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Mismatch Oscillations in High Current Accelerators

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Abstract

When planning the design of high-current FODO transport for accelerators, it is useful to have simple, accurate tools for calculating quantities such as the phase advances σ_0 and σ given the lattice and beam parameters. Along with the KV beam model, the smooth approximation is often used. It is simple but not very accurate in many cases. Although Struckmeier and Reiser [1] showed that the stable oscillation frequencies of mismatched beams could be obtained accurately, they actually used a hybrid approach where σ_0 and σ were already known precisely. When starting instead with basic quantities such as quadrupole dimensions, field strength, beam line charge density and emittance, the smooth approximation gives substantial errors. Here we derive a simple modification of the smooth approximation formula that improves the accuracy of the predicted frequencies by a factor of five at $\sigma_0 = 83^\circ$.

1. Unmatched Envelope Equations - Linearization and Expansion

Review of Results for Matched Beams

In reference [2] we analyzed the envelope equations for a K-V beam with space charge using an expansion in a small parameter followed by integrations. The alternating gradient focusing field profile given as a function of axial length was assumed to be symmetric in z but otherwise arbitrary. Starting with the beam current (or line charge), emittance, and peak focusing field, we found the envelopes $a(z)$ and $b(z)$ and obtained $\langle a \rangle$, $a_{\max}$, and the phase advances σ and σ_0 . We gave explicit results for various truncations of the expansion. The zeroth-order results corresponded to those from the smooth approximation [3]. The first-order results, involving simple correction terms, gave 3 to 10 times improvement in accuracy, e.g., $\sim 1\%$ at $\sigma_0 = 70^\circ$. Our simple, convenient design tools were applied to the design, construction, and testing of ESQ accelerators at LBNL.

K-V Envelope Equations

As in Ref. [2], we consider a non-relativistic beam, although it is easy to generalize to β of order unity. The beam has a uniform current density (K-V distribution) and is transported by a series of linear, symmetric quadrupoles. The paraxial equations for the envelopes a and b are:

$$\begin{aligned} a'' &= -K(z)a + \frac{\epsilon^2}{a^3} + \frac{2Q}{a+b}, \\ b'' &= +K(z)b + \frac{\epsilon^2}{b^3} + \frac{2Q}{a+b}, \end{aligned}$$

where $K(z)$ represents the alternating quadrupole gradient and ϵ the emittance (we assume $\epsilon_x = \epsilon_y$). Q is the normalized perveance, defined nonrelativistically by

$$Q = \frac{I}{4\pi \epsilon_0} \sqrt{\frac{m}{2qV^3}},$$

with m the ion mass, I the beam current, and qV the ion energy. The quadrupole cells have full period length $2L$.

Unmatched Beam: Linearization and Expansion

The above envelope equations have matched period solutions (period $2L$) which we will call $a_0(z)$ and $b_0(z)$. Defining averages $\langle \dots \rangle$ over cell length $2L$, we have $\langle b_0 \rangle = \langle a_0 \rangle = A$, the mean matched radius. A is calculated from the focusing force, emittance, and current using a matching equation. Reference [2] discusses ordering in a small parameter (ϵ), which is proportional to K , and which yields A to third order in ϵ .

We consider small perturbations,

$$a(z) = a_0(z) + x(z), \quad x \ll \langle a_0 \rangle, \quad (1)$$

$$b(z) = b_0(z) + y(z), \quad y \ll \langle b_0 \rangle. \quad (2)$$

substituting forms (1) and (2) into the envelope equations and linearizing in x and y yields

$$x'' = -K(z)x - \frac{3\epsilon^2}{a_0^4}x - \frac{2Q}{(a_0 + b_0)^2}(x + y), \quad (3)$$

$$y'' = +K(z)y - \frac{3\epsilon^2}{b_0^4}y - \frac{2Q}{(a_0 + b_0)^2}(x + y). \quad (4)$$

we write matched solutions in term of A and a small ripple ρ as

$$a_0(z) \cup A[1 + \rho(z)], \quad (5)$$

$$b_0(z) \approx A[1 - \rho(z)]. \quad (6)$$

As in Ref. [2], it is convenient in the following to define the symbol $\iint$ by

$$\iint \Phi \equiv \int_C^z dz' \int_0^{z'} \Phi(z'') dz'',$$

where the lower limit C of the outer integral is chosen so that the average value of the integral over $2L$ vanishes.

