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A Dynamic Model of Consumer Behavior

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A Dynamic Model of Consumer Behavior

A Dissertation submitted in partial satisfaction
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by

Craig Bernhard McLaren

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DEDICATION:

To: Dr. Robert Bruce McLaren - my father and mentor, Without whom this work could not have been accomplished.

To Professor Prashanta Pattanaik and the members of my committee- A thank you for your encouragement and patience

To my immediate family, my church family, and the crew of the Brig Pilgrim who got me through the writing of this. A hearty "Hip-Hip: Hooray!"

ABSTRACT OF THE DISSERTATION

A Dynamic Model of Consumer Behavior

by

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Doctor of Philosophy, Graduate Program in Economics
University of California, Riverside, March 2012
Dr. Mason Gaffney, Chair

This dissertation presents a dynamic mathematical theory of consumer behavior, starting from basic assumptions, and building through the tatonnement processes by which exchange (general) equilibria can be achieved in real time.

The purpose of the dissertation is to revise consumer theory so that consumer behavior can be studied as a function of observable (demographic) variables instead of non-observable quantities such as preference. With the interpersonal comparison of preference no longer a barrier, general equilibrium models can be used to study the impact of the distribution of demographic factors, most notably wealth, on the demand for goods, and on aggregate well-being.

Chapters 2 and 3 discuss the problems with utility, preference, and their measurement as they appeared in the history of economic thought. Chapter 4 develops a theory of the consumer based on his or her marginal prices. The consumer's marginal price for a good is defined as the

maximum s/he would be willing to pay for one more unit of it, given all the goods s/he possesses at the time. The consumer's complete set of marginal prices constitutes a vector function, requiring the theory to be built built using vector analysis.

The consumer is regarded as acquiring her wealth through many small decisions made over time. Since the bundle s/he holds at any given time is the result of past decisions, the consumer is modeled as dynamically interacting with his/her environment. A dynamic general equilibrium model in which an arbitrary number of traders exchange an arbitrary number of goods is presented in Chapter 5.

Chapter 6 develops a method of empirically aggregating consumer's marginal prices to determine how consumers <u>in general</u> would be expected to behave, given their socioeconomic circumstances. The behavior of a demographically diverse community is modeled in the general equilibrium framework, by assuming all consumers have common (aggregate) marginal price functions, yet are differentiated by the bundles they hold. Such bundles are the surrogate for their socioeconomic circumstances.

The theory presented in this dissertation is intended to facilitate incorporation of theories and data from Psychology, Sociology, and other social sciences, as well as those of experimental economics.

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CHAPTER 1: INTRODUCTION

"Truth is one
Sages perceive it differently"
Hindu Dharma

The purpose of this dissertation is to provide the analytic tools needed to study the dependence of macroeconomic factors, such as aggregate demand, on the distribution of socioeconomic resources within an economy. To many, it might seem intuitively obvious that such distribution would have a marked impact. Economists however have been reticent to discuss this matter, arguing that such discussion would require interpersonal comparison of individual's preferences. The capability of interpersonally comparing the choices consumers make is crucial to our study. Acquisition of this capability will therefore be a goal of the tools to be developed here.

Obstacles to the development of the needed tools are rooted deep in the foundation of neoclassical microeconomic theory, and have been accepted as insurmountable since the time of Pareto. Beginning with the Marginal Revolution of the 1870's, economists have presumed that consumer's choices must be explained by unobservable, subjective phenomena (utility) within the consumer's head. The obstacles disappear when such subjective phenomena are dismissed, and the theory is based on the observed choices *themselves*. The approach is *dynamic* in that the consumer's choices are made in increments, each increment being based on present circumstances resulting in part from choices he or she has made previously. As result, the tatonnement processes by which equilibria are reached consist of dynamic interactions between market participants occurring in real time.

The difficulty with this approach is not conceptual but technical. If the consumer chooses among n goods as guided by a single phenomenon (i.e. her utility), her behavior can be formally described in terms of a single (scalar) utility function of n variables. The choices themselves however, must be described in terms of a vector of interdependent *Marginal Rates of Substitution* (MRS), each of which being a function of all n variables. A theory that begins with the MRS is therefore most efficiently expressed in terms of *Vector and Tensor*[1] *Analysis*, a branch of mathematics not widely used outside of the physics community until well after the marginal revolution was completed. A large portion of this dissertation therefore will be devoted to expressing a MRS based consumer theory in terms of vector analysis.

1.1) What Interpersonal Comparability Can Tell Us

The ability to make interpersonal comparisons will allow economists to empirically observe behavioral trends among individuals facing common socioeconomic conditions. Marketing professionals have studied consumer behavior as a function of such demographic circumstances for decades. Retail firms have long relied on these to determine how their products are to be marketed. Due to the affluence of the residents of Beverly Hills, California, one can find retail branches of Tiffany's, Cartier, and De Beers within a three-block stretch of Rodeo Drive. In contrast, it is difficult to find large retailers of any kind in many inner city neighborhoods.

There is evidence that individuals prioritize their consumption. Poorer individuals spend a higher portion of their income on the most urgently needed goods such as food and shelter. As that their need for these goods is nearly satisfied, wealthier individuals spend a larger share of their income

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[1] The use of tensors will be very minimal

on goods that satisfy less urgent wants. If confirmed empirically, this evidence would speak volumes regarding the forms a "typical" consumer's MRS functions may take.

1.2) Evidence that the Distribution of Resources Influences the Macro Economy

As will be discussed in Chapter 7, prioritization of consumption will tend to cause an individual's demand for nearly all goods to grow less rapidly than her wealth. If such prioritization is common to all individuals, aggregate demand will likely decrease as the distribution of wealth becomes increasingly unequal. If an individual's incentive to produce is tied to his incentive to consume, increasing inequality is likely to reduce aggregate output as well. This may well explain the inverse relationship observed between the productivity of land and the wealth of its owner, shown in Table 1.2-1.

Distribution of Agricultural Land in COLOMBIA	**Mini-Fundio**	**Single Family Farm**	**Multi Family Farm**	**Lati-Fundio**
% Of total Farmland	5	25	25	45
% Of Agricultural GDP Produced	21	45	19	15
Relative Productivity (% GDP Produced) ÷ (% Farmland used)	4.2	1.8	0.76	0.33

Minifundio: Farm not large enough to support 2 individuals[2]

Single Family Farm: Farm large enough to support 2 to 4 individuals

Multi Family Farm: Farm large enough to support 4 to 12 individuals

Latifundio: Farm large enough to support 12 to 10,000 individuals

Table 1.2-1 Colombia as an example of the Latifundio-Minifundio pattern of land use[3].

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[2] Assumes typical income, market prices, level of technology, and capital typical to the region. See Todaro and Smith (2003) p.430

[3] Data in this table derived from Todaro and Smith (2003) pp. 430-31

In the *latifundio-minifundio pattern* of land tenure, widely evident throughout Latin America and elsewhere in the world[4], the poorest, or *minifundio* farmers must devote their marginal resources exclusively to production of goods satisfying their survival needs. Since basic needs for the wealthiest *latifundio* farmers have long since been met, they are free to use their marginal resources to address wants related to emotional satisfaction. Hence, to minifundio farmers, an additional unit of land is chiefly a source of food, while for latifundio farmers; it becomes a source of prestige.

In *The problem of Modern Economics* Roger Backhouse lays blame for Russia's tortured transition to a market economy at the feet of an economic theory that could not consider the impact of the distribution of resources such as property rights[5]. Citing the Coase theorem, Russia's foreign advisors argued that, in the absence of transaction costs, ownership of such resources would not influence their use[6]. Russia's leadership was therefore advised to privatize state property as quickly as possible, with little regard for who got it and how. As Deputy Prime Minister Anatoly Chubais commented on the privatization process:

> They steal and steal. They are stealing absolutely everything and no one can stop them. But let them steal and take their property. They will become owners and decent administrators of this property[7].

Chubais could not have been more wrong. Those who stripped the Russian state of its resources quickly converted their newly acquired wealth to assets that could be moved out of the country[8].

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[4] Todaro and Smith (2003) pp.430-32

[5] Backhouse (2010) p.49

[6] Backhouse (2010) p.49

[7] Freeland (2000) p.70 Quoted in Backhouse (2010) p.45

1.3) Consumer Behavior and Economic Value

The problem with neoclassical consumer theory is not that it is "wrong" but that it's logic is constructed backwards. To understand the problem, we need to look at the theory's structure vs. what it seeks to accomplish.

Humans, like all species, seek to address a common set of survival needs. Our ability to organize, so as to address such needs through the production and exchange of goods and services, is a trait that distinguishes us uniquely as human. By communicating our needs in terms of economic value, we have the ability to coordinate production with consumption. Since the time of Aristotle, scholars have recognized that we each assign goods a *use-value*, and from such individual values, society derives fair exchange or *market values* through the process of commercial activity. The great achievement of the Marginal Revolution was discovery of the relationship between the *rate of change* of the use-value one places on a quantity of a good, and the price one would be willing to pay for an additional unit of it - this price depending on the stock of goods one holds at the time.

This marginal relationship, however, can be articulated in two equivalent ways: If the function describing the consumer's use-value were known, the "price," or marginal rate at which the consumer would substitute one good for another, could easily be derived from her use-value by differentiation. If the consumer's use-value function is unknown, one must undertake the more challenging approach of deriving it from the consumer's MRS by integration.

[8] Backhouse (2010) p. 45

Following William Stanley Jevons[9], the economics profession has followed the former approach. Jevons' theory was a product of the psychophysiology of his time. Jevons regarded utility as a phenomenon that *actually existed in nature*. To Jevons, utility was a physiological sensation of satisfaction that the consumer experiences from consumption. Since publication of Paul Samuelson's *Foundations of Economic Analysis*, economists have recognized that utility cannot be seen as anything more than an abstract formalism[10]. Utility (and its ordinal equivalent preference) are still however implicitly scaled in terms of unobservable forces within the consumer's mind, and therefore cannot be compared interpersonally. As Lionel Robbins observed: "There is no means of testing the magnitude of A's satisfaction as compared with B's…Introspection does not enable A to measure what is going on in B's mind, nor B to measure what is going on in A's."[11]

Since utility cannot be measured, economists have relegated it to simply "tastes" which vary arbitrarily between individuals[12]. This restricts consumer theory to the study of a *single idealized consumer,*[13] whose preferences are unknowable. With regard to form of the consumer's utility function, the analyst can assume nothing beyond what is necessary to guarantee that the utility maximization problem has a solution. Since all conclusions must be drawn from the assumption of utility maximization alone, such conclusions are rendered general enough to hold for *any* consumer, regardless of the tastes or factors that might influence his judgment. Although the

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[9] Menger and Walras, who also introduced the marginal paradigm did not use utility in the same way, as will be discussed in Chapter 3

[10] Samuelson (1961) p.91

[11] Robbins (1935) p.139-40, See Also Hands (2001) p.36-37

[12] See Silberberg and Suen (2001) p.5,6

[13] Samuelson (1947) p.96

conclusions of such work are undoubtedly true, they represent only the beginning of what one could discover if he had the means of measuring trends in consumer behavior as a function of observable circumstances. Additionally, utility maximization is by definition static, rendering dynamic modeling of consumer behavior impossible.

Neoclassical consumer theory has, in essence, become a logical envelope within which the result of empirical analysis must fit. Even for a system as basic as Newtonian mechanics, there are no means by which Newton's laws of motion can be deduced from mere assumptions. There is no logic by which one might deduce that the force applied to a physical body, divided by its mass, will determine its acceleration. Without observation, one could not conclude that the gravitational force between two bodies is inversely proportional to the distance between them.[14]

Without content derived from empirical study, the mine of information extractable from the neoclassical model of consumer behavior was exhausted long ago. Samuelson's *Foundations,* for which he won the Nobel Prize, was not only the first concise formulation of consumer theory but also nearly its last. Discussions of consumer theory given in modern textbooks[15] are substantially the same as those given by Samuelson.[16] The difficulties in aggregating demand, presented in Mas-Collel's 1995 textbook[17] are essentially the same as those discussed by Alfred Marshall a century earlier[18]. Historian Ivan Moscatti concluded as recently as 2007, that the

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[14] Newton's laws are derived from empirical studies made by Galileo, Kepler, and Huygens over the previous century, see Boyer (1991) p.391-93

[15] See Mas-Collel (1995) pp.23-28 and Varian (1992) pp. 116-124

[16] Samuelson (1947) pp. 96-116

[17] This refers to the problem of non-constant wealth effect. See Mas Collel, et al. (1995) pp.106-08

[18] Marshall (1997) p.95

proceedings of the 1971 Minnesota Symposium: *Preferences, Utility, and Demand*[19] still represent consumer theory's state of the art[20]. Additionally, realistic tatonnement processes have never been discovered and the stability of equilibria have never been adequately demonstrated.

While one's satisfaction cannot be measured, the choices one makes can be. There are no barriers to making interpersonal comparisons between individual's MRS, though the meaning of such comparisons must be carefully considered. The use-value function derivable from the consumer's MRS is functionally equivalent to a utility function, though it's meaning is somewhat different. The use-value function is indeed an abstract formalism as Samuelson described, in the sense that it exists only by virtue of its mathematical definition. It does however have concrete meaning, its functional form is unique, and its value is cardinally expressible in terms of observable quantities. The function's intuitive meaning is best given in Carl Menger's words as: "judgments [that] economizing [individuals] make about the importance of the goods at their disposal for the maintenance of their lives and well being[21]. Much more will be said with regard to this in Chapters 2 and 3.

While economists may be unfamiliar with such "man made" quantities, physics is replete with them. Physicists define potential energy in terms of the (vector) electric and gravitational forces in the same way that use-value will be defined here from the MRS[22].

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[19]Chipman (1971)

[20] Moscatti (2007)

[21] Menger (2003) p.446

[22] Irving Fisher was the first to recognize the analog between utility and energy, though he did very little with it.

1.4 The Dynamic Model in Summary

The economy is modeled as containing $n+1$ commodities. The exchanges n of these commodities $x_i \quad i \in (1,2,\ldots n)$, using numeraire commodity x_N as "money"[23]. As will be discussed in Chapter 4, the numeraire commodity is subject to certain restrictions that allow it to be used as the standard of measure. The symbol $\vec{x} \triangleq (x_1, x_2, \ldots x_n, x_N)$ is used to represent the bundle of commodities the consumer holds. The set of all possible bundles defines an n-dimensional vector space[24].

In the dynamic model, the consumer acquires his or her bundle of goods through many differentially small transactions occurring over time. In a unilateral exchange (see Chapter 5), where a single consumer exchanges goods with "the market" at a fixed price, the consumer can be modeled as receiving income $I = dx_N/dt$ as a stream of numeraire increments over time. As each increment is received, the consumer exchanges it for goods so as to acquire a differential bundle $d\vec{x}$ as shown in Figure 1.5-1. For each transaction, the consumer's decision is based not only on his budget of numeraire and prevailing prices, but also *the stock of goods he has previously acquired*.

With regard to the consumer's behavior, two primary assumptions are made: First it is assumed that *for any bundle of goods a consumer possesses, he knows how much of one good he would be willing to exchange for an additional unit of any other. Furthermore, the*

[23] The numeraire does not have to be the medium of exchange (the consumer is free to barter) but it must serve as the unit of account

[24] There is no axis for the numeraire commodity for reasons that will be explained in Chapter 4.

consumer will tend to buy more of any good for which the price offered is less than he is willing to pay. Conversely the consumer will tend to sell any good for which the price offered is more than he is willing to pay. This assumption embodies the common-sense logic behind utility maximization, without prying into the consumer's thought processes. As will be shown in Chapter 2, such prying is not only unnecessary, but also potentially misleading. It is intuitive apparent that, under this assumption the consumer will attempt to exchange goods until the prices he is willing to pay equal the prices he is offered.

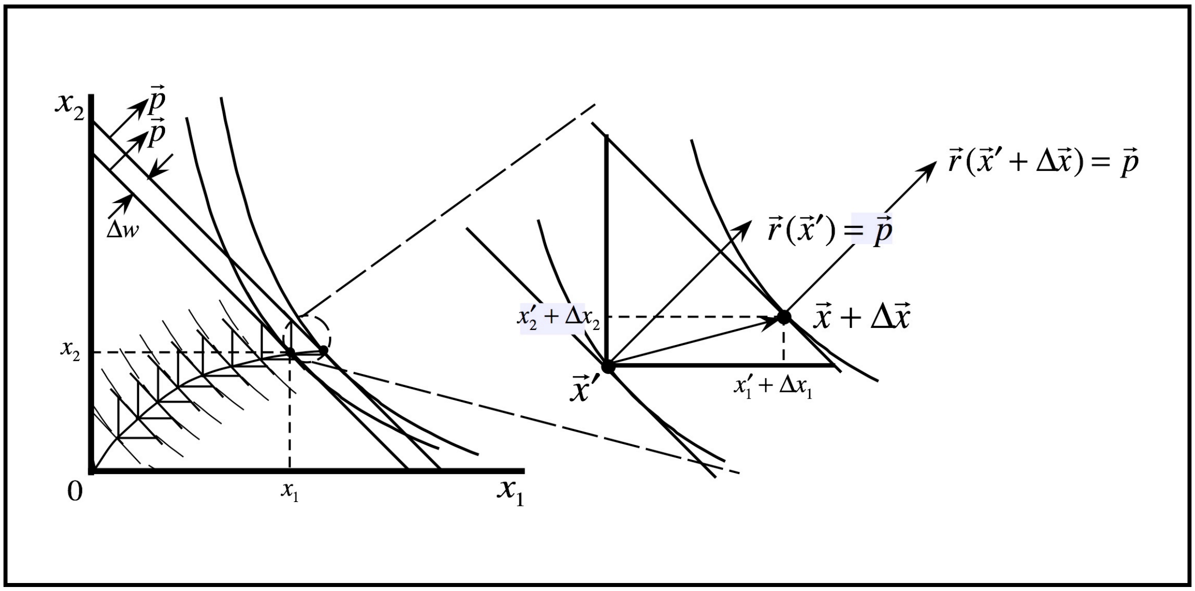

Figure 1.4-1 A Consumer's Bundle of Goods as Acquired Through Many Differentially Small transactions

The second primary assumption is *that the prices the consumer is willing to pay for additional units of goods, sold either individually or in sets, diminishes with the quantity he or she already has*. As will be shown in Chapter 4, this assumption is slightly stronger than *convexity of preferences,* which is generally assumed in current analysis[25]. As will be argued in Chapter 4, violation of this assumption on the part of an individual would manifest itself in the individual's self-destructive behavior as one suffering from an addiction.

Formally, he price the consumer is willing to pay for an additional unit good x_i is his MRS between the numeraire and that good. We will call such MRS, the consumer's *marginal price* r_i for the good in question. Since this marginal price r_i is a function of all goods available, we have:

$$r_i(x_1, x_2, \ldots x_n) = r_i(\vec{x}) \triangleq \frac{dx_N}{dx_i} \qquad (1.4\text{-}1)$$

The complete set of marginal price functions $r_i(\vec{x})$ for all goods x_i form the consumer's marginal price (vector) function $\vec{r}(\vec{x})$.

Since vectors can be represented graphically as arrows attached to the points of which they are a evaluated, vector *functions* can be represented as *streamlines* through such points as shown in Figure 1.4-2b. From basic microeconomic theory, we know that vectors representing the MRS are everywhere perpendicular to the indifference curves they cross, as shown.

In physics, streamline diagrams are used to illustrate the lines of force emanating from an electrically charged particle or from a magnet. Electric and magnetic fields are commonly represented by a combination of streamlines and indifference curves as shown in the figure.

[25] The clause "sold individually or in sets" is what guarantees that the second order, cross partial derivatives behave appropriately, as will be discussed in Chapter 4.

Another example of such diagrams are meteorological maps in which streamlines represent the flow of air currents, and indifference curves represent ridges of high and low pressure.

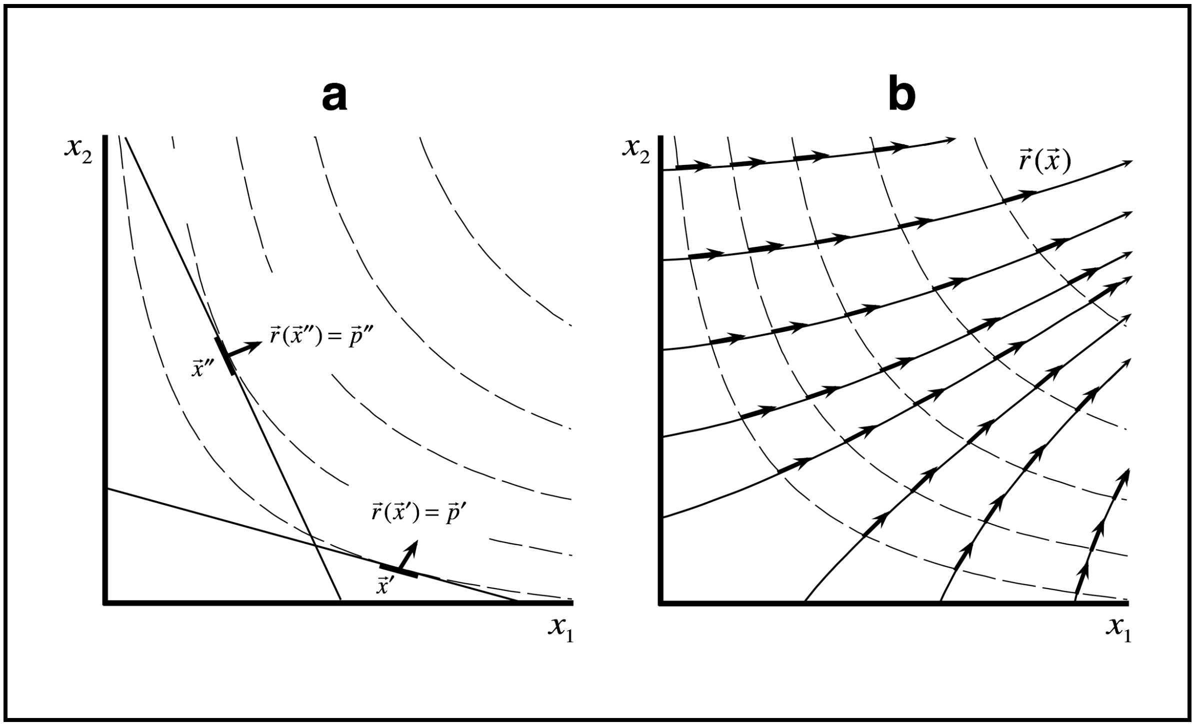

Figure 1.4-2 Stream Lines and Indifference Curves

As will be discussed in Chapter 4, *the use-value function* $V(\vec{x})$ is found by *vector line integrating* $\vec{r}(\vec{x})$ from some reference bundle $\vec{x}_0$ (commonly the origin) to the bundle $\vec{x}$ along some path[26] as shown in Figure 1.4-3. Use-value is the functional equivalent of utility. Though it is not needed to solve the consumer problem, it can be analytically useful. The locus of points x' for which $V(\vec{x}')$ is constant, form an indifference curve.

[26] As will be shown in Chapter 4, in order for the marginal price function to be economically realistic it must satisfy an assumption that guarantees that the integral will be independent of any particular path.

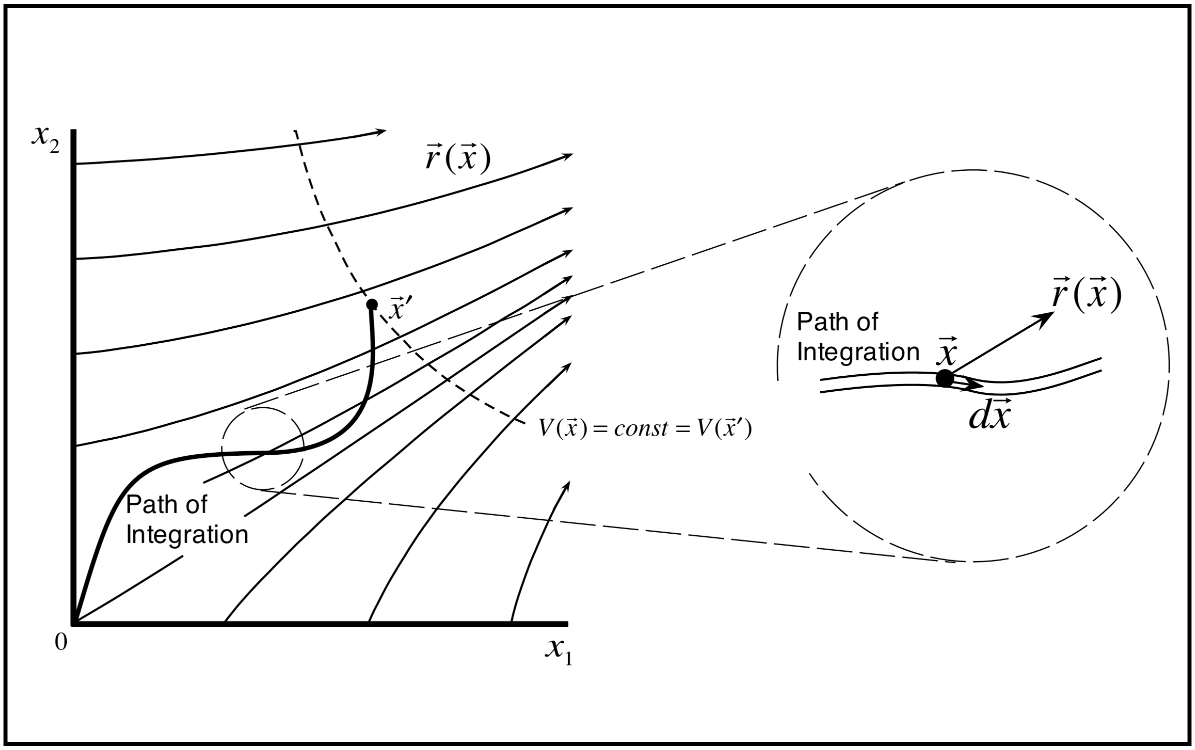

Figure 1.4-3 Path of Integration

1.4.1) Aggregation

Formally, the process of aggregation is very simple. The aggregate marginal price function is simply the average of the functions of the individuals. For a group of m consumers $k \in (1,2,\ldots m)$, each characterized by a marginal price function $\vec{r}^{\,k}(\vec{x})$ the aggregate marginal price function $\vec{R}(\vec{x})$, is given by:

$$\vec{R}(\vec{x}) = \frac{1}{m}\sum_{i=1}^{n} r^k(\vec{x}) \tag{1.4.1-1}$$

What this function means must be interpreted with some care. For any given bundle $\vec{x}'$, the aggregate marginal price vector is the average of the marginal prices each individual would have, *were he or she to currently hold* bundle $\vec{x}'$. The intuitive meaning of this will be discussed more thoroughly in Chapter 6. For the moment it can be said that $\vec{R}(\vec{x})$ represents the marginal price function of a single consumer who typifies the community from which the functions are aggregated.

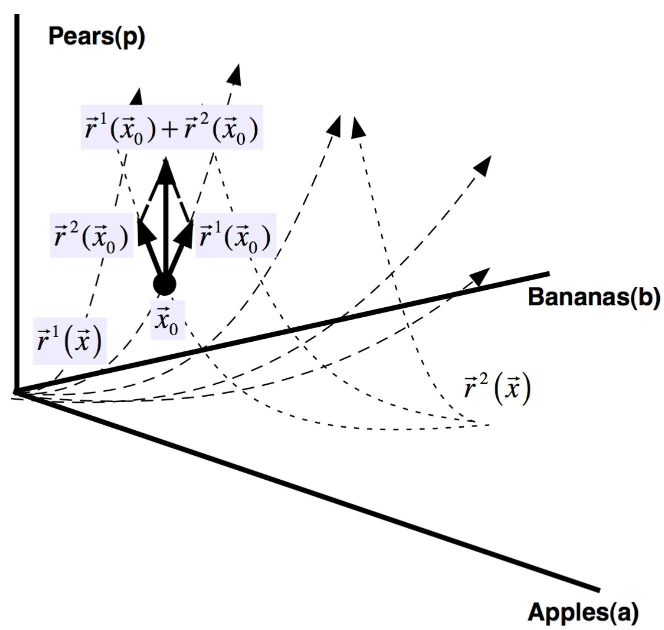

Figure 1.4.1-1. Addition of Marginal Price Functions for Consumers Holding Bundles consisting of Apples, Bananas, and Pears.

The notion of ones “bundle” $\vec{x}$ can be generalized to include not only commodities, but also any identifiable factor that can potentially influence the consumer’s decision-making. This would include demographic factors such as ethnicity, gender, age, and family life cycle[27]. The function $\vec{R}(\vec{x})$ therefore represents how individuals would, on average, respond to whatever social circumstances they might encounter. Each point (bundle) represents a different set of circumstances. The MRS associated with each point indicates the choices an average consumer would make if subject to those circumstances. From such aggregation, whatever common trends each demographic cohort might exhibit would be manifest, while the impact of arbitrary tastes and preferences would average away as random noise. This process of aggregation is simply a formalization of what has been dome by market researchers for many years.

1.4.2) The Aggregate Consumer

The dependence of prices and quantities on the distribution of resources is found by modeling all individuals with a common marginal price function, and subsequently each individual with his own bundle, as shown in Figure 1.4.2-1. Even if all individuals were "identical" and received the same income, the quantities of goods they would demand would differ according to their current circumstances (bundles). The aggregate quantities demanded are of course, the sum of the quantities demanded by the individuals *at the current time*, and thereby dependent on the current distribution of goods. If prices are determined endogenously, they too will depend on the current distribution of socioeconomic resources.

[27] Family life cycle includes stages such as single, married with young children, "empty nesters" and so on.

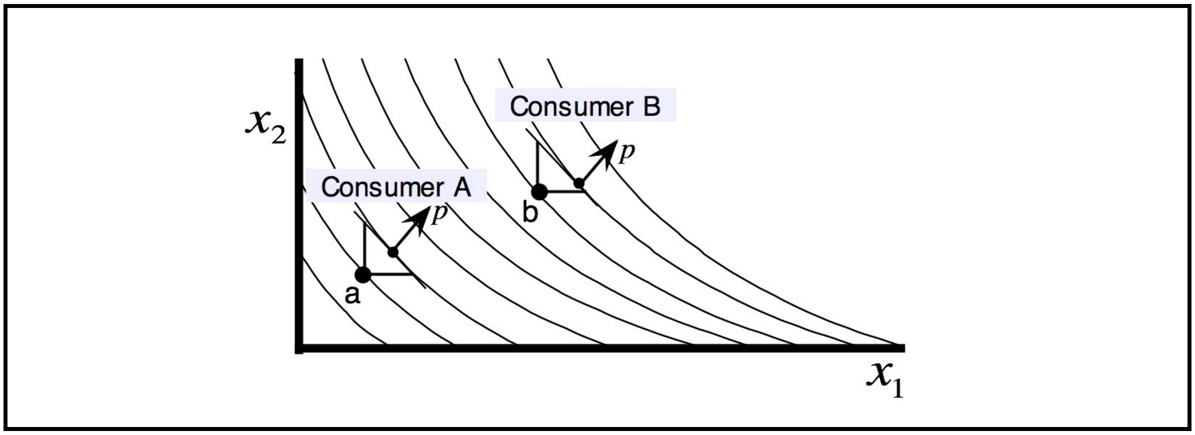

Figure 1.4.2-1 Given identical increments of income, and facing common prices p, consumers A and B, assumed to have identical indifference maps, will make different choices based on their initial bundles a and b.

1.5) How This Dissertation is Organized

Part I, consisting of Chapters 2 and 3 discuss the problems with utility maximization as they have evolved historically. Part II, consisting of Chapters 4 and 5 develop the dynamic model from basic assumptions through exchange equilibria. Part III applies the results of Part II to economies that can be viewed as populated with “aggregate consumers”.

Chapters 2 and 3 address the question: If the vector analytic approach is superior, why was it not adopted earlier. Chapter 2 will address the conceptual problems resulting from the presumption that a consumer’s choices derive from a force within the consumer’s mind. Such view has been taken to imply that satisfaction of such force (utility) determines the consumer’s welfare. Chapter 2 argues that utility can determine neither choice nor welfare.

Chapter 2 contrasts the marginalist view with the Greek and Medieval notion of value. This notion is intuitively similar to use-value defined here. While satisfaction of want was certainly part of the historic concept, one’s assessment of value was subject to the influence of one’s training and character. Since remarkably apt policy prescriptions flowed from the historic understanding of value, this chapter argues that the concepts of utility and preference add nothing themselves to consumer theory. Thus, nothing is lost when they are abandoned.

Chapter 3 traces concepts of value as developed in the Marginal Revolution. While the current concept of utility came from Jevons, his French counterparts were very suspicious of quantities that could not be measured. Walras' predecessors would have preferred to define value in terms of the consumer's willingness to pay as is proposed here. The notion of utility discussed by Jules Dupuit in 1849, was so close to the proposed definition of use-value that he arguably might have made such definition had he considered the multivariate case.

Chapter 4 will develop the dynamic model as it applies to an individual consumer. It begins with the founding assumptions, and proceeds through the formal definition of marginal prices and use-value. It ends by clarifying the relationship between marginal prices and demand, and by defining marginal demand as a new analytic quantity.

Chapter 5 presents models of dynamic exchange equilibria. These begin with the unilateral exchange, which is the foundation of most consumer theoretic analysis. It proceeds to bilateral exchange in which many goods are exchanged between two consumers. It concludes with multilateral exchange, where many goods are traded between many agents. In all three of these cases the equilibria are shown to be unconditionally stable.

Chapter 6 aggregates the results of Chapter 4 for an individual consumer into corresponding functions for a typical or *aggregate* consumer. Functions measuring the demand and welfare of an entire community are defined in terms of the aggregate consumer and the distribution of demographic factors throughout the community. This chapter also discusses practical aspects of aggregation.

In conclusion, Chapter 7 will use the tools developed here to explore the impact of wealth inequality on aggregate welfare and productivity.

PART I:

UTILITY VS. VALUE - HISTORICAL BACKGROUND

CHAPTER 2:
UTILITY, VALUE, AND WELLBEING

"In short, it [the pursuit of wealth] is in Political Economy what gravitation is in physics…the ultimate fact beyond which reasoning cannot go, and of which every other proposition is merely an illustration." [1]

Nassau Senior

"The love of money is at the root of all evil."

1 Timothy 6:10 (KJV)

One who has spent his or her life immersed in neoclassical theory may feel somewhat ill at ease with dismissing so central a quantity as utility. While it will return in a functionally equivalent form called use-value, it will no longer figure as prominently. Instead of being the foundation upon which the theory is built, it will be a somewhat useful quantity existing by virtue of its mathematical definition in terms of other things. One might ask if by demoting utility (preference) anything is lost from the theory. The answer is "no". Even in the hands of Jeremy Bentham, its inventor, utility was no more than an idea, described only in somewhat loose intuitive terms. The same can be said for the notions of economic value that has been with us for centuries. The presumption that utility (preference) actually exists in nature as an identifiable quantity has in fact been very misleading, as has been argued by leading social choice and welfare theorists.

This chapter will examine the problems inherent in the assumption there exists such a thing as utility, which consumers seek to maximize. The most apparent problems with this notion are: first; that it implies that human behavior is exogenously determined by nature to be greedy and hedonistic, and second; that such behavior necessarily maximizes both the individual's and

[1] Senior (1965) p.28

society's well-being. In contrast, Greek and Scholastic thinkers, were able to form an apt understanding of economic processes, without making any such presumption. They believed that humans are truly free to pursue whatever goals and values they choose. On the other hand, there was no presupposition that the path chosen would lead the chooser, or anyone else for that matter, to prosperity or happiness. If one wished to attain well-being, he must learn the proper path of action and discipline his passions in order to follow it. The goal of their teaching was to provide a guide for wise choice on the part of the individual, and advice for policymakers. As will be shown, the reduction of mankind from *homo-agens*, the purposeful creature, to the pleasure driven *homo-oeconomicus*, resulted from inappropriate application of scientific method by political economists of the mid 19th century.

2.1) Bentham's Utility

For a description of utility, we will turn to its author, Jeremy Bentham. Though his work is considered a bit antiquated, it still provides a canvas against which the objections of modern thinkers can be hurled. It should be emphasized that our problem with utility is not with Bentham, but with the way economists have used his ideas since. Bentham was a quintessential maverick; one who challenged the authority of lawmakers, and helped found the University of London as a haven for academics at odds with the Church of England[2]. According to Cambridge Scholar Mary Warnock, Bentham was indeed one of the first to challenge the validity of law from an ethical standpoint. His project was to found a system of "scientific jurisprudence". This would require a criterion by which an observer could evaluate the performance of an institution. Warnock, in her introduction to a volume containing Bentham's *Introduction to the Principles of Morals and*

[2] See Warnock (2003)

Legislation[3], points out that his life's work was twofold: He was concerned first with providing a theoretical foundation for legal systems, and second, with criticizing existing systems in light of that foundation. "In practice, this programme amounted, in large measure, to a testing of existing systems of law by the criterion of the 'principle of utility' "[4]. This principle was inspired by a maxim found in Joseph Priestley's *Essay on Government.* Priestly regarded government as "good" if it provides "the greatest happiness for the greatest number"[5]. As his interest was in a theory of government, not of individual behavior, Bentham was not interested in explaining *why* an individual might behave in such a manner; only *how* such a behavior might impact the well-being of others and thus be of concern to lawmakers. Bentham would have defined the action of an individual intent on improving his own happiness at the expense of society, as "mischief", the opposite of utility[6].

