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Fermion Decay Into Spin-3/2 Fermion Plus Spin-0 Boson

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ABSTRACT

The decay of a fermion of arbitrary spin into an unstable spin-3/2 fermion plus a spinless boson is treated with density-matrix techniques. The formalism described is an extension of that developed by Byers and Fenster for the decay of a fermion into spin-1/2 and spin-0 particles. Decay distributions are completely described for three successive decay processes. Various tests for spin and parity of the parent fermion are suggested.

I. INTRODUCTION

A formalism for treating strong or weak decay of a fermion into a spin-1/2 particle and a spinless boson was developed over a year ago by Byers and Fenster.[1] The purpose of this article is to extend the formalism to treat fermion decay into a spin-3/2 fermion plus a spinless boson. The angular distribution of the decay and also the angular dependence of the fermion's polarization components afford tests for spin and parity hypotheses concerning the parent fermion. In general, the analysis of second-rank tensor polarization of the spin-3/2 fermion is possible; in addition, vector and third-rank tensor polarizations may be analyzable.

II. THE DECAY MATRIX

The decay process

$$X \to Z + B \tag{1}$$

$$(\text{spin:}\quad J \to 3/2 + 0)$$

may be described (in the rest frame of X) by expressing the spin-space density matrix of Z in terms of that for X:

$$\rho_Z = \mathcal{M}\,\rho_X\,\mathcal{M}^\dagger , \tag{2}$$

where $\mathcal{M}$ is the decay matrix.[2] We suppose ρ_X to be given in the usual J, M representation, with some convenient direction defined by X production (e. g., the production normal) as the quantization axis. Further, we wish to treat the decay of X into Z in the system which yields helicity states for the Z. Thus the decay matrix may be considered as having two parts: a rotation matrix which transforms ρ_X into the "helicity system" for Z (with quantization axis along $\hat{Z}$, the direction of "particle" Z in the X rest frame); and a diagonalized transition matrix (A) describing the decay $X \rightarrow Z + B$. That is, $\rho_Z = A(R\rho_X R^\dagger)A^\dagger$, where R represents a rotation operation.

The complete element of the decay matrix may be written

$$\mathcal{M}_{\lambda M} = A_\lambda\,[(2J+1)/4\pi]^{1/2}\,\mathcal{D}^{J*}_{M\lambda}\,(\phi,\theta,0), \tag{3}$$

with A_λ = -3/2, -1/2, 1/2, and 3/2 for the case under discussion here. The $\mathcal{D}^J_{M\lambda}$ is a matrix element for a rotation operator (and may also be referred to as a "symmetrical-top function").[3,4] The A_λ are the helicity amplitudes, the elements of the diagonalized transition matrix describing $X \rightarrow Z + B$. Their form depends on the spin of Z, the spin of X, and the relative X-Z parity.

The helicity amplitudes are obtained as follows. Each A_λ represents the probability amplitude for the breakup of a system of total spin J (with projection λ on the helicity axis) into a system which has spin 3/2 (and helicity component λ) and any allowed orbital angular momentum (with helicity component zero). The A_λ may have contributions from four orbital angular-momentum waves, $\ell = J - 3/2$ through $J + 3/2$; two of these ℓ waves have even parity and two have odd parity. The relative contributions from the different orbital states may be expressed in terms of the complex decay amplitude a_ℓ and the Clebsch-Gordan coefficient for combining ℓ and spin 3/2 to obtain spin J:[5]

$$\begin{aligned} A_\lambda &= \sum_\ell (-)^{\ell-J+3/2}\, a_\ell[(2\ell+1)/(2J+1)]^{1/2}\, C(\ell\, 3/2\, J; 0,\lambda) \\ &= (-)^{\lambda-3/2} \sum_\ell a_\ell\, C(J\, 3/2\, \ell; \lambda, -\lambda)\,. \end{aligned} \tag{4}$$

The second expression for A_λ given in Eq. (4) follows from the first by the use of symmetry properties of the Clebsch-Gordan coefficients; the form of the second expression is reasonable, in that a_ℓ multiplies the coefficient giving the probability amplitude for forming the angular momentum state ℓ. (For the familiar case of decay into spin 1/2, the helicity amplitudes $A_{1/2} = a + b$ and $A_{-1/2} = a - b$ may be found by evaluating $A_\lambda = (-)^{\lambda-1/2} \sum_\ell a_\ell C(J\, 1/2\, \ell; \lambda, -\lambda)$ or by diagonalizing the transition matrix $a + b\, \vec{\sigma}\cdot\hat{p}$. See Appendix I for further discussion.)

It is perhaps more practical to discuss strong rather than weak decay of the X, and to keep the opposite-parity amplitudes separate. The A_λ amplitudes receive contributions from orbital angular momenta $\ell = J-3/2$ (not allowed for $J = 1/2$) and $J+1/2$ if the X has spin and parity, relative to the Z, of $1/2^+$, $3/2^-$, $5/2^+$, etc.; they have contributions from $\ell = J-1/2$ and $J+3/2$ if the X has $J^P = 1/2^-, 3/2^+, 5/2^-$, etc. The two sets of helicity amplitudes have the following forms:

$$A = \begin{bmatrix} a\alpha_J + c\sqrt{3}\,\beta_J & & & \\ & a\sqrt{3}\,\beta_J - c\alpha_J & & \\ & & a\sqrt{3}\,\beta_J - c\alpha_J & \\ & & & a\alpha_J + c\sqrt{3}\,\beta_J \end{bmatrix} \quad \text{and } A' = \begin{bmatrix} b\sqrt{3}\,\alpha_J + d\beta_J & & & \\ & b\beta_J - d\sqrt{3}\,\alpha_J & & \\ & & -b\beta_J + d\sqrt{3}\,\alpha_J & \\ & & & -b\sqrt{3}\,\alpha_J - d\beta_J \end{bmatrix} \qquad (5)$$

for the $1/2^+$ and the $1/2^-$ parity sequences, respectively. All four A_λ elements are actually applicable only to the decay of an X with $J \geq 3/2$, since the $\lambda = 3/2$ and $-3/2$ spin states are not accessible to an initial particle with spin 1/2. The coefficients a, b, c, and d in these matrices represent the complex amplitudes a_ℓ for decay through the channels with $\ell = J-3/2, -1/2, +1/2$, and $+3/2$, respectively; the remaining symbols are dependent on the original X particle's spin, with $\alpha_J = \sqrt{J+3/2}$ and with $\beta_J = \sqrt{J-1/2}$. These A_λ amplitudes are subject to the constraint of normalization (total decay probability being equal to 1):

$$\mathrm{Tr}\, A A^\dagger \text{ or } \mathrm{Tr}\, A' A'^\dagger = 1. \qquad (6)$$