In Ref.[2], Eqs. (21-23), we define the leading order ripple

$$\rho_0(z) = -\iint K,$$

and found through the third power of ϵ

$$\rho = +\rho_0 - \frac{3\epsilon^2}{A^4} \iint \rho_0 + \dots$$

We have dropped terms involving the negligible variations in $a_0(z) + b_0(z)$ so that, through order ϵ^3 ,

$$\frac{2Q}{(a_0 + b_0)^2} = \frac{1}{2} \frac{Q}{A^2}. \quad (7)$$

For the other expansions required for (3) and (4) we only need terms up to order ε^2 :

$$\frac{x}{a_0^4} = \frac{x}{A^4} (1 - 4\rho_0 + 10\rho_0^2), \quad (8)$$

$$\frac{y}{b_0^4} = \frac{y}{A^4} (1 + 4\rho_0 + 10\rho_0^2). \quad (9)$$

Substituting (7)-(9) into (3) and (4) and then adding and subtracting, we obtain

$$s'' = - \left(K(z) - \frac{12\varepsilon^2}{A^4} \rho_0(z) \right) d - \frac{3\varepsilon^2}{A^4} (1 + 10\rho_0^2) s - \frac{Q}{A^2} s, \quad (10)$$

$$d'' = - \left(K(z) - \frac{12\varepsilon^2}{A^4} \rho_0(z) \right) s - \frac{3\varepsilon^2}{A^4} (1 + 10\rho_0^2) d, \quad (11)$$

where

$$s = x + y, \quad (12)$$

$$d = x - y, \quad (13)$$

are the sum and difference terms of the envelope oscillations, which correspond approximately to breathing and quadrupole excitations.

Section 2. Ansatz for Solution

We have found from computer studies that for σ_0 and σ less than about 80° , and for σ_0 up to about 180° with σ 's appropriately chosen, the s or d terms can be separately excited, with relatively small amplitude in the other term. In these well-behaved stable cases the excited mode is observed to be fairly sinusoidal. Our goal here is to analyze the frequencies of these quasi-sinusoidal oscillations.

Notice that in Eqs. (10) and (11) the coefficients of the first terms on the right generally have strong oscillations while the coefficients of the other terms are constant, or, in the case of ρ_0^2 , have even harmonics. In order to solve (10) and (11) we make the *ansatz* that the well-behaved modes are driven mainly by the oscillations at the fundamental period so that all harmonics can be neglected. For example, we keep only the fundamental Fourier component of the focusing term:

where

$$K(z) = \sum K_n \cos\left(\frac{n\pi z}{L}\right) \rightarrow K_1 \cos\left(\frac{\pi z}{L}\right), \quad (14)$$

where

$$K_1 = \frac{1}{L} \int_0^{2L} K(z) \cos\left(\frac{\pi z}{L}\right) dz. \quad (15)$$

Thus,

$$\rho_0(z) \equiv - \iint K \rightarrow \kappa \cos\left(\frac{\pi z}{L}\right), \quad (16)$$

where $K \equiv K_1 L^2 / \pi^2$. Then $\rho_0^2 \rightarrow K^2 / 2$ since the second harmonic is dropped. We also use our matching equation

$$\frac{Q}{A^2} = \left\langle \left[\int K \right]^2 \right\rangle - \frac{\Xi^2}{A^4} (1 + 3\langle \rho_0^2 \rangle), \quad (17)$$

from Ref.[2], so the Eqs. (10) and (11) depend only on Ξ and K .

