilibria GSC, and for which the interval prediction is nontrivial? That is, is it possible to improve on Theorem 5 so that interval predictions are sharper?

Consider the coordination game in Figure 1. I shall show that there is no order on strategies that preserves complementarities, and where the interval prediction is sharper than the whole strategy space. Let

$$b = b_1 \times b_2 : \{\alpha, \beta\}^2 \to \{\alpha, \beta\}^2$$

be the game's best-response function, and $\leq$ be an order on $\{\alpha, \beta\}^2$ such that b is monotone increasing. Now, it must be that (α, β) and (β, α) are incomparable under $\leq$. To see this, let $(\alpha, \beta) < (\beta, \alpha)$, then $b(\beta, \alpha) = (\alpha, \beta) < b(\alpha, \beta) = (\beta, \alpha)$, and b is not increasing. Similarly if $(\beta, \alpha) < (\alpha, \beta)$. Then, for $(\{\alpha, \beta\}^2, \leq)$ to be a lattice, it must be that $(\alpha, \beta) \vee (\beta, \alpha)$ and $(\alpha, \beta) \wedge (\beta, \alpha)$ equals either (α, α) and (β, β) or, respectively, (β, β) and (α, α). But $(\alpha, \beta) \vee (\beta, \alpha) \neq (\alpha, \beta) \wedge (\beta, \alpha)$, hence either (α, α) is the smallest strategy profile and (β, β) is the largest, or vice versa.

I should stress that this does not depend on the coordination game being $2X2$, it depends on the cycles in best-responses between the non-equilibrium strategy profiles

(α, β) and (β, α). In fact, consider the $4X4$ coordination game in Figure 4. In this game, best-responses have three cyclical orbits (besides the fixed-point cycles $(\alpha, \alpha) \to (\alpha, \alpha)$ and $(\delta, \delta) \to (\delta, \delta)$):

$$(\beta, \alpha) \to (\alpha, \beta) \to (\gamma, \alpha) \to (\alpha, \gamma) \to (\beta, \alpha),$$
$$(\delta, \gamma) \to (\beta, \delta) \to (\delta, \beta) \to (\gamma, \delta) \to (\delta, \gamma),$$
$$\text{and } (\beta, \beta) \to (\gamma, \beta) \to (\gamma, \gamma) \to (\beta, \gamma) \to (\beta, \beta).$$

Then, if a partial order $\leq$ makes the $4X4$ coordination game into a GSC, no strategy that belongs to a cyclical orbit can be the smallest, or the largest, strategy profile of the game. To see this, suppose, for example, that (β, α) is the smallest strategy profile. Then $(\beta, \alpha) < (\alpha, \beta)$, so if b is monotone increasing, $b^3(\beta, \alpha) \leq b^3(\alpha, \beta)$. But $(\beta, \alpha) = b^3(\alpha, \beta)$ and $(\alpha, \beta) = b^3(\beta, \alpha)$. So it must be that the smallest and largest element in the strategy space is (α, α) and (δ, δ), as a finite lattice has a smallest and a largest element.

The coordination game shows that it is *in general* impossible to sharpen the Milgrom-Roberts interval predictions. And, if there is a class of GSC for which the interval prediction is non-trivial, the coordination game is not a member of this class. The coordination game is generally considered one of the main examples of strategic complementarities (Cooper 1999), therefore it is not likely that Theorem 5 can be improved upon in an interesting direction.

5 $2X2$ Games

As an application of Theorems 3 and 5 I show that, generically, $2X2$ games either have no equilibria or have complementarities. Essentially, then, $2X2$ games are either isomorphic to "Matching Pennies", or they are GSC.

