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On the detection of a nonlinear damage in an uncertain nonlinear beam using stochastic Volterra series

Luis GG Villani[1], Samuel da Silva[1], Americo Cunha[2] and Michael D Todd[3]

Abstract

In the present work, two issues that can complicate a damage detection process are considered: the *uncertainties* and the *intrinsically nonlinear behavior.* To deal with these issues, a *stochastic version* of the Volterra series is proposed as a baseline model, and novelty detection is applied to distinguish the condition of the structure between a reference baseline state (presumed "healthy") and damaged. The studied system exhibits nonlinear behavior even in the reference condition, and it is exposed to a type of damage that causes the structure to display a nonlinear behavior with a different nature than the initial one. In addition, the uncertainties associated with data variation are taken into account in the application of the methodology. The results confirm that the monitoring of nonlinear coefficients and nonlinear components of the system response enables the method to detect the presence of the damage earlier than the application of some linear-based metrics. Besides that, the stochastic treatment enables the specification of a probabilistic interval of confidence for the system response in an uncertain ambient, thus providing more robust and reliable forecasts.

Keywords

Uncertainties, damage detection, stochastic Volterra model, nonlinear behavior

Introduction

Structural health monitoring (SHM) techniques aim to reduce the maintenance cost and increase the reliability and security of aerospace, civil, or mechanical engineering structures.[1] Moreover, within the hierarchy of complexity that SHM methodologies may achieve, damage *detection* is the first step, and its performance is fundamental to the success of the subsequent application of higher forms of SHM in the hierarchy.[2] In this sense, many authors have studied and developed various damage detection techniques to be implemented for different structures and applications.[2–7] Of course, there is no general approach that can be used to detect damage in all real systems, mainly when the intrinsically nonlinear behavior of many systems[8] and the data variation related to uncertainties[9,10] are counted in the analysis.

Otherwise, many linear structures can exhibit nonlinear phenomena induced by the presence of damage, and in this situation, any form of nonlinearity detector is akin to detecting damage.[11] Such damage that produces nonlinear behavior, for example, delamination,[12,13] rubbing and unbalance in rotor systems,[14–16] and opening cracks,[17–19] may be detected through the observation of nonlinear behavior in the measured responses. However, as mentioned before, many structures fundamentally present nonlinear behavior even in the reference condition,[20] causing confusion in the nonlinearity (as a proxy for damage) detection process.[8] In addition, systems' measured output can exhibit data variability from sources such as environmental or input load variation, aleatoric noise, changes in boundary conditions, variations in the fabrication processes (i.e.

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unit-to-unit variability), and others.[21–23] These variations all confound the damage detection process, suggesting the use of probabilistic tools,[24–26] regression models,[5,27] machine learning algorithms,[28] probabilistic model selection approaches,[29–32] outlier analysis,[33,34] and novelty detection methods.[5] Considering the data variation problem is important in reducing the number of false alarms,[10,35] although there are situations where variability can mask positive detections as well.

Villani et al.[36] introduced an extension of the deterministic Volterra series approach used by Shiki et al.[37] to detect damage in intrinsically nonlinear systems based on input/output measurements, recognizing data variation. The authors examined the identification of the nonlinear data-driven model several times, using Monte Carlo simulations, to create a stochastic reference model capable of predicting the nonlinear performance and the fluctuations in the response at the same time. Two main strategies were suggested to detect the presence of a crack in an initially nonlinear beam, considering simulated data, based on the random kernel's contributions and random kernel's coefficients. The results presented an adequate performance in distinguishing the intrinsically nonlinear behavior and the data variation from the nonlinearity caused by the damage. Although the simulated results showed a promising performance of the method, experimental practice suggested the use of a stochastic Volterra expansion to explore the model space, particularly in the more challenging present problem of inherent nonlinearity, rather than induced nonlinearity. The authors also hypothesized that a formulation using the kernels' coefficients and contributions concomitantly in the same index could be attractive to enhance the robustness of the method, leaving the method to detect damage with diverse characteristics without loss of performance. In addition, in the simulations performed, only the variation of linear parameters was considered in the data variation scenario, a simplification that might not reflect the behavior of structures that operate in the nonlinear regime of motion under the presence of uncertainties. This simplification caused a low variation of the high-order kernel coefficients, suggesting that a more realistic application encompassing the variation of the nonlinear components could complicate the application of the approach.

Then, in Villani et al.,[38] the authors showed the experimental application of the approach based on Volterra kernels' contributions to detect damage in an intrinsically nonlinear beam, considering data variation related to the reassembly of the experimental setup. However, the damage simulated was associated with the loss of mass that reflected in the variation of the natural frequencies of the equivalent linear system. In this situation, the damage did not have an influence in the estimation of kernels' coefficients. These results pointed to the demand for developing a hybrid approach capable of making use of the kernels' contributions and coefficients together in a robust index. Moreover, an experimental application considering damage with direct influence on the nonlinear components of the response could improve the performance of the approach with regard to the differentiation between the intrinsically nonlinear behavior and damage-induced nonlinear behavior.

Hence, this article aims to cover issues that were not yet considered in both previous works: (a) an experimental application of the stochastic Volterra series methodology with the presence of damage that produces nonlinear behavior to the system—a breathing crack emulation; (b) the observance of variation in the intrinsic nonlinearity of the structure including the data variability, thus not only considering its realization in the linear components; (c) the generalization of the approaches studied before with the development of a new hybrid method that considers both the kernels' coefficients and contributions simultaneously in the damage index; and, finally, (d) the construction of a theoretical distribution of the damage index calculated in the reference condition to reduce the number of experimental realizations needed to estimate the threshold value based on the kernel density method used before (a practical problem when we consider real-world experiments). To the best of the authors' knowledge, this is the first article that assumes nonlinear changes associated with an experimental mechanism of damage with the assumption of reference already nonlinear, but unlike Bornn et al.,[8] this paper considers the inherent uncertainties in the experimental setup to perform a rigorous stochastic SHM method.

In this context, an intrinsically nonlinear beam is analyzed, with natural data variation in the full experimental context, to investigate the performance of the proposed methodology. The variation simulated in the data reflects merged changes both in linear and nonlinear components of the system response. Furthermore, the novelty detection is reshaped to hold the random kernels' contributions and coefficients in the same damage index, using principal component analysis (PCA) and Mahalanobis distance metrics. A formal hypothesis test is presented to create a more robust damage detection methodology, based on a theoretical distribution for the Mahalanobis distance determined in the reference condition of the structure. The results exposed in this work have demonstrated the beneficial performance of a nonlinear metric to detect damage in this situation and the capability of the stochastic Volterra series to predict the data variation in a probabilistic framework, improving the statistic confidence of the method.