III. FINAL DENSITY MATRIX

Expressions for the angular distribution and for the polarization components of the Z "particle" are obtained by expressing ρ_X in matrix form and carrying out the transformations of Eqs. (2) and (3). We take

$$\rho_X = (2J_X+1)^{-1} \sum_{L=0}^{2J_X} \sum_{M} (2L+1)\, t^*_{LM}\, T_{LM} \tag{7}$$

in the manner of Byers and Fenster.[6] It is convenient to use the irreducible tensors T_{LM} as basis operators in spin space.[3,7] (These are traceless and symmetric tensors; and hence their use simplifies the satisfaction of normalization and hermiticity requirements for the density matrix. Further, they combine naturally with the orthogonal $Y_{LM}(\theta,\phi)$ tensors in decay distributions derived from the density matrix.) These tensors have forms in spin space which correspond to those of the Y_{LM} in coordinate space; e.g., $Y_{11} \propto (x+iy)/r$ and $T_{11} \propto (S_x+iS_y)/|S|$, where the S_x and S_y are spin operators and $|S|$ is $\sqrt{J(J+1)}$. The T_{LM} tensors obey the symmetry relation $T_{L,-M} = (-)^M T^\dagger_{LM}$.

In Eq. (7), the T_{LM} operators have a maximum rank (L) equal to $2J_X$. The t_{LM} represent the expectation values $\langle T_{LM}\rangle$ which describe the X initial spin state. The expression for an element of the final Z density matrix becomes, by the use of Eqs. (2) and (3),

$$\rho_Z\Big]_{\lambda\lambda'} = \sum_{M'',M'} A_\lambda\, \mathcal{D}^{J*}_{M''\lambda} \left[\sum_{L,M} (2L+1)\, t^*_{LM}\, T_{LM}\right]_{M''M'} \mathcal{D}^{J}_{M'\lambda'}\, A^*_{\lambda'}/4\pi. \tag{8}$$

Then, as shown by Byers and Fenster, Clebsch-Gordan coefficients may be substituted for the matrix elements of the T_{LM}:

$$T_{LM}\Big]_{M''M'} = C(J\,L\,J;\, M'M) \text{ with } M'' = M' + M \tag{9}$$

for the representation where T_{L0} is diagonal. This substitution yields

$$\rho_Z\Big]_{\lambda\lambda'} = (A_\lambda A^*_{\lambda'}/4\pi) \sum_{M'',M'} \sum_{L,M} (2L+1)\, t^*_{LM}\, C(J\,L\,J; M'M)\, \mathcal{D}^{J^*}_{M''\lambda}\, \mathcal{D}^{J}_{M'\lambda'} . \tag{10}$$

With the use of various properties of the $\mathcal{D}$ functions and of Clebsch-Gordan coefficients, this reduces to[6, 8]

$$\rho_Z\Big]_{\lambda\lambda'} = (-)^{J-\lambda'} (A_\lambda A^*_{\lambda'}/4\pi) \sqrt{2J+1}$$
$$\times \sum_{L,M} \Big[\sqrt{2L+1}\, C(J\,J\,L; \lambda, -\lambda')$$
$$\times\, t^*_{LM}\, \mathcal{D}^{L^*}_{M,\lambda-\lambda'}(\phi,\theta,0)\Big] . \tag{11}$$

Evidently the derivation of Eq. (11) is a general one, which is valid for any spin of X or of Z (integer as well as half-integer).

The elements of the ρ_Z density matrix may be used to derive theoretical expressions for decay distributions after simplification of terms. It is convenient to define the symbol[9]

$$n^{(2\lambda)}_{L\mathfrak{m}} \equiv (-)^{J+\mathfrak{m}-\lambda}[(2J+1)/4\pi]^{1/2}\, C(J\,J\,L; \lambda, \mathfrak{m}-\lambda), \tag{12}$$

where λ assumes the usual values from +3/2 to -3/2 and where $\mathfrak{m}$ has a value of 0, 1, 2, or 3 ($\mathfrak{m}$ being $\lambda-\lambda'$). The diagonal elements of the density matrix then may be expressed as

$$\rho_{\lambda\lambda} = |A_\lambda|^2 \sum_{L,M} n^{(2\lambda)}_{L0}\, t^*_{LM}\, Y_{LM}(\theta,\phi)$$
$$= |A_\lambda|^2 \sum_{L,M} n^{(2\lambda)}_{L0}\, t_{LM}\, Y^*_{LM}(\theta,\phi) \tag{13}$$

with $Y_{LM}(\theta,\phi)$ replacing $\sqrt{(2L+1)/4\pi}\, \mathcal{D}^{L}_{M0}(\phi,\theta,0)^*$. The three elements just above the diagonal of ρ_Z are similar, but contain $\mathcal{D}^{L^*}_{M1}$; the two elements above these contain $\mathcal{D}^{L^*}_{M2}$; etc. The density matrix thus has the following form [with $D^L_{M\mathfrak{m}}$ replacing the orthonormal function $\sqrt{(2L+1)/4\pi}\, \mathcal{D}^{L}_{M\mathfrak{m}}(\phi, \theta, 0)$]:

$$\rho_Z = \sum_{L,M} \begin{bmatrix} t^*_{LM}\left|A_{\frac{3}{2}}\right|^2 n^{(3)}_{L0} Y_{LM} & t^*_{LM} A_{\frac{3}{2}} A^*_{\frac{1}{2}} n^{(3)}_{L1} D^{L*}_{M1} & t^*_{LM} A_{\frac{3}{2}} A^*_{-\frac{1}{2}} n^{(3)}_{L2} D^{L*}_{M2} & t^*_{LM} A_{\frac{3}{2}} A^*_{-\frac{3}{2}} n^{(3)}_{L3} D^{L*}_{M3} \\ \text{———} & t^*_{LM}\left|A_{\frac{1}{2}}\right|^2 n^{(1)}_{L0} Y_{LM} & t^*_{LM} A_{\frac{1}{2}} A^*_{-\frac{1}{2}} n^{(1)}_{L1} D^{L*}_{M1} & \text{———} \\ \text{———} & \text{———} & t^*_{LM}\left|A_{-\frac{1}{2}}\right|^2 n^{(-1)}_{L0} Y_{LM} & \text{———} \\ t_{LM} A_{-\frac{3}{2}} A^*_{\frac{3}{2}} n^{(3)}_{L3} D^{L}_{M3} & \text{———} & \text{———} & t^*_{LM}\left|A_{-\frac{3}{2}}\right|^2 n^{(-3)}_{L0} Y_{LM} \end{bmatrix}. \quad (14)$$

As the density matrix is self-adjoint, all terms of ρ_Z below the diagonal are easily obtained from the terms above.