Proposition 6 *Generically, a $2X2$ game either has no pure-strategy Nash equilibrium, or it is a GSC.*

Proof: Consider the game in Figure 5 on the left. Suppose that a, b, c, etc. are such that (β, β) is the unique Nash equilibrium, and (β, β) is a strict equilibrium, so.

$$
\begin{array}{c|c|c}
 & \alpha & \beta \\
\hline
\alpha & a,b & c,d \\
\hline
\beta & e,f & g,h \\
\end{array}
\qquad\qquad
\begin{array}{c|c|c}
 & \alpha & \beta \\
\hline
\alpha & 1,1 & 0,1 \\
\hline
\beta & 0,1 & 1,0 \\
\end{array}
$$
$$\Gamma_3 \qquad\qquad\qquad\qquad \Gamma_4$$

Figure 5: Γ_3 used in the proof of, and Γ_4 illustrates, Proposition 6

$c < g$ and $f < h$. I will rule out all possible cycles in best-response dynamics, except for the cycle involving (β, β). Then Theorem 2 implies that the game is a GSC.

Then (α, α) is not a best response to (α, β) or to (β, α). This rules out the cycles $(\alpha, \alpha) \to (\alpha, \beta) \to (\alpha, \alpha)$, $(\alpha, \beta) \to (\alpha, \alpha) \to (\alpha, \beta)$, $(\alpha, \alpha) \to (\beta, \alpha) \to (\alpha, \alpha)$, $(\beta, \alpha) \to (\alpha, \alpha) \to (\beta, \alpha)$, $(\alpha, \alpha) \to (\beta, \alpha) \to (\alpha, \beta) \to (\alpha, \alpha)$, and $(\alpha, \alpha) \to (\alpha, \beta) \to (\beta, \alpha) \to (\alpha, \alpha)$.

Finally, the cycle $(\beta, \alpha) \to (\alpha, \beta) \to (\beta, \alpha)$ requires that $a \geq e$ and $b \geq d$. But then (α, α) would be an equilibrium, contradicting that (β, β) is the unique Nash equilibrium. Similarly for $(\alpha, \beta) \to (\beta, \alpha) \to (\alpha, \beta)$. These are all possible cycles.

We have shown that all $2X2$ games with a unique, strict Nash equilibrium are GSC. All games with more than one equilibrium are GSC by Theorem 5. The property that a unique equilibrium is strict is generic in the class of $2X2$ games. ■

Proposition 6 does not extend to more complex games than $2X2$, see the example in Figure 2 on the left. The "genericity" qualification in Proposition 6 is necessary. Consider the game on the right in Figure 5. This game has a unique Nash equilibrium (α, α) in pure strategies, and a cycle, $(\alpha, \alpha) \to (\alpha, \beta) \to (\beta, \alpha) \to (\alpha, \alpha)$. The cycle is produced by player 2's indifference when 1 chooses α, which is non-generic.

6   Comparative Statics

In a parameterized GSC, if a parameter t is complementary to players' choices, there are selections of equilibria that are monotone increasing in t (Lippman, Mamer, and McCardle 1987, Sobel 1988, Villas-Boas 1997, Milgrom and Roberts 1990, Milgrom and Roberts 1994, Milgrom and Shannon 1994, Echenique 2001). By theorems 3 and 5, most games can be rationalized as GSC. A natural question is: can any comparative statics conclusion be rationalized as coming from a parameterized GSC?

The answer is no, comparative statics conclusions act as an "identifying condition" that restricts the choice of an order on strategy spaces.

Using the framework in Milgrom and Roberts (1994), the question can be phrased as follows. Let $f_t, f_{t'} : X \to X$, let e be a fixed point of f_t, and e' a fixed point of $f_{t'}$. Is there an order on X such that $e < e'$ and such that $f_t(x) \le f_{t'}(x)$ for all x?

In general the answer is no. Consider the following example, let $x, e, e' \in X$ with $e = f_t(e)$, $e' = f_{t'}(e')$ and $e \ne e'$. Suppose that $e' = f_t(x)$ and $e = f_{t'}(x)$. If there is an order $\le$ such that $f_t(z) \le f_{t'}(z)$ for all $z \in X$, then $e = f_t(e) \le f_{t'}(e) = e'$ and $e < e'$ because $e \ne e'$. But $e' = f_t(x) \le f_{t'}(x) = e$, a contradiction.