Section 3. Solution for *s* Mode

For the approximately sinusoidal modes that we are analyzing, we put

$$s = \cos\left(\frac{u\pi z}{L}\right),$$

with u to be determined. We will insert this into (11) and integrate twice, noting that the d term on the right is of higher order so that iteration can be applied (App. A). The result is an equation containing $\cos(u\pi z / L)$ in all three terms. For this to be valid for all values of z , the three coefficients must satisfy an algebraic equation (A5). Solving it for u gives

$$u^2 = \left[K^2 + \frac{2\Xi^2 L^2}{\pi^2 A^4} \left(1 + \frac{3}{2} K^2 \right) \right] \left(1 - \frac{3}{2} K^2 \right)^{-1}. \quad (18)$$

In the above approximation,

$$K_{eff} \equiv \left\langle \left[\int K \right]^2 \right\rangle \rightarrow \frac{1}{2} K^2 \frac{\pi^2}{L^2} = \frac{\pi^2}{L^2} \langle \rho_0^2 \rangle, \quad (19)$$

so that K^2 is proportional to $\langle \rho_0^2 \rangle$ and to K_{eff} . We make replacements in Eq. (18) that put our final result in our standard form [1] (see discussion at the end of App. A):

$$\sigma_s^2 = (2L)^2 \left[2K_{eff} + 2\frac{\Xi^2}{A^4} (1 + 3\langle \rho_0^2 \rangle) \right] \left(1 - 3\langle \rho_0^2 \rangle \right)^{-1}, \quad (20)$$

where $\sigma_s + 2\pi u$ is the phase advance for the sum mode. Results from (20), plotted in Fig. 1a, are seen to agree well with exact results, as will be discussed in Section 5.

Section 4. Solutions for *d* Mode

Again using the model stated in Section 2, we put

$$d = \cos\left(\frac{v\pi z}{L}\right).$$

We follow the previous procedure, solve for v , and obtain, with $\sigma_d = 2\pi v$,

$$\sigma_d^2 = (2L)^2 \left[K_{eff} (1 + \langle \rho_0^2 \rangle) + 3\frac{\Xi^2}{A^4} \left(1 + \frac{8}{3} x \langle \rho_0^2 \rangle \right) \right] \left(1 - 3\langle \rho_0^2 \rangle \right)^{-1}. \quad (21)$$

Some results from (21) are plotted in Fig. 1b.

Section 5. Comparison with Smooth Approximations

Figures 1a and 1b show that Eqs. (20) and (21) closely match the exact results. For the last points shown ($\sigma_0 = 179^\circ$, $\sigma = 89^\circ$), we obtain $\sigma_s = 225^\circ$ and $\sigma_d = 199^\circ$ with accuracy of 1.5% and -1.6%, respectively.

Dropping all the $\langle \rho_0^2 \rangle$ correction terms in (20) and (21), we get the smooth approximation

$$\sigma_{s,smooth}^2 = (2L)^2 \left(2K_{eff} + 2\frac{\epsilon^2}{A^4} \right). \quad (22)$$

$$\sigma_{d,smooth}^2 = (2L)^2 \left(K_{eff} + 3\frac{\epsilon^2}{A^4} \right) \quad (23)$$

We can express these zero-order results in terms of zero-order σ_0 and σ . Neglecting our correction terms for σ_0 and σ , [1] gives the smooth approximations

$$\sigma_{0,smooth}^2 = (2L)^2 K_{eff}, \quad (24)$$

$$\sigma_{smooth}^2 = (2L)^2 \frac{\epsilon^2}{A^4}. \quad (25)$$

Equation (24) is given in Ref. [4] (10.92), and equation (25) is in Ref. [1], Eq. (6). We can write to zero order

$$\sigma_{smooth}^2 = 2\sigma_{0,smooth}^2 + 2\sigma_{smooth}^2, \sigma_{d,smooth}^2 = \sigma_{0,smooth}^2 + 3\sigma_{smooth}^2, \quad (26)$$

in agreement with [1].

However, these results, plotted in Figs. 1a and 1b, are in error by as much as 18% and 24%, respectively. Our correction terms in (20) and (21) improve the accuracy by more than an order of magnitude for large σ_0 .

