Title
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A SINGULAR SOLUTION OF THE CAPILLARY EQUATION, II: UNIQUENESS.

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DISCLAIMER

A singular solution of the capillary equation, II: uniqueness.*

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We discuss here a uniqueness question for the singular solution $U(r)$ of the capillary equation

$$(1) \qquad \operatorname{div} Tu = -(n-1)u \ , \quad Tu = \frac{1}{W} \nabla u \ , \quad W = \sqrt{1 + |\nabla u|^2} \ ,$$

constructed in the paper [1] directly preceding. There is some evidence that $U(r)$ is - up to trivial transformations - the only solution of (1) with an isolated singularity. We have as yet no proof for that assertion, even in the symmetric case considered in [1]. Our intention in the present work is to show that any symmetric solution $u(r)$ with a (non-removable) isolated singularity at $r = 0$ is asymptotic to $U(r)$ as $r \to 0$. Precisely, we intend to prove:

<u>Theorem 1</u>: <u>Let</u> $u(r)$ <u>be a solution of</u>

$$(2) \qquad \left(\frac{r^{n-1} u_r}{\sqrt{1 + u_r^2}} \right)_r = -(n-1) u r^{n-1}$$

<u>in an interval</u> $0 < r < R$. <u>Then either</u> $u(r)$ <u>can be defined at</u> $r = 0$ <u>so as to satisfy</u> (1) <u>in the entire open ball</u> $0 \le r < R$, <u>or for any two constants</u>

$$\lambda_0 > \frac{\pi + \sqrt{2}}{\sqrt{n-1}} \ , \qquad \lambda_1 > \sqrt{\frac{2}{n-1}} \ ,$$

<u>there holds</u>

(3) $$- \lambda_0 r < U(r) + |u(r)| < \lambda_1 r$$

for all sufficiently small r.

Theorem 2: Under the conditions of Theorem 1, either u(r) can be defined as a solution in the entire ball $0 \leq r < R$, or for any $\gamma > \frac{(\pi+\sqrt{2})^2}{2}$ there holds

(4) $$\sin \psi(r) \equiv \frac{1}{W}|u_r| > 1 - \gamma r^4$$

for all sufficiently small r.

We note that $\psi(r)$, defined by (4), is the angle between the line tangent to the solution curve and the (positively directed) r axis.

We prove these theorems in several steps:

i) Suppose $u(r) = o(r^{-1})$ as $r \to 0$.

We have for any $\varepsilon > 0$

$$\int_\varepsilon^r \left(\frac{\rho^{n-1} u_\rho}{W}\right)_\rho d\rho = - \int_\varepsilon^r \rho^{n-1} u \, d\rho$$

and since $W^{-1}|u_\rho| < 1$ we obtain

(5) $$\frac{r^{n-1} u_r}{W} = - \int_0^r \rho^{n-1} u \, d\rho = o(r^{n-1})$$

as $r \to 0$. Thus, $\lim_{r\to 0} u_r = 0$, and we conclude also $u_0 = \lim_{r\to 0} u(r)$ exists. Defining $u(0) = u_0$, we find from (5) that $u'(0)$ exists, and $u'(0) = \lim_{r\to 0} u_r(r) = 0$. Putting this information into (5) and integrating by parts now yields

$$\frac{r^{n-1}\ u_r}{W} = -\frac{r^n}{n} u_o + o(r^n)$$

from which

$$\frac{1}{r}\frac{u_r}{W} = -\frac{1}{n} u_o + o(1) , \qquad r \to 0 .$$

From (2) we now obtain

$$\frac{1}{W^3} u_{rr} + \frac{n-1}{r}\frac{u_r}{W} = -(n-1)\ u(r)$$

and hence there exists $\lim_{r\to 0} u_{rr} = -\frac{1}{n} u_o$.

