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Abstract

Fostering Embodied Coherence: A study of the Relationship Between Learners' Physical Actions and Mathematical Cognition

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When students first learn properties of number (e.g., order, magnitude, and basic arithmetic), they do so with respect to natural numbers. Learning fractions, however, necessitates recalibration of previously developed integer schemas (Gelman & Williams, 1998; Fuson, 2009). For example, when one multiplies two positive integers, their product is greater than either of them, but when one multiplies proper fractions, the reverse is true. Faced with such contradictions many students discard their emerging intuitions about number and instead rely on memorizing rules and procedural algorithms (Freudenthal, 1985). Consequently, they fail to develop a grounded, connected understanding of fractions and, perhaps worse yet, become discouraged from or unable to engage with more advanced mathematics (Ma, 2000; NCTM, 2000; Wilensky, 1993).

To date, pedagogical efforts to improve students' understanding of fractions complement the introduction of fraction notation and algorithms with activities in which students are guided to create or transform objects such as diagrammatic figures or blocks in order to see, appreciate, and articulate relationships between unit wholes and constituent unit parts. The rationale is to ground otherwise rote operations on symbols in direct actions on objects. While such experiences have proven effective for many learners, national assessments suggest that more work is still needed.

My dissertation adopts an embodied cognition perspective (Barsalou, 1999; Glenberg, 1997; Lakoff & Nunez, 2000) in order to examine the challenges and opportunities that students encounter when participating in artifact-mediated fraction instruction. In adopting this theoretical framework, I attempt to identify possible tensions between: (a) learners' physically embodied, multi-modal, goal-oriented actions; (b) meanings that the learners assign to novel semiotic forms the instructor introduces as symbolizing these actions (Abrahamson, 2009); and (c) the conceptual foundations that we attempt to build learners' initial understanding of fractions upon.

Taking a design-based research approach (Brown, 1992), I developed *Water Works*, a novel activity sequence involving the iteration of measured volumes of water into a vessel. In its pedagogical rationale, Water Works bears some similarity to the Elkonin–Davydov approach to teaching natural numbers as well as curricular designs favoring the number line (e.g., Carraher, 1993; Kalchman, Moss, & Case, 2000). The design begins by presenting students with a "whole" (e.g., one-cup) and labeled unit parts (e.g., ½, ¼ cups, each marked with a corresponding symbol). Students are guided to engage in a set of activities of filling the one-cup. They are to make sense of part-to-whole fractional relationships and fraction arithmetic in terms of observable and reversible physical actions (Piaget, 1971).

Vitally, Water Works has been designed to foster coherence between learners' pre-existing schemas for whole number and emerging understanding of fraction arithmetic by preserving one-to-one correspondence between learners' physical actions and the resulting mathematical outcomes. For example, the arithmetic operation 4 x ¼ is enacted as four physical iterations of a ¼ cup of water. The resulting product—both multiplicative and literal—is 1 cup of water that can visually be perceived as 4 times greater in magnitude than the original ¼ unit measure.

Over the course of seven months, students from two 4^{th} grade general-education public-school classrooms (*n* = 40) participated in a scripted sequence of individually administered problem-solving tasks and written-assessments. Qualitative data from videotaped clinical interviews and quantitative comparisons of written assessments are used to develop a model of artifact-mediated cognitive interaction, and implications are drawn for scaling up this design.

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Last, but not least, I wish to acknowledge the two fellow members of my advisee cohort, Dragan and José. We did it!

Dedication

This work is dedicated to my wife Dana, my son Honor, my parents Bob and Pat, and to my daughter (thank you for still being in the womb).

CHAPTER I: INTRODUCTION

1.1 The Challenge and Complexity of Teaching Fraction Concepts

When students first learn properties of number (e.g. order, magnitude, and basic arithmetic), they do so with respect to natural numbers. Learning fractions, however, necessitates recalibration of previously developed integer schemas (Fuson, 2009; Gelman & Williams, 1998). For example, when children encounter situations involving arithmetic with fractions, they often misapply the rules/heuristics for whole-number algorithms, such as adding the numerators and denominators, respectively, of unlike fractions (i.e. a/b + c/d = (a+b)/(c+d)). While such responses are reasonably logical attempts to reconcile the basic rules of whole-number arithmetic with an unfamiliar (to the student) notational form, they are from a strictly mathematical perspective, incorrect.

Children's initial experience with whole number arithmetic operations thus appears to pose challenges for their learning of fraction arithmetic (see Siegler et al., 2011). Indeed, another naïve heuristic that learners must learn to reconcile is the expectation that 'multiplication makes more.' When one multiplies two positive integers, their product is greater than either of the factors—but when one multiplies proper fractions, the reverse is true. Faced with these and other contradictions, it hardly comes as a surprise that many students choose to discard their emerging intuitions about number and instead rely on memorizing rules and procedural algorithms (Freudenthal, 1985).

One obvious problem with relying on rote algorithm is that while an individual might be able to provide a "correct answer," the reasoning underlying their solution procedure may be flawed (e.g., Ma, 2000). Another, is that this type of understanding cannot be readily applied to model personally meaningful situations in life, let alone in STEM education disciplines or professions in which rational number explicit knowledge is routinely called upon and required (NCTM, 2006). Therefore, to more effectively "ground" students' understanding of rational number concepts, teachers will often turn to the use of pedagogical tools such as diagrams or blocks in order to communicate a mathematical relationship to learners by representing said mathematics in terms of object/action/outcome relationships that are (presumably) familiar to the learner.

1.2 Conventional Strategies for Teaching Fraction Concepts

Researchers of mathematics education generally agree that different media present students with different affordances for supporting the learning of a given concept (e.g., Ball, 1993, Dienes, 1962, 1971). In classrooms today, educators will teach fractions using some combination of three types of instructional representations heretofore referred to as "conventional pedagogical tools": (1) Area models (including, but not limited to circles, rectangle, etc.) that are meant to visually demarcate and show part-to-whole relations; (2) Number lines, which show the measurement model for fraction, and (3) Objects (physical or drawn) which help to communicate the notion of fractions within sets (see also Behr, Harel, Lesh, 1992, van de Walle, 2011), along with numerical and symbolic representations.[1]

Pictorial representations allow early elementary students who may not yet have learned schemes for multiplication or division—to harness their visual modality and naïve schemas in order to reason about a mathematical concept such as "one-half" or "one-fourth" of some "thing." An operating assumption is that children will map between the visual diagram and some analogous lived experience in life. For example, circular area models are commonly used to reference food objects such as "pizzas." The pizza and its enumerated 'slices' are used to visually represent the concept of 'one whole' and 'unit fractions' thereof respectively. A drawn number line can quickly be partitioned to visually indicate and/or compare fractional subunits of a whole.

Curricula inspired by the Elknonin-Davydov methodology that employ the number line, use non-standardized units of measurement to support the expansion of mathematical knowledge beyond integer-based conceptualizations of quantity as enumeration (see Schmittau, 2003). In practice, this might involve using a string or block as the unit for measuring and comparing the magnitudes of different objects. Students are tasked with making qualitative observations such as, "Object X is equal, greater, or lesser than the measuring unit." The underlying rationale of these measuring activities is to expand students' integer based understanding of number (derived from counting activities) and guide them to translate qualitative comparisons of said units into real-number identities such as "1 ½" or "1.5" (see Figure 1). In other words, such activities train students to look at the qualitative *relationship* between the part and the whole, as opposed to simply enumerating the quantity of parts present in a diagram of some whole.

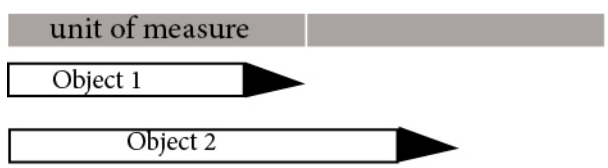

Figure 1. In Elknonin-Davydov curricula, children are tasked with comparing real-world objects with fixed units of measure. In the example above, one would observe that Object 1 is equal to the unit of measure, whereas Object 2 is longer (1.5 times). The rationale is that children develop a qualitative understanding of ratio/proportion that becomes the basis for real-number (e.g., fractional) understanding.

Another popular type of representational strategy involves the use of concrete objects such as physical blocks, bars, and ropes (e.g., Dienes, 1971)—and, increasingly, computational instantiations thereof (e.g., Brown, McNeil, & Glenberg, 2009; Olive, 2000; Sarama & Clements, 2009). Unlike drawings, such concrete or virtually concrete objects may literally be "manipulated" (from the Latin *manus*, hand).

Manipulatives allow the learner to effect structural transformations of the material situation. The belief is that it allows them to make use of their physically embodied experiences derived from interacting with physical objects in the world and more authentically instantiate (to a knowledgeable observer) the concepts/schemes to be learned in terms of actions and expected outcomes.

[1] It should be noted that a recent meta-analysis by Lewis et al. (2010) indicates up to 15 different representations used in US textbooks.

The basic rationale for complementing mathematics instruction in general, and fraction instruction in particular, with instructional artifacts is to ground students' rule-based operations on symbols in direct relation to their visual perception and/or physical transformations of objects. Given that many children learn whole number arithmetic operations by physically aggregating and disaggregating groups of countable objects, it does stand to reason that modeling mathematical concepts/operations through the physical manipulation of objects can likewise support learning. The perceived effectiveness of instructional artifacts has led to their widespread adoption by educators the world over.

1.3 Cognitive Challenges Arising From Interpreting Conventional Designs

From a cognitive standpoint, conventional tools and their accompanying teaching approaches typically rely on the principle of "splitting" some arbitrary whole unit into two distinct objects (Confrey, 1994). To elaborate, one super unit is mentally and/or physically divided into equally sized sub-units. For example, the notion of "one-half" conceptually emerges through splitting a whole unit (area model, number line, or grouping of blocks) into two sub-units. Likewise, grouping objects into sets and then partitioning into two equal subsets prepares learners to apply fractional concepts and operations beyond units of 1 (i.e. ½ of 10 = 5).

The general approach described above might seem perfectly logical from the perspective of an informed adult attempting to communicate this concept *to* a child. We establish the identity of some whole, and define a half as one of two equal parts of that whole. From the *child's* perspective, there are arguably a number of inconsistencies that must first be reconciled before the mathematical convention of "half" and "whole" can be properly integrated into their current system of counting. First and foremost is the fact that whether one splits a number line, area model, or uses two blocks to represent 1/2, there are *always* two visible units. Two units pushed together to ostensibly form a whole still leaves two units. This may be problematic if the child fails to reconcile their subordinate relationship to the whole, and enumerates the units as individual wholes in and of themselves. Suffice it to say, without proper guidance and support, learners can and do struggle to interpret even seemingly simple mathematical artifacts (Ball, 1995; Uttal, Scudder, & Deloache, 1997; Uttal, Liu, & Deloache, 2006).

Consider the common instructional task of presenting students with two area models in order to communicate that the unit-fraction 1/3 is greater than the unit-fraction ¼ (Figure 2). Each circle is *intended* to represent a continuous whole that has been split into discrete, equipartitioned component units.

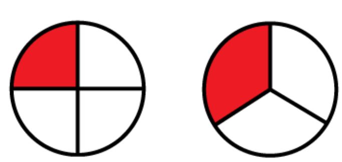

Figure 2. Two area models are intended to highlight: (a) the part-to-whole relationship of unit fractions (1/4 and 1/3, respectively); and, juxtaposed, (b) the relative magnitudes of unit fractions (1/3 < 1/4).

When confronted with an area model such as in Figure 2, above, however, many students will actually conclude that 1/4 > 1/3, because they may enumerate the visually distinct units and make a reasonable comparison based on their knowledge that 4 > 3, that is, the natural number 4 implies a greater quantity than the natural number 3. Another common error of interpretation occurs when students attend to the *ratio* of shaded versus un-shaded objects and label the drawings for 1/4 and 1/3 respectively as "1/3" (1:3) and "1/2" (1:2) (i.e., 1 red to 3 whites; or 1 red to 2 whites). In both cases, children attempt to establish a correlation between the discrete, countable units of the diagram and what they believe to be the quantity indicated by the numerical symbol in the denominator. In short, they appear to draw upon their prior understanding of whole number and familiar problem solving strategies in order to make sense of the new problem situation (Mack, 2001).

Another example where disambiguating between the super-unit and the sub-units can prove problematic occurs when students are tasked with interpreting a set of discrete objects. Consider a "six-pack" of soda-cans comprised of six, clearly distinct sub-units. The task is contingent on learners recognizing that there is a whole unit (the six pack). A teacher might shade in two of the cans to communicate the notion of a fractional subset of a super-unit (i.e., 2 cans represents 1/3 of the six-pack, see Figure 3).

Figure 3. An object model representation of 2/6 (1/3). A common way to misinterpret this figure is to observe that 2 greys, 4 whites = 2/4. Also note that the cans each have their distinct identity, and that a child will always perceive 6 cans, even though this grouping may be used to represent one whole group comprised of 6 equal parts.

The objective of such a task and representation might be to extend children's reasoning about fractions (beyond the simple case of one-whole pizza), or to teach fraction arithmetic operations that involve integers (i.e. 6 x 1/3 = 2). From a child's perspective there are 6 cans, and 2 of them happen to a different color from the other 4. Nevertheless, we cannot assume that learners will recognize the subordinate relationship between 2 shaded units to the original super-unit of 6 (2/6 = 1/3). Students will often observe that there are 2 shaded cans and 4 un-shaded ones and interpret the representation as 2/4.

Finally, let us consider the case of visually representing an "improper" or mixed number fraction. Consider the fraction depicted in the area model in Figure 4 below. Again, it must be emphasized that while *a knowledgeable adult* can easily interpret this as a pictorial representation of 6/4 (1 ½), a child could just as easily interpret this figure to represent 6/8. After all, there clearly are eight, equally sized squares visible, and six of them are shaded in!

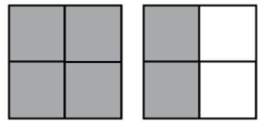

Figure 4. This is an area-model representation of an improper fraction, 6/4. Consider how a child could easily misinterpret this as a representation of “6/8”.

This last example serves to reinforce my earlier point regarding how many children will automatically draw upon their existing additive schemes and enumerate the visible objects in the representation. It also brings to mind another subtle, but important limitation of drawn area models and number lines as a tool for modeling arithmetic when the fractional values are greater than one-whole. To elaborate, let us return to figure 4. Consider how this representation might have been used to model the end result of an arithmetic problem such as ¾ x 2. How might you as the student or teacher end up drawing this representation?

Well first, you might logically start by drawing the super-unit whole (Figure 5).

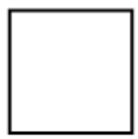

Figure 5. A child draws a super-unit of one whole

Then, it stands to reason that you would split the figure into 4 parts (Figure 6).

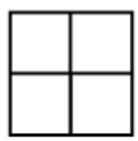

Figure 6. A child splits the one-whole super unit into 4 sub-units.

Next, you would shade in 3 of the 4 parts to represent the fraction ¾ (Figure 7).

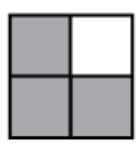

Figure 7. A child shades in 3 of the 4 sub-units of the super-unit.

Repeating this process, you might then have two figures as below (Figure 8).

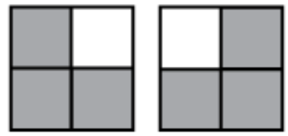

Figure 8. Repeated the earlier steps to represent ¾ x 2

However, imagine how when confronted with the problem ¾ x 2 the child had also naively applied a multiplication strategy wherein s/he multiplied both the numerator the numerator and denominator by the factor, 2? Note again how the visual representation might appear to support the incorrect calculation that ¾ x 2 = 6/8.

Suffice it to say, these and other common errors are not insurmountable. Students can and do learn fractions using the conventional tools widely available today. Moreover, I believe that a conscientious and determined instructor can and will guide students to learn regardless of the instructional tool at their disposable. But, while many students eventually learn to reconcile and integrate these seeming contradictions into a more sophisticated understanding of number, national indicators suggest that many do not (Lamon, 2007). The greater concern of course, is that many students learn to distrust the intuitions and modalities of reasoning that these representations were meant to cultivate in the first place! Given the contradictions between what their senses perceive and the mathematical schemes for arithmetic students may have learned previously, it should hardly come as a surprise when many students abandon any attempts at conceptually reasoning about a problem and rely solely on procedural certainty.

An important takeaway for educators/researchers to consider is that the pedagogical utility of any tool is both enabled and constrained by the learner's prior mathematical knowledge as well the perceptual evidence available to them at the time of instruction. How an intervention can be effectively utilized is therefore predicated upon yet restricted by how well the individual learner identifies and coordinates their prior mathematical understanding with the instructional artifact and/or activity.

1.4 Statement of the Problem and Research Questions

The fact that children and adults alike struggle with rational-number topics has already been well established. Part of this difficulty appears to result from trying and failing to integrate seemingly contradictory properties of whole and rational numbers (Siegler, et al., 2011). Recall that in the case of fractions students are forced to construct *reverse* mappings between denominator magnitude and overall fraction size. The numerical symbol in the denominator position—which hitherto fore represented a whole number—takes on a new meaning. A lesser value in the denominator position (particularly in the case of unit fractions of the form 1/n) now indicates a greater numerical magnitude. Recall also that multiplying a number by a fraction results in a product that is less than the operands. In both cases, students are required to formulate the mapping "less is more" for the novel symbolic representations for unit fractions and their pre-existing schemes for numerical magnitude.

The term 'scheme' was reintroduced by Piaget to describe the sequential organization of a child's sensory and perceptuo-motor activity in the world. It is important to note that a scheme in the Piagetian sense is not merely a unit of knowledge (schema) or 'concept,' but rather, a coordination of action, perception, and pre-existing knowledge (Vergnaud, 2009). For example, when young children attempt to count a small set of objects, they typically attempt to some coordination between their eye-movements, pointing gestures, and the numerical words. Here, the existence—or lack thereof of—of pre-existing conceptual structures also plays an important role. As Vergnaud observes,

> "The efficacy of the (counting) scheme depends on the one-to-one correspondence between these three activities and the set of objects in the physical world. It also relies on the ability to conclude the episode by wording the cardinal of the set, which is more than the last element of the set: cardinals can be added whereas last elements cannot. The concept of number is characterized by the additive property of cardinals, a property that equivalence and order relationships do not have." (Vergnaud, 2009, pg.85)

Returning to the case of early fraction instruction, a similar argument can be made that the ability of child to construct an efficacious scheme for fraction concepts and operations is contingent on their ability to coordinate and in many cases reorganize there pre-existing schemes for whole number (Gelman & Gallistel, 1979).

This leads to another part of the difficulty with learning fractions—learning to make sense of the conventional tools and strategies commonly being used in classrooms today. To a learner, the fractional notation and arithmetic for fractions is seemingly at odds with all their experiences/metaphorical schemes developed for whole number counting and arithmetic (i.e. Lakoff & Johnson, 2000). The crux of the issue is that children become conditioned to a specific convention for interpreting number (i.e. "the later the number appears on the sequential 'counting poem' the greater the number of countable objects") that interferes on a cognitive level with their appropriation of a novel numeric system. This conditioning is rooted directly in their sensory motor interactions with the world (Gelman & Gallistel, 1979) and interferes with their ability to interpret conventional representations in the manner that the instructor intends. The challenge for educators interested in improving early fraction understanding is to resolve these potential conflicts and guide learners to integrate their existing understanding of number with formal conventions for rational number manipulation (Siegler et al., 2011).

Given students' prior experiences with counting whole numbers and enumerating objects, it comes as little surprise when they apply many of their previously learned heuristics towards interpreting the representations used to support early fraction instruction. While logical, their naïve heuristics oftentimes result in what are deemed mathematically incorrect answers. Moreover, I believe that after children construct specific problem solving schemes/heuristics based on a given representation, they soon learn to distrust and eventually abandon them given the design limitations of said representations. Many students learn to distrust their intuitions and begin to doubt their ability to "understand math." Consequently, many learners resort to rote memorization of procedural operations and rules without developing a correct conceptual understanding of the fundamental mathematics (Freudenthal, 1985, Ma, 2000).

This dissertation is rooted in the classic Piagetian tradition known as "constructivism" (Piaget, 1970) and further espouses an embodied cognition perspective (Barsalou, 1999; Glenberg, 1997; Lakoff & Núñez, 2000), which posits that human reasoning in general, and mathematical reasoning in particular, draw on multi-modal, goal-oriented perceptuomotor interactions in the phenomenological world. While educators have long been acquainted with using physical (and increasingly, computational) *objects* to support instruction (i.e. Papert, 1980), my work places special emphasis on exploring the relationship between learners' *actions* and the development of learners' mathematical understanding.

In order to understand the pedagogical utility of a given object, I believe that one must attempt to carefully understand the actions that learners perform, and perhaps more importantly, how learners interpret the outcomes of said performances. Here, I also draw upon Vergnaud's neo-Piagetian notion of 'conceptual fields,' defined in his own words,

> "(A conceptual field) is at the same time a set of situations and a set of concepts tied together. By this, I mean that a concept's meaning does not come from one situation only but from a variety of situations and that, reciprocally, a situation cannot be analysed with one concept alone, but rather with several concepts, forming systems." (Vergnaud, 2009, pg. 86)

In the case of early fraction concepts and operations—which Vergnaud would describes as part of the multiplicative conceptual field—the implication is that the mathematical *schemes* (e.g. problem solving strategies such as counting discrete objects, visual comparisons of area, algorithms for arithmetic, etc.) that students apply and/or construct are inextricably linked to the *situations* (contexts) in which they are/were constructed in. In the case of early fraction instruction in schools, this necessarily implicates the curricular materials and ultimately, the representational media.

Applied to the study of early fraction learning, I explore how the functional interplay between students' prior knowledge, goal-oriented activity, and the phenomenological outcomes/products of physically embodied actions contributes to the meanings students construct for novel semiotic forms (Abrahamson, 2009). My dissertation proposes to revisit the challenges that young students experience with learning early fraction concepts and arithmetic operations through a careful examination of student–teacher artifact-mediated pedagogical interactions—or in Vergnaud's terms—how students' pre-existing schemes interact with pedagogical situations to engender/extend their emerging fraction schemes.

An operating assumption is that many of the instructional artifacts and activities (situations) commonly used to cultivate early fraction understanding inadvertently contribute to the difficulty that students have in integrating their prior understanding of number (schemes based on whole number counting and arithmetic) with rational numbers. If children's individual construction of whole number sense is grounded in their actions with objects in the world, it stands to reason that integrating their understanding of whole number concepts and operations with rational number may be contingent on fostering new relationships with objects in the world as well.

Disambiguating this relationship between scheme and situation may prove useful to instructional designers. After all, the objective of an instructional design is to support learners' development of specific concepts/schemes. Design practice, in turn, is the creation of situations that will steer learners along a particular cognitive trajectory by instantiating specific rules and causal relationships between user inputs and perceptual output with the intention that the user detect, adopt, implement, articulate, reflect on, and generalize said rules, per Piaget's model of learning as reflective abstraction. If as Vergnaud posits, the schemes a learner possesses are inherently bound to a situation—then it stands to reason that educators can shape the trajectory of learners' construction of new schemes by purposefully *designing* said pedagogical situations.

Such an examination necessarily extends to the design of the instructional artifacts themselves, as different types of artifacts obviously lend themselves to different modalities of interaction and cognitive affordances (i.e. Norman, 1988). However, in speaking of design, I take a systemic, situated perspective aligned with Vergnaud's notion of the conceptual field, by which I critique and configure relations among schemes, tasks, and resources. That is, I am interested not only in the pedagogical artifacts' physical form per se or their standalone functions. Rather, the emphasis is placed on how the artifacts' affordances pertain to and shape the cognitive task demands—and by extension learning—that can be engendered when learners are provided with a given set of objects and goal-oriented objectives. I would argue that an awareness of the dynamic interplay between learners' cognitive schemes and the instructional situation is the key to designing effective pedagogy. In other words, we must holistically examine the activity sequence to understand how the pre-existing schemes learners possess would interact with and are ultimately be transformed by and through the pedagogical situation.

Three, interrelated conjectures guide my examination of the problem of learning basic fraction concepts and arithmetic operations. The first conjecture is based on the observation that the representational forms typically used to introduce early fraction concepts will evoke schemes and/or heuristics based upon learners' prior understanding of whole number properties and algorithmic operations. When students attempt to make sense of ill-designed activity sequences, I suspect, they are liable to experience difficulty in building coherence among the following three critical factors: (a) *previously learned* arithmetic operations for adding/multiplying whole numbers; (b) the *transformative actions* and resulting *perceptual outcomes* of the artifact-mediated manipulation; and finally (c) the *formal mathematical procedures* that model these manipulations. Failure to build this coherence between the mathematical procedures with some mental/physical model, I contend, impedes, if not totally undermines, learners' construction of effective schemas for operating with fractions. The evidence suggests that students fail to construct a deep *conceptual understanding* (Freudenthal, 1985; Lamon, 2007; Ma, 1999) and learn to rely on rote memorization of fractional algorithms and procedures.

My second conjecture—which builds upon a long history of incorporating manipulatives in mathematics education—is that providing students with an artifact/activity designed to coordinate their pre-existing counting schemes with physically embodied actions may alleviate some of the ambiguities and confusion that students typically experience with fractions (e.g., Ball, 1995; Dienes, 1962, 1971; see also Radford, Edwards, & Azarello, 2009). To briefly elaborate, recall that students initially learn that multiplying whole numbers (e.g., 4 x 4 = 16) results in a product with a greater numerical magnitude. In the case of fractions, multiplying a unit fraction by a whole number appears to contradict this naïve expectation (e.g. ¼ x 4 = 1). How, then, might students accommodate their pre-existing schemes for counting and arithmetic so as to assimilate the unique properties of fractions? How might we learn and improve upon earlier designs? I believe that a more productive approach would be to create designs which foster coherence among students' prior understanding of natural number, the physical transformations they perform, and the aforementioned formal procedures.

My third conjecture, which builds upon the prior two, is that an activity sequence for supporting early fraction understanding should necessarily build upon students' *additive*, as opposed to *multiplicative*, reasoning (c.f. Confrey, 1994, on the "splitting conjecture"). While it is undeniable that multiplicative relationships are inherent to the structure of ratio and proportion, it is equally undeniable that early elementary students are most often exposed to—and therefore most comfortable with—additive reasoning tasks based upon natural/counting numbers. Indeed, Norton (2008) has provided evidence to suggest that many patterns of reasoning ("schemes") typically associated with solving fraction problems can be developed by students without application of a splitting strategy. Similarly, I believe that in order to foster greater continuity between students' prior knowledge and fraction routines, educators should leverage learners' stable, pre-existing additive reasoning structures (see also Fuson, Kalchman, Abrahamson, & Izsák, 2002).

To be clear, I am not questioning whether or not splitting is, as Confrey (1994) has termed, a "conceptual primitive." Instead, my design analyses suggests that relying upon multiplicative reasoning as the conceptual bases for children's early understand of fraction concepts and arithmetic can be problematic given the characteristics of conventional tools such as drawn area models and number lines.

A key principle of effective manipulative design that I will attempt to elaborate is that the actions learners perform and the outcomes of said actions must cohere with the currently existing cognitive structures possessed by the student in order for them to successfully construct a targeted pedagogical concept.

In order to test my conjectures, I will employ a design-based research (Brown, 1992) approach wherein I evaluate a novel activity sequence for supporting early fraction learning of my design, Water Work which attempts to reconfigure the introductory activities so that the mapping between learners' pre-existing schemes for number/arithmetic, such as iterated-adding, and the new targeted schemes involving fractional units/arithmetic is direct rather than inverse.

1.5 Water Works: A Physically Embodied Design that Builds on Additive Reasoning

The Waterworks design involves the use of standard measuring cups, such as a 1/2 cup or 1/3 cup, and varying volumes of water to measure and/or pour. Whereas the use of manipulatives—much less measuring cups—to support early fraction learning is certainly not novel, the contributions of this dissertation will be to elaborate and further explicate otherwise subtle interactions between learners' cognitive structures, the actions they are tasked with performing while manipulating material objects, and learning gains. A core pedagogical strategy employed in the Water Works design is to build upon young learners' pre-existing mental structures for counting and additive reasoning with the aid of physical manipulatives (e.g., Ball, 1995; Fuson, 2009; c.f. Confrey, 1994).

Here I believe it will prove useful to operationalize the term "pedagogical enactment," which I define as the purposeful implementation of some dynamic situation that the designer/instructor believes will help the learner to assimilate and/or accommodate a given conceptual scheme. Similarly, I will use the term "guided co-enactment" to characterize and describe pedagogical enactments whereby an instructor purposefully observes and regulates the learner's artifact mediated activity with the likewise intention of guiding them to either assimilate the novel scheme or accommodate their pre-existing to scheme(s) to accommodate said situation (e.g. Werstch, 1979).

In the context of early fraction instruction, enactment extends beyond standard definitions such as representing or modeling a concept to a learner. Typically, we think of a representation as a static instantiation of some object or a concept. For example, a drawing of a collection of objects can serve as a static representation of some numerical quantity. Likewise, a number line may be used to represent a concept such as one half. Interestingly, a drawing can be used to represent a dynamic concept such as distance over time. However, in that particular case the added dimension of meaning arises as a result of the observer's ability to interpret and lend meaning to the representation, as opposed to any in-the-moment experience or perception of events.

Modeling a mathematical concept to students[2] entails a more dynamic reproduction of some concept or operation. For example, a teacher might model the concept/operation of addition to students by physically combining a set of objects as students observe. Increasingly, educators have turned to computer simulations that can be used to visually render complex concepts in terms of visually observable phenomena (see Charoenying, Gaysinsky, Ryokai, 2012). It is important to bear in mind however, that modeling something necessarily implies a pre-existing understanding of the concept to be modeled. In other words, one cannot model what one does not know.

Consequently, my use of the term enactment differs from both representation and modeling in that it accounts for and encompasses the *naïve* coordination, production, and/or perception of goal-oriented outcomes that influences the development of learners' cognitive schemes. A representation is simply a static artifact. Modeling a concept presumes an understanding said concept. In contrast, a pedagogical enactment entails the student performing and/or observing a dynamic series of actions with the intent that they ultimately construct meaning from the resulting outcomes and/or transformations of the instructional situation.

Imagine the case of attempting to teach the concept of one-half to a learner. When conventional media are used, the pedagogical enactment might involve guiding them to draw a square area model, equipartitioning it into two, and then physically shading in one of the two sections. Using the Water Works design, the same concept might be enacted by asking students to physically iterate two ½ cup volumes of water into a one-whole cup measure. In both situations, students are essentially asked to observe and infer meaning from a dynamic sequence of events. In both scenarios, the learners have enacted the production of the representation/model of 1/2.

In the following section, I describe the Water Works design and then compare how it is utilized in pedagogical co-enactments to standard approaches for teaching part-to-whole fraction relationships and arithmetic (e.g., shading area models or number lines divided into discrete fractional units).

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[2] The use of term modeling in the context of mathematics education should not be confused with a creating "mathematical model," a technique for simulating/predicting real-world scenarios with equations.

1.5.1 Water Works: A pedagogical design for fostering embodied coherence

The *Water Works* design presents students with a single operational metaphor—the physical act of iterating volumes of water using standard kitchen measuring cups (see Figure 1, i.e. one-whole, ½, 1/3, and ¼ of a U.S. standard cup). Following the hypothesis that children's number sense is based on physically embodied interaction in the world (i.e. Abrahamson, 2009; Lakoff & Nunez, 2000; Piaget, 1970, 1971), the underlying rationale for Water Works is to allow learners to perceive and manipulate fractional and whole number units in a manner that fundamentally coheres with—rather than contradicts—their pre-existing notions of number. A goal is to build upon the *similarities* between arithmetic with integers and fractions, as opposed to the differences, and to foster coherence between learners' manipulation of material objects and the mathematical operations that can be used to describe said manipulations. In plainer language, the design attempts to build upon children's understanding of whole numbers and whole number arithmetic, and provides contexts that allow children to physically model said arithmetic in a manner that is non-contradictory with their pre-existing schemes.

The design relies upon a century-old tool that routinely appears in household kitchens across the world—but less so in the classrooms—standard measuring cups (e.g. Figure 9).

Figure 9. A set of standard kitchen measuring cups.

Students are presented with different problem-solving activities that require them to pour and measure different volumes of water. Through a premeditated sequence of problem posing and facilitated hands-on inquiry, students are guided to perform and report perceptual judgments, determine quantitative properties of the situation, and uncover patterns. Concepts such as part-to-whole relations or equivalence, mixed-numbers and improper fractions, as well as algorithmic operations such as addition, subtraction, and division of fractions are embodied by the designer in the form of physical activities of measuring and comparing volumes of water (see Appendix).

Similar to concrete manipulatives such as blocks, the Water Works approach allows students to physically instantiate both quantitative and qualitative relationships (between volumes of water) that can later be assigned as the meanings of rational-number notations or arithmetic operations. Specifically, these relationships are instantiated as the reversible products of their own dynamic activity (see Piaget, 1970,).

An interesting property of water is that it seamlessly blends into itself. If one were to iterate *n* measures of water into the same vessel, an observer would still see one continuous volume of water. Secondly, if one were to use a 1/*b* cup measure and iterate it *a* times, the resulting fraction *a/b* can be interpreted by learners as the result of *a* actions. To quickly explicate, 3/4 can be directly perceived as the result of pouring a ¼ cup measure, three times. Note how in comparison to any drawn diagram or set of blocks, there is considerably less visual ambiguity with regards with the persistence of visually discrete subunits or what units to enumerate (Figure 10).

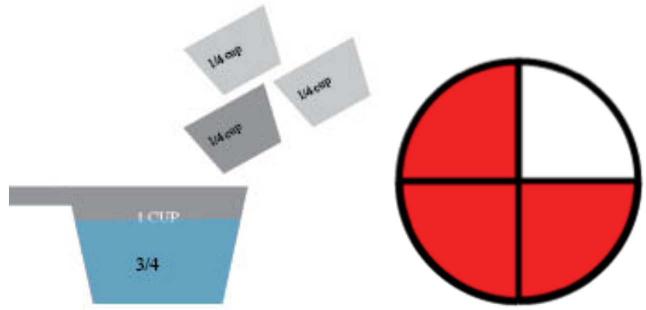


Figure 10. Using an area model, ¾ is represented by dividing a unit into 4 equal parts and shading in 3 of them Using the Water Works activity, the fraction ¾ is represented by physically iterating ¼ measures of water, three times into a one-cup measure to create ¾ of a cup of water.

One simply sees a single volume of water (that importantly, can be visually compared to other volumes of water) and one enumerates one's own actions, as opposed to arbitrary abstractions of whole/fractional objects as is true in the case of a diagram. [3]

Implicit to the activity design is that learners begin with fraction arithmetic even before explaining what a fraction per se is. Thus, the activity sequence attempts to build directly upon the learners presumably robust schemes for addition ('more is more') in order to eventually derive the meaning for what each unit fraction 'is' per se.

To help illustrate, consider a mainstream approach for introducing the unit fraction ½. Typically, the teacher will begin with some 'whole' such as 'one' pizza or 'one whole' number-line split (i.e. Confrey, 1994) into two sub-units. Here the unit fractions are formulated through operating on the 1 whole—i.e. a "1/2" is what you get when you partition 1 into 2 equal parts. The problem as already noted is that for most learners, their scheme of number is based on enumeration. For *them*, the diagram now represents two objects, as opposed to two sub-units that constitute the whole. The notion that ½ + ½ = 1 is not immediately apparent. Rather, the learner sees 1 + 1 = 2.

Waterworks operates in reverse. To introduce the concept of "½" students are guided to either iterate ½ cups of water into the one-cup measure, twice; or conversely, pour water from the one-cup measure back into two, ½ cups (Figure 11). A key difference of this approach is that the unit fraction is defined in terms of the additive/subtractive actions required to construct/deconstruct the one whole cup unit. In this respect, learners begin with the premise that ½ + ½ = 1 cup, as opposed to the premise of ½ being part of some whole.

[3] While one could certainly assemble slices of pizza into a whole, the fact remains you still see all the discrete slices. When volumes of water are combined, one only sees one continuous volume of water.

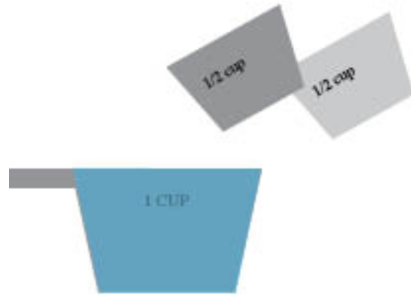


Figure 11. One "whole" cup created by iterating ½ cup volumes twice; or filling the ½ cup measure twice from the volume of water contained in one "whole" cup. Note that different shades of gray for the ½ cup in this static diagram indicates that there is only one ½ unit cup—it is simply being re-used, twice.

The activity and experiences used to establish the relationship between unit fraction cups and the whole (i.e. 1/2 must be iterated twice to fill 1 cup) is then formally mapped directly to the corresponding arithmetic operations (i.e. 1/2 + 1/2 = 1). Prior to learning a formal definition for fractions, the student is literally guided by an instructor to (de)construct the relationship between a fractional unit measure (1/x cups) and some whole-unit (1 cup) by counting their own physical iterations of volumes of water from one vessel to another. The dynamic physical actions and their resulting perceivable outcomes are intended to directly correspond with the mental operations that competent students perform when they are reasoning about fraction arithmetic (e.g., Steffe, 2002); and to furnish the pre-symbolic notions that the student is subsequently to objectify in the form of the novel semiotic inscriptions (e.g., "1/2", see Radford, 2003).

Because the meaning for ½ is established by enumerating the number of physical iterations required to fill the one whole cup vessel, an instructor can easily preempt an initial student misconception such as ½ < ¼. An instructor could present a ½ cup measure and a ¼ cup measure side-by-side. This provides the student with an unambiguous visual comparison of volume. Furthermore, the student can then be asked to physically iterate volumes of water from the ¼ vessel to fill the 1 whole cup and discover that the ¼ cup vessel is "less" than the ½ cup vessel because it takes "more" iterations to fill one whole cup. This experience provides an alternative, and arguably more powerful meaning to the denominator of a unit fraction as compared to the rule-based heuristic that, "The greater the denominator the lesser the magnitude."