The $n_{L\mathcal{M}}^{(2\lambda)}$ coefficients may be related by the use of symmetry properties and recursion relations for the Clebsch-Gordan coefficients; e. g.,

$$n_{L0}^{(-2\lambda)} = (-)^L n_{L0}^{(2\lambda)} \text{ for any } L, \lambda \tag{15}$$

and $$n_{L0}^{(3)} = \left\{1 - L(L+1)[(J+3/2)(J-1/2)]^{-1}\right\} n_{L0}^{(1)} \text{ for even } L.$$

See Appendix II for other $n_{L\mathcal{M}}^{(2\lambda)}$ expressions.

IV. DECAY DISTRIBUTIONS

The angular distribution and all possible polarization distributions for the Z may be found by taking the expectation values of all spin operators required to describe the Z spin state: T_{00} (the identity), T_{10}, T_{20}, T_{21}, T_{22}, T_{30}, T_{31}, and T_{32}. These are the same tensor operators as those described above; but here they have a dimensionality of 4 (are represented by 4 by 4 matrices) rather than $2J_X+1$, as above. The theoretical expressions for the expectation values are derived (in terms of the t_{LM} describing the original X spin state) by taking the trace of $\rho_Z T_{LM}$ (and normalizing through division by $\mathrm{Tr}\,\rho_Z$).[10]

The angular distribution of the Z in the X rest frame is found, with the use of Eq. (15), by evaluating[11]

$$\mathrm{Tr}(\rho_Z T_{00}) = \mathrm{Tr}\,\rho_Z \equiv I(\theta,\phi)$$

$$= \sum_{L_e}^{2J-1} \sum_M [(A_{3/2}^2 + A_{-3/2}^2)\, n_{L0}^{(3)} + (A_{1/2}^2 + A_{-1/2}^2)\, n_{L0}^{(1)}]\, t_{LM} Y_{LM}^*(\theta,\phi). \tag{16}$$

The index L_e takes on only even values because the combination of

λ and $-\lambda$ elements of ρ_Z causes odd-L contributions to cancel (for strong decay). The appropriate forms of the A_λ amplitudes and of the n_{L0} coefficients may be substituted to predict the angular distribution for any spin and parity of the initial X system. As an example, the distribution for $J^P = 3/2^-$ is (if the production normal is the polar axis)

$$I(\theta, \phi) = 4(a^2+c^2)\, n_{00} t_{00} Y_{00} - 8\, \mathrm{Re}\, a^* c\, n_{20}^{(1)} [t_{20} Y_{20} + 2\, \mathrm{Re}(t_{22} Y_{22}^*)] \; ; \qquad (17)$$

whereas that for $J^P = 3/2^+$ is

$$I(\theta, \phi) = 20(b^2+d^2)\, n_{00} t_{00} Y_{00} - 8(2b^2 - 2d^2 + 3\, \mathrm{Re}\, b^* d)\, n_{20}^{(1)} [t_{20} Y_{20} + 2\, \mathrm{Re}(t_{22} Y_{22}^*)] . \qquad (18)$$

Because of the normalization requirement of Eq. (6), the first terms (or average cross sections) are identical in these two cases. (Each is equal to $n_{00} t_{00} Y_{00} = 1/4\pi$.) The complexity of the $I(\theta, \phi)$ distribution demanded by experimental data of course gives information on J, the X spin. Polarization determinations are necessary to establish the X parity, as well as to obtain more information on the spin.

Although $\langle T_{10} \rangle_Z$, $\langle T_{11} \rangle_Z$, and $\langle T_{1,-1} \rangle_Z$, the components of "vector polarization" of the Z, are produced by the $X \rightarrow Z$ decay process, strong decay of the Z cannot serve for analysis of this polarization. A tensor component of Z polarization which will be found in the angular distribution of Z decay is $\langle T_{20} \rangle_Z \propto \langle 3\, S_Z^2 - \overline{S}^2 \rangle$. (This is the Z spin alignment <u>along its direction of flight</u>, as the density matrix ρ_Z used to derive $\langle T_{20} \rangle = \mathrm{Tr}(\rho\, T_{20})$ is in the helicity representation.) Further contributors to the Z decay distribution are $\langle T_{2,\pm 1} \rangle_Z$ and $\langle T_{2,\pm 2} \rangle_Z$; however, these are observable only if azimuthal as well as polar decay angles (relative to $\hat{Z}$) are investigated. The expressions for these tensor polarization components are given by the following:

$$I \left\langle T_{20} \right\rangle_Z = \mathrm{Tr}(\rho_Z T_{20}) = (1/5)^{1/2} \sum_{L,M}$$

$$\times \left[(A^2_{3/2} + A^2_{-3/2})\, n^{(3)}_{L0} - (A^2_{1/2} + A^2_{-1/2})\, n^{(1)}_{L0} \right] t_{LM} Y^*_{LM}(\theta, \phi)$$

$$I \left\langle T_{21} \right\rangle_Z = (2/5)^{1/2} \sum_{L,M} \left[-A_{1/2} A^*_{3/2} + A_{-3/2} A^*_{-1/2} (-)^{2J+L} \right]$$

$$\times n^{(3)}_{L1} t_{LM} \left[(2L+1)/4\pi \right]^{1/2} \mathcal{D}^L_{M1}(\phi, \theta, 0) \qquad (19)$$

$$I \left\langle T_{22} \right\rangle_Z = (2/5)^{1/2} \sum_{L,M} \left[A_{-1/2} A^*_{3/2} - A_{-3/2} A^*_{1/2} (-)^{2J+L} \right]$$

$$\times n^{(3)}_{L2} t_{LM} \left[(2L+1)/4\pi \right]^{1/2} \mathcal{D}^L_{M2}(\phi, \theta, 0) .$$

As these are unnormalized, they represent $I(\theta, \phi)$ times $\left\langle T_{\ell m} \right\rangle_Z (\theta, \phi)$. [All of the relations in Eq. (19) may be readily derived with the use of the $T_{\ell m}$ matrices for spin 3/2, which can be calculated from Eq. (9). These are presented in matrix form in Ref. 12].