The mean value theorem yields immediately the existence of $u_{rr}(o) = -\frac{1}{n} u_o$. <u>Thus, if</u> $u(r) = o(r^{-1})$, <u>any singularity at</u> $r = 0$ <u>is removable</u>.

ii) Suppose there were a sequence $r_k \to 0$ along which $|u(r_k)| \geq r_k^{-1} + \lambda_1 r_k$. By restricting attention to a suitable subsequence and (if necessary) replacing u by -u in (2), we may suppose

(6) $$u(r_k) \leq -r_k^{-1} - \lambda_1 r_k \quad .$$

<u>Lemma 1</u>: <u>Let</u> $\alpha(\rho) > 0$ <u>satisfy</u> $\lim_{r\to 0}\int_r^{r_o} \alpha(\rho)\ d\rho = \infty$. <u>Under the hypotheses of</u> ii), <u>a sequence</u> $\hat{r}_k \to 0$ <u>can be found, at which</u> (6) <u>holds and also</u>

(7) $$u'(\hat{r}_k) \geq \hat{r}_k^{-2} - \lambda_1 - \alpha(\hat{r}_k) \quad .$$

<u>Proof</u>: Set $f(r) = -r^{-1} - \lambda_1 r$. Let k be the smallest integer ≥ 2, such that the function

$$v(r) = f(r) - \{f(r_k)-u(r_k)\} - \int_{r_k}^{r} \alpha(\rho)\, d\rho$$

will satisfy $v(r_1) \le u(r_1)$. In the interval $r_k \le r \le r_1$ we then have $v(r) \le f(r)$, and $v'(r) = r^{-2} - \lambda_1 - \alpha(r)$. Since $v(r_k) = u(r_k)$, $v(r_1) \le u(r_1)$, there must be at least one point $\hat{r}_1$, $r_k \le \hat{r}_1 \le r_1$, with $u(\hat{r}_1) = v(\hat{r}_1)$, $u'(\hat{r}_1) \ge v'(\hat{r}_1)$. This determines the first point of the new sequence.

The n^{th} point $\hat{r}_k$ having been determined, let k_{n+1} be the smallest k such that $r_{k_{n+1}} < \hat{r}_n$. Repeating the procedure with r_1 replaced by $r_{k_{n+1}}$ yields an $(n+1)^{st}$ point $\hat{r}_{n+1} \le r_{n+1}$. An induction completes the construction.

In what follows, we use the original notation r_k to describe the sequence $\hat{r}_k$.

Lemma 2: If $u(r) < 0$ in an interval $r_k < r < R \le \infty$, then $u'(r) > 0$ in this interval.

Proof: We have, in such an interval,

$$\frac{r^{n-1}\, u'(r)}{W} = \frac{r_k^{n-1}\, u'(r_k)}{W} - \int_{r_k}^{r} \rho^{n-1}\, u(\rho)\, d\rho > 0 .$$

We conclude from Lemma 2 that a segment of the solution curve passing through $(r_k, u(r_k))$ projects simply on the u - axis, covering at least the interval $u(r_{k_o}) - u(r_k)$, for any fixed k_o. In particular, for any fixed $\sigma > 0$, an interval of length at least σ will be covered, for all sufficiently large k.

We now write (2) in the form

$$(8) \qquad \frac{\sin\psi}{r} - \frac{1}{n-1}(\cos\psi)_u = -u$$

which splits the mean curvature of the solution surface into latitudinal and meridional components.

We integrate (8) from u_k to $u_k+\sigma$; noting that on this interval, $0 < \sin\psi < 1$, $r > r_k$, $\cos\psi > 0$, we find

$$(9) \qquad \frac{1}{r_k}\sigma + \frac{1}{n-1}\cos\psi_k > -u_k\sigma - \frac{1}{2}\sigma^2$$

In Lemma 1, we may choose $\alpha(\rho)$ so that $r^{-2} - \lambda_1 - \alpha(r) = r^{-2}(1 - \varepsilon(r))$ with $\varepsilon(r) = o(r)$ (a possible explicit choice is $\varepsilon = r/\ln r^{-1}$); we then obtain $\cos\psi_k < \frac{r^2}{1-\varepsilon}$. Using again (6), we find from (9)

$$-\lambda_1 r_k \sigma + \frac{r_k^2}{(n-1)(1-\varepsilon)} + \frac{1}{2}\sigma^2 > 0 .$$

If $\lambda_1^2 > \frac{2}{n-1}$, the choice $\sigma = \lambda_1 r$ yields a contradiction, for all r sufficiently small that $\varepsilon(r) < 1 - \frac{2}{(n-1)\lambda_1^2}$.