Bentham begins his *Principles* with a flamboyant depiction of mankind forever chained between the pillars of pleasure and pain[7]. His style leads one to believe that his concept of utility is both simpler and more sensual than he actually intends. He spends the entirety of Chapter 5 of his *Principles* describing the different aspects of utility, which he defines as "interesting perceptions"[8] of various pleasures, and the absence of their opposing pains. The pleasures he describes can be grouped into *Physical Satisfactions*, and *Psychic Satisfactions* as shown in

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[3] Bentham (2003) p. 45

[4] Warnock (2003) p.4

[5] Warnock (2003) p.1

[6] Warnock (2003) p.5

[7] Flamboyance consistent with having his remains preserved by a taxidermist to be ever present at board meetings at the University of London.

[8] Bentham (2003)

Figure 2.1-1. The physical satisfactions can be broken into additional categories that include *Satisfaction of Wants*, and Bentham's *Pleasure of Relief.* The satisfaction of wants include Bentham's P*leasures of Sense*; the gratification that follows from consuming food, entertainment and the like; and the *Pleasures of Wealth*[9] which include the security one enjoys from possessing the means of insuring his own safety and comfort. Modern psychologists such as Abraham Maslow would regard this as satisfaction of a basic human need that goes beyond mere sensual experience[10].

Bentham's Pleasure of Relief is an "interesting perception" indeed. It is the "pleasure which a man experiences, after he has been enduring pain of any kind for a certain time, when it comes to cease, or to abate."[11] This is a temporary experience, resembling what Amartya Sen calls the "small mercies" that the battered wife, the hopeless beggar, and the hardened unemployed enjoy.[12] Sen questions the efficacy of such a transient sensation as an indication of the person's long-term welfare.

The *Psychic Satisfactions* are among Bentham's more subtle pleasures. These can be grouped into three categories. The *Pleasures of Thought*, which include Bentham's *Pleasures of Good Memories*, *Pleasant Expectations of the Future*, *Pleasant Associations Between Objects and Past Happy Experiences*, and *Pleasures of the Imagination*[13]. Among Bentham's

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[9] Bentham (2003) p. 45

[10] Maslow (1943)

[11] Bentham (2003) p. 47

[12] Sen (1987) p.45

[13] Bentham (2003) p. 47

Psychic satisfactions are his *Pleasures of Benevolence*, and *Pleasure of Malevolence,* which are of particular interest. Bentham's Pleasures of Benevolence include the pleasure one obtains from the knowledge that one has acted compassionately or ethically. This of course relates to what Pattanaik and Harsanyi's refer to as ethical, as opposed to subjective, preferences, which were discussed in Chapter 1.

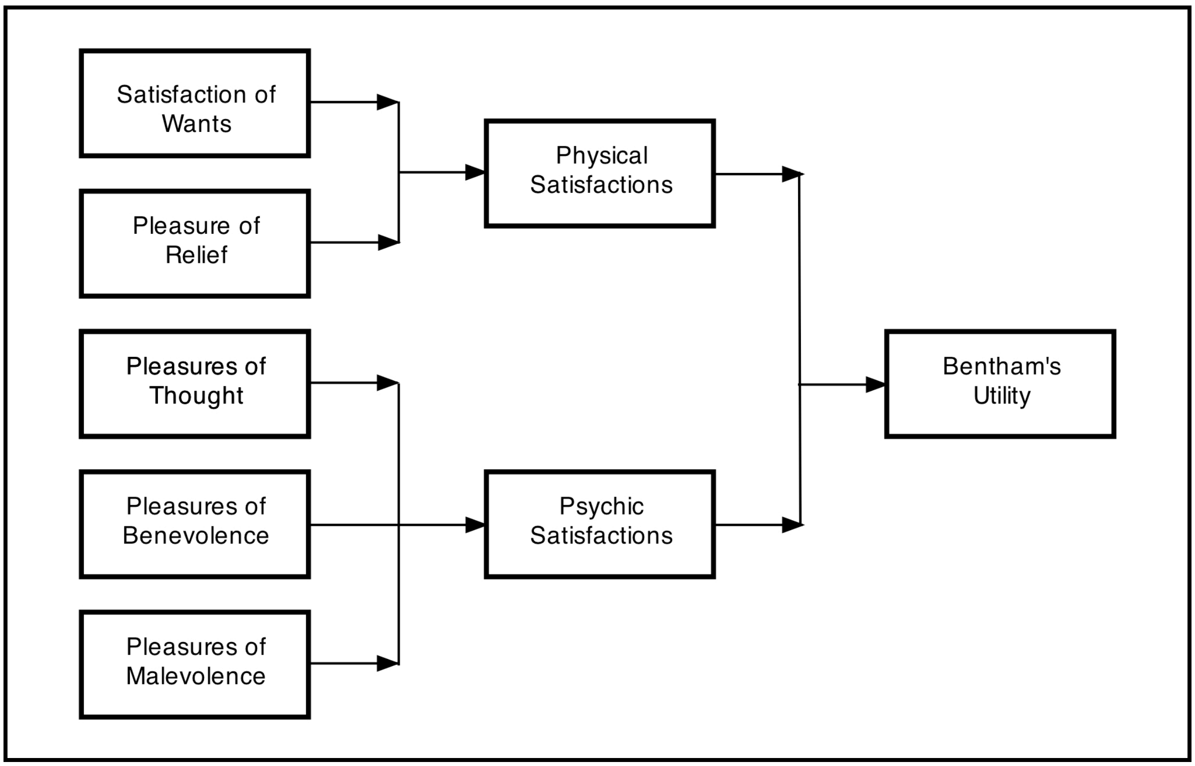

Figure 2.1-1) A Simplified Taxonomy of Bentham's Utility

The *Pleasures of Malevolence* are what one enjoys when seeing harm come to "one who is the object of the observer's malevolence"[14]. This pleasure is one that might be derived from witnessing a public execution. For our purposes, we include Bentham's *Pleasure of Power*, i.e.

[14] Bentham (2003) p. 46

the “being in the condition to dispose people by means of their hopes and fears”[15] among the Pleasures of Malevolence.

From the preceding discussion, one can easily see how such perceptions can influence one’s decision making. One can also see how the experience of such sensations can often indicate the state of one’s welfare. Problems arise when we presume that such *always* determine ones choices and well-being, or that they are the *only* determinants of such. If both such presumptions are made, the circumstance one is observed to choose must be taken as that which best enhances his well-being. Should a man choose a greasy hamburger and an order of French fries over a vegetable plate, an economist would have to assume the burger and fries are the better option, although it is doubtful that the man’s cardiologist would agree!

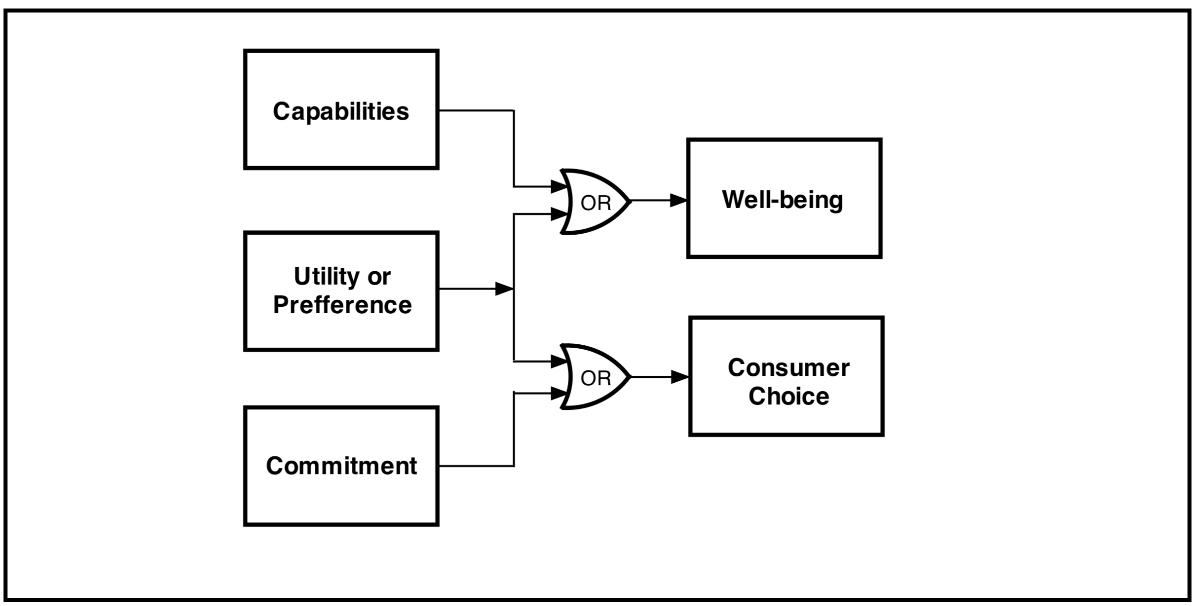

Figure 2.1-2 Utility contributes to both a consumer choice and his well-being but uniquely determines neither.

[15] Bentham (2003) p. 46

The argument against this use of utility is shown diagrammatically in Figure 2.1-2. Maximization of utility is shown as contributing to both an agent's choice and well-being, but not uniquely to either. Pareto was among the first to criticize the notion that such perceptions imply well-being. Pareto dismissed utility as being "*simple ophelimity*", a measure of an individual's arbitrary passions. Amartya Sen carries that argument even further. According to Sen: "Well being is ultimately a matter of valuation, and while happiness and fulfillment of desire may well be valuable for a person's well being, they cannot - on their own or even together – adequately reflect the value of well-being."[16]

When utility maximization is taken as the sole source of well-being, Pattanaik identifies two classes of problem: one of *inclusion* and one of *exclusion*. Problems of inclusion occur when the assumption of utility maximization would cause choices to be included that should not be considered as welfare enhancing[17]. An example would be pursuit of Bentham's Pleasure of Malevolence. This can be seen as a special case of the pleasure one might derive from being "nosey". In *Collective Choice and Social Welfare,* Sen asks if the welfare of an agent A is really enhanced if his neighbor B sleeps on his side as opposed to his back, should the A have preferences over B's behavior[18]. Extension of this argument to the enjoyment one might obtain from discrimination against minorities is quite apparent.

Pattanaik's problems of exclusion occur when the assumption of utility maximization might cause options to be excluded that should be considered as welfare improving. An example Pattanaik

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[16] Sen (1987) p.46. See also Sen (1980 and 1985)

[17] Pattanaik (2009) p.328-334

[18] Sen (1970) p. 79

discusses is the case where one may not prefer classical music it one is not so educated[19]. If one is deprived of opportunities to hear such music, are we to consider his loss of well-being the same whether or not he has been taught to appreciate such music? A less subjective example would be that of an aboriginal tribesman who has no access to, or knowledge of, western medical care. We would clearly regard such an individual as deprived, whether he was aware of his deprivation or not.

Both Pattanaik and Sen comment on the need for a standard of well-being that is determined objectively by expert observers, rather than subjectively by the agents themselves. Sen's version of such is a broadly defined set of *capabilities* that the individual may achieve. These include longevity, freedom from morbidity, access to education, and the like. While individuals may naturally desire capabilities such as longevity, the choices necessary to their achievement (such as the foregoing of French fries) may not bring happiness to the individual in the short term.

While it is now clear that neither happiness nor utility, no matter how broadly defined, can be considered synonymous with well being, one might still argue that utility maximization is what drives an agent's choices. The *Satisfaction of Wants* and the *Pleasures of Thought* clearly motivate consumer behavior. Marketing managers have long recognized that Bentham's *Pleasures of Memory* motivate one's purchase of photographic equipment while his *Pleasures of Association* motivate the purchase of souvenirs and memorabilia. Bentham's *Pleasure of Expectation* is recognized by Revlon Co., which defines its business as selling "hope" rather than cosmetics. [20]

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[19] Pattanaik (2009) p.336-337

[20] Kotler (1994) p.69

To answer the question of whether utility is the *only* motive that drives a consumer's choice, we compare what Bentham calls the Pleasure of Benevolence, to what Sen calls *Commitment*. If one could make the argument that people behave ethically because it gives them pleasure to do so (or pain in the form of guilt if they do not), then one might be able to assert that utility maximization is the sole motivator. Sen does not accept this. He leaves open the possibility for agents to make choices based on ethical commitment, which he defines as involving "the counter-preferential choice"; of an option that is not "…better than (or at least as good as) the others for the person choosing it"[21]. Sen distinguishes between commitment and benevolence or *sympathy*. The well-being of a sympathetic person is directly impacted by the well being of another. For example, "If knowledge of the torture of others makes you sick, it is a case of sympathy; if it does not make you feel personally worse off, but you think it is wrong and you are ready to do something to stop it, it is a case of commitment."[22] If we allow for the possibility of committed behavior, we can no longer consider utility to be the sole determinant of an agent's choices.

2.2) Value in Greek and Scholastic Thought

Greek and Scholastic thinkers were well aware of the complex aspects of human decision making to which Pattanaik and Sen refer. The fact that the choices one makes are the product of one's experience and training is the motivation for much of their teaching. In Henry Spiegel's *Growth of Economic Thought*, Aristotle's position is interpreted as saying that "People can be changed by the proper environment, by suitable institutions, and by the power of persuasion, and if they

[21] Sen (1977) p.328

[22] Sen (1977) p.326

become better men, the economic problem of pervasive scarcity of material goods will be less oppressive."[23]

The notion that an individual necessarily makes choices to enhance his own wealth or pleasure is one that the Aristotle and his colleagues would have soundly rejected. Solon (Chief Magistrate of Athens c. 590 BC) cautioned his citizens against the destructive power of greed as follows:

> But men of the city themselves, hearkening to the call of wealth, are minded by their folly to destroy a mighty city…For they no not know how to check their greed or to order the good-cheer that they have, in the quiet enjoyment of the feast[24].

Epicurus (341- 270 BC), a successor of Aristotle and an anticipator of Bentham's calculus of pleasure and pain, taught his followers to discipline their minds so as to temper their desires with judgment. "If you wish to make a person wealthy, do not give him more money, but diminish his desire."[25]

The goal of Greeks and Scholastic economic thought was to influence public policy as well as individual behavior to promote social well-being. Lionel Robbins points out that Aristotle's most significant discussions of economic principles were made in context of their relevance to the rest of society. His most significant writings on the subject appear in his *Politics*,[26] which, like Bentham's *Principles,* was intended to be a guide for good government. Aristotle regarded the state as an extension of the household. What contributes to good household management also

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[23] Spiegel (1991) p.25

[24] Gordon (1975) p.8

[25] Spiegel (1991) p.38

[26] Robbins(1998) p.18

contributes to good government. The term *Oeconomicus,* from which the word "economics" is derived, literally translates as household management. The term, which comes from two Greek words *οlχοζ* (house) and *νο'μοζ* (rule) refers to the art of arraignment of the family's material goods for providing "the good life". According to Aristotle, "Property is part of the household, and the art of acquiring property is a part of managing the household; for no man can live well, or indeed live at all, unless he is provided with necessaries."[27]

In keeping with his distaste for greed, Aristotle scorned the art of acquiring wealth for its own sake, or *chrematistics* as "unnatural" [28] since one's need is finite, one's greed is not. Aristotle's view of economics was later to be given a Christian ethical twist by the Scholastics. According to Sir Alexander Gray, the scholastic position was that: "We are all brothers and should behave as brothers, respecting each other's right and position in life. Each should receive that to which he is entitled. … No one under any circumstances should take advantage of his neighbor." To Gray, "this is the sum and substance of medieval teaching."[29]

It is apparent from the preceding discussion that Aristotle, and the Scholastics who followed him, would have rejected any attempt to attribute value to a single causal source such as utility, no matter how broadly it is defined. To the Greeks, value assessments, whether they were made by an individual or society as a whole, are a matter of judgment. One should (and therefore one is able to) discipline one's passions with wisdom based on experience, training, and ethical commitment.

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[27] Aristotle (2003a) p.7

[28] Spiegel (1991) p.25

[29] Gray (1959) p.46

It has been popular among authors with viewpoints as diverse as Robert Heilbronner[30] and Jacob Viner to regard writers of this period as mere "anticipators" of economics theory, if they regard at all. This is partially due their "pre-scientific" style of reasoning. As will be argued at the end of this chapter, it is the misapplication of scientific method that has brought about the identified problems.

Writers of Aristotle's day cannot be viewed as necessarily naive with respect to commercial matters. Schumpeter argues that the reason they did not engage more in economic theorizing is that they found it mundane and obvious. As Barry Gordon points out in *Economic Analysis Before Adam Smith*: "Almost all the phenomena associated with modern market economies were present: commercial agriculture, manufacturing, business consortiums and monopoly trading." Commodity speculation was a feature of economic life in the port cities as banking companies "exerted powerful influence". Such banking firms "received deposits, made payments for clients, undertook debt recovery, issued letters of credit, and invested in business ventures."[31]

2.2.1 Aristotelian Use-Value

The concepts of value upon which economic reasoning of this era was built were common sense notions of what an article is "worth". Concepts similar to use-value and exchange-value as used here make their appearance in Aristotle's *Politics*. To Aristotle, the use-value of any item follows from its "primary purpose", its usefulness to its owner or prospective purchaser. Such item carries exchange value by virtue of the fact that it may be exchanged in the market: "…a shoe is used for wear and it is used for exchange. He who gives a shoe in exchange for money or food

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[30] See Heilbronner (1999) pp.22-23

[31] Gordon (1975) p.11

to him who wants one, does indeed use the shoe as a shoe, but this is not its proper or primary purpose, for a shoe is not made to be an object of barter."[32]

In his *Rhetoric*[33] Aristotle mentions qualities such as durability, security, and capacity to serve men in all seasons as sources of use-value. In 1905, Oskar Krauss[34] suggested that Aristotle realized that use-value diminishes marginally. For this he relies on Aristotle's *Topics*, 118: "A thing is more desirable if, when added to a lesser good, makes the whole a greater good. Likewise, you should judge by means of subtraction: for the thing upon whose subtraction the remainder is [made] a lesser good may be taken to be a greater good."

In *Rhetoric,* Aristotle also recognizes the interplay of usefulness and scarcity in a manner that resembles Adam Smith's Water-Diamond Paradox. "What is rare is a greater good than what is plentiful. Gold is a better thing than iron though it is less useful: it is harder to get and therefore better worth getting. Reversely, it may be argued that the plentiful is a better thing than the rare, because we can make more use of it. For what is often useful surpasses what is seldom useful, whence the saying, 'The best of things is water.'"[35] Beyond such intuitive concepts of serviceability, Aristotle does not attempt to define use-value. He leaves it as a common sense notion.

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[32] Aristotle (2003a) p.7

[33] Aristotle (1952) p.593

[34] See Footnote #2 in Spengler (1955) p.371 for the citing of the German language text of Krauss' work.

[35] Spengler (1955) p.376-77

2.2.2) Exchange and Exchange Value

For Aristotle, the exchange-value of a good is established from the use-values assessed it by parties involved in the exchange. in his *Nichomachean Ethics* Aristotle introduces the notion of *reciprocity* or the "exchange of equals."[36] Here Aristotle defines exchange-value in terms of the use-values of the respective traders. Exchanges take place between unequal individuals with complementary talents. "For it is not two doctors that associate for exchange but a doctor and a farmer, or in general people who are different and unequal; but these must be equated."[37] It is quite apparent according to Soudek and Spiegel[38] that what must be equal is the degree of want satisfaction the exchanged goods provide. "All goods must be measured by some one thing…now this unit is in truth *need* or *demand* [39], which holds all thing together."[40]

Aristotle's notion of reciprocity is given in the following example: "Let A be a builder and B a shoemaker, C a house and D a shoe. The builder must get from the shoemaker the latter's work, and must himself give him in return his own. If first, there is a proportionate equality of goods, and then reciprocal action takes place, then the result we mention will be affected.[41]

[36] Aristotle (2003b) p.14

[37] Aristotle (2003b) p.14

[38] See Soudek (1952) p.46 and Spiegel (1991) p.32

[39] The word for "need" used here is η΄ χρει΄α which can also be translated as "demand". In this case Soudek feels that "need" is a better translation. See Soudek (1952) p.60

[40] Aristotle (2003b) p.15

[41] Aristotle (2003b) p.14

Aristotle's example is usually written symbolically as $A : B :: C : D$, or[42]:

$$\frac{A}{B} = x\frac{D}{C} \quad . \qquad (2.2.2\text{-}1)$$

The x term on the right is readily seen as the exchange ratio of shoes for houses. It is the term on the right, the ratio of the builder to the shoemaker, that classical political economists interpreted as the ratio of the labor expended by the relative parties. Recent writers[43] point out that it is labor's *product* that is exchanged. The ratio on the right of Equation 2.2.2-1 is of the *needs of the respective traders*. We can thus rewrite Equation 2.2.2-1 as:

$$\frac{V(L_A)}{V(L_B)} = x\frac{D}{C} \quad . \qquad (2.2.2\text{-}2)$$

where $V(L_A)$ is understood as the use value that A's labor produces. If, for example one house normally exchanges for a thousand shoes in the market, the use value a builder's labor produces is a thousand times that of a shoemaker[44]. Soudek identifies this relationship as a "value theory of labor". In commenting on *Ethics*, Aquinas said that the criterion "which measures all truthfully is need, because it embraces all exchange goods insofar as they are related to need. … The price of saleable things does not depend on their rank in nature … but on their usefulness to man." [45]

The ethic behind Aristotle's "exchange of equals" principle is reflected by Aquinas in *Summa Theologica.* There he writes; "Purchase and sale are seen to have been introduced for the

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[42] Spiegel (1991) p.31
[43] See Soudek (1952) p. p.46 and Spiegel (1991) p.32

[44] Aristotle does not consider the time spent by the laborers on their respective tasks.

[45] See footnote 41 in Gordon (1975) p.176

common utility of both parties, since one needs the goods of the other … but what was introduced for the common good ought not be more of a burden on the one than the other; and so the contract between them ought to be established according to an equality."[46]

When such exchange involves many traders, the scholastics saw the process as bringing about a consensus as to the value of the goods considered. According to Bernadine of Sienna (1380 – 1444), "Price is a social phenomenon and is set not by the arbitrary decision of individuals but by the community."[47] The phrase *communis aestimatio* used of "community estimation" was used interchangeably by the scholastics with *aestimatio fori* or "market valuation". [48]

2.2.3) Scholastic Policy and the "Just Price"

The scholastic thinkers of the middle ages based their thought on Aristotle, applying to it the ethical norms of Christianity. Their innovation over Aristotle's work was their concept of the "just price" which reflects a form of exchange value. Aquinas is interpreted as defining the just price to be: "the [price], which at a given time, can be gotten from buyers, assuming common knowledge and in the absence of all fraud and coercion."[49] From their arguments however it is apparent that prices determined by supply and demand were not appropriate under all circumstances. The goal of the scholastic doctor's foray into economics was to craft policies that protected the marketplace from speculative trading practices, and to insure price stability in times of famine or glut. Even without the influence of scientific method that shapes neoclassical economic theory, the

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[46] Summa Theologica II-II, Q77, art. 1, quoted in Gordon (1975) p.174

[47] De Roover (1958) p.423

[48] De Roover (1958) p.424

[49] This is De Roover's interpretation of Aquinas. See De Roover (1958) p.423

scholastics were remarkably perceptive in their policy prescriptions. Economic chaos resulting from the collapse of speculative bubbles plagues us whenever our regulations give way to the practices that the scholastics forbade.

Attempts to manipulate the market were a common problem during the scholastic period. There are many recorded prosecutions of traders who engaged in practices such as "*engrossing, forestalling, and regrating. Engrosing* refers to the accumulation of a commodity in attempt to corner the market, while *forestalling* is the purchase of stocks of goods before they reached the market for which they were intended. *Regrating* is speculative buying in a given market with the intention of selling the same commodities in the same market at a higher price."[50] The scholastics were unanimous in their condemnation of any form of conspiracy to fix prices above or below the competitive level.[51] They also condemned the craft guilds for their tendency to set prices "for their singular profit and to the common hurt and damage to the people."[52]

Additionally, violent price swings were a major source of hardship. J. Gilchrist cites the price of wheat in England as rising ten fold between 1287 (a good year) and 1315, which marked the beginning of a famine. Additionally, Gilchrist cites that in 1497 the price of wheat in Florence nearly doubled, then returned to its original price within the space of a single month.[53] Such price

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[50] Gordon (1975) p.220

[51] De Roover (1958) p.426

[52] De Roover (1958) p.432

[53] Gilchrist (1969) p.87

fluctuations could have a catastrophic impact on the poor, many of whom were reported as committing suicide over inability to buy bread at the prevailing price.[54]

The stabilizing policies, which the scholastics advised were set at the township level. Each township functioned as an independent economy with its own sets of tariffs and trade regulations.[55] In time of dearth, town officials stepped in to place price ceilings on staples as a means of preventing riots. Occasionally, public stores of grain were stocked in times of plenty, for resale to the poor in times of famine[56].

Maintaining price stability for the protection of tradesmen was more important than a modern observer might think. This is due to factor immobility, particularly that of labor. Peasants were tied to the land, and an artisan's occupation was chosen by tradition. Generally, one's trade was the same as his father's and of his father's father regardless or the misfortunes the marketplace might bring him. Aquinas' assertion that the just price should cover "labor and costs" is a policy prescription, lest "the arts …be destroyed if prices are not so determined" [57]. Debates of this period over the effectiveness of regulation bear resemblance to the debates of the present. On one side was British theologian John Duns Scotus (1265 – 1308) who taught that the just price should be sufficient to compensate the producer for his costs, including, transportation and risk undertaken in bringing his goods to market. [58] Duns Scotus was later denounced by Francisco Vitoria (c. 1480 – 1546) of the School of Salamanca. According to Vitoria, "inefficient producers or

[54] Gilchrist (1969) p.87

[55] Spiegel (1991) p.52

[56] Development economists in recent years have advocated this practice in third world countries.

[57] Summa Theologica II-II, Q77, art. 2 and 3, quoted in Gordon (1975) p.176

[58] Gordon (1975) p.223

unfortunate speculators should simply bear the consequences of their incompetence, bad luck, or wrong forecasting."[59] In an argument that anticipates the view of the Chicago School, Martin Azpilcueta (1493 – 1587), also known as Navarrus opposed price regulation as he found it "unnecessary in times of plenty and ineffective or harmful in times of dearth"[60]

2.3) Economic Methodology

The practice of attributing value to a single entity such as utility began with the view of science held by Political Economists of the mid 19th century. According to V. W. Bladen, in his introduction to the Toronto Press edition of Mill's *Principles of Political Economy*, Mill sought to create a "pure scientific theory" of political economy, from which the practical "art" of Political Economy could be informed[61]. As result of this effort came the hypothesis that "considers [man] *solely as a being who desires to possess wealth*, and who is capable of judging the comparative efficiency of means for obtaining that end".[62] This hypothesis has since come to be known pejoratively as *homo-oeconomicus* or "economic man" [63] and is the forerunner of the utility maximization hypothesis. As benign as this assumption might at first seem, it is one that Aristotle would have regarded as not only ludicrous but dangerous. Its presupposition constrains political economy into the realm of chrematistics, ignoring the wisdom that should be central to economics.

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[59] De Roover (1958) p.424

[60] De Roover (1958) p.426

[61] Bladen (1965) pp.xxvii – xxi

[62] Mill (1874) pp.137-9

[63] Mill (1874) pp.137-9

The hypothetico-deductive or *a-priori* methodology has held sway up through Milton Friedman's essay of 1953 which, according to Mark Blaug, "every modern economist has read at some stage in his or her career"[64]. From the viewpoint of a physical scientist, this methodology appears extremely odd, if indeed it can be considered science at all due to its lack of connection with the facts. In Mill's view, scientific knowledge consists of a set of propositions or *laws*, from which future events can be predicted. These laws are discovered by induction from experience. Such experience could consist of controlled experiment, such as in physics, direct observation such as in astronomy, or by the method, *a-priori,* which begins with hypotheses derived from introspection or reason[65]. Modern scientists would have little problem with the first two means of induction, the third is cause for reservation. Unfortunately, it is this third category into which Mill places Political Economy.

> We can go farther than to affirm that the method a-priori is a legitimate mode of philosophic investigation in the moral sciences; we contend that it is the only mode. We declare that …[the experimental method]…is altogether inefficacious in those sciences, as a means of arriving at any considerable body of valuable truth[66].

Those trained in recent years in the physical sciences would have been influenced by Karl Popper. Popper divides statements into two categories: *synthetic* and *analytic*. Synthetic statements are those that can, in principle at least, be verified or falsified by experience. Analytic statements on the other hand, are those whose validity is based on their logical structure[67]. Such are valid if logically consistent with their premises, whether such are "true" in an empirical sense

[64] Blaug (1992) p.90

[65] Hands (2001) p.20

[66] Mill (1874) p.145 (Quoted in Hands (2001) p.21

[67] Blaug (1992) p.12

or not. In Popper's view, it is the body of synthetic statements that constitute science. As Einstein has often been quoted: "Many a beautiful theory has been destroyed by an ugly fact."[68]

The disconnection between Mill's a-priori hypotheses and fact places this method in the analytic category. If Political Economists had been willing to test their hypotheses by requiring that their predicted results be confirmed by observation, their theory would have some synthetic content, though that content would be weak. While one may easily say that some observed consequent (B) *sometimes* results from an assumed antecedent (A), it does not follow that (A) will *always* give rise to (B) or that *only* (A) can produce (B).

Mill places Political Economy in a class with geometry, whose hypotheses or axioms are not subject to question. Today of course, geometry is considered purely analytic, and a part of mathematics, not science. With regard to human nature Mill writes:

> Geometry presupposes an arbitrary definition of a line…Just in the same manner does Political Economy presuppose an arbitrary definition of a man, as a being who invariably does that by which he may obtain the greatest amount of necessities, conveniences, and luxuries with the smallest quantity of labor and physical self denial with which they can be obtained in the existing state of knowledge[69].

Mill is careful to point out that this is an abstraction from the way agents actually behave, "Not that any political economist was ever so absurd as to suppose that mankind are really thus constituted, but because this is the mode in which science must necessarily proceed."[70] Mill regards political economy as an "abstract science "which is true in the abstract but will only be

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[68] Clarke (1971)

[69] See Bladen 1965

[70] Mill (1874) pp.137-9

true in concrete cases with inclusion of the proper specific allowances."[71] D. Wade Hands in his *Reflections without Rules* points out that the "laws" of political economy must be regarded as *Tendency Laws,* which in real life are riddled with exceptions or *disturbing causes*. Using Ricardo as an example, Mill states that Malthusian population theory and the differential fertility of land will produce a *tendency* for the rate of profit to fall, not that it will do so in any given case. A problem that a physical scientist would have with this use of disturbing causes is that the Political Economist is too quick to write them off as unknowable. This places the assumptions upon which political economy (and subsequently economics) are based beyond refutability. In his *Methodology of Economics*, Mark Blaug complains that whenever predictions based on the assumptions of economic theory fail to conform to the facts "diligent research … will always reveal some ad hoc disturbing causes that must bear the blame for the discrepancy."[72]

Had physicists of the late 19th century dismissed their disturbing causes as easily as did political economists, the great discoveries of the early 20th Century would have never been made. Nineteenth century physicists assumed that light waves propagated through an invisible medium, called the *ether*, which filled all of space. Based on that assumption, Newtonian physics predicts that the speed of light will appear to vary with the earth's movement through the ether. The celebrated Michelson and Morley experiment failed to detect any such variation. The "disturbing cause" in this case is explained by Einstein's theory of special relativity.

Physicist's insist that their theories, no matter how counter-intuitive they may be, must fit the facts. This insistence tends to render their theories as transitory as wildflowers in the desert. With every probe that is sent out into the solar system, unexplained phenomena are discovered

[71] Mill as paraphrased in Hands (2001) pp.22-23

[72] Blaug (1992) p.76

rendering present theories obsolete. Minute deviations in the trajectories of those spacecraft from their predicted courses are now calling Einstein's theories into question.

2.4) Conclusion

It appears the scientific methodology inherited from Mill, Senior, Carnes, and others has saddled us with the belief that we must find some natural "law" by which we can explain the values humans attach to their goods and services. It has been shown that the maximization of utility or "rational choice", no matter how we might define it, will not completely account for the choices individuals make, or the well-being they enjoy from such choices. Aristotle and his Scholastic followers were able to develop an apt, though not in our sense "scientific" understanding of economic processes. Finally, we have seen that it is our attempt to make political economy, and henceforth economics a science that causes us problems.

Like the physical scientists, economists have sought to discover natural laws from which observable phenomena can be explained. Unlike physical scientists, economists propose irrefutable hypotheses, rather than abstract laws from the nature we observe. The result is an analytic system that is more science fiction[73] than science. This is not to propose that axiomatic theorizing is without merit; far from it. Nor can we build a theory that is entirely free of hypothetical laws. We can however get closer than we are today.

By basing, our theory on observed behavior, most propositions can, at least in principle be falsified. In the coming chapters there will be many occasions where criteria for choice and well-

[73] A good science fiction story contains a logically consistent set of events that occur according to plausible, but unreal premises; the more plausible the events, the more interesting the story. No one however expects such events to anticipate reality, though they sometimes do as with Jules Verne's work.

being will be hypothesized. Along with these hypotheses, the means by which they can be tested will be presented.

CHAPTER 3:
QUESTIONS OF VALUE IN THE MARGINAL REVOLUTION

"The abstract idea of wealth or value in exchange ...must be carefully distinguished from the accessory ideas of utility, scarcity, and suitability…which the word wealth still suggests in common speech. These ideas are variable and by nature indeterminate, and consequently ill suited for the foundation of a scientific theory."

A. Augustin Cournot (1836) [1]

The perspectives of the marginalist thinkers in Britain and on the continent differed widely. British founders of the Marginal Revolution followed Mill's deductive or a-*priori* method. They sought to explain economic behavior in terms of principles of human nature that could be determined by introspection. This approach is what gave birth to the notion that there exists in nature an essence such as utility that determines human behavior. Mill had made special exception to his generally prescribed *inductive* approach for "sciences of the mind". In his view, these sciences followed laws not subject to inductive study. Introspection was the only way such laws could be discovered.

The French, who began their work long before Mill wrote, were generally suspicious of hypothetical essences. They looked outward to observable social processes that could be studied in terms of collected data. Whatever they might conclude regarding human nature would have to be drawn inductively from the behaviors they observed. Their concept of value retained the Aristotelian notion of a judgment call. In the work of Jules Dupuit, the y came quite close to

[1] Cournot (1960) p.10

actually defining their concept of use-value in terms of observed behavior as this dissertation proposes.

While Carl Menger's work may have been the least quantitative in the formal sense, his version of the law of diminishing marginal utility was the most advanced.

3.1) Utility as the Product of Human Physiology: The Work of Wm. Stanley Jevons and Francis Ysidro Edgeworth

The project of William Stanley Jevons (1835-1882) was to found a theory of political economy on the workings of the human mind. Jevons followed Mill's deductive method, but broke with Mill's extra-physical view of mind. To Jevons, thought was a product of the brain, a physical organ subject to physical laws.

Jevons was keenly interested in the mechanics of human thought. He had studied the binary logic that underlies modern computer science, from its inventor, George Boole.[2] As a means of demonstrating that a biological instrument could carry out reasoning, Jevons constructed his own *Logical Machine*, perhaps the first truly digital computer implementing Boole's logic[3]. This invention can be considered an early foray into what would later come to be known as artificial intelligence.

To model the brain's ability to compare variable quantities such as pleasure and pain, Jevons followed contemporary work in psychophysiology. In the opening passages of his *Theory of*

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[2] Maas (2005) p.123-28

[3] Babbage, whose work Jevons studied is usually credited with building the first computer. His device was more of a mechanical calculator rather than a machine based on binary logic. See Maas (2005) p.128-136

Political Economy, Jevons declares, *"value depends entirely upon utility"*.[4] His utility, however, is not that of Jeremy Bentham, but a physiological sensation such as considered by Richard Jennings. Jennings' *Natural elements of Political Economy* drew from work in psychophysiology from both Brittan and Germany.[5] This work attempted to measure nervous system responses, such as the reaction time to the application of heat, as in the case of Carpenter in Britain, or the rate at which responses to repeated stimuli diminish, as in the case with Fechner in Germany. Jevons' *Degree of Utility*, what Marshall would later call marginal utility, reflects Fechner's diminishing response to stimulus. Jevons introduces his degree of utility with the following illustration:

> "Let us imagine the whole quantity of food which a person consumes on average during a twenty-four hour period to be divided into ten equal parts. If the food be reduced by the last part, he will suffer but little; if a second part be deficient, he will feel the want distinctly; the subtraction of the third tenth part will be decidedly injurious; with every subsequent subtraction of a tenth part his sufferings will be more and more serious until he will at length be on the verge of starvation. Now if we call each of the tenth parts an increment, the meaning of these facts is that each increment of food is less necessary, or possesses less utility than the previous one."[6]

In Figure 3.1-1, reproduced from his *Theory*, Jevons represents the degree of utility obtained from each increment by the area of its corresponding rectangle. The *total* utility experienced is the sum of the areas of the rectangles. In Figure 2.1-2, he argues that if the number of increments becomes arbitrarily large and their sizes arbitrarily small, the total utility becomes the area under the curve corresponding to the degree of utility.

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[4] Jevons (1970) p.77

[5] Maas (2005) p.10

[6] Jevons (1970) p.106

Jevons' great accomplishment was of course recognition of the marginal relationship between total utility and the degree of utility, and that the degree of utility diminishes with consumption. In terms of analytic expression, Jevons does not appear at all wedded to the idea that the degree of utility must be derived from total utility by differentiation. He even implies that the degree of utility is observable, making it the natural starting point. "I hesitate to say that men will ever have the means of measuring the feelings of the human heart…but it is the amount of these feelings that prompts us to buying and selling… *and it is from the quantitative effects of these feelings that we must estimate their comparative amounts*."[7] These "quantitative effects" are the ratios with which the consumer would be willing to exchange one good for another.