To be familiar with the Volterra series model application in damage detection problems, find the motivation to the study of the nonlinear phenomena in the procedure and have more details about the Volterra series reformulation to predict the nonlinear responses considering data variation, the interested reader is referred to seeing Shiki et al.,[37] Villani et al.,[36] and Villani et al.[38] This article fundamentally differentiates from previous works by considering an initially uncertain nonlinear system subject to damage that itself induces nonlinear behavior in the structural response, which hasn't been considered before.

The present paper is organized as follows. Section "The damage detection methodology based on stochastic Volterra series" describes the stochastic mathematical model used to describe the nonlinear systems response and the approach proposed to detect the damage considering the data variation related to uncertainties. Section "Experimental setup" shows the nonlinear structure considered in this work, the damage simulated, and the main characteristics of its behavior. Section "Application of the proposed methodology" shows the application of the methodology proposed and the main results obtained. Finally, section "Final remarks" presents the conclusions of the work.

The damage detection methodology based on stochastic Volterra series

This section outlines the methodology proposed to be practiced in the damage detection problem in initially nonlinear systems, taking into account the data variation related to uncertainties. The development of the stochastic model is briefly described, with more detail in Villani et al.[36] In addition, the reader can obtain more information about the deterministic Volterra series expanded using the Kautz functions in Shiki et al.[37]

The stochastic version of the Volterra series

In order to take into account the uncertainties in the model formulation, a parametric probabilistic approach is assumed. Thus, the model parameters are assumed as random parameters and the model response as a random process.[21–23] Therefore, a probability space $(\Theta, \mathbb{\Sigma}, \mathbb{P})$ is considered, where Θ represents the sample space, $\mathbb{\Sigma}$ is a σ-algebra over Θ, and $\mathbb{P}$ is a probability measure.[36]

In the discrete-time domain, assuming the presence of uncertainties, single system output can be interpreted as a random process realization that is a consequence of a single deterministic input. The relationship between the deterministic input and the random output can be described, using the convolution notion,[39] through the stochastic version of the Volterra series

$$\mathbb{y}(\theta,k) = \sum_{\eta=1}^{\infty}\sum_{n_1=0}^{N_1-1}\cdots \sum_{n_\eta=0}^{N_\eta-1}\mathbb{H}_\eta(\theta,n_1,\ldots,n_\eta)\prod_{i=1}^{\eta}u(k-n_i) \tag{1}$$

where $(\theta,k)\in\Theta\times\mathbb{Z}_{+}\mapsto\mathbb{y}(\theta,k)$ represents the single random output that is consequence of the single deterministic input $k\in\mathbb{Z}_{+}\mapsto u(k)$, $(\theta,n_1,..,n_\eta)\in\Theta\times\mathbb{Z}^{\eta}\mapsto \mathbb{H}_\eta(\theta,n_1,\ldots,n_\eta)$ represents the random version of the η-order Volterra kernel, and $\mathbb{Z}_{+}$ represents the set of integer positive numbers.

The principal benefit of the Volterra series model is the capability to reproduce the system output as a sum of linear and nonlinear contributions

$$\begin{aligned}\mathbb{y}(\theta,k) &= \sum_{\eta=1}^{\infty}\mathbb{y}_\eta(\theta,k) = \\ &= \underbrace{\mathbb{y}_1(\theta,k)}_{linear} + \underbrace{\mathbb{y}_2(\theta,k)+\mathbb{y}_3(\theta,k)+\cdots}_{nonlinear}\end{aligned} \tag{2}$$

where $(\theta,k)\in\Theta\times\mathbb{Z}_{+}\mapsto\mathbb{y}_1(\theta,k)$ is the random output obtained using the first random kernel, $(\theta,k)\in\Theta\times\mathbb{Z}_{+}\mapsto\mathbb{y}_2(\theta,k)$ is the random output obtained using the second random kernel, and so on. In this work, the series will be truncated in the third-order kernel because of the cubic characteristic of the nonlinear system response investigated, and the capability to separate linear and nonlinear contributions in the total response will be used in the damage detection procedure as feature sensitive to the presence of damage.

On the other hand, as broadly addressed in previous works,[36,37,40] the central disadvantage of the approach is the challenge in achieving the convergence when a high number of terms are used. To solve this problem, the Volterra series can be extended utilizing the Kautz functions,[41,42] and the system random output can be represented as

$$\mathbb{y}(\theta,k)\approx \sum_{\eta=1}^{\infty}\sum_{i_1=1}^{J_1}\cdots \sum_{i_\eta=1}^{J_\eta}\mathbb{B}_\eta\left(\theta,i_1,\ldots,i_\eta\right)\prod_{j=1}^{\eta}\mathbb{l}_{\eta,i_j}(\theta,k) \tag{3}$$

where $J_1,\ldots,J_\eta$ represents the number of Kautz functions used in the kernels projections, the η-order random Volterra kernel expanded in the orthonormal basis is represented by the random process $(\theta,i_1,\ldots,i_\eta)\in\Theta\times\mathbb{Z}_{+}^{\eta}\mapsto\mathbb{B}_\eta(\theta,i_1,\ldots,i_\eta)$, and the random process $(\theta,k)\in\Theta\times\mathbb{Z}_{+}\mapsto l_{i_j}(\theta,k)$ is a filtering of the deterministic input signal by the random Kautz functions. The Kautz functions are supposed random because their definition depends on the dynamics of the system and, as the system response is considered as a random process, it is presumed that the Kautz functions will also randomly change.

Conclusively, the coefficients of the kernels can be calculated using the least-squares approximation in a deterministic way,[37] and then, adopting Monte Carlo simulations, the process is repeated until the stochastic model converges. The Monte Carlo method was chosen because it is easier to perform when the deterministic algorithm is known.[23,43] More information about the random Kautz functions and the process of the random Volterra kernels estimation may be found in Villani et al.[36,38]

Damage detection based on novelty detection

In the previous work, two points were analyzed separately in the damage detection process: the kernel's coefficients and the kernel's contributions. Nevertheless, it is difficult for practical applications to decide which feature is better. Thus, the damage detection index used here examines both features jointly. Regarding the stochastic model identified with training data in the reference condition, the damage-sensitive index may be determined in the reference status. First of all, recognizing that the Volterra series will be truncated in the third-order kernel, the kernel's coefficients can be allocated together to be used as damage-sensitive feature

$$\begin{aligned}
\mathbb{\lambda}_{lin}(\theta, i_1) &= \begin{Bmatrix} \mathbb{B}_1(\theta,1) \\ \mathbb{B}_1(\theta,2) \\ \vdots \\ \mathbb{B}_1(\theta,J_1) \end{Bmatrix} \\
\mathbb{\lambda}_{qua}(\theta, i_1 = i_2) &= \begin{Bmatrix} \mathbb{B}_2(\theta,1,1) \\ \mathbb{B}_2(\theta,2,2) \\ \vdots \\ \mathbb{B}_2(\theta,J_2,J_2) \end{Bmatrix} \\
\mathbb{\lambda}_{cub}(\theta, i_1 = i_2 = i_3) &= \begin{Bmatrix} \mathbb{B}_3(\theta,1,1,1) \\ \mathbb{B}_3(\theta,2,2,2) \\ \vdots \\ \mathbb{B}_3(\theta,J_3,J_3,J_3) \end{Bmatrix}
\end{aligned} \tag{4}$$

where the random process $(\theta, i_1) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{\lambda}_{lin}(\theta, i_1)$ represents the coefficients of the first kernel, $(\theta, i_1 = i_2) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{\lambda}_{qua}(\theta, i_1 = i_2)$ represents the coefficients of the diagonal of the second kernel, and $(\theta, i_1 = i_2 = i_3) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{\lambda}_{cub}(\theta, i_1 = i_2 = i_3)$ represents the coefficients of the main diagonal of the third kernel.