Having established both the ½ and "one-whole" cups, improper fractions can be demonstrated by iterating the unit-measure a greater number of times than indicated by the denominator (e.g. 3/2 cups = 3 x ½ cup). Integers greater than "one" (in the case of adding/multiplying fractions to create mixed number values) can in turn be conceptualized in terms of physically iterating a base unit of 1-cup *n* times into a larger container (e.g., 10, 100 cups, etc.).

Examples of other tasks include: establishing the magnitudes of two different unit measure cups in order to compare unit fractions relative to one another (e.g. "What's greater, 1/3 or ¼?"); establishing a partitioning scheme (i.e. Steffe, 2002) for individual fractions (e.g. "Why is ¼ called one fourth?); and learning an iterative strategy for multiplying fractions (e.g. "How do you think about 4 x 1/3?" see Figure 12).

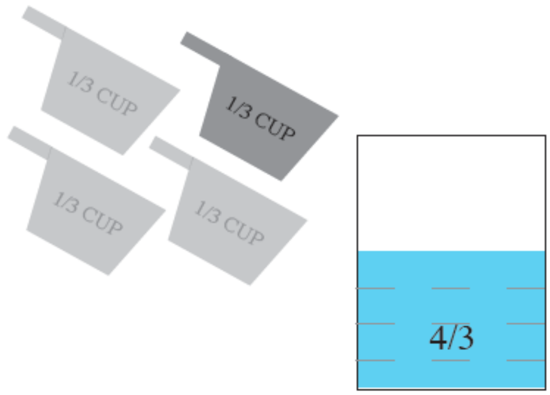


Figure 12. Iterating 1/3 cups of water, four times into a clear, unmarked container to create 4/3 of a cup of water (or to model 4 x 1/3). Note that marks have been added to the illustration in order to assist the reader. In the activity as designed, the students had an unmarked container—the emphasis was placed upon counting the physical iterations as opposed to attending to markings on the container.

1.5.1.1 Notes on the design of the Water Works tools and activity.

Design oriented readers may instinctively question why the decision was made to use standard, opaque unit measuring cups (sets that includes 1 whole, ½, 1/3, and ¼ cups). An obvious constraint of this set is that one cannot easily create/contain volumes of water greater than one cup in each vessel. Another, is that one cannot physically model operations with unit fractions less than ¼ (i.e. 1/5, 1/10, 1/100).

A simple, widely available alternative would have been to combine the measuring cup sets with a graduated cylinder/measuring cup such that students could have compared the water-level with a pre-established marking (see also Moss, 2005). Another option would have been to fabricate unit fraction cups of different, non-standard volumes (e.g., 1/7, 2/5, etc. etc.). Alternately, one could easily have digitized the entire activity sequence and materials and been free programmatically model and visually represent any number of scenarios (e.g., Lamberty & Kolodner, 2004; Resnick, 2002).

Bearing all these possibilities for optimizing the pedagogical utility of the Water Works activity sequences in mind, the decision was nevertheless made to develop the instructional sequence (see Appendix) using the basic tools as described. The primary rationale was to minimize the resources and task demands that a typical elementary school teacher might require when implementing similar activities in their own classrooms. Consequently, digital media—though increasingly widespread and affordable—was ruled out given that not all classrooms would have adequate access to such tools. Likewise, non-conventional tools such as custom-fabricated measurers were also excluded given the high likelihood that the average teacher would not be able to acquire them for his/her own classroom use.

With regards to the use of transparent, graduated measuring cups, these were also excluded from the study given that the cognitive task demands required to use them (measuring) could possibly have detracted from the intended focus of the Water Works activities—namely building students' understanding of fraction concepts and arithmetic competency in terms of additive operations—given that the primary cognitive task demand required to compare volumes of water in cylinders is measuring the relative water levels (as opposed to enumerating iterative actions).

1.5.2 Comparisons to conventional pedagogical tools.

Earlier I have argued that conventional tools and pedagogical strategies commonly used to introduce early fraction concepts can present a number of subtle, but potentially deleterious challenges to young learners. One point is that the number of sub-units a whole object is divided into always remains visually persistent. This may be problematic if the child fails to reconcile their subordinate relationship to the whole, and enumerates the units as individual wholes in and of themselves. A second point is that the actions used by the child to construct a material sub-unit and/or the whole unit from sub-units is not fully commensurate with the resulting outcomes. Again, pushing *n* blocks together to form a fraction of some whole leaves *n* units; the stroking actions with a pencil to draw/fill one half of a whole does not correspond directly with the numerical representation indicated by the fraction.

A third and final consideration is that the subjective conceptualization (visualization) of one whole shifts with referential context. Yes, an instructor will adapt instruction based upon the media available, but the evidence to date remains that many learners will come to rely on rote memorization and arithmetic procedure (Freudenthal, 1985; Lamon 2007; Ma, 1999). This suggests that they are failing to formulate a robust conceptual understanding of fractions. I believe that the inability of students to coherently ground their understanding through the representational contexts provided by their instructors is to blame, and consequently, that the existing status quo bears reevaluating.

To highlight the unique pedagogical affordances of the Water Works design as well as to frame the relationships between design and mathematical cognition that will be the subject of this dissertation, it may prove useful now to methodically compare and contrast Water Works with conventional pedagogical media in terms of cognitive task demands/mental operations required of students.

The comparison is divided into subsections that each treats one parameter thematic to the mathematics education research literature (see Table 1)

Table 1

Categories of Comparison Between Traditional Designs for Fractions and Water Works

	Area Model	Number Line	Water Works
Internal vs. External Fractional Sub-units	Internal	Both	External
Counting vs. Measuring	Counting	Both	Both
Partitioning vs. Additive Constructing	Partitioning	Partitioning	Additive construction
Discrete Units vs. Continuous Volumes	Discrete	Both	Continuous
Levels of Abstraction[a]	*n*	*n*	*n* - 1

[a] The variable *n* represents the number of levels between a concept and its representation. For example, to learn interpret an area model as a representation of "1/2", the child must arguably first learn that the model is itself an abstraction—i.e. a representation of a real world object such as a pizza (see also Uttal, Scudder, & Deloache, 1997, 2006). The argument is that with the Water Works media, there is one *less* level (n - 1) to abstract because the artifacts "representing" the concept are one and the same.

1.5.2.1 "Internal" Vs. "External" Fractional Sub-units

When we conceptualize some quantity as a "part," we perforce conceptualize some other quantity as the "whole." For example, to represent the idea of one-fourth using either a number line or area model, one must first establish an ad hoc unit of "one," and then *internally* demarcate this whole into four sub-units (Figure 13).

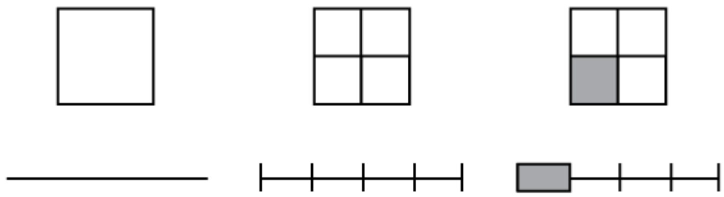


Figure 13. In order to represent the fraction ¼ using either an area model or unit fraction, the representation must first be drawn, internally partitioned into four parts, and finally one of the four parts must be shaded.

In the Water Works design, the unit-fraction measures are *external* to the whole-cup unit, creating an ontological differentiation between the part and the whole (Figure 14). Two distinct objects exist in the world (the measure marked as "1/4" and the other as "1"), and the learner's task is to establish empirically the relationship between them. Similar to Elkonin-Davydov inspired approaches, this may help to eliminate any ambiguity between the super- (the whole cup) and sub-ordinate (unit parts of the whole) structures.

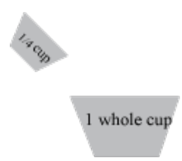


Figure 14. A one-fourth cup is separate and external to a one-whole cup.

An example of an internal representation.

To better illustrate how students might process differently an internal versus external representation, consider the case of attempting to represent the multiplication problem 2/4 x 3. A typical approach using an area model representation is to create a composite unit of 2/4 and iterate it thrice. This procedure might begin by drawing a square, equipartitioning it into four internal sub-units, and then shading in two of the four sub-units (Figure 15).

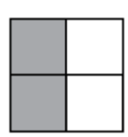


Figure 15. A conventional area model representation of 2/4 (1 x 2/4)

After shading in the first 2/4 (2/4 x 1), a next step would be to shade in another 2/4 of the mode and explain that this corresponds to the arithmetic operation, 2/4 x 2 (Figure 16).

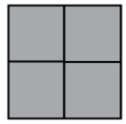


Figure 16. A conventional area model representation of 4/4 (2 x 2/4)

The progression at this point should be reasonably clear to a learner. The student began by shading in 2 out of 4 boxes, and after repeating the operations—the equivalent of 2 x 2/4—the result is that 4 out of 4 of the smaller boxes are shaded, or 1 whole larger square.

Now, in order to complete a visual representation of 2/4 x 3, observe how students can run into a practical constraint of drawn area models and number lines. While area models and number lines are convenient tools for depicting—or in the case of a teacher who is instructing a student, *attempting to convey*—fractional values less than or equal to one, they can introduce a certain degree of visual ambiguity when used to represent fractional values greater than 1. In order to visually represent 2/4 x 3 (6/4) using an area model representation, one must now draw a *second* area model (Figure 17).

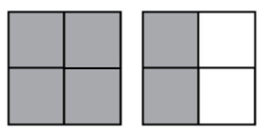


Figure 17. An internal representation of 6/4 (3 x 2/4)

We can now clearly observe a conceptual—and as a result, representational—challenge posed by an internal representation of fractions. In order to visually designate some fractional quantity as a "part," we must perforce conceptualize and represent some other quantity as its "whole." In this specific case, in order to represent the fraction two-fourths using an area model (or number-line), one must first arbitrarily establish a unit of "one," and then *internally* demarcate this whole into four visually discrete sub-units, prior to shading in two of said units. In order to draw a value greater than one whole unit—but less than two whole units—a student must draw two whole units.

Note how the drawing in Figure 17 can easily be misconstrued as a representation of 6/8 (that is, 6/8 of 2 whole squares). While a knowledgeable adult can recognize the drawing in Figure Z to be the product of a dynamic process, one can nevertheless appreciate the potential for ambiguity in interpretation on the part of students.

An Example of an External Representation.

Conceptually speaking, we have described each sub-unit of an area model or number line, as "internal" to it's whole. Let us now compare how the same problem (2/4 x 3) can be presented using an external representation such as the Water Works design.

Consider how the 1/4 unit-fraction measuring cup remains *external* to the whole-cup unit. Unlike the sub-units and the corresponding super-unit of a drawn model, the number of objects under scrutiny is obvious. There are a ¼ cup measure and a 1-cup measure. The one-fourth cup possesses a unique ontological identity apart from the other vessels, given that each vessel can be perceived as its own separate physical entity.

In order to represent 2/4 using a ¼ cup measure, one must physically iterate two volumes of water using said measure. The 2/4 volume of water created as a result of iterating the ¼ cup can now be understood as the product of iterative actions. This volume can subsequently be transferred into the reservoir container, without changing its meaning (it is still a volume equal to 2 iterations of the ¼ cup of water). Observe also that a 2/4-cup volume of water (poured into an appropriately sized holding vessel) is visually continuous.

A practical constraint of a whole cup measure is that it can only contain one whole cup of water without overflowing. Obviously, modeling a multiplication operation such as 2/4 x 3 would necessitate pouring more water into a one-cup measure than could be contained in it. A one-whole cup measure would completely fill with water and then overflow. This is similar to the same dilemma faced by the student attempting to depict the same operation with a drawn area model or number line. Once you have completely filled in a discrete area model representing one whole, you must perforce draw another representation of one whole in order to continue to fully model the operation as a drawing. While there are any number of pedagogical solutions that can be utilized in this situation—i.e. using multiple, one-cup measures, or using a measure that could hold more than one cup for example—a solution I devised for this activity was to ask the student to pour each 2/4 volume of water in the one-whole cup into a separate, unmarked holding reservoir.

Thus, in order to physically enact a multiplication problem such as 2/4 x 3 using the measuring cups, one might begin by first iterating the ¼ cup measure twice into a one whole cup measure, in order to create a 2/4 cup volume (Figure 18). This volume is then poured into a larger, unmarked vessel. The entire process (measuring then pouring 2/4 cup volumes of water) is repeated a total of three times.

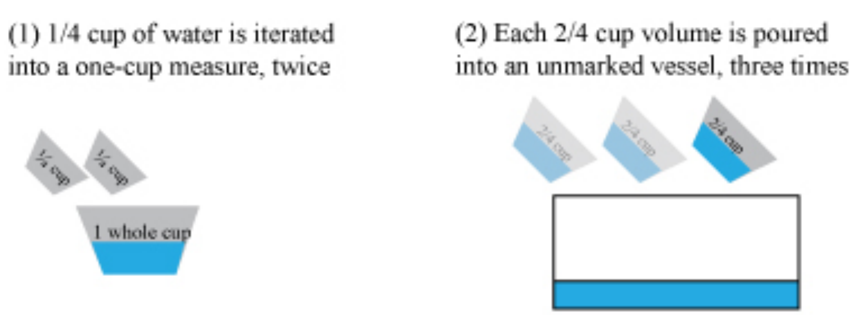


Figure 18. Enacting the multiplication problem 2/4 x 3 using the Water Works design.

To clarify, after the first sequence (1/4 x 2 = 2/4) the student now has the equivalent of 2/4 cup of water inside of a one-whole cup measure. The student is then asked to transfer this volume of water into the reservoir-container. The student is then asked to iterate another 2/4 cups of water into the one whole cup. Again, I instruct the student to pour the volume of water in the half-filled one-cup measure into the unmarked vessel containing 2/4 cups of water. Here, I indicate to the student that we have performed the equivalent of the multiplicative operation, 2/4 x 2.

After a third iteration of 2/4, the end result is a visually indeterminate volume of water. Of course, one problem of an unmarked vessel filled with water is that there are no visual indicators that can be enumerated. Instead, the student must resort to keeping track of the iterated-adding process and its cumulative results in the unmarked vessel. And yet it is entirely possible for a young student to lose track of how many times they have poured water. Consequently, as is true of most pedagogical activities, mindful-supervision on the part of the instructor is required. In this case, I remind students, as warranted, about the number of iterations they have completed by verbally enumerating the count of their actions. Furthermore, I oftentimes instruct the student to confirm the running total by performing a reversal of the operation and enumerating how many times they can fill (and discard) a ¼ cup measure with the volume of water (6/4 cups) contained in the unmarked vessel. By completing this reverse operation, the students physically also enact a situation that accurately corresponds to the arithmetic for subtracting and dividing unit fractions.

Admittedly, the written description of multiplication provided above may lead the reader to believe that the Water Works design is more complicated for students and teachers than it actually is. As we shall see in the qualitative analyses provided in Chapter 5, it is a surprisingly trivial task for students to coordinate meaning between their actions and the concept of multiplying a unit fraction.

1.5.2.2 Counting Vs. Measuring

When young students are presented with an area model divided into sub-units or a set of blocks, the mental operation that is typically evoked is counting. Students enumerate visually discrete units, and are typically tasked with comparing the ratio of shaded to un-shaded units. A basic critique of such counting-based tasks is that they do not require students to expand their understanding of number beyond that of whole-number integers. Pre-partitioned representations, I would argue, predispose learners more towards enumeration (counting) of the perceptually discrete sub-units at the expense of utilizing and developing their additive and/or measuring schemes. This is perhaps best exemplified in the classic error case where students conclude that a representation for 1/4 is greater than a representation for 1/3 by simple virtue of the fact the former has "more" sub units (Figure 19).

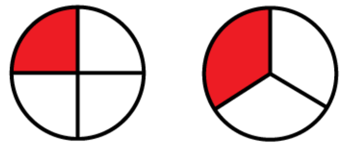


Figure 19. First time learners typically count the visibly discrete subunits of drawn representations for fractions and ignore the relative surface areas.

In recent years a number of leading scholars in the field of early mathematics education have championed the use of the number line based on both its greater perceived pedagogical utility (e.g., Carraher, 1993; Moss & Case, 1999; Saxe et al., 2009; Siegler et al., 2011, 2013) as well as its successful track record in Elknonin-Davydov inspired curricula (Schmittau, 2003). Recall that a key feature of many number-line curricula is the use non-standardized units of measurement to train students to discover qualitative *relationships* parts and the whole, as opposed to simply enumerating the quantity of parts present in a diagram of some whole (see Schmittau, 2003). The argument can also be made that the presentation of lines and line-like objects might naturally lend themselves to visual comparisons of relative length (Figure 20).

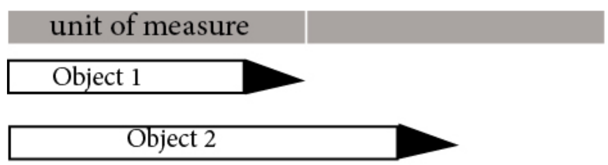


Figure 20. There are clearly two distinct objects, and one object is clearly longer than the other. Note also how there are no sub-units to enumerate as in the case of the demarcated area model in Figure 20

Similar to the Elknonin-Davydov curricula, *Water Works* allows students to build upon their ability to make visual comparisons of magnitude as well their robust counting schemes so as to understand and express non-integer magnitudes. For example, the identities and relationships of the unit-fraction cups (1/2, 1/3, and 1/4) are established by first comparing the relative sizes of each vessel, then counting the number of physical iterations of water poured from the unit-fraction measures into the one-whole cup.

Figure 21. Two sets of standard household measuring cups

Taking the case of 1/2 and one-whole (see Figure 22), once two measures of water were poured, the student would observe a full vessel, and understand that pouring any more water would overflow the container and thus be superfluous to the task.

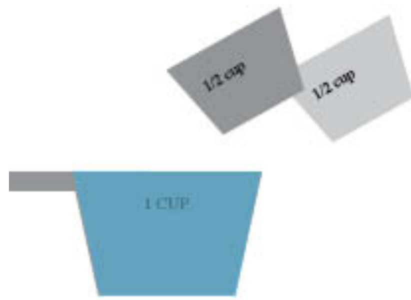


Figure 22. One “whole” cup created by iterating ½ cup volumes twice; or filling the ½ cup measure twice from the volume of water contained in one “whole” cup.

The resulting concepts of “one-whole cup” and “1/2” emerge from *both* iterating (repeatedly adding) and enumerating (counting) students’ own physical actions *and* observing and comparing the volumes of water (measuring. This deliberate structuring of learners’ attention towards objects, actions, and the resulting perceptual outcomes, may result in a higher degree of coherence between the novel mathematical procedure (i.e. adding or multiplying fractions) and their pre-existing additive schemes.

1.5.2.3 Partitioning vs. additive constructing

Virtually all curricula for fractions begin by establishing some arbitrary unit as a 'whole' (e.g., a circle, a rectangle, a line). This whole is then partitioned into equal parts—as many parts as indicated by the value of the denominator *b* in the *a/b* fraction—and then *a* of these are marked/shaded as the *a/b* fractional quantity of the whole. For example, to visually illustrate a fraction such as 1/4, some whole is established, then split/partitioned into four equal sections, and one of the four is shaded.

Recall however, that it is not uncommon for learners initially to focus on the ratio of shaded to un-shaded parts in an area model (or segmented sections of a number-line) and identify a representation for 1/4 as 1/3. In such cases, students appear to be enumerating only the discrete units and perhaps making an inference based on their comparison to the shaded/segmented parts. These students may also be failing to integrate the visually internal, sub-ordinate parts into the broader super-ordinate structure (the “whole”)—a developmental challenge Piaget characterized as one of “class inclusion” (see Flavell, 1963). In this case, they do not understand that both the shaded and un-shaded parts together constitute the whole, that is, that the shaded parts must be enumerated twice: once as the numerator constituent, and then again as part of the overall whole, the denominator, that includes the numerator constituent previously counted (see also Abrahamson, 2000).

By contrast, in Water Works, there are no overt acts of partitioning. There is a container labeled as "one cup," and there are smaller units, containers labeled as "1/2", "1/3", “1/4” etc. Thus the Water Works design begins by presenting students with a fixed “whole” (e.g., one-cup) *and* constituent unit parts (e.g. ½, ¼ cups, etc., see Figure 23). For example, the instructor will ask a student how many times a 1/4 cup must be filled and poured into the one-whole cup to fill it up. A knowledgeable observer will note that these smaller units are pre-partitioned unit fractions of the one-cup, for example, 1/3 means that the small cup has a capacity that is one-third of the large cup, so that it would take three of these 1/3 volumes to fill the one-cup to the brim.

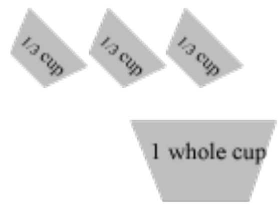


Figure 23. Three physical iterations of water from a 1/3 cup measure are required to fill one cup.

1.5.2.4 Discrete units vs. continuous volumes

In standard area models and number lines, sub-units are discrete entities, and so they remain perceptually articulated even after they are joined in the fraction composite (Figure 24). This is useful in that it allows students to draw upon their already robust counting schemes in order to identify relationships between the numerator and denominator of a fraction symbol and the discrete representation. Students can immediately see that the total number of discrete units corresponds to the denominator and that the shaded sub-units correspond to the numerator.

The problem with simply counting, however, is that it may not induce students to construct the partitioning and/or part-to-whole schemes (Steffe, 2002, Norton, 2008) and so doing, extend and transform their cognitive model of number (Siegler, et al 2009).

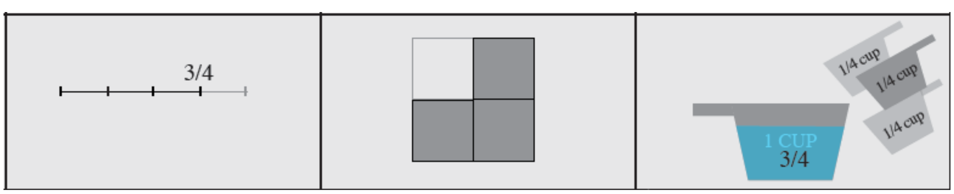


Figure 24. A comparison of how the fraction ¾ is represented across different media. Observe how in the number line and area model, the discrete units remain visible. By contrast, water blends into a continuous product of discrete actions (three iterations of the ¼ cup measure).

Water, similar to an unmarked number line and unlike a pre-partitioned representation, possesses an amorphous perceptual quality. Sub-unit are each *discrete* volumes of water prior to being iterated into the vessel but thereafter they seamlessly merge with one another to form a *continuous* volume.

Continuous-quantity fraction models, as compared to discrete-quantity models, may preempt the confounding effects of whole-number knowledge on integrating fraction schemes. Recall that an area model partitioned into sub-units can evoke children's whole number counting schemes and lead them to enumerate the visually discrete units in unexpected ways (e.g., to conclude that 1/4 is greater than 1/3). The absence of any visible demarcations in the water itself or upon its containing vessel may eschew entirely this cognitive conflict.

And yet, how are the volumes measured in this activity? That is, once a child has poured several discrete volumes of water into an unmarked containing vessel, there is no longer any visual trace of the number of volumes poured, so how is this cumulative continuous quantity of water quantified? The student can tell by the rising water level that there is "more," but they must complete the teacher-guided task of physically iterating volumes of water from the unit-fraction measure into the whole cup.

In other words, it is the actions of pouring that are enumerated, and so it is the actions that ultimately should lend meaning to the fraction notation. The emphasis shifts from enumerating visible objects, to the additive iteration of physical actions. For example, a child who is iterating the ¼ cup into the larger vessel counts off each pouring actions, so that the *product* of iteration—that is, the cumulative volume of water—is conceptualized in terms of the iteration *process*. ¾ is literally what you get when you pour in a "1/4" cup three times, that is, when you aggregate three "1/4" cups.

1.5.2.5 Levels of abstraction

A final category of comparison between the Water Works design and traditional representations concerns what I will describe as the "levels of abstraction" between the perceptual/physical activity performed by the learner and the targeted concepts to be learned.

In the broader context of mathematics education, an ongoing point of contention exists with regards to the virtues of "realistic" versus "abstract" instructional representations—a debate without clear-cut answers or guidelines (but see Kaminsky, Sloutsky, & Heckler, 2008; Uttal, Scudder, Deloache, 1997; Uttal, Liu & Deloache, 2006). The crux of the debate rests on the tension between mapping learners' real-world experiences/perception to mathematical concepts that can be abstracted beyond the immediate situational context. While we will explore aspects of this debate—specifically the contributions of physically embodied interactions and objects towards mathematics learning (see Abrahamson, 2009)—it is beyond the scope of the current section to do so. Instead, my task analysis will attempt to explore the "number" of cognitive task demands that a learner must coordinate between some mathematical notation, some representation, and some pedagogical actions in order to understand a given mathematical concept/operation, such as reconciling fractional notation with previously learned arithmetic operations.

To better elaborate upon how this notion of levels of abstraction might apply in the context of a learner interpreting representations meant to depict a concept such as distance over time. In order to represent a distance travelled over-time such as "four miles per hour," one might begin with a symbolic ratio such as 1:4 to indicate that in 1 hour, a person might be able to walk 4 miles. This can also be represented graphically using a linear plot and a coordinate system. To a mathematically sophisticated individual, there is a clear mapping between each of 3 levels: some lived experience walking between point A and point B (level 1); the numerical ratio (level 2); and the Cartesian graphical representation (level 3)

Returning to the case of learning fractions, let us consider again a rudimentary problem such as: 1/4 + 2/4 = ¾.

While this problem is trivial from the perspective of a mathematically adept adult, recall that it presents a number of conceptual hurdles for the young learner to overcome. A commonly observed problem-solving strategy among students will be to apply heuristics for whole number addition to the numerators and denominators respectively. A typically incorrect answer might be 3/8. Note that we have not even begun to discuss the conceptual mapping of this problem to a “real life” situation.

Now, let us consider how a teacher might represent this problem in order to develop students’ conceptual understanding as well as the correct arithmetic procedures.

When area models or number lines are used, one general approach would be to visually demonstrate that ¼ of some whole unit + 2/4 of some whole unit results in ¾ of said whole unit (see Figure 25). Note that the “whole unit” is itself an abstraction—it is a drawing of “something” such as a pizza or rope, fence post, etc. Finally, this relationship is then mapped back to the algorithm (1/4 + 2/4 = ¾).

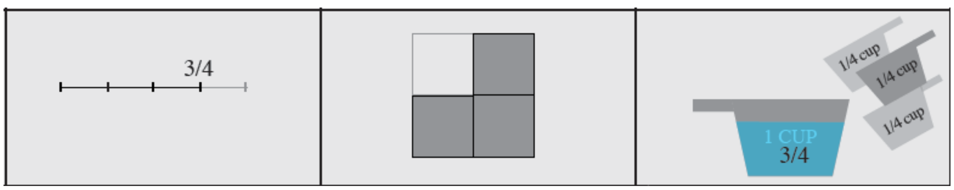


Figure 25. Consider how an instructor might repurpose a representation for a fraction such as “¾” to communicate the algorithm (1/4 + 2/4 = ¾)

I argue that in the case of using an area model/number line to communicate the concept of fraction addition, there are at least three levels of abstraction. First, there is the symbolic notation. The second level involves the translation of the drawn representation into abstract actions, objects, and outcomes (i.e. shading in sections of an area model/number line). Lastly, in order to develop what researchers such as Ma (1999) have described as a ‘deep conceptual understanding’ of the mathematics a learner must be able to map the mathematical concept and representation back to a third level—that of literal objects and actions that they have encountered and observed in their lives (such as pizzas/ropes and cutting/eating, etc. etc.). By contrast in the Water Works design, I would argue that because of the inherent physical reality of the measuring cub objects, there is *one fewer* (n-1) level of abstraction for the learner to draw connections between because the artifacts representing the concept are one and the same. The objects, actions, and outcomes involved in the specific example of adding the unit fractions 1/4 and 2/4 is only one level of abstraction removed from combining 1/4 cups of water with 2/4 cups.

What I wish to highlight with the above example is that there are multiple cognitive task demands required of the child when attempting to reconcile problems that otherwise appear simple to knowledgeable adults. For a child, it arguably involves multiple steps, including, but not limited to: (1) mapping the mathematical notation for *fractions* to a pictorial representation of *objects;* (2) mapping the mathematical operation towards a pictorial representation of *actions*; (3) abstracting that the actions performed *upon the pictorial representations* correspond to actions performed *upon real objects*; and (4) lastly, translating the transformations performed upon the pictorial objects back into symbolic notation. Given the complexity of the cognitive task demands, it really should be of little surprise that learning fractions has proven so difficult for so many students! Another important point to bear in mind is this—while a mapping between *addition/subtraction*, a drawn diagram, and conceptualizing a real world scenario (adding/removing objects such as slices from a pizza...) is reasonably straightforward, the complexity literally multiplies when attempting to use the same diagrams to conceptually map operations such as multiplication and division!

1.5.3 Summary

The Water Works design is a novel instructional activity sequence based on the physical iteration of volumes of water between U.S. standard kitchen measuring cups. The design may be considered as falling within the general category of instructional media broadly referred to as "physical manipulatives." In comparison to "conventional" media such as drawn diagrams such as area models and number lines that are routinely used in classrooms and curricula, I believe that the Water Works presents sufficiently unique properties so as to warrant a research study that evaluated for its potential advantages for the teaching and learning of fraction concepts and arithmetic (see Table 1). In brief, I believe that the physical and perceptual affordances of the design—similar to Elkonin-Davydov inspired curricular approaches—could lead students to establish qualitative relationships between unit parts and unit whole beyond rote enumeration. Moreover, the design allows students to continue to build upon their additive reasoning competencies through enumerating their own physically embodied actions. I argue that the design promotes an understanding of fraction operations and concepts that is consistent with mathematical convention and coherent with learners' already well-developed schemes of counting. In order to evaluate the pedagogical affordances of the Water Works design, I have positioned it as an alternative to what I have collectively referred to in terms of traditional or "conventional" representational media. Such media typically take the form of drawn inscriptions or the aforementioned manipulatives. They are part and parcel of virtually all elementary textbooks and curricula, and are widely accepted as proven and effective pedagogical tools and long accepted as "standards" (e.g. NCTM, 2000).

Nevertheless, numerous national indicators continue to suggest that fractions are a problematic subject matter for students of all ages (Lamon, 2007). Therefore, it stands to reason to explore alternatives in light of current accepted practices, and the explication of the Water Works design falls squarely in this tradition. Towards this end, I revisit the Water Works design in light of empirical data collected in this dissertation project via implementing the design with participating students, and I compare the design to commonly used designs such as the number line and area model.

The dissertation was not conceived with the intention of suggesting that traditional media are fundamentally "flawed" nor, conversely, that the Water Works design is definitively "more effective." Rather, by highlighting both positive and negative affordances of a given medium and delineating dimensions of comparison, I attempt to identify otherwise tacit aspects of mathematical cognition that may take place through artifact mediated interaction (see also Abrahamson, 2009). Moreover, by focusing on the relationship between action and fraction learning, I hope to advance the discourse surrounding learners' use of pedagogical artifacts and their resulting appropriation of mathematical concepts beyond historical notions such as "concrete vs. abstract" or "physical manipulatives."

1.6 Research Questions

In the context of many elementary school mathematics curriculums, children are typically exposed to simple unit fractions such as 1/2 , 1/3, 1/4, etc. in 2nd and 3rd grades. Concurrently, they learn arithmetic operations involving whole numbers. By grade 4, they are challenged to expand their concept of number and arithmetic in light of the existence of rational numbers. A basic research objective guiding this dissertation is to further identify (and document) cognitive challenges and transitions that transpire as students attempt to solve arithmetic problems involving rational numbers. To approach this question, I analyze video-recorded student-instructor tutorial interactions as well as written worksheets in order to document and identify generalizable patterns.

A more theoretical objective of this project will be to further explore the relationship between students' artifact-based physical interactions and the learning of mathematical content matter. Working from an "embodied design" approach (Abrahamson, 2009, 2012, 2014), special emphasis will be placed on exploring how learners' coordinated multi-modal perceptions and interactions mediate the development of mathematical concepts. In particular, I am interested in the relationship between students' physical actions, the phenomenological outcomes of said actions, and the effect of this coincidence on trajectories of learning.

With specific regards to the learning of fractions, I consider how part-to-whole relationships and basic algorithms for fraction can be mapped to discretely observed and countable physical actions. Given children's prior experience with arithmetic is based on whole number arithmetic, I also consider whether designs that rely upon additive reasoning as opposed to notion of 'splitting' (Norton, 2008; Steffe, 2002) would better support learners' understanding of fractional concepts and arithmetic.

A thematic guideline of this work will be Lamon's (2007) observations that early fraction curricula should stress: (a) ideas of unit and equivalence of fractions; (b) techniques for comparison (so that students can judge the relative size of fractional numbers); and (c) comfort and flexibility in fraction related thinking that transcends rules and rote algorithm. A pragmatic objective of this project will be to introduce a low-cost, physically embodied instructional activity for supporting the teaching and learning of rational-number concepts and operations that can be widely disseminated in schools today.

As a design-based researcher, I believe that educational research should be conducted within real-world classroom conditions (Brown, 1992; Confrey, 2005). For better and worse, however, classroom-based education research often requires a disruption of the normal patterns and routines of teachers and students alike. Thus, a final methodological objective will be to describe how researcher-led instructional interventions can be more ecologically integrated into the life of the classroom. I believe that greater sensitivity to the dynamics of day-to-day classroom life are likely to foster healthier short-term relationships with research participants, including students, teachers, and principals, and by extension, long-term relationships and access to longitudinal data.

1.7 Outline of Dissertation

This dissertation will be divided into six chapters and an appendix section. The first chapter has provided a brief introduction about the challenges of early rational number instruction; reflected upon the roles of instruction and pedagogical representation in juxtaposition to prevailing views about rational-number development; reexamined what it means to ground mathematical understanding in the classroom; and concluded with the statement of the problem and core research questions.

Chapter 2 describes the research methods. Here I present the rationale for the instructional activities and artifacts, the experimental design, forms of data collection, coding schemes, and methods of analyses. I conclude by demonstrating how the design-based research paradigm can be adapted towards more ecologically sensitive observations of student learning in classrooms over time.

Chapter 3 will briefly summarize the initial pre-assessment administered to the general education students participating in the primary Water Works study. In Chapter 4 each of the pedagogical interventions administered to the treatment groups will be presented alongside both quantitative and qualitative analyses. Chapter 5 provides an in-depth qualitative comparison of the two treatment groups. The final chapter will summarize results and reflect upon the implications of this dissertation.

CHAPTER II: METHODS

The Water Works study was structured in the following manner: First, a pilot study was conducted with special education students (age 8-11, n=7) in a self-contained public school classroom (combined grades 3-5). Second, a quasi-experimental research design involving 40, Grade 4 general education students was developed, in which two randomly assigned groups of students (n = 19, and n=20) underwent tutorial sessions using the Water Works tools and 'convention tools' respectively. Finally, a third group of Grade 4 students who did not receive any tutorial sessions (n = 26) from classrooms using the same mathematics curriculum (Everyday Mathematics, Bell, et al, 2012) was designated as a control group. Mixed-method techniques including micro-genetic analysis and grounded theory are used to interpret the qualitative data. In the following sections, I describe the rationale for and implementation of the study design.

2.1 Overview: Design-based Research

Given the dual objectives of the Water Works project—1.) To better understand students' mathematical cognition in relation to embodied activity, and 2.) To contribute artifacts/activities that improve the overall efficacy of classroom pedagogical practice—the overarching methodological approach for Water Works is design-based research (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

A defining characteristic of the design-based research (DBR) methodology is that the researcher actively structures and designs the context of interaction for subjects. Applied research on cognition and learning, the term "design" oftentimes refers to the selection of artifacts as well as the scope of the subjects' interactions with said artifacts.

The ability of the researcher to carefully observe and adapt the activity—e.g. the scope of the interactions between learner and artifacts—positions the researchers to better elicit and observe specific learning outcomes. Combined with traditional assessment methods such as written/verbal assessments, clinical interview tasks, qualitative analyses of audio/video recordings, etc. etc., DBR may allow researchers to capture and observe how specific artifacts and interactions inform the trajectory of students' insight and reasoning—sometimes at the very moment of conception.

DBR is by definition a highly flexible and inherently iterative approach for the study of learning phenomena. One does not begin by defining a study protocol and then rigidly adhering to it. Instead, the researcher is continually tasked with documenting, evaluating, and when deemed necessary, adapting his/her tools and/or methods. Invariably, the researcher may also elect to change a given parameter of interaction for the research subjects based on reactions/feedback from earlier subjects. The justification for such adaptations in the study is that the "thing" to be scrutinized is the germ of human cognition, a seed that would be otherwise unobservable unless circumstances elicited it.

The flexibility of protocol afforded to practitioners of DBR, to be sure, is a double-edged sword. On the one hand, it may help the researcher to collect experimental data that can be closely correlated to adaptations to the study design. Given that no two students, teachers, classrooms, or schools are ever the same, the ability to responsively adapt a research design in light of real-world contingencies arguably positions the researcher to uncover insights that would otherwise be obscured by a more rigid study.

On the other hand, the looseness of the DBR approach also opens the study design to legitimate concerns about muddiness, opaqueness, and lack of rigor. After all, if aspects of the experiment are continually being changed—how can one be sure that a given effect is actually the result of the purported cause? To address this concern, classroom research conducted under the umbrella of DBR requires careful and systematic documentation on the part of the researcher. Much like fieldwork conducted by anthropologists—that by its very nature cannot be replicated—the context and particulars of the study must be described as transparently as possible. Thus, while the design-based research approach is increasingly viewed as a legitimate form of scientific inquiry within education research (and other disciplines), it continues to behoove the researcher adopting this methodological approach to ensure rigorous measures for substantiating the validity of their findings (Schoenfeld, 2005; see also Guba & Lincoln, 1994).

Finally, DBR studies are somewhat different from action research—teachers' experimental use of innovative design—in that a core commitment is to generate insight into and theory of mind and/or learning as opposed to innovation for the sake of pedagogy per se (Reeves, Herrington, & Oliver 2005, Wang & Hannafin, 2005).