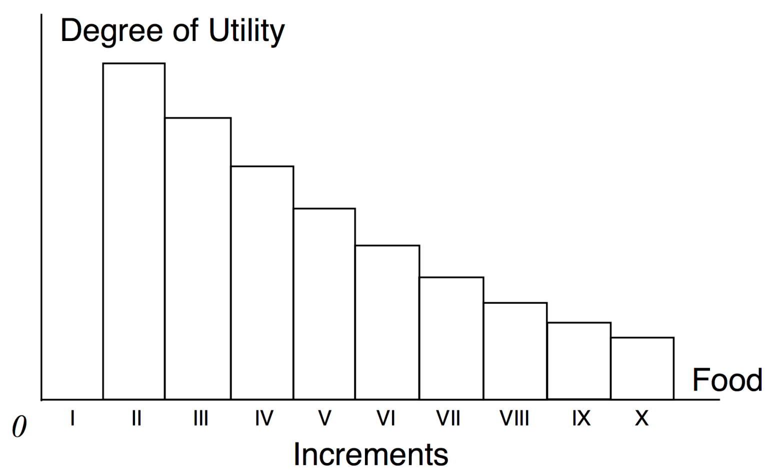

Figure 3.1-1[8] Jevons' Degree of Utility

> "Imagine that there is one trading body possessing only corn, and another possessing only beef. … Suppose for a moment, that the ratio of exchange is approximately that of ten pounds of corn for one pound of beef: then if, to the trading body that possesses corn, ten pounds of corn is less useful than one pound of beef, that body will desire to carry the exchange further. Should the other body possessing beef find one pound less useful than ten pounds of corn, this body will also be desirous to continue the exchange. Exchange will go on

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[7] Jevons (1970) p.83 [emphasis is in original text]

[8] Figure is adapted from Jevons (1970) p.107

> until each body has obtained all the benefit that is possible, and loss of utility would result if more were exchanged. Both parties, then, rest in satisfaction and equilibrium, and the degrees of utility have come to their level, as it were." [9]

By introducing a non-observable quantity to explain the behavior of observables, his system would always have one more variable than knowable quantities. Had he not been able to eliminate this unknown from the final equation of exchange, his work would never have gotten past the raised eyebrows of his contemporaries. Had Jevons really believed that the consumer's degree of utility for some good A could be measured in terms of his willingness to exchange it for another good B, he could have defined the consumer's degree of utility for a unit of A in terms of B. That however would have required him to regard utility as merely an explanatory idea as did the French. By presuming that utility is a phenomenon *that exists in nature*, he could not assume that it varies predictably with any other quantity.

The physiological interpretation of utility becomes particularly problematic in the work of Francis Y. Edgeworth. In his *Mathematical Psychics,*[10] Edgeworth references the same psycho-physiological work, as does Jevons.[11] Edgeworth emphasized the likelihood that physiological responses vary unpredictably between individuals. According to Edgeworth, "If we were to follow Bentham's precepts would this not mean that one's compensation does (or should) be in proportion to his capacity to experience pleasure?" Edgeworth speculates that some, in fact, do receive higher wages than do others due to their inherently greater capacity for enjoyment.[12]

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[9] Jevons (1970) p.139

[10] Edgeworth (1967)

[11] Edgeworth (1967) pp. 56-63,

[12] Edgeworth (1967) p. 64

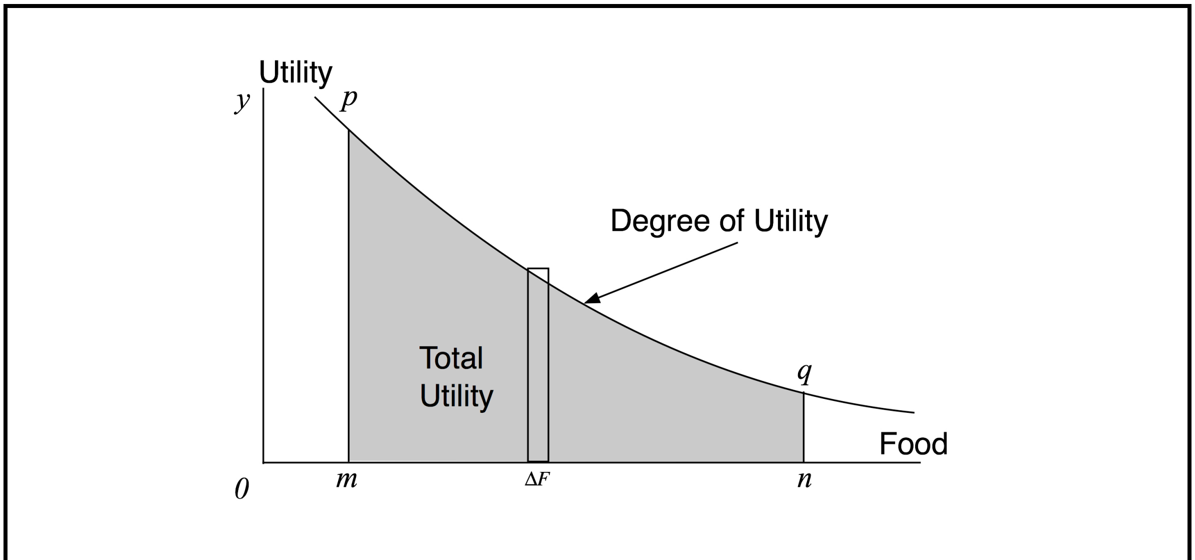

Figure 3.1-2[13] **Jevons' Degree of Utility vs. His Total Utility**

3.2) Use-Value as Derived From Demand: The Marginal Revolution in France

The French precursors upon whom Walras drew were primarily interested measuring social phenomena without speculation into the psychology of why they happen. Leon Walras (1834-1910) and his predecessors used the term "utility" to express the loose concept of use-value, as understood since Aristotle. By leaving use-value as an abstract notion, they allowed for it to be defined operationally and measured indirectly in terms of demand. Walras himself made little use of utility per-se, building his theory on demand directly. In his *Elements of Political Economy-Pure,* Walras argued that there would normally be a unique set of prices, for which the demands for all goods in the marketplace would simultaneously equal their supply. It was not until after he had demonstrated the validity of this argument that he made any mention of utility[14].

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[13] Figure adapted from Jevons (1970) p.108

[14] Ingrao and Israel p.93

Ingrao and Israel trace the origins of Walras' approach back to the Physiocrats of the late Eighteenth Century.[15] Due to their concern with practical matters, these writers were more empirically oriented than were Jevons and Edgeworth. Early social choice theorists such as the Marquis de Condorcet (1743-1794), whose work will be discussed later in this paper, pioneered collection of demographic data regarding healthcare and the like. Walras' precursors were often members of the French Engineering School.[16] Such engineers were responsible for the maintenance of roads, bridges and other public works, and it was their responsibility to secure the necessary resources from the population for their construction. As result, they were concerned that the benefits warranted the costs, particularly when such costs were paid in the form of the hated *corvée,* a form of conscripted labor[17].

From the time of the Physiocrats, these thinkers avoided speculating about human motivation. Turgot commented: "I do not wish to investigate how pleasure and pain…influence the determination of the will. I merely say that we find in experience only one principle productive of movement, and that is that the will of intelligent beings which is not primitively determined but determines itself.[18]"

Rather than focusing on individual actors, these investigators studied interpersonal processes. They followed the flow of value through the economy, drawing analogies between it and the flow of blood through the body. According to Nicholas-François Canard:

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[15] Ingrao and Israel pp.42-46

[16] Blaug (1996) p.303

[17] Jupp (1999) p.vi

[18] Ingrao and Israel (1990) p.49

> "The product of labor circulates in all channels of this system of ramifications like a liquid and everywhere attains its own equilibrium. Each vessel causing the product of labor to circulate and is accompanied by an analogous vessels causing money to circulate in the opposite direction; and the system of the circulation of money and labor as a whole resembles the circulation of blood.[19]"

Anne-Robert Jacques Turgot (1721-1787) was among the first to challenge the notion that a good's exchange-value (market value) was determined by its cost of production.[20] Turgot's argument was conceptually the same as Jevons' law of exchange, though far less detailed. In Turgot's model, each party to an exchange assesses in his own mind the relative use-values of the goods in question. When traders meet, each is willing to give up what he values less in exchange for what he values more. It is through the process of bargaining that the relative exchange-value is determined[21].

Turgot's model was mathematically formalized somewhat by Achylle-Nicholas Isnard (1749-1801), an engineer, and Nicholas-François Canard (1750-1833), a high school teacher of mathematics. In their models, the value that traders placed on the goods in question came to be measured implicitly by their willingness to pay for them. Canard postulated that for an exchangeable good, there would be a *maximum* price the purchaser would pay, and a *minimum* price for which the seller would part with it. Between these prices was bargaining latitude within which the sale price will be negotiated.[22] If, as Jevons were to later propose, the traders were to exchange the commodity in increments, the latitude would diminish with each increment

[19] Quoted in Ingrao and Israel (1990) p.68

[20] Jupp (1999) pp.i-x

[21] Turgot (1999) pp. 14—15

[22] Ingrao and Israel (1990) pp. 66-72

exchanged[23]. Canard's model would thus reduce to Jevons' law of exchange, but without reference to Jevons form of utility.

A. Augustin Cournot (1801-1877) was likely the most influential of the French writers, and the one who placed French political economy on a solid scientific foundation. Walras referred to Cournot as his mentor and referenced his work frequently[24]. Jevons and Marshall were also significantly influenced by his work, with Jevons referring to himself as the first Englishman to have read Cournot.[25]

A message that pervades Cournot's 1836 *Researches into the Mathematical Principles of the Theory of Wealth* is that science is "about" discovery of relationships between measurable quantities. His anticipation of Mill's inductive, as opposed to deductive, method is evidenced by his emphasis on observation. "Observation must be depended on for furnishing the means of drawing up between proper limits a table of the corresponding values of [sales quantities] and [prices]; after which by well known methods of interpolation or by graphic processes, and empiric formula or curve can be made to represent the function in question.[26]" Cournot's dependence on mathematical functions, " which may not be capable of algebraic expression", was his means of articulating exactly what was, as opposed to what was *not*, knowable in the absence of data.

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[23] Ingrao and Israel (1990) pp.71,72

[24] Ingrao and Israel (1990) p.91

[25] Ingrao and Israel (1990) p.78

[26] Cournot (1960) pp. 47-48

Cournot's *Law of Demand* was essentially a law of sales, since "…we do not see for what reason theory need take account of any demand which does not result in a sale"[27]. This law was an empirically verifiable relationship between the prices and the quantities of goods sold.

Cournot devoted the second chapter of his *Researches* to the relative nature of measurement. Empirically, one cannot speak of some quantity A without there being at least one other quantity B against which it is measured. If there are n quantities, there are at most $n-1$ independent comparisons that can be made between them. As long as measurements can be made consistently, such a set of quantities can be measured relative to one another. One quantity is chosen as the standard, and the values of the other quantities are given in terms of it.[28] Cournot would have rejected Jevons' attempt to attribute use-value to a physiological force as unnecessary, were it possible to define use value operationally in terms of quantities already observed. Such definition is what Dupuit was to propose a decade later.

Jules Dupuit (1804-1866) was a civil engineer charged with estimating the value provided by public works. Dupuit argued, as had J.B Say previously[29], that the value one placed on the consumption of a good was measured by his willingness to pay for it. This value is generally greater than the price one is *actually* is required to pay. In his paper of 1849, he offered a spirited defense of the measurability of use-value, as he conceived it:

> "Here is a person who needs a kilogram of meat, that is, who is willing to make a sacrifice to obtain it. The butcher says the kilogram of meat is worth one franc. Two things can happen: either he buys it or he does not. If he does, I shall ask him whether he would have bought it at twenty-one sous, then at twenty two, at

[27] Cournot (1960) p.46

[28] Cournot's chapter does not express this idea in quite this way.

[29] Ingrao and Israel (1990) p.74

twenty three, and so on. I do not think it would be an abuse of the word "obviously" if I say that, by so doing, I would gradually ascertain the maximum price at which he would be willing to buy his kilo of meat."[30]

If, as is shown in Figure 3.2-1, the toll for a bridge were to be lowered from p to p', the public would certainly use it more. Dupuit argued that the public would be willing to pay *over and above* what they were actually charged, the amount given by the shaded area under the demand curve in Figure 3.2-2. This quantity, which is what Alfred Marshall would later call the consumer's surplus[31] can be measured, at least in principle, by having the consumer purchase his goods from a perfectly discriminating monopolist.

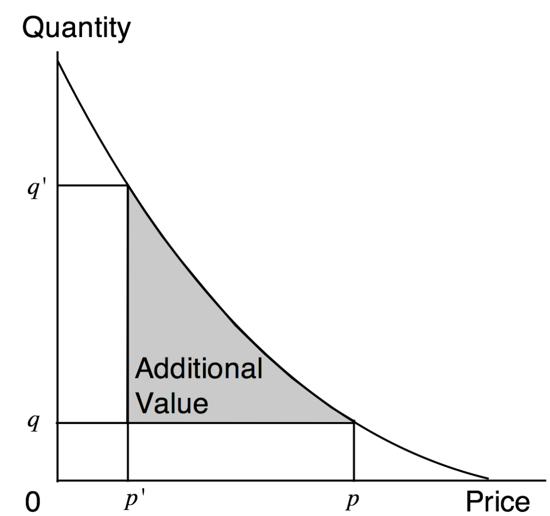

Figure 3.2-1[32] DuPuit's Use-Value

[30] Ingrao and Israel (1990) p.76

[31] In his development of consumer's demand, Marshall references Dupuit's work. See Marshall (1997) p.101

[32] Adapted from Blaug (1996) p.305

As is the case with Marshall's consumer's surplus, Dupuit has implicitly defined this additional value to be the integral of demand curve from p' to p, less the toll paid $q \bullet (p' - p)$.

Dupuit's value is essentially, a quantity that exists by virtue of its mathematical definition. We understand it in terms of the added opportunity cost a consumer would be willing to bear, but we have no knowledge of (or concern for) whatever "force" might prompt the consumer to bear the additional cost. It can be intuitively interpreted as the use-value understood since the time of Aristotle.

3.3) The Reconciliation that Never Was

The end of the 19th century saw a glimmer of reconciliation between the two different concepts of value. While Alfred Marshall downplayed the idea of utility as a physiological phenomenon, it remained in the analysis as a distinct quantity separate from those that could be observed. Though efforts to derive use-value in terms of demand were attempted, lack of understanding of the necessary mathematics resulted in much confusion and little progress. Ultimately, the concept of use value disappeared, and utility was replaced with ordinal preferences. While the adoption of preferences might reduce one's qualms with regard to utility's unquantifiablity, they do nothing to improve the analysis. They in fact introduce their own set of problems

3.3.1) Marshall's Synthesis

Alfred Marshall (1842 -1924) combined the work of his predecessors with his own original thought, but was not able to reconcile the differing motions of value. Marshall retains Jevons' notion of utility as "correlative to Desire or Want",[33] yet he concerns himself with it only as far as it

[33] Marshall (1997) p.92

produces an observable result[34]. Marshall does not assume, as might Jevons, that one can predict a consumer's behavior from an understanding of his thought processes. He in fact chastised Jevons for "…having led many of his readers into confusion between the provinces of Hedonics and Economics…"[35]

Marshall renames Jevons' degree of utility as *marginal* utility since it measures the utility provided a consumer by the *marginal purchase* made when "he is on the margin of doubt whether it is worth his while to incur the outlay required to obtain it."[36] Marshall defines the price the consumer is just willing to pay for his marginal purchase as his *marginal demand price*. He then states his law of diminishing marginal utility: "The larger amount of a thing that a person has, the less, other things being equal …will be the price he will pay for a little more of it: or in other words his marginal demand price for it diminishes."[37]

Marshall was not quite willing to go as far as Dupuit did. Had he followed Dupuit's lead, he could simply have replaced Jevons' degrees of utility, shown in Figure 2.1-1 with the corresponding marginal demand prices r_i[38] that the consumer is observed to pay for each increment. The total utility of the food considered in Figure 2.1-1 would simply be the sum of the marginal demand prices for each respective addition as given by Equation 3.3.1-1

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[34] See Marshall (1992) p.16

[35] Marshall (1992) footnote on p.101

[36] Marshall (1997) p.93

[37] Marshall (1997) p.95

[38] r is used for marginal demand prices to distinguish them from market prices p

$$U = \sum_{i=II}^{X} r_i \qquad 3.3.1\text{-}1$$

If the increments of food F are taken arbitrarily small as shown in Figure 1.1-2, and we replace the incremental prices r_i with a continuous marginal price function $r(F)$, we would have the integral:

$$U(F) = \int_m^n r(F)dF \qquad 3.3.1\text{-}2$$

Had Marshall taken these steps, his utility would have simply been Dupuit's notion of value, with no necessary connection to a sensation of satisfaction. Following this line of reasoning, from Jevons' law of exchange, we know that the rate at which the consumer is willing to exchange good x_n for good x_i (given by dx_n/dx_i) is the inverse of the ratio of their marginal utilities MU_i/MU_n. At equilibrium, this equals the market price p_i of x_i in terms of x_n.

$$\frac{dx_n}{dx_i} = \frac{MU_i}{MU_n} = p_i \qquad 3.3.1\text{-}3$$

If x_n is used as the numeraire, dx_n/dx_i is also, by definition, the consumer's marginal demand price r_i. If there are n goods available, the consumers marginal demand prices would equal the market prices, or:

$$\frac{r_1}{p_1} = \frac{r_2}{p_2} = \cdots = \frac{r_{n-1}}{p_{n-1}} = 1 \qquad 3.3.1\text{-}4$$

Marshall, however, wasn't willing to go as far as to establish any given numeraire as a standard of measure. His *equimarginal rule* went only so far as to say that the consumers marginal utilities are *proportional* to market prices, and the constant of proportionality is the (unknown) marginal utility of the numeraire:

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \cdots = \frac{MU_{n-1}}{p_{n-1}} = MU_n \qquad 3.3.1\text{-}5$$

Marshall was not quite ready to abandon the notion of utility as a phenomenon that must be measured against its own scale. Marshall leaves us open to consider that the marginal utility of whatever numeraire we might choose, varies with respect to some external standard.

Marshall recognizes two practical problems with using a numeraire good as the standard of measure. The first problem he was able to solve, and the second he was not. The first problem is that, depending on the numeraire chosen, the law of diminishing marginal utility may be violated. We can see why this might be so, from a thought experiment provided by Irving Fisher. In this experiment, a consumer is asked to purchase increments of bread using milk as payment. [39] Depending on whether the consumer's need for milk or bread diminishes faster, the consumer's marginal utility for bread, as measured with respect to milk, may increase or increase.

Allowing only fiat money, or currency, to be used as numeraire, solves the problem.

Such "money" M is a commodity that has properties distinct from other goods. Since its only value to the consumer is in its ability to be exchanged for other goods, its value is determined by whatever goods the consumer may wish to purchase. If the law in question can be reworded to

[39] Blaug (1996) pp.313-314

say that the marginal utility of any good diminishes *with respect to consumption in general*, then the use of money as numeraire solves this problem.

The question now arises: Can the diminishing marginal utility of money *itself* have any meaning? If so, *with respect to what* might the marginal utility of money diminish, and how can we detect it? [40] In Marshall's view, it appears to vary with respect to other goods as the consumer's wealth changes. While this *income effect*, familiar from basic microeconomics is very real; Marshall's interpretation of it is an illusion.

Marshall observes that poorer individuals are much less willing to spend money on luxuries than richer ones. "We have seen how a clerk with £100 a year will walk to business in a heavier rain than a clerk with £300 would."[41] He recognizes that an extra schilling can meet a greater need for the man who is poorer. "A rich man in doubt whether to spend a shilling on a single cigar, is weighing against one another smaller pleasures than a poor man, who is doubting whether to spend a schilling on a supply of tobacco that will last him a month. …If a poorer man spends [£1], he will suffer more from the want of it afterwards than the richer would.[42]…A stronger incentive will be required to induce a person to pay a given price for anything if he is poor than if he is rich".[43] From this assessment, Marshall concludes that: "… every diminution of his

[40] If the author says something in the forest, and his girlfriend is not there to hear him, is he still wrong?

[41] Marshall (1997) p.95

[42] Marshall (1997) p.19

[43] Marshall (1997) p.19

resources increases the marginal utility of money to him, and diminishes the price he would be willing to pay for any benefit."[44]

The illusion of the diminishing marginal utility of money is due in part to Marshall's additive utility functions, which could not consider substitution effects. Marshall assumed that the utility gained from consuming any one good did not depend on the quantities of other goods consumed. Carl Menger on the other hand, anticipated the interplay of income and substitution effects, and in a very novel way. In Menger's model, the rich man would *appear* to value money less, because he placed less marginal value on the goods he sought to acquire, as result of the larger stock of goods he has *already acquired*. For a clearer view, we consider the illustrations in Menger's *Principles of Economics.*

<u>3.3.2) Use-Value and Hierarchical Need: The Work of Carl Menger</u>

Menger's version of the law of diminishing marginal utility anticipates the work of mid 20^{th} century psychologist Abraham Maslow[45], Menger predicts that all humans first seek goods that satisfy their survival needs, followed by others that improve the quality of life. Finally they seek goods that provide only a passing pleasure. The first quantities of the most urgent goods consumed such as food, satisfy survival needs, while further acquisitions of these same goods satisfy less urgent needs. According to Menger:

> "Men consume food for several reasons: Above all, they take food to maintain life; above this they take further quantities to preserve health, since a diet sufficient to maintain life is too sparing, as experience shows, to avoid organic

[44] Marshall (1997) p.96

[45] Maslow (1943)

> disorders; finally, having consumed quantities sufficient to maintain life and preserve health, men further partake of food simply for the pleasure derived from their consumption."[46]

Menger's example with food closely parallels that given by Jevons (See Figure 1-1). Menger's extension of this argument to the *types* of goods individuals seek introduces the interdependence between goods.

> "We observe that men fear the lack of food, clothing, and shelter much more than the lack of a coach, a chessboard, etc. … The maintenance of life depends neither on having a comfortable bed nor having a chessboard, but the use of these goods contributes, and certainly in very different degrees, to the increase of our wellbeing. Hence there can be no doubt that, when men have a choice between doing without a comfortable bed or doing without a chessboard, they will forgo the latter much more readily than the former."[47]

Menger illustrates this idea graphically with the table 3.2-1. Goods such as food (I) that meet the most basic of needs can provide a maximum use-value of 10. Housing (II), though not absolutely necessary for survival, is certainly vital to life's quality and hence can provide a maximum use-value of, say, nine. A comfortable bed (III) might provide a maximum value of eight; a chessboard (VII) would provide four, while an evening at the opera (IX) might provide maximum value of only two. The key to understanding the tradeoffs that consumer's make between goods can be found in the way the marginal values of all goods diminish with consumption. A destitute individual would first seek only food, obtaining a value of ten for his first increment. The second increment of food would provide a value of only nine, equaling the value of the first increment of bedding. Hence, after the first increment of food is consumed, the individual would be indifferent between an additional unit of food and an initial unit of bedding, and would most likely consume both in equal quantities.

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[46] Menger (2003) p.449-50

[47] Menger (2003) p.449

To illustrate how this concept addresses Marshall's diminishing marginal utility of money, consider his example of the clerk whose frequency of cab rides depends on his income. Assume that a cab ride (in the worst possible weather) provides the clerk a maximum value of seven, placing it under Column IV in the table. It does not sustain life as well as food, shelter, bedding or the like, but cab rides in inclement weather can provide some protection against pneumonia. To even consider a cab ride, the clerk must have sufficient funds to purchase all goods upon which he places a value of 8 or higher. Only at that point would he consider a cab ride, to which he would be indifferent to a *second* increment of bedding, a *third* increment of housing, or a f*ourth* increment of food. If all goods were priced at $1 per unit, the clerk would need an income of $10 as shown by wealth line A in Figure 3.3.2-1[48]

If the consumer's wealth were increased to $28, he would be indifferent to taking a third cab ride as opposed to purchasing a chessboard. It is not that he would value a dollar any less if he has $28 as opposed to $10. It is rather the fact that his un-met needs would be fewer. For a man with $10, the opportunity cost of a cab ride (his first in this case) is a fourth increment of food. For a man with $28, the opportunity of a cab ride (his fourth) is a *seventh* increment of food, or his first chessboard. From modern indifference curve analysis one can readily see that the income effect on any one good depends on the other goods present. For a given consumer, goods A and B might appear inferior in the presence of another good C, while appearing to be normal when they are the only goods present.

[48] Note that in this illustration, the budget line has a different interpretation that in standard microeconomic analysis. The budget indicated by the line must be sufficient to purchase all goods within the set, i.e. to above and to the left of the budget line.

	I	II	III	IV	V	VI	VII	VIII	IX	X
	10	9	8	7	6	5	4	3	2	1
	9	8	7	6	5	4	3	2	1	0
	8	7	6	5	4	3	2	1	0	
	7	6	5	4	3	2	1	0		
A ($10)	6	5	4	3	2	1	0			
	5	4	3	2	1	0				
	4	3	2	1	0					
B ($28)	3	2	1	0						
	2	1	0							
	1	0								
	0									

Assume the price of one increment of any good is $1

Figure 3.3.2-1[49] Menger's Diminishing Use-Value

3.3.3) Antonelli, Pareto, and the Problem of Integrability

Efforts to actually derive a measure of use use-value from observable phenomena as Dupuit had suggested, were sporadic at best and little progress was made. To integrate a set of marginal demand functions $r_1(x_1, x_2, \ldots x_m)$, $r_2(x_1, x_2, \ldots x_m)$, $\ldots$ $r_m(x_1, x_2, \ldots x_m)$ into a single use-value (or utility) function $U(x_1, x_2, \ldots x_m)$ requires the technique of *vector line integration*. This technique had been developed a few years earlier, but saw little use outside the physics community. Except for Irving Fisher, no one in the economic community appears to have been aware of it until well into the 20th century.

To perform such an integral, a vector is formed from the marginal demand functions, and integrated over a curve through commodity space between endpoints serving as the limits of integration, as will be discussed in Chapter 4. In general, the value of the integral will depend on the curve chosen. In economics, such path would correspond to the order in which the goods are acquired or consumed. By what is known as *Stokes' Theorem*, however, this integral will be

[49] Menger (2003) p.451

independent of the path if the marginal price functions form a *complete differential* of the utility function, i.e.:

$$dU \equiv \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + \cdots + \frac{\partial U}{\partial x_m}dx_m = r_1 dx_1 + r_2 dx_2 + \cdots + r_m dx_m \qquad (3.3.3\text{-}1)$$

From Young's Theorem we know that this requires:

$$\frac{\partial^2 U}{\partial x_i \partial x_k} \equiv \frac{\partial^2 U}{\partial x_k \partial x_i} \quad \Rightarrow \quad \frac{\partial r_i}{\partial x_k} \equiv \frac{\partial r_k}{\partial x_i} \qquad \forall i,k \in (1 \ldots m) \qquad (3.3.3\text{-}2)$$

This was essentially the message of G. B. Antonelli's paper of 1886, which began discussion of the so-called *Problem of Integrability*[50]. This problem was with identifying what the conditions given by the right side of Equation 3.3.3-2 might mean economically, and with determining whether or not one may assume that these conditions would always be met.

Antonelli was the first to investigate the integration of inverse demand functions, which are essentially Marshall's marginal demand prices. Antonelli's paper was largely an exercise in the solution of differential equations. There is little explanation of the intuitive meaning of the equations, and no discussion of the meaning of his results. It comes as no surprise that his paper languished in obscurity for nearly 60 years.[51]

In that paper Antonelli worked the demand functions into a system of differential equations. He then argued that there existed a function $U(x_1, x_2, \ldots x_n)$ that would solve this system, provided

[50] Antonelli (1971)

[51] Chipman (1971) p.321

the conditions (right side of Equation 3.3.3-2) were satisfied. In the economics literature these conditions became known as the *Antonelli Conditions*, though they were already well known to mathematicians.

Vilfredo Pareto (1848-1950) began his work on the integrability issue in 1892, apparently unaware of Antonelli's earlier contribution[52]. His project was to find a measure of *ophelimity,* utility as understood in Jevons' sense, by integrating inverse demand functions. Pareto approached the question in a manner similar to that of Antonelli but got less far. He was not aware of the Antonelli conditions until mathematician Vito Volterra pointed them out to him in a 1906 review of his *Manual Of Political Economy* [53]. In the review, Volterra had expressed hope that Pareto would try to find the economic significance of the Antonelli conditions. Instead, Pareto addressed the order of consumption that determined the path of integration. In his *Ophelimity in Nonclosed Cycles*[54], Pareto tried to offer an economic explanation for cases where the Antonelli conditions were *not* met. His efforts earned the derision of Hicks and Allen and later Samuelson[55]. These criticisms, however, point to the confusion of the critics. Samuelson comments that

> "… there is confusion between the 'order of consumption' and the 'dependence of certain integrals on the path' between two points. It must be emphasized that the paths along which I as an economic scientist choose to evaluate the man's preference have absolutely nothing to do with the order in which the human guinea-pig consumes the goods. … I don't know whether it even makes sense to say that he enjoys his food before he enjoys his shelter. …Rather we should

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[52] Chipman (1971) p.324

[53] See Chipman (1971) p. 324 and Volterra (1971) p.367

[54] Pareto (1971a)

[55] Chipman (1971) p.324

> always regard the budget of goods at [a point] as a steady flow of consumption per unit time, optimally patterned to the consumer's tastes."[56]

Samuelson's "economic scientist" is in a bit of a quandary as to which path to choose, since the result he obtains will depend on the choice. If the order in which the consumer acquires his goods is not relevant, what rationale does the economic scientist have for choosing one path over another? Samuelson's argument only makes sense if one can assume that the Antonelli conditions are always satisfied. This is in fact the conclusion at which Samuelson arrives, but by other means. His conclusion is based on the work of Hendrik Houthakker who essentially re-proves a portion of Stokes's Theorem for the case of ordinal preferences[57].

Eugen Slutsky had provided an economic interpretation of the Antonelli conditions as far back as 1915[58]. Slutsky observed that such conditions implied that net substitution effects between goods must be mutual. If some good A is a substitute (compliment) for another good B, then B must likewise be a substitute (compliment) for A.

Even if Pareto or his contemporaries had understood the problem of integrability, is not likely that it would have resulted in adoption of the broad notion of use-value discussed here. Pareto's notion *ophelimity* resembles that of Jevons and Edgeworth. Though Pareto mentions Marshall, Cournot, and Fisher in his *Manual* he made no use of their notions of utility or use-value. Pareto defines ophelimity as satisfaction of emotional desire without regard to whether or not the

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[56] Samuelson (1950) p.361

[57] Houthakker (1950)

[58] Samuelson (1950) pp.356-7

consumer's wellbeing is enhanced. To an addict, Pareto regards morphine as economically useful, "even though it is unhealthful, because it satisfies one of his wants"[59].

By reverting to a purely sensual notion of utility, hope of finding an empirical yardstick by which it can be measured is foregone. Pareto was left in the same position, as were Jevons and Marshall, though he stated the problem more forcefully. Any given utility function implies that more is known regarding the consumer than is actually the case. With any utility function of m variables $U(x_1, x_2, \ldots x_m)$, comes the implication that there are m relationships $\partial U / \partial x_i$ that can be determined, when in fact there are only *m-1* relationships $\left(\partial U / dx_i\right) / \left(\partial U / \partial x_k\right) = dx_k / dx_i$ that can ever be observed. As result of the additional degree of freedom, there are *many* utility functions from which the same set of relationships dx_k / dx_i can be derived. In the appendix to his *Manual,* Pareto demonstrates this in the following manner:

For any utility function $U\left(x_1, x_2, \ldots x_m\right)$ the consumer's family of indifference curves is given by:

$$k_i = U\left(x_1, x_2, \ldots x_m\right), \qquad \text{3.3.3-3}$$

where k_i represents a set of constants. Let $F\left\{U\left(x_1, x_2, \ldots x_n\right)\right\}$ be another function that is a positive transformation of U as shown in Figure 3.3.3-1. The equation defining the indifference curves represented by $F\{U\}$ is given by:

$$j_i = F\left\{U\left(x_1, x_2, \ldots x_n\right)\right\} \qquad \text{3.3.3-4}$$

[59] Pareto (1971) p.111

The “shape” of the curves represented by U and $F\{U\}$ can be deduced from the differentials of Equations 3.3-3 and 3.3-4, which Pareto shows to be equivalent:[60]

$$
\begin{aligned}
0 = d\left[F\left\{U\left(x_1, x_2, \ldots x_n\right)\right\}\right] &= \frac{dF}{dU}\frac{\partial U}{\partial x_1}dx_1 + \frac{dF}{dU}\frac{\partial U}{\partial x_2}dx_2 + \cdots + \frac{dF}{dU}\frac{\partial U}{\partial x_n}dx_n \\
&= \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + \cdots + \frac{\partial U}{\partial x_n}dx_n \qquad \text{3.3.3-6} \\
&= d\left[U\left(x_1, x_2, \ldots x_n\right)\right]
\end{aligned}
$$

As figure 3.3.3-1 shows, transformation of U simply shifts the positions of the indifference curves along the U axis without changing their shape. What Pareto has shown is that, as long as the marginal rates of substitution are all that is considered important, utility functions *themselves* are no more significant than the magnitudes of the utilities they propose to represent. What matters is the shape of the indifference curves and their relative order. This information is contained in the consumer’s ordinal preferences. While such preferences may be *described* by a utility function, such a description is only an analytic convenience.

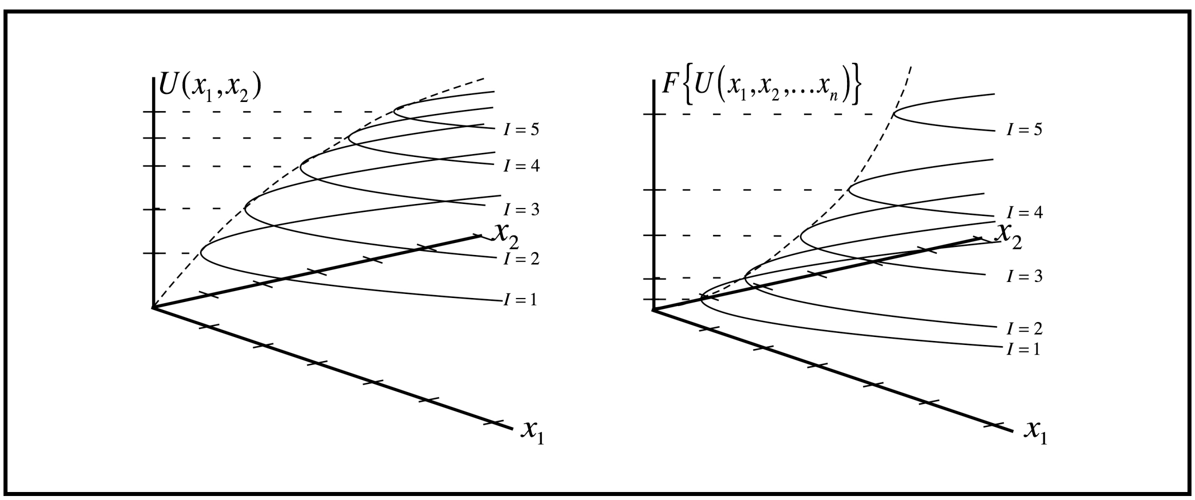

Figure 3.3.3-1 Monotonic Transformation of a Utility Function

[60] See Pareto (1971) pp. 392-393

3.4) Conclusion and Epilogue

With the advent of ordinal preferences, consumer theory begins to take the shape it has held ever since. Consumers are presumed to act as if they were maximizing some quantity (utility) subject to a constraint. The task of defining that quantity is as illusive as ever. While the Greeks use-value as a descriptive idea, positive economists regard utility (preference) as an abstract quantity that obeys certain rules. Welfare economists, however still tend to regard utility as a measure of "happiness" though there is still considerable debate as to whether any such entity can be used as an indicator of well-being[61].

The concept of use-value as a quantity that can be defined analytically in terms of observed behavior has been lost. Samuelson's theory of revealed preferences comes close. By saying, however, that integration of demand can be used to "recover" or "reveal" a consumer's preference admits belief that such a quantity *exists in nature*. The fact that there is no scale by which it can be measured requires economists to treat it as an ordinal quantity.

While Economists from Pareto to Samuelson have persuaded their colleagues that a cardinal measure of utility is unnecessary[62], lack of such severely limits the kinds of empirical studies that can be done. As Robbins argued (and Arrow later proved) aggregation of ordinal preferences is impossible. While Arrow's Impossibility Theorem is quite extensive, the core of its reasoning is visible in Condorcet's Paradox of Voting, illustrated as follows: Consider three individuals, Fred, Mary, and George, who are asked to rank, according to their preference, three bundles of goods,

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[61] See Sen (1977)

[62] Samuelson (1961) pp.93, 94

a, b, and c. Fred prefers a to b to c (written $a \succ b \succ c$) Mary prefers b to c to a, and George prefers c to a to b. These orderings are summarized below:

$$\begin{aligned} &Fred: && a \succ b \succ c \\ &Mary: && b \succ c \succ a \\ &George: && c \succ a \succ b \end{aligned}$$

The three are asked to vote their preference among each pair of alternatives. By a two-thirds majority, Fred and George prefer a to b. By the same majority, Fred and Mary select b over c. Finally, Mary and George select c over a. We find that the social "ordering" (shown below) is no ordering at all. There is alternative which is clearly "best" or 'worst" Additionally, there is no utility function by which it can be represented.