In addition, the contribution of the kernels can be calculated as

$$\begin{aligned}
\mathbb{y}_{lin}(\theta,k) &\approx \sum_{i_1=1}^{J_1} \mathbb{B}_1(\theta,i_1)\,\mathbb{l}_{1,i_1}(\theta,k) \\
\mathbb{y}_{qua}(\theta,k) &\approx \sum_{i_1=1}^{J_2}\sum_{i_2=1}^{J_2} \mathbb{B}_2(\theta,i_1,i_2)\,\mathbb{l}_{2,i_1}(\theta,k)\,\mathbb{l}_{2,i_2}(\theta,k) \\
\mathbb{y}_{cub}(\theta,k) &\approx \sum_{i_1=1}^{J_3}\sum_{i_2=1}^{J_3}\sum_{i_3=1}^{J_3} \mathbb{B}_3(\theta,i_1,i_2,i_3)\ldots \\
&\ldots\ \mathbb{l}_{3,i_1}(\theta,k)\,\mathbb{l}_{3,i_2}(\theta,k)\,\mathbb{l}_{3,i_3}(\theta,k)
\end{aligned} \tag{5}$$

where $(\theta, k) \in \Theta \times \mathbb{Z}_+ \mapsto y_{lin}(\theta, k)$ is the linear contribution, $(\theta, k) \in \Theta \times \mathbb{Z}_+ \mapsto y_{qua}(\theta, k)$ is the quadratic contribution, and $(\theta, k) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{y}_{cub}(\theta, k)$ is the cubic contribution. Therefore, to reduce the order of the classification problem, the PCA[26,44–46] can be applied to the kernel contributions

$$\begin{aligned}
\mathbb{y}_{lin}(\theta,k) &\gg \text{PCA} \gg \mathbb{C}_{lin}(\theta,1),\ \ldots,\mathbb{C}_{lin}(\theta,n_{pca}) \\
\mathbb{y}_{qua}(\theta,k) &\gg \text{PCA} \gg \mathbb{C}_{qua}(\theta,1),\ \ldots,\mathbb{C}_{qua}(\theta,n_{pca}) \\
\mathbb{y}_{cub}(\theta,k) &\gg \text{PCA} \gg \mathbb{C}_{cub}(\theta,1),\ \ldots,\mathbb{C}_{cub}(\theta,n_{pca})
\end{aligned} \tag{6}$$

where $(\theta, n_{pca}) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{C}_{lin}(\theta, n_{pca})$ represents the principal components of the linear contribution, $(\theta, n_{pca}) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{C}_{qua}(\theta, n_{pca})$ represents the principal components of the quadratic contribution, $(\theta, n_{pca}) \in \Theta \times \mathbb{Z}_+ \mapsto \mathbb{C}_{cub}(\theta, n_{pca})$ represents the principal components of the cubic contribution, and n_{pca} is the number of principal components considered. The number of components was defined based on the contribution of each component in the construction of the covariance matrix.[44]

After the calculation of the kernel's coefficients and the principal components of the kernel's contributions, the damage index can be defined in the reference condition

$$\begin{aligned}
\mathbb{I}_{lin} &= [\mathbb{\lambda}_{lin}(\theta,i_1)\ \ \mathbb{C}_{lin}(\theta,n_{pca})]_{(N_s \times (J_1+n_{pca}))} \\
\mathbb{I}_{nlin} &= [\mathbb{\lambda}_{qua}(\theta,i_1=i_2)\ \ \mathbb{\lambda}_{cub}(\theta,i_1=i_2=i_3)\ \ldots \\
&\ldots\ \mathbb{C}_{qua}(\theta,n_{pca})\ \ \mathbb{C}_{cub}(\theta,n_{pca})]_{(N_s \times (J_2+J_3+2n_{pca}))}
\end{aligned} \tag{7}$$

where $\mathbb{I}_{lin}$ and $\mathbb{I}_{nlin}$ are the linear and nonlinear indices in the reference situation, respectively. The linear and nonlinear indices will be assessed with the aim of comparison between the linear and nonlinear methodologies performance. Therefore, in the reference condition, $\mathbb{I}_{lin}$ is a $N_s \times (J_1 + n_{pca})$ matrix and $\mathbb{I}_{nlin}$ is a $N_s \times (J_2 + J_3 + 2n_{pca})$ matrix, being N_s the number of observations used in the training phase of the reference stochastic model.

Now, with the structure in an unknown ("test") situation, a new deterministic model can be identified, and the indices may be estimated in the unknown status

$$\begin{aligned}\mathcal{I}_{lin} &= [\lambda_{lin}(i_1)\ \ \mathcal{C}_{lin}(n_{pca})]_{(1\times(J_1+n_{pca}))}\\ \mathcal{I}_{nlin} &= [\lambda_{qua}(i_1=i_2)\ \ \lambda_{cub}(i_1=i_2=i_3)\ \ldots\\ &\ldots\ \mathcal{C}_{qua}(n_{pca})\ \ \mathcal{C}_{cub}(n_{pca})]_{(1\times(J_2+J_3+2n_{pca}))}\end{aligned} \tag{8}$$

where $\mathcal{I}_{lin}$ and $\mathcal{I}_{nlin}$ are, respectively, the linear and nonlinear indices in an unknown condition. As the indices calculated in the reference condition are matrices composed by more than one single feature, the novelty detection has to be performed considering multivariate data. Therefore, the comparison between the indices calculated in the reference and an unknown condition can be made considering $\mathcal{D}^2$ Mahalanobis distance[33,47]

$$\mathcal{D}_m^2 = [\mathcal{I}_m - \mu_{\mathbb{I}_m}]^T\, \Sigma_{\mathbb{I}_m}^{-1}\, [\mathcal{I}_m - \mu_{\mathbb{I}_m}] \tag{9}$$

where $m = lin$ or $m = nlin$; $\mu_{\mathbb{I}_m}$ and $\Sigma_{\mathbb{I}_m}$ are, respectively, the mean vector and the covariance matrix of the index calculated in the reference condition. The simple machine learning method based on Mahalanobis distance is used here because the classification is done between two possible conditions (healthy and damaged) and the goal is to examine the performance of the Volterra kernel characteristics as damage-sensitive features and not to investigate the differences between refined classification methods. Other classification methods can be used in the future to improve the methodology depending on the real application confronted.