Thus,

$$-u(r) - r^{-1} < \lambda_1 r$$

for all sufficiently small r. From the result of [1] we have

$$(10) \qquad U(r) = -r^{-1} + O(r^3)$$

as $r \to 0$. These two relations establish the right side of (3).

iii) We establish the left side of (3) by reducing the problem to the case just discussed. We shall show that <u>if the singularity is not removable</u>, <u>then the existence of a sequence</u>

$r_k \to 0$ <u>for which</u>

(11) $$u(r_k) > -r_k^{-1} + \lambda_o r_k$$

<u>implies the existence of a sequence</u> $\hat{r}_k \to 0$, <u>with</u> $u(\hat{r}_k) < -\hat{r}_k - \lambda_1 \hat{r}_k$.

We note first that if a sequence $r_k \to 0$ exists for which (11) holds and if the singularity is not removable, then by i) there exists a sequence - which we again label r_k - for which (11) holds and for which $u(r_k) < -C\, r_k^{-1}$, for some fixed $C > 0$.

For any fixed $k > 1$, define

$$v(r) = \max\left\{u(r_k) + \frac{u(r_1)-u(r_k)}{r_1-r_k}(r-r_k),\ -r^{-1}+\lambda_o r\right\} .$$

If k is sufficiently large, there will be exactly two points at which v(r) is not differentiable, and the lower derivate D^-v of v(r) will satisfy (at all points)

$$D^-v > -r_1^{-2} + \lambda_o .$$

Since v(r) is continuous, there must exist at least one point $\hat{r}_1$, $r_k \le \hat{r}_1 \le r_1$, at which $u(\hat{r}_1) = v(\hat{r}_1) \ge \hat{r}_1^{-1} + \lambda_o r_1$, and $u'(\hat{r}_1) \ge \hat{r}_1^{-2} + \lambda_o$.

The n^{th} point $\hat{r}_n$ having been determined, let k_{n+1} be the smallest integer for which $r_{k_{n+1}} < \hat{r}_n$; repeating the procedure with r_1 replaced by $r_{k_{n+1}}$ yields a point $\hat{r}_{n+1} \le r_{k_{n+1}}$, with $u(\hat{r}_{n+1}) \ge \hat{r}_{n+1}^{-1} + \lambda_o \hat{r}_{n+1}$ and $u'(\hat{r}_{n+1}) \ge \hat{r}_{n+1}^{-2} + \lambda_o$, and $\hat{r}_{n+1} \to 0$, $u(\hat{r}_{n+1}) \to -\infty$. <u>We conclude by induction the existence of a sequence</u> $\hat{r}_k \to 0$ <u>with those properties</u>.

We now revert (for the same sequence) to the original notation r_k, and consider, for fixed k, the auxiliary equation

$$(12) \qquad \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{W}\right)_r \equiv \frac{1}{r^{n-1}}(r^{n-1}\sin\psi)_r = -(n-1)u_k = nH_k$$

for a surface u(x) of constant mean curvature H_k.

Lemma 3: Given r_k, u_k, α_k with $0 < -\frac{n-1}{n} r_k u_k \le \alpha_k < 1$, there is a unique solution v(r) of (12), for which $\sin\psi(r_k) = \alpha_k$, and which has the properties: the solution is defined in an interval $I_k = (a_k, b_k)$, $0 < a_k < -u_k^{-1} < b_k < -\frac{n}{n-1} u_k^{-1}$, and satisfies $\sin\psi(r) > 0$ in I_k, $\lim_{r\to a_k, b_k} \sin\psi(r) = 1$.

If $n = 2$, then $a_k + b_k = -2u_k^{-1}$; if $n > 2$, this relation holds asymptotically as $a_k \to u_k^{-1}$ or $b_k \to u_k^{-1}$.

Proof: In what follows, we suppress the index k. For the most general real solution of (12), there holds

$$\sin\psi(r) = Hr + A\,r^{1-n}$$

for some constant $A < \frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1} H^{1-n}$.