Without the capability of aggregating preferences, there is no means of determining how individual might generally behave as a function of circumstances beyond the most basic of cases. There would be no way of empirically verifying whether or not individuals generally prioritize their consumption as Menger supposes. In order to argue that aggregation of preferences is unnecessary, Samuelson had to limit the scope of consumer theory to one that considers only a single *hypothetical* consumer, whose preferences are unknowable. Regarding the consumer's preferences, nothing can be presumed beyond what was necessary to guarantee that the utility maximization problem has a solution. From the assumption of utility maximization alone, the conclusions derived are general enough to hold for *any* possible consumer. Though such conclusions are undoubtedly true, they represent only the beginning of what could be learnt if empirical observation were possible.

PART II:
MODEL OF THE INDIVIDUAL CONSUMER

CHAPTER 4:
USE VALUE AND MARGINAL PRICES

This Chapter will provide the foundation of the dynamic model by deriving the consumer's use-value function from his or her MRS. The following discussion is intended for economists in general, who may not be familiar with vector analysis.

4.1) Basic Assumptions

This sub-section will introduce the basic assumptions needed to insure that the dynamic model will provide results that are *at least* as robust as those based on utility or preference.

This section will also present the formal results that can be derived from the assumptions without resorting to vector calculus. We begin with an informal overview of the assumptions.

(1) For any bundle of goods an agent might hold, he knows how much of any one good he would be willing to exchange for an additional unit of any other good. Furthermore, he will seek to *buy* commodities for which he would be willing to pay *more* than the price he is asked for. He will seek to *sell* commodities for which he would be willing to pay *less* than the price he is offered.

(2) The agent's exchange rates are *consistent*; (i.e. if she would exchange one A for two B, and one B for two C, then she would exchange one A for four C).

(3) There exists at least one commodity, in terms of which can be measured the value consumers place on other goods.[1]

(4) For any given bundle of goods the consumer may hold, the complementary effects between them are *mutual* (i.e. if good A is a net-compliment (or substitute) for good B, then B must be a net-compliment (or substitute) for A.

(5) There is no commodity, or linear combination of commodities, to which the individual is *addicted* (i.e. there is no commodity for which the more of it she possesses, the greater the price she would pay for an additional unit it).

Assumptions (1) through (3) allow the consumer's MRS to be stated in terms of a single (vector valued) function which will be called the consumer's *marginal value* (or *marginal price*). These terms, which will be used interchangeably as best fits the context, come from what Alfred Marshall called the consumer's *marginal value price*. This is the money price the consumer would be "just willing to pay" for a good that "he is on the margin of doubt whether it is worth his while to incur the outlay required to obtain it." [2] Assumptions (1) and (2) simply state that the MRS exist and are internally consistent. Assumption (3) indirectly introduces money (broadly defined) as the standard of measure. Until now, the standard of measure in consumer theory has implicitly been satisfaction or "happiness", even if only ordinaly so in terms of preference. Even Marshall, who suggested the use of money as standard of measure, abandoned that approach, as that money was not a reliable index of *satisfaction*. Here the entire discussion of satisfaction is banished in favor of quantities that can be observed.

[1] i.e. Medium of exchange, unit of account, and store of value.

[2] Marshall (1997) p.93

Assumption (4) allows use-value to be defined in terms of the integral of the marginal price function, and is intimately tied up with the historic *problem of integrability*. Specifically, Assumption (4) guarantees that the consumer's use-value does not depend on the order in which goods are acquired or consumed. It will be shown that a violation of Assumption (3) would produce results that are intuitively absurd.

Assumption (5) is the analog of the law of diminishing marginal utility. By including *linear combinations* of goods, this assumption requires that the indifference surfaces of the use-value function be convex to the origin, as is required for the maximization problem to have a unique answer. Assumption (5) is stated in terms of obsessive behavior on the part of the consumer to highlight the fact that it is a social necessity rather than simply an analytic convenience. Should there be a consumer for whom the assumption is violated with respect to any given good, his behavior would become self destructive as well as socially dangerous. Most societies have developed institutions to constrain the availability of addictive goods, as well as to restrain the behavior of addicted individuals.

Our formal discussion begins at a basic level, in order to introduce notation:

Definition: Marginal Rate of Substitution (MRS)
Given an economy with n commodities, x_i where $i \in (1,2,\ldots n)$. For any bundle of commodities $(x_1, x_2 \ldots x_n)$ that the consumer might possess, and any pair of commodities x_j and x_k within that bundle, the consumer's MRS_{j-k} is given by:

$$MRS_{j-k}(x_1, x_2, \ldots x_n) \triangleq \frac{dx_k}{dx_j} \tag{4.1-1}$$

Given this definition, we can state the first part of Assumption (1) as follows:

Assumption (1a) [Existence of the MRS]
For all non-negative quantities of commodities x_i and x_k all MRS_{i-k} exist with non-negative real values.

Assumption (1a) is obvious (statement of the second part of the assumption (1b) will be postponed until we discuss the consumer choice problem in Chapter 5). If the assumption were not satisfied, there would be bundles for which the consumer becomes "confused" as to the value she would place on one or more of the goods it contains, and would thus unable to engage in exchange[3].

Assumption (2), which is simple transitivity, is stated formally as follows:

Assumption (2)
Given an economy with n commodities, for each set of commodities x_i, x_k, x_l where $i, k, l \in (1, 2, \ldots n)$

$$MRS_{k-i} = MRS_{k-l} \bullet MRS_{l-i} \quad (4.1\text{-}2)$$ [4]

We now assume that there is a commodity that can be used as a standard of measure. As will become evident not every commodity can serve this purpose effectively. As the following analysis will predict, the commodities a society chooses to use for money, such as precious metals and currency, tend to be used for little else.

[3] In spite of this, analysts often consider ordinal preferences for which the MRS do not exist for all possible bundles. Such orderings can complicate matters considerably. Assumption (1) argues that we can disregard such cases as irrelevant to the behavior of a real consumer.

[4] This would be a mathematical identity if a functional relationship between the goods had been established. Such relationship will not be established until later.

Assumption (3) [Standard of Measure]

Given an economy of n+1 commodities, and given that there is some function $V(x_1, x_2, \ldots x_{n+1})$, which measures the value the consumer places on his bundle of goods in terms of a numeraire x_M. I.e. it must be true that $dV/dx_M \equiv 1$ for all possible bundles held by the consumer.

For the remainder of the discussion, M will be used to denote this numeraire instead of x_M.

Using Assumptions (1) through (3), we can define the consumer's marginal price function for a single good. This is the maximum price, in terms of numeraire, that the consumer would be willing to pay for one additional unit of that good, given his holdings of all goods. Formally, this quantity is the MRS between the chosen numeraire and the good in question.

Definition: Marginal Price (of the ith good)

For an economy with n goods $(x_1, x_2, \ldots x_n)$ and numeraire M, the consumer's *marginal price* for good x_i is a scalar function $r_i(.) : \Re^{n+1} \rightarrow \Re^1$ of the goods and numeraire the consumer holds, defined by:

$$r_i(x_1, x_2, \ldots x_n, M) \triangleq \frac{dM}{dx_i} \tag{4.1-3}$$

This defines a set of n functions, which contain all the information that can be empirically determined regarding the consumer's choice behavior.

By taking a close look at our method of measurement, it can be shown that the use-value function must be cardinally measurable, even before it is formally defined. Consider the typical illustration used to show that any given set of MRS could be derived from a myriad of utility functions. Consider a utility function $U(x_1, x_2, \ldots x_n)$ and a monotonically increasing transform of it $f[U(x_1 x_2, \ldots x_n)]$. The MRS derived from both functions must be the same since:

$$MRS_{j-k} = \frac{\frac{\partial U}{\partial x_j}}{\frac{\partial U}{\partial x_k}} = \frac{\frac{\partial f}{\partial U}\frac{\partial U}{\partial x_j}}{\frac{\partial f}{\partial U}\frac{\partial U}{\partial x_k}} = \frac{\frac{\partial\left[f(U)\right]}{\partial x_j}}{\frac{\partial\left[f(U)\right]}{\partial x_k}} \qquad (4.1\text{-}4)$$

This follows from both $U(.)$ [5] and $f[U(.)]$ being implicitly defined in terms of measures of satisfaction: (perhaps "utils" for the former and "futils" for the latter). Solving the problem by resorting to an ordinal scale merely begs the measurement question. Satisfaction can be replaced with an observable standard of measure as follows: Consider the function $V(x_1, x_2, \ldots x_n, M)$, which is a transform of $U(x_1, x_2, \ldots x_n, M)$ that satisfies Assumption (3). Since the transform is monotonic, its inverse exists so we may write:

$$U\left(x_1, x_2, \ldots x_n, M\right) = f^{-1}\left[V\left(x_1, x_2, \ldots x_n, M\right)\right] \qquad (4.1\text{-}5)$$

The total differential of Equation 4.1-5 is:

$$dU = \frac{dU}{dV}\left(\frac{\partial V}{\partial x_1}dx_1 + \frac{\partial V}{\partial x_2}dx_2 + \cdots + \frac{\partial V}{\partial x_n}dx_n + dM\right) \qquad (4.1\text{-}6)$$

Consider now the consumer's marginal prices $r_i(x_1, x_2, \ldots x_n, M)$. From Assumption (3) we have:

$$r_i\left(x_1, x_2, \ldots x_n, M\right) \triangleq \frac{dM}{dx_i} = \frac{MU_{x_i}}{MU_M} = \frac{\frac{\partial U}{\partial V}\frac{\partial V}{\partial x_i}}{\frac{\partial U}{\partial V}\frac{\partial V}{\partial M}} = \frac{\partial V}{\partial x_i} \qquad (4.1\text{-}7)$$

[5] The notation (.) is shorthand for the arguments previously, (and commonly) used with that function.

where MU_{x_i} and MU_M are the consumer's marginal utilities of x_i and M respectively.

Substituting Equation (4.1-7) into Equation (4.1-6) gives:

$$dU = \frac{dU}{dV}\left(r_1 dx_1 + r_2 dx_2 + \cdots + r_n dx_n + dM\right) = \frac{dU}{dV} dV \qquad (4.1\text{-}8)$$

The function $V(x_1, x_2, \ldots x_n, M)$ must therefore be "that function for which the marginal price functions form a complete differential or *gradient*. Since the $r_i(.)$ can be empirically measured, *V(.)* can be evaluated to within constants of integration determined by the numeraire and a "reference bundle of goods" which will be discussed later. The unobservable way in which satisfaction may vary with respect to *V(.)* is captured in dU/dV. By replacing *U(.)* with *V(.)* the term dU/dV, along with its associated problems with measurement, are discarded. From Equation (4.1-8) it is apparent that there are certain restrictions that the various $r_i(.)$ must follow. From Young's theorem we have:

$$\frac{\partial V}{\partial x_i \partial x_k} \equiv \frac{\partial V}{\partial x_k \partial x_i} \Rightarrow \frac{\partial r_k}{\partial x_i} \equiv \frac{\partial r_i}{\partial x_k} \qquad (4.1\text{-}9)$$

The right hand side of Equation (4.1-9), which shall become Assumption (4) momentarily, appears in the economics literature as the *Antonelli Conditions* that must be satisfied by any set of $r_i(.)$ to be *integrable*. The economic meaning of these conditions was the subject of debate for the first half of the Twentieth Century. We shall touch on that debate briefly when we discuss the meaning of *vector line integration* in the next section. Meanwhile, notice that $\partial r_i/\partial x_k$ indicates the degree to which the consumer's holdings of x_k influences the price she would be willing to pay for an increment of good x_i. If $\partial r_i / \partial x_k > 0$, x_i is a compliment of x_i, (if $\partial r_i / \partial x_k < 0$ x_k is a substitute for x_i). We therefore make the following definition:[6]

> **<u>Definition: Complementarily</u>**[7]
> For a given consumer holding bundle $(x_1, x_2, \ldots x_n, M)$, the complementary effect of her possession of good x_k on the marginal value $r_i(.)$ she places on another good x_i is defined to be: $\partial r_i(.) / \partial x_k$

Using this definition, we can now formally state Assumption (4)

> **<u>Assumption (4) [Mutual Complementarily]</u>**[8]
> For any given bundle $(x_1, x_2, \ldots x_n, M)$ the consumer might hold, the complementary effect of his possession of some good x_i on the marginal value $r_k(.)$ he places on another good x_k is equal to the complementary effect of his possession of good x_k on the marginal value $r_i(.)$ he places on good x_i. I.e.:
>
> $$\frac{\partial r_i}{\partial x_k} = \frac{\partial r_k}{\partial x_i} \qquad (4.1\text{-}10)$$

From Assumptions (3) and (4) which gives us the cardinality of *V(.)* we can draw a testable hypothesis regarding the nature of M: Since $\partial V / \partial M = r_M$ is by definition unity, $\partial r_M / \partial x_i \equiv 0$ for all x_i. Then by Assumption (4) $\partial r_i / \partial M \equiv 0$ for all r_i. Thus, none of the consumer's $r_i(.)$ can depend on M. From Equation (4.3-5) the differential of *V(.)* can be written:

$$dV = dM + \sum_{i=1}^{n} r_i \left(x_1, x_2, \ldots x_n \right) dx_i \qquad (4.1\text{-}11)$$

---

[7] This refers to net-complementarity.

[8] Eugen Slutsky recognized this as a testable hypothesis that must be true if demand functions were integrable See Samuelson (1950) p.357

The function *V(.)* is thus of the form:

$$V\left(x_1,x_2,\ldots x_n,M\right)=v\left(x_1,x_2,\ldots x_n\right)+M \qquad (4.1\text{-}12)$$

Equation (4.1-12) implies that the commodity a society chooses to use as numeraire, (i.e. money) is one that is valued for that purpose alone. It is neither a substitute nor compliment for other goods ($\partial r_i/\partial M \equiv 0$ for all *I*). Its value to the consumer is derived from the use-value of goods it could be used to acquire rather than from its own characteristics. This would explain why most societies use either currency or some commodity such as gold or silver for money.

Depending on the analysis one is attempting, it may or may not be necessary to include M explicitly in the argument of *V(.)*. Even then, it appears implicitly as the unit of measure by virtue of the definition of $r_i(.)$. As result there will be a full n functions $r_i(.)$ corresponding to a given $V(x_1,x_2,\ldots x_n)$ where there are only *n-1* MRS that can be derived from a given $U(x_1,x_2,\ldots x_n)$. This is because the MRS are measured relative to each other rather than to an external reference *M*.

As we proceed, the function *V(.)* will be defined in terms of the *vector line integral*:

$$V(\vec{x}'-\vec{x}^0)=\int_{\vec{x}^0}^{\vec{x}'}\vec{r}(\vec{x})\bullet d\vec{x} \qquad (4.1\text{-}13)$$

Development of both the concepts and notation needed to interpret this will be the topic of the next section.

4.2.) Vector Analysis (A Digression)

While the notation of a linear array is often used to represent vectors, that is not strictly speaking what a vector is. A vector is a *type of number* having two distinct properties, geometrically interpreted as *magnitude* and *direction*. While some physical quantities such as mass and temperature can be expressed as *scalars* (numbers having only magnitude) quantities such as velocity and momentum must be expressed as vectors. The velocity of a vehicle leaving Los Angeles along Interstate Highway 10 is not merely traveling at "65 mph," it is traveling "65 mph. *in an easterly direction.*[9]" The force of impact when two automobiles collide depends on both their speed and relative *direction* of travel. Graphically, vectors can be represented as arrows, giving a clear intuitive picture of the physical situation.

Motion of extended, elastic bodies such as fluids are described by *fields,* represented by vector *functions*. The motions of particles suspended within a fluid vary with their position within the field. The velocity of a particle suspended in a stream of water will be a function of where it is relative to the riverbank. Particles closer to the shore will move more slowly and with a trajectory that follows curves in the riverbank, while particles near the center will move faster and in more of a straight line. Figure 4.2-1 illustrates of a vector field showing the velocity of exhaust gas as it escapes from an automotive tailpipe.

As Figure 4.2-1 shows, it is often convenient to represent vector fields graphically with continuous streamlines (the dotted curves) rather than a set of arrows associated with individual points. Streamline diagrams are commonly used to depict the fields of air currents and weather patterns.

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[9] Of course, given LA traffic, it is not likely moving at 65mph either.

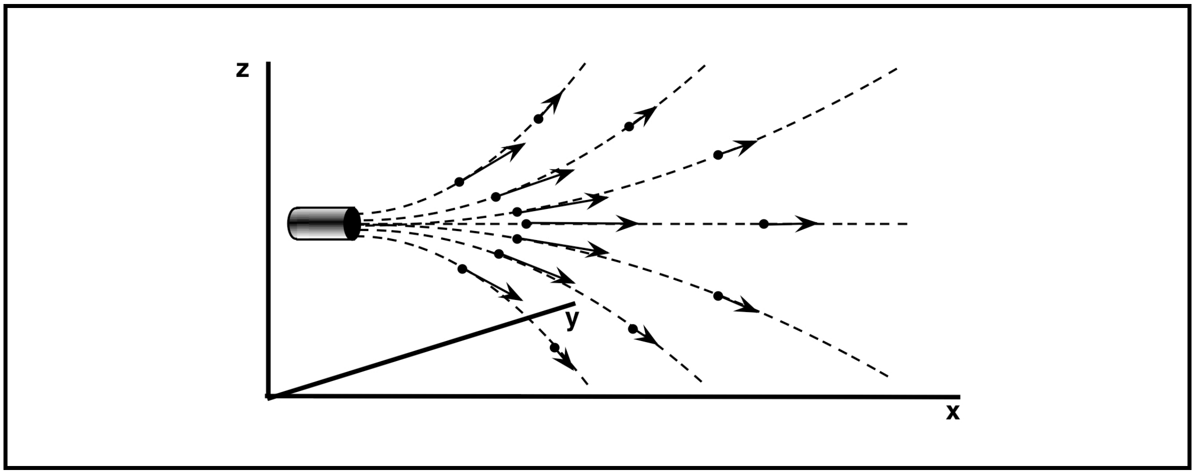

Figure 4.2-1 Vector field depicting the velocity of gas escaping from a pipe.

Vector addition can be visualized as shown in Figure 4.2-2. Consider a boat attempting to travel eastward with engine speed of 4 knots, moving perpendicular to a current moving southward at 3 knots. The boat's velocity relative to the shore is the vector sum of its engine speed and the current. This can be found graphically by placing the tail of one vector against the head of the other as shown. The actual speed of travel is found from the Pythagorean theorem to be 5 knots.

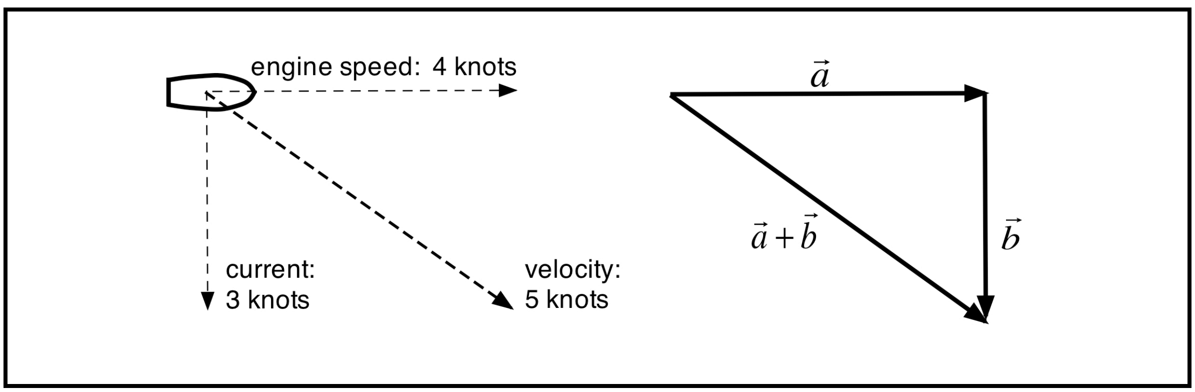

Figure 4.2-2: The velocity of a boat is the sum of its engine speed and ocean current.

The most common (of several) forms of vector multiplication can be used to show the extent to which one vector may act in the direction of another. When a force $\vec{F}$ applied to an object causes it to move a distance $\vec{s}$, we say that "work" W is done on the object. If the force is constant, the work done is the product of the force applied and the distance moved. To understand how such a product is defined, consider a child pushing a toy train along its track with his finger. If the child pushes the train from behind and in the direction of the track, the full magnitude of the applied force will contribute to moving the train forward. On the other hand if the child pushes on the train at an angle θ as shown in Figure 4.2-3, only *some* of the force exerted will go into moving the train, the remainder will push the train sideways against the track, tending to topple it over. To find the work done, we resolve the force into two components: one $\vec{F}_t$ tangent (parallel) to the track, the other $\vec{F}_n$ normal (perpendicular) as shown.

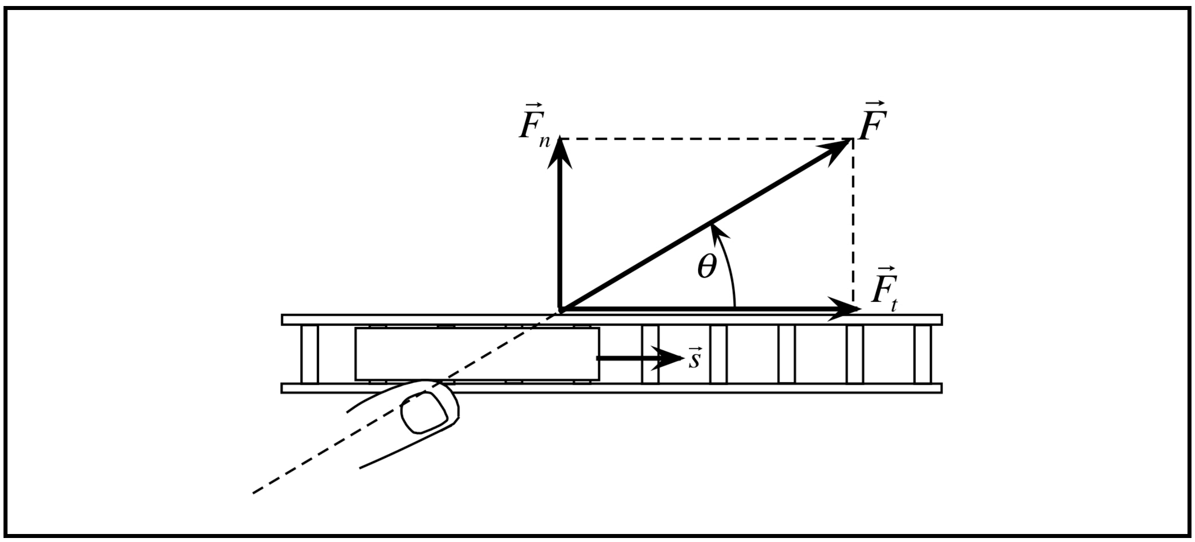

Figure 4.2-3: Vectors can be resolved into components to facilitate analysis.

The magnitudes of components $\vec{F}_t$ and $\vec{F}_n$ (written $|F_t|$ and $|F_n|$) are given as follows:

$$
\begin{aligned}
|F_t| &= |F|\cos\theta \\
|F_n| &= |F|\sin\theta \qquad (4.2\text{-}1) \\
\vec{F}_n + \vec{F}_t &= \vec{F}
\end{aligned}
$$

As argued above, only the tangent component F_t of the force contributes to the work. Since F_t and $\vec{s}$ are parallel we know that $W = |F_t||s|$. We substitute the original force into the equation using Equations 4.2-1 and find: $W = |F||s|\cos\theta$. We express this as a *scalar product*:

$$W = \vec{F} \bullet \vec{s} \triangleq |F||s|\cos\theta \tag{4.2-2}$$

The scalar product gives us the right to express vectors in matrix form familiar in economics. Any three-dimensional vector quantity can be resolved into components parallel to the axes of any coordinate system we may choose. We define the coordinate system with a set of *basis vectors,* each one having unit magnitude and direction parallel to its respective coordinate axis. For the familiar three-dimensional Cartesian system with axes labeled x-y-z, the basis vectors would be denoted: $\hat{\varphi}_x, \hat{\varphi}_y, \hat{\varphi}_z$ respectively as shown in Figure 4.2-4. For a given vector $\vec{A}$ its three respective components are:

$$\begin{array}{lclcl} |A_x|\hat{\varphi}_x & & a_x\hat{\varphi}_x & & a_x = \vec{A} \bullet \hat{\varphi}_x \\ |A_y|\hat{\varphi}_y & \text{or simply:} & a_y\hat{\varphi}_y & \text{where} & a_y = \vec{A} \bullet \hat{\varphi}_y \\ |A_z|\varphi_z & & a_z\hat{\varphi}_z & & a_z = \vec{A} \bullet \hat{\varphi}_z \end{array} \tag{4.2-3}$$

We can write the vector in analytic form as the sum of its components i.e.:

$$\vec{A} = \quad \vec{A}_x + \vec{A}_y + \vec{A}_z = \quad a_x\hat{\varphi}_x + a_y\hat{\varphi}_y + a_z\hat{\varphi}_z \tag{4.2-4}$$

The magnitude or "length" of vector $\vec{A}$ is given by:

$$\left|\vec{A}\right| = \left(a_1^2 + a_2^2 + a_3^2\right)^{\frac{1}{2}} \tag{4.2-5}$$

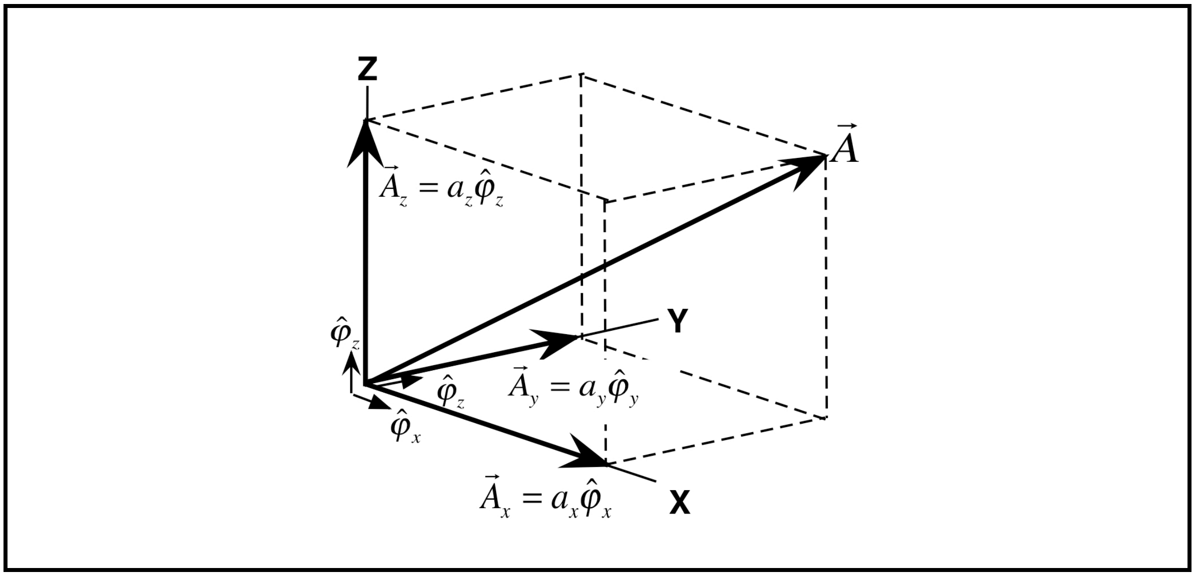

Figure 4.2-4 Components of a Vector

It is the scalar coefficients $\left(a_x, a_y, a_z\right)$ that comprise the familiar matrix notation. This notation is merely a shorthand for the right hand term in Equation 4.2-4 and has meaning only with respect to the coordinate system defined by $\hat{\varphi}_x, \hat{\varphi}_y, \hat{\varphi}_z$.

Whereas the familiar *scalar* function $f(x_1, x_2, \ldots x_n)$ assigns or "maps" a single value f to each set of values of its independent variables, a vector function $\vec{v}(x_1, x_2, \ldots x_n)$ assigns a *vector* $v_1(x_1, x_2, \ldots x_n)\hat{\varphi}_1 + v_2(x_1, x_2, \ldots x_n)\hat{\varphi}_2 + \cdots + v_n(x_1, x_2, \ldots x_n)\hat{\varphi}_n$. To a particle suspended in a stream of water at location (x_1, x_2, x_3) which shall be given the vector notation $\vec{x}$, the velocity (vector) function assigns the vector $v_1(\vec{x})\hat{\varphi}_1 + v_2(\vec{x})\hat{\varphi}_2 + v_3(\vec{x})\hat{\varphi}_3 = \vec{v}(\vec{x})$, as is shown in Figure 4.2-5.

Returning to our example of the child pushing the train (Figure 4.2-3), the question arises: How do we calculate the work done if the force exerted varies with the displacement s? The answer is that we simply integrate the tangential component $F_t(s)$ over the path traveled from the starting point s^0 to the endpoint s^1 I. e.:

$$W = \int_{s^0}^{s'} F_t(s)\,ds \tag{4.2-6}$$

If, in our example, the child's finger rotates, or he presses less hard as the train moves from starting point s_0 to point s_1, the tangential force will vary with both the magnitude and direction of the applied force, hence:

$$F_t(s) = |F(s)|\cos\{\theta(s)\} \tag{4.2-7}$$

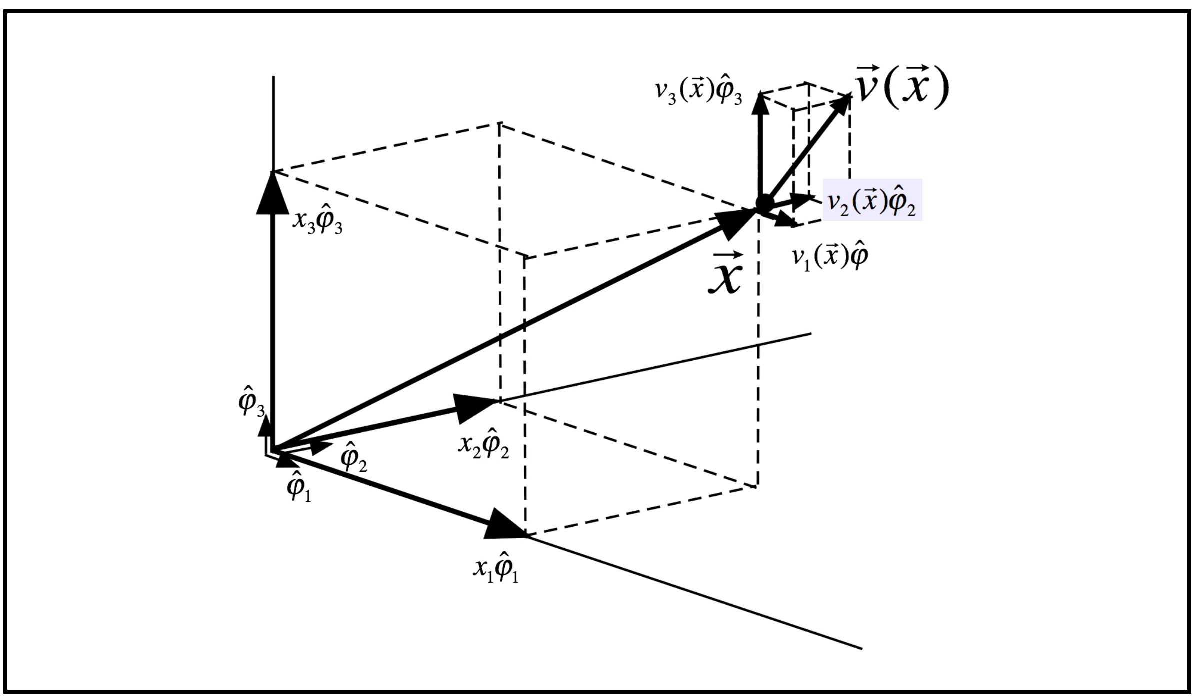

Figure 4.2-5 Components of a Vector Function

We substitute Equation 4.2-7 into Equation 4.2-6. Note that the integrand is now the dot product of the force function and a vector representing an element of the displacement path.

$$W = \int_{s^0}^{s'} \left|F_t(s)\right| \cos\left\langle \theta(s) \right\rangle ds = \int_{\vec{s}^0}^{\vec{s}'} \vec{F}(\vec{s}) \bullet d\vec{s} \tag{4.2-8}$$

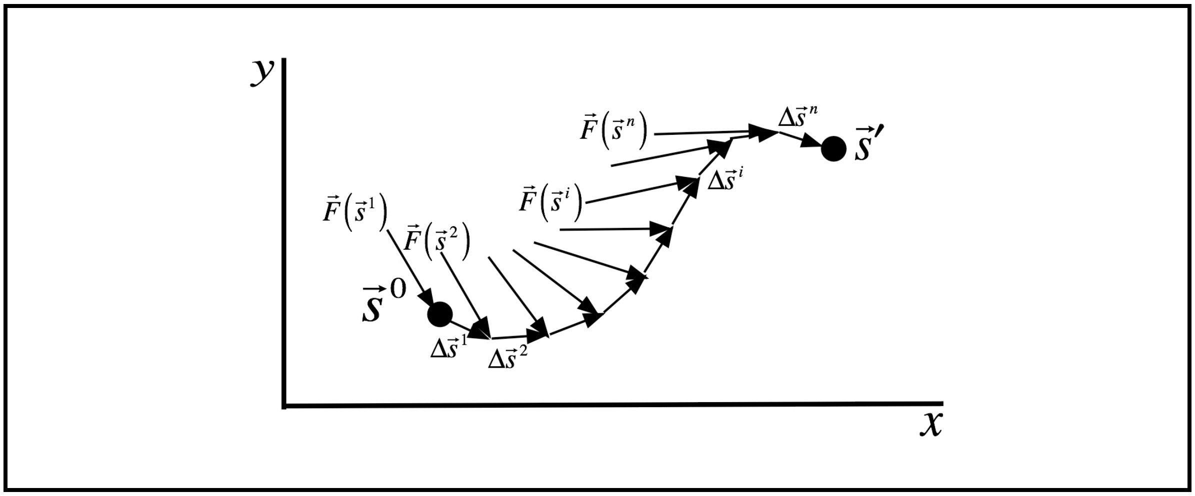

Figure 4.2-6 Displacement Path

One might ask what happens to this integral if the path were not constrained to a straight track running along the number line? In spaces of more than one dimension, many paths can generally be taken between $\vec{s}^0$ and $\vec{s}'$. Since W depends on the direction of the force relative to the path taken, the value of the integral will generally depend on the path taken. (A person sliding a box across a floor will work less hard if she pushes the box along a straight line between points $\vec{s}^0$ and $\vec{s}'$, than if a circuitous path were used as shown in Figure 4.2-6).

Work done is not generally an easily definable function unless the force is exerted by a *field*, i.e. the force is a function $\vec{F}(\vec{x})$ of the position coordinates $\vec{x}$. If the components of $\vec{F}(\vec{x})$ satisfy

$\partial F_i / \partial x_k \equiv \partial F_k / \partial x_i$, we know that they form a complete differential of another function we can call $E(\vec{x})$ which is entirely a function of position $\vec{x}$. The force of gravity provides perhaps the simplest illustration of such a force. In a "small" region near the surface of the earth, the force is directed uniformly downward. If a rock were to be lifted from some point $\vec{x}_0$ to a point $\vec{x}_1$. The work done *against* the force of gravity would be proportional to the vertical distance through which the rock was raised.

$$W = \int_{x_0}^{x_1} \vec{F}(\vec{x}) \bullet d\vec{x} \tag{4.2-9}$$

If the rock were dropped, an equivalent amount of work would be done *by* gravity as the rock falls back to $\vec{x}_0$:

$$W = \int_{x_1}^{x_0} \vec{F}(\vec{x}) \bullet d\vec{x} = -\int_{x_0}^{x_1} \vec{F}(\vec{x}) \bullet d\vec{x}$$

We say that $E(\vec{x}_1 - \vec{x}_0)$ represents the gravitational potential energy that is stored when the rock is lifted. This energy would be released (on your toe should it be so unfortunate to reside at $\vec{x}_0$) when the rock is dropped. The work done (energy released) when the rock is dropped from $\vec{x}_1$ to $\vec{x}_0$ does not depend on the path through which the rock was lifted (see Figure 4.2-7). Because of the dot product $\vec{F}(\vec{x}) \bullet d\vec{x}$, only the vertical component of the path elements $d\vec{x}$ contribute to the integral.