With the squared Mahalanobis distance calculated, a hypothesis test may be proposed. In this work, this distance calculated in the reference condition is modeled with a chi-square distribution, which may be calculated based on the assumption of independence and normality in the underlying multivariate features from which the squared Mahalanobis distance is calculated.[48,49] Ideally, *sampling* distribution of the Mahalanobis distance is desirable, but no such analytical form is known to exist, so the theoretical model is fit to the (limited) empirical data obtained. This approximation has satisfactory performance, as will be shown further along. Therefore, the hypothesis is proposed

$$\begin{cases} H_0 : \mathcal{D}_m^2 \sim \mathcal{X}^2 \\ H_1 : \mathcal{D}_m^2 \nsim \mathcal{X}^2 \end{cases} \tag{10}$$

where $\mathcal{X}^2$ is the chi-square distribution, H_0 is the null-hypothesis (healthy condition) and H_1 is the alternative hypothesis (damaged condition).

Besides, the probability of a distance value calculated to be included in the theoretical chi-square distribution can be computed integrating its probability density function (PDF)[50]

$$p_m = F(\mathcal{D}_m^2|\nu) = \int_0^{\mathcal{D}_m^2} \frac{t^{(\nu-2)/2} e^{-t/2}}{2^{\nu/2}\Gamma(\nu/2)} dt \tag{11}$$

where p_m is the probability of the value $\mathcal{D}_m^2$ belonging to the chi-square distribution, $\Gamma(.)$ is the Gamma function, and ν is the number of degrees of freedom. Finally, a sensitivity value can be determined depending on the application and probability of false alarms tolerated, and the hypothesis test can be rewritten

$$\begin{cases} H_0 : p_m \geqslant \beta \\ H_1 : p_m < \beta \end{cases} \tag{12}$$

where β represents the sensitivity chosen for the hypothesis test. The definition of this parameter depends on the practical application and, as an experimental laboratory setup is utilized, several values will be examined to study the performance of the method. Figure 1 shows a flowchart of the damage detection approach. On the left-hand side of Figure 1, the training phase is observed, with the identification of the stochastic reference model and the estimation of the damage indices in the reference condition. On the right-hand side of Figure 1, the identification of a new model in an unknown status and the calculation of the new indices are represented. Then, the indices are correlated using the squared Mahalanobis distance, and finally, the hypothesis test is applied to classify the condition of the structure between healthy and damaged.

Experimental setup

The experimental setup used is presented in Figure 2. The structure monitored is formed by a clamped-free beam, that is constructed by gluing four thin beams of Lexan together, 2.4×24×240 (mm^3) each one, with the intention of emulating a damage propagation that is described further on. At the free boundary, two steel masses are affixed and interact with a magnet, generating a nonlinear behavior in the system response, even in the reference condition due to added magnetic potential. Moreover, the setup includes the following:

- A National Instruments acquisition system:

  CompactDAQ Chassis (NI cDAQ-9178);
  A C Series Sound and Vibration Input Modules (NI-9234);
  A C Series Voltage Output Module (NI-9263).

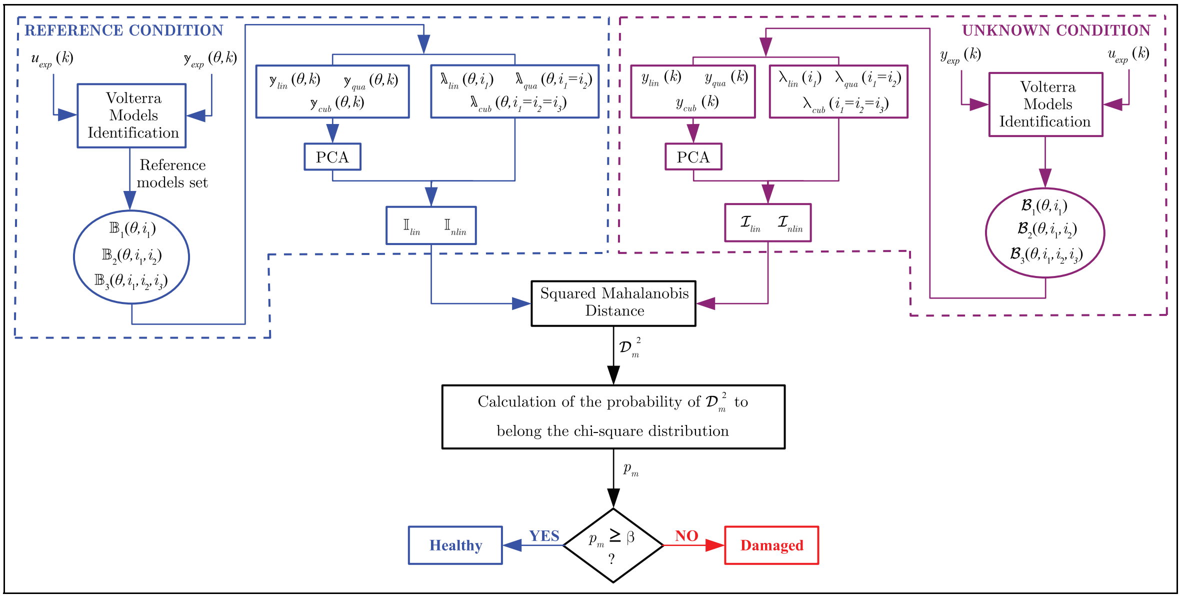

Figure 1. Flowchart of the damage detection methodology proposed.

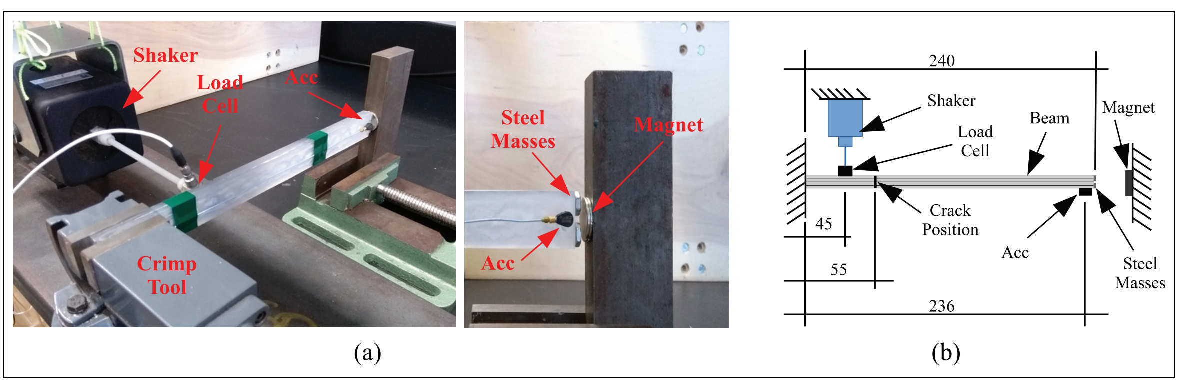

Figure 2. Experimental apparatus: (a) photos and (b) scheme (dimensions in millimeters).