In order for the polarization components of Z to be analyzed, the nature of Z decay must be examined. The simplest possibility is the strong decay

$$Z \rightarrow F + b \qquad (20)$$

$$(\text{spin: } 3/2 \rightarrow 1/2 + 0),$$

where F may be an unstable fermion (Ξ or Λ) or a stable one (p or n). The original Byers-Fenster formalism may be applied to obtain angular and polarization distributions for F (in the Z rest frame) in terms of $\left\langle T_{\ell m} \right\rangle_Z$ parameters described above. (If the fermion F has spin 3/2 rather than 1/2, the expressions developed above for ρ_Z should be reapplied to determine ρ_F and hence the various $I \langle T_{\ell m} \rangle_F$.)

The formalism predicts that the angular distribution of F (spin 1/2) in the Z rest frame is (with azimuthal angle ignored)

$$\mathcal{I}(\psi) = \frac{1}{4\pi} I(\theta, \phi) [1 - \langle T_{20} \rangle_Z \sqrt{5}(3\cos^2\psi - 1)/2] . \qquad (21)$$

Here the angle ψ must refer to the angle between $\hat{F}$ and $\hat{Z}$ ($\hat{Z}$ being now defined as the direction of transformation into the Z rest frame), a "correlation" angle; this is required by the interpretation given $\langle T_{20} \rangle_Z$ in deriving Eq. (19). Equation (21) has a particularly simple form if the direction of Z (specified by angles θ and ϕ) is averaged over, as all terms then vanish in $I(\theta, \phi)$ and $I \langle T_{20} \rangle$ except for the L, M = 0, 0 terms of Eqs. (16) and (19); thus $I(\theta, \phi)$ becomes equal to $\mathrm{Tr}\, A A^+/4\pi = 1/4\pi$ and $I \langle T_{20} \rangle_Z$ becomes a constant dependent on the helicity amplitudes for $X \to Z$. As the helicity amplitudes are functions of J_X and these functions depend on the X parity, some spin-parity information may be extracted from a simple $\hat{F} - \hat{Z}$ correlation analysis. If only the lower ℓ wave [amplitude a or b of Eq. (5)] is included, the expected distribution is[13]

$$\mathcal{I}(\psi) \propto a^2[4J - (1/2)(-2J+3)(3\cos^2\psi - 1)] \propto [1 + \left(\frac{2J-3}{2J+1}\right) \cos^2\psi] \qquad (22)$$

for the $3/2^-$, $5/2^+$, $7/2^-$, etc. parity sequence ($\ell = J - 3/2$); and it is

$$\mathcal{I}(\psi) \propto b^2[4J+4 - (J+5/2)(3\cos^2\psi - 1)] \propto [1 - \left(\frac{6J+15}{10J+13}\right) \cos^2\psi] \qquad (23)$$

for the $3/2^+$, $5/2^-$, $7/2^+$, etc. parity sequence ($\ell = J - 1/2$). For the case of $J_X = 1/2$, there is no parity discrimination; the correlation distribution for $1/2^+$ or $1/2^-$ is

$$\mathcal{I}(\psi) \propto [1 + (3/2)\cos^2\psi] . \qquad (24)$$

If all angles are observed in the X → Z → F decay chain, the angular distribution of the spin-1/2 F may be expressed as follows:

$$\mathcal{J}'(\theta,\phi;\psi,\zeta) = \frac{1}{4\pi} I(\theta,\phi)\{1 - \langle T_{20}\rangle(\theta,\phi)\sqrt{5}(3\cos^2\psi - 1)/2$$
$$+2(15/2)^{1/2}[\mathrm{Re}\langle T_{21}\rangle(\theta,\phi)\cos\zeta + \mathrm{Im}\langle T_{21}\rangle(\theta,\phi)\sin\zeta]\sin\psi\cos\psi$$
$$-(15/2)^{1/2}[\mathrm{Re}\langle T_{22}\rangle(\theta,\phi)\cos 2\zeta + \mathrm{Im}\langle T_{22}\rangle(\theta,\phi)\sin 2\zeta]\sin^2\psi\}, \quad (25)$$

where θ, ϕ angles give the direction of $\hat{Z}$, and ψ, ζ angles give the direction of $\hat{F}$ (in the X and Z rest frames, respectively). The experimental evaluation of these angles should make use of "direct Lorentz transformations" to move reference axes from one rest frame to the next.[14] Further, the aximuthal angle ζ of $\hat{F}$ should be referred to the x axis used for constructing ρ_Z; this is found by taking $\hat{x} = \widehat{Z \times (Z \times \mathfrak{z})}$ where $\mathfrak{z}$ is the polar axis for θ (probably the production normal).

If the fermion F is unstable, its decay provides an analyzer for the (vector) polarization components of the F. It is only in these polarizations that the odd-ℓ $\langle T_{\ell m}\rangle_Z$ appear, if the F has been produced by strong decay. The expressions for these $\langle T_{\ell m}\rangle_Z$ in terms of the t_{LM} parameters describing the original X spin state are (with L_0 taking only odd values)[15]

$$I\langle T_{10}\rangle_Z \equiv \mathrm{Tr}(\rho_Z T_{10}) = (1/15)^{1/2} \sum_{L_o, M}$$
$$\times \left[3(A^2_{3/2} + A^2_{-3/2}) n^{(3)}_{L0} + (A^2_{1/2} + A^2_{-1/2}) n^{(1)}_{L0}\right] t_{LM} Y^*_{LM}(\theta,\phi)$$

$$I\langle T_{11}\rangle_Z = -(2/15)^{1/2} \sum_{L, M} \left\{[A_{1/2}A^*_{3/2} + A_{-3/2}A^*_{-1/2}(-)^{2J+L}]\sqrt{3}\, n^{(3)}_{L1} \right. \quad (26)$$

$$+2A_{-1/2}A^{*}_{1/2}n^{(1)}_{L1}\Big\}\ t_{LM}[(2L+1)/4\pi]^{1/2}\mathcal{D}^{L}_{M1}(\phi,\theta,0) \qquad \text{cont. (26)}$$