Set

$$(13) \qquad A = \frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1} H^{1-n}(1-\varepsilon), \qquad r = \frac{n-1}{nH}(1+\sigma)$$

If $0 < \varepsilon < 1$, the solution will exist on an interval (a,b) determined by the two real roots σ_a, σ_b of

$$(14) \qquad \varepsilon = n\left\{\sigma(1+\sigma)^{n-1} - \frac{(1+\sigma)^n - 1}{n}\right\}.$$

There holds $-1 < \sigma_a < 0 < \sigma_b < \frac{1}{n-1}$, $|\sigma_b| < |\sigma_a|$; the minimum of σ_a and the maximum of σ_b are attained in the limit as $\varepsilon \to 1$. Asymptotically as $\varepsilon \to 0$,

(15) $$\varepsilon \sim \frac{n(n-1)}{2} \sigma^2 = \binom{n}{2} \sigma^2$$

If $n = 2$, (15) becomes an equality, and in that case $\sigma_a + \sigma_b = 0$.

The initial condition will be satisfied by the choice

(16) $$\varepsilon = n(\sigma+1-\alpha)(1+\sigma)^{n-1} + 1 - (1+\sigma)^n$$

where σ is the value corresponding to r_k. We note $\sigma(r_k) \to 0$, $\varepsilon \to n(1-\alpha)$ as $r_k u_k \to -1$. One verifies immediately that a solution of the type indicated is possible, with $0 < \varepsilon \leq 1$, for all α in the range

$$\frac{n-1}{n}(1+\sigma) \leq \alpha < 1$$

This relation yields the condition of the lemma.

<u>Lemma 4</u>: <u>Under the conditions of Lemma</u> 3, <u>there hold</u>, <u>if</u> $n > 2$,

(17) $$-\sqrt{1-(1-\varepsilon)^{1/n}} \left\{ \sqrt{\frac{2}{n-1}}(1-\varepsilon)^{1/2n} + \sqrt{1-(1-\varepsilon)^{1/n}} \right\} < \sigma_a < -\sqrt{\frac{2\varepsilon}{n(n-1)}}$$

(18) $$\sqrt{1-(1-\varepsilon)^{1/n}} \left\{ \sqrt{\frac{2}{n-1}}(1-\varepsilon)^{1/2n} - \sqrt{1-(1-\varepsilon)^{1/n}} \right\} < \sigma_b < \sqrt{\frac{2\varepsilon}{n(n-1)}}$$

<u>In particular</u>, $\lim_{\varepsilon \to 0} \left| \frac{\sigma_b}{\sigma_a} \right| = 1$. <u>If</u> $n = 2$ <u>then</u> $-\sigma_a = \sigma_b = \sqrt{\varepsilon}$.

<u>Proof</u>: Differentiating (14) yields

(19) $$\frac{d\varepsilon}{d\sigma} = n(n-1)\sigma(1+\sigma)^{n-2} .$$

Thus, if $\sigma \geq -1$, (14) has the unique solution $\sigma(0) = 0$.

If $n > 2$ then $\varepsilon'(\sigma) \geq n(n-1)\sigma$, equality holding only at $\sigma = 0$. From this estimate the right sides of (17), (18) follow directly.

To obtain the remaining inequalities we start with (16), which yields, for fixed ε,

$$(20) \qquad \frac{d^2\alpha}{d\sigma^2} = (n-1)\,\frac{1-\varepsilon}{(1+\sigma)^{n+1}}$$

Denoting by σ_c, α_c the values of σ, α at which $\alpha'(\sigma) = 0$, we find

$$(21) \qquad \alpha_c = 1 + \sigma_c = (1-\varepsilon)^{1/n}$$

Hence if $\sigma < \sigma_c$, then $\alpha'(\sigma) < 0$ and $\alpha''(\sigma) > (n-1)(1-\varepsilon)^{-1/n}$. From this we conclude

$$(22) \qquad \frac{d\alpha}{d\sigma} < \frac{n-1}{(1-\varepsilon)^{1/n}}\,(\sigma-\sigma_c)$$

and noting that $\alpha(\sigma_a) = 1$, we obtain the left side of (17) from (22) after a further integration.