Bearing the strengths and limitations of design-based research in mind, in the following sections I describe the data sources used in the Water Works study, the methods and rationale behind the data collection, and my plan for analyses. Additionally, I describe a novel data-collection technique that attempts to combine aspects of large-scale controlled experimental design with the ethos of design-based research.

2.2 Data Sources

2.2.1 Piloting

Data for the Water Works study were drawn from two sources. The first data source is a pilot study in a grade 3-5 self-contained, special education classroom (n=7) from a public elementary school located in the San Francisco Bay Area.

In order to gain further insight into the students' current levels of understanding and to develop a more formalized sequence of activities for the primary study, I conducted a rudimentary assessment of their initial fraction understanding.

Figure 26. Special needs students engaged in small group activity.

Students were verbally guided through a 20-item written assessment (see Appendix). Items on the assessment were fraction problems adapted from the annual statewide mathematics proficiency exams for grades 3, 4, 5, and 6. Examples of problems included verbally explaining what "1/2" or "1/4" meant; comparing fractional magnitudes (i.e. what is greater, 1/2 or 1/5); and arithmetic with unit fractions or mixed numbers (e.g. a ¼ + ½ = ___; 1 ¾ ÷¾ = ___). The students were advised to answer the questions they felt comfortable answering or guessing, and to skip those they did not understand. Pre-test data revealed that many of the students possessed a rudimentary understanding of fractional concepts that had yet to be formally aligned with the notational symbols and procedural algorithms for adding, subtracting, multiplying or dividing with fractions. One student for example, when asked what "one-half" meant, answered, "half of a number." In fact, said student could even calculate one half of numerical values (e.g. half of 20) and had demonstrated a developed capacity for mental calculations of basic arithmetic. However, when asked what 1/3 + 1/3 means, the same student curiously answered, "one whole." This revealed that basic operations involving addition with the symbolic form for fractions were still unfamiliar, even though the student was familiar with concepts and operations such as halving and doubling.

Following the pre-test, I then developed a series of small-group tutorial sessions with the students in the special education class. For two weeks, I acted as teacher-researcher in 20-30 minute sessions with groups of 2-3 students. Each session was video recorded and facilitated under "live" classroom conditions (see Figures 15a). This involved working with some of the students at a "mathematics station" (with the help of the classroom paraprofessional aide), while the classroom teacher conducted her normal lesson with the remainder of the class.

A typical tutorial session began with a planned sequence of tasks, conducted in the format of a guided tutorial session. Similar to Socratic teaching methods as well as the clinical interview tradition (Ginsburg, 1997; Piaget, 1970, 1971) an initial question would be posed by the researcher/tutor (e.g., "Why is this measure labeled 1/4"?). Vitally, and however much as possible, explicit answers to said questions were never provided to the students. Instead, the tutor would attempt to guide the students to reach the desired conclusion by prompting them to perform measurement tasks. For example we might first begin by iterating water using the ½ cup measure into the one whole cup measure until it was filled. The question with regards to the ¼ cup measure would then be repeated.

Examples of other activities that were performed were: using the measuring cups to visually compare the unit fractions 1/3 and ¼ by placing the respective measuring cups side by side, or "discovering" how many iterations of pouring water from each of the respective unit fraction cups were required to fill a one cup vessel. Following each activity, the question would be posed again, and the post-activity answers recorded.

Through the data collection period, field notes and video results from each session were analyzed, and the subsequent days lesson adapted in response to researcher observations. Based on the perceived ease or difficulty that students appeared to have with the activities and/or questions, adaptations would be made. Some examples of data-driven adaptions to the research protocol included rearranging the sequence of an activity or the wording of questions/tasks. A question that was promptly and confidently answered by the majority of students was rated as too easy. Conversely, if the majority of students appeared hesitant and/or unable to answer a given problem, it was deemed as likely too difficult.

After completion of the initial pilot collection, a grounded theory approach (Corbin & Strauss, 2008) based on content analyses of both video and written assessments was used to generate preliminary categories of observed problem-solving strategies for fraction-arithmetic tasks. Specific emphasis was placed on instructional artifacts as units of analysis, and specifically triangulating students' recurring error types around an artifact and the artifact's apparent cognitive affordances. The working assumption guiding the error analyses was that types of errors that students perform provide insight into their current understandings of rational-number structures. By tracking changes in students' error patterns, the presumption is that researchers may gain useful insight into their cognitive shifts as well. Finally, in keeping with the grounded theory tradition, the pilot findings were then used to structure and analyze the Water Works study as well.

2.2.1 Rationale for piloting with a special needs population.

Given that the broader study would be conducted with general education students, other researchers may question the decision to begin piloting with a special needs population for work intended for general education classrooms. In selecting a special education class for the initial piloting, I both intended to call attention to the needs of this particular population of students and to demonstrate the power and efficacy of the Water Works activities.

My decision to work with a special needs population for the pilot study was motivated in large part by practical considerations. Again, based upon my own past experiences as both a special- and general-education classroom teacher, I believe that working only with typical to high performing students allows teachers/researchers to take many aspects of human cognition and learning for granted. A typically functioning student oftentimes (but not always) possesses many of the prerequisite cognitive constructs required to learn a given age- and developmentally appropriate concept. The instructor is allowed to build upon many assumptions. For example, if attempting to teach a "4th grade" mathematics concept/operation, the instructor can assume that the student has been adequately equipped with skills that would have been taught in 1st, 2nd, and 3rd grades. The instructional process can then appear effortless—an instructor attempts to "teach" something, and the typical to high- performing student "learns."

In the terminology of the cognitive-linguist Herbert Clark (2000), there is a high degree of "common ground" between teacher and typically functioning student—shared assumptions, beliefs, and knowledge. In other words, more things can be assumed and taken for granted. In this case, it is assumed that the knowledge base from prior academic years has been accurately learned and retained.

With special needs students this is not always the case. One *cannot* assume that they have retained the same academic knowledge. Indeed—given their restrictive classroom conditions—one cannot assume that they have been exposed to the material at all! Given these constraints, the researcher/teacher must search for the absolute common ground in order to devise interventions that truly help them grasp what might otherwise be elusive concepts. This can involve carefully scrutinizing one's choice of language, evaluating the perceptual features of a design, and/or considering the activity sequence. A special education group can thus serve as the proverbial "canaries in the mine." Any difficulties faced by these subjects with a lesson design may indicate a subtle, if not significant shortcoming of lesson design. Designing a lesson that is effective for these students requires a high degree of clarity in materials, activity, and instruction. Such improvements to the materials and/or lesson design will only make the material more accessible to general education students as well.

I would therefore argue that if an intervention can be effective for cognitively atypical students, the effectiveness could be greatly amplified for higher functioning students. In other words, if Water Works would work for students who were already far behind and struggling with mathematics, it would be even more likely to be effective for higher functioning and/or academically sophisticated students. Indeed, developmental psychologists Ceci and Papierno (2005) have observed that educational interventions originally designed for socio-economically disadvantaged students ironically have the effect of widening achievement gaps when applied to students from more privileged backgrounds. While this effect could likewise apply to an intervention such as Water Works, I believe that any risk of broadening the achievement gap that might arise from improving the understanding in early fractions for *all learners* is acceptable given the established difficulty that the subject matter already poses.

Finally, the decision to pilot with a special needs population was motivated by both ethical concerns and historical precedence. Historically speaking, the tradition of deriving insight into cognition and learning from working with special needs populations can be traced to the writings of scholars such as Vygotsky and Feuerstein (1969, 1980). Moreover, many of the most influential thinkers and pioneers of educational reform, such as Montesssori and Freire, began by working with children from impoverished and/or under-represented social, cultural, and economic backgrounds. The viewpoint shared by many of these visionaries is that education is a universal right and a means to lift the children of the downtrodden of society—and by extension, society as a whole.

Following the ethos that access to quality education and meaningful learning was a right for all learners, I wished to draw added attention to the needs of populations that are oftentimes ignored both by the broader educational research community and the school systems meant to support them. Students with special needs encompass a broad spectrum of characteristics that require cognitive, emotional, or physical restrictions that cannot be accommodated for in a general education setting. Thus, in the same "special education classroom" we might find a child whose cognitive functions are below age-appropriate markers, as well as a child who functions at or above age-level but needs physical and/or emotional supports, as in the case of children with epilepsy or bi-polar disorder respectively. It is my belief—based on anecdotal experience as a teacher and observations of classrooms as a researcher—that many children assigned to special education classrooms at an early age fall behind in terms of academic milestones not for lack of cognitive ability, but for want of stimulation and exposure given the (oftentimes necessary) restrictive conditions of the special education setting.

2.3 Water Works

Based on the encouraging progress made by students during the pilot, the decision was made to scale the study upward using students from two Grade-4 general education classrooms (n=40) from the same public elementary school to test an experimental tutorial sequence. A Grade 4 classroom from another public school using the same curriculum, Everyday Mathematics, would serve as a control (n = 26). The study would introduce an instructional sequence for early fraction concepts and arithmetic based upon additive reasoning. In one treatment condition the Water Works tools would be used. In another treatment condition, more conventional representations such as area models and number lines would be used. The control group would receive no direct interventions apart from their participation on three assessments.

Data collection spanned 7 months of the students' 4th grade academic year, and coincided with the scheduled introduction of fraction concepts/arithmetic according established by the mathematics curriculum already in use in the classrooms.

2.3.1 Pre-Assessment

Similar to the pilot conducted with the special needs students, I began by administering a pre-test to gauge the general education students' current levels of fraction understanding. Questions were adapted from the pilot assessment (derived from items typically found on statewide proficiency examinations for grades 3, 4, 5, and 6). The selection of items was further informed by the responses of students during the initial pilot and feedback from the participating teachers. Given that the subjects were currently in Grade 4, the expectation was that students would easily answer some of the questions (i.e. the Grade 3 items), but would not be able to solve problems that were typically taught in later grades, such as those involving multiplication/division with fractions.

The rationale for selecting problems not normally taught at grade level was to partially control for the effects of instruction. If a child's primary exposure to the problem-solving skills necessary for tackling such problems was the intervention, then it stands to reason that intervention was likely to have contributed towards this development. For example, if a child could correctly answer Grade 5 or 6 level problems *after* the intervention, then it could be assumed that the intervention—more than classroom instruction focused on grade 4 problems—had an effect.

The assessment was presented as a worksheet that was also orally administered to each student individually. Unlike the multiple choice items typically found on statewide exams, the majority of the items required students to formulate their own answer as opposed to random guessing (students were still able to guess on items that involved comparing the relative magnitudes of two fractions). Students were allowed to either write down their answer or to verbally explain it. The interaction was video recorded such that both verbal utterances (if any) and written inscriptions (if any) could be used in conjunction to determine a given student's understanding of a given problem.

The pre-test data revealed a heterogeneous range of student knowledge with respect to early fraction understanding. Moreover, teacher feedback revealed a range of academic tracking that included students selected for "gifted and talented" enrichment at one extreme, and students with individualized education plans for learning disabilities at the other. This heterogeneity is typical of many elementary school classrooms, and adds to the pedagogical challenge for teachers and curricular designers alike.

After the students had completed pre- and subsequent assessments, I asked them to elaborate upon their thinking for a given problem. Examples of prompts included, "Can you explain how you got this answer?" or "How were thinking about this problem?" Eliciting their problem solving strategies (where possible) would theoretically provide insight into their informal, non-procedural problem solving approaches to novel problems, and ultimately to compare and contrast how the different instructional artifacts (Water

Works tools versus conventional media) informed their thinking and problem-solving heuristics.

2.3.2 Experimental assignment & interventions.

Students from the two intervention classrooms were then randomly assigned to one of two treatment groups. The conventional treatment group would engage in the assessments and tutorial sessions using whichever conventional tools they felt comfortable with such as area models and number lines. The experimental group would engage in the similar activities albeit instruction would be complemented with the Water Works material activities.

The majority of instructional activities were conducted as one-on-one, tutorial sessions between the researcher and student. The basic format involved allowing students to answer prompted items from a worksheet on their own, followed by explicit instruction on occasions if the students could not correctly answer. In this manner, both the "spontaneous" and "guided" responses of students could be captured (Ginsburg, 1997).

Students in both conditions received instruction relating to specific fraction concepts and arithmetic operations without being taught the formal procedural algorithms. The instructional sequences were directly adapted from and informed by the activities used during the pilot study. Instruction in both conditions was similarly oriented around building students' mathematical competencies on fraction concepts and operations that they encountered in the initial worksheets. The presumption of the study design was that inferences about students' mathematical growth could be measured, contrasted, and compared between the two conditions over time.

To reiterate, students in both conventional treatment and experimental treatment conditions were provided the same assessment items. The point of differentiation would be in the instructional intervention provided to each group. To be more precise, a student in the conventional treatment group would be allowed to choose the instructional artifact (number line or area model) they personally felt most comfortable with. For a given instructional sequence, the student in the conventional treatment group might then be taught to reason about adding or multiplying unit fractions in terms of repeatedly iterating their choice of the a drawn vector or unit of an area model. A student in the experimental condition for the same lesson on adding or multiplying fractions would be taught using the measuring cups and volumes of water,

2.4 Overview of Logistical Considerations

A logistical challenge that ultimately shaped my data collection plan involved accommodating the day-to-day demands and requirements of the participating teachers. While both teachers and the school administration were interested and supportive of the study's objectives, neither could/would commit more than an hour of instructional time on any given day. This was understandable because many teachers feel pressured by their seemingly never ending stream of curricular responsibilities and the limited time available each school day to address them all. Moreover, the teachers were participating in the research on a purely voluntary basis, with no compensation. Thus, in order to maintain access to the subjects, I needed to preserve a good working partnership with the teachers themselves.

Regular, whole class instruction was deemed to be logistically impractical given that students in both classes had been randomly assigned to one of the two treatment groups. It would have been impractical to repeatedly ask the two classroom teachers to separate and divide their classes in order to accommodate my study. Moreover, it was methodologically inappropriate given that the study concerned individual students' learning trajectories of specific mathematical concepts as they interacted with artifacts. The small group activity used in the pilot was also deemed unsuitable for the given research objectives, given that the general education classrooms were much more crowded than the special education one. Although an objective of the research was to test the efficacy of artifacts and curricula that can be used in normal classroom conditions, the concern was that students not immediately involved in the activity would overhear, appropriate, and build upon the mathematical utterances of students engaged in the tutorial sessions. Thus, routine whole class—as well as small group—instruction would distort a researcher's attempt to isolate particular learning phenomena in individuals. The compromise was made to introduce the intervention for each treatment group as a whole class activity/intervention. Each subsequent intervention was then conducted by working one-on-one in with students in the hallway directly outside their classroom door.

2.4.1 Study timeline: scope & scale

Another legitimate concern with qualitative studies conducted under the banner of design-based research is that the data are often drawn from either rather small sample sizes, or from very limited time-scales. Understandably, working with qualitative data is extremely time-intensive both in terms of collection time and analyses. For example, to collect 1 hour of continuous video footage per student with an n=54 requires a minimum of 54 hours, not including setup and transitions between students. Given that the amount of time devoted to mathematics instruction at the school site was approximately one hour each day, this allows for a window of 5 hours per week to collect this data. If one were limited to operating at the rate of one student per hour, the entire process could take close to two months to complete (this also assumes optimal operating conditions and no unplanned delays). Of course, increasing the number of data collectors would decrease the amount of time, but to do so may require additional resources, which may or may not be immediately available to the research team.

A methodological concern that arises when the data collection period is protracted is that the parameters of experimental control may become compromised. This is particularly true in cases of classroom learning, where students might encounter the constructs under investigation through their normal interactions with peers, teachers, or their friends, parents, and siblings. Thus, while students at the start of a study might genuinely benefit from a particular intervention, those at an extended tail end may acquire the same concepts from alternate sources in the interim, which would weaken the strength of claims that could be made about an intervention.

A more time-efficient strategy for managing larger sample sizes is to utilize written assessments. However, while entire schools of measurement support their utility, the tradeoff of using such forms of assessment is that they fail to capture brief, ephemeral moments such as gestures and utterances that may represent pivotal moments in the ontogenesis of student insight and ideas (see Abrahamson, 2009; Schoenfeld, 2007).

As a result, many published design-based studies will present summative quantitative analyses from all the participants yet then proceed to elaborate only on selected paradigmatic case-study episodes from only a small number of subjects that intensively focus on extremely small windows of time (even minutes of an hour). Some researchers take further measure to ensure research transparency by making the data available to scrutiny by the broader research community. Still, these measure do not address any power constraints causes by the n-power issue.

2.4.2 Longitudinal microgenetic clinical snapshots

The mixed-methods approach I applied combined both clinical tasks and written assessments to capture multiple, brief snapshots of student captured learning over the longitudinal arc of a school-calendar year. While the overall premise is certainly not new, this approach may nonetheless be of practical and pragmatic utility to researchers interested in studying learning phenomena settings such as schools where access to subjects can be limited.

A key aspect of the approach is that both tasks and assessments were designed to fit into approximately 5-10 minute sessions. In this case, rather than a single, hour long clinical interview as the primary evidence of a particular student's learning, I collected multiple "snap-shots," or data points at roughly the same total cost in time per student as a one hour-long interview. While restricting the window of observation clearly restricts both the amount of data and the claims that can be made per student, the belief was that carefully tailoring the instructional interaction would allow the researcher to focus on specific aspects of cognitive development, in this case the relationship between representational media and learners' *initial* conceptualizations of fraction notation/operations.

The notion of focusing on limited windows of interaction partially follows a precedent established in mathematics educations for the "microgenetic" analysis of brief intervals of video data (Schoenfeld, et al.,1993), as well as the Piagetian tradition of clinical tasks interviews. In classic microgenetic analyses, large amounts of data are progressively winnowed down to highlight representative moments of cognitive interaction/discovery that can be re-evaluated across the broader corpus. In the snap-shot approach, the restricted windows of recorded activity are winnowed down and informed by prior field work, in this case, from analyzing data from the initial, more loosely structured piloting sequences.

An important affordance of the snapshot approach is that each individual interview can be completed in a relatively short amount of time. Working in this manner, I was able to individually video record a given instructional interaction with each of 50+ students over the course of 2-3 school days. The schedule below (see Table 2) demonstrates the cost in time of the micro-genetic snapshot technique for subject "LU," one of the 63 participants in the study.

Table 2
"Snapshots" to date collected for subject "LU."

Date	Nov 20	Dec 14	Jan 15	Jan 25	Mar 19	May 2
Activity	Pre Test	Intervention 2	Assessment 1	Assessment 2	Assessment 3	Post Test
Time	14 min	4.5 min	10 min	9 min	11 min	10 min

One can observe from this table that six unique data points have been generated for this specific student, at a total cost in time of approximately one hour. Note that I am able to track students' learning trajectory over time as well, as the data set includes both qualitative video recordings and written assessments of students' mathematical understanding at each unique point in time. One planned objective in structuring such brief interventions in this protracted manner is to demonstrate long-term and long-lasting effectiveness of the Water Works intervention—and by extension—its pedagogical utility as a means for encouraging mathematical development.

Due to various factors beyond the control of the researcher, such as student absences, teacher decision, etc. etc., 14 of the treatment subjects did not complete all of the tasks. Therefore, the final sample that will be analyzed for the dissertation involves a total of 40 students (conventional treatment, n = 19; experimental treatment, n = 21). Note also that the control students were not individually assessed in this manner.

The instructional objective of the tutorial sessions would be to help the students to formulate a *qualitative* understanding of individual unit fractions and the part-to-whole concept. At the minimum, the goal was to ensure that they clearly understood basic fractional concepts, notations, and operations that they would encounter in their grade-level assessments. When deemed appropriate—and as proof-of-concept of the instructional potential of the water works media—connections to more advanced fractional concepts/operations would be attempted.

2.5 Limitations

An acknowledged limitation of this study design is that in both conventional treatment and experimental groups, a compounding factor is the quality of instruction that each subject receives for a given intervention. This can call to question the reliability of the results, as one can easily imagine a supremely skilled instructor conducting a tutorial session with conventional media and the students far exceeding the gains made by students in an experimental setting conducted by a mediocre instructor, or vice versa. A variation of this problem concerns the differences in students' backgrounds, extra-curricular opportunities, and individual abilities. Students come to a study with more/less exposure to a given subject matter and varying levels of innate ability. Evidence of this is reflected by the handful of students in each of the study groups who demonstrated a reasonable to high level of proficiency with fraction arithmetic at the onset of the study.

Another legitimate concern of the study design is the potential for researcher bias. Given that I was the individual conducting and designing the intervention, the possibility that I would consciously or sub-consciously bias the results of one design in favor of another is a reality that must be acknowledged and addressed.

In speaking to the first concern, student/teacher quality/ability is a reality confronting virtually all intervention-based education research (as well as a measurable fact of life in real-world classrooms the world over). These are simply the realities confronting educational research that all studies acknowledge and must account for. Typically, this can be addressed by a more tightly controlled, large-scale-design that is beyond the scope and resources available in the current study.

With regards to the second, and more pressing concern of researcher bias—which suffice it to say is a concern across virtually all social-science research involving direct researcher intervention—the best recourse is transparency in method such as the involvement of a larger community of reviewers of both the original data (i.e. access to original audio/video recordings, transcripts) or relying upon inter-rater consensus when developing the coding schemes for example.

To account for the fundamental difference between the Water Works activities and the comparative looseness of the more "traditional" instructional approach selected for the tutorial sessions, special care will be taken when interpreting both quantitative and qualitative results in the study. In other words, the nature of the claims to be presented in the study will not be presented in the form of "A is better than B," but rather, "Here are some interesting *differences* between A and B."

A third and final important limitation of the overall study design is that students in the control group completed three of the assessments (the pre-test, penultimate assessment, and post-test), but did not receive any comparable pedagogical interventions. As of consequence, it would not be very surprising if the students in the treatment groups receiving instruction outperformed the control subjects on tasks involving fraction arithmetic. To use a crude example, a group of students taught to sing a particular song would likely outperform students who were not taught said song on a test of the songs lyrics. A superior study design would also include a traditional algorithm group that, at the very minimum, received instruction on the formal algorithms typically taught at higher grades. However, this was not possible during the initial data collection given personnel and resource constraints.

Despite this oversight in the study design, the control group, as constituted, nevertheless provides an important baseline for evaluating the potential efficacy of the treatment. This is particularly the case if one bears in mind that the total time per student for the pedagogical interventions used in the treatment groups—brief 5-10 minute sessions conducted at lengthy, irregular intervals, in which formal algorithms are *never* explicitly taught to students—effectively totaled less than two hours, or approximately the amount of time that would be used to provide classroom lessons! In this light, the claim that the fundamental design rationale for the pedagogical treatments—namely, building upon students' additive reasoning skills as the bases for fractional arithmetic understanding—was an effective pedagogical strategy becomes more plausible.

CHAPTER 3

To recap, the overall study employs a quasi-experimental design wherein subjects divided into one of three conditions: (1.) a conventional treatment group (n = 19), (2.) the Water Works treatment group (n = 21), and (3.) a control group receiving no treatment (n = 26). Students in all three conditions were from public elementary schools that currently use the Everyday Mathematics curriculum. This chapter presents an overview of the pre-test assessment administered to the general-education students.

3.1 Overview of the Pretest

The pre-test involved twenty items adapted from fraction arithmetic problems that typically appear on grade 3, 4, 5, and 6 statewide mathematics assessments (see Table 4 for a detail of items and a breakdown of student results). The first 10 items of the assessment included problem types and concepts that students would normally encounter in grades 3 and 4. The remaining items (11-20) included concepts and operations such as subtracting, multiplying, and dividing unit and mixed-number fractions that most students are not typically exposed to until grades 5 and 6. The final problem in the pre/post-assessment—1 ¾ ÷ ½ —is borrowed from Li Ping Ma's (1999) work with elementary school *teachers*. It is also worth noting that the unit fractions on the assessment were commonly occurring ones such as ½, ⅓, and ¼.

The expectation for the pre-test was that the majority of students would be able to answer many, if not all of the first ten, grade-level appropriate items. Conversely, it was assumed that the majority of students would have difficulty with the latter half of assessment—items 11-20—given that fraction arithmetic is not introduced in the curriculum at this juncture in students' academic progression (Fall semester, grade 4).

3.2 Results

Table 3 below presents a side-by-side comparison of the student averages for problems 1-10, 11-20, and for all items combined, for each of the respective groups. Table 4 provides a comprehensive list of all 20 pre-test items and a breakdown of how students performed on each item by treatment group.

Table 3
Pretest Results

	Items 1 - 10	Items 11 - 20	All Items Combined
Conventional Treatment	70.5%	21.1%	45.8%
Experimental Treatment	64.2%	26.1%	45.2%
Total Treatment	**67.2%**	**23.8%**	**45.5%**
Control Group	66.5%	19.2%	42.9%

Table 4

Pre-Assessment Items & Percentages of Students Who Answered Correctly

#	Question	Conventional (n=19)	Water Works (n=21)	Combined (n=40)	Control (n=26)
1	What does ½ mean?	100%	95%	98%	100%
2	What does ¼ mean?	95%	95%	95%	96%
3	What does 1 ½ mean?	95%	90%	93%	96%
4	What does 2 ¼ mean?	89%	71%	80%	92%
5	½ + ½ =	79%	86%	83%	88%
6	1/3+1/3 =	58%	38%	48%	42%
7	1/4 + 1/4+1/4	58%	38%	48%	42%
8	1/2 + ¼	37%	24%	30%	15%
9	1/2 > 1/5	79%	67%	73%	73%
10	1/2 = 2/4	37%	38%	38%	27%
11	2 1/5 < 2 ½	74%	71%	73%	73%
12	5/4 < 4/2	58%	52%	55%	50%
13	1/2 x 6	16%	29%	23%	12%
14	1/4 x 8	16%	29%	23%	12%
15	1/2 div ¼	0%	10%	5%	12%
16	10 div ½	5%	14%	10%	8%
17	2 1/2 + 4 ¼	16%	14%	15%	12%
18	4 1/2 x 2	11%	24%	18%	12%
19	4 1/2 div 2	5%	14%	10%	0%
20	1 3/4 div 1/2	0%	0%	0%	0%
	Percentage Correct	Conventional 45%	Water Works 46%	Combined 45.5%	Control 42.9%

Collectively, students in the treatment groups (conventional & Water Works combined, n=40) averaged 9.1 correct answers or 45.5% of the problems on the assessment (median = 8, mode = 8, standard deviation: 3.815, variance 14.55). Breaking this down in terms of the easier items (1-10), students answered 67.2% of the easier items correctly. Focusing only on the more challenging items (11-20), students in the treatment groups could initially only answer 23.8% of the items.

When we compare the results of the two treatment groups with one another, the pre-test scores would appear to suggest that both groups (conventional and Water Works) appeared evenly matched in terms of their familiarity with fraction concepts and arithmetic operations. On average, students in the conventional (n = 19) and experimental (n=21) group correctly answered 9.2 (45.8%) and 9.0 (45.2%) of all items respectively.

On average, students in the control groups (n=26) performed similarly to the students in the treatment group. Control subjects averaged 8.6 correct answers or 42.9% of the problems on the assessment (median = 8, mode = 8, standard deviation: 2.762, variance 7.63). Breaking this down in terms of the easier items (1-10), students in the control group answered 66.5% of the "easier" items correctly. Students in the control groups also had difficulty with the more challenging arithmetic items (11-20). On average, students could only answer 19.2% of the more challenging pre-test items involving fraction arithmetic correctly.

When we compare the overall pre-test results of the treatment groups (45.5% correct) to the control group (42.9%), we can observe that the control group scored slightly lower on average. However, performing an unpaired t-test to compare the pre-test results of the treatment and control groups reveals that t-statistic is not significant at the 0.05 critical alpha level, $t(64)=6.00$, $p=0.55$.

Using a boxplot to visually the pre-test results (Figure 27), we observe that the results for all three groups are indeed clustered around similar medians.

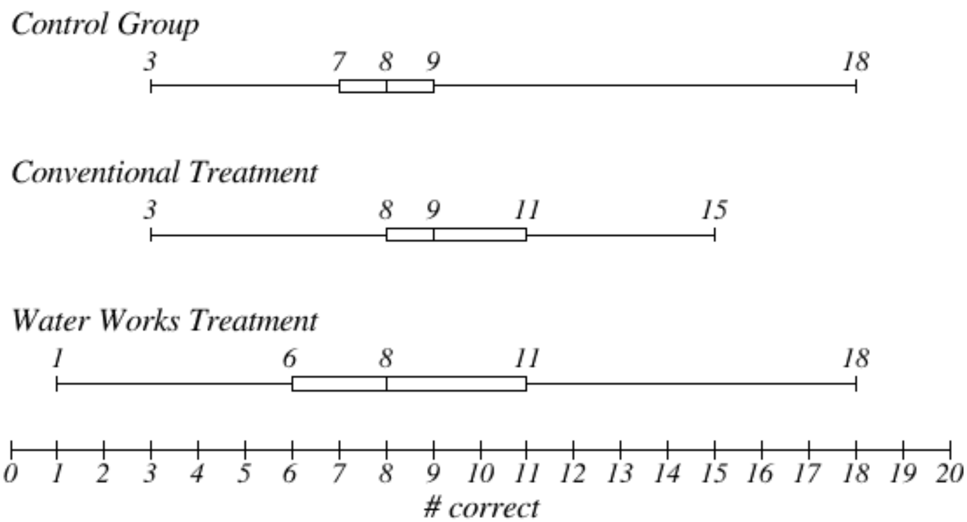


Figure 27. Overall Pre-Assessment items Answered Correctly (items 1-20)

Thus, while students in the two treatment groups both scored higher on average than the control on the pre-test, we can see a reasonably comparable distribution centered around a similar means of 8 or 9 across all three groups on the pre-test items. The skew in number of correct answers is not particularly prominent in any of the groups. Students' range of responses was most variable in the Water Works treatment group. As might be expected of a heterogeneous population that included both ends of the special-education spectrum (high-need learners and gifted-talented), there were a few outliers, with one student correctly answering 18 out of 20 of the items, whereas one student could only answer 1.

Focusing on the distribution of results for pre-assessment items 11-20 exclusively (Figure 28), we see that all three groups appear evenly matched in terms of their competency (or in this case, lack thereof) with regards to fraction arithmetic tasks. This result confirms the expectation that the majority of students in the study would have difficulty with these items for the simple fact that they have not been exposed to formal problem solving heuristics for fraction arithmetic problems. Once again, outliers on the high end can be observed—clearly some students have been to fraction arithmetic strategies outside of their regular classroom activities.

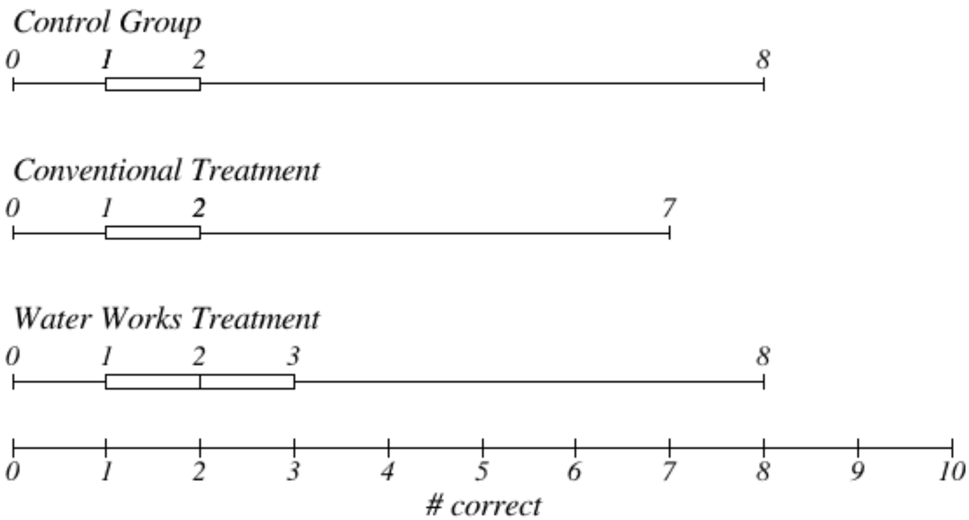


Figure 28. Pre-Assessment items 11-20 Answered Correctly

3.3 Arithmetic proficiency scheme

To further parse the data, a categorical scheme was developed to rate the initial results of students on the pre-test as either not proficient, proficient, and highly proficient[4] based upon their ability to accurately answer the more difficult (for their grade level) arithmetic items on the pre-test (11-20). These categories were established based upon the mean combined average (2.3) and standard deviation (2.6) of results for items 11-20 on the pretest. Students who could answer less than 50% of the items (4 or fewer out 10) correctly were rated as not proficient (at fraction arithmetic tasks); students scoring approximately 1-standard deviation higher than the mean (50-79% accuracy) were rated as proficient; and finally students scoring more than 2 standard deviations above the mean were designated as "highly proficient" (80 – 100% accuracy) on the fraction arithmetic tasks.[5]

Applying the proficiency metric to the pre-test results, we initially observe a higher percentage of students in the Water Works group performing at the "proficient" or "highly proficient" level than in the other groups; and that there were a greater percentage of proficient/highly proficient students in the treatment groups combined than in the control group.

[4] Note that the baselines percentage levels for not proficient, proficient, and highly proficient were fixed according to the pre-test assessment mean and standard deviation (and not recalibrated per assessment).
[5] Reader's more familiar with Kilpatrick, et al.'s (2001) definition of proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) will note that I am measuring only procedural fluency with this metric.

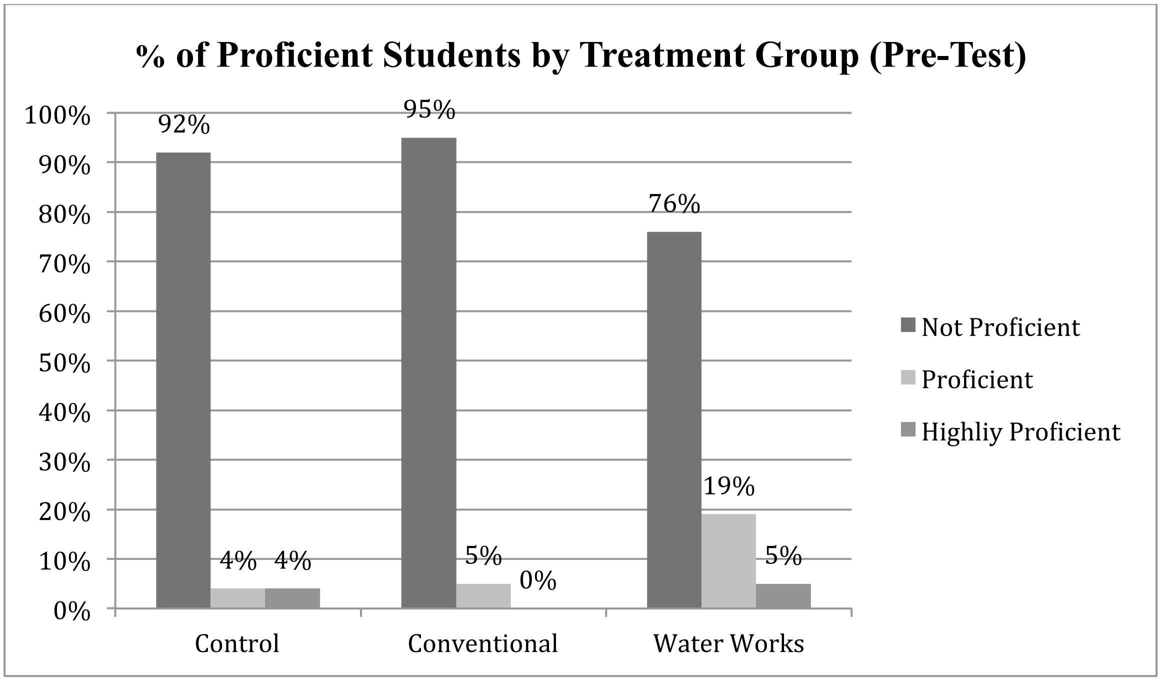


Figure 29. The proficiency categories were based upon the mean combined average (2.3) and standard deviation (2.6) of results for items 11-20 on the pretest. Students who could answer less than 50% of the were rated as not proficient; students scoring approximately 1-standard deviation higher than the mean (50-79% accuracy) were rated as proficient; and finally students scoring more than 2 standard deviations above the mean were designated as "highly proficient" (80 – 100% accuracy) on the fraction arithmetic tasks

3.4 Discussion

The pre-tests were used to establish a baseline for future comparison between groups in order to evaluate the efficacy of the instructional interventions. For the most part, all three groups (control and experimental groups 1 and 2) appeared evenly matched in terms of their overall results on the pre-test. As expected, when presenting students with subject matter that has not yet been taught in the classroom curricula, the majority of students struggled. A handful of students appeared to have previous exposure to the subject matter, with a higher percentage of these students having been randomly assigned to the Water Works treatment. The safe assumption in such cases is that they were taught the concepts/operations outside of the classroom context.

3.4.1 A Closer Look

Typically speaking, students displayed a range of naïve heuristics based upon the rules of whole number arithmetic. Bearing in mind that the vast majority of students had not yet been taught to solve fraction arithmetic problems, it may prove useful to examine some of the naïve problem solving strategies on display.

Examples of typically occurring errors would be to sum the numerators and the denominators when adding unit fractions (i.e. ½ + ½ = 2/4, see Table 5 for typical examples of naïve student problem-solving heuristics). Based upon consistently recurring patterns of student errors, a coding scheme was developed with the aid of the classroom teachers and feedback from fellow researchers. This coding scheme was then reapplied to the full corpus of student responses.

Table 5

Examples of Erroneous Problem Solving Strategies for Fraction Arithmetic

Typical Errors for Fraction Addition		
	Adds numerators, adds denominators	1/6 + 1/6 = 2/12
	Adds denominator only	1/6 + 1/6 = 1/12
	Adds numerations + Regrouping of denominator	1/6 + 1/6 = 3/12
Typical Errors for Fraction Multiplication		
	Adds fraction to whole number	9 x ½ = 9 ½
	Multiplies by adding (incorrectly)	1/2 x 1/2 = 2/4
	Multiplies num/denominator with whole number	1/3 x 6 = 18 (6 x 1 x 3)

In figure 30 below we can observe how student MC (Water Works Treatment) applies an addition heuristic whereby the numerators and denominators are summed. Interestingly, MC skips Item #7, even though it would have been a simple calculation to (in)correctly sum 1/3 + 1/3 as 2/6 as was done with items #5, #7, and #8.