$$I\left\langle T_{30}\right\rangle_Z = (1/35)^{1/2}\sum_{L,M}\Big\{[A^2_{3/2}+A^2_{-3/2}]$$

$$\times\ n^{(3)}_{L0} - 3[A^2_{1/2}+A^2_{-1/2}]\ n^{(1)}_{L0}\Big\}\ t_{LM}Y^{*}_{LM}(\theta,\phi)$$

$$I\left\langle T_{31}\right\rangle_Z = 2(1/35)^{1/2}\sum_{L,M}$$

$$\times\left[-A_{1/2}A^{*}_{3/2}n^{(3)}_{L1}+\sqrt{3}\,A_{-1/2}A^{*}_{1/2}n^{(1)}_{L1}-A_{-3/2}A^{*}_{-1/2}n^{(-1)}_{L1}\right]t_{LM}\mathcal{D}^{L}_{M1}(\phi,\theta,0)$$

$$I\left\langle T_{32}\right\rangle_Z = (2/7)^{1/2}\sum_{L,M}\left[A_{-1/2}A^{*}_{3/2}n^{(3)}_{L2}-A_{-3/2}A^{*}_{1/2}n^{(1)}_{L2}\right]t_{LM}\mathcal{D}^{L}_{M2}(\phi,\theta,0)$$

$$I\left\langle T_{33}\right\rangle_Z = -2(1/7)^{1/2}A_{-3/2}A^{*}_{3/2}\sum_{L,M}n^{(3)}_{L3}t_{LM}\mathcal{D}^{L}_{M3}(\phi,\theta,0).$$

The contributions of these expectation values to the longitudinal and transverse polarization components of the F are given by the following:

$$\mathcal{J}\ \bar{P}\cdot\hat{F} = (4\pi)^{-1/2}\Big\{0.488[\langle T_{10}\rangle Y_{10}+2\,\mathrm{Re}(\langle T_{11}\rangle Y^{*}_{11})]$$
$$-1.34[\langle T_{30}\rangle Y_{30}+\sum_{\nu}2\,\mathrm{Re}(\langle T_{3\nu}\rangle Y^{*}_{3\nu})]\Big\} \qquad (27)$$

$$\mathcal{J}\ (\bar{P}\cdot\hat{x}'+i\bar{P}\cdot\hat{y}') = -\gamma(4\pi)^{-1/2}\Big\{1.27\sqrt{3/4\pi}[\langle T_{10}\rangle\mathcal{D}^{1}_{01}+\langle T_{11}\rangle\mathcal{D}^{1}_{11}$$
$$-\langle T_{11}\rangle^{*}\mathcal{D}^{1}_{-1,1}]-1.55\sqrt{7/4\pi}[\langle T_{30}\rangle\mathcal{D}^{3}_{01}+\sum_{\nu}(\langle T_{3\nu}\rangle\mathcal{D}^{3}_{\nu1}-\langle T_{3\nu}\rangle^{*}\mathcal{D}^{3}_{-\nu,1})]\Big\}$$

where $\hat{x}' = \widehat{F\times(F\times Z)}$ and $\hat{y}' = \widehat{Z\times F}$. The summation index ν runs from 1 to 3. The $Y_{\ell m}$ and $\mathcal{D}^{\ell}_{mm'}$ symbols represent the functions $Y_{\ell m}(\psi,\zeta)$ and $\mathcal{D}^{\ell}_{mm'}(\zeta,\psi,0)$, respectively; also, in both equations, the substitution of $(-)^m\langle T_{\ell m}\rangle^{*}$ for $\langle T_{\ell,-m}\rangle$ has been made. In the second of these equations, the symbol γ is to be taken as +1 or -1 if the relative Z-F parity is such that the angular momentum in Z decay is J-1/2 or J+1/2, respectively. The polarization components of Eq. (27) are determined experimentally by taking $(3/\alpha)\sum \hat{p}\cdot\hat{F}$, $(3/\alpha)\sum \hat{p}\cdot\hat{x}'$, and $(3/\alpha)\sum \hat{p}\cdot\hat{y}'$, where α is the usual asymmetry parameter for F decay and $\overline{p}$ is the decay momentum in the F rest frame; the sums are taken over all events with F at some particular ψ, ζ orientation.

V. EXPERIMENTAL TESTS FOR SPIN AND PARITY

One possible test for parity and spin of the X is to be found in the sign and magnitude of the $\cos^2\psi$ coefficient of Eqs. (22) through (24), which are valid under the assumption that only the lower angular-momentum wave contributes to X decay. Equation (23) is not very sensitive to spin assumption, and Eq. (24) yields no information on parity. Other possible tests, some of them more general than the above, are presented in the following paragraphs.

The $\langle T_{2m}\rangle_Z$ values describing the Z spin state may be determined from the angular distribution observed for the process $Z \to F$. [See Eq. (25).] If F undergoes weak decay, the $\langle T_{\ell m}\rangle_Z$ with $\ell = 1$ and 3 may also be determined. (In principle, a scattering of F with a known analyzing target would also yield the $\langle T_{1m}\rangle_Z$ and $\langle T_{3m}\rangle_Z$.) The experimental evaluations of $I(\theta,\phi)$ for $X \to Z$ and the three $\langle T_{2m}\rangle_Z(\theta,\phi)$ from $X \to Z \to F$ yield a total of four evaluations of each even-L t_{LM} describing the initial X state; further, the two $\langle T_{1m}\rangle_Z$ and the four $\langle T_{3m}\rangle_Z$ yield six evaluations of each odd-L t_{LM} describing the X state. Odd-L (even-L) t_{LM} may also be obtained from $\langle T_{2m}\rangle$ ($\langle T_{1m}\rangle$ and $\langle T_{3m}\rangle$) for $m \neq 0$; but these arise from interference of the two orbital amplitudes permitted for a given X parity [the a and c or b and d amplitudes of Eq. (5)] and thus are probably small