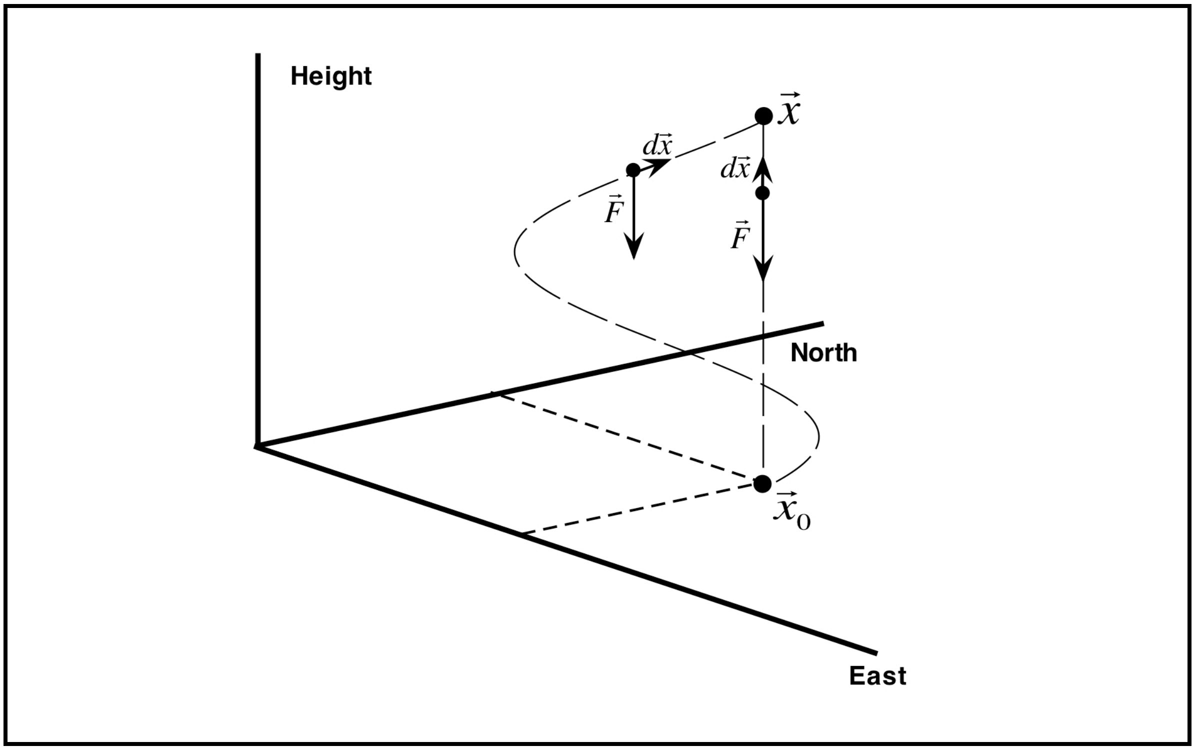

Figure 4.2-7 Displacement Path of a Rock Lifted Against Gravity

Note that the net energy gained when the rock is mover around a closed path, from $\vec{x}_0$ to $\vec{x}_1$ and back again to $\vec{x}_0$ is zero, i.e.:

$$\oint \vec{F}(\vec{x}) \bullet d\vec{x} \equiv 0 \qquad (4.2\text{-}10)$$

Using the notation of Equation 4.2-10 we may now state a result of what is commonly known as *Stokes's Theorem.*[10] According to this theorem, the following three statements are equivalent:

$$\oint \vec{F}(\vec{x}) \bullet d\vec{x} \equiv 0 \quad \Leftrightarrow \quad F_i(\vec{x}) \equiv \frac{\partial E(\vec{x})}{\partial x_i} \ \ \forall i \quad \Leftrightarrow \quad \frac{\partial F_i(\vec{x})}{\partial x_k} \equiv \frac{\partial F_k(x)}{\partial x_i} \ \ \forall i,k \qquad (4.2\text{-}11)$$

[10] A proof of this theorem can be found in many undergraduate physics texts. See Lorrain and Corson (1970) pp.16-22

The middle and right hand terms are familiar from our discussion of complete differentials. The left hand term states that the integral of $\vec{F}(\vec{x})$ around *any* closed path is zero. To express this in a form that can be applied to the problem of integrability, note that *any* two points $\vec{x}_0$ and $\vec{x}_1$ we might choose can be incorporated into a closed path as shown in Figure 4.2-8

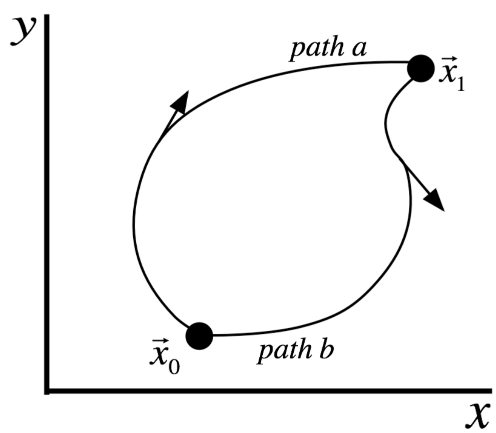

Figure 4.2-8 A Closed Path of Integration

From the left side of Equation 4.2-8, we know that:

$$\oint \vec{F}(\vec{x})d\vec{x} = 0 \quad \Rightarrow \quad \int_{\vec{x}_0 \quad path\ a}^{\vec{x}_1} \vec{F}(\vec{x})d\vec{x} = \quad -\int_{\vec{x}_1 \quad path\ b}^{\vec{x}_0} \vec{F}(\vec{x})d\vec{x} \tag{4.2-12}$$

If the direction of travel on path b were reversed, we would have:

$$\int_{\vec{x}_0 \quad path\ a}^{\vec{x}_1} \vec{F}(\vec{x})d\vec{x} = \quad \int_{\vec{x}_0 \quad path\ b}^{\vec{x}_1} \vec{F}(\vec{x})d\vec{x} \tag{4.2-13}$$

Stokes' Theorem thus implies that when the components of $\vec{F}(\vec{x})$ satisfy the right hand statement of Equation (4.2-12), its integral from any point $\vec{x}_0$ to another point $\vec{x}_1$ will be independent of the path taken.

4.3) The Marginal value and Use Value Functions

We now have the mathematical tools needed to properly define the (vector) marginal value and use-value functions. For an economy in which n commodities are present, the set of all possible bundles an agent might possess can be represented as an n-dimensional space, with basis vectors $\hat{\varphi}_i$ defining the axes against which the x_i and $r_i(.)$ are measured. The vector function $\vec{r}(\vec{x})$ is defined formally as follows:

<u>**Definition: Marginal Price**</u>
For a consumer possessing a bundle $x_1, x_2 \ldots x_n = \vec{x}$, and marginal prices $r_i(x_1, x_2, \ldots x_n) = r_i(\vec{x})$ for each commodity x_i, the consumer's marginal price function $\vec{r}(\vec{x})$ is defined by:

$$\vec{r}(\vec{x}) = r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 + \cdots + r_n(\vec{x})\hat{\varphi}_n \tag{4.3-1}$$

We now define the use-value function, which is analytically equivalent to the utility function, though its intuitive interpretation is somewhat different. Defining the use-value function is quite easy. Developing an intuitive understanding of what the definition means will take some effort.

<u>**Definition: Use-Value**</u>
Given a consumer with marginal prices given by $\vec{r}(\vec{x})$, the components of which satisfy Assumption (4), the use-value a consumer places on a bundle of goods $\vec{x}'$, measured with respect to the value she places on some other bundle $\vec{x}^0$ is defined to be:

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\vec{x}^0}^{\vec{x}'} \vec{r}(\vec{x}) \bullet d\vec{x} \qquad (4.3\text{-}2)$$

where integral is evaluated over any path between $\vec{x}^0$ and $\vec{x}'$.

Note that this is a *definite integral* from which a function $V(\vec{x}')$ is defined as measured with respect to a reference bundle $\vec{x}^0$. This is important in empirical work because in practice, identifying a consumer who has no goods at all would be difficult to do. Measurements can thus be made with respect to a minimum, or subsistence reference bundle of the analyst's choosing. The locus of points for which $V(\vec{x}')$ equals some constant is an *iso-value* curve (or surface), which is equivalent to an indifference curve. Depending on the analysis, it may be convenient to represent the consumer's characteristics with a network diagram showing both his marginal prices and his indifference curves as given in Figure 4.3-2

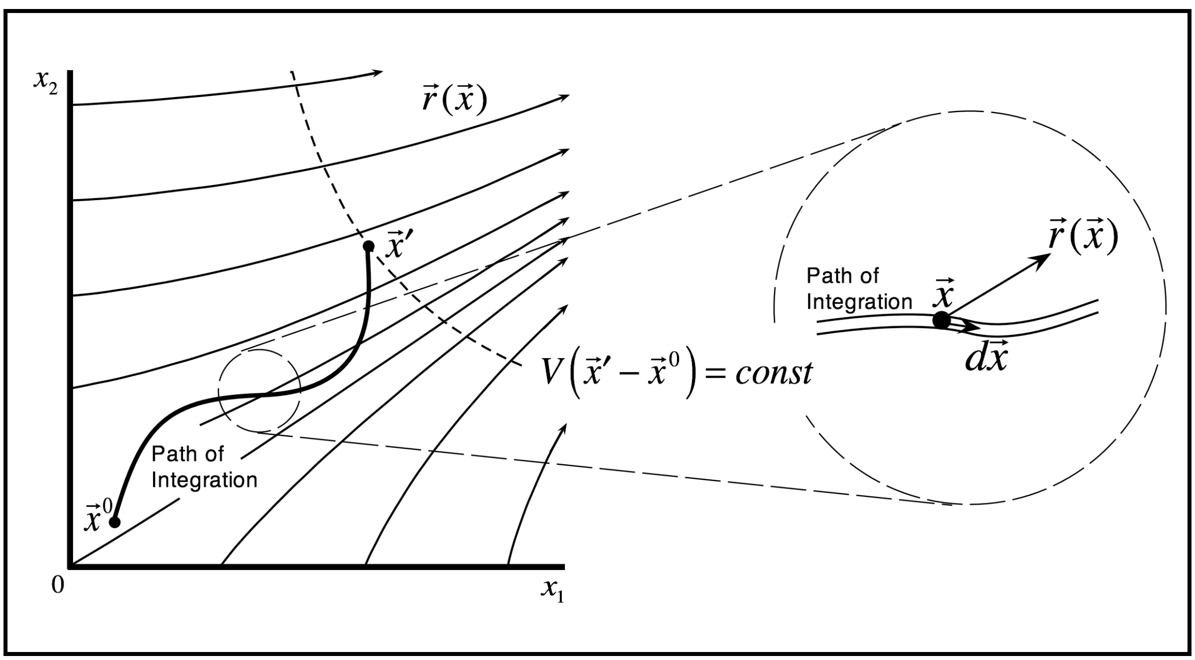

Figure 4.3-1 Path of Integration to an Indifference Curve

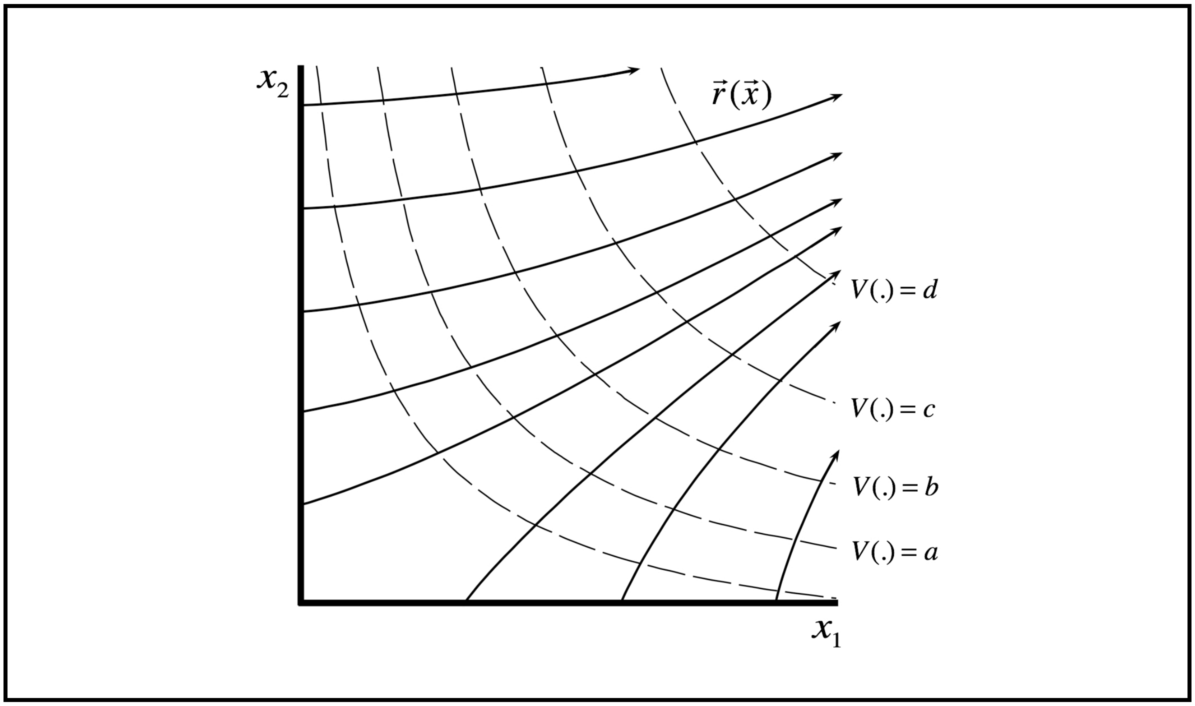

Figure 4.3-2 Stream Lines and Indifference Curves

The path of integration corresponds to the order in which the consumer acquires his goods. Whether or not such order should have any economic meaning was a topic of debate from the time Pareto first introduced it in 1906 [11] until Samuelson and Houthakker put the matter to rest in 1950.[12] To gain an intuitive understanding of what the path of integration means, consider an economy consisting of two goods: x_1 and x_2 plus numeraire M. The consumer begins with some initial bundle $\vec{x}^0 = x_1^0\hat{\varphi}_1 + x_2^0\hat{\varphi}_2$. The consumer is then given a small amount of M, which he immediately uses to purchase a differential increase to his bundle $d\vec{x}$. Depending on the consumer's marginal prices $\vec{r}(\vec{x}^0)$ and market prices $\vec{p}$, the consumer may spend all his

[11] See Pareto (1971)

[12] See Samuelson (1950) and Houthakker (1950)

numeraire on x_1 (i.e. $d\vec{x} = dx_1\hat{\varphi}_1$), he may spend it all on x_2, or some combination of both (i.e. $d\vec{x} = dx_1\hat{\varphi}_1 + dx_2\hat{\varphi}_2$). This process repeats indefinitely.

As time goes on, his bundle $\vec{x}$ grows and his marginal prices $\vec{r}(\vec{x})$ shift with each new acquisition. This process, plus changes in market prices over time causes the mix of goods purchased to change with each purchase, producing the path of integration shown in Figure 4.3-1[13]. At some point in time, when the consumer's bundle has grown to $\vec{x}'$, we stop the process to determine the use-value he places on all that he has acquired since the process began. By equation 4.3-2, this value is simply the sum of the values placed on each incremental bundle acquired. In general, this integral could be very difficult to calculate unless the path of consumption was known to follow a convenient shape. If, however, Assumption (4) was satisfied, Stokes' Theorem (Equations 4.2-8) indicates that the value the consumer places on his goods is independent of the order in which the goods are acquired. This is of course the same conclusion at which Samuelson and Houthakker arrived, using an argument remarkably similar to Stokes' Theorem. In addition to answering questions of integrability, Stokes' Theorem provides the means by which evaluation of these integrals can be made tractable. It allows the analyst to choose *any* path between $\vec{x}^0$ and $\vec{x}'$. Usually these paths can be broken into "legs" that run parallel to the coordinate axes. Along each leg, the consumer acquires only one good, thus reducing the integral along that leg to one of a single variable. In the following example, consider a consumer with marginal prices given by:

[13] Samuelson has argued that this path of consumption takes place "behind the scenes of the market" and is thus analytically irrelevant (See Samuelson (1950) p. 361). His model of market processes is different from what is used here.

$$\vec{r}(\vec{x}) = r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 = \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}\hat{\varphi}_1 + \left(\frac{x_1}{x_2}\right)^{\frac{1}{2}}\hat{\varphi}_2 \qquad (4.3\text{-}3)$$

Assumption (4) is satisfied since: $\dfrac{\partial r_1}{\partial x_2} = \dfrac{\partial r_2}{\partial x_1} = \dfrac{1}{2}\left(x_1 x_2\right)^{-\frac{1}{2}}$ (4.3-4)

Since $\overline{r}(\vec{x})$ is not defined at the origin we choose a point $\vec{x}^{\varepsilon} = \varepsilon_1\hat{\varphi}_1 + \varepsilon_2\hat{\varphi}_2$, differentially close to it as shown if Figure 4.3-3. The consumer's final bundle is $\vec{x}' = x_1'\hat{\varphi}_1 + x_2'\hat{\varphi}_2$. We choose the path of integration to be as shown in Figure 4.3-3. Along Path Leg 1, the consumer acquires only x_1 proceeding from $\vec{x}^{\varepsilon}$ to $\vec{x}^1$. Since this leg is parallel to the x_1 axis, its path element is $d\vec{x} = dx_1\hat{\varphi}_1$. Along Path Leg 2, the consumer acquires only x_2 as she proceeds from $\vec{x}^1$ to $\vec{x}'$. The path element for the second leg is $d\vec{x} = dx_2\hat{\varphi}_2$.

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\vec{x}^0}^{\vec{x}^1}\left[r_1\left(\vec{x}\right)\hat{\varphi}_1 + r_2\left(\vec{x}\right)\hat{\varphi}_2\right] \bullet dx_1\varphi_1 + \int_{\vec{x}^1}^{\vec{x}'}\left[r_1\left(\vec{x}\right)\hat{\varphi}_1 + r_2\left(\vec{x}\right)\hat{\varphi}_2\right] \bullet dx_2\varphi_2 \qquad (4.3\text{-}5)$$

Multiplying out the dot product, using $\hat{\varphi}_1 \bullet \hat{\varphi}_1 = \hat{\varphi}_2 \bullet \hat{\varphi}_2 = 1$ and $\hat{\varphi}_1 \bullet \hat{\varphi}_2 = 0$, we see that $r_2(\vec{x})$ contributes nothing to the first integral (since it is perpendicular to the path), while $r_1(\vec{x})$ contributes nothing to the second integral. We thus have:

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\substack{x_1 = \varepsilon \\ x_2 = \varepsilon}}^{\substack{x_1 = x_1' \\ x_2 = \varepsilon}} r_1\left(x_1, x_2\right) dx_1 + \int_{\substack{x_1 = x_1' \\ x_2 = \varepsilon}}^{\substack{x_1 = x_1' \\ x_2 = x_2'}} r_2\left(x_1, x_2\right) dx_2$$

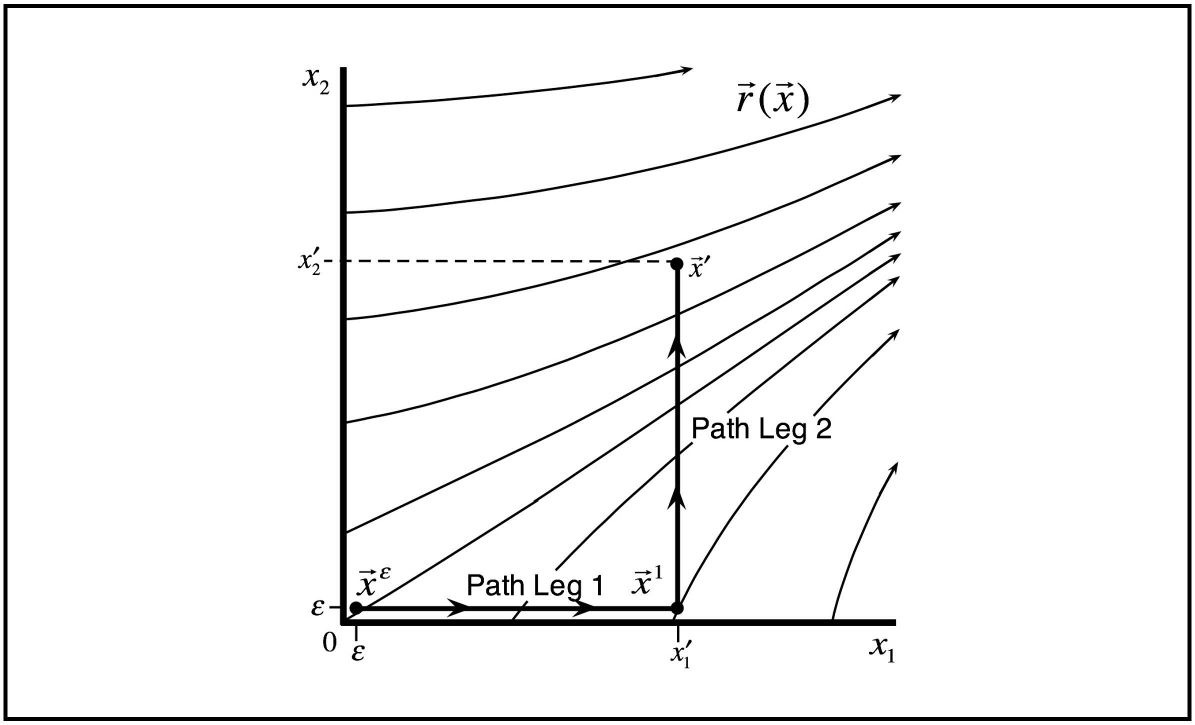

Figure 4.3-3 One Path of Integration

Using the fact that x_2 is a constant equal to ε in the first integral, and x_1 is a constant equal to x_1' in the second, we proceed with the solution:

$$
\begin{aligned}
V\left(\vec{x}' - \vec{x}^0\right) &= \int_{\substack{x_1=\varepsilon \\ x_2=\varepsilon}}^{\substack{x_1=x_1' \\ x_2=\varepsilon}} \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}} dx_1 \quad + \int_{\substack{x_1=x_1' \\ x_2=\varepsilon}}^{\substack{x_1=x_1' \\ x_2=x_2'}} \left(\frac{x_1}{x_2}\right)^{\frac{1}{2}} dx_2 \\
&= \sqrt{\varepsilon} \int_{x_1=\varepsilon}^{x_1=x_1'} \frac{dx_1}{\sqrt{x_1}} \quad + \sqrt{x_1'} \int_{x_2=\varepsilon}^{x_2=x_2'} \frac{dx_2}{\sqrt{x_2}} \\
&= 0 \quad + \frac{1}{2}\sqrt{x'x_2'}
\end{aligned}
$$

In the last step, we have used the result that ε is approximately zero.

We now have a simple Cobb-Douglas use -value function $V(\vec{x}')$ that gives the value the consumer places on any bundle $\vec{x}'$, measured with respect to the origin. Notice how the complementary effects between the goods influence the integrals. To place value on either good, the consumer must have at least some of the other. On the first leg of the path, the consumer's holding of x_2 is essentially zero. This effectively suppresses any value he might place on x_1; hence the use-value accumulated along Path Leg 1 is zero. However, as the consumer proceeds to the second leg, his holdings of x_1 begin to compliment the value he places on increments of x_2 as they are acquired.

Now that we have demonstrated the means by which these path integrals can be evaluated, we can use them to show that violations of Assumption (4) produce results that are intuitively absurd.

Consider a consumer whose marginal price function is given by:

$$\vec{r}(\vec{x}) = r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 = \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}\hat{\varphi}_1 + \left(\frac{1}{x_2}\right)^{\frac{1}{2}}\varphi_2 \qquad (4.3\text{-}8)$$

We see that x_2 compliments x_1 since $r_1(\vec{x})$ contains a quotient of both variables. The reverse, however, is not the case, which violates Assumption (4). Integrating $\vec{r}(\vec{x})$ over the path used above we find:

$$
\begin{aligned}
V\left(\vec{x}'-\vec{x}^{0}\right) &= \int_{\substack{x_1=\varepsilon\\ x_2=\varepsilon}}^{\substack{x_1=x_1'\\ x_2=\varepsilon}} r_1\left(x_1,x_2\right)dx_1 &&+ \int_{\substack{x_1=x_1'\\ x_2=\varepsilon}}^{\substack{x_1=x_1'\\ x_2=x_2'}} r_2\left(x_2\right)dx_2 \\
&= \int_{\substack{x_1=\varepsilon\\ x_2=\varepsilon}}^{\substack{x_1=x_1'\\ x_2=\varepsilon}} \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}} dx_1 &&+ \int_{\substack{x_1=x_1'\\ x_2=\varepsilon}}^{\substack{x_1=x_1'\\ x_2=x_2'}} \left(\frac{1}{x_2}\right)^{\frac{1}{2}} dx_2 \\
&= \sqrt{\varepsilon}\int_{x_1=\varepsilon}^{x_1=x_1'} \frac{dx_1}{\sqrt{x_1}} &&+ \int_{x_2=\varepsilon}^{x_2=x_2'} \frac{dx_2}{\sqrt{x_2}} \\
&= 0 &&+ 2\sqrt{x_2'}
\end{aligned}
\qquad (4.3\text{-}9)
$$

In this case, we find our consumer to be somewhat of a "Consumer in Wonderland" facing circumstances that become "curiouser and curiouser."[14] As before, the near-zero value of x_2 over the entirety of Path Leg 1 prevents the consumer from realizing any value from his acquisition of x_1. Since x_1 does *not* compliment x_2 ($r_2(\vec{x})$ is independent of x_1), the value the consumer derives along Path Leg 2 is *not* influenced by his prior consumption of x_1, as was the case in the previous example. The consumer thus places no value on his quantity of x_1, simply because he acquired it *before* he acquired any x_2.

We now reverse the order of consumption by integrating over the path shown in Figure 4.3-4 . Here the consumer acquires only x_2 along Path Leg 3, making $d\vec{x} = dx_2\hat{\varphi}_2$ along that leg. Along Path Leg 4 he acquires only x_1, thus $d\vec{x} = dx_1\hat{\varphi}_1$. Multiplying out the dot product we have:

[14] After all, Lewis Carroll was a mathematician!

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\vec{x}^0}^{\vec{x}^2} \left[r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 \right] \bullet dx_2\varphi_2 + \int_{\vec{x}^2}^{\vec{x}'} \left[r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 \right] \bullet dx_1\varphi_1$$

$$= \int_{\vec{x}^0}^{\vec{x}^2} r_2(x_1, x_2)dx_2 \qquad + \int_{\vec{x}^2}^{\vec{x}'} r_1(x_1)dx_1 \tag{4.3-10}$$

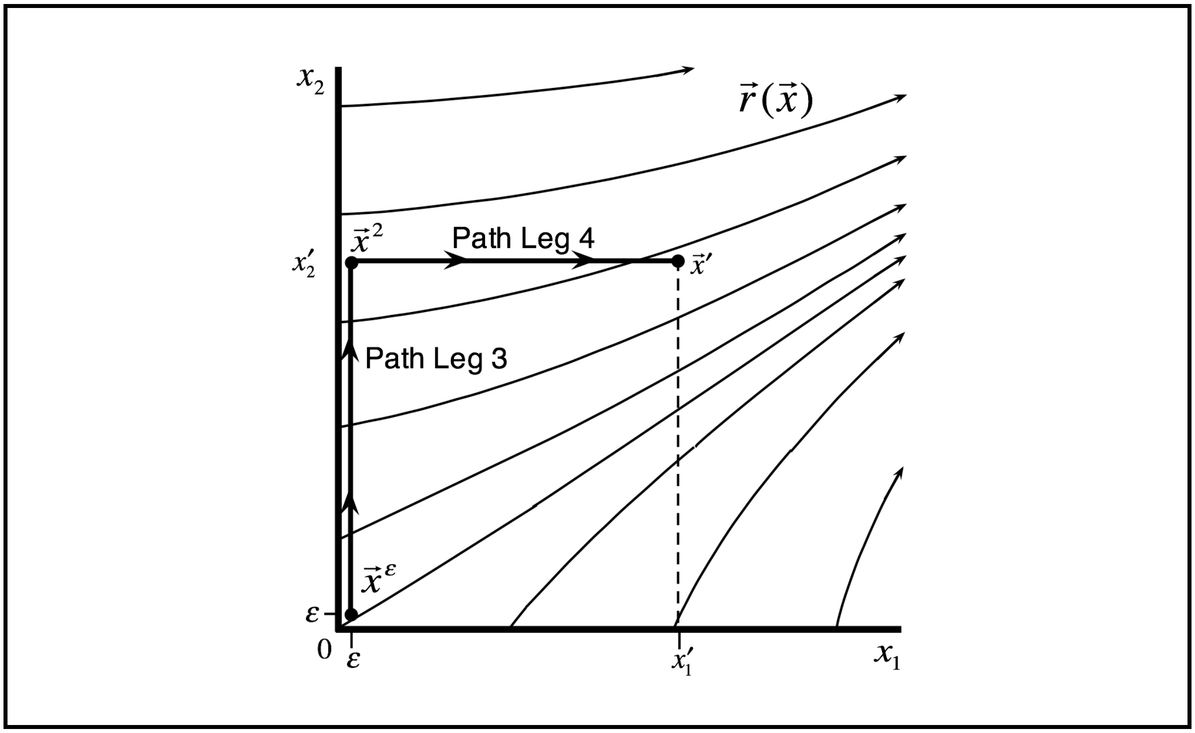

Figure 4.3-4 An Alternate Path of Integration

Completing the problem we find:

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\substack{x_1=\varepsilon \\ x_2=\varepsilon}}^{\substack{x_1=\varepsilon \\ x_2=x_2'}} \left(\frac{1}{x_2}\right)^{\frac{1}{2}} dx_2 \quad + \int_{\substack{x_1=\varepsilon \\ x_2=x_2'}}^{\substack{x_1=x_1' \\ x_2=x_2'}} \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}} dx_1$$

$$= \int_{x_2=\varepsilon}^{x_2=x_2'} \frac{dx_2}{\sqrt{x_2}} \qquad + \sqrt{x_2'} \int_{x_1=\varepsilon}^{x_1=x_2'} \frac{dx_1}{\sqrt{x_1}} \tag{4.3-11}$$

$$= 2\sqrt{x_2'} \qquad + 2\sqrt{x_1' x_2'}$$

The consumer realizes a value of $2\sqrt{x_2'}$ along Path Leg 3 since it is unaffected by the consumer's lack of x_1. Since he has acquired x_2' *before* acquiring x_1, the presence of x_2 augments the value he places on x_1 as it is acquired along Path Leg 4. In the end, the order of consumption has provided the consumer considerably more value along the second path than the first. To make matters even "curiouser", consider what would happen if the consumer acquired his goods along Path Legs 3 and 4, then "un-acquired" them along Path Legs 2 and 1. (Integrating along a path in the opposite direction gives the value lost as goods are taken away from the consumer.) The value surrendered as x_1 is given up along Path Leg 2 just equals the value gained as it was acquired along Path Leg 3. Since the value of x_2 that can be gained (or given up) along Path Legs 1 and 4 are complimented by the presence of x_1, the consumer loses no value as he surrenders his x_2 along Path Leg 1 since his holding of x_1 is already gone! Our consumer, who began with next to nothing, is left in the same condition at the end, yet he assigns his impoverishment a value of $V(x^{\varepsilon}) = 2\sqrt{x_1'x_2'}$. This is completely absurd[15]. By the right and left hand statements of Stokes' Theorem (Equation 4.2-8) Assumption (4) is equivalent to saying that the value one places on a bundle of goods is a *function of the goods in the bundle*, not on how or when they were acquired. Thus, if a consumer who begins with some bundle $\vec{x}^0$ has goods given to him and then taken away again, the value he places on the initial bundle should be unchanged[16]. This latter statement is analogous to Samuelson's Strong Axiom of Revealed Preferences (SARP) from which the equivalent of Assumption (4) can be derived. A more

[15] Except for perhaps on Wall Street

[16] This later statement is essentially Samuelson's Strong Axiom of Revealed Preferences (SARP) from which the equivalent of Assumption (4) can be drawn. See Houthakker (1950) and Samuelson (1950) p.367

straightforward derivation of the equivalence of these two statements can be found in several undergraduate texts in Mathematics and Physics.[17]

4.4) The Assumption of Non Addiction

Assumption (5) generalizes the law of diminishing marginal utility to address complementary effects between goods. It is similar to, yet slightly stronger than, the convexity condition commonly used currently. All functions $\vec{r}(\vec{x})$ that satisfy Assumption (5) will necessarily produce a $V(\vec{x})$ with convex indifference surfaces. The reverse however will not necessarily be the case. As we will see, convexity (essentially a weak form of Assumption (5)) will be adequate to prove most of the results to be obtained in Chapter (5), but will not suffice in all cases.

We begin by stating the traditional law of diminishing marginal utility in terms of marginal prices:

$$\frac{\partial V}{\partial x_i} > 0, \quad \frac{\partial^2 V}{\partial x_1^2} < 0 \qquad \Rightarrow \qquad r_i(\vec{x}) > 0, \quad \frac{\partial r_i}{\partial x_i} < 0 \tag{4.4-1}$$

Consider the partial derivative of $r_i(\vec{x})$ in terms of its definition. Let the agent begin at point A with some bundle $\vec{x}_0 = x_1, x_2, \cdots x_n$ as shown in Figure 4.4-1. We now give him a small quantity Δx_i of the i[th] good. This number represents a positive displacement along the x_i axis, bringing him to point B with a bundle $x_0 + \Delta x_i$.

[17] See Lorrain and Corson (1970) pp.16-22

By Equation (4.4-1) it must be true that: $r_i(\vec{x}_0) > r_i(\vec{x}_0 + \Delta x_i)$. If all other components $r_i(\vec{x})$ remained unchanged, the vector $\vec{r}(\vec{x}_0 + \Delta x_i)$ would be rotated with respect to $\vec{r}(\vec{x}_0)$ back in the direction of $\vec{x}_0$ as shown in Figure 4.4-1. (Since $\vec{r}(\vec{x})$ is normal to the indifference surfaces it passes through, the element of the surface at B must be "tilted" with respect to its orientation at A, as shown.)

If the quantity Δx_i had been taken away from the agent rather than added, the agent would be at point C, for which $r_i(\vec{x}_0) < r_i(\vec{x}_0 - \Delta x_i)$. In this case, $\vec{r}(\vec{x}_0 + \Delta x_i)$ would be tilted *forward* with respect to $\vec{r}(\vec{x}_0)$, again in the direction of $\vec{x}_0$. In both cases, the change in $\vec{r}(\vec{x})$ resulting from the displacement Δx_i is in the *opposite* direction of the displacement.

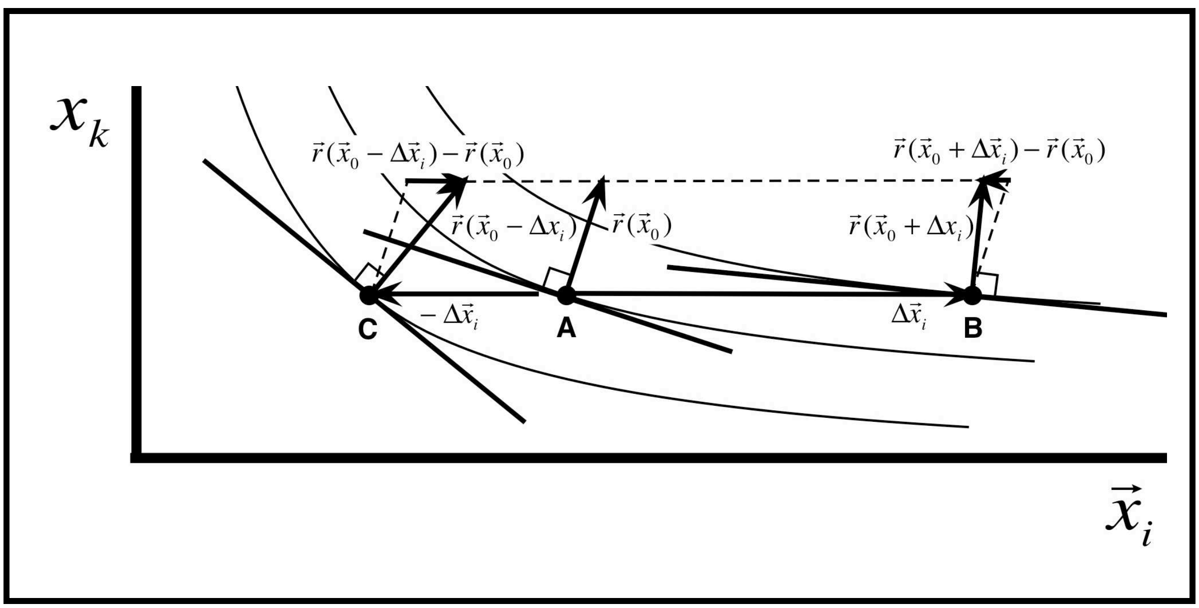

Figure 4.4-1 Marginal Price Vectors Illustrating Non-Addiction

We express this in terms of vector algebra as:

$$\left[r_i(\vec{x}+\Delta x_i) - r_i(\vec{x}) \right] \Delta x_i < 0 \qquad \forall x_i \tag{4.4-2}$$

Equation 4.4-2, like Equation 4.4-1 only considers displacements parallel to one of the coordinate axes. This corresponds to the consumer's having acquired an incremental quantity of only one good. The consumer may however be acquiring quantities of several different goods, and we need to insure that he is not addicted to them *in any combination*. An example of such a combination might be a cocktail that an alcoholic is unable to resist, while he might find the constituent ingredients alone noxious enough to stay away from. Another example might be a lifestyle requiring consumption of goods as a group. The assumption, described geometrically by Figure 4.4-1 must hold for a displacement $\Delta\vec{x}$ in any direction, not simply those Δx_i parallel to a coordinate axis. To account for this we substitute the vector $\Delta\vec{x}$ for Δx_i in the argument of $\vec{r}(\vec{x})$. We can now state Assumption (5) in formal terms, after we make the following definition:

Definition (Addiction)

For a consumer possessing a bundle $x_1, x_2 \ldots x_n = \vec{x}$, and with marginal prices $\vec{r}(\vec{x})$, the consumer is said to be *addicted* to some good x_i, or set of goods, $(x_i x_k \ldots)$ if an incremental bundle,

$$\Delta\vec{x} = \sum_{i=1}^{n} \Delta x_i \hat{\varphi}_i \tag{4.4-3}$$

can be constructed so that the consumption of which would cause the consumer's marginal price for some good x_l to be non-decreaseing[18] i.e.:

$$\left[r_l(\vec{x}+\Delta\vec{x}) - r_l(\vec{x}) \right] \Delta x_l \geq 0 \tag{4.4-4}$$

[18] To be completely rigorous, strong addiction and weak addiction should be defined in terms of whether or not the inequality in Equation 2.4-6 is strict. That detail is omitted here, as it does not contribute significantly to the argument.