Figure 4 in Ref. [1] shows curves stated to “represent the smooth approximation results” with high accuracy for $\sigma_0 \leq 90^\circ$. Actually, those curves were obtained from exact numerical values of σ_0 and σ , not from smooth approximations – which would be (24) and (25) in our notation. Reference [3] plots what we will call hybrid values:

$$\sigma_{s,hybrid}^2 = 2\sigma_{0,exact}^2 + 2\sigma_{exact}^2, \sigma_{d,hybrid}^2 = \sigma_{0,exact}^2 + 3\sigma_{exact}^2. \quad (27)$$

We use the term “hybrid” because these are not exact formulas even though they employ exact numerical values for σ_0 and σ . Eqs. (27) are useful if one already knows those exact values, but if the data is in the form of quadrupole voltage, emittance, and current (or line charge), one could use (22) and (23). Much better accuracy is obtained from our Eqs. (20) and (21). The three alternatives are compared in Fig. 2, with $\sigma_0 = 83^\circ$, and in Fig. 3, where $\sigma_0 = 112^\circ$.

One sees that results from the smooth-approximation formulas do not come close to the exact results anywhere. Our formulas give good accuracy for all values of σ if σ_0 is less than about 80° . For σ_0 larger than 90° there is an unstable region. This is seen in Fig. 3 for $44.5^\circ < \sigma < 86.3^\circ$. Outside of the unstable region there are regions where the mismatch oscillations are well behaved and satisfy our model (Fig. 4a) and regions where the opposite is the case (Fig. 4b). Examining Fig. 3a, we notice that for σ around 89° , where the regular oscillations satisfy our ansatz, our formulas give very good accuracy for σ_s and σ_d . For $\sigma = 109^\circ$, with poorly behaved oscillations, the results are less accurate but still a big improvement over the smooth approximation (Fig. 3b). The hybrid formula, of course, approaches perfection at the far right of Fig. 3c, where space charge vanishes, since the exact σ_0 is given in advance.

Conclusions

In all cases our formulas improve over the smooth approximation. In the example where $\sigma_0 = 83^\circ$, the improvement is a factor of 5 for any value of σ .

For large σ_0 such as 112° , where there are unstable or irregular oscillations, the improvement is small except in a limited region of large σ . This region where oscillations are decoupled and regular may be of limited practical interest, but the good accuracy of our formulas there seems to vindicate our ansatz.

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Appendix A

Here we give details for the solution of Eqs. (10) and (11) in the main text. We use the abbreviations

$$K_1 \frac{L^2}{\pi^2} \equiv \kappa, \quad \frac{\pi z}{L} \equiv \theta, \quad \beta \equiv \frac{3 \epsilon^2 L^2}{\pi^2 A^4}.$$

Then

$$\rho_0(z) \rightarrow \kappa \cos \theta,$$

and

$$\rho_0^2 = \frac{1}{2} \kappa^2 (1 + \cos 2\theta) \rightarrow \frac{1}{2} \kappa^2 = \langle \rho_0^2 \rangle,$$

since we ignore harmonics. Equation (17) in the main text becomes

$$\frac{Q}{A^2} = \frac{1}{2} \kappa^2 \frac{\pi^2}{L^2} - \frac{\epsilon^2}{A^4} \left(1 + \frac{3}{2} \kappa^2 \right).$$

Changing the independent variable from z to θ in Eqs. (10) and (11), we have

$$s_{\theta\theta} = -\kappa(1 - 4\beta)\cos(\theta)d - \left(\beta + 5\beta\kappa^2 + \frac{1}{2}\kappa^2 - \frac{1}{3}\beta - \frac{1}{2}\beta\kappa^2 \right)s,$$

or

$$s_{\theta\theta} = -\kappa(1 - 4\beta)\cos(\theta)d - \left(\frac{2}{3}\beta + \frac{1}{2}\kappa^2 + \frac{9}{2}\beta\kappa^2 \right)s, \quad (\text{A1})$$

$$d_{\theta\theta} = -\kappa(1 - 4\beta)\cos(\theta)s - \beta(1 + 5\kappa^2)d. \quad (\text{A2})$$

Everything now depends on κ and β , which represent the focusing strength and emittance pressure, and are therefore closely related to σ_0 and σ .