The example above shows that we cannot rationalize comparative statics conclusions by a model where the parameter is complementary to players' choices. Alternatively, is there an order on X such that $e < e'$ and such that f_t and $f_{t'}$ are monotone increasing functions? That is, can we rationalize a comparative statics conclusion as coming from parameterized GSC, even though the parameter may not be complementary to the strategies? The following example shows that this need not be the case. Let $x, e, e' \in X$ with $e = f_t(e)$, $e' = f_{t'}(e')$ and $e \ne e'$. Suppose that $x = f_t(e')$, $e = f_{t'}(x)$ and $e' = f_{t'}(e)$. Then $e < e'$ implies that $f_t(e) \le f_t(e')$, so $e < x$. Then $f_{t'}(e) \le f_{t'}(x)$ implies that $e' < e$, a contradiction.

7 Proof of Theorem 3

I prove the second statement in Step 1. Steps 2 and 3 prove the first statement when X is finite—Step 2 constructs an order such that ϕ is weakly increasing, and Step 3 checks that this is a total order. Steps 4 and 5 prove the first statement for arbitrary X by non-standard methods. The idea is to embed X in a hyperfinite set, apply the result for finite sets to get an order that works in the hyperfinite set, and then restrict the order to X.

STEP 1. Let $\le$ be a total order on X such that ϕ is increasing in the strong set order, and let $x \ne e$. Suppose, by way of contradiction, that there is $\{x_m\}_{m=0}^{K} \subseteq X$ such that $x = x_0 = x_K$ and $x_m \in \phi(x_{m-1})$, $1 \le m \le K - 1$. Note that we must have $x_m \ne e$ for all m because $\phi(e) = \{e\}$ would imply that $x = e$.

Since X is totally ordered by $\le$, either $x < x_{K-1}$ or $x_{K-1} < x$, as $x = x_{K-1}$

is ruled out because x is not a fixed point. Suppose first that $x < x_{K-1}$, then $\phi(x) \ni x_1$ is smaller than $\phi(x_{K-1}) \ni x$ in the strong set order, so $x_1 \wedge x \in \phi(x)$. Now, $x_1 \wedge x \in \{x_1, x\}$ because $\leq$ is a total order, and $x \notin \phi(x)$ because x is not a fixed point—so we must have $x_1 < x$. It remains to show that if $x_m < x_{m-1}$, then $x_{m+1} < x_m$, for $1 \leq m \leq K-1$, which implies that $x_K < x_{K-1} < \ldots < x_0 = x$, a contradiction. If $x_m < x_{m-1}$, then $\phi(x_m) \ni x_{m+1}$ is smaller than $\phi(x_{m-1}) \ni x_m$ in the strong set order. Then $\{x_{m+1}, x_m\} \ni x_{m+1} \wedge x_m \in \phi(x_m)$, but $x_m \notin \phi(x_m)$ as $x_m \neq e$, so $x_{m+1} < x_m$.

If, instead, $x > x_{K-1}$ we can apply an analogous argument to reach a contradiction. So there cannot be a K with $x \in \phi^K(x)$.

STEP 2. Let X be finite. Let $\phi : X \twoheadrightarrow X$ be a correspondence with a unique fixed point $e \in X$, and no cycles besides e. For all $x \in X$ there is at least one sequence $\{x_m\}_{m=0}^{K}$ with $x_0 = x$, $x_m \in \phi(x_{m-1})$, $m = 1, 2, \ldots K$, and $x_K = e$; that is a *path* connecting x and e; K is the length of the path. To see that such a path must exist, note that, by absence of cycles, if $x_m \neq e$ for $m = 1, 2, \ldots K$, then $\{x_m\}_{m=0}^{K}$ are all distinct, so X has at least $K+1$ elements. X is finite, so we must have, for K large enough, that $x_K = e$. For each $x \in X$, let K_x be the smallest K such that there is a path of length K connecting x and e. Clearly, $K_e = 0$. Fix, for each x a path of minimal length $\{x_m\}_{m=0}^{K_x}$ connecting x and e. Note that if $K_{x_1} = K_x - 1$, as the K's are minimal.