- Electrodynamic Transducer Labworks Inc. (ET-132);
- Amplifier MB Dynamics (SL500VCF);
- Load cell PCB PIEZOTRONICS (208C02);
- Accelerometer PCB PIEZOTRONICS (352C22).

The electrodynamic transducer is employed to excite the structure with different signals and considering two levels of input (low—1 V root mean square (RMS) and high—6 V RMS). The output data are measured by the accelerometer positioned close to the free extremity of the beam, because the authors are only interested in the region of the first mode shape of the structure (see Figure 2). The input signal analyzed is the voltage signal applied in the electrodynamic transducer. As a single-input/single-output (SISO) model is considered, this pair of signals is enough to identify the Volterra kernels and monitor the structure health. All the acquisition parameters, signals considered, and equipment used were the same in the experiments performed considering the different structural conditions.

Intrinsically nonlinear behavior

The mechanical system used exhibits nonlinear operation even in the reference condition, without the presence of damage. Figure 3(a) shows the results obtained during the stepped sine test applied considering two levels of input. When the input applied has a low level of amplitude (1 V RMS), the output signal shows linear characteristics for both up-sweep and down-sweep inputs. However, when the input signal is at a

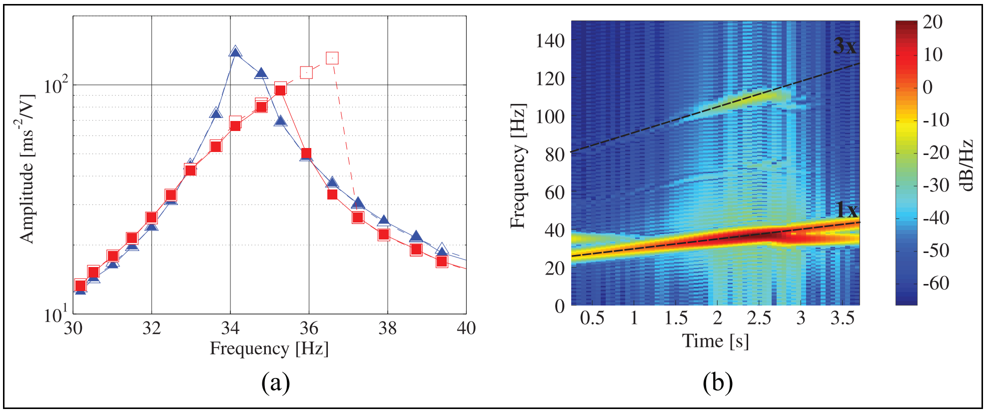

Figure 3. Intrinsically nonlinear behavior of the system. (a) Stepped sine test, where –△–, –▲–, –□–, and –■– represent, respectively, frequency up and frequency down considering low level of input, and frequency up and frequency down considering high level of input. (b) Time–frequency diagram of the system response.

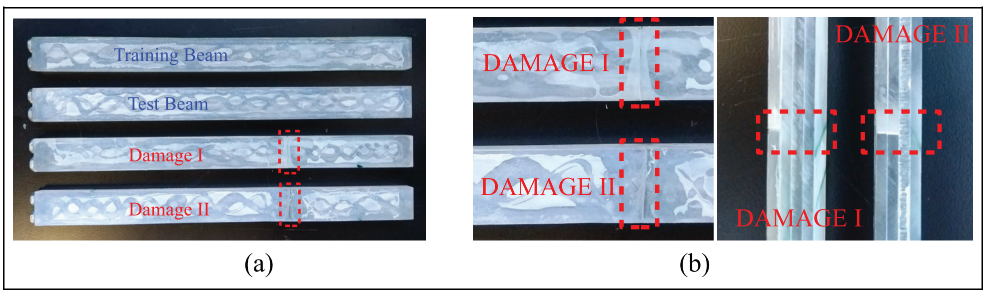

Figure 4. Structural conditions considered: (a) all beams constructed and (b) damaged beams (zoom).

sufficiently high amplitude (6 V RMS), the nonlinear phenomena can be seen with the jump presented in the test. In addition, Figure 3(b) shows the time–frequency diagram of the system response considering a high level of a chirp input in the region of the first mode shape of the structure. The presence of the cubic harmonic in the response confirms the nonlinear characteristic of the response caused by the interaction between the magnet and the steel masses.

Therefore, the system studied presents nonlinear behavior, subject to sufficient level of input, even when the structure is healthy. This characteristic is obtained because the magnetic force changes nonlinearly with distance from the end masses to the magnet. This intrinsically nonlinear behavior can be confused with the simulated damage that causes a distinct nonlinear characteristic to the system response, as will be seen next.

Damage simulated

The damage imposed on the structure aims to simulate a breathing crack present in the system. In this sense, four different beams were built to be used in the application of the damage detection methodology:

- Training beam: beam constructed with four intact Lexan beams and used in the training phase of the algorithm (see Figure 4(a)).
- Test beam: beam constructed with four intact Lexan beams and used in the test phase of the algorithm (see Figure 4(a)).
- Damage I: beam constructed with three intact and one cut beam (see Figure 4(a) and (b)).
- Damage II: beam constructed with two intact and two cut beams (see Figure 4(a) and (b)).

The cut in the beams is positioned close to the excitation point (see in Figure 2(b)). This spot was chosen to obtain the required nonlinear behavior to test the performance of the algorithm. The damage condition might be judged severely, but the position and excitation combined were defined to generate a condition that is difficult to detect—mainly in the condition damage I—as will be shown further along. Figure 5 shows the time–frequency diagram of the system response

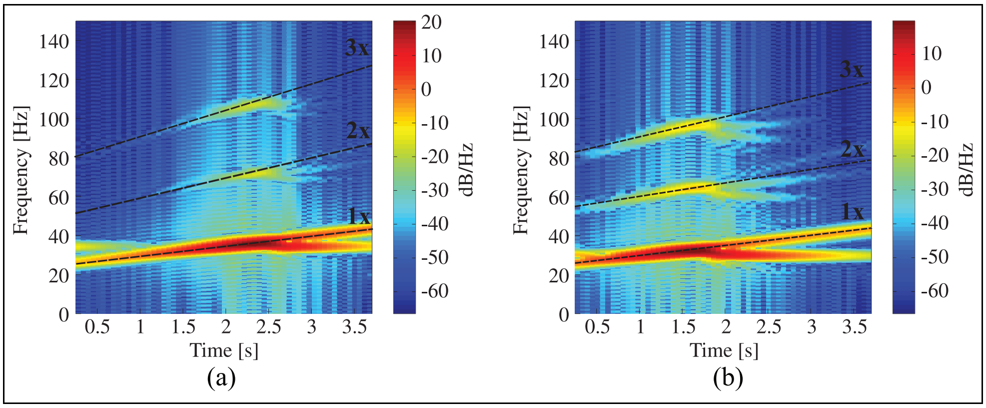

Figure 5. Time–frequency diagram of the system response obtained in damaged conditions: (a) damage I and (b) damage II.