Solution for s Mode

For our quasi-sinusoidal modes, we put

$$s = \cos(u\theta), \quad (\text{A3})$$

insert this into (A2) and integrate twice, using the zero-average operator $\iint$ define in the main text, except that here we replace the variable z by θ :

$$\iint \cos(\theta) \cos(u\theta) = -\frac{1+u^2}{(1-u^2)^2} \cos \theta \cos u\theta - \frac{2u}{(1-u^2)^2} \sin(\theta) \sin(u\theta).$$

We assume that the period of envelope oscillations is longer than the focusing period, expand the fractions, and drop higher powers of u . Further, we will get

$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta)$, a second harmonic which we have agreed to omit. We drop the sine term now to save writing:

$$\iint \cos(\theta) \cos(u\theta) \rightarrow -(1 + 3u^2) \cos(\theta) \cos(u\theta).$$

Then the lowest approximation to (A2) is

$$d_{(0)} = +\kappa(1 - 4\beta)(1 + 3u^2) \cos(\theta) \cos(u\theta).$$

In the following iteration, we consider β , K^2 and u^2 to be $\sim \varepsilon^2$ and drop terms which would produce coefficients in (A2) higher than ε^4 . Then

$$\begin{aligned} d_{(1)} &= d_{(0)} - \beta \iint d_{(0)} = d_{(0)} + \beta \kappa (1 - 4\beta) \cos(\theta) \cos(u\theta) \\ &= \kappa (1 - 4\beta) (1 + \beta + 3u^2) \cos(\theta) \cos(u\theta). \end{aligned}$$

or

$$d_{(1)} = \kappa (1 - 3\beta + 3u^2) \cos(\theta) \cos(u\theta). \quad (\text{A4})$$

Putting (A4) into (A1) gives

$$s_{\theta\theta} = -\kappa^2 (1 - 7\beta + 3u^2) \cos^2(\theta) \cos(u\theta) \left(\frac{2}{3}\beta + \frac{1}{2}\kappa^2 + \frac{9}{2}\beta\kappa^2 \right) \cos(u\theta).$$

We note that $\cos 2\theta = [2 \cos^2(\theta) - 1]$, where we discard the second harmonic, and that by Eq. (A3), $s_{\theta\theta} = -u^2 \cos(u\theta)$. Then, if the above equation is to be valid for all θ .

$$u^2 - \frac{1}{2}\kappa^2 (1 - 7\beta + 3u^2) + \frac{2}{3}\beta + \frac{1}{2}\kappa^2 + \frac{1}{2}\beta\kappa^2. \quad (\text{A5})$$

Collecting terms,

$$u^2 = \kappa^2 \left(1 + \frac{3}{2}u^2 \right) + \frac{2}{3}\beta \left(1 + \frac{3}{2}\kappa^2 \right).$$

Solving for u^2 ,

$$u^2 = \left[\kappa^2 + \frac{2}{3}\beta \left(1 + \frac{3}{2}\kappa^2 \right) \right] \left(1 - \frac{3}{2}\kappa^2 \right),$$

which is Eq. (18) in the main text.

Solution for d Mode

Using the model stated in Section 2, we put

$$d = \cos(v\theta).$$

We follow the previous procedure, dropping $\sin(\theta)\sin(v\theta)$ terms and keeping only the appropriate powers of ε :

$$\begin{aligned} s_{(0)} &= \kappa (1 - 4\beta) (1 + 3v^2) \cos(\theta) \cos(v\theta), \\ s_{(1)} &= s_{(0)} - \left(\frac{2}{3}\beta + \frac{1}{2}\kappa^2 \right) \iint s_{(0)} = s_{(0)} + \left(\frac{2}{3}\beta + \frac{1}{2}\kappa^2 \right) \kappa (1 - 4\beta) \iint \cos(\theta) \cos(v\theta), \\ &= \kappa \left(1 - 4\beta \left(1 + 3v^2 + \frac{2}{3}\beta + \frac{1}{2}\kappa^2 \right) \right) \cos(\theta) \cos(v\theta), \end{aligned}$$

or

$$s_{(1)} = \kappa \left(1 - \frac{10}{3}\beta + 3v^2 + \frac{1}{2}\kappa^2 \right) \cos(\theta) \cos(v\theta).$$