Every $n_{Lm}^{(2\lambda)}$ coefficient appearing with a t_{LM} in the $I(\theta,\phi)$ or $\langle T_{\ell m}\rangle$ distributions may be expressed as $n_{L0}^{(1)}$ times some factor containing J_X and L. (See Appendix II.) The A_λ helicity amplitudes also depend on J, if J is $\geq 3/2$. Thus, by comparison of the $A_\lambda A_{\lambda'}^* n_{Lm}^{(2\lambda)}$ coefficients of $Y_{\ell m}^*$ or $\mathcal{D}_{mm'}^{\ell}$ from one distribution with those in another distribution, various tests of J may be made. One way of estimating J is to construct a function similar to a χ^2 which compares values of $n_{L0}^{(1)}$ t_{LM} obtained in two or more distributions and to treat J as a variable parameter in this function;[16] another possible approach is construction of a general likelihood function treating all stages of decay and maximizing of this function for various J assumptions. A final and possibly very useful method is evaluation of a J-dependent function multiplying some $n_{L0}^{(1)}$ t_{LM} by taking ratios of terms in various distributions.[17]

A possible approach in setting up a general spin test function might be the following. Let the definition of "moment" be the coefficient of Y_{LM}^* or $[(2L+1)/4\pi]^{1/2}\mathcal{D}_{MM'}^{L}$ (projected out of a distribution by weighting that distribution with Y_{LM} or $[(2L+1)/4\pi]^{1/2}\mathcal{D}_{MM'}^{L*}$ and summing over all events). If $t_{LM}(1)$ stands for the L, M moment obtained from one distribution, and if $f_J(1)$ is the function of J which must be divided into this moment to obtain $n_{L0}^{(1)}$ t_{LM}, and if $t_{LM}(2)$ represents a similar term from a second distribution, etc., then a comparison can be made of the four evaluations of even-L t_{LM}'s by calculating the following for various J values. (A minimum "χ^2" yields the best J estimate.)[16]

$$"\chi_J^2" \equiv \sum_{\substack{L,M,\\ L'M'}} \sum_{i,j} \{[t_{LM}(i)/f_J(i)] - \langle t_{LM}(i)/f_J(i)\rangle\} G^{-1}_{LM(i),\,L'M'(j)} \times \{[t_{L'M'}(j)/f_J(j)] - \langle t_{L'M'}(j)/f_J(j)\rangle\}. \quad (28)$$

The indices i and j designate the four evaluations; and $\langle\ \rangle$ represents an average of these. The symbol G stands for a variance matrix. {That is, $G_{LM(i),\, L'M'(j)}$ is the second-moment matrix, the average value of $[t_{LM}(i) - \langle t_{LM}(i)\rangle]\ [t_{L'M'}(j) - t_{L'M'}(j)]$. } A "$\chi^2$" for the six evaluations of odd-L moments may be developed by analogy with Eq. (28).

Construction of a likelihood function is not difficult; the proper distribution function is $\mathcal{J}(\psi, \zeta)$ if the fermion F does not decay, and is $\mathcal{J} \times [1+\alpha \mathbf{P}_F \cdot \hat{p}]$ if it does decay. Either of these distribution functions would be most useful if expressed in terms of $n_{L0}^{(1)} t_{LM}$ times the $f_J(i)$ function of J discussed above. A high-spin form of the likelihood function, one appropriate for the maximum J assumed, might be used; then a maximum could be sought as a function of J, $n_{L0} t_{LM}$, and the ℓ-wave amplitudes (without changing the form of the likelihood function).

Finally, spin functions may be evaluated by taking ratios of a particular L, M moment found in one experimental distribution to the corresponding L, M moment found in another distribution. For example, after substitution of expressions for helicity amplitudes and for $n_{Lm}^{(2\lambda)}$ coefficients (see Appendix II), the ratio of an even-L moment in $I\langle T_{20}\rangle$ to the same moment of $I(\theta, \phi)$ yields [from Eqs. (19) and (16)][18]

$$\frac{I\langle T_{20}\rangle \text{ moment}}{I \text{ moment}} \equiv \langle\langle T_{20}\rangle\, Y_{LM}\rangle / \langle Y_{LM}\rangle \approx \frac{(3-2J)(2J-1) - 2L(L+1)}{4J(2J-1) - 2L(L+1)} . \quad (29)$$

Also, the ratio of moments from $I\langle T_{22}\rangle$ and $I\langle T_{21}\rangle$ yields, for even L and $J \geqslant 3/2$,

$$\left| \langle\langle T_{22}\rangle \mathcal{D}_{M2}^{L*}\rangle / \langle\langle T_{21}\rangle \mathcal{D}_{M1}^{L*}\rangle \right| = (J+1/2)/[(L+2)(L-1)]^{1/2}. \quad (30)$$

Similar spin tests may be constructed from odd-ℓ $I\langle T_{\ell m}\rangle$ if they can be evaluated experimentally. Care must be taken in the interpretation of these J estimations; the ratio of two normally distributed quantities is itself <u>not</u> normally distributed.[12]

A simple test may be made for the X - Z relative parity for any J assumption, the test being the determination of the <u>relative sign</u> of a moment in $I \langle T_{22} \rangle$ with respect to the corresponding moment in $I \langle T_{21} \rangle$. (The helicity amplitudes for these moments are the same except for a sign dependent on the parity.) Thus if $\Gamma = +1$ or -1 for $J^P = 3/2^-, 5/2^+$, etc. or $3/2^+, 5/2^-$, etc., respectively, for even L and $J \geq 3/2$,

$$\left\langle \left\langle T_{22} \right\rangle \mathcal{D}'^{L*}_{M2} \right\rangle / \left\langle \left\langle T_{21} \right\rangle \mathcal{D}^{L*}_{M1} \right\rangle = \Gamma (J+1/2)/[(L+2)(L-1)]^{1/2}. \qquad (31)$$

A similar test may be found in the odd-ℓ $I \langle T_{\ell m} \rangle$. (These tests are analogous to the test which may be made for the parity of a particle decaying strongly into a spin-1/2 fermion: the determination of the sign of any odd-L moment in the $I \langle T_{11} \rangle$ or $I(\overline{P} \cdot \hat{x} + i\, \overline{P} \cdot \hat{y})$ distribution, relative to the sign of the same moment in the $I \langle T_{10} \rangle$ or longitudinal polarization of the spin-1/2 fermion.) A χ^2 which tests the equality of corresponding moments with $\Gamma = +1$ or -1 is easily constructed.

In the course of analysis, it may be convenient to study the odd-L moments from the $I \langle T_{2m} \rangle$ distributions and the even-L moments from $I \langle T_{1m} \rangle$ or $I \langle T_{3m} \rangle$ distributions, as these are proportional to $2\, \mathrm{Im}\, A_\lambda A^*_{\lambda'}$ and may give a measure of the interference of the higher ℓ-wave.