considering a chirp input with a high level of amplitude and the structure in damaged conditions. Figure 5(a) shows the appearance of a quadratic harmonic when compared with Figure 3(b), that is a consequence of the crack, without significant alterations in the behavior of other components of the response (first and third harmonics look similar to the ones observed in Figure 3(b)). With the propagation of the damage (damage II), the quadratic and cubic harmonics grow up and the resonance frequency changes (Figure 5(b)).

Therefore, it is recommended that the initial propagation of the damage has influence in the quadratic harmonic of the system and when the extension of the damage is more significant, all linear and nonlinear components of the response are affected. The damage detection approach applied has to be able to detect the appearance of the nonlinear behavior caused by the damage, without confusing this evolution with the cubic nonlinear behavior caused by the presence of the magnet. In addition, the problem becomes more complicated when the data variation is assumed, requiring a strategy to separate these effect as shown in the next section.

Data variation

In order to study the performance of the strategy in the presence of data variation or other uncertainties, the distance between the magnet and the steel masses (see in Figure 2(b)) was varied from 2 to 3.5 mm during the tests, repeated on different days to obtain a total of 200 experimental tests for each beam constructed.

Figure 6 illustrates the variation of the system response during the tests. The results consider only the first mode shape frequency range, as only this range will be examined in the damage detection process. It is clear that it is easy to classify the structural condition when the extension of the damage is even more severe (damage II), but it is not plausible to observe visually the deviation between the reference and damage I conditions when the frequency response function (FRF) and the linear modal parameters are considered. These modal parameters were estimated considering a line-fit method[51] and the experimental realizations of the FRF.

These results show how hard it is to detect the presence of the damage when the data variation is considered in the analysis. In the scenario considered in this work, with the experimental data measured, the model proposed has to be able to detect the presence of the damage in the uncertain ambient state without confusions between the nonlinear behavior caused by the presence of the magnet and that one caused by the presence of the damage. In this sense, the next section presents the main results obtained with the application of the methodology based on stochastic Volterra series.

Application of the proposed methodology

This section matches the application of the methodology described in section "The damage detection methodology based on stochastic Volterra series" to detect damage admitting the experimental setup described in section "Experimental setup." The main results obtained are shown, and the performance of the linear and nonlinear analysis is compared.

Reference model identification

The first step to the utilization of the methodology is the estimation of the stochastic reference model. As mentioned before, the first three Volterra kernels were considered in the analysis. The number of Kautz functions and the Kautz parameters related to each Volterra kernel were defined as described in the previous work.[36] Therefore, the number of functions used here are $J_1 = 2$,

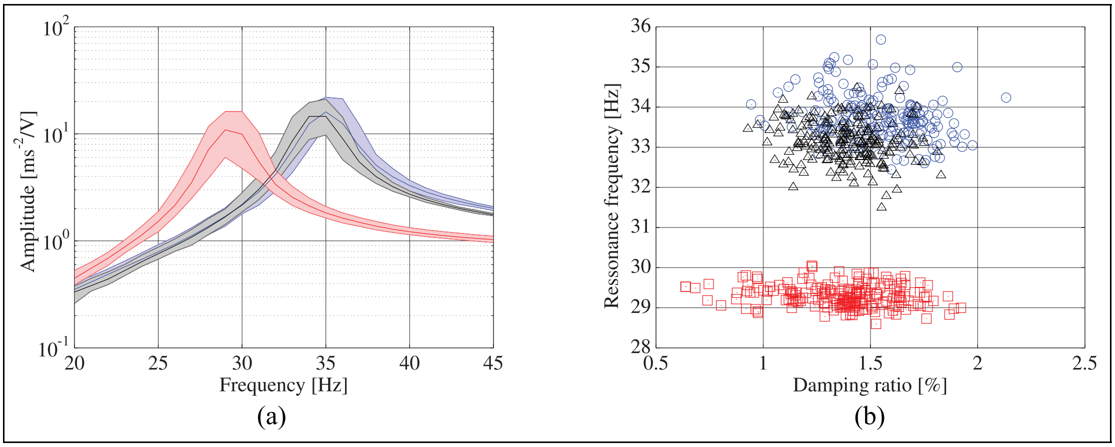

Figure 6. Variation of the data measured. (a) Frequency response function, where the continuous lines represent the mean values and the color zones represent the curves obtained with 99% of confidence: ■ - reference; ■ - damage I; and ■ - damage II. (b) Variation of linear modal parameters, where ○ represents the reference condition, Δ represents damage I, and □ represents damage II.

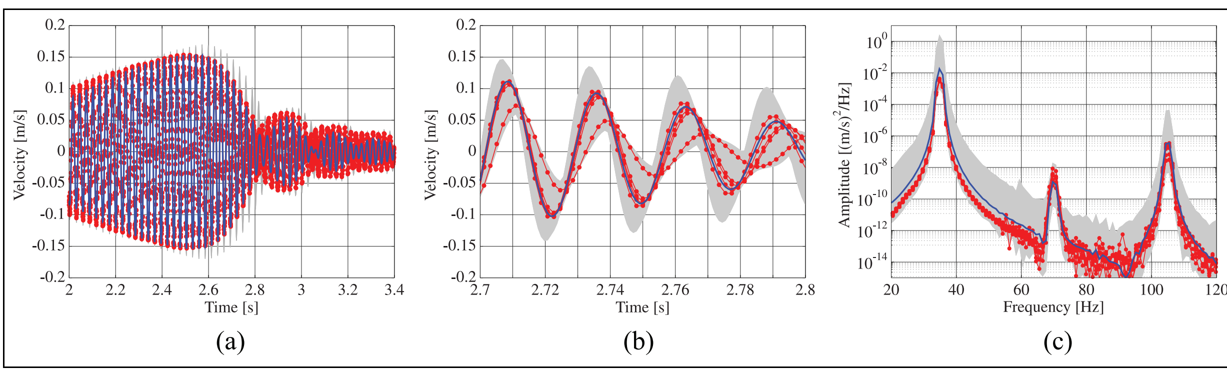

Figure 7. Verification and validation of the reference model. (a) Model response in the time domain considering a chirp excitation. (b) Model response in the time domain considering a chirp excitation (zoom). (c) Model response in the frequency domain considering a sine excitation. ■ represents the 99% model response confidence bands, —— represents the model response mean, and – ● – represents five realizations of the experimental data.