Assumption (5)
There is no good or set of goods to which the consumer is addicted. In other words for all goods $x_i \; i = (1,2,\ldots n)$ the following inequality must hold:

$$\left[r_i\left(\vec{x}+\Delta\vec{x}\right)-r_i(\vec{x})\right]\Delta x_i < 0 \qquad \forall x_i \tag{4.4-5}$$

To demonstrate that inequality 4.4-5 addresses complementary effects, we expand the bracketed expression in equation 4.4-5 using the mean value theorem[19].

$$\left[r_i\left(\vec{x}+\Delta\vec{x}\right)-r_i\left(\vec{x}\right)\right]\Delta x_i = \Delta x_i \sum_k \frac{\partial r_i\left(\vec{x}+\theta\Delta\vec{x}\right)}{\partial x_k}\Delta x_k \qquad 0<\theta<1 \tag{4.4-6}$$

Since $\theta\Delta\vec{x}$ represents a very small displacement from $\vec{x}$, we can ignore it. Equation 4.4-5 becomes:

$$\left[r_i\left(\vec{x}+\Delta\vec{x}\right)-r_i\left(\vec{x}\right)\right]\Delta x_i = \Delta x_i \sum_k \frac{\partial r_i}{\partial x_k}\Delta x_k < 0 \tag{4.4-7}$$

The relationship between Assumption (5) and the assumption of convexity is given in the following proposition.

Proposition 4.4
For a consumer possessing a bundle $x_1, x_2 \ldots x_n = \vec{x}$, and with marginal prices $\vec{r}(\vec{x})$, If Assumption (5) holds for $\vec{r}(\vec{x})$ the indifference surfaces of $V(\vec{x})$ will be convex towards the origin.

[19] See Taylor and Mann (1983) p.204

Proof:

If inequality 4.4-5 holds for all goods, the sum of such inequalities taken over all goods x_i must be negative as well. Recognizing that such a sum represents a dot product we have:

$$\sum_{i=1}^{n}\left[r_i\left(\vec{x}+\Delta\vec{x}\right)-r_i(\vec{x})\right]\Delta x_i=\left[\vec{r}\left(\vec{x}+\Delta\vec{x}\right)-\vec{r}(\vec{x})\right]\bullet\Delta\vec{x}<0 \tag{4.4-8}$$

As before, we apply the mean value theorem to the bracketed term in the middle of Equation 4.4-8 obtaining:

$$r_i\left(\vec{x}+\Delta\vec{x}\right)-r_i\left(\vec{x}\right)=\sum_k\frac{\partial r_i\left(\vec{x}+\theta\Delta\vec{x}\right)}{\partial x_k}\Delta x_k \qquad 0<\theta<1 \tag{4.4-9}$$

Ignoring $\theta\Delta\vec{x}$, we substitute Equation 4.4-9 into the right side of Equation 4.4-8 obtaining:

$$\sum_i\sum_k\frac{\partial r\left(\vec{x}\right)}{\partial x_k}\Delta x_i\Delta x_k=\sum_i\sum_k\frac{\partial^2 V\left(\vec{x}\right)}{\partial x_k^2}\Delta x_i\Delta x_k<0 \tag{4.4-10}$$

The quadratic form on the right side of Equation 4.4-10 is the condition for convexity of $V(\vec{x})$ which is commonly expressed as a negative semi-definite matrix of second order partial derivatives.

QED

The dot product expression (Equation 4.4-8) is much easier to use than either Inequality 4.4-10 or Assumption (5). Since satisfaction of Assumption (5) implies inequality 4.4-8, the latter can be used in place of Assumption (5) in most cases.

4.5) Demand and Marginal Demand

The consumer choice or *unilateral exchamge* problem, in which a consumer trades goods with a "market" at predetermined prices, will be treated in detail in Chapter 5. In order to introduce the notions of Demand and Marginal Demand, the consumer choice problem will be revisited here briefly by applying the familiar LaGrange method to a use value function. Using the language of this chapter, the Consumer Choice problem is stated as:

$$\underset{\vec{x}}{MAX}\left\{V\left(\vec{x}\right)\right\} = \underset{x}{MAX}\left\{\int_{0}^{\vec{x}} \vec{r}(\vec{x}) \bullet d\vec{x}\right\} \quad \text{subject to:} \quad \vec{p} \bullet \vec{x} = w \tag{4.5-1}$$

The LaGrangian is:

$$L(\vec{x}, w) = \left\{\int_{0}^{\vec{x}} \vec{r}(\vec{x}) \bullet d\vec{x}\right\} - \lambda\left\{\vec{p} \bullet \vec{x} - w\right\} \tag{4.5-2}$$

Since all prices and values are measured in terms of a numeraire that is external to the problem, the LaGrange multiplier λ is 1. The first order conditions for the i^{th} good are:

$$\begin{gathered} 0 = \frac{\partial V}{dx_i} - p_i = r_i(\vec{x}^*) - p_i \\ 0 = \vec{p} \bullet \vec{x}^* - w \end{gathered} \tag{4.5-3}$$

where $\vec{x}^*$ is the maximizing bundle. Expressing the first order conditions for all goods as a single vector gives us $\vec{r}(\vec{x}^*) = \vec{p}$. The optimizing bundle can be expressed as a function of the consumer's wealth and market prices and is traditionally written $\vec{x}^* = \vec{x}^*(w, \vec{p})$. Using this, we define the wealth expansion path as follows:.

Definition (Wealth Expansion Path)
For a consumer described by a marginal price function $\vec{r}(\vec{x})$, facing constant market prices $\vec{p}$, his or her wealth expansion path is the vector function stating the optimal bundle of goods the consumer will choose as a function of his or her wealth. The wealth expansion path $\vec{x}^*(w)$ is the function that solves:

$$\vec{r}\left(\vec{x}^*\right) - \vec{p} = 0 \tag{4.5-4}$$

Geometrically, this path is represented as a curve through commodity space. This is a *parametric* curve, meaning that each component of the vector $\vec{x}^*(w)$ is a function of the single parameter w, i.e.:

$$\vec{x}^*(w) = \sum_{i=1}^{n} x_i^*(w)\hat{\varphi}_i \tag{4.5-5}$$

This will prove extremely useful in Chapter 6.

Since the time of Walras, the consumer has been usually modeled as receiving his entire stock of wealth w at once, and as purchasing his optimal bundle $\vec{x}^*(w)$ in a single decision. In his review of Walras' *Manuale,* Henri Poincaré severely criticized Walras for requiring that his consumer possess the "infinite foresight" need to purchase a lifetime of goods in a single instant[20]. In the dynamic model, The consumer can begin with some bundle $\vec{x}'$, with market value only slightly

[20] See Ingrao and Israel (1990) p.(195)

less than the budget constraint under consideration. To acquire his optimal bundle he needs only a small increment of wealth received in the form of income $\Delta w = \vec{p} \bullet (\vec{x}^* - \vec{x}')$ to acquire it.

In the dynamic model, the consumer is portrayed as receiving income as a stream of wealth $I = dw/dt$ acquired over time. The consumer is free to spend any received increment of wealth $\Delta w = I \Delta t$ (a paycheck if you will) on an incremental allotment of the n available goods $\Delta x_i \; (i = 1,2,\ldots n)$ such that:

$$\Delta w = \sum_{i=1}^{n} p_i \Delta x_i \qquad (4.5\text{-}6)$$

If we assume all goods are infinitely durable, the consumer's total wealth at any time t is represented by his accumulated bundle $\vec{x}[t]$. For the moment, we are not so much interested in how the consumer's behavior varies with time but with wealth. If all goods are durable, the consumer's wealth accumulates as a known function of time $w = w[t]$. If at each point in time the consumer posesses his optimal bundle, we can write: $\vec{x}^* = \vec{x}^*(w[t])$. This depicts the consumer as sliding outward along his wealth expansion path as shown in Figure 4.5-1.

It is apparent from the figure that the consumer's optimal bundle for any given wealth $\vec{x}^*(w[t])$ does not depend on whether it was acquired through several transactions or all at once. This quantity however does not necessarily represent the proportions of goods the consumer will obtain *at any given time*. Unless the consumer's use value functions are homothetic (i.e. have wealth expansion paths that are straight lines through the origin), the proportions of goods the consumer seeks to obtain at a given time will be given by $d\vec{x}^*/dw$ which will almost never be proportional to $\vec{x}^*(w)$. As will be discussed later, homothetic functions will rarely if ever match empirical reality. The market phenomena of interest depend on how consumers behave at a point in time. As will be shown, such behavior is best described in terms of first derivative of $\vec{x}^*(w)$ as will now be argued.

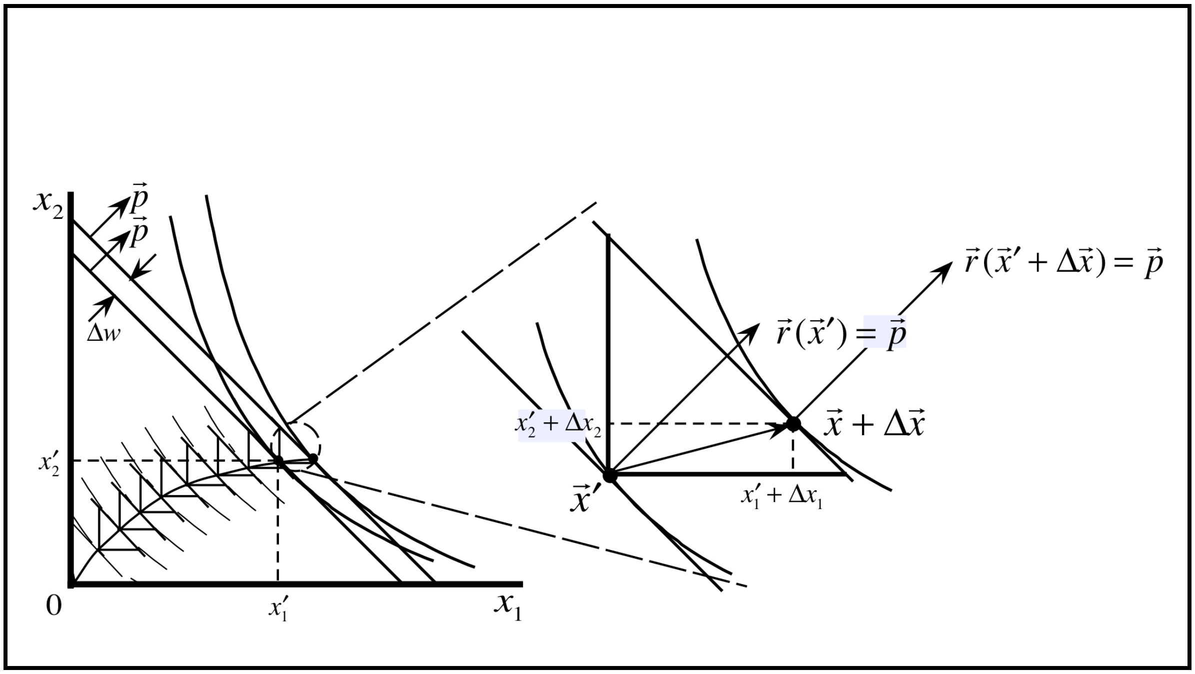

Figure 4.5-1 A Consumer's Bundle of Goods as Acquired Through a Series of Incremental Transactions

The consumer is modeled as receiving income I as a constant stream of wealth increments Δw in regular time intervals Δt. Prior the beginning of each pay period Δt, the consumer's wealth is represented by her bundle $\vec{x}^*[t]$. Each payment provides a budget $\Delta w[t]$, from which the consumer purchases an optimal incremental bundle $\Delta\vec{x}^*[t]$. This incremental bundle represents her demand during that time period. The goods purchased are presumed durable and are added to her bundle. Her bundle at the beginning of the next pay period $t+1$ is therefore $\vec{x}^*[t+1] = \vec{x}^*[t] + \Delta\vec{x}^*[t]$. The goods demanded in time period t can be expressed as:

$$\Delta\vec{x}^*[t] \quad = \left(\vec{x}^*[t] + \Delta\vec{x}^*[t]\right) - \vec{x}_t^* \quad = \vec{x}^*(w^*[t] + \Delta w^*[t]) - \vec{x}^*(w^*[t]) \qquad \text{(4.5-7)}$$

If $\Delta\vec{x}^*$ is "small" with respect to $\vec{x}'$, we can, in any time period make the approximation:

$$\Delta\vec{x}^* \quad = \frac{\Delta\vec{x}^*}{\Delta w}\Delta w \quad = \Delta w \ \frac{\partial}{\partial w}\left(\vec{x}^*\left(w, \vec{p}\right)\right) = \Delta w \ \frac{\partial}{\partial w}\left(\vec{x}^*\left(w \ , \vec{p}\right)\right) \qquad \text{(4.5-8)}$$

We can thus formally define *marginal demand* as follows:

Definition (Marginal Demand)
Given a consumer holding the optimal bundle $\vec{x}^*(\vec{p}, w[t])$ corresponding to his or her wealth $w[t]$, and market prices $\vec{p}$, and who is receiving income $I = dw/dt$. The consumer's marginal demand $\Delta\vec{x}^*(w[t], \vec{p}, I, \Delta t)$ is defined to be:

$$\Delta\vec{x}^* \triangleq I\Delta t \frac{\partial}{\partial w}\left[\vec{x}^*\left(\vec{p}, w[t]\right)\right] \qquad \text{(4.5-9)}$$

Notice that Δw has been replaced with $I\Delta t$, since the income the consumer receives may be independent of his stock of wealth. Additionally, Δt is included to allow the researcher to specify the time period over he wishes to observe the consumer's behavior (month, quarter, year, etc.)

4.6 Conclusion

From the way use value is defined, we see that it, as well as all other quantities used in analysis can be measured in terms of quantities that are readily observable. Since these quantities may be measured, the barriers to their comparison between individuals is removed. We will make extensive use of this in Chapter 6. Additionally, by using marginal prices as the foundation of analysis, consumer behavior can be analyzed as a sequence of events occurring over time, rather than as a single maximization decision. This will become the basis for the dynamic models of exchange equilibria to be presented in Chapter 5.

As we conclude, we need to look back on the notion of "rationality" which was much the topic of Chapter 2. Here we have assumed only that the consumer knows the value he or she places on things. We have made no assumption as to how he or she has arrived at those values. Such value may be the product of calculation intended to maximize wealth, or it may be the product of emotion, or both. The consumer may seek to maximize his own wealth (or pleasure), that of another, or seek some entirely different goal. To the model proposed here, it makes no difference, so long as the values the consumer assesses contain the logical consistency required by the five assumptions upon which the model is based.

CHAPTER 5:
DYNAMIC CONSUMER CHOICE AND EXCHANGE EQUALIBRIUM

We now turn our attention to the problem of consumer choice, and how such might bring about exchange equilibria. Here the dynamic aspects of the model become critical. With regard to the theory of General Equilibrium, the dynamic aspect of the model presented here allows us to grant the so-called Warasian Auctioneer a well-deserved (and long since needed) retirement.

Since the time of Walras, the tatonnement process, by which exchange equilibrium is achieved, has been modeled as occurring in an imaginary market, whose participants exchanged goods at prices called out by a virtual auctioneer. The first set of prices called out would usually result in unsold surpluses of some goods, and shortages of others. The auctioneer then adjusts the prices so as to at reduce the surpluses and shortages before initiating a new round of exchanges. Through repetition, this tatonnement, (groping) process allows the auctioneer to eventually arrive at a single set of prices that would “clear” the market, with no goods unsold or in short supply[1].

Such a hypothetical process may be sufficient to show that equilibria exist, but provides no real insight as to how they might be achieved in an actual market. As result, general equilibrium theory can say nothing about how markets that are out of equilibrium might behave, or even guarantee that equilibria would be stable once received.

The dynamic model presented in this dissertation solves these problems by breaking the achievement of equilibria into many small steps that occur in real time. The market is always “clear” in that all goods are owned by someone, while the marginal prices at which the consumers

[1] Groping now-days is generally grounds for arrest.

would be willing to trade goods may vary widely between individuals. In each step, buyers and sellers seek each other out and exchange marginal bundles of goods. Following their transactions, they each move on to different trading partners with whom they transact additional business. The process of exchange redistributes goods so as to equalize all the consumer's marginal prices into a common set of "market" prices. This is much the way random collisions of gas molecules in a vessel redistribute kinetic energy until equilibrium temperature and pressure are reached[2].

A distinct advantage of the dynamic tatonnement process is that it is unconditionally stable. Exchanges stop as soon as equilibrium is reached, and do not begin again unless the equilibrium is disturbed. Such disturbance may occur either through an exogenously induced change in the distribution of goods, or a change in the consumers' marginal prices. Should such disturbance occur, the tatonnement process begins again and continues until equilibrium is reestablished.

The structure of the propositions (or theorems) describing dynamic equilibria is as shown in Figure 5-1. The case of Multilateral Equilibrium (exchange of many goods among many consumers) is built up from the bilateral case, which in turn is built on the unilateral or "fixed price" case. Each case is modeled as a tatonnement process consisting of a series of marginal exchanges. The propositions describing the marginal exchanges show that each exchange will increase the use value enjoyed by the consumers involved, and adjust their marginal prices according to Assumption (5). The propositions describing the tatonnements simply show that the repeated marginal price adjustments will cause the consumer's marginal prices to converge to a market price.

[2] See Reiff [1965]

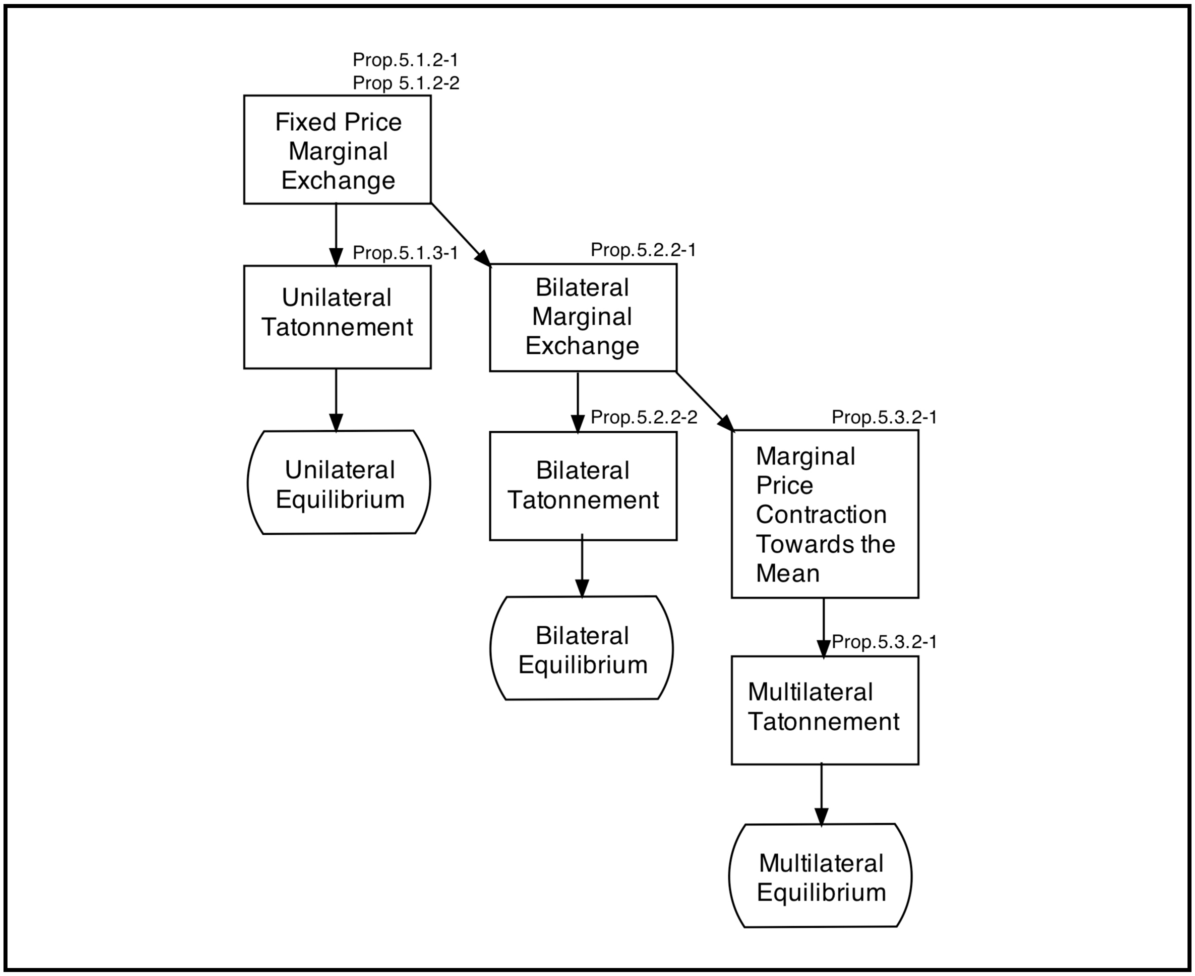

Figure 5-1 Structure of the Proofs

In the unilateral case, a single consumer exchanges marginal quantities of goods with a “market” at pre-determined fixed prices. The consumer buys goods for which his or her marginal price is higher than the fixed price, and sells goods for which her marginal price is lower. As result of the exchange, the consumer’s marginal prices for goods she has purchased falls, while her prices for goods she sold rises. In both cases, the consumer’s marginal prices contract towards the fixed price. In the tatonnement process, such exchanges continue until the consumer’s marginal prices match the market prices.

In the Bilateral case, two consumers meet and agree to exchange goods at mutually beneficial prices. Using the reasoning from the fixed price case, we show that such exchange draws the consumer's marginal prices closer together. In the tatonnement process, continual exchange of marginal bundles at constantly renegotiated prices causes the consumer's marginal prices to ultimately converge.

In the multilateral case, consumers meet in pairs that engage in a single bilateral marginal exchange, before paring off with different partners for a subsequent exchange. In addition to being drawn together, each exchange partner's marginal prices are drawn closer to the mean of the marginal prices for all consumers. In the tatonnement process, exchanges among different members of the community cause the marginal prices of all consumers to contract to the (continuously adjusting) mean, which becomes the set of "market" prices.

5.1) Unilateral Exchange

We begin our discussion with the case of a single consumer exchanging goods with a "market" that allows him or her to exchange as much of any good as he or she desires at fixed "market" prices.

After making the necessary definitions, we begin by showing that whenever the consumer's marginal prices $\vec{r}(\vec{x})$ do not equal $\vec{p}$, the consumer will benefit by exchanging a differentially small bundle of goods $d\vec{x}$. As result of the exchange, the consumer's marginal prices will be brought "closer" to $\vec{p}$ as shown in Figure 5.1-1. We then show that the consumer will continue to make these marginal exchanges until $\vec{r}(\vec{x}) = \vec{p}$. Finally, we show that the bundle $\vec{x}^*$, for which

$\vec{r}(\vec{x}^*) = \vec{p}$ is the one that offers the consumer the greatest use-value. This is done by showing that it the consumer were to continue making marginal exchanges once $\vec{x}^*$ has been acquired, she would begin to loose the use-value she had previously gained.

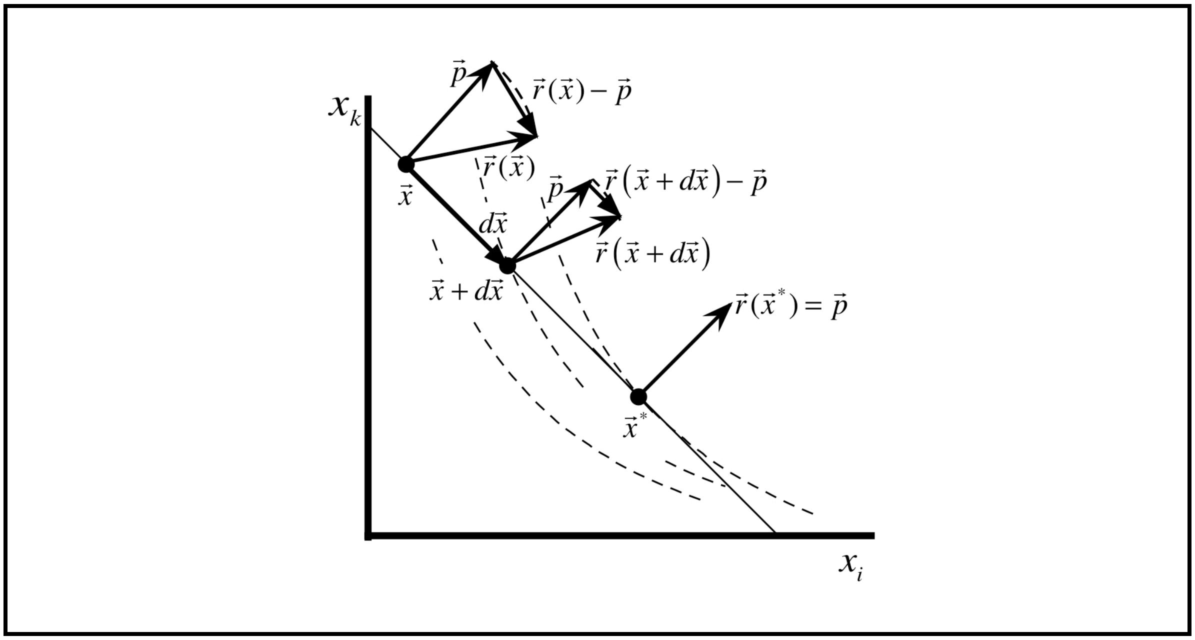

Figure 5.1-1 Unilateral Exchange

5.1.1) Definitions and Assumptions

In chapter 4 we formally stated only the first part of Assumption 1; that the consumer knows what his marginal prices are. We now need to state the second part, which indicated how he would respond to an opportunity for exchange. Intuitively, we want to say that the consumer will take advantage of a “good deal”, or will try to get the most benefit per unit of numeraire spent.

We begin by defining the “benefit” or “deal” that the consumer seeks to obtain by making a marginal exchange. This is simply his or her consumer’s marginal surplus, exactly as Dupuit envisioned it (see Section 3.2). It is the difference between what a consumer would be *willing* to pay for a marginal amount of a good, and what he is *required* to pay by an exchange partner.

Definition: Consumer's Marginal Surplus (for a single good)
For a consumer described by a marginal price function $\vec{r}(\vec{x})$, and holding a bundle $\vec{x}$, the marginal surplus the consumer would enjoy from purchasing (or selling) a differential quantity dx_i of some good x_i is given by:

$$\left[r_i(\vec{x}) - p_i\right]dx_i \tag{5.1.1-1}$$

Notice that if the consumer would be willing to pay more for the good than its market price, the consumer would gain surplus by acquiring the good. In this case, both $[r_i(\vec{x}) - p_i]$ and dx_i are positive, and so is the surplus. If the consumer values a good less than does the market, he gains surplus by selling some of it. In this case both $[r_i(\vec{x}) - p_i]$ and dx_i are negative, and the surplus is again positive.

We will assume that the consumer will try to maximize the surplus obtained for each transaction. This requires that he adjust the relative quantities of the goods dx_i bought and sold, which will be reflected in the direction of the vector $d\vec{x}$.

Assumption 1b
Given, a consumer described by a marginal price function $\vec{r}(\vec{x})$, and holding a bundle $\vec{x}$. For all goods x_i (and only for such goods) for which the consumer's marginal price $r_i(\vec{x})$ differs from the price p_i he or she is offered, the consumer will buy quantities dx_i, or sell quantities $-dx_i$ as necessary to gain the maximum total marginal surplus, subject to the budget constraint. $p_1dx_1 + p_2dx_2 + \cdots + p_ndx_n = \vec{p} \bullet d\vec{x} = 0$

By assumption 1b, the consumer solves the following problem:

$$\underset{\Delta x}{Max}\left\{\left(\vec{r}(\vec{x})-\vec{p}\right)\bullet d\vec{x}\right\}=\underset{\Delta x}{Max}\left\{\vec{r}(\vec{x})\bullet d\vec{x}-\vec{p}\bullet d\vec{x}\right\} \text{ s.t. } \vec{p}\bullet d\vec{x}=0 \qquad (5.1.1\text{-}2)$$

We can substitute the budget constraint into the objective function, re writing the problem as:

$$\underset{\theta}{Max}\left\{\left|\vec{r}(\vec{x})\right|\left|d\vec{x}\right|\cos\theta\right\} \qquad (5.1.1\text{-}3)$$

As can be seen from Figure 5.1.1-1, the objective function is maximized when the angle θ between $\vec{r}(\vec{x})$ and $d\vec{x}$ is minimized. This occurs when $d\vec{x}$ lies in the intersection of the budget plane, determined by $\vec{p}\bullet d\vec{x}=0$ and the plane determined by $\vec{r}(\vec{x})$ and $\vec{p}$. The exchange bundle $d\vec{x}$ has differential magnitude $\left|d\vec{x}\right|$, and has direction parallel to $\vec{r}(\vec{x})-\left(\vec{r}(\vec{x})\bullet\vec{p}\right)\vec{p}$. The marginal exchange bundle can therefore be written as:

$$d\vec{x}=\frac{\vec{r}(\vec{x})-\left(\vec{r}(\vec{x})\bullet\vec{p}\right)\vec{p}}{\left|\vec{r}(\vec{x})-\left(\vec{r}(\vec{x})\bullet\vec{p}\right)\vec{p}\right|}\left|d\vec{x}\right| \qquad (5.1.1\text{-}4)$$

This of course is simply the projection of the price difference $\vec{r}(\vec{x})-\vec{p}$ into the budget plane as shown in Figure 5.1.1-1.

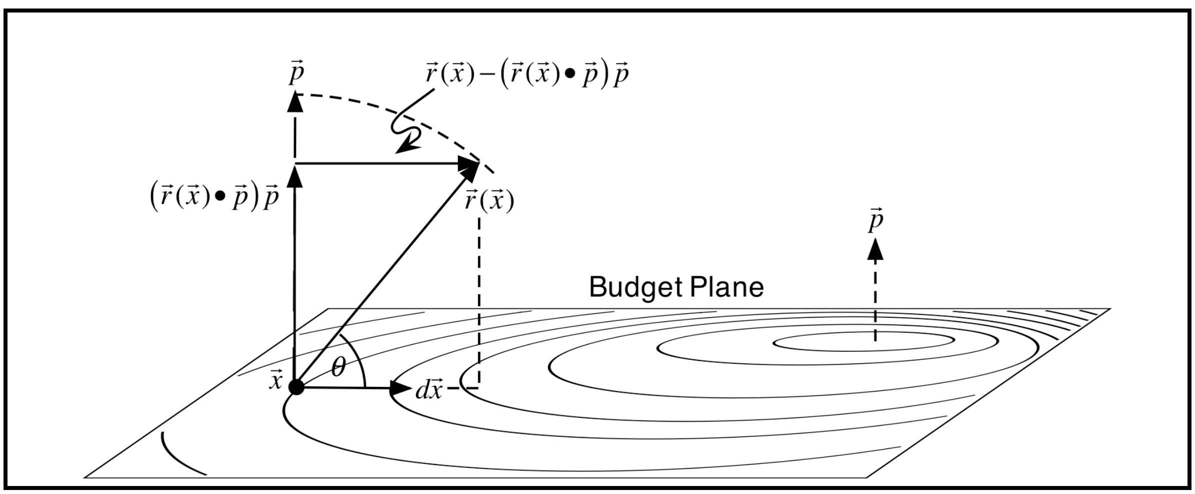

Figure 5.1.1-1 Orientation of $d\vec{x}$ Within the Budget Plane

5.1.2) Propositions and Proofs

The following two propositions describe the exchange of a marginal bundle between a consumer and “the market” at predetermined prices. The first of these propositions states in essence that whenever a consumer’s marginal prices differs from those he is offered, benefit from exchange is possible and the consumer will engage in a marginal exchange. When benefit from exchange is not possible, the consumer will refrain from exchange. This will be useful later in proving that equilibria are stable.

The second proposition indicates that, the consumer’s marginal prices will “contract” towards $\vec{p}$ with each marginal exchange, The collective difference between the consumer’s marginal prices and $\vec{p}$ is measured by the magnitude of the difference between the vectors. Before the consumer exchanges $d\vec{x}$ the difference between the prices is $\left|\vec{r}(\vec{x})-\vec{p}\right|$ while after the exchange it is $\left|\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right|$ as shown in Figure 5.1.2-1

<u>Proposition 5.1.2-1 (Benefit from Marginal Exchange)</u>
Given a consumer who is described by marginal price function $\vec{r}(\vec{x})$ and who possesses a bundle $\vec{x}$. If (and only if) the consumer is given the opportunity to exchange goods at a price $\vec{p}$ for which $\vec{r}(\vec{x}) \neq \vec{p}$, the following will result:

<u>a)</u> The consumer will exchange a small bundle $d\vec{x}$ constructed such that:

$$\left[r_i(\vec{x})-p_i\right]dx_i > 0 \quad \forall i \tag{5.1.2-1}$$

and

$$p_1dx_1 + p_2dx_2 + \cdots + p_ndx_n = \vec{p} \bullet d\vec{x} = 0\,. \tag{5.1.2-2}$$

b) Such exchange will increase the use value of the consumer’s holdings, i.e.

$$V(\vec{x}+d\vec{x}) > V(\vec{x})\,. \tag{5.1.2-3}$$

Proof of Part a:

By definition, $\vec{r}(\vec{x})$ and $\vec{p}$ are of unit magnitude, hence $\left[\sum_{i=1}^{n} r_i(\vec{x})^2\right]^{\frac{1}{2}} \equiv \left[\sum_{i=1}^{n} p_i^2\right]^{\frac{1}{2}} \equiv 1$.

If (and only if) there exists some good x_i for which $r_i(x) > p_i$ then there must be at least one good x_k for which $r_k(x) < p_k$. Assuming that goods are divisible, and the consumer already possesses some of the good (or goods) x_k, the consumer is able to devise an exchange bundle $d\vec{x}$ containing only goods, the exchange of which will grant the consumer a positive surplus while satisfying the budget condition $\vec{p} \bullet d\vec{x} = 0$. By Assumption 1b, the consumer will exchange this bundle. Therefore, the consumer will make a marginal exchange whenever his or her $\vec{r}(\vec{x}) \neq \vec{p}$, and will refrain from making an exchange when $\vec{r}(\vec{x}) = \vec{p}$. This completes the proof of part (A).

Proof of Part b:

The increase in use value that a customer holding a bundle $\vec{x}$ would gain by exchanging a bundle $d\vec{x}$ is by definition:

$$V(\vec{x} + d\vec{x}) - V(\vec{x}) = \int_{0}^{\vec{x}+d\vec{x}} \vec{r}\left(\vec{x}\right) \bullet d\vec{x} - \int_{0}^{\vec{x}} \vec{r}\left(\vec{x}\right) \bullet d\vec{x} = \vec{r}\left(\vec{x}\right) \bullet d\vec{x} \qquad (5.1.2\text{-}4)$$

By Assumption 1b, the marginal surplus the consumer gains from the exchange of all goods in $d\vec{x}$ is positive, hence:

$$0 < \sum_{i=1}^{n} \left[r_i\left(\vec{x}\right) - p_i\right] dx_i = \left[\vec{r}\left(\vec{x}\right) - \vec{p}\right] \bullet d\vec{x} = \vec{r}\left(\vec{x}\right) \bullet d\vec{x} - \vec{p} \bullet d\vec{x} \qquad (5.1.2\text{-}5)$$

Since the last term on the right is zero we have:

$$0 < \vec{r}\left(\vec{x}\right) \bullet d\vec{x} \tag{5.1.2-6}$$

Hence from Equation (5.1.2-4) we have $V(\vec{x}+d\vec{x}) > V(\vec{x})$ for every exchange. This completes the proof of Part B.

QED.