If $\sigma > \sigma_c$ then $\alpha'(\sigma) > 0$, $\alpha''(\sigma) < (n-1)(1-\varepsilon)^{-1/n}$; thus, (22) holds also in this case. Integration and use of (21) yields the left side of (18).

We note also the expressions, for $\sigma = \sigma_a$ or $\sigma = \sigma_b$,

$$(23) \qquad \varepsilon = \sigma^2 \frac{d}{d\sigma}\left\{\frac{1}{\sigma}\left[(1+\sigma)^n - (1+n\sigma)\right]\right\}$$

$$(24) \qquad \varepsilon = \binom{n}{2}\sigma^2 + 2\binom{n}{3}\sigma^3 + \ldots + n\binom{n}{n}\sigma^n$$

<u>Lemma 5</u>: <u>Under the conditions of Lemma</u> 3, <u>suppose</u> $-r_k u_k < 1$, <u>and let</u> σ_k <u>correspond to</u> r_k, σ_b <u>to</u> r_b. <u>Then the change in height</u> δv <u>between the values</u> σ_k <u>and</u> σ_b <u>satisfies</u>

$$(25) \qquad -u_k\delta v < \frac{\pi}{\sqrt{n-1}}\left\{\frac{1}{2}\,\frac{(1-\varepsilon)^{\frac{n+2}{2n}}}{(1+\sigma_k)^{\frac{n+1}{2}}}\left(\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\frac{1-\alpha_k}{1-\alpha_c}\right) + \frac{1}{2}\,\frac{(1+\sigma_b)^{n+1}}{(1-\varepsilon)^{\frac{2n+1}{2n}}}\right\}$$

Proof: We may write

$$\delta v = \frac{dr}{d\sigma}\int_{\sigma_k}^{\sigma_b} \frac{\alpha}{1-\alpha^2}\,d\sigma$$

$$= -\frac{1}{u_k}\left\{\int_{\alpha_k}^{\alpha_c} \frac{\alpha}{\sqrt{1-\alpha^2}}\frac{d\sigma}{d\alpha}\,d\alpha + \int_{\alpha_c}^{1} \frac{\alpha}{\sqrt{1-\alpha^2}}\frac{d\sigma}{d\alpha}\,d\alpha\right\} .$$

To estimate $\frac{d\sigma}{d\alpha}$ we start with (16). If $\sigma < \sigma_c$ then $\alpha''(\sigma) > (n-1)(1-\varepsilon)(1+\sigma_c)^{-(n+1)}$, thus $\alpha'(\sigma) < (n-1)(1-\varepsilon)(1+\sigma_c)^{-(n+1)}(\sigma-\sigma_c)$; on the other hand, $\alpha''(\sigma) < (n-1)(1-\varepsilon)(1+\sigma_k)^{-(n+1)}$, which with the above estimate and (21) yields

$$(26) \qquad 0 < -\frac{d\sigma}{d\alpha} < L(\alpha-\alpha_c)^{-1/2}, \quad L = \frac{1}{\sqrt{2(n-1)}}\,\frac{(1-\varepsilon)^{\frac{n+2}{2n}}}{(1+\sigma_k)^{\frac{n+1}{2}}}$$

Similarly, if $\sigma > \sigma_c$ we find

$$(27) \qquad 0 < \frac{d\sigma}{d\alpha} < R(\alpha-\alpha_c)^{-1/2}, \quad R = \frac{1}{\sqrt{2(n-1)}}\,\frac{(1+\sigma_b)^{n+1}}{(1-\varepsilon)^{\frac{2n+1}{2n}}}$$

Thus,

$$-u_k\delta v < \frac{1}{\sqrt{2}}\left\{L\int_{\alpha_c}^{\alpha_k} \frac{1}{\sqrt{1-\alpha}\sqrt{\alpha-\alpha_c}}\,d\alpha + R\int_{\alpha_c}^{1} \frac{1}{\sqrt{1-\alpha}\sqrt{\alpha-\alpha_c}}\,d\alpha\right\}$$

since $\frac{\alpha}{\sqrt{1+\alpha}} < \frac{1}{\sqrt{2}}$. Formal integration yields the stated estimate.