$J_2 = 4$, and $J_3 = 6$. To obtain the input/output signals used in the kernels estimation, the structure was excited admitting a chirp input signal varying the excitation frequency from 25 to 40 Hz (first mode shape region) and considering two levels of amplitude (1 V RMS—linear system behavior and 6 V RMS—nonlinear system behavior). The chirp input is attractive because it excites the linear and nonlinear components of the system response with enough energy, leading to a better estimation of the high-order kernels.[19,37,52] The output signal considered (velocity vibration signal) is obtained through the integration of the acceleration signal measured by the accelerometer (see Figure 2). The velocity signals are used because of the previous implementation of the model identification procedure considering this type of output. As done before,[36,37] the kernel estimation is performed in two steps, that is, the first kernel is identified considering the underline linear behavior of the system and then, the second and third kernels are identified considering the nonlinear components of the system response.

The deterministic model is identified several times, considering the 200 experimental realizations obtained from the healthy training beam, to construct the stochastic reference model. Figure 7 presents the verification and validation of the model identified. Figure 7(a) and (b) shows the stochastic model output with 99% of confidence bands, in the time domain, considering the same chirp input used in the model estimation with a high level of amplitude (6 V RMS), in comparison with experimental data measured. It is observed that the stochastic model can predict the system output considering the data variation. In addition, Figure 7(c) shows the stochastic model output with 99% of confidence

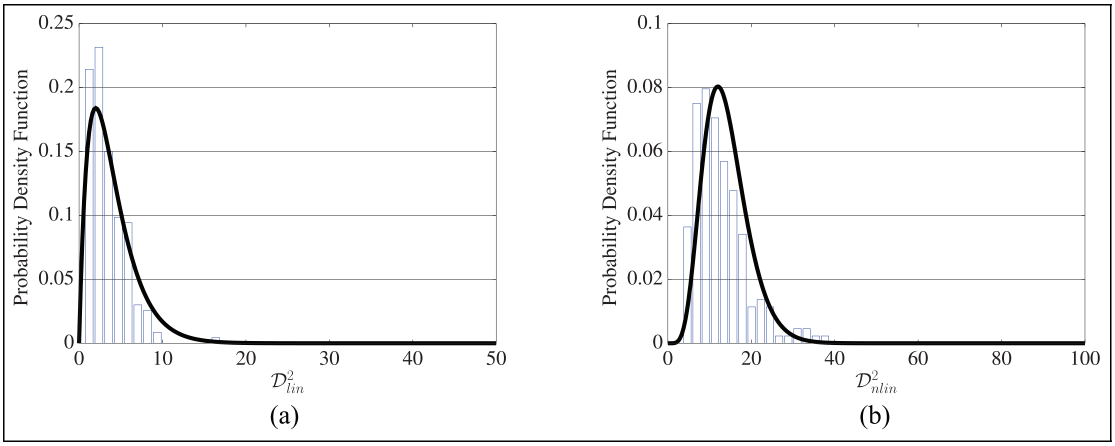

Figure 8. Comparison between the theoretical and experimental distributions. (a) Distance calculated based on the linear index in the reference condition. (b) Distance calculated based on the nonlinear index in the reference condition. The histogram represents the experimental data and the black line represents the theoretical chi-square distribution.

bands, in the frequency domain, holding a single sine input with excitation frequency close to the system resonance frequency ($\approx$35 Hz) and a high level of amplitude (6 V RMS), in comparison with experimental data measured. As can be seen, the stochastic model can predict the harmonic components of the response. This characteristic is unusual in the sense that as the damage considered produces variations in the nonlinear components of the response, it is expected that the model could be sensitive to these variations with adequate performance to detect the damage faster than the linear approach. With the stochastic reference model estimated, the damage detection procedure can be applied and the results obtained are shown in the next section.

Damage detection performance

First of all, it is interesting to observe whether the Mahalanobis distance computed considering the indices obtained in the reference condition ($\mathbb{I}_{lin}$ and $\mathbb{I}_{nlin}$) belongs to the chi-square distribution, as proposed in the methodology. Figure 8 shows adjustment between the histograms obtained from the Mahalanobis distance, calculated with the indexes in the reference condition, and the chi-square theoretical distribution. Of course, the theoretical approximation is not perfect but satisfactory, considering the limited amount of data available in the analysis. Moreover, this approximation overcomes the problems involving the empirical estimation of distributions based on experimental data, and the Kernel Density Estimator used before, related to the needed amount of experimental data and the choices of the kernel and smoothing parameter.[47,53] This aspect of the indices distribution allows the application of the hypothesis test proposed based on the probability of the distance calculated belonging to the chi-square distribution.

The evolution of the distance calculated from the indices with the propagation of the damage is presented in Figure 9. It is observed that the linear index is not sensitive to the presence of the damage at the beginning of the propagation, that is, damage I condition (see Figure 9(a)). The nonlinear index presented more sensitivity to the presence of damage, showing an adequate separation between damage I condition and the reference condition (see Figure 9(b)). Besides, the nonlinear index calculated from the data measured considering the test beam presents some outliers that will be reflected, probably, in false positives, depending on the threshold value used. These outliers are a consequence of the high sensitivity of the nonlinear components to structural variations. These results are not unexpected since the test and training beams are nominally the same, but not identical. The use of more than one training beam could improve the methodology performance in a real application. Another interesting aspect of the index used is the increase of the distance values with the propagation of the damage that, in the future, may be well correlated to the severity of the damage.

Considering the distances determined, the hypothesis test proposed may be applied. Table 1 shows the results obtained for both linear and nonlinear indices. It is clear that the nonlinear index presented a higher capability to detect the presence of the damage. The false alarms (false detection) are also higher considering the nonlinear index. However, this value can be reduced without loss of performance to detect the damage with the decrease of the test sensitivity as perfomed, for example, by Avendaño-Valencia and Fassois.[31] The