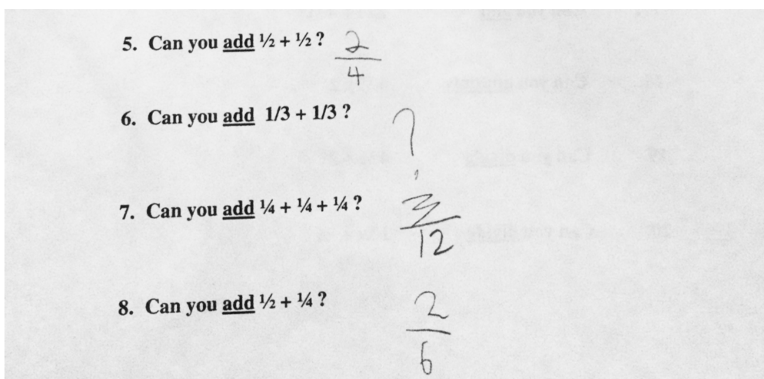


Figure 30. Pre-Test items for Student MC

Now, let us compare MC's sample with student SM, in the control group (Figure 31). Unlike MC, student SM shows some familiarity with the basic rules for fraction addition with like denominators. Indeed, SM successfully correctly applies the heuristic for summing fractions with like denominators for items 5,6, and 7. However, confronted with problem number 8 in which two unlike denominators suddenly appear, SM also opts to sum both the numerators and the denominators. This is contrary to the rule of "not adding denominators" that SM had already displayed some level of awareness of in "correctly solving" the previous problems.

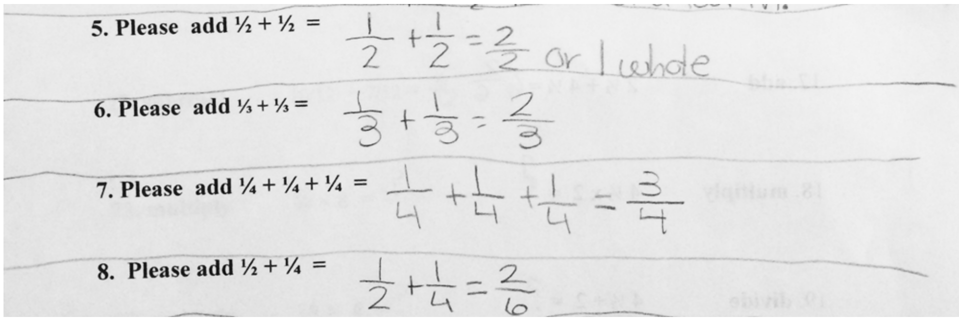


Figure 31. Pre-Test items for Student SM

What we observe here with both students MC and SM encapsulates the challenge facing early fraction educators. Confronted with novel circumstances involving mathematic problems, students instinctively apply tried and true problem solving strategies they have previously learned. Such strategies should be considered logical—albeit misguided—given that they are applications of proven of arithmetic (i.e. 2 + 4 = 6) that students have learned to date.

An interesting counterpoint to students who attempt to apply the rules of whole number arithmetic to solve fraction addition problems can be seen in the work of AJ (conventional treatment). AJ is able to sum Item #5 (½ + ½ = 1) and to accurately draw area-model representations of thirds and quarters. Like most of the other students however, it is apparent that AJ has not yet learned the procedural rules for adding fractions.

For Item #6, AJ draws two area models and partitions each model internally into thirds in order to visually represent the two addends. Note how the drawing for 1/3 on the left is also divided into halves, while the drawing for 1/3 on the right has been subtly demarcated into quarters (Item #6, Figure 32). After scrutinizing his own drawing, AJ's answer to 1/3 + 1/3 is "*about* ½ and one ¼" (emphasis mine). This approximation is reasonably close to the actual surface area indicated by problem. Interestingly, for items #7 and #8 (¼ + ¼ + ¼ and ½ + ¼, respectively), AJ also answers "one half and a quarter."

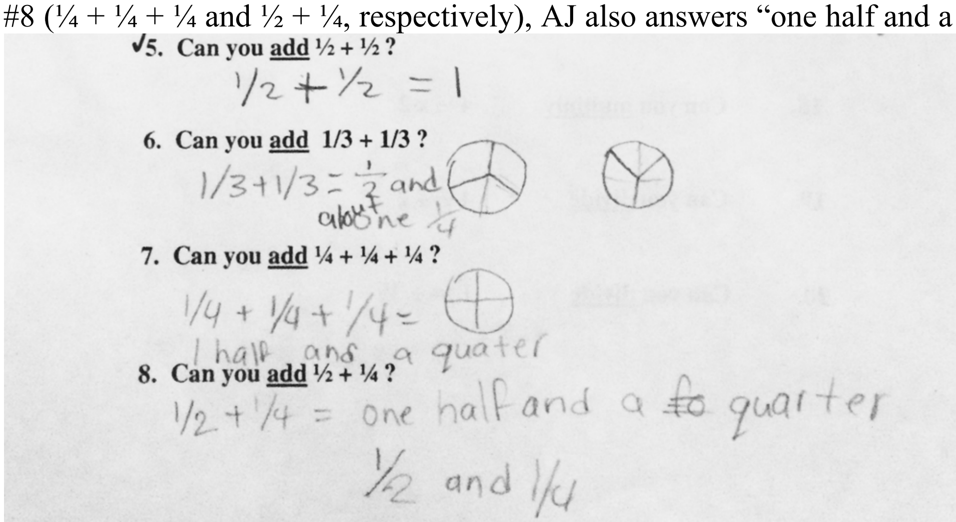


Figure 32. Pre-Test items for Student AJ

AJ's answers to items #6, #7, and #8 seem to indicate some degree of conceptual understanding with regard to the cognitive task demands required to solve the problem.

AJ understands that the surface area of each model will increase as a result of the additive function. AJ also understands the relationship between ¼, ½ , and one whole. For items #7 and #8, AJ's responses correspond with three-quarters. The reason that AJ cannot provide a "correct" answer 3/4 does not appear to be a conceptual one per se. Instead, it appears to stem from a lack of development with respect to the mathematical conventions for deriving and expressing the sum of two fractions.

Now let us look at Item #5 from the pre-test work for BK (Figure 33). Like AJ, BK already understands on some level that ½ + ½ is equal to one whole. BK answers Item #5 by drawing a complete area model circle. Moving onto assessment Item #6 however, observe how BK, unlike AJ, lacks the fine motor skills to accurately demarcate an area model representation into thirds. BK initially attempts to draw two models, but then erases them in order to draw a single circle that presumably will encapsulate an area equal to two-thirds of the circle.

At first glance, it seems as if BK—like AJ—understands on some level that 1/3 + 1/3 would encompass approximately two-thirds of a "one-whole" area model.

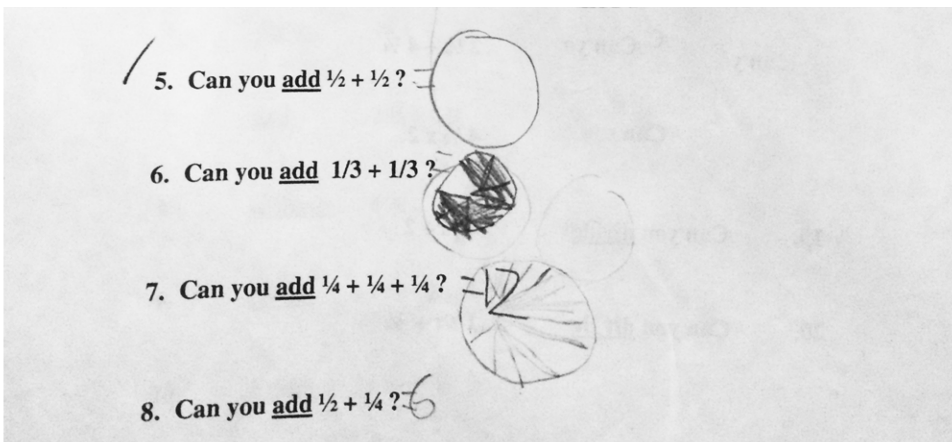


Figure 33. Pre-Test items #5-8 for Student BK

However, a closer inspection of BK's drawn representation of 1/3 + 1/3 (Figure 22) reveals that BK has also demarcated each of the two, shaded "third" portions into three subsections each, for a total of 6 sub-sections (Figure 34). It appears as if BK's problem solving strategy is contingent on summing the denominators of the fractions.

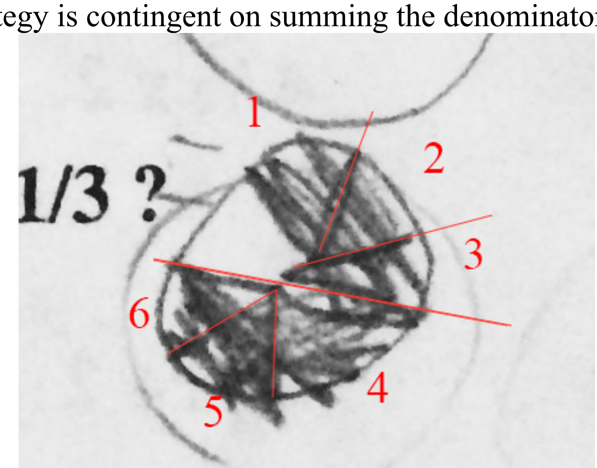


Figure 34. Close up of Pre-Test item #6 for Student BK

If we take BK's answers to problem items #7 and #8 into account, we can safely conclude that this was indeed the case. For Item #7, BK attempts to partition the area model into 12 discrete parts. For Item #8, BK simply sums the denominators and writes the number 6. Revisiting BK's drawing for Item #6, an alternate interpretation emerges: Unlike AJ—whose problem solving strategy was based on adding the ½ and ¼ units of drawn area models —the inscription was a drawn out thought-experiment in which BK attempted to visually render a naïve arithmetic strategy of summing the denominators.

What these (incorrect) answers reveal are systematic attempts by students to apply their previously learned problem-solving heuristics within a novel situation. More importantly, they would also suggest that students have yet to develop any alternative means and/or mental models for reasoning about problems involving fractional notation apart from arithmetic operations.

On the one hand, too much weight or importance need be placed on these observations given that for the majority of students, this may very well be the first time they have been tasked to solve problems of this type. On the other hand, the well-documented fact that many adults rely solely on procedural algorithm to reason about fraction problems underscores the importance of helping students to cultivate alternate problem solving heuristics and/or mental models of reasoning. But, when students invariably receive feedback that their answer is "wrong," the element of doubt in their own abilities to "do math" can begin to creep undermine their productive disposition (Kilpatrick, et al. 2001) and sense of self-efficacy with respect to the broader domain of mathematics (i.e. Zimmerman, 2000). It does not appear to be coincidental that many students appear to develop negative associations about mathematics once fractions are introduced to the curriculum.

Finally, let us examine student BD's (control group) correct problem solving strategy for Item #9 on the pre-test (circle the fraction that is "greater"). BD begins by drawing an area model representation of one whole, internally partitions it into two discrete parts, and then shades in one-part to visually represent ½ (Figure 35).

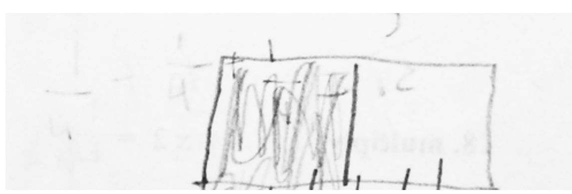


Figure 35. Close up of Pre-Test item #9 for Student BD

Next, BD draws a second rectangle underneath the representation of ½. BD initially attempts to partition it into 5 sections by drawing 5 lines—which inadvertently rest produces 6 subsections. BD realizes the mistake, and erases the fifth line. While it is evident that BD has a reasonably developed understanding of how to represent unit fractions, it is interesting to observe how the construction of a drawn area model diagram can be a challenge in and of itself.

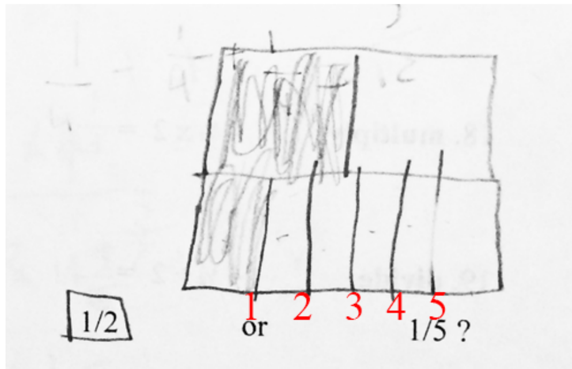


Figure 36. Another close up of Pre-Test item #9 for Student BD. BD has stacked two representations for ½ and 1/5 on top of one another in order to compare the relative magnitudes of the two fractions. Note how BD first draws 5 lines to demarcate the rectangle on the bottom into fifths before realizing that the end result is actually six partitions.

3.5 Summary

In summary, students in the control, conventional treatment, and Water Works treatment groups all performed similarly on the pre-test. Most students could answer most of items #1-10. The majority of students was still unfamiliar with basic arithmetic operating involving unit fractions and struggled with the pre-test items #11-20.

Students applied a variety of naïve problem solving strategies based upon their prior understanding of whole number arithmetic. Many students also attempted to use drawings to help them reason about fraction arithmetic. The ability of students to draw accurate diagrams varied in direct relation to the fine-motor skills of individual students. Likewise, the ability to *interpret* their own drawings varied widely, and appeared to be a function of their current level of fraction understanding.

A small number of students in each group displayed a moderate to high degree of proficiency with solving fraction arithmetic problems. Consider the work example from student KG (control group) in Figure 37.

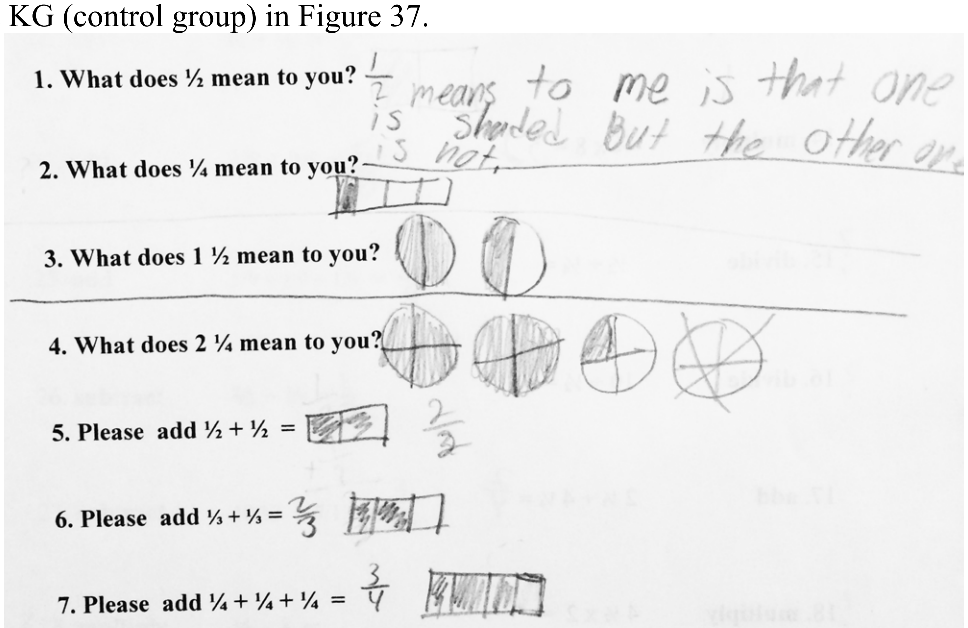


Figure 37. Pre-Test items #1-7 for Student KG

KG correctly sums the fraction addition problems and is also able to accurately represent the operations in terms of drawn area models. While ideal, responses such as KG's to the pre-assessment items were exceptions from the norm.

Overall, a higher number of students in the Water Works treatment group demonstrated proficiency with fraction arithmetic. However, there were no statistically significant differences in performance across the three groups. In the following chapters, we will describe the interventions used in the conventional and Water Works treatment groups and examine the results.

CHAPTER 4

Following the pre-test, students assigned to the two treatment groups participated in a series of pedagogical interventions. The general principle guiding the design of the intervention activities was to explicitly build continuity between learner's pre-existing numerical schemes and fraction concepts.

Readers familiar with the Everyday Mathematics (EM) Curriculum already in use at the study locations will recognize that this general principle is also used to teach multiplication at the end of fourth grade. In EM, multiplication is introduced to students by guiding them to additively iterate the internal partitions of area models and number lines (See Figure 38a and 38b). Students are also taught the formal algorithms alongside numerous other strategies such as converting fractions to decimals and/or percentages.

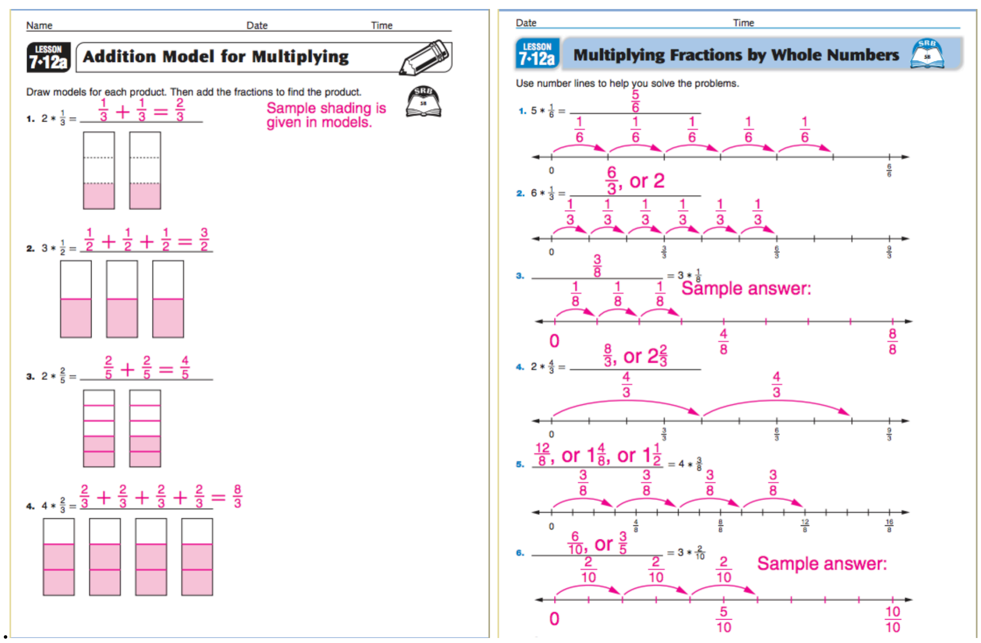


Figure 38a and 38b. Sample lessons from Everyday Mathematics fourth grade curriculum (Bell, et al, 2012). In EM, multiplication is likewise introduced to students by guiding them to additively iterate the internal partitions of area models and number lines.

However, for my overall lesson design—and to emphasize the argument that additive reasoning should be as the conceptual bases for early fraction instruction—the more ambitious goal would be to introduce addition, subtraction, multiplication, and division with fractions concurrently, and without any explicit reference to algorithms. Arithmetic operations and concepts involving fractions would be verbally explained and physically modeled in terms of an additive transformation. Students would likewise engage in pedagogical activities that explicitly build upon their competencies with additive reasoning. The operating hypothesis was that such an approach would compliment if not enhance current best practices. Proof of concept would be demonstrated by gains in student proficiency on fraction arithmetic tasks that would normally be introduced at the end of the 4th, if not 5th grade curricular years.

In the conventional treatment group, the activities were designed using area models and number lines. For example, in order to teach students how to multiply unit fractions and whole numbers (i.e. 1/n x Z) I would: (1) Draw a representation for one-whole; (2) partition it into *n* discrete units; and finally, (3) repeatedly shade in the area indicated by the fractional unit, *Z* times. No mention would be made of the basic algorithm formal algorithm.

In the experimental treatment group, the same lessons were further adapted to incorporate the use of kitchen measuring cups and water. Returning to the multiplication example, instead of drawing and shading a diagrammatic representation, a student would physically pour volumes of water from a fractional unit cup, Z number of times. The operating conjecture distinguishing the Water Works group from the conventional treatment group was that the unique affordances of the measuring cups and water—outlined in Chapter 1 and compared to traditional, drawn representations such as area model and/or number lines—would provide specific affordances for children learning fractions. In brief, I felt that the physically embodied actions and outcomes that arise through the pedagogical enactment of the Water Works activity design would help foster a higher degree of embodied conceptual coherence between students pre-existing numerical schemes and fractional arithmetic operations, particularly when compared to the use of conventional drawn media. The expectation was that students in the experimental treatment group would outperform students in the conventional treatment using the same measures—and that students in both treatment groups would outperform control subjects.

In the following sections, I present a high-level overview and detailed description of each of the interventions; quantitative results from student work; and where appropriate, in-depth qualitative analyses of student interactions to highlight interesting and/or important findings.

4.1 Intervention 1 / Assessment 1

4.1.1 Overview

The first intervention was a whole class activity conducted one month after the initial round of pre-tests. The objective of the intervention would be to introduce additive-based reasoning strategies for fraction arithmetic to students. Recall that there were two classrooms participating in the intervention. Students from two 4th grade classrooms were each randomly assigned to one of the two treatments groups, referred to from this point onwards as conventional or the Water Works (media) treatment.

For this first intervention, the students from each of the two treatment groups were separated into their two respective groups and administered separate whole-class lessons. Students in the control group did not receive a comparable pedagogical intervention.

Each treatment group received one, approximately 30 minute long, lesson wherein I introduced and explained the following fraction concepts/operations:

- The conventional unit of 1 whole and the unit fractions available thereof;
- The part-to-whole relationship between 1 whole and unit fractions;
- The equivalence ratio between 1/2 and 2/4 by iterating the 1/4 unit;
- A non-algorithmic strategy for adding/multiplying fractions by iterating units;
- A non-algorithmic strategy for subtracting fractions;

- A non-algorithmic strategy for dividing with fractions.

To summarize, addition, multiplication, subtraction and division were described in terms of an additive iteration. No explicit mention of the formal algorithmic operations was used. Given that a key element of the Water Works intervention design involved direct interaction with and transformations using physical objects, an "Elmo" document projector system was used to facilitate the demonstration (Figure 39). In this case, the document projector was used to display my interactions with the measuring cups and the resulting volumes of water to the entire class by projecting the enacted pedagogical demonstration onto a white-board.

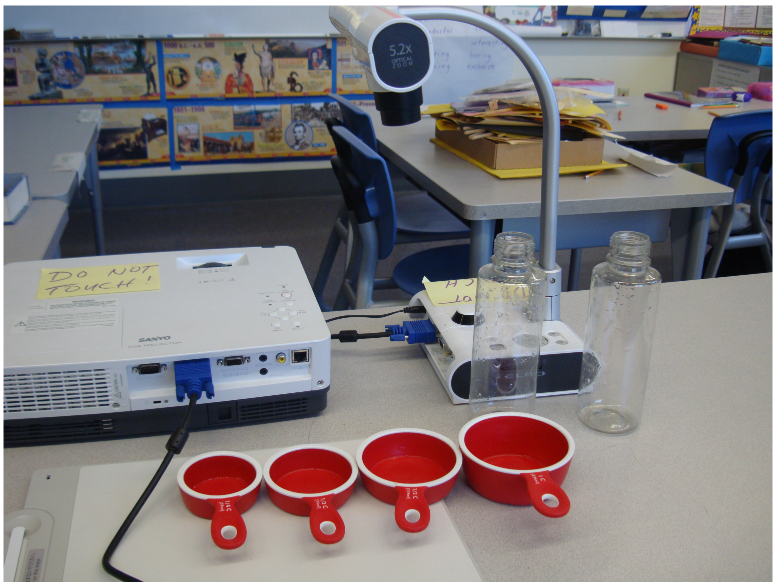


Figure 39. An Elmo document projector system is used to project the researcher led pedagogical enactment alongside the measuring cups and two transparent containers.

To help elaborate how arithmetic operations can be taught without the use of formal algorithms, consider the case of constructing one whole cup of water. In order to help students in the Water Works treatment group reason about the part-to-whole relationship between unit fractions and the whole cup measure, I would begin by asking, "Why is 1/3 cup called 1/3?" I would then demonstrate that three iterations of the 1/3 cup are required to fill the one whole cup. Students would literally watch me pour water from a 1/3 cup measure into the one-whole cup measure, three times.

To reinforce the point, I would then present two identical, transparent, cylindrical containers side by the side. I would then iterate three volumes of water using the 1/3 cup into one of the cylinders (see Figure 41). I would then iterate one volume of water using the one-cup measure into the other cylinder. I would then ask the students to compare the two volumes of water side by side (as projected onto the white board).

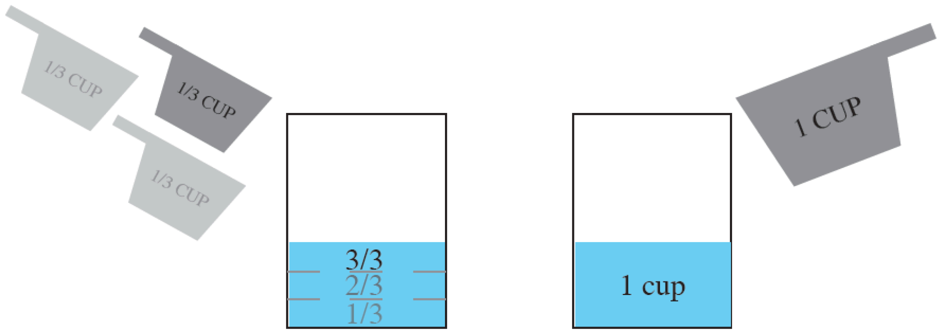


Figure 40. A side by side comparison of one cup volumes of water.

A similar demonstration would be performed for the conventional media group using *both* a drawn area model and number line representation (and shading in the corresponding sub-units). However, instead of observing me iterating volumes of water on a projector screen, the pedagogical enactment would entail me first drawing, and then repeatedly shading-in segments/sub-sections of a representation directly onto the white-board using markers.

To demonstrate equivalence between 1/2 and 2/4 to the Water Works group, I would iterate volumes of water from a ¼ cup measure into a 1/2 cup measure, twice; or shade in the appropriate units/vectors on a discrete area model or number line. I would repeat this performance using a side-by-side comparison of the water levels using the two matched, transparent cylindrical containers (Figure 41).

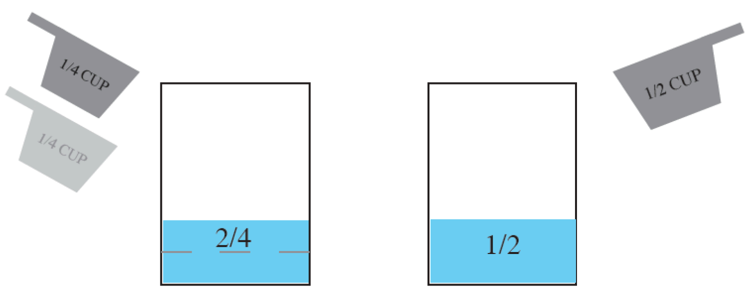


Figure 41. A side by side comparison of 1/2 cup volumes of water.

The enacted performance was intended to serve as the model for addition/multiplication. Subtraction was demonstrated as the reverse of such operations (reversing the additive iteration by scooping water out or erasing some shaded in drawing). Note also that by, by overflowing a one-whole cup (e.g. with four scoops of water using the 1/3 cup measure) or completely filling an area model/number line, one could demonstrate the concept of mixed-number fractions greater than 1 whole cup.

The arithmetic operation for 1/3 + 1/3 + 1/3 + 1/3 = 4/3, or 4 x 1/3 = 4/3 was presented to students in terms of repeatedly pouring water or repeatedly drawing an object (Figure 42).

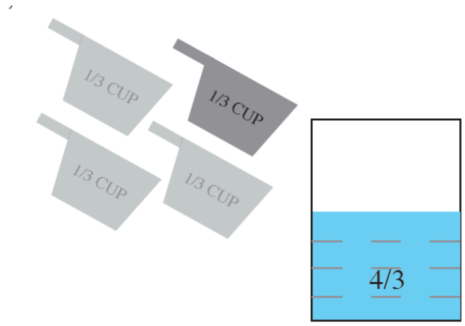


Figure 42. Using four iterations of 1/3 into a container to demonstrate addition/multiplication.

Finally, a division problem, 1 1/3 ÷ 1/3 = 4, was explained in roughly the following terms: “Given a volume of water equal to 1 1/3 cups (4/3). How many 1/3 cup measures could I fill?” To demonstrate, a 4/3 volume of water was first poured into an unmarked vessel (said volume was established by iterating the 1/3 measure, four times in front of the students). Then this volume was poured out to fill a 1/3-cup vessel, four separate times (Figure 43).

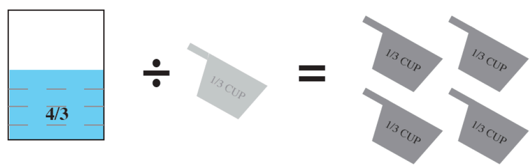


Figure 43. Using four reverse iterations of 1/3 from the container to demonstrate division/subtraction.

For the conventional media group, this operation was demonstrated by first drawing a discrete representation of 4/3, and then asking how many 1/3 sized units the area model and number line unit could be decomposed into. This is similar to the approach used to introduce fraction division in the fifth grade by the EM curriculum (Figure 44).

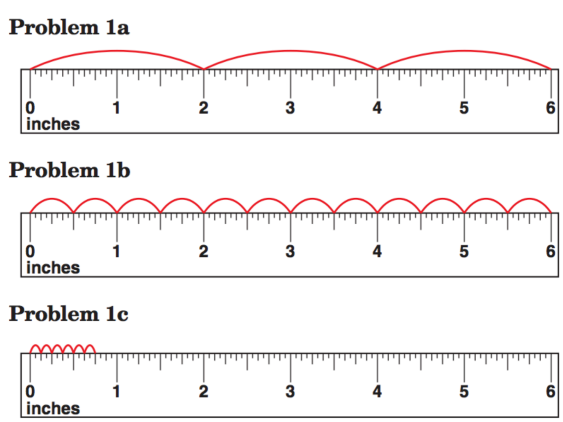


Figure 44. This sequence of drawings is used alongside the following prompts: 1a) How many 2s are in 6?, 1b) How many ½ are in 6. How many ⅛s are in ¾. (Source: Everyday Mathematics, pg. 681)

In EM, the division lesson begins by asking students to consider how many vectors of a given magnitude can be created from a greater given magnitude. Note how the visual representation—a ruler—is for all intents and purposes a number line. The conceptual progression is to move from division with highly familiar whole units and half-units, to fractions of fractional units. Observe also, how even though this is a division problem, the fundamental conceptual mechanisms for reasoning about this problem are essentially counting and measuring.

4.1.2 Results

Following this introductory lesson, a new assessment closely patterned after items 11-20 from the pre-test was then administered. A summary of the problems and how students in each of the treatment groups performed is provided below in Table 7.

Table 6
Intervention 1—Assessment Items & Results

#	Question	% Correct Conventional Treatment (n = 19)	% Correct Experimental Treatment (n = 21)	% Correct Combined Treatment (n = 40)
1	2 1/5 < 2 ½	89%	86%	88%
2	5/4 < 4/2	53%	57%	55%
3	1/2 x 6	79%	38%	58%
4	1/4 x 8	74%	38%	55%
5	½ ÷ ¼	26%	10%	18%
6	10 ÷ ½	11%	10%	10%
7	2 1/2 + 4 ¼	37%	38%	38%
8	4 1/2 x 2	42%	33%	38%
9	4 1/2 ÷ 2	21%	19%	20%
10	1 3/4 ÷ ½	0%	5%	3%
% Correct All Items		Conventional 48%	Experimental 32%	Combined Treatment 38%

Students in the conventional (n = 19) correctly answered 4.3 out of 10 items correctly. On the other hand, students in the Water Works group (n = 21) correctly answered 3.2 out of 10 items correctly. Using a boxplot in (Figure 45) to compare the intervention results for the conventional treatment and water works groups suggests that the Conventional treatment was more effective.

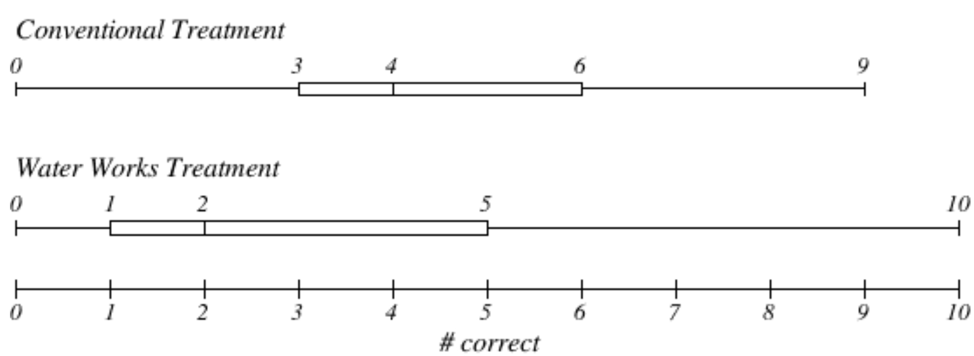


Figure 45. Intervention 1 Assessment items answered correctly.

A t-test was performed to compare the conventional group's performance on the pre-test items #11-20 with their performance on the Intervention 1 assessment to determine if the difference was significant. The t-statistic was highly significant at the .05 critical alpha level, $t(36)= 3.53$, $p=.0001$. Therefore, we can reject the null hypothesis that the intervention had no effect.

Likewise, a t-test was performed to compare the Water Work group's performance on the pre-test items #11-20 with the Intervention 1 assessment to determine if the difference was significant. The t-statistic was not significant at the .05 critical alpha level, $t(36)= 0.79$, $p=0.22$.

4.1.3 Discussion

For the conventional treatment group, the single, whole-class thirty-minute lesson in which fraction concepts and operations were presented in terms of additive operations appeared to have had a significant, measurable effect on students' performance on fraction arithmetic problems. On the other hand, a similar intervention using the Water Works media did not appear to produce any significant effects.

One possible explanation for the significant improvement for the conventional treatment groups is that students' prior familiarity with representations such as area models and number lines allowed them to more effectively focus on the lesson at hand. Rather than trying to make sense of a novel instructional artifact *and* learn novel concepts/operations, they could simply focus on the latter.

The students in the Water Works treatment group on the other hand, had an increased cognitive load. Not only were they learning novel concepts/operations, but they were also tasked with understanding how said concepts/operations applied with and/or "transferred" to an entirely new contextual system.

The notion of transfer has typically been defined in terms of an ability to apply knowledge or procedures learned in one context to new contexts (Lobato, 2006; Mestre, 2002). In recent years, the distinction is commonly made between "near transfer," the initial ability to apply learning in one setting to another that is closely related; and "far transfer" which refers to the ability to apply what learns in one setting to seemingly different and/or novel problems that share common underlying structures (Barnett & Ceci, 2002).

In the case of students in the conventional treatment group, it would appear that many of the students were able to near transfer what they had learned from the lesson. They were able to process the lesson that was presented to them—and more importantly, synthesize an appropriate problem solving strategy—when presented with fraction arithmetic problem notation. To channel both Piaget and Vergnaud, the students were able to adapt their fraction schemes because the novel information to be assimilated was presented within the context of a familiar pedagogical situation (conventional representations such as area models and number lines). Indeed, if we apply Vergnaud's (2009) conceptual pairing between scheme and situation we arrive at a simple, yet elegant way to describe the phenomenon of transfer: the ability to recognize that a conceptual scheme that originated in one situation is also applicable in another.

For the conventional treatment group, the argument can be made that because the situational context (in this case, the representational model used to illustrate the concepts) remains the same, the task demand—between how they are able to interpret the representational context with a familiar arithmetic operation—was less challenging. In the case of the group receiving this instruction on fraction arithmetic using the Water Works media, the context has shifted dramatically. The conceptual elements are arguably the same—arithmetic, the notion of unit fraction parts and wholes—but the context that introduces the pairing between the physical and mathematical operations is entirely novel.

This hypothesis appears to be tangentially supported by the case of subject CNC from the experimental group. What is interesting about CNC is that his initial conceptualization of fractions was based upon *time* and *musical notation*. During the pre-test, when asked to explain what one and-a-half meant for him, he answered:

> "I play piano—for a piano keynote if you play one whole, it's a whole note, which is four beats...like, four seconds. If it's one and-a-half it would be like… six beats. Six seconds on a piano. So you would hold it. So you would hold the key for six seconds."

To elaborate, he conceptualized fractions in terms of units of time signified by a given musical note, in his case, equating one-quarter with a temporal duration of one second. Moreover, he could apply his informal conceptualization to a mixed number fraction. When asked to explain what 2 ¼ meant to him, he answered,

> "Two wholes and quarter. On a piano key it means. 8, 9 beats, seconds to hold. Which is sort of long. You keep holding it till 9 seconds is past..."

In the specific case of CNC, fractional units such as quarter and half notes could be mentally converted into familiar, whole numbered units of time (seconds). These units could then be comfortably manipulated arithmetically, and transposed again in terms of the desired numerical form (some answer containing fraction notation). CNC was able to further apply his informal reasoning to accurately solve many of the fraction arithmetic problems in the pre-test, particularly on the assessment items focused primarily on fraction comparisons and arithmetic (numbers 11-20). On the pre-test, he correctly answered 5 of these items—well above the mean of 2.3.

Given that his mental model of fractions as units of time had helped him infer "correct" answers prior to acquiring any formalized algorithms for fraction arithmetic, it was a surprise to discover that CNC had performed much worse on similar assessment items after the first intervention. On the Intervention-1 assessment—which comprised 10 items focused on fraction comparisons and arithmetic patterned after the pre-test—he only answered one item correctly.