Eq. (A2) gives

$$d_{\theta\theta} = -\kappa^2 \left(1 - \frac{22}{3}\beta + 3v^2 + \frac{1}{2}\kappa^2 \right) \cos^2(\theta) \cos(v\theta) (\beta + 5\beta\kappa^2) \cos v\theta;$$

then

$$\begin{aligned}
v^2 \frac{1}{2} \kappa^2 \left(1 + 3v^2 + \frac{1}{2} \kappa^2 \right) + \beta + \frac{4}{3} \beta \kappa^2, \\
v^2 \left(1 - \frac{3}{2} \kappa^2 \right) = \frac{1}{2} \kappa^2 \left(1 + \frac{1}{2} \kappa^2 \right) + \beta \left(1 + \frac{4}{3} \kappa^2 \right). \tag{A6}
\end{aligned}$$

Solving for v^2 , writing $\sigma_d = 2\pi v$, and making substitutions as before, we get Eq. (21). As mentioned under Eq. (19) in the main text, our abbreviation κ^2 is proportional $\langle \rho_0^2 \rangle$ and also to K_{eff} . We made the choices that put (A5) and (A6) into the forms of Eqs. (20) and (21) in the main text. These are analogous to Eqs. (32) and (33) in Ref. 2. All of these equations involve basic quantities from the smooth approximation, modified by correction terms involving the mean-squared zero-order ripple $\langle \rho_0^2 \rangle$.

In the main text, Section 2-4, these correction terms were derived with the temporary help of a Fourier approximation, Eq. (14). For consistency with out other work, we revert to the basic definitions $\rho_0(z) \equiv -\iint K$ and $K_{eff} + \langle \left[\int K \right]^2 \rangle$ in applying Eqs. (20) and (21). (Actually, the difference is usually negligible.) These basic definitions were used, for example, in producing Fig. 2 and 3.

Fig 1. Mismatch phase advances: (a) σ_s with $\delta a = \delta b$ (even mode); (b) σ_d with $\delta a = \delta b$ (odd mode), showing smooth approximation [3] vs. our method. Quadrupole lattices parameters from Ref. [2]. Voltage ranged between 5.0 and 30.2kV so that $19.2^\circ < \sigma_0 < 179.1^\circ$. The measured σ_s or σ_d were obtained by integrating the K-V envelope equations and counting mismatched oscillations (see Fig. 4) over 20 full lattice periods. For each σ_0 up to about 80° , our formulae were applied over a wide range of σ gave modes not well-decoupled, regular, or

Fig. 2. Envelope-oscillation phase advances σ_s and σ_d vs. $V_Q = 20kV (\sigma_0 = 83^\circ)$. Lattice parameters as in Table 1, Ref. [2]. The exact frequencies were found by numerical integrations followed by eigenvalue analysis. (a): Our methods [Eqs. (32) and (36)], is accurate to about 2% over the whole range of σ . (b): Smooth approximation [Eq. (40)] is in error by about 10%. (c): Hybrid formula, used in Ref. [1], requires advance knowledge of exact values of σ and σ_0 .

Fig. 3 Envelope-oscillation phase advances σ_s and σ_d vs. σ for $V_Q = 25kV (\sigma_0 = 112^\circ)$. Lattice as in Table 1, Ref. [1]. Numerical integrations followed by eigenvalue analysis yielded the exact frequencies, which coincide over the unstable region. (a): Our methods [Eqs. (20) and (21)], give good accuracy somewhat to the right of the unstable region, where decoupled sinusoidal oscillations exist. (b): Smooth approximation [Eq. (40)] shows poor accuracy everywhere. (c): Hybrid formula, using exact numerical results for σ and σ_0 , agrees with exact curves at the extreme right, where space change vanishes.

Fig. 4. (a) and (b). Envelope perturbations: sum $x+y$ (solid line) and difference $x-y$ (dotted line) for initial perturbations $x(0) = y(0) = 1.0 \times 10^{-5}$ cm. Standard lattice as in Table 1, Ref. [1]. For both (a) and (b), $V_Q = 25kV$, so that $\sigma_0 = 112^\circ$. (a): with $\sigma = 89^\circ$, sum and difference modes are well decoupled (note that the dotted line has tiny amplitude) and Eqs. (20) and (21) give accurate results, confirmed in Fig. 3. (b): with $\sigma = 109^\circ$, the strong coupling between modes violated the model in Section 2; Fig. 3 shows decreased accuracy there.