If the spin J of "particle" X is 1/2, only the quantities I, $I \langle T_{20} \rangle$, $I \langle T_{10} \rangle$, $I \langle T_{11} \rangle$, $I \langle T_{30} \rangle$, and $I \langle T_{31} \rangle$ can be non-zero, as $\rho_Z]_{\lambda\lambda'}$ is zero for $|\lambda|$ or $|\lambda'| > 1/2$. Parity of a spin-1/2 X may be found by comparing the L, M = 1, 0 moment in $I \langle T_{11} \rangle$ with the corresponding moment in $I \langle T_{10} \rangle$, or by comparing a moment of $I \langle T_{31} \rangle$ with the corresponding one in $I \langle T_{30} \rangle$, etc. [See Eqs. (26) and (5).]

VI. APPLICATIONS

Some of the tests described above are being applied to the decay sequence $\Xi^*(1820) \rightarrow \Xi^*(1530) + \pi$, $\Xi^*(1530) \rightarrow \Xi + \pi$, $\Xi \rightarrow \Lambda + \pi$. Unfortunately, the number of useful events is small and the background is appreciable.[19] Other processes to which the formalism for spin J decay into spin 3/2 might be applicable are (a) higher lying $N^* \rightarrow N^*_{33}$ and (b) $Y^*(1815) \rightarrow Y^*_1(1385)$.

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APPENDICES

I. Helicity Amplitudes

The general form of a helicity amplitude is

$$A_\lambda = (-)^{\lambda - S} \sum_{\ell} a_\ell \, C(J S \ell; \lambda, -\lambda) \tag{32}$$

for the decay of a particle of spin J into one with spin S plus a spinless boson, where J and S may be either integral or half-integral spins. The helicity amplitudes may be readily constructed for the case of spin J → [spin J plus spinless boson] by taking combinations of irreducible tensors in the spin space of dimensionality 2J+1. In analogy with the construction of the transition matrix for decay into a spin-1/2 object, the matrix for decay into spin J would seem required by invariance arguments to be

$$A = \sum_{L}^{2J} \sum_{M} (p^{\dagger}_{LM} T_{LM} \times \text{complex coefficient}), \tag{33}$$

where p_{LM} represents an irreducible tensor formed from components of decay momentum. In the helicity representation, only the p_{L0} terms are nonzero; these in fact become constants because $p_Z = p$. With the absorption of p_{L0} factors, the transition matrix becomes, in the helicity representation,

$$A = a + c\,T_{20} \text{ for one parity}$$

$$\text{and } A' = b\,T_{10} + d\,T_{30} \text{ for the opposite parity of decay.} \tag{34}$$

However, when the initial and final spins differ in a decay, factors dependent on initial spin modify the various elements of A; and these must be calculated by a prescription similar to that of Eq. (4). Decay into final spin 1/2 is an exception to this statement, as there is only one initial-spin factor which is common to $A_{1/2}$ and $A_{-1/2}$, and this is absorbed in the normalization of the amplitudes.

II. Relations Among $n^{(2\lambda)}_{Lm}$ Coefficients

The derivation of general expressions for

$$n^{(2\lambda)}_{Lm} \equiv (-)^{J-\lambda'} [(2J+1)/4\pi]^{1/2} C(J\,J\,L;\, \lambda, -\lambda') ; \quad \lambda' = \lambda - m \qquad (35)$$

in terms of $n^{(1)}_{L0}$ is useful because comparison of experimental distributions containing these coefficients may test for J, the spin of X.[9] The identity $C(J\,J\,L;\, \lambda_1 \lambda_2) = (-)^{2J+L} C(J\,J\,L;\, \lambda_2 \lambda_1)$ permits derivation of relations between the λ and $-\lambda$ forms of $n^{(2\lambda)}_{L0}$ and also relations between the $n^{(2\lambda)}_{Lm}$ coefficients in the (λ, λ') and $(-\lambda', -\lambda)$ elements of ρ_Z [Eq. (14)]:

$$n^{(-2\lambda)}_{L0} = (-)^{2J+L+1}\, n^{(2\lambda)}_{L0}$$

$$n^{(-1)}_{L1} = (-)^{2J+L}\, n^{(3)}_{L1}$$

$$n^{(1)}_{L2} = (-)^{2J+L+1}\, n^{(3)}_{L2} \qquad (36)$$

Recursion relations for Clebsch-Gordan coefficients (p. 39, Edmonds, Ref. 3) may be utilized to obtain the following:

$$n^{(3)}_{L0} = (1/X)\, \{X - L(L+1) + (J+1/2)^2 [1 + (-)^{2J+L}]\}\, n^{(1)}_{L0} \qquad (37)$$

which becomes $n^{(3)}_{L0} = (1/X)[X - L(L+1)]\, n^{(1)}_{L0}$ for even L

and $n^{(3)}_{L0} = (1/X)[3J(J+1) - 1/4 - L(L+1)]\, n^{(1)}_{L0}$ for odd L,

where $X = (J+3/2)(J-1/2)$. Further, by use of the same recursion relations,

$$-n^{(1)}_{L1} = (2J+1)[L(L+1)]^{-1/2}\, n^{(1)}_{L0}\, [1+(-)^{2J+L}]/2 \qquad (38)$$

$$n^{(3)}_{L1} = X^{-1/2}\, \{L(L+1) - (J+1/2)^2 [1+(-)^{2J+L}]\}\, [L(L+1)]^{-1/2}\, n^{(1)}_{L0} \qquad (39)$$

$$n^{(3)}_{L2} = X^{-1/2}(J+1/2)\, \{[1+(-)^{2J+L}] - L(L+1)\}\, [(L+2)(L-1)L(L+1)]^{-1/2}\, n^{(1)}_{L0} \qquad (40)$$

$$-n^{(3)}_{L3} = X^{+1/2}\, [1+(-)^{2J+L}]\, [(L+3)(L-2)]^{-1/2}\, n^{(3)}_{L2} \qquad (41)$$

FOOTNOTES AND REFERENCES

* Work performed under the auspices of the U. S. Atomic Energy Commission.