How might we explain the case of CNC's assessment declines, and by juxtaposition, the overall gains demonstrated by students in the conventional treatment group? One plausible explanation can be inferred from Piaget's classic description of learning as an adaptation of existing mental structures—an ongoing, dynamic process that involves the accommodation of existing mental structures and the assimilation of new concepts and ideas. In the case of the students participating in the study possessed, we can safely assume that they each possessed—to varying degrees of sophistication—some working mental model of fraction and/or whole number arithmetic constructed over the course of their time in school as well as through participating in mundane practices outside of school.

For the conventional treatment group, the context of instruction—in this case, the representational tools—was familiar. As a result, the students could more readily focus on assimilating the new problem solving heuristics/schemes for fraction arithmetic that I was introducing. In the case of the Water Works group as a whole, and CNC in particular, the novelty of the pedagogical situation may have forced them to devote more of their cognitive resources to accommodating their pre-existing conceptualization of fractions to include the mapping between fractions and fraction arithmetic with the measuring cups and changes to the volume of water.

Recall that CNC had originally equated unit-fractions and the notion of one whole with units of time. Consider his reaction to the Intervention 1 lesson that now asked him to equate unit fractions and the concept of one-whole with some volume of water. While the overarching relationship between unit fractions and whole may appear obvious to a knowledgeable to an adult—at this point in students such as CNC's understanding of fractions, the understanding had originally been grounded within a specific context. For CNC the grounding context was musical notation; for the majority of the other students, conventional representations such as area models and number lines. And, once the context was shifted to a novel medium—some degree cognitive restructuring would be required before the students could reconcile their existing knowledge base in order to make sense of the instruction being provided.

As a result, the students in the Water Works treatment group may have initially been unable to assimilate the problem solving heuristics introduced during the intervention as effectively as the students in the conventional treatment. While it is obvious that an unfamiliar context becomes more familiar with repeated exposure, a simple, but important takeaway for educational research is that novel interventions—no matter their potential ultimate efficacy—require time for students to adapt to them.

The overall performance of the Water Works treatment group in comparison to the conventional treatment group serves as a reminder of complexity of learning and calls to mind debates on the notion of transfer. As critics and proponents of transfer (as a form of assessing knowledge) alike all argue, pre-existing cognitive structures—in this case, ways of reasoning about fractions and problem-solving heuristics—appear to be context dependent (Engle, 2006; Engle, et al., 2012; Hershkowitz, Schwarz, & Dreyfus, 2001). Learning does not occur in a vacuum—it is contextually situated. Moreover, it would appear as if some degree of familiarity with the context of instruction is also an essential prerequisite for new learning.

4.1.4 Conclusion

For the majority of students, their initial conceptualization of fractions was predicated on either an area model and/or number line representation as introduced in and reinforced by their mathematics curriculum. Consequentially, I suspect that it may have been comparatively easier for students in the conventional treatment group to build upon their preexisting mathematical foundation—which included comfort and familiarity with additive reasoning situated within the specific situation/medium—to help them construct/formulate their individual strategies for fraction arithmetic.

What we have arguably observed by way of comparing the conventional and Water Works groups is precisely how much situation matters. In the former, the situation was familiar. As a result, students were more successful at adapting their pre-existing schemes to accommodate the novel heuristics. For the latter, both the pedagogical situation (context of instruction) and the schemes to be learned were somewhat novel. This added to the complexity of the task. In neo-Piagetian terms, if a learner's conceptual schemes arise through interactions with a given situation, it may not always be obvious to the learner how said schemes apply to an analogous, albeit novel situation (e.g. Vergnaud, 2009).

We therefore arrive at a potential dilemma faced by any developers of novel instructional curriculum. The conceptual and mathematical schemes available to students are invariably constructed in a given pedagogical situation. Given that students must also learn to map their pre-existing schemes to a novel instruction situation, a novel paradigm may not initially show any advantage or evidence of successful transfer when compared to the status quo. The issue then, is whether the schemes engendered through interaction with the novel instructional situation will prove more robust than the conventional approach? To help answer this question with respect to the Water Works design, a more in-depth examination of students' artifact-mediated cognitive interactions is required.

4.2 Intervention 2 / (No Assessment)

4.2.1 Overview

Following the whole-class introductions for both of the respective pedagogical treatment groups, the second and subsequent interventions were conducted individually with each student, in the format of one-to-one, semi-structured tutorial sessions conducted in the hallway directly outside of the classroom. The rationale for this strategy as previously explained in Chapter 2, was both to prevent the other students from overhearing any instruction until it was their turn and minimize disruption to the classroom teachers' regular instructional schedule and classroom routines. So doing, I could maintain good working relationships with the teachers and access to the students.

The second intervention begins by presenting students with a drawn inscription of ½. In the case of the conventional treatment group, the students observe drawings of both an area model and a number line (Figure 46a and 46b). In the case of the Water Works treatment, the students are presented with a drawn representation of the ½ cup measure and a one whole cup measure (Figure 47). A corresponding set of unit measuring cups is also presented to students (Figure 48). In both scenarios, students would be guided through a pedagogical co-enactment intended to support their development of a given fraction concept/operation.

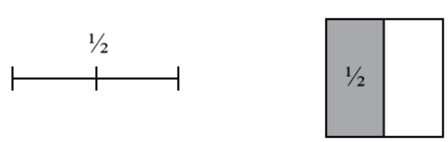


Figure 46a & 46b. Number line and Area model representations of ½ and 1 whole

Figure 47. A drawn representation of ½ cup and 1 whole cup measures.

Figure 48. Two different sets of standard household measuring cups

After presenting the materials to the students, I then inform them that I will be teaching them how I like to think about fractions. I next ask each individual student, "How many one wholes" would be made if "1/2 was repeated four times." I wait for the students to formulate their own answer. I then guide them perform a visual transformation of some representational object to either confirm or disconfirm their initial conjecture.

For the conventional treatment group, this might involve asking them to iteratively draw a ½ representation using either a number line or area model (the choice of which determined by the student) until two complete "wholes" were constructed. In Figure 49 below, we can observe that the student first draws representation for one whole, divides it into two-halves, and then shades each half until the one whole is completely shaded. This operation is repeated a second time. Conceptually, this series of actions is intended to serve as a proxy for the mathematical operation of multiplying the unit fraction ½ by four.

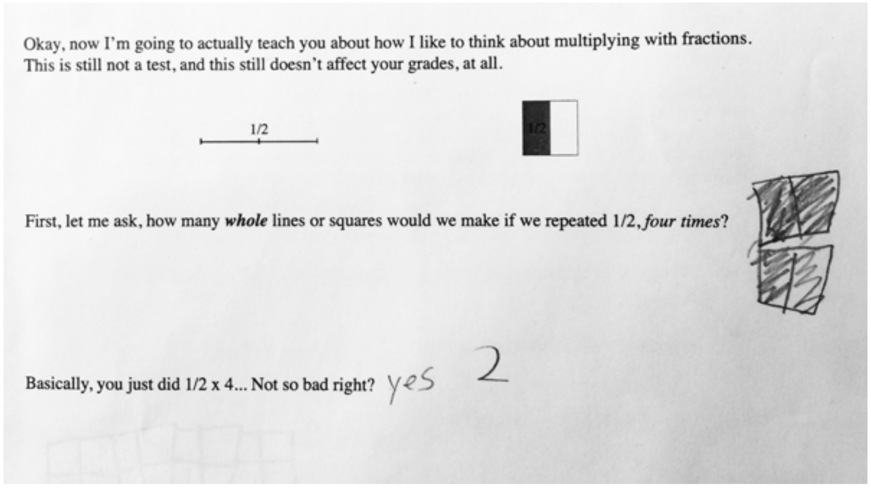


Figure 49. Example of Conventional treatment student's final drawing when asked to repeat ½ four times.

The treatment for students in the Water Works group differed from the conventional media group with respect to the actions the students were tasked with performing (Figure 50). Instead of iteratively shading in sections/segments of a drawing, the students physically iterated volumes of water from an unmarked reservoir (of indeterminate volume) using the ½ cup measure directly into a one whole cup measure. After two iterations, the one-whole-cup measure was filled. The student would then pour the full cup of water into an empty, unmarked "holding" reservoir, before proceed to iterate another two ½ cup measures of water into the one whole cup measure again. This second cup of water is poured into the reservoir containing one cup of water. I then ask the student how many whole cups of water are in the "holding" reservoir (2 cups total).

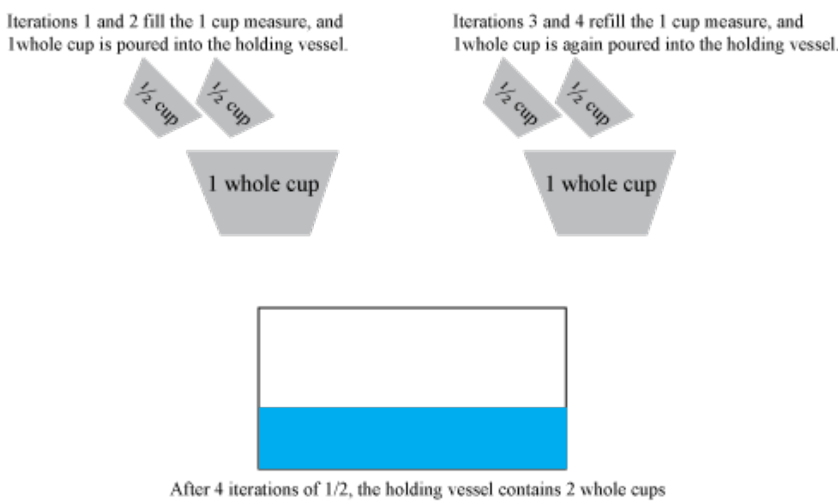


Figure 50. In this example, the process of multiplying ½, four times (½ x 4), is enacted in terms of repeatedly iterating ½ cups of water into a one-whole cup measure. Because a one-whole cup measure can obviously only hold one cup, once it is filled, the water is collected into a separate reservoir. Once four iterations of the ½ cup have been performed, I might then ask students how many one-whole cups (the final "product") can be filled from the reservoir. Alternately, one can iterate the ½ cup measures into a reservoir before asking how many one-whole cups can be filled.

Finally, after they have completed the first item, I inform the students that they have performed the equivalent of the mathematical operation " ½ x 4 = 2." I then prompt each of the students to attempt to complete the remaining three multiplication items on the worksheet, ½ x 6, ½ x 7, and ⅕ x 6.

4.2.2 Results

Students in both treatment groups responded overwhelmingly positively to the intervention (Table 7). Students in both treatment groups averaged a combined 98% items correct. The problems were not especially difficult given by that by grade 4, students have repeatedly encountered problems involving the unit fraction ½ and its relationship to one-whole unit. Nevertheless, the interesting point for consideration is that the students were taught to effectively reason about multiplication as an additive iteration with little to no effort, and to extend this reasoning to the unit fraction ⅕.

Table 7
Intervention 2 Items and Results

#	Question	% Correct Conventional Treatment (n = 19)	% Correct Experimental Treatment (n = 21)	% Correct Combined Treatment (n = 40)
1	½ x 4	100%	100%	100%
2	½ x 6	100%	100%	100%
3	½ x 7	100%	100%	100%
4	⅕ x 6	89%	95%	93%
% Correct All items		97%	99%	98%

4.2.3 Discussion

Judging by the results, the task was easy to comprehend and students were able to immediately apply the lesson toward problems involving multiplication with unit fractions. That 37 out of 40 (93%) of the students were able to successfully answer the last item (⅕ x 6) was also particularly encouraging given that for the Water Works group, there was no corresponding measure for the unit fraction ⅕. Likewise, for the conventional group, drawing a representation of ⅕ was considerably more awkward and difficult than drawing a corresponding representation for ½ or ¼. While some students nevertheless attempted to draw an area model or number-line representation for 1/5, the majority of students in both groups were forced to abandon the use of a drawn and/or physical enactment of the mathematical operation in order to arrive at a correct answer.

It is worth noting again that the arithmetic algorithm for multiplication with fractions had not yet been formally introduced—an observation corroborated by the fact that only 23% of the students could correctly answer the problem ½ x 6 on the pre-test. After the lesson however, 93% of the student in the combined treatments group could correctly answer the more challenging problem, ⅕ x 6. Could this one simple direct intervention have long-lasting effects on the students' understanding of fraction multiplication and addition? In order to test the effectiveness of the intervention on the students a follow-up assessment was performed.

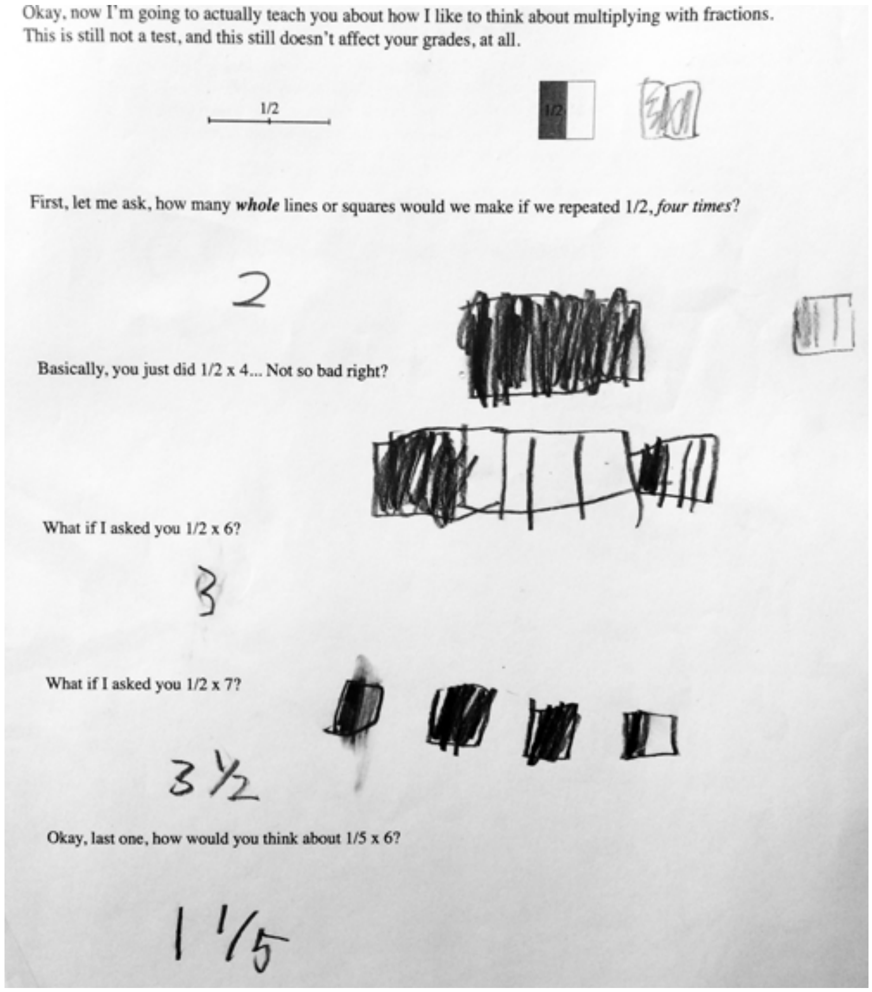


Figure 51. Student TR's completed worksheet (conventional treatment group). Note the initial difficulty that the student had with drawing representations for ½ x 6. Note also that while no attempt is made to draw a representation for ⅕ x 6, the student is able to formulate a correct answer.

4.3 Intervention 3 / Assessment 2

4.3.1 Overview

The reader will recall that previously, in Intervention 2, students were provided with guided instruction on four items involving the multiplication of a unit fraction. While extremely brief in duration, Intervention 2 had appeared extremely promising, as all the students were able to follow along and successfully complete most of the items.

The subsequent Intervention 3 was an assessment administered approximately 1 month after Intervention 2. The assessment contained 10 items involving addition and multiplication with unit fractions (Table 8).

Table 8
Assessment-2 Items and Results

#	Question	% Correct Conventional (n = 19)	% Correct Water Works (n = 21)	% Correct Combined Treatment (n = 40)
1	1/8 x 9	36.8%	42.9%	40.0%
2	1/3 x 6	47.4%	42.9%	45.0%
3	1/10 x 12	31.6%	38.1%	35.0%
4	1/5 x 11	31.6%	38.1%	35.0%
5	2/5 x 3	21.1%	23.8%	22.5%
6	1/6 + 1/6	42.1%	47.6%	45.0%
7	⅕ + ⅕+⅕	47.4%	47.6%	47.5%
8	1/7 + 3/7	36.8%	42.9%	40.0%
9	4/6 + 3/6	42.1%	38.1%	40.0%
10	3 2/8 + 5 1/8	42.1%	42.9%	42.5%
% Correct All items		**37%**	**40%**	**39%**

4.3.3 Results

Overall, students in both treatment groups correctly answered 39% of the items correctly (3.9 out of 10 problems on average). This time, students in the Water Works treatment groups slightly outperformed students in the conventional media group as a whole, averaging 40% versus 37% of the items correct, respectively.

If one were to compare the pre-test results with Assessment 2 results, the data would appear to favor Water Works. For the Water Works treatment group the mean pretest score was 2.62, and the mean intervention 3 score was 4.05. A one-tailed matched-pairs t-test would reveal a t-statistic that was significant at the .05 critical alpha level, t(40)= 1.84, p=.040. By comparison, the conventional treatment group the mean pretest score was 2.11, and the mean intervention 3 score was 3.74. A t-test would reveal that t-statistic was not significant at the .05 critical alpha level, t(36)= 1.84, p=.0525.

It would be disingenuous of course, to make any claims strictly based on a comparison of the pre-test and Assessment-2 alone. A critical point to highlight is that the ten items on Assessment 2 were not perfectly matched with items 11-20 on the pre-test. The pre-test Assessment Items 11-20 contained two fractional comparison items, four division problems, one subtraction problem, and three multiplication problems. The Assessment 2 items were comprised of 5 multiplication problems and 5 addition problems. Therefore, comparing students' average assessment scores alone would be misleading. In order to better ascertain whether or not one treatment was proving to be more effective than the other, a more careful parsing of the data would be required.

The proficiency rating system based upon students' results on the pre-test was developed to rate students as either not proficient, proficient, and highly proficient[6] on fraction arithmetic tasks. These categories were based upon the mean combined average (2.3) and standard deviation (2.6) from items 11-20 on the pretest. Students who could answer less than 50% of the items (4 or fewer out 10) correctly were rated as not proficient (at fraction arithmetic tasks); students scoring approximately 1 standard deviation higher than the mean (50-79% accuracy) were rated as proficient; and finally

[6] Note that the baselines percentage levels for not proficient, proficient, and highly proficient were fixed according to the pre-test assessment mean and standard deviation (and not recalibrated per assessment).

students scoring more than 2 standard deviations above the mean were designated as "highly proficient" (80 – 100% accuracy) on the fraction arithmetic tasks. The results from parsing the data in this manner can be viewed in Tables 9a and 9b below.

Table 9a

Conventional Treatment Group Proficiency Trend

	Pre Test	Assessment 1	Assessment 2
not proficient	94.7%	52.6%	57.9%
proficient	5.3%	42.1%	15.8%
highly proficient	0%	5.3%	26.3%
proficient or higher	**5.3%**	**47.4**	**42.1**

Table 9b

Water Works Treatment Group Proficiency Trend

	Pre Test	Assessment 1	Assessment 2
not proficient	76.2%	71.4%	52.4%
proficient	19.0%	19.0%	14.3%
highly proficient	4.8%	9.5%	33.3%
proficient or higher	**24.8%**	**28.6%**	**47.6%**

One important pattern to notice is that a higher percentage of students in the Water Works treatment group displayed some initial level of proficiency with fraction arithmetic at the beginning of the study in comparison to students in the conventional treatment group (24.8% compared to 5.3%). Following the first intervention—whole group instruction—the percentage of students displaying some level proficiency increases marginally for the Water Works group from 24.8 to 28.5% (+3.7%), but increases dramatically for the conventional treatment group from 5.3 to 47.4% (+42.1%). As was discussed in the previous section, the results at that particular junction in time would appear to strongly favor the use of conventional materials to support fraction arithmetic instruction. However, following the 2nd intervention, there was a large increase in the percentage of students in the Water Works group performing at as "proficient" level or higher from 28.5 to 47.6% (+19.1%). A graph showing the growth in number of proficient students over the course of the intervention is shown in Figure 52.

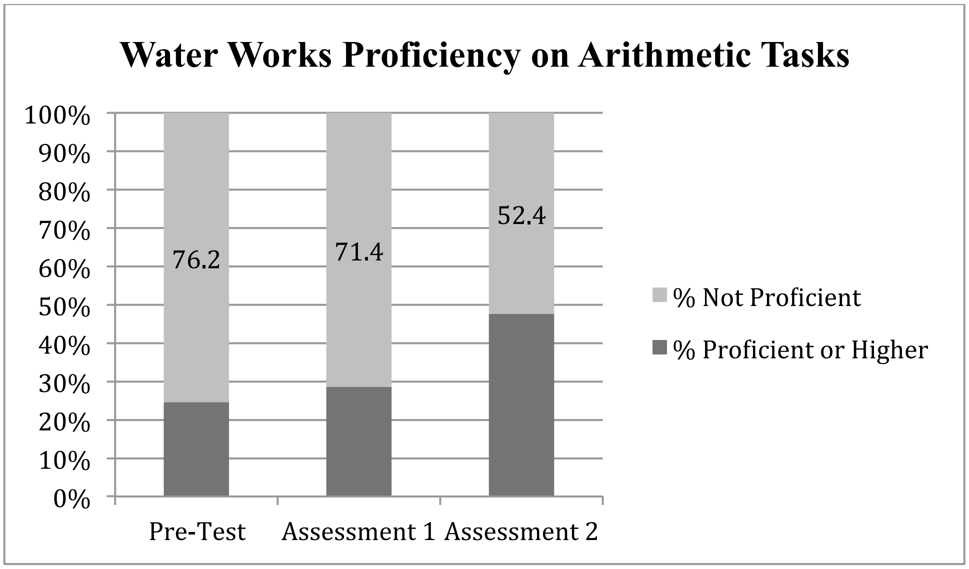


Figure 52. This graphic shows the percentage of students in Water Works group demonstrating proficiency on fraction arithmetic tasks. Note the small increase in proficiency on fraction arithmetic tasks between the Pre-Test and Assessment 1 (+3.7%) following the whole class intervention, but a larger increase (+19.1%) between Assessment 1 and Assessment 2 following the second intervention.

The conventional treatment group on the other hand, saw an overall *decrease* in the number of students performing at a proficient level or higher from 47.4 to 42.1% (-5.3%). Interestingly, both conventional and waterworks treatment saw increases in the number of students performing at a *highly proficient* level (+21% and +22.8% respectively). Nonetheless for the conventional group, these gains were more than offset by the overall decrease in students performing at a proficient level.

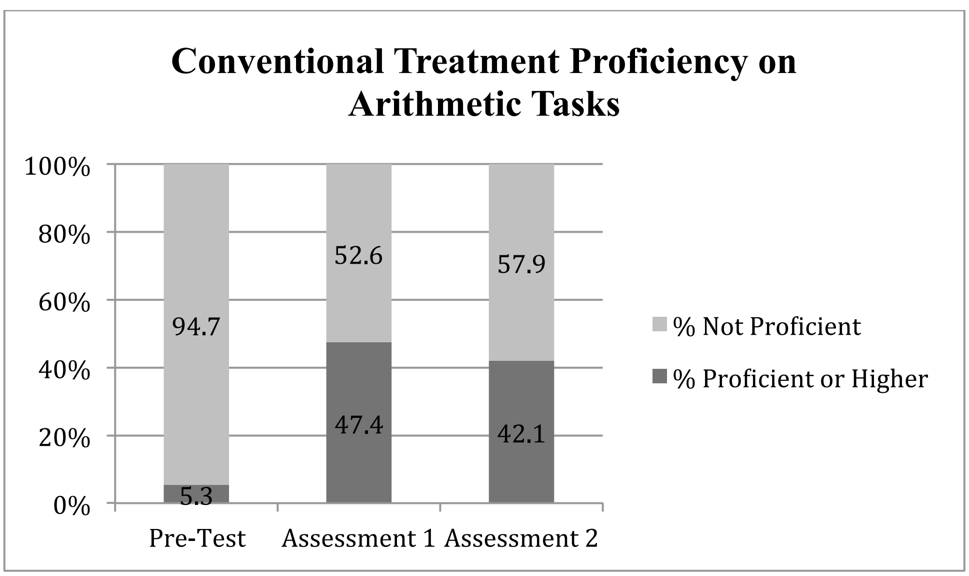


Figure 53. This graphic shows the percentage of students in the Conventional Treatment group demonstrating proficiency on fraction arithmetic tasks. Note the large increase in proficiency on fraction arithmetic tasks between the Pre-Test and Assessment 1 (+42.1%) following the whole class intervention, but a decrease (-5.3%) between Assessment 1 and Assessment 2 following the second intervention.

4.3.4 Discussion

How might we interpret the results of Assessment 2? The most important trend to notice is that over the course of three, relatively brief interventions (the whole group, and two one-on-one activities)—there was an overall positive increase in the number of students displaying proficiency on fraction arithmetic tasks across both treatment groups. While the number of students scoring at a "proficient" levels on the assessments remained less than 50%, this was nevertheless a marked improvement compared to overall student performance to the start of the intervention (see Tables 10a and 10b for individual students gains).

This positive increase suggests that the underlying premise of the treatment—teaching students to effectively reason about fraction arithmetic in terms of iterative–additive transformations of perceptual media—would appear to be a promising approach for building students' understanding of and competency in early fraction operations. One can reasonably project that more consistent classroom instruction using this approach could lead to a greater percentage of students performing at a proficient level on fraction arithmetic tasks involving addition and multiplication.

Table 10a

Individual Student Results for Conventional Treatment Group

Student	Pre-Test	Assessment 1	Assessment 2
V.G.	10%	40%	40%
J.M.	40%	90%	90%
K.C.	10%	30%	0%
M.I.	20%	30%	0%
E.R.	40%	60%	60%
K.D.L	20%	60%	10%
AJ	0%	30%	0%
NB	20%	30%	50%
KIS	20%	60%	60%
JY	10%	60%	80%
DC	10%	30%	100%
DL	20%	60%	0%
TR	20%	20%	0%
JHW	70%	70%	0%
SL	20%	40%	0%
LB	20%	40%	90%
KAD	10%	0%	0%
KC	30%	60%	100%
C.T.	10%	10%	30%
Total	**21.5%**	**43.2%**	**37.4%**

Table 10b
Individual Student Results for Water Works Treatment Group

Student	Pre-Test	Assessment 1	Assessment 2
EK	10%	10%	0%
AY	60%	60%	100%
BK	20%	10%	0%
CNC (check video)	50%	10%	0%
EA	0%	10%	0%
GK	10%	20%	100%
LU	30%	40%	70%
RK	10%	30%	0%
CN	70%	100%	100%
SC	20%	20%	0%
MC	20%	60%	20%
ML	10%	10%	50%
AK	10%	10%	0%
DB	10%	10%	0%
EF	10%	30%	0%
JK	60%	70%	100%
JVR	30%	50%	50%
RKL	10%	0%	0%
RTL	80%	80%	90%
CN2	10%	30%	80%
SHC	20%	20%	90%
Total	**30%**	**32.4%**	**40.5%**

While we can observe an increase for the Water Works group between Assessment 1 and Assessment 2 (from 32.4% to 40.5%) and a *decrease* for the Conventional Treatment group on the assessments (43.2% to 37.4%) we must bear in mind that assessment-1 and assessment-2 differed in terms of the types of problems (assessment 1 was matched to the pre-test, whereas assessment-2 contained only addition and multiplication items). Nevertheless, based on the respective changes in percentage scores of students in both groups, we can begin to suggest that the Water Works approach was just as, if not more, effective than the conventional media at as a medium for teaching addition and multiplication.

Similarly, if we compare the overall percentage of students scoring at a proficient level or higher between assessment 1 and assessment-2, we can observe that there was a large increase in the percentage of students scoring at a proficient level in the Water Works group, but a slight decrease in that number for students in the conventional media group. This could indicate that the initial comparative advantage of the conventional media—students' comfort level and familiarity with the tools—was becoming less of a factor after students became better accustomed to the Water Works tools through hands-on experimentation and discovery.

4.4 Assessment 3

4.4.1 Overview

This assessment was administered after Assessment 2. Recall that the Assessment 2 was conducted in conjunction with Intervention 3. In other words, after assessing what students retained from Intervention 2 conducted one-month prior, students were then explicitly taught how to solve the various problems on the Assessment 2 itself.

To determine whether or not the instructional treatment was effective, Assessment 3 was administered 1 month after Intervention 3. Note that a month is a considerable time span in the life of the average 4th grader. Students this age tend to forget things that are taught the previous week—let alone the previous day—unless they are repeatedly reinforced.

The extended interval was justified on the basis of attempting to demonstrate retention of learning. If students could still reason about and solve similar problems a month or more after the original intervention, the argument that the intervention promoted a deeper understand of the mathematical concepts and relationships could be advanced.

Assessment 3 included ten items involving fraction comparisons and arithmetic operations (Table 11). Unlike the previous assessments however, each item also included two check boxes labeled "help" and "no idea." The activity was designed such that the student would first complete the assessment items by his/herself. If the student literally had "no idea" how to complete a given item, he/she would simply check the box to indicate that the problem was still unfamiliar. This feature was used mainly to alleviate the level of frustration students might experience after repeatedly being administered items not typically taught at their grade level (alternatively, students could also leave blank assessment items they did not feel comfortable completing).

In the previous assessments, students were essentially solving the problems on their own with minimal feedback from the researcher. For this assessment however, following the students' first un-assisted attempt at completing an assessment item, I would attempt to scaffold their problem solving by guiding them to co-enact a similar, albeit simpler arithmetic problem.

For example, if a student was unable to determine whether or not 1/5 or 1/10 was greater, I might ask a student in the conventional treatment group to recall an instructional activity we had conducted with an area model or number line and how many ½ and ¼ sections it had contained. In the case of the Water Works treatment group, I might ask them to recall, "How many times would you have to pour water from a 1/*x* cup to fill a one cup measure?" Similarly, if the problem involved multiplication, I would present a simpler multiplication problem and then guide them to iteratively draw/shade in or scoop volumes of water.

Administered in this fashion, the assessment could be used to reveal two levels of insight about the child: first, the unassisted initial response from each student—what clinical interviewers might term the "spontaneous conviction"; and second, the assisted, "liberated conviction" of the child (see Ginsburg, 1997; Piaget, 1928) through what modern educators might refer to in terms of scaffolding (see Wertsch, 1979). From a Vygotskian perspective, this technique allows us to analyze both "the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Moreover, by observing the effects of instruction, the instructional component of the assessment would allow us to compare and contrast the relative pedagogical effectiveness of the instructional media.

While admittedly unorthodox, this approach was deemed warranted given the subjects (school age children) and context (their school environment). Students in schools are buffeted by a veritable wealth of different activities and information each school day. Given the brief duration of my interactions with each student—and just as, if not more importantly—the extended gaps in time *between* said interactions, it would hardly come as a surprise that students forgot the concepts/operations we had worked on.[7]

Thus, in order to more accurately determine whether or not the intervention had been effective, it was deemed reasonable to attempt to further probe students' reasoning and understanding through the use of guiding clues. This was made practical because of the methodological decision to conduct the intervention in a one-to-one fashion with each individual student. Finally, if a student failed to answer an item correctly even after assistance was provided, I would explicitly walk the student through a problem solving procedure based on the treatment group. In this regards, Assessment #3 also served as an pedagogical intervention for the treatment group students.

To more clearly illustrate the nature of the assistance, I shall now compare and contrast how assistance was provided in the two treatment conditions, respectively.

4.4.2 Conventional Assistance

In the event that a student in the conventional group had checked the "help" box, the assisted prompt would be to guide the student through a similar, but less difficult problem using drawn area models and/or number lines. Here, we will examine the assistance provided to student SL.

SL has scored at a non-proficient level for fraction arithmetic tasks on the pre-test and previous assessments. Perhaps not surprisingly, SL is unable to correctly add fractions. After successfully answering the first two items (comparing unit fractions) on his own, SL informs the researcher that he needs assistance on Item #3, 1/6 + 1/6.

SL: I need help.
R: Alright, you can check help. What are you thinking about with this problem? What's concerning you?
SL: There's six parts, and there's one part filled out of the six.
R: Okay. So what are you not sure about? Do you just not know how to do this?
SL: No.

[7] The well-documented phenomenon of students forgetting what they have learned in school after extended breaks in time is part of the justification for proponents of year-round schooling (as opposed the current system of lengthy summer vacations—a relic of agrarian times when children were expected to help on the farm during the harvest season).

While SL appears able to accurately equate the notation 1/6 with a part-whole model (presumably of some area model), it is evident that at this juncture in time he has yet to learn an algorithm for adding unit fractions.

In order to assist him, I instruct him to draw an area model representation for 1/4.

R: Okay, so I'll give you a clue. I want you to draw a square. And cut it into fourths....

I then guide him to iteratively shade in ¼ sections, two times, and then query him on the finished drawing (Figure 53).

R: ...And shade in one fourth.... Okay...Shade in another fourth.
R: How many fourths do you have now?
SL: 2

Figure 53. SL's drawing of ¼ + ¼.

I ask SL to revisit the problem 1/6 + 1/6 again.

R: Can you do this one now? (Item #3).

Student SL immediately sets to work drawing a rectangular area model (Figure 54a & 54b). He begins by drawing a rectangle, equipartitions it into six sections, and shades in one of the sections to represent 1/6. Next—instead of simply shading in a second section of the representation he has drawn—SL writes a + symbol to indicate addition, and proceeds to draw a second area model.

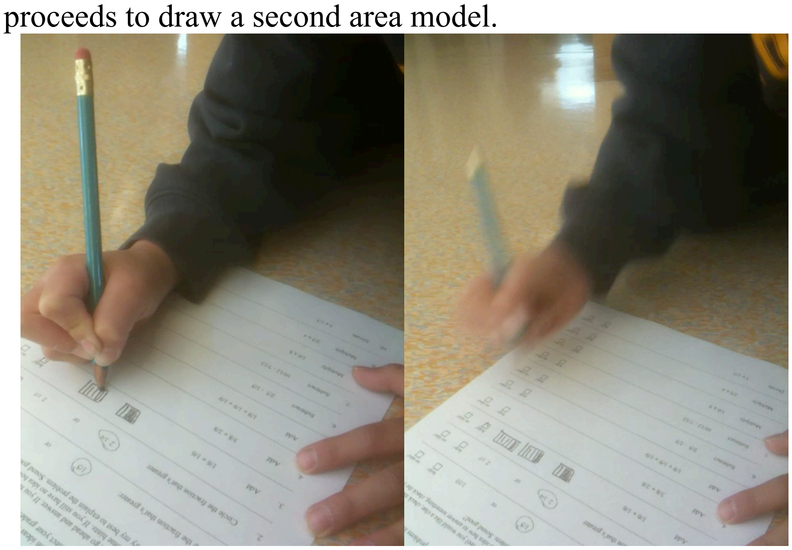


Figure 54a & 54b. SL begins to draw 1/6 + 1/6.

This decision by SL recalls the earlier discussion in Chapter 1 about the internal characteristics of conventional representations such as area models. The concept of 1/6 appears to only exist internally for SL with respect to the each individual area model representation of one-whole. A knowledgeable adult might simply shade in a second 1/6 subsection of the initial drawing in order to represent the operation 1/6 + 1/6 = 2/6. However, SL proceeds to draw a second area model, which he once again equipartitions into sixths. It is as if he cannot conceptualize 1/6 apart from this particular relationship to the whole (a single sub-area on the left of the containing area). This interpretation appears to be supported by the fact that he draws an addition symbol between the two representations.

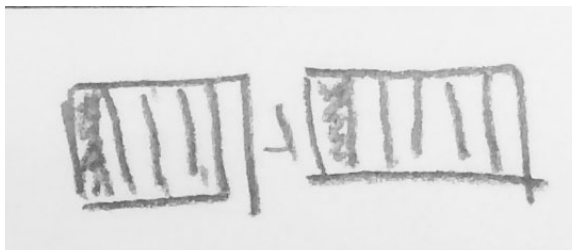


Figure 55. Close up of SL's drawing of 1/6 + 1/6.

At this point in the activity, it appears as if SL now understands the problem. It comes as somewhat of a surprise, then, when he answers as follows.

SL: So it's (1/6 + 1/6 =) 2 fourths?

Where does the reference to fourths come from? Perhaps he is looking at the first drawing of fourths he was guided to make? Perplexed, I attempt to probe his thinking.

R: Why would you say it's two fourths? These are, what? (I point to the two representations SL has just made for 1/6).
SL: I don't know...

Because SL has now voiced his uncertainty/inability to solve Item #3 with the initial clue, I determine that it is time to provide more direct instruction.

R: Okay... Check (the box indicating) "no idea."
R: So what's confusing you here? This is one-sixth, and that's another one-sixth, so how many (1/6's) do you have?
SL: 2...
R: So that's the answer...
SL: Just 2?
R: 2 over 6.
SL: Oh. Now I get it.
R: You get it now? Remember this one? (the drawing of 2/4) There's one fourth, and now there's two fourths.

My clarification of the problem appears to have had an effect. SL is subsequently able to transfer and apply what I have just taught him in the context of Item #3 to correctly solve Items #4 and #5 (likewise addition with fractions). All told, SL asks for a clue once, and is able to answer 5 problems correctly.

4.4.3 Water Works Assistance

Let us now compare how similar assistance (for Item #3 specifically) is provided in the context of the Water Works design. Like student SL from the previous vignette, student EK has scored at a non-proficient level on each of the preceding fraction arithmetic assessments. Indeed, across three assessments EK has only managed to answer 2 out of 30 items correctly.

Clearly, EK has yet to learn an algorithm for adding fractions with like denominators. For Item #3, she naively multiplies the denominators as her guess.

EK: Is it 36? (6 x 6)
R: No. Check (the box) over there (to indicate) that you needed help...

In order to assist EK, I place the ¼ cup and 1 whole cup measures in front of her (Figure 56).

R: How much is that (I point to the one-fourth measuring cup)...
...(after no response) This is one fourth.