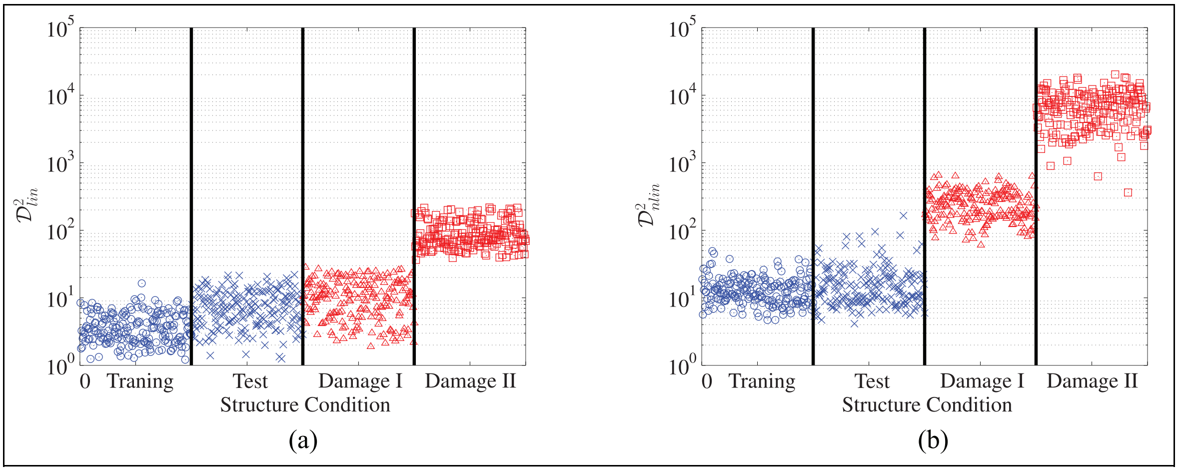

Figure 9. Evolution of the Mahalanobis distance calculated with the progression of the damage. (a) Distance calculated based on the linear index. (b) Distance calculated based on the nonlinear index. ○ represents the reference training beam, × represents the reference test beam, △ represents damage I condition beam, and □ represents damage II condition beam.

linear index presented problems to detect damage I condition, even with the variation of the test sensitivity. In real applications, the value of the test sensitivity (β) has to be determined depending on the level of security/confidence required and on the previous knowledge about the monitored structure. In some applications, this value can also be optimized to present a better performance in the damage detection process, based on measured data obtained from damaged structures.[31]

However, a better way to analyze the damage detection capability of the methodology is computing the receiver operating characteristics (ROC) curve. This curve relates the false alarm ratio with the true detection ratio obtained applying the hypothesis test for different values of sensitivity (β). As a result, the closer to the point (0,1) is the curve, and better is the performance of the index because it presents a higher probability of detecting the damage with a low level of false alarms. This curve is used here to study the discrepancies between the performance of the linear and nonlinear approaches proposed. Figure 10 shows the curves obtained after the application of the hypothesis test using different threshold values, considering both indices (linear and nonlinear) and all structural conditions studied (training beam, test beam, damage I and damage II). In the figure, it can be seen that the nonlinear index presents a better performance since the red curve is close to the point (0,1). This result was expected as the linear components of the response are not sensitive to the initial propagation of the damage (see Figure 9 and Table 1).

Moreover, the performance of the nonlinear index (almost perfect, but this is admittedly a consequence of a finite data set), with the curve very close to the point (0,1), was achieved in an experimental laboratory application. Even with the consideration of the data variation simulated during the experimental tests, these results do not reflect a real-structure application. It is supposed in a real-world application a higher number of confounding effects, different types of damage occurring coincidentally, and other aspects that may decrease the methodology performance. However, the better capability of the features related to the nonlinear kernels to detect the presence of the damage considering the confounding effects caused by the nonlinearities and uncertainties must be preserved.

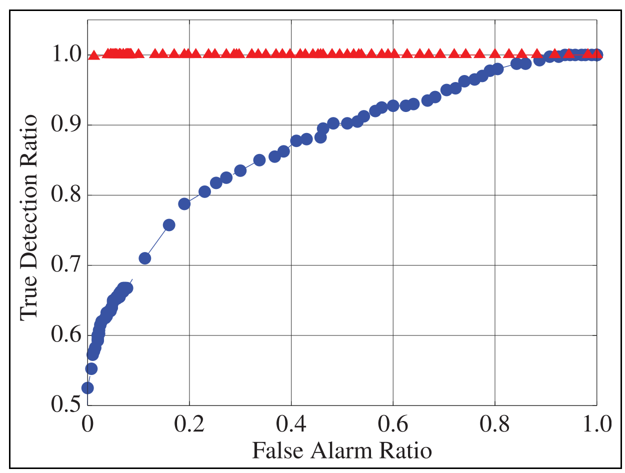

Figure 10. Receiver operating characteristics (ROC) curve. ● represents the linear index and ▲ represents the nonlinear index.

Table 1. Results obtained through the application of the hypothesis test.

	Hypothesis test sensitivity (β)	Percentage of false detection (%)		Percentage of true detection (%)	
		Training beam	Test beam	Damage I	Damage II
Linear index	10^{-2}	0.5	15.5	34.5	100
	10^{-4}	0	0	5	100
	10^{-6}	0	0	0	100
	10^{-12}	0	0	0	75
Nonlinear index	10^{-2}	4.5	12.5	100	100
	10^{-4}	1	7	100	100
	10^{-6}	0	4.5	100	100
	10^{-12}	0	1.0	96.5	100

Final remarks

The problem of damage detection in an intrinsically nonlinear system, regarding the data variation associated with uncertainties, subjected to the presence of damage that causes a nonlinear characteristic in the system response, was investigated in this article. In this sense, an initially nonlinear beam was analyzed, and the data variation was emulated by the random variation imposed in the experimental setup (variation of the distance between the magnet and the beam). The damage studied was a breathing crack that affects the system to exhibit a nonlinear operation with a distinct character of the initial one. In this condition, the methodology has to be adequate to distinguish the intrinsically nonlinear operation and the data variation to the presence of the damage. A method based on a stochastic version of the Volterra series, with the use of the random kernel's coefficients and contributions as damage detection features, combined with a novelty detection technique, was applied to solve the problem.

Unlike what has been shown in the previous works published,[36,38] the kernel coefficients and the kernel contributions approaches were applied together, considering a unique index to monitor the structural condition, aiming to augment the robustness of the method. Moreover, a theoretical distribution was introduced to the Mahalanobis distance computed in the reference condition to reduce the possible problems related to the use of the kernel density estimator previously assumed. In addition, for the first time, the methodology was examined through an experimental application considering an intrinsically nonlinear beam subordinated to the presence of damage that produces nonlinear aspects to the system response, all this considering data variation that reflects changes in the linear and nonlinear components of the response.

The results obtained revealed that the monitoring of the nonlinear components of the system response, denoted by the high-order kernel's coefficients and contributions considered as damage detection features, is a helpful method to be implemented in damage detection problems when the nonlinear system response is present. Again, the nonlinear metric confirmed to be more sensitive to the appearance of the damage and showed better performance considering the data variation compared with the monitoring of linear components. Finally, the use of the stochastic reference model, combined with the novelty detection technique and the hypothesis test, exhibited satisfactory performance to overcome the problem related to the data measured variation, providing the metric for detecting the presence of the damage with probabilistic confidence even in an uncertain ambient. Based on the results achieved, although the application was performed considering an experimental laboratory setup, the authors believe that in real-world applications the nonlinear metric will also demonstrate higher performance than the linear one for this kind of problem.

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