1. N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963).
2. For discussion of the density matrix, see L. Wolfenstein, Ann. Rev. Nucl. Sci. 6, 43 (1956), and U. Fano, Rev. Mod. Phys. 29, 74 (1957).
3. For descriptions of the $\mathcal{D}^J_{MM'}(\alpha, \beta, \gamma)$ functions and the T_{LM} tensors, see A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1957) or M. E. Rose, Elementary Theory of Angular Momentum (John Wiley & Sons, Inc., New York, 1957).
4. M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959) show that the matrix element connecting a J, M representation of a state with a helicity representation is $[(2J+1)/4\pi]^{1/2}\mathcal{D}^{J*}_{M\lambda}(\phi, \theta, -\phi)$. The third argument is not observable in processes discussed here.
5. The first expression in Eq. (4) follows from Eq. (B5) of Jacob and Wick, Ref. 4. These Clebsch-Gordan coefficients are written in the form used by Jacob and Wick, $C(j_1 j_2 j; m_1 m_2)$.
6. See reference 1 and also the unpublished appendix of "Determination of Spin and Decay Parameters of Fermion States," N. Byers and S. Fenster, unpublished report of Dept. of Physics, University of California, Los Angeles, May 27, 1963.
7. These T_{LM} tensors have been previously utilized to describe the spin state of the deuteron in scattering processes. See W. Lakin, Phys. Rev. 98, 139 (1955).

8. The useful relations, found in Rose and Edmonds (Ref. 3), are

$$\mathcal{D}^{j_1}_{\mu_1 m_1}\mathcal{D}^{j_2}_{\mu_2 m_2} = \sum_j C(j_1 j_2 j;\mu_1\mu_2)C(j_1 j_2 j;m_1 m_2)\mathcal{D}^{j}_{\mu_1+\mu_2,\, m_1+m_2}$$

and $\mathcal{D}^{j*}_{m'm}(\alpha\beta\gamma) = (-)^{m'-m}\mathcal{D}^{j}_{-m',\,-m}(\alpha\beta\gamma)$ and also

$\sum_{m_1 m_2} C(j_1 j_2 j;m_1 m_2)\,C(j_1 j_2 j';m_1 m_2) = \delta_{jj'}$. The first relation here holds also for products of spherical harmonics Y_{LM}, as $Y_{LM}(\theta,\phi) \propto \mathcal{D}^{L}_{M0}(\phi,\theta,0)^*$.

The $\mathcal{D}^{L}_{MM'}$ functions are described in Jacob and Wick[4] as well as in Rose and Edmonds; some are tabulated in the former reference in terms of simple θ and ϕ functions. For evaluation of Eq. (11), symmetry properties of the T_{LM} (see $T_{L,-M}$ expression in text) and $\mathcal{D}^{L}_{MM'}$ are useful $[\mathcal{D}^{L}_{-M,\mu}(\phi,\theta,0) = (-)^{L+\mu}\mathcal{D}^{L}_{M\mu}(\phi,\pi-\theta,0)^*]$.

9. An alternate definition, more convenient for calculation, is

$$n^{(2\lambda)}_{Lm} \equiv \left[\frac{2L+1}{4\pi}\right]^{1/2} C(JLJ;\lambda-m,m).$$

10. Taking $\mathrm{Tr}(\rho\, T_{\ell m})$ is equivalent to finding $\langle \chi_n | T_{\ell m} | \chi_n \rangle$ for each spin state n and summing over all spin states with proper weighting.

An alternate derivation of the distributions for particle F may be used which does not demand the calculation of the I $\langle T_{\ell m} \rangle$ quantities for Z. This is the transforming of the density matrix ρ_Z by use of a transition matrix ($\mathcal{M}'$) for the Z → F decay; i.e., the calculation of $\rho_F = \mathcal{M}'\rho_Z\mathcal{M}'^{\dagger}$ from the expression for ρ_Z in Eq. (8). The transition matrix $\mathcal{M}'$ here involves the well-known $\mathcal{D}^{3/2}_{mm'}$ functions and the helicity amplitudes for spin 3/2 decay into spin 1/2 plus spin 0. Although this is a more elegant derivation, it does not provide so clearly the means for making spin and parity tests as does the method presented in the text.

11. The θ and ϕ angles must be referred to axes defined by vectors in the X production process. If the normal serves as polar axis, all t_{LM} with odd M are zero.

12. See appendix of J. Button-Shafer and D. W. Merrill, "Properties of the Ξ Hyperon," Lawrence Radiation Laboratory Report UCRL-11884, December 1964 (unpublished).

13. Prof. Charles Zemach has derived these same distributions by the use of an entirely different formalism. Charles Zemach, (University of California, Berkeley), private communication, 1964.

14. A direct Lorentz transformation means the translating of axes so that their orientation relative to the direction of the usual Lorentz transformation is maintained. See H. P. Stapp, Relativistic Transformation of Spin Directions, University of California Radiation Laboratory Report UCRL-8096, December 1957. (unpublished).

15. Here the first two expressions represent longitudinal and transverse polarization components for Z; i. e., $T_{10} \propto \overline{S} \cdot \hat{Z}$ and $T_{11} \propto (\overline{S} \cdot \hat{x} + i \overline{S} \cdot \hat{y})$.

Some relations from Appendix II have been utilized to simplify expressions.

16. This function cannot be interpreted as a true χ^2, but should yield an unbiased estimate of J. An example of the application of a "χ^2" test for variable J is presented in an analysis of the $Y^*(1385)$, Janice B. Shafer and Darrell O. Huwe, Phys. Rev. <u>134</u>, B1372 (1964); the χ^2 of Eq. (19) and Fig. (2) tests the relation γ $\gamma(2J+1)\, t(1) = [L(L+1)]^{1/2}\, t(2)$, where t(1) and t(2) represent moments from longitudinal and transverse components of polarization, respectively.

17. An example is given in the calculation of $(2J+1)^2$ from moments for Ξ decay suggested by M. Ademollo and R. Gatto, Phys. Rev. 133, B531 (1964), or in the calculation of 2J+1 for strong decay suggested by Byers and Fenster, Ref. 1.

18. Equation (29) is valid only if the a_ℓ amplitude of higher orbital angular momentum can be ignored relative to the amplitude of lower angular momentum (i. e., $c \ll a$ or $d \ll b$).

19. As no general formalism exists for the treatment of background or interference problems, the experimenter confronted with these problems can at best (a) throw away events in portions of resonance bands showing interference (by using strong-decay symmetry and splitting an X resonance band at $\hat{Z} \cdot \hat{X} = 0$); (b) treat background near resonance separately and compare results; and (c) try to find tests least sensitive to background.