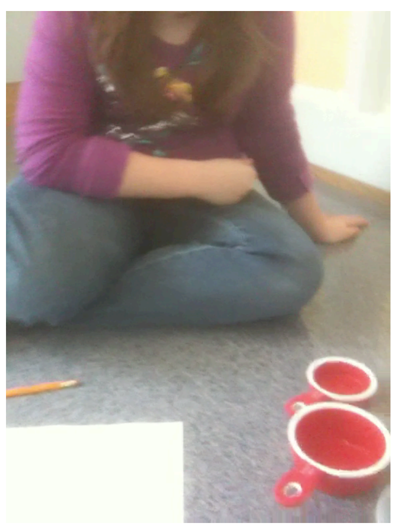


Figure 56. Student EK observing the ¼ cup and 1 whole measures

Before proceeding, let us briefly observe how in the case of the measuring cups, the fractional measure for ¼ cups is physically external to and distinct from the one-whole cup measure. Also, note how there are no visually discrete sub-units to count as in the case of a partitioned area model of number line. We will now return to the narrative.

I instruct EK to iteratively pour water from the one-fourth measure into the one-cup measure.

R: Pour one-fourth into here... (EK pours the water from the one-fourth cup measure directly into one cup measure)... So, how much water is in there right now?
EK. Um. One fourth.

I instruct EK to pour a second ¼ cup volume into the one-cup measure,

R: Do it again...
EK: Do I do this six times?
R: No.

Perhaps still thinking about how this relates to the problem of adding 1/6 + 1/6, EK asks if she should repeat the task 6 times, to which I respond in the negative. While there is a relationship between the denominator and the number of iterations that would be required to iteratively fill a one-cup measure using a "1/6" measure, that is not the focus of the task at hand, so I do not affirm this line of reasoning. After her second iteration of the ¼ cup measure, I ask,

R: Now how much water do you have?
EK: Two fourths.

Student EK clearly understands that the total volume of water she has poured is now 2, ¼ cups. To guide her to draw the connection between the clue and the problem, I ask her to imagine having a "one-sixth" cup.

R: That's your clue. Imagine you had a one-sixth cup.
EK. Two, um... Two sixths?

EK proceeds to correctly answer Item #3 as well as the subsequent items involving addition and subtraction (see Figure 57). Note that unlike SL, EK does not need to draw an additional representation on the paper. The prior physical co-enactment(s) performed for the addition problems and the array of unit cups in front of her appear to be sufficient grounds for reasoning. EK is able to correctly answer 9 out of the 10 problems with the aid of 3 clues.

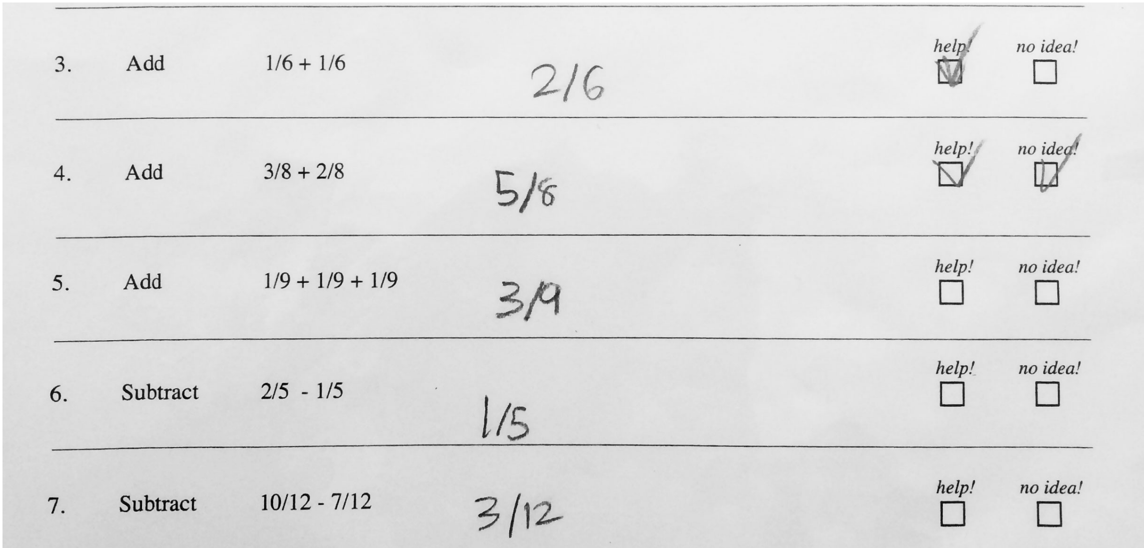


Figure 57. Assessment 3 items #1-7 for Student EK.

4.4.4 Control Group

Students in the control group (n = 26) completed the worksheets individually and without any assistance. Furthermore, they did not receive any additional explicit pedagogical instruction with regards to fraction arithmetic following the completion of the assessment. Consequently, their results will only be compared to the unassisted results for the two treatment groups. This difference should also be taken into account when considering the post-test results.

4.4.5 Results

Given that "assistance" in the form of a guided co-enactment was provided to students in the conventional and water works treatment groups, both "unassisted" and "assisted" results will be presented. To clarify, if a student required assistance to solve a problem, the "help" box was checked to distinguish it from problems that the student was able to complete independently, or "unassisted." The rationale for designing the intervention in this manner was that it simultaneously allows us to assess both students' current proficiency level—and indirectly, the effectiveness of prior interventions—as well as the relative pedagogical efficacy of the media.

One important caveat to note is that assistance on one item could "bootstrap" a student's ability to solve subsequent problems involving the same arithmetic operation. In other words, it is clear that my conceptually explication of one fraction addition item helped the students derive either a conceptual and/or procedural heuristic for solving subsequent addition items.

Table 11 below provides a breakdown of all the assessment items, and the percentages of students in unassisted and assisted conditions who could answer according to treatment group. Of particular interest is the dramatic increase in the number of students in the Water Works group's who can answer items #8, 9 (involving multiplication) and 10 (division) when assistance is provided.

Table 11

Assessment Items and % of Students Answering Correctly

#	Question	Conventional (n=19) *Unassisted / Assisted*	Water Works (n=21) *Unassisted / Assisted*	Control (n=26) *Unassisted*
1	1/5 > 1/10	84% / 100%	86% /95%	77%
2	2 ¼ > 2 1/7	89% / 94%	90% / 100%	69%
3	1/6 + 1/6 =	58% / 78%	71% / 100%	77%
4	3/8 + 2/8	84% / 89%	90% / 95%	88%
5	1/9 + 1/9 + 1/9	95% / 100%	95% / 100%	88%
6	2/5 - 1/5 =	89% / 94%	86% / 100%	77%
7	10/12 – 7/12	84% / 94%	100% / 100%	88%
8	1/6 x 8	39% / 58%	57% / 90%	15%
9	2/5 x 4	42% / 53%	67% / 90%	15%
10	3 ÷ ⅓	11% / 26%	5% / 76%	7%
% Correct		**67% / 75%**	**76% / 95%**	**59.2%**

4.4.5.1 Unassisted Results

On average, students in the conventional group independently solved 67% (6.7 out of 10 items) problems correctly. In comparison, students in the Water Works group solved 75% of the problems without any assistance. Students in the control group managed to solve 5.6 problems. The boxplot in Figure 58 below shows the overall distribution.

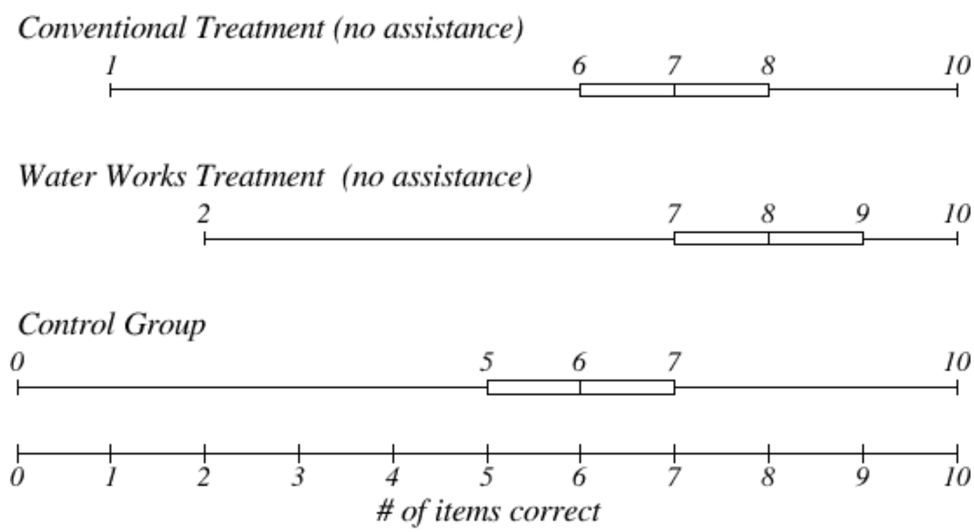


Figure 58. Assessment 3 items answered correctly without assistance.

Students in both treatment groups scored higher than the control. This fact alone is not surprising given that the objective of the interventions was to build students' proficiency with fraction arithmetic.

Of interest is the observation that students in the experimental treatment condition answered more items correctly than both the control and conventional treatment groups. This could suggest that the students were more effectively retaining the information learned from the prior intervention. However, there was no significant difference between the scores for conventional (M=6.79, SD=2.04) and water works treatment groups (M=7.48, SD=1.97) conditions; t (37)=-1.08, p = 0.287. How do the numbers change when we factor in the "assisted results"?

4.4.5.2 Assisted Results

Recall that the assisted condition involved supplying a "clue" in order to scaffold the student's problem solving. Again, the rationale behind the methodology was to ascertain what the students might have retained from the previous interventions, as well to compare the relative efficacy of the Water Works tools with conventional representations as instructional media.

On average, students in the conventional group solved an additional 1.2 problems correctly (7.95 total). In comparison, students in the Water Works group solved an additional 2.0 problems correctly with assistance (9.48). The boxplot in Figure 59 below provides a visual comparison of the overall distributions for the two assisted treatment groups alongside the unassisted results of the control group.

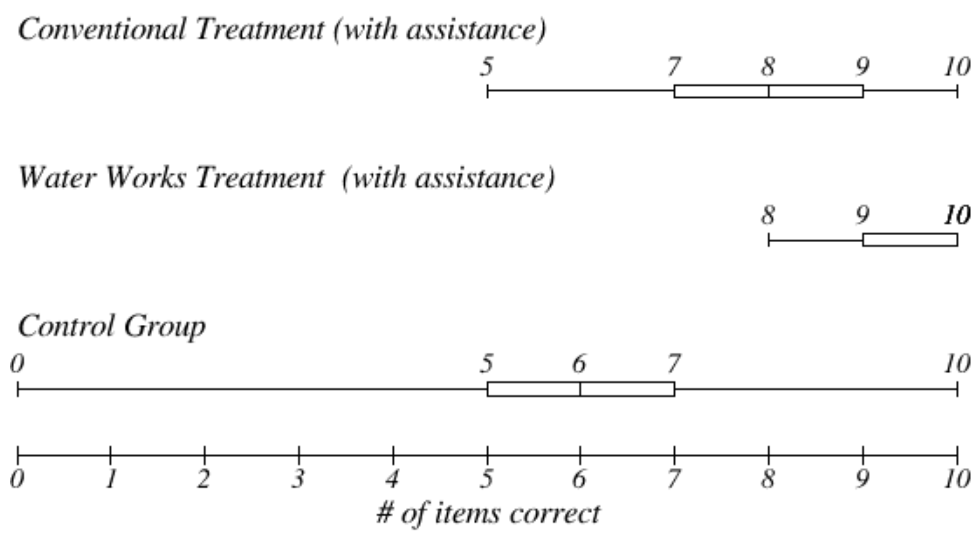


Figure 59. Assessment 3 items answered correctly with assistance. Recall that the unlike the two treatment groups, the control group did not receive any assistance.

A t-test was once again used to compare the assisted scores for the conventional and Water Works groups. This time, there was an extremely significant difference in the average scores for the Water Works treatment group (M=9.48, SD=0.75) in comparison to the conventional group (M=7.95, SD=1.58); t (25)=-3.84, p = 0.00073. These results, which we will explore more in-depth through qualitative analyses in Chapter 7, provide the most compelling evidence so far for the use of Water Works as a tool for teaching fraction arithmetic.

4.4.6 Discussion

To summarize, during this intervention students in both intervention groups were tasked with completing 10 problems involving fraction arithmetic (addition, subtraction, multiplication, and division). Students were first asked to complete each item on their own, without assistance. This allowed us to document and observe their actual, or current level of proficiency fraction arithmetic. If a student correctly answered a problem, s/he would be prompted to move on to the next one. If a student struggled to and/or failed to answer a given problem, the student would have the option of checking one of two boxes—either they had "no idea" or they wanted "help" with the problem. If they checked the box for "help," the facilitator would guide them through a similar, albeit less challenging example problem with the intention of providing just enough information for them to bootstrap their own problem solving heuristics for the assessment items. Taking a Vygotskian perspective, this would allow us to document and observe a student's zone of proximal development.

When the results from the unassisted tasks are compared, there was no significant difference in student performance. This would seemingly support the null hypothesis that there is no benefit from the Water Works design and activities over conventional media, such as area models or number lines for teaching early fraction arithmetic.

However, when the researcher provided assistance, we observe that the students using the Water Works tools significantly outperformed the students using conventional media. This would appear to be strong evidence in favor of the Water Works design over traditional media for the teaching of early fraction concepts and arithmetic operations. Table 12a and 12b provide a complete breakdown of the number of items each student was able to correctly answer in the unassisted and assisted conditions for both the conventional and Water works treatment groups, respectively.

As promising as the assisted results for the Water Works group may appear in comparison to the conventional media, before making any rush judgments as to the efficacy of the Water Works design/media it is important to address the possibility that the researcher conducting each intervention may have biased the results. If one were to play the role of the devil's advocate, a primary concern would be that the researcher conducting the intervention, the very same researcher who has stock in the design that bore a greater effect, may have consciously or unconsciously biased the problem-solving clue or the instruction in favor of one group or another.

Here, I will preemptively note that both treatment groups not only significantly outperformed the control group on the final post-assessment, but also that the conventional treatment group actually performed slightly better on average than the Water Works group on a number of metrics. This last point bears repeating, as it helps corroborates the claim that an honest effort was made to support both treatment groups! I am preemptively presenting these results—at the expense of the narrative— in order to support the position that the pedagogical interventions for both the conventional and

Water Works were ethically conducted with learners' best interests in mind. In other words, the goal was to effectively teach them, regardless of the medium used.

Table 12a

Number of Items Conventional Treatment Students Answered Correctly

Student	Unassisted	Assisted	Total Correct
V.G.	7	0	7
J.M.	9	1	10
K.C.	7	2	9
M.I.	7	0	7
E.R.	7	0	7
K.D.L	5	2	7
AJ	5	2	7
NB	7	1	8
KIS	6	2	8
JY	10	0	10
DC	8	1	9
DL	9	1	10
TR	7	0	7
JHW	8	2	10
SL	4	1	5
LB	8	1	9
KAD	1	4	5
KC	8	1	9
C.T.	6	1	7
Average # correct	**6.8**	**1.2**	**7.9**

Table 12b

Number of Items Water Works Students Answered Correctly

Student	Unassisted	Assisted	Total Correct
EK	6	3	9
AY	9	1	10
BK	2	6	8
CNC	8	2	10
EA	7	1	8
GK	9	1	10
LU	8	1	9
RK	6	3	9
CN	9	1	10
SC	4	4	8
MC	7	2	9
ML	9	1	10
AK	8	2	10
DB	5	5	10
EF	7	3	10
JK	9	1	10
JVR	9	1	10
RKL	8	1	9
RTL	10	0	10
CN2	9	1	10
SHC	8	2	10
Average # correct	**7.5**	**2**	**9.5**

4.5 Post-Test

4.5.1 Overview

The final post-test was administered near the end of academic year. To recap, during the intervening seven-month time-span that had elapsed after the pre-test was first administered, a number of brief pedagogical interventions were conducted with each of the subjects in the two treatment group. By contrast, the control group received no interventions in the form of these one-on-one tutorial sessions apart from having opportunities to take a pre-, post-, and Assessment 3 themselves.

4.5.2 Results

The results for each of the three treatment groups post-test are summarized below in Table 13a. For comparisons sake, the initial pre-test results are also provided again in Table 13b.

Table 13a
Post Test Results

	Items 1 - 10	Items 11 - 20	All Items Combined
Conventional Treatment	94.7%	60.5%	77.6%
Water Works Treatment	95.2%	58.1%	76.7%
Control Group	82.7%	25.4%	54.0%

Table 13b
Pretest Results

	Items 1 - 10	Items 11 - 20	All Items Combined
Conventional Treatment	70.5%	21.1%	45.8%
Water Works Treatment	64.2%	26.1%	45.2%
Control Group	66.5%	19.2%	42.9%

On average, students in the control group could answer 54% (10.8 out of 20) of the assessment items correctly. Students in the conventional treatment group correctly answered 77.6% (15.5 of out 20) of the post-test items, while those in in the Water Works group answered 76.7% (15.3 out of 20). The boxplot in Figure 60 below provides a visual summary the distributions.

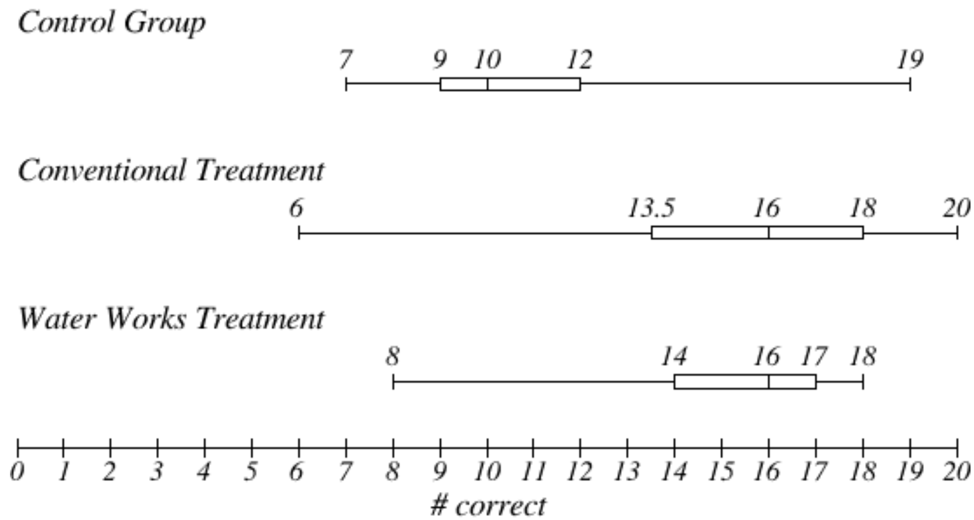


Figure 60. Post-Test items #1-20 answered correctly

The post-test results for items #11-20 in particular indicate the effectiveness of the two treatment groups in comparison to the control. On average, students in the conventional and Water Works treatments correctly answered 60.5% and 58.1% of these items respectively. In comparison, students in the control group could only answer 25.4% of these items correctly. The box plot in Figure 61 below summarizes the distributions of these scores.

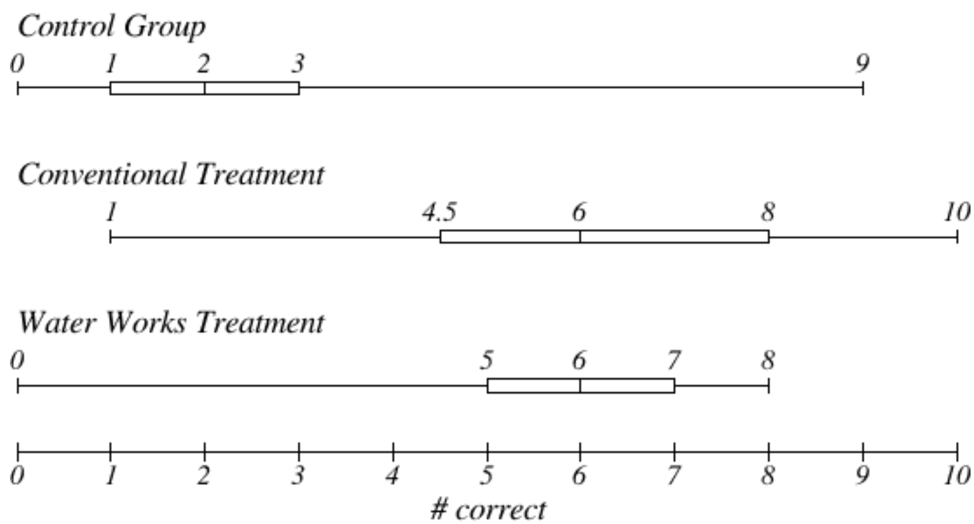


Figure 61. Post-Test items #11-20 answered correctly

Applying our proficiency metric to students' performance on items #11-20, we can observe a dramatic difference in the number of non-proficient versus proficient (and highly proficient) students between the control and the two treatment groups (Figure 62). Observe that only a combined 15% of the students in the control group are able to score at a proficient or highly proficient level on the post-test. By contrast, 74% and 82% of the students in the Conventional and Water Works groups, respectively, are able to demonstrate moderate to high levels of proficiency on these fraction arithmetic items.

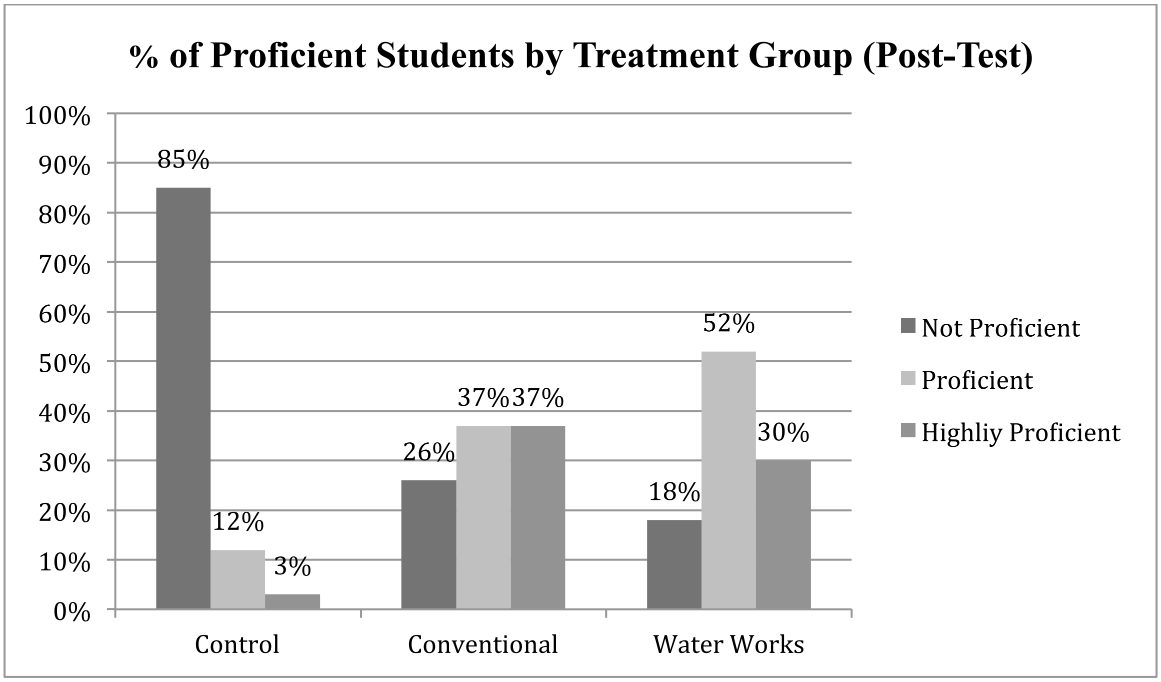


Figure 62. Breakdown of students demonstrating proficiency on fraction arithmetic problems across treatment groups after Post-Test

4.5.3 Discussion

On the one hand, the gains of the treatment groups in comparison to the 4th grade students in the control group should not come as a surprise. Recall that the assessment items #11-20 involved problems that are typically not introduced to students in the 5th and 6th grades. A basic tenet of teaching is that students can only learn to apply what they have been taught. Thus, the fact that the students exposed to problem solving strategies for fraction problems outperformed students who were not is in and of itself unremarkable. It could be reasonably argued that had the control teachers taught students the traditional algorithms for fraction arithmetic, similar gains might have been made.

On the other hand, there are many reasons to be optimistic about the gains made by the students in the treatment groups. For one thing, the amount of actual instruction was minimal. Recall that the pedagogical interactions with students were limited to short, 5-10 minute tutorial sessions, conducted at week(s) long to month(s) long intervals. All told, my interactions with each individual student may have totaled between 1-2 hours over the span of seven months! The fact that these 4th graders could easily be taught to add, subtract, and multiply fractions is itself something of an accomplishment.

At first glance, the results also would indicate that the Water Works interventions provided no value added when compared to conventional media. After all, both groups appeared to perform approximately as well, with the conventional media group actually boasting a slightly higher percentage of correct items.

Individually, a greater number of students (see Tables 14a and 14b) were able to transition from a non-proficient to proficient or highly proficient level of performance in the Conventional Treatment group (76%) than in the Water Works group (52%).

While this might seem to indicate that the Conventional Treatment was actually superior the superior of the two, this number should be interpreted with a grain of salt given a number of factors: First, the initial number of students demonstrating proficiency on items #11-20 in the Conventional group (5%) was actually lower than the Water Works group (24.8%). Even if the two treatments were equally effective, it stands to reason that a greater percentage of students in the conventional population would stand to benefit.

Second, and more importantly, we must bear in mind that conventional representations are already a standard and daily component of the Everyday Mathematics curriculum. In other words, students were already being regularly exposed to problem solving strategies for fractions using traditional media that could complement and reinforce the lessons learned over the course of the pedagogical interventions. Indeed, by the time the post-test was administered, the teachers had begun to cover the curricular units involving fraction arithmetic. Not surprisingly, both groups faired extremely well when compared to the control group (Table 14c & 15). In this light, the gains displayed by the Water Works group compare favorably to those made by students in the conventional group

Finally, given both my observations in the field and the statistically significant findings from Assessment 3 (unassisted versus assisted condition) I would argue that concluding that the Conventional media were equally if not more effective that the Water Works design would be premature without a closer examination of the qualitative data. In the next chapter we shall provide an in-depth, a side-by-side comparison of the two interventions using data from Assessment 3.

Table 14a

Individual Student Results for Conventional Treatment (items #11-20)

Student	Pre-Test	Post-Test
V.G.	10%	90%
J.M.	40%	80%
K.C.	10%	50%
M.I.	20%	90%
E.R.	40%	60%
K.D.L	20%	30%
AJ	0%	60%
NB	20%	60%
KIS	20%	70%
JY	10%	100%
DC	10%	20%
DL	20%	80%
TR	20%	40%
JHW	70%	80%
SL	20%	80%
LB	20%	30%
KAD	10%	50%
KC	30%	70%
C.T.	10%	10%
Total	**21.5%**	**60.5%**

Table 14b

Individual Student Results for Water Works Treatment (items #11-20)

Student	Pre-Test	Post-Test
EK	10%	30%
AY	60%	70%
BK	20%	50%
CNC	50%	60%
EA	0%	80%
GK	10%	80%
LU	30%	50%
RK	10%	70%
CN	70%	80%
SC	20%	40%
MC	20%	60%
ML	10%	60%
AK	10%	30%
DB	10%	0%
EF	10%	80%
JK	60%	60%
JVR	30%	70%
RKL	10%	40%
RTL	80%	70%
CN2	10%	80%
SHC	20%	60%
Total	**30%**	**58.1%**

Table 14c

Individual Student Results for Control Group (items #11-20)

Student	Pre-Test	Post-Test
2	10%	10%
3	10%	30%
4	30%	30%
5	80%	90%
6	30%	30%
7	50%	70%
8	10%	10%
9	20%	20%
10	10%	10%
11	40%	40%
12	10%	10%
13	10%	10%
14	20%	20%
15	10%	10%
16	0%	10%
17	10%	10%
18	10%	20%
19	10%	10%
20	10%	10%
21	0%	10%
22	40%	60%
23	20%	20%
24	20%	60%
25	0%	0%
26	20%	40%
27	20%	20%
Total	**19.2%**	**25.4%**

Table 15

Post-Test Items & Percentages of Students Who Answered Correctly

#	Question	Conventional (n=19)	Water Works (n=21)	Control (n=26)
1	What does ½ mean?	100%	100%	100%
2	What does ¼ mean?	100%	100%	100%
3	What does 1 mean?	100%	100%	100%
4	What does 2 ¼ mean?	100%	100%	100%
5	½ + ½ =	94%	100%	88.5%
6	1/3+1/3 =	84%	100%	88.5%
7	1/4 + 1/4+1/4	89%	95%	92.3%
8	1/2 + ¼	84%	86%	26.9%
9	1/2 > 1/5	100%	95%	96.2%
10	1/2 = 2/4	94%	71%	34.6%
11	2 1/5 < 2 ½	100%	95%	92.3%
12	5/4 < 4/2	44%	62%	50%
13	1/2 x 6	89%	86%	23.1%
14	1/4 x 8	94%	86%	23.1%
15	1/2 div ¼	58%	48%	19.2%
16	10 div ½	21%	29%	7.7%%
17	2 1/2 + 4 ¼	84%	81%	15.4%
18	4 1/2 x 2	68%	52%	15.4%
19	4 1/2 div 2	53%	38%	3.8%
20	1 3/4 div ½	11%	0%	0%
% Correct All items		**77.6%**	**76.7%**	**54.04%**

CHAPTER 5

The original argument driving the design of Water Works was that the tools offered compelling enough cognitive affordances over conventional representations that would warrant their use in classroom instruction. At the end of the preceding chapter, we observed a significant difference between the Water Works and conventional treatment group's performance on the assisted Intervention 3 fraction arithmetic tasks. These promising results were partially tempered by the fact that there were no significant differences in performance between the two treatment groups on the final post-test, administered approximately 7 months after the pre-test. In light of the post-test results, are we left to conclude that the Water Works design offers no advantage over conventional tools?

I would argue that the answer is both yes and no. Given that the mathematics curriculum in use emphasized conventional representations—it should come as little to no surprise that students would be most familiar and comfortable using said representations. Also, recall how the first introductory whole-group lesson on fraction arithmetic was far more effective when conventional representations were used. The observation could be made that students in the initial Water Works condition were faced with a two-fold task: learning novel fraction concepts and arithmetic, as well as learning to make sense of an entirely new representational context. It stands to reason that is far easier to extend children's existing understanding of fractions based on a given representational media, then it is to guide them to reformulate their understanding to fit a new context. A more tightly controlled study in which students' entire curricular exposure to fractions was adapted to use either conventional media or the Water Works tools would have been ideal—but was not feasible at the time. We are thus left to conclude that if students' day-to-day instruction and curricular material on fractions emphasizes the use of conventional representations, then yes, brief, intermittent instruction with the Water Works design will not offer any comparative advantage.

On the other hand, I believe that there is sufficiently compelling evidence to make the counter-argument that the Water Works design does offer an advantage over conventional tools. In particular, I return to data from Assessment 3—which the reader will recall involved the assisted and unassisted interactions. We will now examine how the respective pedagogical enactments differed according to the representational media used in greater depth and detail.

5.1 Assessment 3 Revisited – A Closer Look

The observation has already been made that students in the Water Works group outperformed those in the conventional media group when pedagogical assistance was provided on the assessment.

Therefore, in addition to comparing the number of items that students answered correctly, it may also prove useful to examine the relative effectiveness of the assistance for each condition in terms of the % of items students failed to answer even after receiving assistance from the instructor (Table 16).

Table 16
Assessment Items and % Left Unanswered After Assistance

#	Question	Conventional (n=19)	Water Works (n=21)
1	1/5 > 1/10	0%	5%
2	2 ¼ > 2 1/7	5%	0%
3	1/6 + 1/6 =	21%	0%
4	3/8 + 2/8	5%	5%
5	1/9 + 1/9 + 1/9	0%	0%
6	2/5 - 1/5 =	5%	0%
7	10/12 – 7/12	5%	0%
8	1/6 x 8	42%	10%
9	2/5 x 4	47%	10%
10	3 ÷ ⅓	74%	29%
% Unanswered After Assistance		**21%**	**5%**

All told, students in the conventional group were unable to answer 21% of all items on Assessment 3 after receiving assistance. By comparison, students in the Water Works condition were only unable to answer 5% of the items after receiving assistance.

To better understand these percentages, consider the difference in outcomes between the two treatment groups for item #10, which involved dividing a whole number by a unit-fraction. Of the 17 (out of 19) students in the conventional group who asked for assistance in solving item #10, 74% were still unable to solve the problem. By contrast, of the 20 (out of 21) students who requested assistance in the Water Works group, only 29% were unable to arrive at a correct solution. This would suggest that the guided pedagogical co-enactment was more effective using the Water Works design for this particular problem.

Parsing these numbers in greater detail (Table 17), we can see that students in the conventional treatment received help for approximately 33% of the items (62 out of 190). Dividing the number of items answered with assistance (22), by the total number of times assistance was requested (62) provides us with what I will refer to as the 'effective conversion rate'—an informal metric of the pedagogical utility of instruction —of 35.5%.

Students in the Water Works condition received assistance on 25% of the items (52 out of 210). However, and in striking comparison to the conventional media group, the effective conversion rate on the assisted items for this group was 77%. Again, if we interpret the effective conversion rate as a metric for how easy/difficult it is for students to internalize and transfer a fraction arithmetic concept using a given representational medium, then the higher effective conversion rate demonstrated by students in the Water Works group is strong evidence in support of my claims regarding the relative pedagogical efficacy of the design

Table 17
Effective Conversion Rate for Assistance on Assessment 3

Group	Requests for Assistance	Answered With Assistance	Effective Conversion Rate (Answers / Assists)
Conventional	62	22	36%
Water Works	53	41	77%

5.2 Qualitative Analyses

In order to help the reader better appreciate the differences between the Water Works design and conventional forms of fraction instruction, I will now provide a side-by-side comparison of two students' problem solving, EK (Water Works) and SL (conventional treatment). These two students were selected by virtue of their comparable initial pre-test results on fraction arithmetic items (both non-proficient) and the relative clarity and representativeness of their observed responses. One caveat to note is that SL was unable to complete the items involving multiplication and division, even with assistance. Therefore, examples from student LB of the conventional group will also be provided in order to complete the comparison with EK. Where appropriate, I will also frame my examination in terms of the categories for comparison between Water Works and conventional media as outlined in the cognitive domain analyses from Chapter 1.

5.2.1 Adding Unit Fractions

Both EK and SL were able to answer items #1-2 independently. However, both display a lack of an appropriate problem-solving strategy for solving item #3 (1/6 + 1/6).

5.2.1.1 SL (Conventional)

After briefly scrutinizing the problem on his own, Student SL's reaction is to immediately state,

SL: I need help.

I begin by attempting to verbally elicit his reasoning about the problem.

R: All right, you can check help. What are you thinking about with this problem? What's concerning you?
SL: There's six parts, and there's one part filled out of the six.

SL has essentially described an area model representation of 1/6. It is evident that he can correctly interpret the fraction notation "1/6" as designating one part of some whole equipartitioned into six parts. However, it is clear that he is unsure of what it means to add two fractions together. Recall that in the case of an internal fraction representation, the identity of the part is visually inseparable from its whole. Therefore, it may not be obvious to the student what it mean to add two area models together.

R: Okay. So what are you not sure about? Do you just not know how to do this?
SL: No.

In order to assist SL with this problem, I instruct him to draw an area model representation partitioned into fourths. Then I guide him to first shade in one-fourth, and subsequently, a second fourth. The objective is for him to understand the addition of unit fractions in terms of iteratively shading in the equipartitioned sub-units of an area model.

R: Okay, so I'll give you a clue. I want you to draw a square... And cut it into fourths.... And shade in one fourth.... Okay.... Shade in another fourth...

The resulting drawing is as follows (Figure 63).

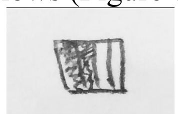


Figure 63. Close up of SL's drawing of ¼ + ¼

R: How many fourths do you have now?
SL: Two...
R: Can you do this one now? (1/6 + 1/6)

After affirming that he has understood our guided co-enactment of ¼ + ¼, I prompt SL to attempt to solve item #3 again. SL immediately sets to work and accurately draws and equipartitions two area model representations of 1/6 (Figure 64). Observe how in order to represent the problem 1/6 + 1/6 the student feels the need to create two separate representations. This is a function of using a discretely partitioned internal representation such as an area model. In order to depict the addition of two unit fractions, the student believes that two separate whole units must be drawn. He then proceeds to add the number of discrete partitions that are shaded in. Note how he places the addition symbol in between the two drawings.

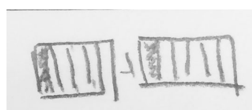


Figure 64. Close up of SL's drawing of 1/6 + 1/6

Judging by the drawing alone, it would be easy to presume that SL now has a grasp of the problem. It therefore comes as somewhat a surprise when he answers,

SL: So it's 2 fourths?
R: Why would you say it's two fourths? These are, what? (I point to the inscription for 1/6
SL: I don't know...

I infer from SL's response that he cannot solve Item #3 even after being guided to co-enact the problem, ¼ + ¼. Evidently, he is unable to mentally coordinate between the example we have co-enacted together, his two drawn representations of 1/6, and the problem at hand (adding 1/6 + 1/6). This was not all that unusual in the conventional condition, as 21% of students in the treatment group had difficulty with Item #3 even after some assistance was provided.

Looking at SL's side-by-drawings of fraction representations for Item #3 (Figure 65), there are two possible explanations for his answer. The first is that he simply misspoke while looking at the drawings of fourths. However, this seems unlikely given that the inscription for 1/6 is clearly printed on the problem. A second possible explanation is that SL observed that the area of the two shaded rectangles in his two side-by-side drawings of 1/6 corresponded to the two shaded rectangles his drawing of 2/4 (Figure 65). My in-the-moment judgment is that he is simply guessing now, so I decide to more explicitly guide him.

Figure 65. Assessment 3 Item #3 for student SL. The representation of 2/4 was created with the researcher's assistance. The two drawings representing 1/6 were the product of SL's own initiative.