Define $X_m = \{x \in X : K_x = m\}$, $m = 0, 1, \ldots M$, where M is such that $X = \cup_{m=0}^{M} X_m$. The collection $\{X_m\}_{m=0}^{M}$ is a partition of X because it covers X and has disjoint elements, as the K_xs are minimal.

I shall define recursively an order $\leq_m$ on X_m, $m = 0, 1, \ldots K$. Let $\preceq$ be a total order on X such that e is the largest element of X (for example, embed X in $\mathbf{N}$ such that e is mapped to a number larger than any other element in X, and take the relative order on X). Let $\leq_0$ on X_0, be the restriction of $\preceq$ to X_0. Given a total order $\leq_m$ on X_m, let $\leq_{m+1}$ on X_{m+1} be defined by, $x \leq_{m+1} y$ if $x_1 <_m y_1$ or if $x_1 = y_1$ and $x \preceq y$. Note that $\leq_m$ is in fact a total order because it is a lexicographic order.

Finally, define a total order $\leq$ on X by $x \leq y$ if either $k_y < k_x$ or if $k_y = k_x$ and $x \leq_{k_x} y$. Note that e is the largest element in X.

STEP 3. I shall check that ϕ is weakly increasing. Let $x, y \in X$ and $\{x_m\}_{m=0}^{K_x}$,

$\{y_m\}_{m=0}^{K_y}$ be the paths of minimal length connecting x and y to e from step 2.

Let $x \leq y$. If $K_y < K_x$ then there is $x_1 \in \phi(x)$ and $y_1 \in \phi(y)$, $x_1 \in \{x_m\}_{m=0}^{K_x}$, $y_1 \in \{y_m\}_{m=0}^{K_y}$, with $K_{y_1} = K_y - 1 < K_x - 1 = K_{x_1}$ because the K's are minimal. Then, $x_1 \leq y_1$. If $K_y = K_x$ then there is $x_1 \in \phi(x)$ and $y_1 \in \phi(y)$, $x_1 \in \{x_m\}_{m=0}^{K_x}$, $y_1 \in \{y_m\}_{m=0}^{K_y}$ with $x_1 \leq_{K_x+1} y_1$. Then $K_{y_1} = K_y - 1 = K_x - 1 = K_{x_1}$, because the K's are minimal. Then, $x_1 \leq y_1$. In both cases there is $x_1 \in \phi(x)$ and $y_1 \in \phi(y)$ with $x_1 \leq y_1$, thus proving that ϕ is weakly increasing

To prove that $\leq$ is a total order, note first that $x \leq x$ because x is in some X_m, and $x \leq_m x$, as $\leq_m$ is an order; thus $\leq$ is reflexive. If $x \leq y$ and $y \leq x$ then we must have $K_x = K_y$, and then $x = y$ because $\leq_{K_x}$ is an order. So, $\leq$ is antisymmetric. Let $x \leq y$ and $y \leq z$. If $K_y < K_x$ or $K_z < K_y$ then $K_z < K_x$ and $x \leq z$. If $K_x = K_y = K_z$ then $x \leq z$ follows from transitivity of $\leq_{K_x}$. Thus $\leq$ is transitive. Finally, $\leq$ is total because if $x, y \in X$ then there is m, n such that $x \in X_m$ and $y \in X_n$. If $m < n$ or $n < m$, x and y are ordered. If $n = m$, x and y are ordered because $\leq_m$ is a total order.

STEP 4. Suppose first that $\phi(e) = \{e\}$. Let H be a hyperfinite set with $X \subseteq H \subseteq {*}X$. For all $x \in X$, $\phi(x)$ is finite, then ${*}\phi$ takes hyperfinite values, and therefore ${*}\phi(H)$ is hyperfinite, as it is the hyperfinite union of hyperfinite sets. Let $\tilde{H} = H \cup {*}\phi(H)$, $\tilde{H}$ is hyperfinite.