Lemma 6: Let $u(r)$, $v(r)$ satisfy

$$(28) \qquad (r^{n-1}\sin\psi)_r = -(n-1)r^{n-1}u, \quad \sin\psi = \frac{u_r}{\sqrt{1+u_r^2}},$$

(29) $$(r^{n-1}\sin\hat{\psi})_r = -(n-1)r^{n-1}u_k\ , \quad \sin\hat{\psi} = \frac{v_r}{\sqrt{1+v_r^2}}$$

in $r_k \le r < r_b$. Suppose $u(r) > u_k$ in this interval, and $\sin\hat{\psi}(r_k) \ge \sin\psi(r_k)$. Then $u(r) < v(r)$, $u'(r) < v'(r)$ in $r_k < r < r_b$.

Proof: Integrating the difference of (28), (29) yields

$$r^{n-1}(\sin\psi(r) - \sin\hat{\psi}(r))$$

$$= -(n-1)\int_{r_k}^{r} \rho^{n-1}(u-u_k)\,d\rho + r_k^{n-1}(\sin\psi(r_k) - \sin\hat{\psi}(r_k))$$

$$< 0 .$$

iv) We return to consideration of the singular solution $u(r)$, for which we have supposed the existence of a sequence $r_k \to 0$ with $u_k = u(r_k) \ge -r_k^{-1} + \lambda_o r_k$, and have shown as a consequence that the sequence can be chosen so that

(30) $$u(r_k) \to -\infty$$

(31) $$u'(r_k) \ge r_k^{-2} + \lambda_o$$

By Lemma 2, $u(r) > u(r_k) = u_k$ in any interval $r > r_k$ in which $u(r) < 0$. We consider the solution $v(r)$ of (29) with $v(r_k) = u_k$, $v'(r_k) = u'(r_k)$. By Lemma 3, this solution exists and has the properties indicated in that lemma, for all k sufficiently large that

$$\frac{u'(r_k)}{\sqrt{1+u'^2(r_k)}} > \frac{n-1}{n}(1 - \lambda_o r_k^2) .$$

By Lemma 5, $v(r) < 0$ on $a < r < b$ for large k, hence by Lemmas 2 and 6, $u(r) < v(r)$ on $r_k < r < b$. From Lemma 5 we then conclude that the change in height δu on this interval satisfies, for any $\eta > 0$,

$$-u_k \delta u < \frac{\pi}{\sqrt{n-1}} + \eta \tag{32}$$

for all k sufficiently large.

We have by hypothesis

$$\sigma_k = -(1 + u_k r_k) \leq -\lambda_o r_k^2$$

and $\sigma_a < \sigma_k$. Referring again to

$$\frac{d\varepsilon}{d\sigma} = n(n-1)\sigma(1+\sigma)^{n-2} \tag{19}$$

we find

$$\frac{d\varepsilon}{d\sigma} > \frac{n(n-1)}{2^{n-2}} \sigma$$

in the range $-\frac{1}{2} \leq \sigma \leq 0$. If $-1 \leq \sigma < -\frac{1}{2}$ there holds in any event $\varepsilon'(\sigma) > 0$; thus

$$\varepsilon(\sigma_a) > \frac{n(n-1)}{2^{n-1}} \min\{\sigma_a^2, 2^{-2}\} \geq \frac{n(n-1)}{2^{n+1}} \sigma_a^2 .$$