R: Okay... Check (the) "no idea" (box).
R: So what's confusing you here? This is one-sixth, and that's another one-sixth, so how many (sixths) do you have?
SL: 2...
R: So that's the answer...
SL: Just 2?
R: 2 over 6.
SL: Oh. Now I get it.
R: You get it now? Remember this one? (I point to the drawing he made earlier of 2/4.) There's one fourth, and now there's two fourths. Now try this one (Item #4)

Student SL indicates that he now understands how to solve the fraction addition problems. He sets to work solving Item #4 (3/8 + 2/8) by carefully drawing two separate area model representations, respectively for 3/8 and 2/8. (Figure 66) and is able to provide a correct answer of 5/8.

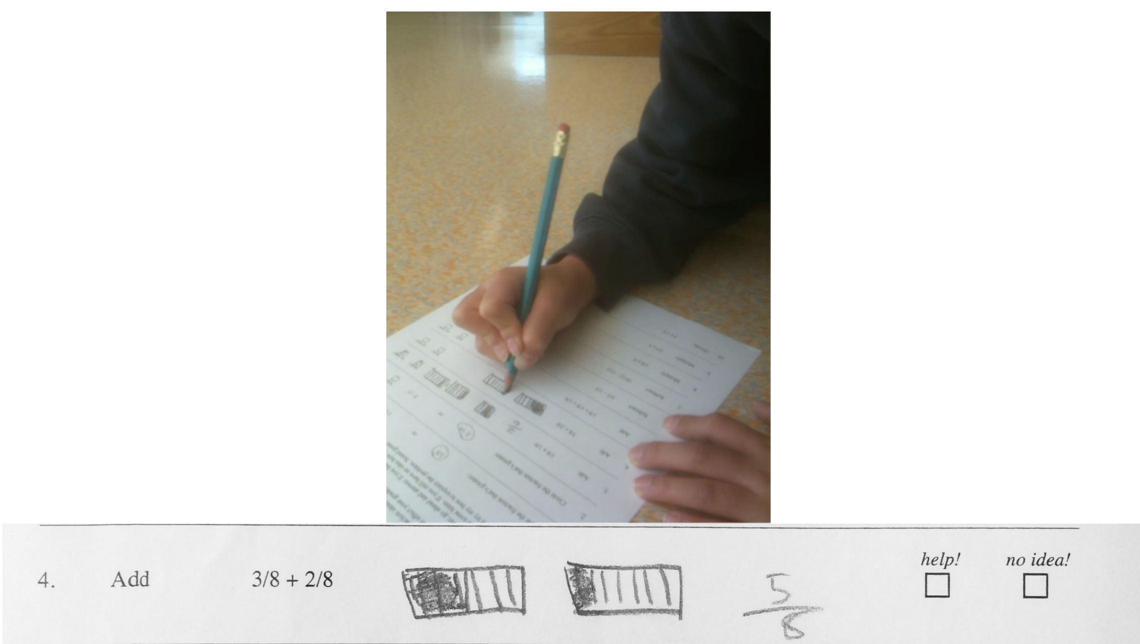


Figure 66. Assessment 3 Item #4 for student SL.

Similarly, for item #5 (1/9 + 1/9 + 1/9, Figure 67) SL begins by carefully drawing a rectangle, partitions it into nine subunits, and shades in one of the nine. However, unlike his approach to the previous two items, SL no longer feels the need to produce a corresponding drawing for each of the subsequent unit fraction inscriptions. Once again he is able to correctly answer the problem independently.

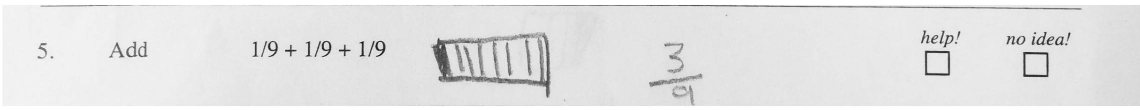


Figure 67. Assessment 3 Item #5 for student SL

5.2.1.2 EK (Water Works)

Now let us examine how EK in the Water Works group proceeds through the assessment items involving addition.

R: You need a clue on this one? Or do you think you got it?
EK: Is it 36?
R: No. Check over there that you needed help...

EK's first naïve strategy is to multiply the denominators 6 x 6. Observing this, I initiate our guided co-enactment with the measuring cups. I place the one-cup measure and one-fourth cup measures in front of her.

Figure 68. The researcher showing ¼ and 1 cup measures to EK

R: How much is that (The one cup measure)? This is one fourth (indicating the one-fourth cup measure).

After affirming that she understands the distinction between the two measures, I instruct EK to iterate one-fourth cup volume of water in the one-cup measure.

R: Pour one-fourth into here... (into the one cup measure)
R: So How much water is in there right now?
EK: Uhm. One fourth.

Note how in this particular instance, the Water Works design offers a number of significant advantages to drawn conventional models. For one, the student is not required to painstakingly draw and partition a representation. For another, because the unit fraction is a distinct, external object relative to the one-whole cup, its relationship to the whole unit—in this case, one-cup—can be conceptualized independently from the whole itself.

I then instruct her to enact the physical operation a second time. Recall that in order to visually represent the addition of two unit fractions (that may or may not result in a sum less than1), students using the conventional tools felt the need to draw two separate representations of one whole, equipartition those wholes, and then shade in the appropriate sub-unit. With Water Works the student can perform the addition operation simply by iterating and observing the outcome of their own pouring actions.

R: Do it again...
EK: Do I do this six times?

Interestingly, EK asks if we need to do this six times. Clearly she is still attempting to map between the activity at hand and the problem to solved (1/6 + 1/6).

R: No. (EK pours). Now how much water do you have?

After correcting EK, I check to see that she understands the result of her two iterative pours with the one-fourth measure.

EK: Two fourths.

Satisfied, I reframe Item #3 in terms of the measuring cup iteration we have just enacted.

R: That's your clue. Imagine you had a one-sixth sup.
EK: 2, uhm. 2/6?
R: Does that make sense?
EK: Yes.

In comparison to SL, EK's physical and cognitive demands appear to have been significantly reduced by the Water Works activity. There appears to be a direct cognitive mapping between the enacted physical activity and the arithmetic. Compared to the act of translating a problem into a drawn representation, and then mapping said representation back to the arithmetic, EK only needs to map between the (guided) act of iterating water and the arithmetic. It is in this respect that I argue that the Water Works design requires students to navigate one less 'level of abstraction.'

Moving on to item #4 (2/5 + 3/5), EK again indicates she would like assistance.

EK: Can I have a clue? (Checks the box)
R: Yep. I want you think of everything as water now...
Ok. So take your scooper, and put in two-fourths (of water into an empty container)

Anticipating that we may pour a volume of water greater than one-cup, I instruct EK to iterate her volumes of water into a clear, unmarked holding container (Figure 69)

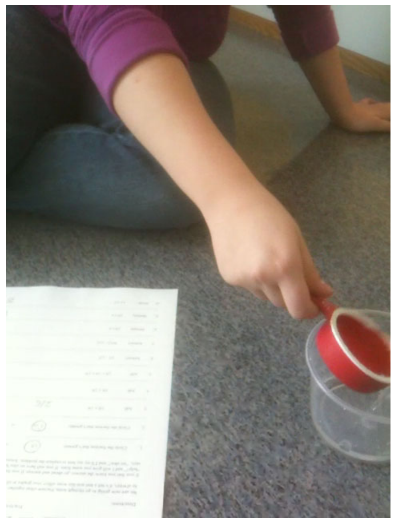


Figure 69. EK pouring ¼ cup iterations of water into an unmarked container.

R: That's one fourth... (EK iterates a second 1/4) so how much do you have?
EK: Two fourths.
R: Add one more fourth... Two-fourths plus one fourth...
EK Is three fourths?
R: Can you do that now (2/8 + 3/8)?
EK: I don't have any idea... (She checks the box indicating 'no idea')

Clearly, EK is able to add unit fractions of the form 1/n by enumerating the number of pouring actions. However, she is still uncertain about how to conceptualize and add non-unit fractions. I attempt to assist her with this problem by asking her to reflect on the volume of water she has just physically iterated.

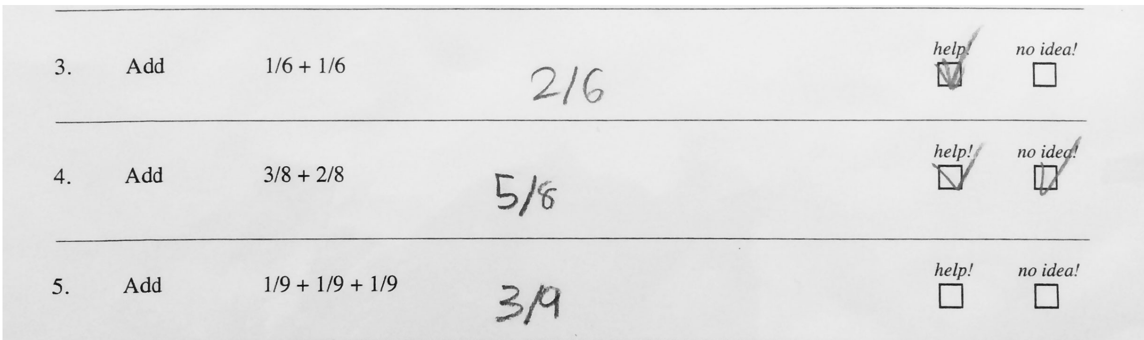
R: Ok. How much is in here? (I point to the volume of water EK has created)
EK: Uhm. 3 fourths.

I then fill the 1 fourth cup vessel again, and place it next to the clear container contain containing ¾ cups of water.

R: Three fourths plus one fourth is how many fourths?
EK: Four fourths? It's also equal to a whole right?
R: Right. Ok. 3/8ths plus 2/8s is it going to be how many eights?
EK: Five eights? Okay, I get it now!

Similar to SL, EK appears to have constructed a heuristic for adding fractions (with like denominators) from our guided co-enactment. And like SL, EK is able to provide an answer without dynamically enacting the problem (by drawing or pouring).

R: If you get it you're going to get number 5...
EK: 3/9?
R: Yeah.



3.	Add	1/6 + 1/6	2/6	help! ☑	no idea! ☐
4.	Add	3/8 + 2/8	5/8	help! ☑	no idea! ☑
5.	Add	1/9 + 1/9 + 1/9	3/9	help! ☐	no idea! ☐

Figure 70. Assessment 3 Items #3-5 for student EK

5.2.2 Subtracting Unit Fractions

5.2.2.1 SL (Conventional)

SL begins to answer item #6 (2/5 – 1/5) by drawing area model representations for 1/5 and then 2/5. Observe how SL accidentally makes a representation for 2/6 instead of 2/5. Observe how one might misinterpret the figure on the right to be a representation for 1/5 as well. The drawing error does not appear to affect him at the moment and SL writes an answer of 3/5 (a surprisingly common mistake).

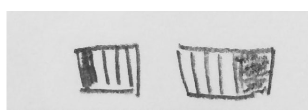


Figure 71. Close up of Assessment 3 Item #6 for student SL

R: Oh no. This one's [a] subtraction [problem].
SL: Oh, I didn't see that.

After being informed that the problem involves subtraction, not addition, SL appears to think for a while before being unable to proceed.

R: You're not sure what to do?
SL: Not sure.
R: So check help.

I then ask him to revisit the drawings he has just created.

R: What did you draw here? How many fifths were you trying to draw (I point to the drawing on the right)?
SL: 2 fifths.
R: What would happen if you minused one of them? Subtract one of the boxes?
SL: This one? (He points to the drawing of 1/5 he has made)

In the moment, I anticipate that he will erase the drawing on the left, which would leave him with a single representation of 2/5. I change tactics by asking him to draw a representation for 2/4 instead.

R: Let's do another example... Draw a square. Cut it into fourths. Shade two of them. How many are shaded?
SL: 2 fourths.
R: Subtract one them. Erase one of the boxes. How many fourths is that?
SL: one fourth.
R: So two fourths, minus one fourth, is equal to...? It's in the picture... Two fourths minus one fourth is equal to...?

At this point student SL has drawn a square, equipartioned it, shaded in two of the sub-sections (to represent 2/4), and lastly, erased one of the shaded sub-sections (Figure CVF). It therefore comes as a surprise when SL answers:

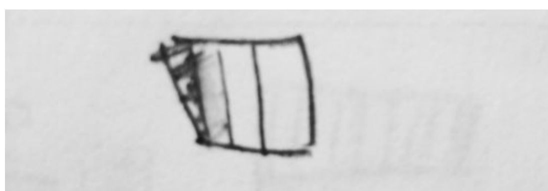


Figure 72. A drawn representation of 2/4 – ¼ by student SL

SL: Three fourths?

Obviously, a knowledgeable observer would expect that the student answer one-fourth. While perplexing in the moment, there is a plausible explanation for SL's answer. Observe how there are 3 *unshaded* boxes that are visible. This error may again be attributable the discrete nature of drawn area models—it is as if students almost cannot help themselves from enumerating the visible sub-sections.

Because I am unable to infer the logic underlying his reasoning in the moment, I decide to explicitly instruct SL how to solve the problem and begin by drawing my own representation for 2/5 using the same vertical rectangle convention SL has used earlier (Figure 73a and 73b).

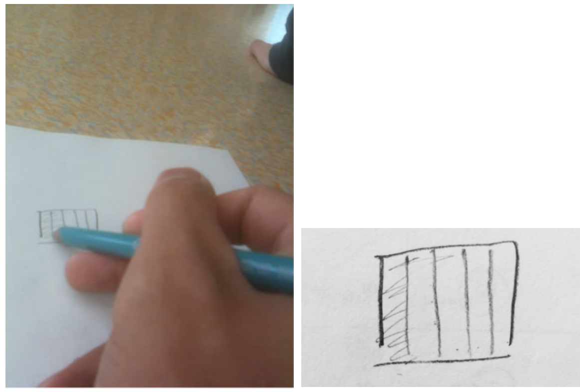


Figure 73a & 73b. The researcher drawing in and shading a representation of 2/5 before erasing one the shaded sections to demonstrate subtracting 1/5.

R: I'm going to subtract one fifth. How many are left?
SL: One fifth.
R: Does that make sense?
SL: Yes.
R: So two fifths minus one fifth... is equal to one fifth. But I had to (explicitly) show you, so mark that as wrong.

After providing some verbal indication of understanding, SL proceeds to Item #7 (10/12 – 7/12). First, SL tries to draw a representation for 10/12. He then attempts to draw a representation for 7/12 (Figure TR). Seemingly, the very act of attempting to draw and count causes him to lose track of the objective of the activity.

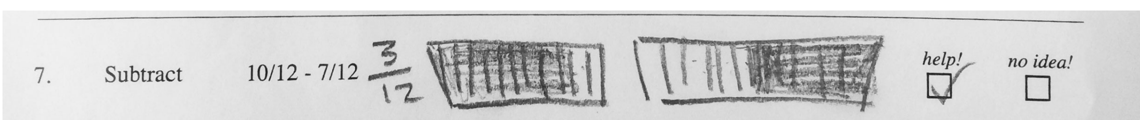


Figure 74. Assessment 3 Item #7 for student SL

SL: I think I messed up...
R: What were you trying to do?
SL: 7/ 12.

In order to reorient him, I simply ask him to reflect upon the drawings he has already made.

R: So here you drew what? (pointing at 10/12) What's this supposed to be.
SL: 10/12
R: And what's this supposed to be?
SL: 7/12:

I then restate the problem in terms of finding the difference between two numbers, and the difference in the number of shaded sub-sections in his two drawings..

R: So when you subtract, you're also finding the difference. So what's the

difference between these two (drawings)?
SL: Uhm, 3?
R: Yeah, that's the answer.
SL: I need to check (the box indicating I received) help?
R: Yeah.

5.2.2.2 EK (Water Works)

Similar to SL, EK initially applies an addition heuristic to solve the subtraction problem (2/5 – 1/5).

EK: 3/5? (which would be correct for addition)
R: It's subtraction...

However, after apprising her of the actual operation, there is a dramatic difference in how EK handles the problems. Unlike SL who painstakingly draws a series of representations and is still unable to answer on his own, EK appears to only need to briefly think about the problem before answering:

EK: One five? (1/5)
R: (I nod to indicate my assent) You can say "one fifth."

Moving on to item #7 (10/12 – 7/12) the exact same situation repeats itself. EK initially adds the two fractions (correctly), before being reminded that it is a subtraction problem. Once again, she solves the subtraction problem correctly.

EK: 17/12? (twelfths)
R: It's subtraction...
EK: I keep forgetting that!
R: If we were adding you would have been right.
EK: Three twelfths?

In dramatic comparison to SL, EK's problem solving is accurate and seemingly effortless. Bear in mind that both students faired equally poorly on the pre-test and prior assessments on fraction arithmetic items, and required assistance to solve the initial items involving addition with fractions. How might we explain difference displayed by EK?

5.2.3 Multiplying & Dividing Unit Fractions

The final three items from Assessment 3 (#8-10) involved multiplying and dividing unit fractions. After being guided through item #8, SL becomes mentally fatigued to the point where he opts not to even attempt to complete items #9-10. We therefore conclude the conventional group narrative with the work of student LB, who, like SL and EK, scored at a non-proficient level on the pre-test items #11-20.

5.2.3.1 SL (Conventional), item #8

After observing SL thinking about the problem but not answering, I ask.

R: So can you multiply?
SL: No:
R: Check help.

I then begin to assist him to reason about the problem (1/6 x 8) with a simpler example (1/4 x 4).

R: So when you multiply, it just means you repeat something. So draw a square. I'll draw it for you, that will be faster.

Observing what I interpret as increasing mental fatigue, help expedite part of the process by drawing an square and equipartitioning it into four sub-sections (Figure Square). I then instruct him to iteratively shade in each of the four sub-sections.

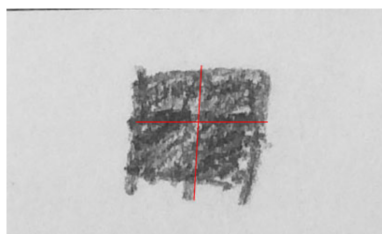


Figure 75. A drawn in representation of 2/4 x 2.

R: Shade one fourth. That's one-fourth times one. Shade another fourth. That's one fourth times what?
SL. 2
R: Shade another one., Times...
SL: 3...
R: Do another one.
SL: ¼ times four.
R: So when you multiply, you repeat. How many times are you going to repeat 1/6?
SL: 8
R: Okay, figure out an answer... Any idea...?

SL recognizes that the multiplication problem requires eight repetitions, but is still unable to come up with an answer. After demonstrating how to solve item #8 to him (Figure Help), I ask him to think about items #9 and item #10. By observing his body language, I infer that he is no longer able or willing to proceed with these final assessment items so I allow him to leave.

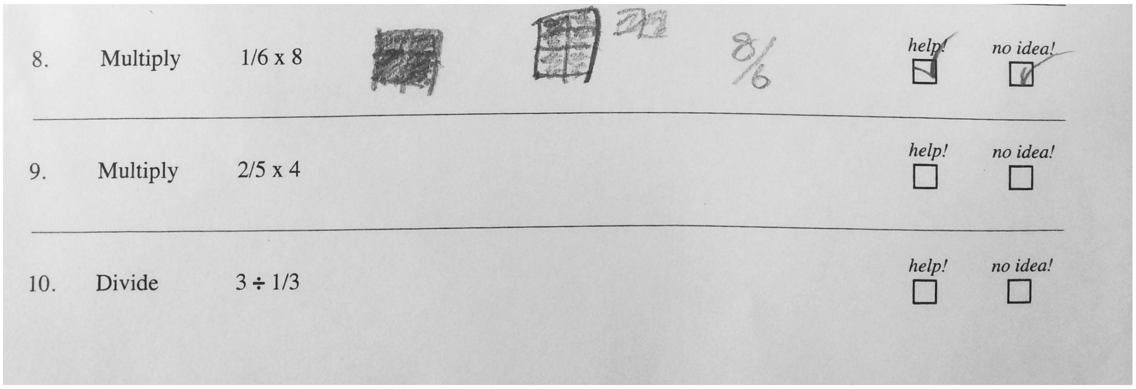


Figure 76. Assessment 3 Items #8-10 for student SL

5.2.3.2 LB (Conventional), Items #8-10

Resuming the conventional group narrative, student LB—like SL before her—is unable to solve item #8 without assistance. Her initial naïve strategy to is multiply both the numerator and denominator by a factor of 8 (1/6 x 8 = 8/48).

I instruct LB to flip the paper over and state:

R: Multiplication is repeating something. You understand that?
LB: Yeah.
R: So, why don't you draw a square and divide it into fourths...

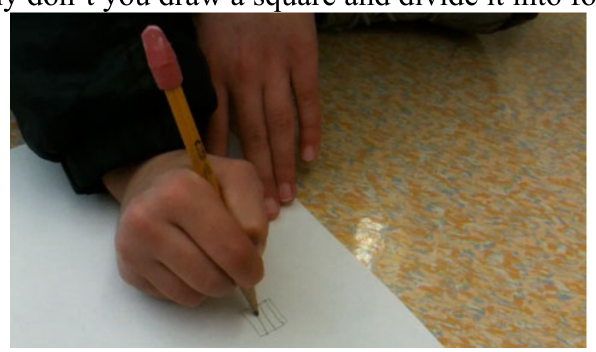


Figure 77. Student LB drawing a representation and dividing it into 4 sections.

R: Why don't you shade in one of them... That's like saying, one-fourth times one... Now why don't you shade in another one. That's like saying, one fourth times...
LB: Two...
R: And another one...
LB: One fourth times three...
R: ...Is?
LB: ...three fourths
R: Now shade another one
LB: ...four fourths...

It appears that LB is following along with the intent of my activity. Based on my experience with SL just a few moments prior, I decide to slightly modify the assisted prompt on item #8 and ask for a fifth iteration in order to ensure that the product is a fraction greater than one-whole.

R: Shade another one...

LB is momentarily unsure of how to proceed because the first representation has already been completely filled. Anticipating this, I tell her.

R: Start another box...

LB draws a second square, demarcates it into fourths, and shades in one of them (Figure 78).

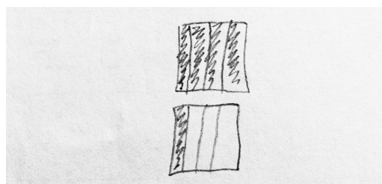


Figure 78. Student LB's completed representation of ¼ x 5.

R: How many fourths have you shaded?
LB: Five.
R: Because you did one-fourth times 5. Is this enough to help you?
LB: Yes.

LB begins by drawing an area model. She partitions it into sixths, and then shades in all six sub-units. She draws draw a second model equipartitions it into sixths again, shades in two more sub-units, and then correctly answers 8/6.

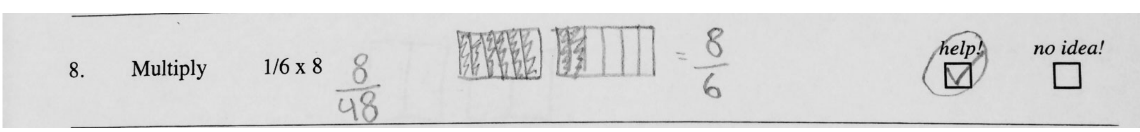


Figure 79. Assessment 3 Items #8 for student LB

Unprompted, LB moves onto item #9 and applies a similar strategy. She begins by drawing an area model and partitions it into fifths. Next she shades in the first two subsections to represent 2/5 (Figure 80).

LB: 2/5's times six... (to herself)

Figure 80. Student LB drawing and partitioning a representation into fifths.

Sensing some hesitation but that she is on the right track, I offer some positive reinforcement.

R: You're on to something!

LB continues to shade in blocks of 2 subsections. In order to complete the third iteration of 2/5 (6/5), she draws a second representation and partitions it into fifths. In order to keep track of her progress, she visually demarcates each 2/5 pair (see Figure CARAT). Finally, LB shades in a fourth iteration, and arrives at the answer 8/5.

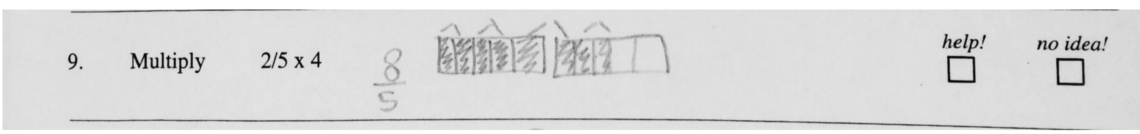


Figure 81. Assessment 3 Item #9 for student LB.

For the final problem of the series, #10 (3 ÷ ⅓), LB indicates she needs help.

R: You want to talk about it?
LB: Yeah.
R: Okay, I want you to think about division as, "How many of something, can I make from something else..." Are you following?
LB: Yeah.
R: Okay. How many one-thirds, can I make from three?
LB: Three? Three one thirds... Wait...
R: I'll give you an example. Draw two squares. (I wait for LB to draw two squares). How many halves can you make from two squares?
LB: Four...
R: Okay, cut them in half. (partition the two squares into halves)
LB: (Partitions the two squares into 4 halves) You can make four...
R: So. How many thirds can you make from 3? You should draw circles. That might be easier.

She draws a single circle, and asks:

LB: That’s a third?

My inference is that she is now guessing, so I explicitly guide her through the problem.

R: No. Draw three circles. Now, cut each of them into thirds...

LB is also uncertain how to equipartition the circle, so I tell her to “draw a Y.”

R: Draw a Y... You got it now?

After she completes three circles, each equipartitioned into thirds (Figure Third), I ask her:

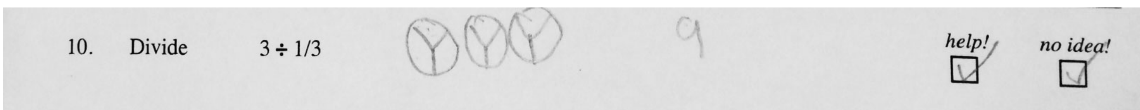


Figure 82. Assessment 3 Item #10 for student LB

Figure 83. LB counting the total number of discrete partitions contained in the three representations she has drawn for item #10.

5.2.3.3 EK (Water Works), Item #8-10

Returning to item #8 (1/6 x 8) in the Water Works condition, student EK similarly begins with a naïve multiplication strategy and multiplies the denominator 6 by 8.

EK: 48?
R: No. Mark the help box. What you did was say 8 times 6.
EK: And then I multiplied it...
R: Erase that (naïve heuristic) from your head okay. What I want you think about multiplication is that we repeat something... One sixth. How many times do we repeat it.
EK: Eight... Uhm... (pauses to think)... Eight sixths?
R: Yeah. Does it make sense?
EK. Yeah. It just took a while, cause...
R: Hey, this is fifth grade, sixth grade stuff.
EK: Really?
R: Yeah, in fourth grade, they are happy if you can do this (I point to the fraction comparison problems). I'm asking you to multiply and divide.

Remarkably, there was no need to demonstrate or re-enact a similar situation for EK. After drawing the analogy between multiplication and a repeated iteration, she is able to draw upon our prior enacted activity (in the context of the addition problems) to reason about multiplication as an additive operation. Unprompted, EK is able to immediately solve Item #9 (2/5 x 4).

EK: eight fifths
R: yeah.

We conclude our quantitative analyses by examining the interaction with EK for Item #10 (3 ÷ ⅓). Per her previous pattern, EK appears to apply her most recently learned heuristic (multiplication) to the problem before noticing that it involves division.

EK: One? (3 x 1/3 = 1)
R: Not quite. Need help? Let me review. When you multiply, you repeat something...

First, I acknowledge and affirm that her multiplication strategy was valid. Then, I establish the mental model we will use for division.

R: When you divide, you say, "how many of something, can I make from something else?"

I place the one-cup measure in front of her, and ask.

R: So here's a one-cup. If I were to divide 1 by one-fourth, how many fourths can I make from this (1 cup)

EK: Four
R: How many one-thirds can I make from this?
EK: Three

The objective of this exchange is to reframe division in terms reverse-iterating unit-fraction volumes of water out a larger volume.

R: How many one thirds can I make from three cups?
EK: 3

Interestingly, EK's response at this juncture is identical to LB's from the conventional group. She is able to reason that one can make three thirds from one cup, but once I switch to three wholes, she likewise answers that '3' one-thirds can be made from three cups. As with LB, I decide to explicitly guide EK through this problem. We begin by iterating three one-cup volumes of water into an unmarked container.

R: We're going to prove it. Pour me three cups of water please? (I hand EK the one-cup vessel).

Figure 84. EK pouring 3 cups of water into an unmarked vessel.

R: One. And it's just going to reach the top, so you'll be okay. (meaning the vessel that will contain the three cups of water won't overflow.
R: Two. How do I know this? Because we just tested it out! And if there's extra we'll just pour it over here (in an extra container)
EK: There's a little extra.
R: Okay, dump it in here. So the challenge is, how many of these can we make? (I point to the 1/3 cup.) We'll start scooping out of here (the 3 cup volume of water with the 1/3 cup measure)

With my assistance, EK begins the process of reverse iterating from the 3 cup volume using the 1/3 cup measure. I further assist by counting each iteration out loud.

R: 1. 2. Keep going... 3... (I recombine the overflow into the vessel she's working with)4... 5... 6.... 7... 8... 9...So the answer is?
EK: 9...
R: Does that make sense?
EK: yes.

5.3 Building the Case For Embodied Coherence

The underlying design rationale behind the Water Works design was to allow learners to physically enact situations that would allow them to readily integrate fractional concepts into their pre-existing schemes for counting and whole number arithmetic.

In order to demonstrate addition with fractions, EK is instructed to iterate and combine unit-fraction cup volumes of water. The iterations can be directly observed and counted, and each iteration results in a greater volume of water. More importantly, the physically embodied activity coheres with her prior understanding and expectations for addition. In other words, there is a direct mapping between her physical activity and her prior understanding of and/or expectations for addition operations. Just as she has learned from earlier experiences counting and adding physical objects that $1 + 1 = 2$, the activity allows her to experientially confirm that one ¼ cup volume plus another, one ¼ cup volume results in two ¼ cup volumes.

By contrast, recall that in order to begin to conceptualize a problem involving addition, SL felt compelled to draw area model representations of the fractions. Indeed, much, if not most of his cognitive and physical activity appeared to be focused on creating the actual representations themselves. And, although his drawings were for the most part accurate representations of the fractional addends, it was less obvious to him how said drawings could be combined to represent an operation such as addition. Here, I once again implicate the perceptual features present in area models, which according to my earlier cognitive domain analyses, can be characterized as possessing visually discrete, internal fractional representations. Unlike volumes of water—which can be combined together seamlessly simply by pouring—it is impossible to combine two drawn representations. Instead, separate—arguably less coherent—operations such as shading, erasing, and re-shading a drawing are required to enact a dynamic arithmetic operation.

Indeed, while SL was ultimately able to incorporate my assistance and derive a problem solving strategy for the subsequent fraction addition items, when presented with a subtraction problem we observed him once again unable to infer how the arithmetic operation of subtraction corresponded to the representations he had drawn. This is representative of the experience of many students, who learn to abandon conceptual reasoning about fractions (in SK's case, in terms of the construction and transformation of drawn representations) in favor of the relative efficacy of rote arithmetical algorithms.

EK on the other hand, was able to arrive at a mathematically correct answer for the unit fraction subtraction items without any additional assistance. Evidently, the enacted performances for adding unit fractions were sufficient. The argument can thus be made that the physically embodied activity required to construct a drawn area model representation was less coherent with respect to both the addition algorithms than the act of physically iterating volumes of water. Conversely, we may state the Water Works design offered a greater degree of embodied coherence for the cognitive task demand of learning to integrate fractional units with a preexisting addition scheme.

The contrast between EK and the conventional treatment students on the multiplication items was particularly striking. Unlike SL, LB, and the majority of students in the conventional treatment group, EK (and others in the Water Works group) was able to answer the multiplication problems without any assistance apart from a verbal reframing of the multiplication operation as “repeating something.” EK is able to draw upon the prior enactment of addition in order to reason about multiplication as an additive operation.

Vitally, the difference in student performance on fraction multiplication was not unique to EK alone. In the Water Works condition, the effective conversion rate—our measure for the utility of the pedagogical assistance—on multiplication items was 80%. By comparison, students in the conventional group only arrived at a correct solution 26% of time after assistance was provided. Given that an additive model for reasoning about multiplication was provided in both treatment conditions, we may safely presume that the higher degree of embodied coherence afforded by the various Water Works co-enactments contributed to this difference.

Finally, while the students in the qualitative examples provided could not complete the division problem (3 ÷ ⅓) without assistance, the quantitative numbers speak for themselves (Table 18). In the conventional treatment group, assistance for Item #10 was provided for 17 out of the 19 students. Of those 17 students, only 3 could solve the problem, resulting in an effective conversion rate of 18%. In the Water Works condition, assistance on item #10 was provided for 20 of the 21 students. Remarkably, 14 of those students were able to solve the division problem, an effective conversion rate of 70%.

Table 18

Effective Conversion Rate for All Assessment 3 Items

#	Question	Conventional (n=19) # of requests	Conventional Conversion Rate	Water Works (n=21) # of request	Water Works Conversion Rate
1	1/5 > 1/10	3	100%	3	66%
2	2 ¼ > 2 1/7	2	50%	2	100%
3	1/6 + 1/6 =	8	50%	6	100%
4	3/8 + 2/8	3	33%	2	50%
5	1/9 + 1/9 + 1/9	1	100%	1	100%
6	2/5 - 1/5 =	2	50%	3	100%
7	10/12 – 7/12	3	67%	0	-
8	1/6 x 8	12	33%	9	78%
9	2/5 x 4	11	18%	7	71%
10	3 ÷ ⅓	17	18%	20	70%
Total		**62**	**33%**	**53**	**77%**

CHAPTER 6

This dissertation opened with the assertion that the conventional fraction representations—e.g. static, drawn diagrams such as area models and number lines—widely used in classrooms today should be implicated in the struggles that many students continue to display with fraction concepts and operations. Three conjectures guided my examination of the problem of learning basic fraction concepts and arithmetic operations. The first conjecture was that part of the difficulty in learning fractions stems from students' inability to coordinate their prior knowledge about whole numbers arithmetic operations with both the artifact-mediated manipulation and the formal mathematical conventions used to describe these manipulations. My second conjecture was that in order to effectively guide students to construct the desired schemes, one must account for and eliminate where possible the potential contradictions that a given activity might introduce. The third conjecture, which builds upon the prior two, was that an activity sequence for supporting early fraction understanding should necessarily build upon students' *additive*, as opposed to *multiplicative*, reasoning (c.f. Confrey, 1994, on the "splitting conjecture").

In order to examine and test my assertions, I devised a cognitive design analysis outlining the potential challenges that students had in interpreting conventional tools such as area models and number lines, and developed an alternate design for teaching fractions—Water Works.

Water Works is built upon what I have termed the principle of embodied coherence. I have argued that in order to foster coherence between what is taught and what is actually learned, an instructor must deliberately consider the interplay between learners' perception of pedagogical artifacts, the actions they are tasked to perform, and the resulting outcomes of the dynamically enacted activity sequence. In the context of early fraction arithmetic, a highly coherent design would build upon and leverage children's pre-existing understanding of whole numbers and whole number arithmetic. In direct contraposition to this principle of coherence, I believe that many of the pedagogical conventions for teaching fractions introduce a number of conceptual and perceptual contradictions that may impede, rather than advance, learners' understanding of fraction concepts and operations.

Water Works was designed to allow children to physically model fraction arithmetic in a manner that purposefully leverages and builds upon their competency with additive operations involving whole numbers. A key aspect of the design was to preserve a one-to-one correspondence between learners' physical actions and the mathematical concepts to be learned. Furthermore, the unique physical properties of water—in particular, the ability to seamlessly combine or separate volumes of water—appeared to alleviate many of the errors that typically arise when students first attempt to interpret and make sense of traditional representational media.

In order to test my claims as well as the pedagogical efficacy of the Water Works design, I conducted a mixed-methods design-based research study. Piloting began with students in a self-contained, Grade 3-5 special education classroom. Based on the promising results with this population, the study was scaled up, and 4th grade students from three general-education classrooms were assigned into control, Water Works treatment, and conventional treatment conditions. The study compared how students in the various treatment groups performed on fraction arithmetic tasks with the aide of different representational media. A unique aspect of the resulting snapshot methodology was that it relied primarily upon multiple, brief (5-10 minute) one-on-one interactions collected for an extended time frame (approximately 7 months). Combined with traditional assessment tools, this approach allowed for both a quantitatively driven overview of the data as well as in-depth qualitative analyses of students' in the moment thought processes and development.

6.1 Summary of Findings

A naïve presumption of the study was that students using the Water Works media would outperform students using conventional media on fraction arithmetic tasks. This cautious optimism was fueled in large part by the positive results observed during the initial piloting work with the special needs population. Not surprisingly, a closer examination of the data revealed a more nuanced and complex story.

After the very first intervention, which involved a brief whole class lesson introducing fraction arithmetic operations, students in the conventional treatment group showed significant improvements on a fraction arithmetic assessment, whereas the students in the Water Works condition did not. The results appeared to fly in the face of expectations, and would at first glance suggest that the Water Works design offered little to no advantage when compared to conventional media.

Such an interpretation would be premature given that over time, students in the Water Works groups could be observed making statistically significant improvements—matching, and by Assessment 3 eventually surpassing, the relative gains displayed by the conventional treatment groups.

One possible explanation for these results is that the initial advantage that students in the conventional group possessed was a pre-existing familiarity with the context of instruction. To elaborate, students in the conventional treatment group had been exposed to conventional representations such as area models and number lines for months—if not years—by virtue of their classroom mathematics curriculum. In contrast, students in the Water Works condition were not only learning new arithmetic concepts and operations, but they were also learning to interpret an entirely new representational context. As of consequence, students in the conventional treatment group were more readily able to learn the novel concepts and operations being taught. Following this line of reasoning, the subsequent gains made by the Water Works group could then be attributed to the superior cognitive affordances of the Water Works design. Indeed, the compelling qualitative data as well as the quantitative results from Assessment 3 (as presented in the preceding chapter) would appear to strongly support this claim.

However, by the final post-assessment, students in both treatment groups performed at the same level on fraction arithmetic tasks, a finding that would once again suggest that the Water Works design offered no advantage over conventional media. How can we reconcile our contradictory findings?