Proposition 5.1.2-2 (Price Contraction from Marginal Exchange)
Given a consumer described by marginal price function $\vec{r}(\vec{x})$ and possessing a bundle $\vec{x}$. If such consumer, who is given the opportunity to exchange goods at prices $\vec{p}$, exchanges a marginal bundle $d\vec{x}$ as defined by Assumption 1b, the differences between the consumer's $\vec{r}(\vec{x})$ and $\vec{p}$ will contract, i.e.:

$$\left|\vec{r}(\vec{x}) - \vec{p}\right| > \left|\vec{r}(\vec{x}+d\vec{x}) - \vec{p}\right| > 0 \tag{5.1.2-7}$$

Proof:

By convexity, (Equation 4.4-8) we have:

$$\left[\vec{r}(\vec{x}+d\vec{x}) - \vec{r}\left(\vec{x}\right)\right] \bullet d\vec{x} < 0 \tag{5.1.2-8}$$

Since $d\vec{x}$ is very small, we can assume from Equation 4.5-8 that $\vec{r}(\vec{x}+d\vec{x}) \bullet d\vec{x}$ is positive whenever $\vec{r}(\vec{x}) \bullet d\vec{x}$ is positive, thus:

$$\vec{r}\left(\vec{x}\right) \bullet d\vec{x} > \vec{r}(\vec{x}+d\vec{x}) \bullet d\vec{x} > 0 \tag{5.1.2-9}$$

Since $\vec{p} \bullet d\vec{x} = 0$ we can subtract it from all terms in Equation (5.2.1-8) without altering the inequality, leaving:

$$\left(\vec{r}(\vec{x})-\vec{p}\right)\bullet d\vec{x} > \left(\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right)\bullet d\vec{x} > 0 \tag{5.1.2-10}$$

Substituting $d\vec{x}$ from its value given in Equation (5.1.1-4) and cancelling the denominator, we have:

$$\begin{aligned}&\left(\vec{r}(\vec{x})-\vec{p}\right)\bullet\left(\vec{r}(\vec{x})-\left(\vec{r}(\vec{x})\bullet\vec{p}\right)\vec{p}\right) > \\ &\quad \left(\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right)\bullet\left(\vec{r}(\vec{x}+d\vec{x})-\left(\vec{r}(\vec{x}+d\vec{x})\bullet\vec{p}\right)\vec{p}\right) > 0\end{aligned} \tag{5.1.2-11}$$

Multiplying out the dot product and collecting terms leaves:

$$\left|\vec{r}(\vec{x})\right|^2 - \left(\vec{r}(\vec{x})\bullet\vec{p}\right)^2 > \left|\vec{r}(\vec{x}+d\vec{x})\right|^2 - \left(\vec{r}(\vec{x}+d\vec{x})\bullet\vec{p}\right)^2 > 0 \tag{5.1.2-12}$$

From the Pythagorean theorem we know that:

$$\begin{aligned}&\left|\vec{r}(\vec{x})\right|^2 - \left|\vec{r}(\vec{x})\bullet p\right|^2 = \left|\vec{r}(\vec{x})-\vec{r}(\vec{x})\bullet p\right|^2 = \left|\vec{r}(\vec{x})\right|^2\left(\sin\theta_1\right)^2 \\ &\left|\vec{r}(\vec{x}+dx)\right|^2 - \left|\vec{r}(\vec{x}+dx)\bullet p\right|^2 = \left|\vec{r}(\vec{x}+dx)-\vec{r}(\vec{x}+dx)\bullet p\right|^2 = \left|r(x+dx)\right|^2\left(\sin\theta_2\right)^2\end{aligned}$$

Since $\left|\vec{r}(\vec{x})\right| \equiv \left|\vec{r}(\vec{x}+dx)\right| \equiv 1$, from Equation 5.1.2-12 we have:

$$\sin\theta_2 < \sin\theta_1 \quad \Rightarrow \quad \theta_2 < \theta_1 \tag{5.1.2-13}$$

Since $\vec{p}$ also has unit magnitude, the decrease in angle indicates that $\vec{r}(\vec{x}+d\vec{x})$ is “closer” to $\vec{p}$ than is $\vec{r}(\vec{x})$, thus $\left|\vec{r}(\vec{x})-\vec{p}\right| > \left|\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right| > 0$ as claimed. This completes the proof.

<u>**QED**</u>.

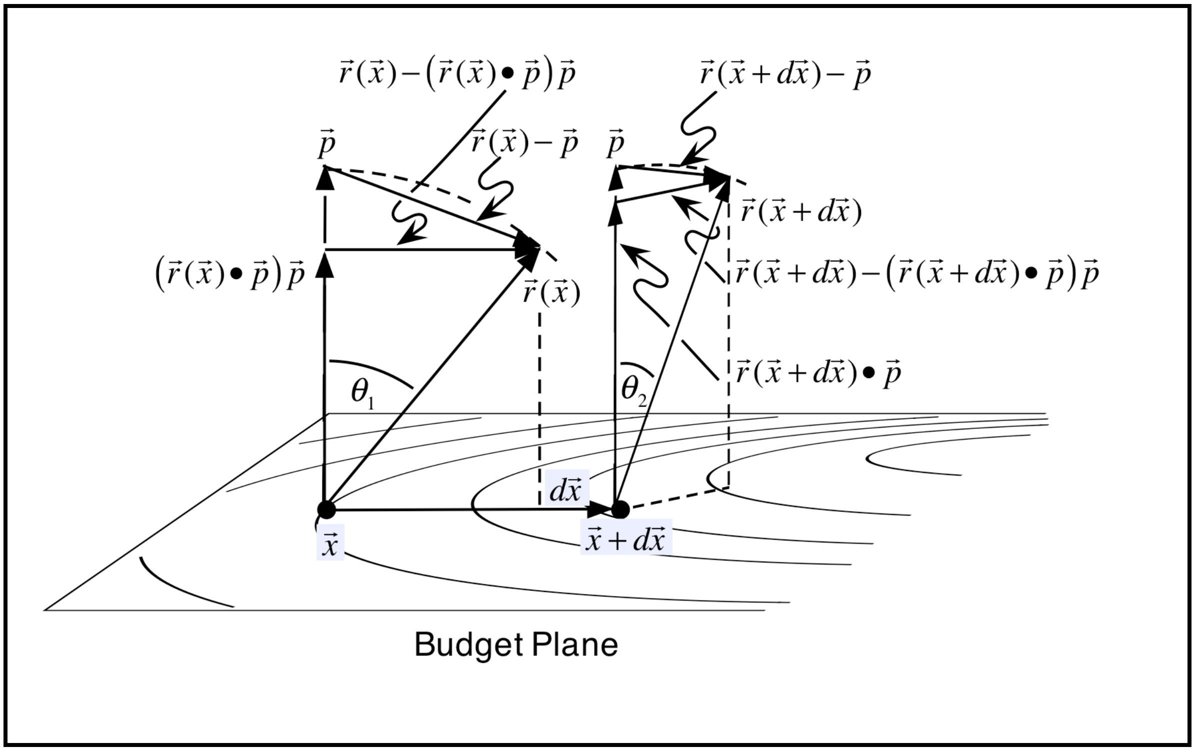

Figure 5.1.2-1 Marginal Price Contraction in a Unilateral Exchange

The final proposition of this section models the tatonnement process as a sequence of marginal exchanges made over time. Since the difference between the consumer's marginal prices and $\vec{p}$ reduce with each exchange, they muse eventually reach zero.

<u>Proposition 5.1.2-3: Unilateral Tatonnement</u>

Given a consumer described by marginal price function $\vec{r}(\vec{x})$, and at time t_0 possesses an initial bundle $\vec{x}[t_0]$. Given also that the consumer is given the opportunity to exchange any number of marginal bundles at a fixed prices $\vec{p}$. The consumer will, at t_0, and in future time periods t_0+n, exchange marginal bundles $d\vec{x}[t_0+n]$, until he attains a bundle $\vec{x}[t_0+z]$ for which $\vec{r}(\vec{x}[t_0+z])=\vec{p}$. Furthermore, the total use-value $V(x[t_0+z]-x[t_0])$ gained by the consumer will be the maximum available to him at prices $\vec{p}$ given his wealth.

Proof:

For every time period $t_0 + n$ for which the consumer's marginal prices $\vec{r}(\vec{x}[t_0 + n])$ do not equal $\vec{p}$, Proposition 5.1.2-1 implies that the consumer will exchange a marginal bundle $d\vec{x}[t_0 + n]$. As result, the use value the consumer enjoys will have increased, i.e.: $V(x[t_0 + n] + dx[t_0 + n]) > V(x[t_0 + n])$. Per Proposition 5.1.2-2 we know that for every time period we have $\left|\vec{r}(\vec{x}[t_0 + n]) - \vec{p}\right| > \left|\vec{r}(\vec{x}[t_0 + n] + d\vec{x}[t_0 + n]) - \vec{p}\right| > 0$.

At the beginning of every time period $t_0 + n + 1$, the consumer's bundle is simply the one he held previously, adjusted by the bundle exchanged, $\vec{x}[t_0 + n + 1] \triangleq \vec{x}[t_0 + n] + d\vec{x}[t_0 + n]$. Per Propositions 5.1.2-1 and 5.1.2-2 we thus have:

$$V(x[t_0 + n + 1]) > V(x[t_0 + n]) \tag{5.1.2-14}$$

$$\left|\vec{r}(\vec{x}[t_0 + n]) - \vec{p}\right| > \left|\vec{r}(\vec{x}[t_0 + n + 1] - \vec{p}\right| > 0 \tag{5.1.2-15}$$

From Equation 5.1.2-15 it is apparent that:

$$\underset{n \to \infty}{\text{limit}} \left|\vec{r}(\vec{x}[t_0 + n]) - \vec{p}\right| = 0 \tag{5.1.2-16}$$

For practical purposes, we will choose some number ε, that is negligibly close to zero. Since Equation 5.1.2-16 approaches zero monotonically, there must be some number $0 < z < \infty$ such that:

$$\left|\vec{r}(\vec{x}[t_0 + z]) - \vec{p}\right| < \varepsilon \tag{5.1.2-17}$$

Therefore, at least for practical purposes, $\vec{x}[t_0 + z]$ is the bundle for which the consumer's marginal prices equal $\vec{p}$.

According to Proposition 5.1.1-1 exchange will stop at this point, and will not restart as long as $\vec{p}$ or $\vec{x}[t_0 + z]$ remain unchanged.

To show that $V(\vec{x}[t_0 + z])$ provides the maximum use value available at prices $\vec{p}$, we assume for a moment that it does not. By Assumption 5, the indifference curves of $V(\vec{x})$ are convex. Thus, if $V(\vec{x}[t_0 + z])$ is not the maximum, there is some marginal bundle $d\vec{x}'$ the consumer could exchange, for which $V(\vec{x}[t_0 + z] + d\vec{x}') > V(\vec{x}[t_0 + z])$. If the consumer were to make such exchange, his marginal price vector $\vec{r}(\vec{x}[t_0 + z] + d\vec{x}')$ must satisfy Equation 4.4-5 (Assumption 5), thus:

$$\left[\vec{r}(\vec{x}[t_0 + z] + d\vec{x}') - \vec{r}(\vec{x}[t_0 + z])\right] \bullet d\vec{x}' < 0 \qquad (5.1.2\text{-}18)$$

Since at equilibrium $\vec{r}(\vec{x}[t_0 + z]) = \vec{p}$, and $\vec{p} \bullet d\vec{x}$ is always zero, Equation (5.1-18) becomes:

$$\vec{r}(\vec{x}[t_0 + z] + d\vec{x}') \bullet d\vec{x}' < 0 \qquad (5.1\text{-}19)$$

If the consumer, who now holds $x[t_0 + z] + d\vec{x}'$ were to reverse his exchange of $d\vec{x}'$, he would gain a positive surplus since: $\vec{r}(\vec{x}[t_0 + z] + d\vec{x}') \bullet (-d\vec{x}') > 0$. We thus have $V(x[t_0 + z] + d\vec{x}) < V(x[t_0 + z])$ which contradicts our temporary assumption. We have thus shown that $V(\vec{x}[t_0 + z])$ is the maximum value available to the consumer. This completes the proof. **<u>QED.</u>**

5.2) Bilateral Exchange

Bilateral exchange, or the exchange of goods between two individuals, is modeled as an extension of the of the fixed price exchange. Two individuals with marginal different prices $\vec{r}^{\,1}(\vec{x}^{1})$ and $\vec{r}^{\,2}(\vec{x}^{2})$ which are functions of the bundles $\vec{x}^{1}$ and $\vec{x}^{2}$ they respectively hold, engage in a sequence of bilateral marginal exchanges. In each round, the individuals agree to a price $\vec{p}$ that lies "between" their marginal prices, at which the bundle $d\vec{x} = d\vec{x}^{1} = -d\vec{x}^{2}$ is to be exchanged. Once the price has been agreed upon, the remainder of the exchange is, to each consumer, no different from a fixed price exchange. We know therefore that each marginal exchange benefits each consumer, and causes his marginal prices to contract towards $\vec{p}$. Since $\vec{p}$ is "between" $\vec{r}^{\,1}(\vec{x}^{1})$ and $\vec{r}^{\,2}(\vec{x}^{2})$, we show that these marginal prices have contracted towards each other. In the tatonnement process, we show that the contraction continues until the consumer's marginal prices merge into the equilibrium price. Finally we show that the use value enjoyed by each consumer is the maximum available to them given the bundles they started out with, and the equilibrium price.

5.2.1) More Definitions

We begin by defining what it means for a vector to lay "between" another pair of vectors. There are two different notions of "between-ness" that we will have occasion to use. The first applies to a vector that lies in the same plane as the vectors it is "between". Such a vector can be described algebraically in terms of the other vectors. The second notion of "between-ness" applies to a vector whose components lie between the components of the bounding vectors. Such a vector lies in the hyper-rectangular region of space defined by the vectors it is said to be between, as shown in Figure 5.2.1-1.

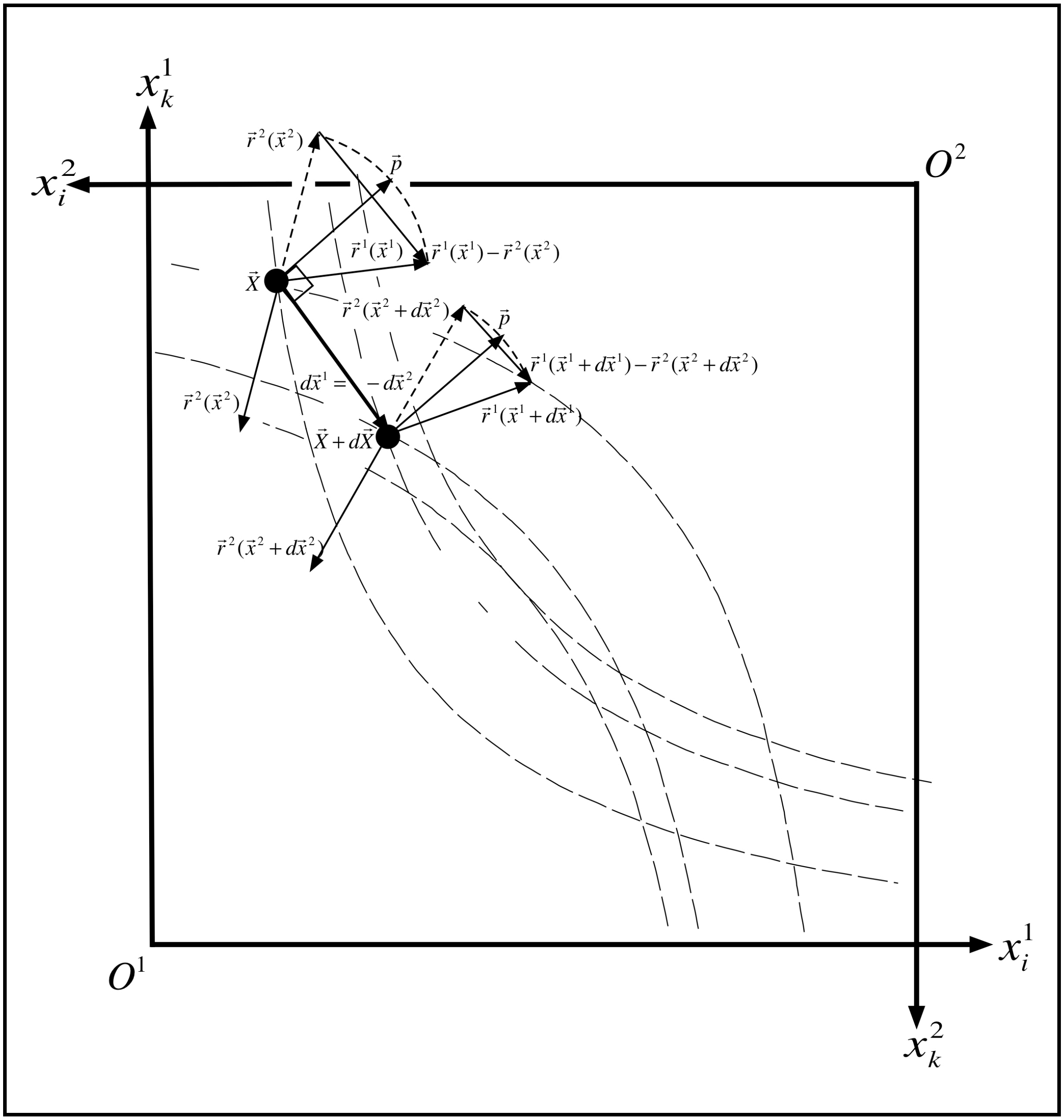

Figure 5.2-1 Edgeworth Box Diagram of a Bilateral Exchange

<u>Definition [A vector that lays “Between” a pair of vectors]</u>

Given Three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, each of n components: Vector $\vec{B}$ lays “between” $\vec{A}$ and $\vec{C}$, if and only if $\vec{B}$ can be expressed in the form: $j\vec{B}=\vec{C}+k(\vec{A}-\vec{C})$ where. $0<k<1$ and $j>0$.

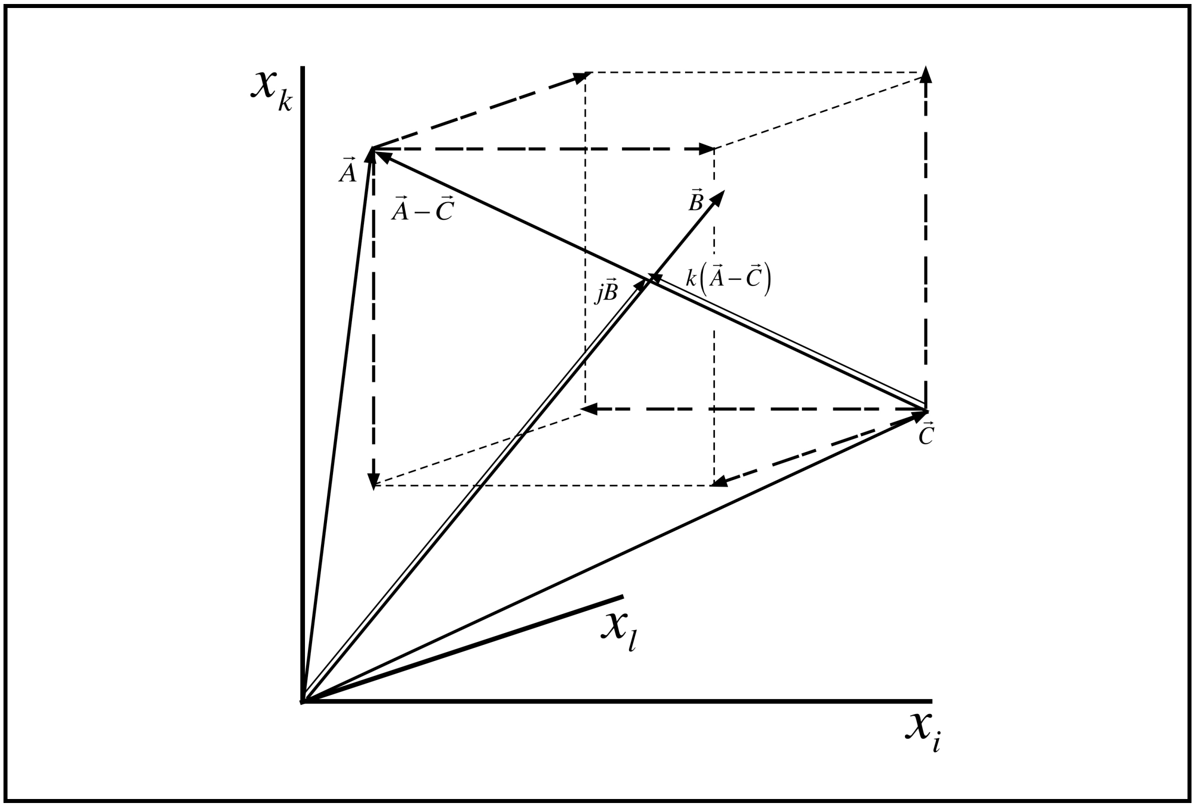

Figure 5.2.1-1 Vector B Lays "Between" Vectors A and C.

Definition [Box Defined by Two Vectors]

Given pair of vectors $\vec{A} = A_1\hat{\varphi}_1 + A_2\hat{\varphi}_2 + \cdots + A_n\hat{\varphi}_n$ and $\vec{C} = C_1\hat{\varphi}_1 + C_2\hat{\varphi}_2 + \cdots + C_n\hat{\varphi}_n$
The Box defined by these vectors consists of the set $\aleph$ of all vectors $\vec{\chi}$ such that for all components χ_i :

$$A_i > C_i \Rightarrow A_i \geq \chi_i \geq C_i \qquad or \qquad C_i > A_i \Rightarrow C_i \geq \chi_i \geq A_i \tag{5.2.1-1}$$

5.2.2) More Propositions and Proofs

Proposition 5.2.2-1 Bilateral Marginal Exchange

Given two consumers who are described by marginal price functions $\vec{r}^{\,1}(\vec{x}^1)$ and $\vec{r}^{\,2}(\vec{x}^2)$ and possess bundles $\vec{x}^1$ and $\vec{x}^2$ respectively.

If and only if $\vec{r}^{\,1}(\vec{x}^1) \neq \vec{r}^{\,2}(\vec{x}^2)$, the consumers will agree to exchange a marginal bundle $d\vec{x} = dx^1 = -d\vec{x}^2$ of goods at a price $\vec{p}$ that lies between $\vec{r}^{\,1}(\vec{x}^1)$ and $\vec{r}^{\,2}(\vec{x}^2)$. As result of the exchange:

a) The use value of both consumers will have increased $V^1(\vec{x}^1 + d\vec{x}^1) > V^1(\vec{x}^1)$ and $V^2(\vec{x}^2 + d\vec{x}^2) > V^2(\vec{x}^2)$.

b) The marginal prices of the consumers will have contracted together: $\left|\vec{r}^1(\vec{x}^1) - \vec{r}^2(\vec{x}^2)\right| > \left|\vec{r}^1(\vec{x}^1 + d\vec{x}^1) - \vec{r}^2(\vec{x}^2 + d\vec{x}^2)\right| > 0$

Proof:

Since $\vec{r}^1(\vec{x}^1)$ and $\vec{r}^2(\vec{x}^2)$ are of unit magnitude and $\vec{r}^1(\vec{x}^1) \neq \vec{r}^2(\vec{x}^2)$, there exists at least one good x_i for which $\vec{r}_i^1(\vec{x}^1) > \vec{r}_i^2(\vec{x}^2)$ and at least one other good x_k for which $\vec{r}_k^2(\vec{x}^2) > \vec{r}_k^1(\vec{x}^1)$. Therefore there exists at least one price $\vec{p}$ that lies between $\vec{r}^1(\vec{x}^1)$ and $\vec{r}^2(\vec{x}^2)$. Therefore, by Proposition 5.1.2-1 there is an opportunity for both consumers to benefit from an exchange.

Per Assumption 1b the consumers will attempt to negotiate the price $\vec{p}$ and the contents of marginal bundle $d\vec{x}$ so as to satisfy:

$$\begin{aligned} consumer\,\#1: \quad & d\vec{x}^1 = \vec{r}^1(\vec{x}^1) - \left(\vec{r}^1(\vec{x}^1) \bullet \vec{p}\right)\vec{p} \\ consumer\,\#2: \quad & d\vec{x}^2 = \vec{r}^2(\vec{x}^2) - \left(\vec{r}^2(\vec{x}^2) \bullet \vec{p}\right)\vec{p} \\ & d\vec{x}^1 = -d\vec{x}^2 = d\vec{x} \end{aligned} \tag{5.2.2-1}$$

This will cause the consumers to choose a price that lies exactly half way between $\vec{r}^1(\vec{x}^1)$ and $\vec{r}^2(\vec{x}^2)$, i.e. with $k = 1/2$. To show this, we combine Equations 5.2-1 giving:

$$\begin{aligned} d\vec{x} = \vec{r}^1(\vec{x}^1) - \left(\vec{r}^1(\vec{x}^1) \bullet \vec{p}\right)\vec{p} &= \left(\vec{r}^2(\vec{x}^2) \bullet \vec{p}\right)\vec{p} - \vec{r}^2(\vec{x}^2) \\ \Rightarrow \vec{r}^1(\vec{x}^1) + \vec{r}^2(\vec{x}^2) &= \left(\left[\vec{r}^1(\vec{x}^1) + \vec{r}^2(\vec{x}^2)\right] \bullet \vec{p}\right)\vec{p} \\ \Rightarrow \vec{p} = \frac{\vec{r}^1(\vec{x}^1) + \vec{r}^2(\vec{x}^2)}{\left[\vec{r}^1(\vec{x}^1) + \vec{r}^2(\vec{x}^2)\right] \bullet \vec{p}} &= \frac{\vec{r}^1(\vec{x}^1) + \vec{r}^2(\vec{x}^2)}{\left|\vec{r}^1(\vec{x}^1) + \vec{r}^2(\vec{x}^2)\right|} \end{aligned} \tag{5.2.2-2}$$

In the last step of Equation 5.2.2-2 we have recognized that the denominator is a scalar, hence the direction of $\vec{p}$ is given by the numerator. Since $\vec{p}$ is by definition a unit vector, its dot product with any vector parallel to it is simply the magnitude of the parallel vector.

By Proposition 5.1.2-2, the marginal prices of both consumers will contract towards $\vec{p}$, i.e.

$$\begin{aligned} &\left|\vec{r}^{\,1}(\vec{x}^1)-\vec{p}\right| > \left|\vec{r}^{\,1}(\vec{x}^1+d\vec{x}^1)-\vec{p}\right| > 0 \\ &\left|\vec{r}^{\,2}(\vec{x}^2)-\vec{p}\right| > \left|\vec{r}^{\,2}(\vec{x}^2+d\vec{x}^2)-\vec{p}\right| > 0 \end{aligned} \tag{5.2.2-3}$$

Since $\vec{p}$ is between $\vec{r}^{\,1}(\vec{x}^1)$ and $\vec{r}^{\,1}(\vec{x}^1)$ as shown in Figure 5.2-2 we must have:

$$\left|\vec{r}^{\,1}(\vec{x}^1)-\vec{r}^{\,2}(\vec{x}^2)\right| > \left|\vec{r}^{\,1}(\vec{x}^1+d\vec{x}^1)-\vec{r}^{\,2}(\vec{x}^2+d\vec{x}^2)\right| > 0 \tag{5.2.2-4}$$

This completes the proof:

QED

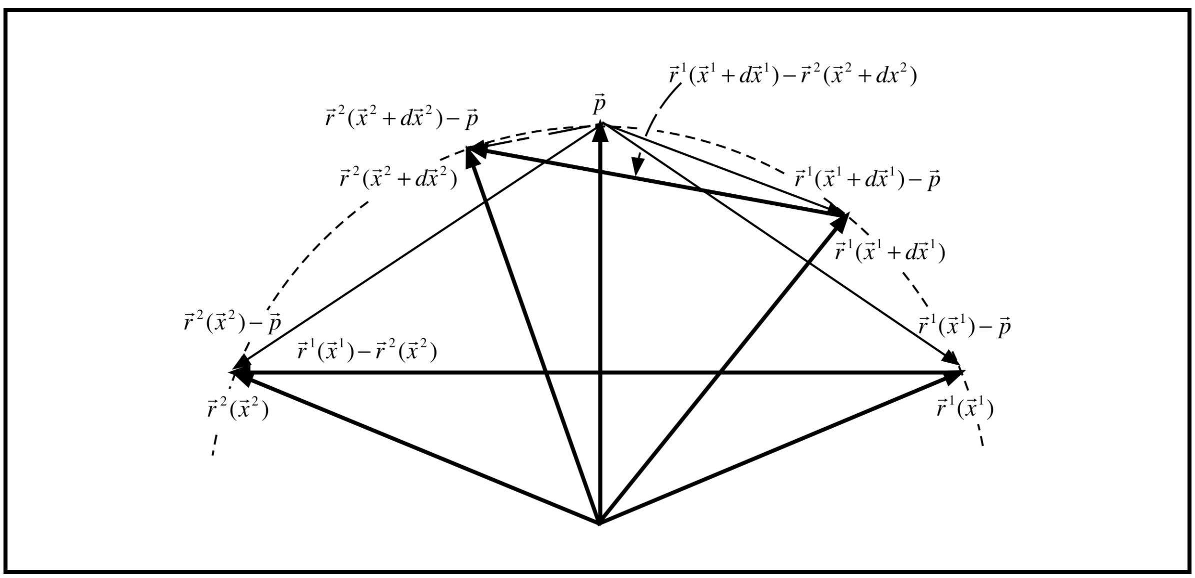

Figure 5.2.2-1 Marginal Price Contraction in a Bilateral Exchang

Proposition 5.2.2-2 Bilateral Tatonnement

Given two consumers described by marginal price functions $\vec{r}^{\,1}(\vec{x})$, and $\vec{r}^{\,2}(\vec{x})$ respectively. At time t_0 they possess respective initial bundles $\vec{x}^1[t_0]$ and $\vec{x}^2[t_0]$. If, in any time period t_0+n the consumers are allowed to exchange marginal bundles $d\vec{x}[t_0+n]$, they will do so until a time period t_0+z in which:

a) The consumers arrive at a common "market" set of prices $\vec{p}[t_0+z]$ where:
$\vec{r}^{\,1}(\vec{x}^1[t_0+z]) = \vec{r}^{\,2}(\vec{x}^2[t_0+z]) = \vec{p}[t_0+z]$

b) The consumers will have obtained the maximum use value available to them at price $\vec{p}[t_0+z]$, given their initial bundles $\vec{x}^1[t_0]$ and $\vec{x}^2[t_0]$.

Proof:

At any time t_0+n, where $n=0,1,2,\ldots$ unless the marginal prices of the two consumer's are already equal, Proposition 5.2-1 indicates that they will exchange a marginal bundle $d\vec{x}[t_0+n]$ at a mutually agreed price $\vec{p}[t_0+n]$. After the exchange is completed, the consumer's marginal prices will have contracted towards each other, i.e.:

$$\left|\vec{r}^{\,1}(\vec{x}^1[t_0+n]) - \vec{r}^{\,2}(\vec{x}^2[t_0+n])\right| > \left|\vec{r}^{\,1}(\vec{x}^1[t_0+n+1]) - \vec{r}^{\,2}(\vec{x}^2[t_0+n+1])\right| > 0$$

Where:

$$\vec{x}^1[t_0+n+1] = \vec{x}^1[t_0+n] + d\vec{x}$$
$$\vec{x}^2[t_0+n+1] = \vec{x}^1[t_0+n] - d\vec{x}$$

Since this applies to every time period, we must have:

$$\underset{n\to\infty}{\text{limit}}\left|\vec{r}^{\,1}(\vec{x}^1[t_0+n]) - \vec{r}^{\,2}(\vec{x}^2[t_0+n])\right| = 0 \qquad (5.2.2\text{-}5)$$

As before, we choose some number $\varepsilon > 0$, that is negligibly close to zero. Since Equation 5.2.2-5 approaches zero monotonically, there must be some number $0 < z < \infty$ such that:

$$\left|\vec{r}^{\,1}(\vec{x}^{\,1}[t_0+n])-\vec{r}^{\,2}(\vec{x}^{\,2}[t_0+n])\right|<\varepsilon \qquad (5.2.2\text{-}6)$$

Thus equilibrium is achieved (and exchange stops) at time t_0+z when: $\left|\vec{r}^{\,1}(\vec{x}^{\,1}[t_0+z])=\vec{r}^{\,2}(\vec{x}^{\,2}[t_0+z])\right|$ as claimed. This completes the proof of (a)

To prove part (b) We know from the definition of use value and from Assumption (4) that the use value each consumer gains through the course of their marginal exchanges does not depend on the exchanges themselves. For Consumer 1 $V^1(\vec{x}^{\,1}[t_0+z])-V^1(\vec{x}^{\,1}[t_0])$ would be the same whether he acquired $\vec{x}^{\,1}[t_0+z]$ through bilateral marginal exchanges or through fixed price exchanges made at $\vec{p}[t_0+z]$. The maximization result follows from the unilateral tatonnement (Proposition 5.1.2-3). This completes the proof of part (b)

QED.

5.3) Multilateral Exchange

The multilateral case, where goods are exchanged among many consumers, is broken into many bilateral marginal exchanges. Each exchange brings the marginal prices of the trading partners closer to the averages for the whole community. The partners to any given marginal exchange do not necessarily continue making exchanges with each other. They may meet and exchange only a single marginal bundle before moving on to find other partners. If we were to plot the marginal price vectors for each member of the community as points in any coordinate plane, they would appear as a random cluster that is collapsing onto its center, as shown in Figure 5.3-1. With each marginal exchange, the mean shifts to compensate. The mean to which the points collapse is therefore constantly readjusting so as to maintain its central position in the cluster.

Consumers would be expected to “shop around” for partners with whom trade will provide the greatest benefit. These are of course those individuals whose marginal prices differ the most from their prospective partners. We model this by only considering exchanges between consumers whose marginal price vectors define a “box” that contains the current mean. This indicates that while one partner’s marginal price for a given good is at or above the mean, the other’s marginal price is at or below the mean. Thus individuals will choose partners whose marginal prices are somewhat “across the cluster” in Figure 5.3-1, as opposed to nearby neighbors.

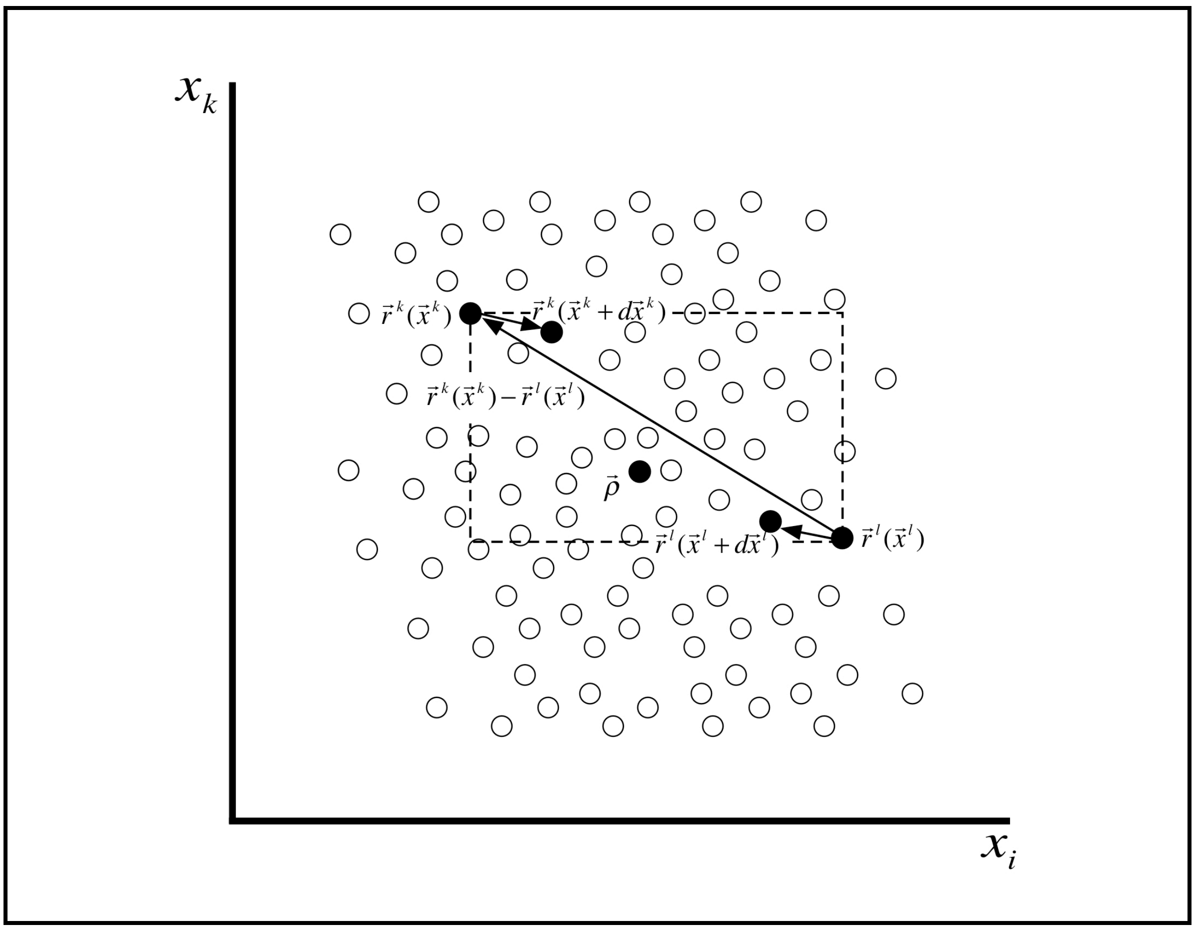

Figure 5.3-1 Marginal Price Contraction Towards the Mean in a Multilateral Exchange

5.3.1 Definitions

DEFFINITION: Mean Marginal Price

Given a community of m consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of n goods $i \in (1,2,\ldots n)$. Given that each consumer μ holds a specific bundle $\vec{x}^{\mu}$ and is described by his or her individual marginal price function:

$$\begin{aligned}\vec{r}^{\,\mu}(\vec{x}^{\mu}) &= r_1^{\mu}(x_1^{\mu},x_2^{\mu}\ldots x_n^{\mu})\hat{\varphi}_1 + r_2^{\mu}(x_1^{\mu},x_2^{\mu}\ldots x_n^{\mu})\hat{\varphi}_2 + \cdots + r_n^{\mu}(x_1^{\mu},x_2^{\mu}\ldots x_n^{\mu})\hat{\varphi}_n \\ &= \sum_{i=1}^{n} r_i^{\mu}(\vec{x}^{\mu})\hat{\varphi}_i \end{aligned} \qquad (5.3.1\text{-}1)$$

The Mean Marginal Price is a vector $\vec{\rho} = \rho_1\hat{\varphi}_1 + \rho_2\hat{\varphi}_2 + \cdots + \rho_n\hat{\varphi}_n$, each component ρ_i of which is the mean of the marginal prices $r_i^{\mu}(\vec{x}^{\mu})$ of the consumers μ for the good x_i given by:

$$\rho_i \triangleq \frac{1}{m}\sum_{\mu=1}^{m} r_i^{u}(\vec{x}^{\mu}) \qquad (5.3.1\text{-}2)$$

DEFFINITION: Deviation from the Mean Marginal Price

Given a community of m consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of n goods $i \in (1,2,\ldots n)$. Given that each consumer μ holds a specific bundle $\vec{x}^{\mu}$ and is described by his or her individual marginal price function $\vec{r}^{\,\mu}(\vec{x}^{\mu})$. The Deviation from the Mean Marginal Price by the marginal price of the μ^{th} consumer is the vector $\vec{s}^{\,\mu} = \vec{r}^{\,\mu}(\vec{x}^{\mu}) - \vec{\rho}$

DEFFINITION: Average Deviation from the Mean Marginal Price

Given a community of m consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of n goods $i \in (1,2,\ldots n)$. Given that each consumer μ holds a specific bundle $\vec{x}^{\mu}$ and is described by his or her individual marginal price function $\vec{r}^{\,\mu}(\vec{x}^{\mu})$. The Average Deviation from the Mean Marginal Price σ is the average of the magnitudes of the deviations from the from the mean marginal price given by:

$$\sigma \triangleq \frac{1}{m}\sum_{\mu=1}^{m}\left|\vec{s}^{\,\mu}\right| = \frac{1}{m}\sum_{\mu=1}^{m}\left(\sum_{i=1}^{n}\left(\vec{r}_i^{\,\mu}(\vec{x}^{\mu}) - \vec{\rho}_i\right)^2\right)^{\frac{1}{2}} \qquad (5.3.1\text{-}3)$$

5.3.2) Propositions

For a pair of consumers whose marginal prices define a box that contains the mean, the following proposition indicates that a bilateral marginal exchange between them will result in both of their marginal prices drawing closer to the mean. We show this in two steps: First we show that the Assumption of Non-Addiction in its strongest form will cause the marginal prices for both consumers to shift into the interior of the box. We do this by considering the impact of Assumption (5) on each component of the marginal price vectors individually. As result of the exchange the shifted marginal prices $\vec{r}^{\,k}(\vec{x}^{k}+d\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l}+d\vec{x}^{l})$ will lie in the corner regions of the box near $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l})$ respectively.