If $\sigma > 0$, (19) yields, since $\sigma < \frac{1}{n-1}$,

$$\frac{d\varepsilon}{d\sigma} < \frac{n^{n-1}}{(n-1)^{n+1}} \sigma$$

from which

$$\varepsilon < \frac{n^{n-1}}{(n-1)^{n+1}} \frac{\sigma_b^2}{2} .$$

We conclude, for the solution considered,

$$\sigma_b^2 > \frac{(n-1)^{n+2}}{2^n n^{n-2}} \sigma_a^2 . \tag{33}$$

We have

$$\sigma_a < \sigma_k = -(1 + r_k u_k) \quad .$$

From the given relation

$$\lambda_o r_k^2 - r_k u_k - 1 < 0$$

we conclude

$$2\lambda_o r_k < -u_k \left\{ \sqrt{1 + \frac{4\lambda_o}{u_k^2}} - 1 \right\}$$

so that, for any $\beta < 1$,

$$r_k < -\frac{1}{u_k} + \frac{\beta\lambda o}{u_k^3}$$

for all $|u_k|$ sufficiently large, depending on β. Thus

$$\sigma_a < -\frac{\beta\lambda_o}{u_k^2}$$

so that from (33)

$$\sigma_b > \frac{(n-1)^{\frac{n+1}{2}}}{2^{\frac{n}{2}}\, n^{\frac{n-2}{2}}} \frac{\beta\lambda_o}{u_k^2} = \lambda_o\, C_{n,\beta}\, u_k^{-2}$$

and therefor

$$(34) \qquad -\frac{1}{r_b} = \frac{u_k}{1+\sigma_b} > \frac{u_k}{1 + \lambda_o C_{n,\beta} u_k^{-2}}$$

$$> u_k - \lambda_o \beta C_{n,\beta} u_k^{-1}$$

for sufficiently large $|u_k|$. But by Lemmas 5 and 6,

$$u_b = u(r_b) = u_k + \delta u < u_k - \frac{\pi}{u_k\sqrt{n-1}} - \frac{1}{u_k}\eta$$

so that

$$-\frac{1}{r_b} - u_b > -\frac{1}{u_k}\left(\lambda_o \beta C_{n,\beta} - \frac{\pi}{\sqrt{n-1}} - \eta\right) \quad .$$

Since

$$-\frac{1}{u_k} = \frac{r_b}{1+\sigma_b} < \frac{n-1}{n} r_b$$

there follows finally

$$\frac{1}{r_b} - u_b > \left(\lambda_o \frac{n-1}{n} \beta C_{n,\beta} - \frac{\pi\sqrt{n-1}}{n} - \eta \frac{n-1}{n}\right) r_b$$

and we conclude from the result of ii)

<u>Lemma 7</u>: <u>Let</u> $u(x)$ <u>be a solution of</u> (2) <u>in</u> $0 < r < R$, <u>with an isolated singularity at</u> $r = 0$, <u>such that</u> $u(r_k) < 0$ <u>for some sequence</u> $r_k \to 0$. <u>Then either the singularity is removable or for any</u> β, $0 < \beta < 1$ <u>and</u>

$$\hat{\lambda}_o > \left(\frac{2}{\sqrt{n-1}} + \frac{\pi\sqrt{n-1}}{n}\right) \frac{n}{n-1} \frac{1}{\beta C_{n,\beta}} \tag{35}$$

<u>there holds</u>

$$u(r) + \frac{1}{r} < \hat{\lambda}_o r \tag{36}$$

<u>for all sufficiently small</u> r.

v) We proceed to use Lemma 7 to complete the main results. We note first that the estimate (36), together with the equation (2), yields, for $\tau < r$,

$$r^{n-1}\sin\psi(r) = \tau^{n-1}\sin\psi(\tau) - (n-1)\int_\tau^r \rho^{n-1}u\,d\rho$$

$$> \tau^{n-1}\sin\psi(\tau) + (r^{n-1} - \tau^{n-1}) - \lambda_o \frac{n-1}{n+1}(r^{n+1} - \tau^{n+1}) ,$$

from which, letting $\tau \to 0$,

$$\sin\psi(r) > 1 - \hat{\lambda}_o \frac{n-1}{n+1} r^2 \tag{37}$$

It follows in particular that the condition of Lemma 3 is fulfilled for all sufficiently small r. If the data of that lemma are chosen from the given solution $u(r)$, there follows by (13), (36)

$$(38) \qquad -\hat{\lambda}_o r_k^2 \; < \; \sigma_k \; < \; 0 \quad .$$

From (16) follows, for $\alpha > \alpha_o > 0$ and $|\sigma|$ small,

$$\varepsilon < (n-1)(1+n\sigma+\binom{n}{2}\sigma^2) - n\alpha(1+(n-1)\sigma) + 1 \quad .$$