By the Transfer Principle, $h \in H$ implies $(h \in {*}\phi(h) \Rightarrow h = e)$. Define $\tilde{\phi} : \tilde{H} \to \tilde{H}$ by

$$\tilde{\phi}(h) = \begin{cases} {*}\phi(h) & \text{if } h \in H \\ \{e\} & \text{if } h \in \tilde{H} \backslash H. \end{cases}$$

$\tilde{\phi}$ is internal, and $\tilde{\phi}(e) = {*}\phi(e) = \{e\}$. I show that e is the only fixed point of $\tilde{\phi}$. Suppose $x \in \tilde{\phi}(x)$. First, if $x \in H$, then $\tilde{\phi}(x) = {*}\phi(x)$, so $x \in {*}\phi(x)$ implies that $x = e$ by Transfer. Second, if $x \in \tilde{H} \backslash H$ then $\tilde{\phi}(x) = \{e\}$, so $x = e \in H$, a contradiction.

To show that $\tilde{\phi}$ has no cycles except for e, suppose that $x \in \tilde{\phi}^K(x)$ for some $x \in \tilde{H}$ and $K \in {*}\mathbf{N}$. Note that $x \in H$ because $x \in \tilde{H} \backslash H$ implies that $\tilde{\phi}^m(x) = \{e\}$ for all m, and $e \in H$. There is a sequence $\{x_m\}_{m=0}^{K}$ with $x_0 = x_K = x$ and $x_m \in \tilde{\phi}(x_{m-1})$, $m = 1, 2, \ldots K$. If there is m such that $x_m \in \tilde{H} \backslash H$ then $e = x_l$, $m \leq l \leq K$, so $x = e$. On the other hand, if $x_{m-1} \in H$ for all m, then $x_m \in \tilde{\phi}(x_{m-1}) = {*}\phi(x_{m-1})$ for all m.

Then x is a cycle of $*\phi$, so $x = e$ by Transfer.

Then, $\tilde{H}$ is hyperfinite, $\tilde{\phi} : \tilde{H} \twoheadrightarrow \tilde{H}$ has a unique fixed point e, and no cycles but e. By steps 2 and 3 above and the Transfer Principle, there is a total order $\lhd$ on H such that $x \lhd e$ for all $x \in H$, and such that $x \lhd y$ implies that there is $z \in \tilde{\phi}(x)$ and $z' \in \tilde{\phi}(y)$ with $z \lhd z'$. Let $\leq$ be the restriction of $\lhd$ to $X \subseteq \tilde{H}$, $\leq$ is a total order on X, and e is its largest element.

To verify that ϕ is weakly increasing, let $x \leq y$. Then, $x \lhd y$ implies that there is $z \in \tilde{\phi}(x)$ and $z' \in \tilde{\phi}(y)$ with $z \lhd z'$. But $\tilde{\phi}(x) = *\phi(x) = \phi(x)$ and $\tilde{\phi}(y) = *\phi(y) = \phi(y)$, as $x, y \in X \subseteq H$. Then $z \in \phi(x)$, $z' \in \phi(y)$, and $z \leq z'$ because $z \lhd z'$.

STEP 5. Finally, if $\phi(e) \neq \{e\}$. then let $\phi' : X \twoheadrightarrow X$ coincide with ϕ on $X \backslash \{e\}$, and take the value $\{e\}$ on e. By step 4 there is an order $\leq$ on X such that e is its largest element and such that ϕ' is weakly increasing. Now we check that ϕ is weakly increasing in this order as well. We only need to prove that if $x \in X$ then there is $z \in \phi(x)$ and $z' \in \phi(e)$ with $z \leq z'$. But, since $e \in \phi(e)$ and e is the largest element in X, we can set $z' = e$ and be done. ∎

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