The puzzle, I believe, can be readily explained once again in terms of students' more frequent opportunities for exposure to and, by extension, greater degree of familiarity with conventional representations. Given the mandated requirements of the school district, the participating teachers could not deviate from their mathematics curriculum to satisfy the whims of my study. I could not—nor was I prepared to—fully restructure their entire fractions curriculum to only utilize the Water Works media. Furthermore, given my resource constraints with regards to time and manpower, my interventions and opportunities for data collection were brief and intermittent both by necessity and design. Thus, while students in the Water Works condition may or may not have benefited from a more coherent pedagogical enactment—the fact remains that in the intervening time frames between my interventions students in both conditions would have received supplementary lessons on fraction arithmetic using conventional representations.

6.2 Implications and Conclusion

The Water Works study explored the possible tensions between: (a) learners' physically embodied, multi-modal, goal-oriented actions; (b) the meanings that the learners assigned to the mathematical inscriptions symbolizing these actions; and (c) the conceptual foundations that instructors attempt to build learners' initial understanding of fractions upon.

On a practical level—in light of the evidence collected to date—I believe that a compelling case can be made for integrating the Water Works pedagogical activities into a broader curricular approach for early fraction instruction. Compared to conventional drawn representations already in widespread use such as area models and number lines, the Water Works media appears to be as—if not more so—effective tool for teaching early fraction concepts such as part-to-whole relationships, as well as arithmetic operations such as addition and subtraction.

Indeed, the Water Works design appears to be a particularly powerful tool for teaching multiplication and division with fractions. Unlike drawings, which require students to allocate a number of cognitive resources to carefully illustrate and interpret, the Water Works design allows students to directly enact said mathematical operations in terms of physically embodied, additive actions (and the resulting outcomes).

Generalizing beyond the case of the Water Works design itself, another important takeaway for designers of novel pedagogical interventions to bear in mind is that establishing students' comfort and familiarity with a given context for instruction is a prerequisite before one can make any measurement of its effectiveness. To elaborate, the efficacy of a novel intervention is typically measured in terms of how it compares to the incumbent best practice; or alternately, how well learning in the novel context transfers to a conventional context. As we have observed in the Water Works study, when students are taught a novel concept/scheme in the context of the novel situation, their cognitive task demands are actually increased. Given that an individual's cognitive schemes first emerge through interaction from a specific situation, it should not come as a surprise when learners fail to immediately recognize how the same scheme applies to a different situation. Consequently, in order to measure the utility or effectiveness of a new intervention or curricular unit for example, one must be prepared to account for this possible contingency.

Theoretically speaking, I have suggested that an additive approach to teaching early fractions may be more viable than one based upon the multiplicative principle of splitting. The rationale was that all students' initial experiences with number are rooted in counting and addition. Introducing historically challenging fraction concepts and operations in terms of already established interaction routines is not only sound in principle, but also appears to be supported by the data collected to date.

Finally, I have attempted to advance a cognitive design principle I refer to in terms of embodied coherence. In the case of Water Works, I have suggested that the design fosters coherence between learners' physical actions, the outcomes of said actions, and their prior understanding of whole-number addition and thus supports the learning of fraction arithmetic operations. Can this principle of embodied coherence be applied to other pedagogical design work?

In order to answer this question, we must first step back and consider how pedagogical designers can specify the rules and causal relationships that govern learners' actions, establish the goals of an instructional activity, and pre-determine the resulting perceptual outcomes that learners will observe. In this light, we may state that instructional designers plan the actions/transformations of a dynamically unfolding situation *to be* enacted by a learner—with the assumption that said enactment will result in the learner constructing a specific concept/scheme. Embodied coherence can thus be said to emerge from the learner's anticipated coordination of: (1) Their prior knowledge schemes; (2) their interactions with the instructional situation, which we may characterize here as encompassing goal oriented, artifact-mediated objectives of the designed activity; and finally (3) learners' perceptions of the outcomes. Framed in this matter, it does not seem unreasonable to believe that a thoughtful and purposeful coordination of students' existing cognitive resources and immediate perceptual experiences as mediated through goal-oriented activity should guide and inform all pedagogical design work.

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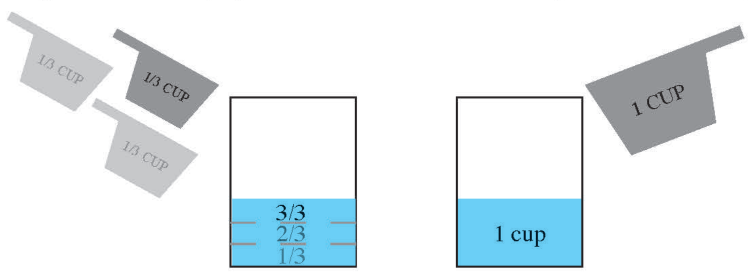
APPENDIX

Activity 1
Establish the conventional unit of "1 whole cup" and the unit fractions available thereof. This allows students to coordinate their visual perceptions with the new semiotic notation of fractions. Critically, it provides visual evidence contradicting their prior (axiomatic) belief about the magnitudes of natural numbers. E.g. 4 > 3, but 1/3 > 1/4

Activity 2
Establish the part-to-whole relationship between unit fractions and the whole cup. E.g. "Why is 1/3 cup called 1/3? Because *three* scoops of the 1/3 cup are required to fill the one whole cup." This will attend students to the role of the denominator. This can be done by iterating the 1/3 cup three times into the one cup measure.

Alternately, one can use two, separate unmarked containers to compare the results from pouring.



This also provides an opportunity to teach the conventional notation, 3/3 (also 4/4 , 2/2, etc.) = 1 whole cup

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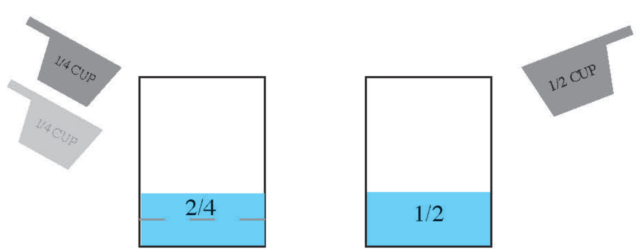
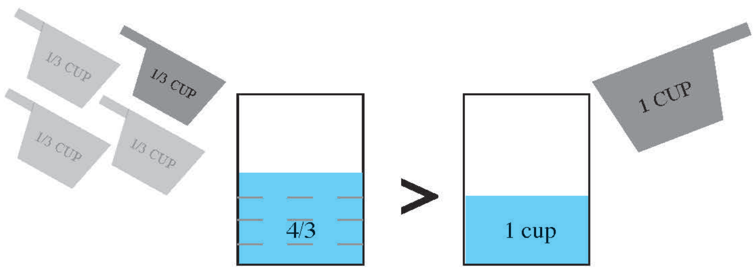
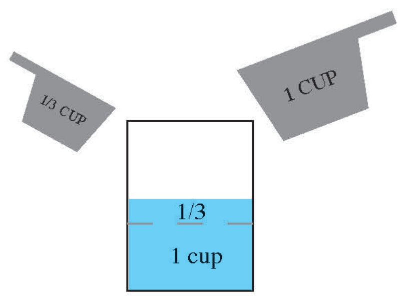


Equivalence relationships such as 2/4 and 1/2 (or by iterating the 1/4 into the 1/2 cup measure, twice)

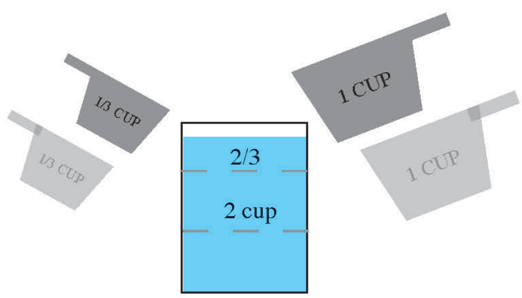
Activity 3
After establishing the unit compositions of one whole cup (e.g. 2/2, 3/3, 4/4), we can introduce students to the notion of improper fractions > 1, e.g. 4/3 as the iteration of four 1/3 cups.



Again, the semiotic forms and the relationship between 4/3 and 1 whole are visually confirmable.



And sets up a conceptual model for mixed numbers such as 1 1/3 = 4/3



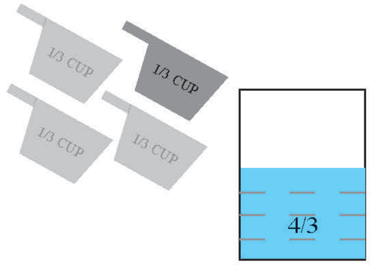
And beyond, e.g. 2 2/3

A key affordance is the semiotic notation now has a corresponding representational form/media that can be manipulated in a manner that is analogous to nearly all arithmetic operations associated with the a/b form.

Procedural Algorithms (1/3)

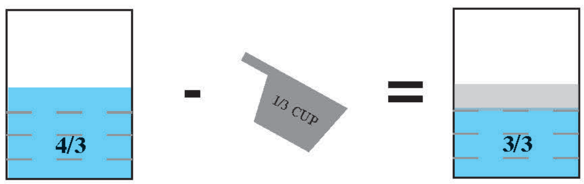
Addition and Multiplication with Unit Fractions

1/3 + 1/3 + 1/3 + 1/3 = 4/3, or 4 x 1/3 = 4/3



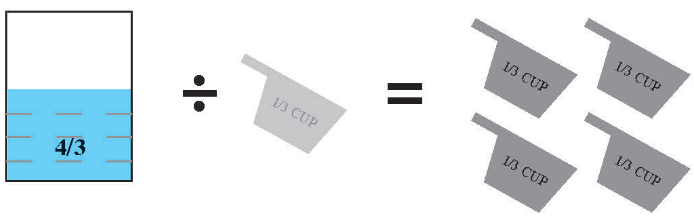
Subtraction with Unit Fractions

4/3 - 1/3 = 3/3



Division with Unit Fractions

4/3 ÷ 1/3 = 4



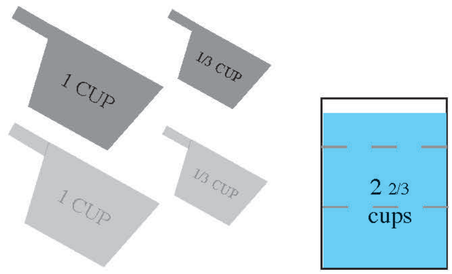
Note that division might be framed by asking, “how many 1/3 cup scoops can be made from 4/3 cups?”

Obviously, we are constrained somewhat by the materials available. Most measuring cups only have units of 1 cup, 1/2, 1/3, and 1/4, but the main point is how a simple case of each operation can be modeled.

Procedural Algorithms (2/3)

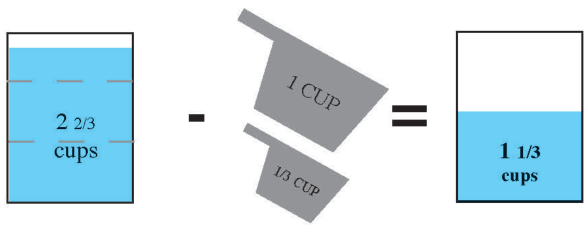
Addition and Multiplication with Non-Unit or Mixed-Number Fractions

1 1/3 + 1 1/3 = 2 2/3 cups, or 1 1/3 x 2



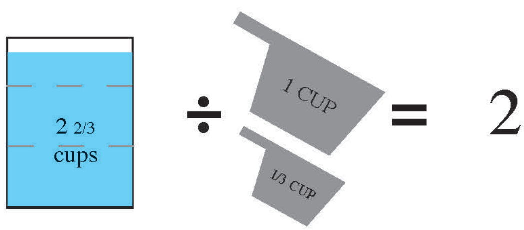
Subtraction with Unit Fractions

2 2/3 - 1 1/3 = 1 1/3



Division with Unit Fractions

2 2/3 ÷ 1 1/3 = 2



In this case, students would be asked, "How many 1 1/3 cups (as a whole) are their in 2 and 2/3 cups?"

Procedural Algorithms (3/3)

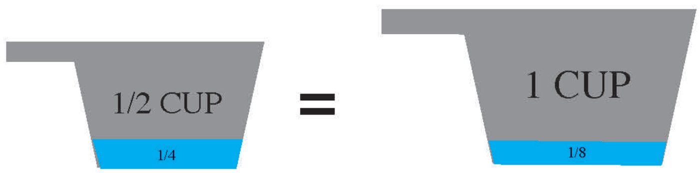
Multiplying fractions by fractions

(Note: this is arguably the most difficult concept to represent with measuring cups, as it requires an additional levels of abstraction in that one must conceptualize two different "wholes"—the 1 cup, and the 1/4 cup)

Consider the case 1/2 x 1/4 = 1/8

One may conceptualize this problem semantically as "one-half of a one-fourth cup is what fraction of a one-whole cup"

(or conversely, "one-fourth of a one-third cup is what fraction of a one-whole cup")



Dividing fractions by fractions

For **1/2 ÷ 1/4 = 2**, we might ask, "How many 1/4 cups can be made from a 1/2 cup?"

Conversely for **1/4 ÷ 1/2 = 1/2**, we might ask, "How many 1/2 cups can be made from a 1/4 cup?"

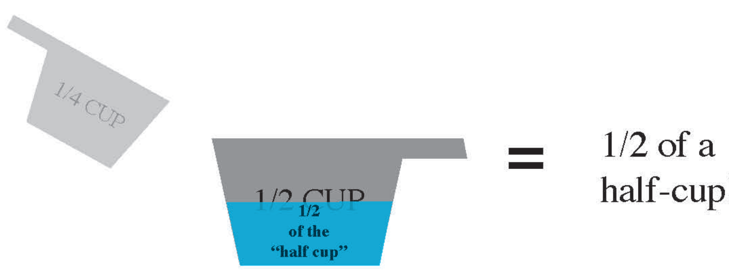
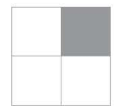
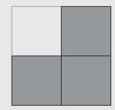
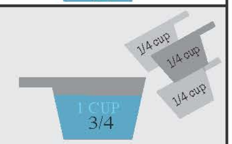
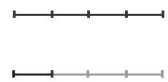
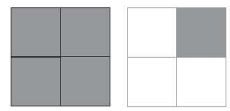
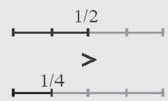
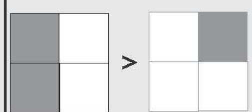
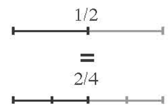
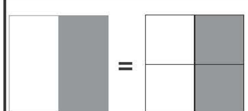
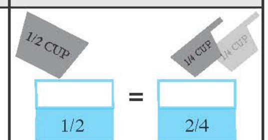
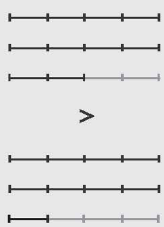
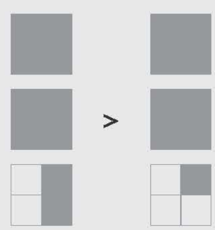
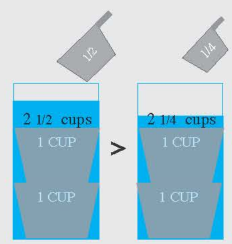


Table 1. Representing and Comparing Fraction Concepts

Concept	Number Line	Area Model	Water + Measuring Cups
(a) Unit Fractions (e.g. 1/4)			
(b) Non-Unit Fractions (e.g. 3/4)			
(c) Mixed Numbers and Improper Fractions (e.g. 5/4, 1 1/4)*			
(d) comparing unit fractions (e.g. 1/2 > 1/4)			
(d) basic equivalent fractions (e.g. 1/2 = 2/4)*			
(e) comparing mixed numbers or improper fractions (e.g. 2 1/2 > 2 1/4)			

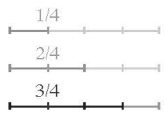
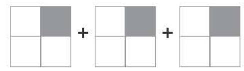
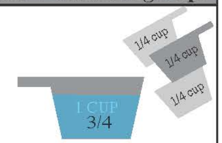
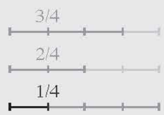
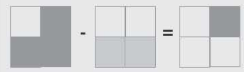
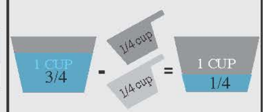
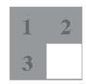
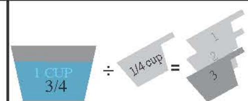
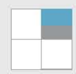
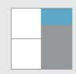
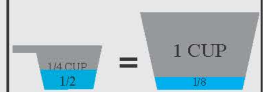
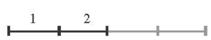
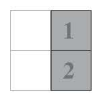
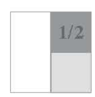
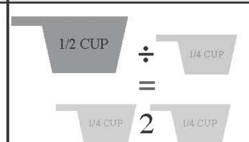
**water can be poured into separate containers to create continuous volumes or to make visual comparisons*

Notes: Because discrete units (e.g. squares and vectors) are distinctly visible to students using area and number line models, students may not readily perceive the implied "whole unit." This may explain commonly observed errors, such as when students describe a fractional representation of "1/4"(one shaded box or vector out of four) as "1/3" instead (e.g. one shaded box and three non-shaded boxes, see example (a) in the table above).

In contrast, the standard unit of one-whole cup provides an explicit "whole unit." Furthermore, the discrete physical iterations of scooping water needed to construct the whole (e.g. pouring from the 1/4 measure to the one whole measure) are visually distinct from the continuous volume of water.

Furthermore, representing fractions > 1 does not require drawing new discrete units (as with the area and number-line) because the water merges together into a new continuous whole (see example (e) and refer to the appendix for a more detailed illustration). This may be facilitated in classroom activity using two clear, congruent containers.

Table 2. Operations for Values *Less* than 1

Fraction Procedures	Number Line	Area Model	Water + Measuring Cups
(f) Iterating Fractions (Adding/Multiplying) (e.g 1/4 + 1/4 + 1/4; 1/4 x 3)			
(g) Subtracting Fractions (e.g. 3/4 - 2/4 = 1/4)			
(h) Dividing by Fractions (e.g. 3/4 ÷ 1/4)			
(i) Multiplying Fractions by Fractions (e.g. 1/2 x 1/4 = 1/8 —or— 1/4 x 1/2 = 1/8)**	 *—or—* 	 *—or—* 	 *—or—* 
(j) Dividing Fractions by Fractions (e.g. 1/2 ÷ 1/4 = 2 —and— 1/4 ÷ 1/2 = 1/2)***	 *—and—* 	 *—and—* 	 *—and—* 

Notes: While area and number line models can be creatively used to represent procedural operations with fractions, they require multiple drawn instantiations. This again may obscure the notion of what is meant by "one whole."

Also, consider a common student error such as 1/4 + 1/4 + 1/4 = 3/12 (see example (f) in the table above). The multiple diagrams that students may produce to visually represent this problem, may also potentially reinforce naive procedural errors such as adding both numerators and denominators. An area model may inadvertently provide visual proof for such an erroneous strategy, given that there are three shaded boxes and twelve small boxes total!

*** Multiplying fractions may be conceptualized as, "one-half of a one-fourth cup is what fraction of a one-whole cup." This can be difficult to model with measures of water, and might perhaps best be explained using drawing and algorithm.*

****For 1/2 ÷ 1/4 = 2, we might ask, "How many 1/4 cups can be made from a 1/2 cup?"*
Conversely for 1/4 ÷ 1/2 = 1/2, we might ask, "How many 1/2 cups can be made from a 1/4 cup?"

Table 3. Operations for Values *Greater* than 1

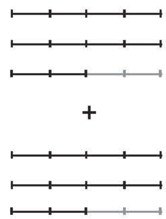
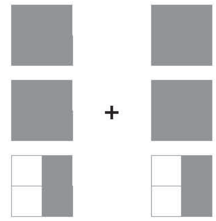
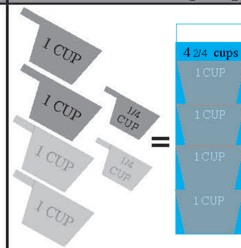
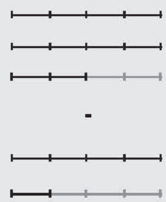
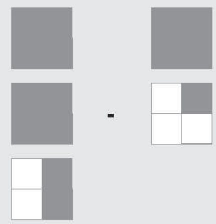
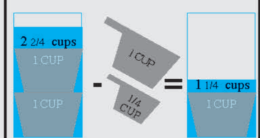
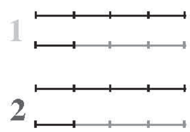
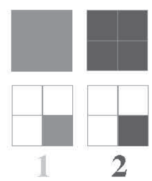
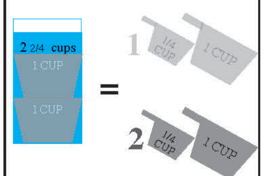
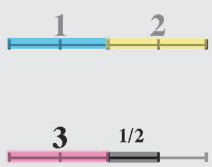
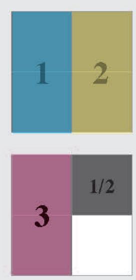
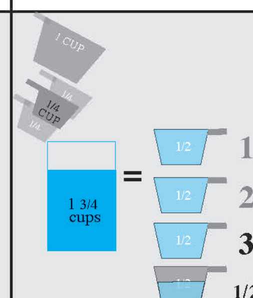
Fraction Procedures	Number Line	Area Model	Water + Measuring Cups
(k) Adding/Multiplying Mixed Numbers or Improper fractions (e.g 2 1/4 + 2 1/4 *or* 2 1/4 x 2)			
(l) Subtracting Mixed Numbers or Improper Fractions (e.g. 2 1/2 - 1 1/4 = 1 1/4)			
(m) Dividing by Mixed Numbers or Improper Fractions (e.g. 2 1/2 ÷ 1 1/4 = 2)			
(n) The "Liping Ma" Teacher Problem 1 3/4 ÷ 1/2 = 3 1/2			

Table 4a. Explanations for Student Pre/Post Assessment Items 1-5

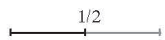
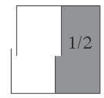
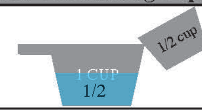
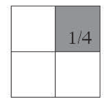
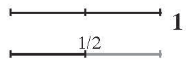
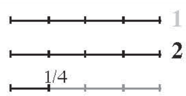
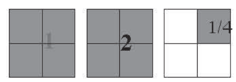
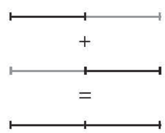
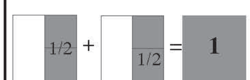
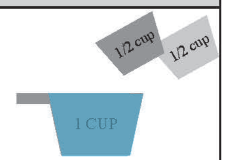
Fraction Procedures	Number Line	Area Model	Water + Measuring Cups
(1) Ways to think about: 1/2			
(2) Ways to think about: 1/4			
(3) Ways to think about: 1 ½			
(4) Ways to think about: 2 ¼			
(5) Ways to think about: 1/2 + 1/2			

Table 4b. Explanations for Student Pre/Post Assessment Items 6-10

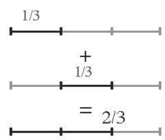
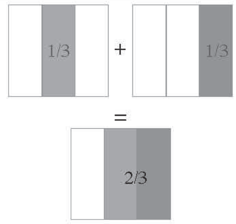
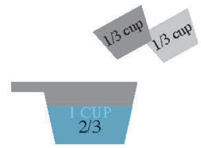
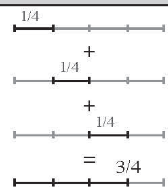
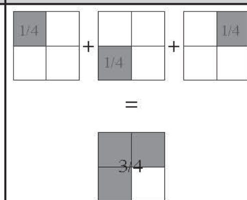
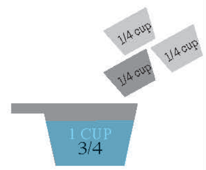
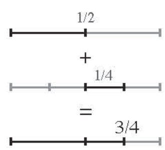
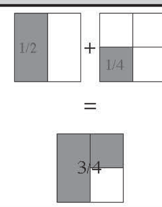
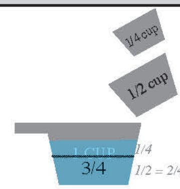
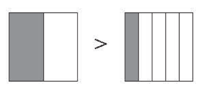
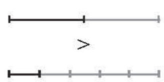
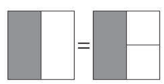
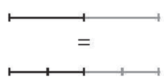
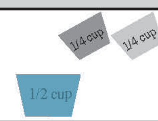
Fraction Procedures	Number Line	Area Model	Water + Measuring Cups
(6) Ways to think about: 1/3 + 1/3			
(7) Ways to think about: 1/4 + 1/4 + 1/4			
(8) Ways to think about: 1/2 + 1/4			
(9) Ways to compare fractions 1/2 > 1/5			
(10) Ways to compare fractions 1/2 = 2/4			

Table 5a. Protocol for Control Group - Student Pre/Post Assessment Items 1-5

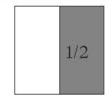
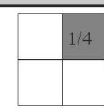
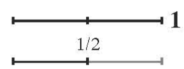
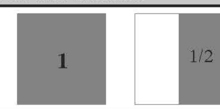
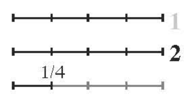
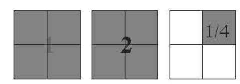
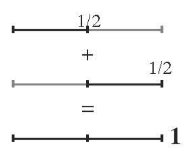
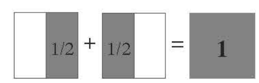
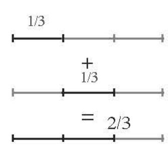
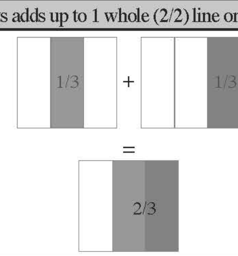
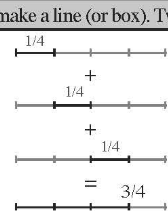
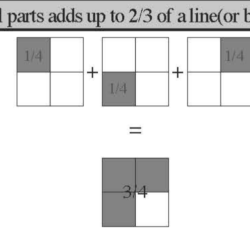
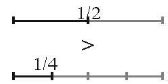
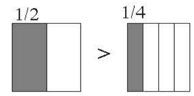
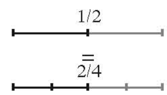
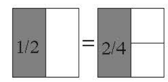
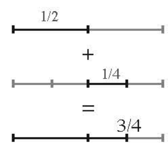
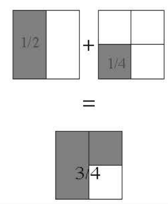
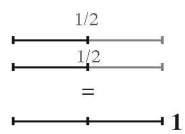
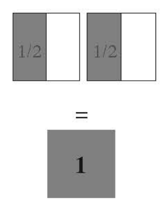
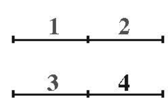
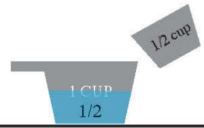
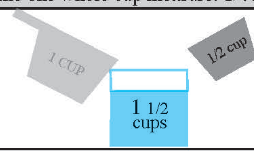
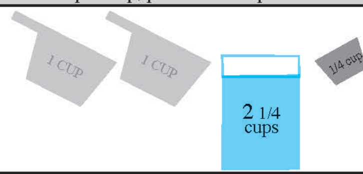
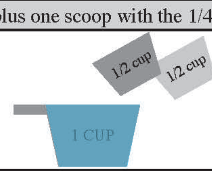
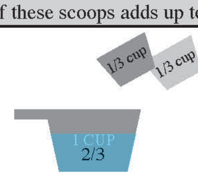
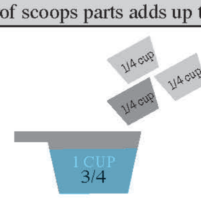
Fraction Procedures	Number Line	Area Model
(1) Ways to think about: 1/2		
I think about 1/2 like this: I imagine a line (or box) with two equal sections. 1/2 is one of the two sections.		
(2) Ways to think about: 1/4		
I think about 1/4 like this: I imagine a line (or box) with four equal sections. 1/4 is one of the four sections.		
(3) Ways to think about: 1 ½		
I think about 1 1/2 like this: I imagine 1 WHOLE line (or box), and then I imagine 1/2 of another line (or box)		
(4) Ways to think about: 2 ¼		
I think about 2 1/4 like this: I imagine 2 WHOLE lines (or boxes), and then I imagine 1/4 of another line (or box)		
(5) Ways to think about: 1/2 + 1/2		
I think about 1/2 as one of two equal parts needed to make a line (or box). Two of these equal parts adds up to 1 whole (2/2) line or box.		
(6) Ways to think about: 1/3 + 1/3		
I think about 1/3 as one of three equal parts needed to make a line (or box). Two of these equal parts adds up to 2/3 of a line(or box).		
(7) Ways to think about: 1/4 + 1/4 + 1/4		
I think about 1/4 as one of four equal parts needed to make a line (or box). Three of these equal parts adds up to 3/4 of a line(or box).		

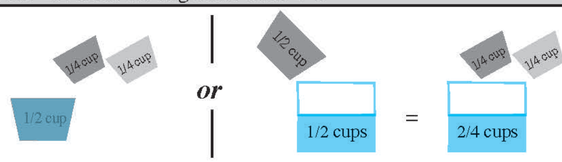
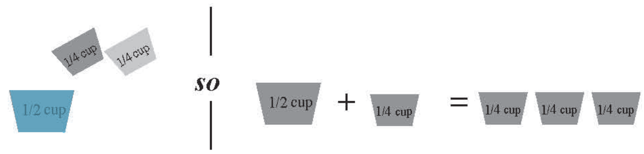
Table 5b. Explanations for Student Pre/Post Assessment Items 8-10 + Multiplication/Divison Models

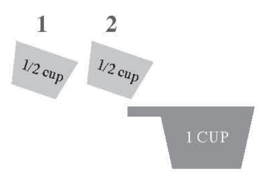
Fraction Procedures	Number Line	Area Model
(8) Ways to compare fractions 1/2 > 1/4		
I can see that the 1/2 part is larger than the 1/4 part. It only takes two equal 1/2 parts to make 1 whole line (or box), but it takes four equal 1/4 parts to make 1 whole line (or box). Therefore, the 1/2 part is greater than 1/4.		
(9) Ways to compare fractions 1/2 = 2/4		
I can see from the picture that when I divide a line (or box) into 2 equal parts and shade one of the two (1/2), it is the same as dividing the line (or box) into 4 equal parts and shading two of the four (2/4).		
(10) Ways to think about: 1/2 + 1/4		
I think about 1/2 as one of two equal parts needed to make a line (or box). I think about 1/4 as one of four equal parts needed to make a line (or box). Looking at the picture I see that 1/2 is equal to 2/4. I now think of the problem as adding 2/4 + 1/4 to get 3/4.		

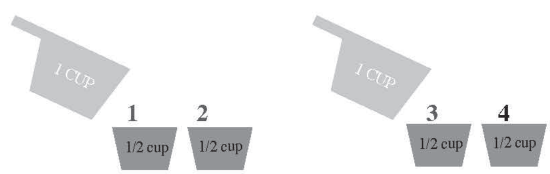
Multiplying with Fractions		
1/2 x 2 = 1		
When I think about the multiplication problem above, I think, "repeat the 1/2, two times." Two 1/2's make one whole, and so the answer is 1.		

Dividing with Fractions		
(10) 2 ÷ 1/2 = 4		
When I think about the division problem above, I ask, "how many 1/2's can I make from two wholes?" Because I know that there are two 1/2's in one whole, I reason that 2 wholes, would have four 1/2's.		

Fraction Procedures	Water + Measuring Cups
(1) Ways to think about: 1/2	
I think about 1/2 like this: It takes 2 scoops from a 1/2 cup to fill the one whole cup measure. 1/2 is one scoop.	
(2) Ways to think about: 1/4	
I think about 1/4 like this: It takes 4 scoops from a 1/4 cup to fill the one whole cup measure. 1/4 is one scoop.	
(3) Ways to think about: 1 ½	
I think about 1 1/2 like this: I imagine scooping 1 whole cup scoop, plus one scoop with the 1/2 cup measurer.	
(4) Ways to think about: 2 ¼	
I think about 2 1/4 like this: I imagine scooping 2 whole cup scoops, plus one scoop with the 1/4 cup measurer.	
(5) Ways to think about: 1/2 + 1/2	
I think about 1/2 as one of two scoops needed to make 1 cup. Two of these scoops adds up to 2/2 or 1 whole cup.	
(6) Ways to think about: 1/3 + 1/3	
I think about 1/3 as one of three scoops needed to make 1 cup. Two of scoops parts adds up to 2/3 of 1 whole cup.	
(7) Ways to think about: 1/4 + 1/4 + 1/4	
I think about 1/4 as one of four scoops needed to make 1 cup. Three of these scoops adds up to 3/4 of 1 whole cup.	

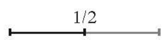
<table>
<tr><th>Fraction Procedures</th><th>Water + Measuring Cups</th></tr>
<tr><td>(8)
Ways to compare fractions
1/2 > 1/4</td><td>
</td></tr>
<tr><td colspan="2">I can see that the 1/2 measure is larger than the 1/4 measure. It only takes two 1/2 scoops to make 1 cup, but it takes four 1/4 scoops to make 1 cup. Therefore, the 1/2 measure is greater than 1/4.</td></tr>
<tr><td>(9)
Ways to compare fractions
1/2 = 2/4</td><td>
</td></tr>
<tr><td colspan="2">I can see from the picture that when I divide a line (or box) into 2 equal parts and shade one of the two (1/2), it is the same as dividing the line (or box) into 4 equal parts and shading two of the four (2/4).</td></tr>
<tr><td>(10)
Ways to think about:
1/2 + 1/4</td><td>
</td></tr>
<tr><td colspan="2">I think about 1/2 as one of two equal scoops needed to make 1 whole cup. I think about 1/4 as one of four scoops needed to make 1 whole cup. Because I can make 1/2 from two 1/4, I know 1/2 = 2/4. I now think of the problem as adding 2/4 + 1/4 = 3/4.</td></tr>
</table>

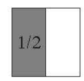
<table>
<tr><th>Multiplying with Fractions</th><th></th></tr>
<tr><td>1/2 x 2 = 1</td><td>
</td></tr>
<tr><td colspan="2">When I think about the multiplication problem above, I think, “pour with the 1/2 cup, two times.”
Two 1/2 cups make one whole cup, and so the answer is 1.</td></tr>
</table>

<table>
<tr><th>Dividing with Fractions</th><th></th></tr>
<tr><td>(10)
2 ÷ 1/2 = 4</td><td>
</td></tr>
<tr><td colspan="2">When I think about the division problem above, I ask, “how many 1/2 cups can I make from two whole cups?”
Because I know that there are two 1/2 cups in one whole cup, I reason that 2 cups would make four, 1/2 cups.</td></tr>
</table>

Initials__________ Date__________ Teacher_____________

Okay, now I'm going to actually teach you about how I like to think about multiplying with fractions. This is still not a test, and this still doesn't affect your grades, at all.





First, let me ask, how many ***whole*** lines or squares would we make if we repeated 1/2, *four times*?

Basically, you just did 1/2 x 4... Not so bad right?

What if I asked you 1/2 x 6?

What if I asked you 1/2 x 7?

Okay, last one, how would you think about 1/5 x 6?

Initials_________ Date_________ Teacher____________

Okay, now I'm going to actually teach you about how I like to think about multiplying with fractions. This is still not a test, and this still doesn't affect your grades, at all.

First, let me ask, how many ***whole cups*** would we make if we repeated the 1/2 cup, *four times*?

Basically, you just did 1/2 x 4... Not so bad right?

What if I asked you 1/2 cups x 6?

What if I asked you 1/2 cups x 7?

Okay, last one, how would you think about 1/5 cups x 6?

Initials_________ Date_________ Teacher____________

Pre and Post Assessment Protocol (FRACTIONS) for Grades 3 - 8
(Researcher will read the following out loud to student)
First of all, this is not a test. Whether you get something right or wrong will not affect your grades or what your teacher thinks of you at all. If you don't want to answer a problem, or are tired and want to stop, that's fine. No one will be upset with you. If you don't know how to do a problem, just write "I don't know." That's fine too, okay?
What I am interested in is how students think about fractions. You will see a list of problems that students from from 3rd grade to 12th grade, and even teachers, sometimes find tricky. Again, if you are not sure how to do something, you don't have to, it will not affect your grade at all.
[Researcher will then say, You are in grade (x) right? Would we expect you to know how to do problems for grade (x + 2)?]
Still ready to try this? (pause). Okay, good. Now before we begin, I will be video-taping and recording you while you do your work. This helps me learn about how you think and learn. No one but my teachers at college and I will watch/listen to this. It will never be on YouTube or the Internet. This is just so we can learn from how you learn. Is that still okay?
Great.

Age: _________ Grade: ________ Initials: _________ Gender (circle): M / F

Fractions

Directions:

Here are twenty questions. Again, this is NOT A TEST. The questions are all about fractions. Some of them will seem easy to you. Some of them may be difficult depending on whether or not you have been taught how to do them before. If you don't know how to do something, you can still try to do it, or you can just say I can't do this one and skip it.

If you want to draw pictures to explain something, that's fine. If you want to explain a problem by writing a sentence or talking, that's fine too. Sometimes I might stop you and ask you why you did what you did… not because you are right or wrong, but because I'm curious as to how you learn. Okay?

1. What does ½ mean to you?

2. What does ¼ mean to you?

3. What does 1 ½ mean to you?

4. What does 2 ¼ mean to you?

5. Can you add ½ + ½ ?

6. Can you add 1/3 + 1/3 ?

7. Can you add ¼ + ¼ + ¼ ?

8. Can you add ½ + ¼ ?

9. Which fraction is greater 1/2 or 1/5 ? (circle)

10. Which fraction is greater 1/2 or 2/4 ? (circle)

11. Which fraction is greater 2 1/5 or 2 1/2 ? (circle)

12. Which fraction is greater 5/4 or 4/2 ? (circle)

13. Can you multiply ½ x 6 ?

14. Can you multiply ¼ x 8 ?

15. Can you divide ½ ÷ ¼

16. Can you divide 10 ÷ ½

17. Can you add 2 ½ + 4 ¼

18. Can you multiply 4 ½ x 2

19. Can you divide 4 ½ ÷ 2

20. Can you divide 1 3/4 ÷ ½