Thus, for the given solution, (37) and (38) imply

$$\varepsilon \; < \; n\hat{\lambda}_o \frac{n-1}{n+1} r^2 + \frac{n(n-1)^2}{2} \hat{\lambda}_o^2 r^4 \quad .$$

In particular, $\varepsilon \to 0$ as $r \to 0$. It follows from Lemma 4 that also $\sigma_a, \sigma_b \to 0$ and

$$(39) \qquad \left|\frac{\sigma_b}{\sigma_a}\right| \; \to \; 1 \quad .$$

Suppose there were a sequence $r_k \to 0$ along which

$$(40) \qquad u_k \; > \; -\frac{1}{r_k} + \lambda_o r_k$$

for fixed $\lambda_o > 0$. Then

$$-\frac{1}{r_b} = \frac{u_k}{1+\sigma_b} = -u_k(-1 + \sigma_b + o(\sigma_b))$$

$$= -u_k(-1 - \sigma_a + o(\sigma_a))$$

$$> -u_k(-1 + \lambda_o r_k^2 + o(r_k^2))$$

as $r_k \to 0$ by (39), since

$$\sigma_a \; < \; \sigma_k \; = \; -u_k r_k - 1 \; < \; \lambda_o r_k^2$$

by (40).

We have again

$$u_b \; < \; -u_k(-1 + \frac{\pi}{u_k^2\sqrt{n-1}} + o\left(\frac{1}{u_k^2}\right) \;)$$

so that, using (36),

$$-\frac{1}{r_b} - u_b > (\lambda_o - \frac{\pi}{\sqrt{n-1}}) r_k + o(r_k) .$$

This inequality contradicts the result of ii) if $\lambda_o > \frac{\sqrt{2}+\pi}{\sqrt{n-1}}$.
The proof of Theorem 1 is thus complete.

vi) Suppose now the existence of a sequence $r_k \to 0$ along which

$$\alpha_k = \sin \psi_k < 1 - \gamma r_k^4 .$$

From (36) follows $\sigma_k \to 0$, while from (16) we obtain

$$\varepsilon > n(1-\alpha_k)[1+(n-1)\sigma_k] > n\gamma r_k^4(1-\lambda_o r_k^2)$$

for all sufficiently small $|\sigma_k|$. By Lemma 4,

$$\sigma_b > \sqrt{\frac{2\varepsilon}{n(n-1)}}\,(1-\varepsilon) - \frac{\varepsilon}{n}$$

and hence

$$(41) \qquad \sigma_b > r_k^2 \sqrt{\frac{2\gamma}{n-1}} - C\, r_k^4$$

for a fixed constant C, as $r_k \to 0$. Again using Lemmas 5 and 6,

$$u_b < u_k \left\{1 - \frac{\pi}{u_k^2 \sqrt{n-1}} - \frac{n}{u_k^2}\right\}$$

while by (36), (41),

$$-\frac{1}{r_b} > u_k \left\{1 - r_k^2 \sqrt{\frac{2\gamma}{n-1}} + C\, r_k^4\right\}$$

so that, by Lemma 7,

$$-\frac{1}{r_b} - u_b > \left(\sqrt{\frac{2\gamma}{n-1}} - \frac{\pi}{\sqrt{n-1}}\right) r_k - C\, r_k^2 .$$

From (16) we find $\varepsilon \to 0$, hence by Lemma 4, $\sigma_b \to 0$. Since

$$\frac{r_k}{r_b} = \frac{1+\sigma_k}{1+\sigma_b} \to 1 ,$$

Theorem 2 now follows from Theorem 1 and from (10).

vii) We remark that a bound from above for $\sin\psi$ in an average sense follows immediately from Theorems 1 and 2. A more precise estimate could be a significant step toward a strict uniqueness proof for the singular solution U(r).

Reference and Footnotes

1. Concus, P. and R. Finn: A singular solution of the capillary equation, I: existence.

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