The second step of the proof will be to show that $\vec{r}^{\,k}(\vec{x}^{k}+d\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l}+d\vec{x}^{l})$ must be closer to any point $\vec{\rho}$ that is interior to the box, than are $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l})$. This is apparent from Figure 5.3.2-1.

> **<u>PROPOSITION 5.3.2-1 Marginal Price Contraction towards the Mean</u>**
> Given two consumers who are described by marginal price functions $\vec{r}^{\,k}(\vec{x})$ and $\vec{r}^{\,l}(\vec{x})$ respectively who are members of a community of m consumers, having mean marginal price $\vec{\rho}$. Given also that $\vec{\rho}$ lies in the box defined by $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x})$. The consumers will exchange a marginal bundle $d\vec{x}=d\vec{x}^{1}=-d\vec{x}^{2}$ after which, their marginal prices will have contracted towards the mean, i.e.: $\left|\vec{s}^{\,k}(\vec{x}^{k}+dx^{k})\right|<\left|\vec{s}^{\,k}(\vec{x}^{k})\right|$ and $\left|\vec{s}^{\,l}(\vec{x}^{l}+dx^{l})\right|<\left|\vec{s}^{\,l}(\vec{x}^{l})\right|$

<u>Proof:</u>
We begin by showing that $\vec{r}^{\,k}(\vec{x}^{k}+d\vec{x}^{k})$ lies within the box defined by $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l})$.

From the proof of Proposition 5.2.2-1 we know that the prices $\vec{p}$ at which the consumers will agree to trade lie between $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l})$. Thus, for any good x_i for which $r_i^{k}(\vec{x}^{k})>r_i^{l}(\vec{x}^{l})$

it must also be true that $r_i^k(\vec{x}^k) > p_i$. Consumer k will therefore purchase a (positive) quantity dx_i^k. After the purchase is completed, Assumption (5) (Inequality 4.4-4) implies that Consumer k's marginal price for x_i will have shifted so that $\left(r_i^k(\vec{x}^k + d\vec{x}^k) - r_i^k(\vec{x}^k)\right)dx_i^k < 0$ since dx_i^k is positive the term in brackets must be negative and:

$$r_i^k(\vec{x}^k) > r_i^k(\vec{x}^k + d\vec{x}^k) > r_i^l(\vec{x}^l) \tag{5.3.2-1}$$

Similarly, for any good x_i for which $r_i^k(\vec{x}^k) < r_i^l(\vec{x}^l)$ it must be true that $r_i^k(\vec{x}^k) < p_i$. In this case Consumer k will sell a (negative) quantity $-dx_i^k$. Again after the purchase is completed, Assumption 5 implies that $\left(r_i^k(\vec{x}^k + d\vec{x}^k) - r_i^k(\vec{x}^k)\right)(-dx_i^k) < 0$ since the exchange quantity this time is negative we must have:

$$r_i^k(\vec{x}^k) < r_i^k(\vec{x}^k + d\vec{x}^k) < r_i^l(\vec{x}^l) \tag{5.3.2-2}$$

From Equations (5.3.2-1) and (5.3.2-2) we know that by definition, $\vec{r}^k(\vec{x}^k + d\vec{x}^k)$ lies within the box defined by $\vec{r}^k(\vec{x}^k)$ and $\vec{r}^l(\vec{x}^l)$. By similar reasoning it can be shown that $\vec{r}^l(\vec{x}^l + d\vec{x}^l)$ lies within the box as well.

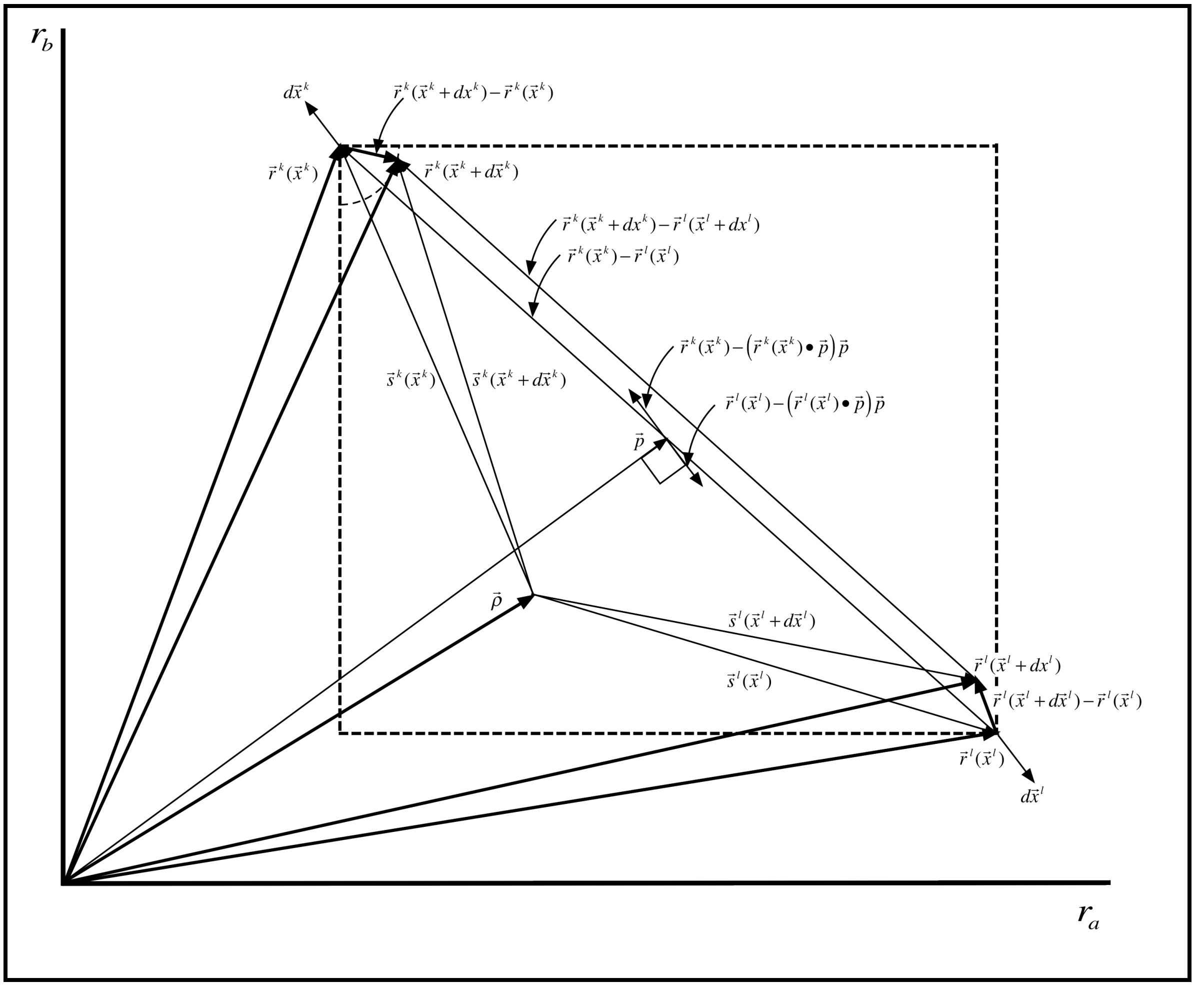

Figure 5.3.2-1 Marginal Price Contraction Towards the Mean

We now show that because these vectors lie within the box, they have contracted towards the mean. Notice from Figure 5.3.2-1 that such box is also defined by the vectors by $\vec{s}^k(\vec{x}^k)$ and $\vec{s}^l(\vec{x}^l)$. Since $\vec{r}^k(\vec{x}^k + d\vec{x}^k)$ and $\vec{r}^l(\vec{x}^l + d\vec{x}^l)$ lie differentially close to $\vec{r}^k(\vec{x}^k)$ and $\vec{r}^l(\vec{x}^l)$ respectively we must have, for all components $s_i^k(\vec{x}^k + d\vec{x})$:

$$\begin{aligned} s_i^k(\vec{x}^k) > s_i^l(\vec{x}^l) \quad &\Rightarrow \quad s_i^k(\vec{x}^k) > s_i^k(\vec{x}^k + d\vec{x}) > \rho_i > s_i^l(\vec{x}^l) \\ s_i^k(\vec{x}^k) < s_i^l(\vec{x}^l) \quad &\Rightarrow \quad s_i^k(\vec{x}^k) < s_i^k(\vec{x}^k + d\vec{x}) < \rho_i < s_i^l(\vec{x}^l) \end{aligned} \tag{5.3.2-3}$$

Without loss of generality, assume components $s_a^k(x^k) < s_a^l(x^l)$ and $s_b^l(x^l) < s_b^k(x^k)$ are as shown in Figure 5.3.2-1. Assume that the only component of $\vec{s}^k(\vec{x}^k + d\vec{x}^k)$ that differs from $\vec{s}^k(\vec{x})$ is the a^th component $\vec{s}_a^k(\vec{x}^k + d\vec{x}^k)$. For all points within the box for which $\vec{s}_a^k(\vec{x}^k + d\vec{x}^k) \neq \vec{s}_a^k(\vec{x}^k)$ we must have $\vec{s}_a^k(\vec{x}^k + d\vec{x}^k) < \vec{s}_a^k(\vec{x}^k)$ in this case:

$$\left|\vec{s}^k(\vec{x}^k + d\vec{x})\right| = \left(\left(s_a^k(\vec{x}^k + d\vec{x})\right)^2 + \sum_{i \neq a}\left(s_i^k(x^k)\right)^2\right)^{1/2} < \left(\sum_{i=1}^{n}\left(s_i^k(x^k)\right)^2\right)^{1/2} = \left|\vec{s}^k(\vec{x}^k)\right| \quad (5.3.2\text{-}4)$$

Using the same reasoning, we can show that Equation 5.3.2-2 will hold for every component $s_i^k(\vec{x}^k + d\vec{x}^k)$ were it the only one allowed to deviate from its corresponding component $s_i^k(\vec{x}^k)$. Since all vectors $\vec{s}^k(\vec{x}^k + d\vec{x})$ contain at least one component that is smaller than their corresponding components of $s_i^k(\vec{x}^k)$ (and no components which are larger), it must generally be true that $\left|\vec{s}^k(\vec{x}^k + d\vec{x}^k)\right| < \left|\vec{s}^k(\vec{x}^k)\right|$. By similar reasoning, it can be shown that $\left|\vec{s}^l(\vec{x}^l + d\vec{x}^l)\right| < \left|\vec{s}^l(\vec{x}^l)\right|$. Thus, the marginal exchange has caused both consumer's marginal prices to have contracted towards the mean as claimed.

QED.

PROPOSITION 5.3.2-2 Multilateral Tatonnement

Given a community of m consumers $\mu \in (1,2,\ldots m)$, with each consumer μ described by a marginal price function $\vec{r}^{\mu}(\vec{x})$ and holding a bundle $\vec{x}^{\mu}[t_0 + n]$ at time $t_0 + n$. Given also that at any time period the community is described by a mean marginal price $\vec{\rho}[t_0 + n]$ and average deviation of marginal prices $\sigma[t_0 + n]$. Given that in any time period in which the consumer's marginal prices are not all equal, a pair of consumers k and l, whose marginal prices $\vec{r}^k(\vec{x}^k[t_0 + n])$ and $\vec{r}^l(\vec{x}^l[t_0 + n])$ enclose the mean, are allowed to exchange a marginal bundle $d\vec{x}[t_0 + n]$. Therefore the following will occur:

a) Marginal exchanges will commence and continue until a time period $t_0 + z$ at which time all consumers arrive at a common set of "market" prices $\vec{p}[t_0 + z]$ where: $\vec{r}^{\mu}(\vec{x}^{\mu}[t_0 + z]) = \vec{p}[t_0 + z] \quad \forall \mu$

b) The use values $V^{\mu}[t_0 + z]$ of each consumer will be the maximum available to that consumer at prices $\vec{p}[t_0 + z]$, given their initial bundles at time t_0.

Proof:

Exchange of the marginal bundle $d\vec{x}$ is implied by Proposition 5.2.2-1. as is the contraction of the consumer's marginal priced toward each other. Proposition 5.3.2-1 implies that the marginal prices of both consumers will contract towards the mean, i.e.:

$$
\begin{aligned}
\left|\vec{s}^{k}(\vec{x}^{k}[t_0 + n] + dx^{k}[t_0 + n])\right| &< \left|\vec{s}^{k}(\vec{x}^{k}[t_0 + n])\right| \\
\left|\vec{s}^{l}(\vec{x}^{l}[t_0 + n] \ + dx^{l}[t_0 + n])\right| &< \left|\vec{s}^{l}(\vec{x}^{l}[t_0 + n])\right|
\end{aligned}
\tag{5.3.2-5}
$$

Since none of the other consumer's marginal prices will have deviated we must have:

$$
\begin{aligned}
\sigma'[t_0 + n] &\triangleq \frac{1}{m}\left(\left|\vec{s}^{k}\left(x^{k}[t_0 + n] + dx^{k}[t_0 + n]\right)\right| + \left|\vec{s}^{l}\left(x^{l}[t_0 + n] + dx^{l}[t_0 + n])\right)\right| + \sum_{\mu \neq k,l}^{m}\left|\vec{s}^{\mu}\left(x^{\mu}[t_0 + n]\right)\right|\right) \\
&< \frac{1}{m}\left(\sum_{\mu \neq k,l}^{m}\left|\vec{s}^{\mu}\left(x^{\mu}[t_0 + n]\right)\right|\right) = \sigma[t_0 + n]
\end{aligned}
\tag{5.3.2-6}
$$

The shift in the marginal prices for the two consumers will in general shift the mean, causing all of the deviations to change slightly. At the beginning of the next time period the new mean will be $\vec{\rho}[t_0 + n + 1]$. For the consumers who were not engaged in the exchange, the new deviations will be:

$$
\vec{s}^{\mu}(\vec{x}^{\mu}[t_0 + n + 1]) = \vec{r}^{\mu}(\vec{x}^{\mu}[t_0 + n]) - \vec{\rho}[t_0 + n + 1] \qquad \mu \neq k,l \tag{5.3.2-7}
$$

For the consumer's involved in the exchange:

$$\vec{s}^{\,k}\left(\vec{x}^{k}[t_0+n+1]\right)=\vec{r}^{\,k}\left(\vec{x}^{k}[t_0+n]+d\vec{x}^{k}[t_0+n]\right)-\vec{\rho}[t_0+n+1]$$
$$\vec{s}^{\,l}\left(\vec{x}^{l}[t_0+n+1]\right)=\vec{r}^{\,l}\left(\vec{x}^{l}[t_0+n]+d\vec{x}^{l}[t_0+n]\right)-\vec{\rho}[t_0+n+1] \quad (5.3.2\text{-}8)$$

Recalculation of the mean may decrease the average deviation but cannot increase it, therefore:

$$\sigma[t_0+n+1]\le\sigma'[t_0+n]<\sigma[t_0+n] \quad (5.3.2\text{-}9)$$

We know therefore, that in every time period the average deviation decreases due to the exchanges, hence:

$$\operatorname*{limit}_{n\to\infty}\sigma[t_0+n]=0 \quad (5.3.2\text{-}10)$$

Using the $\varepsilon-z$ reasoning as was done in Proposition 5.1.2-2 we know that there is some time period t_0+z at which $\sigma[t_0+z]$ differs from zero by a negligible amount. With $\sigma[t_0+z]=0$ we know that all consumer's marginal prices will have converged to the mean, which becomes the market price as claimed, i.e.:

$$\vec{r}^{\,\mu}(\vec{x}^{\mu}[t_0+z]=\vec{\rho}[t_0+z]\triangleq\vec{p} \quad \forall\mu \quad (5.3.2\text{-}11)$$

This completes the proof of (a)

To prove part (b) We know from the definition of use value and from Assumption (4) that the use value each consumer gains through the course of their marginal exchanges does not depend on the exchanges themselves. For Consumer μ $V^{\mu}(\vec{x}^{\mu}[t_0+z])-V^{\mu}(\vec{x}^{\mu}[t_0])$ would be the same whether he acquired $\vec{x}^{\mu}[t_0+z]$ through multilateral marginal exchanges or through fixed price exchanges made at $\vec{p}[t_0 + z]$. The maximization result follows from the unilateral tatonnement (Proposition 5.1.2-3). This completes the proof of part (b)

QED

5.4) Conclusion: Equilibrium Happens!

As flippant as the title of this section may appear, it is apt to the situation. As was the upshot of the Sonnenschein-Mantel-Debreu, theorem, exchange equilibria are ubiquitous. For any set of consumers and for (nearly[3]) any initial distribution, the market process will bring about a stable equilibrium. Government intervention does not change that fact. Taxes, transfers, and government purchases redistribute goods, while government regulation on industry influences a seller's marginal prices through her marginal costs. In any case, these merely shift equilibria, they do not hamper their formation. There is nothing yet in this analysis (or in current neoclassical theory) that establishes any one equilibrium as socially "better" than any another.

To understand what specific equilibria may be reached, or whether any given equilibrium is socially preferable to any other, we need to know more about the consumer's themselves. Analytically, this knowledge will be embedded in the marginal price functions for the individual consumers. Specifically, we need to discern trends in these functions that are common to all consumers. This will be the topic of the next chapter.

[3] The goods have to be distributed such that there are no "corner solutions". These are where there are not enough of some goods to alloy all consumer's marginal prices to equalize. In these events, the tatonnement process will bring about a stable distribution with some traders having none of some goods, and with marginal prices not equalized

PART III:
MODEL OF THE AGGREGATE CONSUMER

CHAPTER 6: AGGREGATION AND THE AGGREGATE CONSUMER

This chapter will address application of the theory developed in Chapter 4 to the analysis of an entire community. We begin by showing that marginal price functions for members of a community can in fact be aggregated to form a single marginal price function representing the entire community. What such function represents however are trends common to all members of the community, allowing it to be treated as the marginal price function of a single representative consumer.

As will be shown, the aggregate marginal value function tells only part of the story. This function will be used in tandem with demographic data profiling the distribution of goods and services throughout the community. By modeling changes in such distribution, one can model the impact of such changes on the demand for various goods, as well as aggregate welfare

6.1) Defining "The Aggregate Consumer" (In Theory)

The aggregate marginal price function of a community of consumer's will be defined simply as the weighted average of the marginal price functions of the community's membership, much as Social Welfare Functions are defined presently. As mentioned earlier, it is such aggregation of utility or preferences that economic theory declares to be impossible. When ordinal preferences are used, it is entirely possible to have a community of individuals whose preferences cannot be aggregated into a consistent function by any means that would be considered "fair" to all members[1]. This is due to the technical byproduct of ordinal numbers, illustrated by Condorcet's Paradox of Voting,

[1] This is the central of Arrow's Impossibility Theorem.

which was discussed in Section 3.4 of Chapter 3. This problem disappears when a cardinal measures of use value is used, such as are derived from marginal prices.

We define the aggregate marginal price function as follows:

Definition: Aggregate Marginal Price Function
Given a community of m consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of n goods $i \in (1,2,\ldots n)$. Given that each consumer is described by his or her individual marginal price function:

$$\begin{aligned}\vec{r}^{\,\mu}(\vec{x}) &= r_1^{\mu}(x_1,x_2\ldots x_n)\hat{\varphi}_1 + r_2^{\mu}(x_1,x_2\ldots x_n)\hat{\varphi}_2 + \cdots + r_n^{\mu}(x_1,x_2\ldots x_n)\hat{\varphi}_n \\ &= \sum_{i=1}^{n} r_i^{\mu}(\vec{x})\hat{\varphi}_i\end{aligned} \qquad (6.1\text{-}1)$$

The aggregate marginal price function $\vec{R}(\vec{x})$ for this community is defined to be:

$$\vec{R}(\vec{x}) \triangleq \sum_{\mu=1}^{m} a_\mu \vec{r}^{\,\mu}(\vec{x}) = \sum_{\mu=1}^{m} a_\mu \left(\sum_{i=1}^{n} r_i^{\mu}(\vec{x})\hat{\varphi}_i \right) \text{ where } \sum_{\mu=1}^{m} a_\mu = 1 \text{ and all } a_\mu \geq 0 \qquad (6.1\text{-}2)$$

By reversing the order of summation in Equation (6.1-2), we see that the components of the aggregate marginal price function are the weighted averages of the corresponding components of each member of the community.

$$\vec{R}(\vec{x}) = \sum_{i=1}^{n} \left(\hat{\varphi}_i \sum_{\mu=1}^{m} a_\mu r_i^{\mu}(\vec{x}) \right) = \sum_{i-1}^{n} R_i(\vec{x})\hat{\varphi}_i \quad \text{where} \quad R_i(\vec{x}) \triangleq \sum_{\mu=1}^{m} a_\mu r_i^{\mu}(\vec{x}) \qquad (6.1\text{-}3)$$

Let's take a moment to understand what Equation 4.5-3 actually means: Aggregation of the $\vec{r}(\vec{x})$ functions for multiple individuals must be done on a bundle-by-bundle basis, among consumers holding the *same* bundle. For two consumers k and l we can compare $\vec{r}^{\,k}(\vec{x}')$ with $\vec{r}^{\,l}(\vec{x}')$, or $\vec{r}^{\,k}(\vec{x}'')$ with $\vec{r}^{\,l}(\vec{x}'')$, but not $\vec{r}^{\,k}(\vec{x}')$ with $\vec{r}^{\,l}(\vec{x}'')$. In concept, each member of society would be presented with some bundle $\vec{x}'$ and her marginal price $r_i(\vec{x}')$ for good x_i determined. The weighted average of each member's responses would be taken as the aggregate marginal price $R_i(\vec{x})$ for good x_i given possession of bundle $\vec{x}'$. After the aggregate marginal prices were determined for all other goods x_k in $\vec{x}'$, the members would be given a new bundle $\vec{x}''$ and the process repeated. This entire process would be repeated for all other bundles $\vec{x}$.

The aggregate marginal price function $\vec{R}(\vec{x}')$ defined in this manner represents the marginal prices community members would *typically* be willing to pay for goods x_i assuming they were to currently possess bundle $\vec{x}'$. As such, the aggregate marginal price function can be thought of as describing a single *aggregate consumer*. It should be noted that the aggregate consumer defined here is somewhat different from representative consumer used in some macroeconomic models. As will become apparent as we progress, the aggregate consumer's behavior does not reflect the behavior of the entire community. Instead, the community will be assumed to behave like a community of identical aggregate consumers, holding different bundles.

There are of course numerous objections that economists might raise regarding the aggregation process described here. These will be addressed in Section 6.2 where we will discuss practical aspects of aggregation and commodity specification.

The remainder of this section will be dedicated to showing that the aggregate marginal price function satisfies Assumptions (1) through (5). This will guarantee that it may be used in the same manner as the marginal price function for an individual consumer.

It is easy to show that, if Assumptions (1) through (5) are satisfied by each individual's $\vec{r}^{\,\mu}(\vec{x})$, they will be satisfied by the aggregate $\vec{R}(\vec{x})$ as well. Satisfaction of Assumptions (1) and (3) are self-evident: Satisfaction of Assumption (1) follows immediately from Equation (6.1-2). The linear sums $R_i(\vec{x})$ exist wherever the corresponding $r_i^{\mu}(\vec{x})$ exist. Existence of a numeraire N, such that $\partial R_i / \partial N \equiv 1$ for all R_i follows from Equation (6.1-3). Satisfaction of Assumption (3) for all consumers μ implies that:

$$\frac{\partial}{\partial N} r_i^{\mu}(x_1, x_2, \ldots x_n, N) \equiv 1 \text{ for all } r_i^{\mu}(\vec{x}). \text{ Thus:}$$

Therefore, Equation 6.1-3 implies:

$$\frac{\partial}{\partial N} R_i(x_1, x_2, \ldots x_n, N) = \sum_{\mu=1}^{n} a_\mu \frac{\partial}{\partial N} r_i^{\mu}(x_1, x_2, \ldots x_n, N) = \sum_{\mu=1}^{n} a_\mu = 1 \qquad (6.1\text{-}4)$$

Satisfaction of Assumption (4) also follows from Equation (6.1-3). Satisfaction of Assumption (4) by all consumers μ implies:

$$a_\mu \frac{\partial (r_i^{\mu})}{\partial x_k} \equiv a_\mu \frac{\partial (r_k^{\mu})}{\partial r_i} \text{ for all } r_i^{\mu}(\vec{x}) \text{ and } r_k^{\mu}(\vec{x}) \qquad (6.1\text{-}5)$$

Summing Equations (6.1-5) for all consumers μ gives:

$$\sum_{\mu} a_\mu \frac{\partial r_i^{\mu}}{\partial x_k} = \sum_{\mu} a_\mu \frac{\partial r_k^{\mu}}{\partial x_i} \quad \Rightarrow \quad \frac{\partial \left(\sum_{\mu} a_\mu r_i^{\mu} \right)}{\partial x_k} = \frac{\partial R_i}{\partial x_k} = \frac{\partial R_k}{\partial x_i} = \frac{\partial \left(\sum_{\mu} a_\mu r_k^{\mu} \right)}{\partial x_i} \qquad (6.1\text{-}6)$$

From equation (6.1-6) we know that $\vec{R}(\vec{x})$ is integrable. Thus the use-value of the aggregate consumer can be defined as follows:

Definition: Use-Value of the Aggregate Consumer

Given a community of m consumers $\mu = (1, 2, \ldots m)$ each characterized by his or her marginal price function $\vec{r}^{\,\mu}(\vec{x})$. If the community's aggregate marginal price function is $\vec{R}(\vec{x})$, the use-value of the aggregate consumer holding bundle $\vec{x}'$, measured with respect to a reference bundle $\vec{x}^0$ is defined to be:

$$B(\vec{x}' - \vec{x}_0) \triangleq \int_{\vec{x}_0}^{\vec{x}'} \vec{R}(\vec{x}) \bullet d\vec{x} \tag{6.1-7}$$

From Equation (6.1-3) it is evident that the aggregate use value obtained using Equation (6.1-7) is the same as would be obtained by aggregating the use values of the individuals directly:

$$B(\vec{x}' - \vec{x}_0) = \int_{\vec{x}_0}^{\vec{x}} \sum_{\mu} a_{\mu} \vec{r}^{\,\mu}(\vec{x}) \bullet d\vec{x} = \sum_{\mu} a_{\mu} \int_{\vec{x}_0}^{\vec{x}'} \vec{r}^{\,\mu}(\vec{x}) \bullet d\vec{x} = \sum_{\mu} a_{\mu} V^{\mu}(\vec{x}' - \vec{x}_0) \tag{6.1-8}$$

Demonstrating that the aggregate marginal value function exhibits non-addiction follows much the same process. If Assumption (5) is satisfied for all consumers μ we must have:

$$a_{\mu}\left[r^{\mu}(\vec{x} + \Delta\vec{x}') - r^{\mu}(\vec{x})\right] \bullet \Delta\vec{x}' \leq 0 \text{ for all } \mu \tag{6.1-9}$$

Summing these equations gives:

$$0 \geq \sum_{\mu} a_{\mu}\left[r^{\mu}\left(\vec{x}+\Delta\vec{x}'\right)-r^{\mu}\left(\vec{x}\right)\right]\bullet\Delta\vec{x}' =$$

$$= \left[\sum_{\mu} a_{\mu}\left[r^{\mu}\left(\vec{x}+\Delta\vec{x}'\right)\right]-\sum_{\mu} a_{\mu}\left[r^{\mu}\left(\vec{x}\right)\right]-\right]\bullet\Delta\vec{x}' \tag{6.1-10}$$

$$= \left[R(\vec{x}+\Delta\vec{x})-R(\vec{x})\right]\bullet\Delta\vec{x}'$$

Since $\vec{R}(\vec{x})$ satisfies Assumptions 1-5, we know that constrained maximization can be done for the aggregate consumer, jus as it was for the individual consumer in Chapter 4. We can therefore define the wealth expansion path, demand, and marginal demand functions for the aggregate consumer.

Definition (Wealth Expansion Path of the Aggregate Consumer)
For the aggregate consumer described by a marginal price function $\vec{R}(\vec{x})$, facing constant market prices $\vec{p}$, his(her) wealth expansion path is defined as the set of bundles $\vec{X}^*$ that solve the equation[2]:

$$\vec{R}\left(\vec{X}^*\right)-\vec{p}=0 \tag{6.1-11}$$

Definition (Demand of the Aggregate Consumer)
For a the aggregate consumer described by a marginal price function $\vec{R}(\vec{x})$, facing constant market prices $\vec{p}$, The demand of the aggregate consumer is the bundle $\vec{X}^*(w,\vec{p})$ on his (her) income expansion path corresponding to his or her current wealth w. I.e.

$$\vec{X}^* = \vec{X}^*\left(w,\vec{p}\right) \tag{6.1-12}$$

Definition (Marginal Demand of the Aggregate Consumer)
Given the aggregate consumer whose demand is $\vec{X}^*(\vec{p},w)$ and who is receiving income I. The consumer's marginal demand $\Delta\vec{x}^*(w,\vec{p},I)$ is defined to be:

[2] An allowable representation for a curve in n dimensions is a set of n-1 functions meeting certain requirements. Recall that equation 4.5-6 is actually a set of n functions. This might over define the curve if it were not already apparent that the components of r were sufficiently consistent. See Kreyszig (1991) pp.17-29

$$\Delta\vec{X}^* \triangleq I\frac{\partial}{\partial w}\left[\vec{X}^*(\vec{p},w)\right] \tag{6.1-13}$$

6.2) The Aggregate Consumer in Practice

To adapt this concept to realistic situations, we begin by thinking of a commodity as *any* measurable factor that potentially influences the consumer's decision. These factors include the demographic factors mentioned earlier. If the consumer is a young, single male, we will regard youth, "singleness," and male gender as commodities that he holds. Such commodities are not necessarily tradable, (although the familiar phrase: "What I wouldn't give to be young again!" illustrates that marginal prices could conceivably be placed on them[3]).

With regard to tradable commodities, it will likely be best to define them in broad categories. The marginal price that any individual might place on a good such as a 2010 Chevrolet Malibu for example, may be subject to more influences than can be accounted for empirically. By defining the commodity in broad terms such as "personal vehicle" or even simply "access to transportation" consistent results may be more easily obtained. This example reflects Amartya Sen's notion of a *capability*.[4] In Sen's view, it is not so much the good that the consumer values, but that which it makes him capable of accomplishing. In this example, "access to transportation," is the capability that can be met either by a personal vehicle or by public transportation. When aggregated over a congested city such as New York, the consumer's marginal price for "access to transportation" might be statistically more significant that the marginal price for either a personal vehicle or access to purely public transportation.

[3] In a very real sense one's "singleness" is exchangeable as an opportunity cost of marriage.

[4] Sen (1999)

By generalizing the notion of commodities, one's endowment bundle becomes representative of one's socioeconomic circumstances. The thought experiment with which this section began is reminiscent of John Rawls famous "veil of ignorance" scenario[5]. In Rawls' thought experiment, all members of the community are placed behind a "veil of ignorance" that prevents them from knowing what place they will occupy in society, their state of health, or what goods they will possess. For each possible situation they might find themselves in, they are asked to reveal the choice they would consider most just.

Market researchers have used a simplification of this process for decades. Using demographic information, researchers are able to identify individuals that currently hold a given endowment bundle $\vec{x}'$. By any one of several methods, they determine a surrogate for this group's marginal prices $R_m(\vec{x}')$ and $R_n(\vec{x}')$ for the m^{th} and n^{th} goods in the bundle x'. For example, consider a textbook study of the behavior of young, single professionals, as opposed to young married professionals with small children[6].

Individuals holding bundle $\vec{x}'$, containing youth, status as a single, education and income commensurate with professional employment, as "commodities, are identified as the first group. Individuals holding a similar bundle $\vec{x}''$, the same as x' with exception that "singleness" is replaced by "married with young children" are identified as the second group. In both endowments we let x_m represent a class of luxury goods that can be considered "tools for the mating game", while we let x_n represent "household appliances". A finding that

[5] Rawls (1999) pp.118-123

[6] Kotler (1994) pp.174-84

$R_m(\vec{x}') > R_m(\vec{x}'')$ and $R_n(\vec{x}') < R_n(\vec{x}'')$ would indicate that young, single professionals tend to spend a significant portion of their incomes on luxury goods that facilitate courtship. Once married, these same individuals spend heavily on the appliances needed by a growing family.

Even with studies as simple as this one, it is possible to refute the hypothesis that an individual's preferences are arbitrary, as opposed to a function of observable factors. If the hypothesis were true, one could not find values for $R_m(\vec{x}')$ and $R_n(\vec{x}')$ that were statistically different from $R_m(\vec{x}'')$ and $R_n(\vec{x}'')$.

6.3) Community Welfare and Pareto Optimality

In this section we will define a measure of a society's welfare based on the use value it's members enjoy. This is in some ways similar to the one used by utilitarian welfare economists for a long time[7], though free of the interpersonal comparison problems they have been unable to overcome. This discussion will show that, while Pareto optimality plays an important role, it is no longer the final standard by which a society's wellbeing may be judged.

The welfare of a small community, whose members can be treated as discrete individuals, can be defined to be a sum of the welfares of the constituent individuals. If the welfare of the aggregate consumer were measured by $B(\vec{x})$, the welfare of the community can be defined by the wellbeing enjoyed by an equivalent number of identical aggregate consumers, each one holding the bundle of a corresponding community member. This is formalized in the following definition.

[7] For an overview of this literature see d'Aspremont and Gevers (2002) pp.465-76

Definition (Welfare – of a discrete community)
Let $\theta = \{\vec{x}^1, \vec{x}^2, \ldots \vec{x}^m\}$ represent the set of bundles $\vec{x}^k$ allocated to a community of m individuals $k = (1,2,\ldots m)$. Each individual is described by an aggregate use-value function $B(\vec{x})$, and each member holds a bundle $\vec{x}^k$. The Welfare $W(\theta)$ of this discrete community is defined by:

$$W(\theta) \triangleq \sum_{k=1}^{M} B(\vec{x}^k) \tag{6.3-1}$$

A large community will need to be treated as a continuous distribution of consumers described by a *demographic density* function. Such function describes the community in terms of the relative sizes of its demographic groups, as identified by the bundles they hold.

Definition (Demographic Density)
For an economy in which n goods are available for consumption and for any region of commodity space laying in the box defined by $\vec{x}'$ and $\vec{x}' + d\vec{x}$, the fraction of the population whose bundles lay within the box is given by: $\psi(\vec{x}')dx_1 dx_2 \cdots dx_n$

If we take $B(\vec{x}')$ would be a measure of the welfare of the aggregate consumer, were he to hold bundle $\vec{x}'$, the wellbeing enjoyed by the entire cohort holding that bundle is $\psi(\vec{x}')B(\vec{x}')$. The welfare of the entire community is found by integrating this over all possible values of $\vec{x}$. Since $\vec{x}$ is a vector, of n goods, this will be a multiple integral taken over all goods within the bundle. A measure of the community's welfare W is defined as follows:

Definition (Welfare – of a Large Community)
Given a community of agents described by an aggregate use-value function $B(\vec{x})$, to whose bundles $\vec{x} = x_1, x_2, \ldots x_n$ are distributed according to $\psi(\vec{x})$. The community's total welfare as determined by use-value, enjoyed by its membership is defined to be:

$$W(\psi) = \int_{x_1}\int_{x_2}\ldots\int_{x_n}\int_{N} \psi(x_1, x_2, \ldots x_n) B(x_1, x_2, \ldots x_n)\, dx_1, dx_2, \ldots dx_n \tag{6.3-2}$$

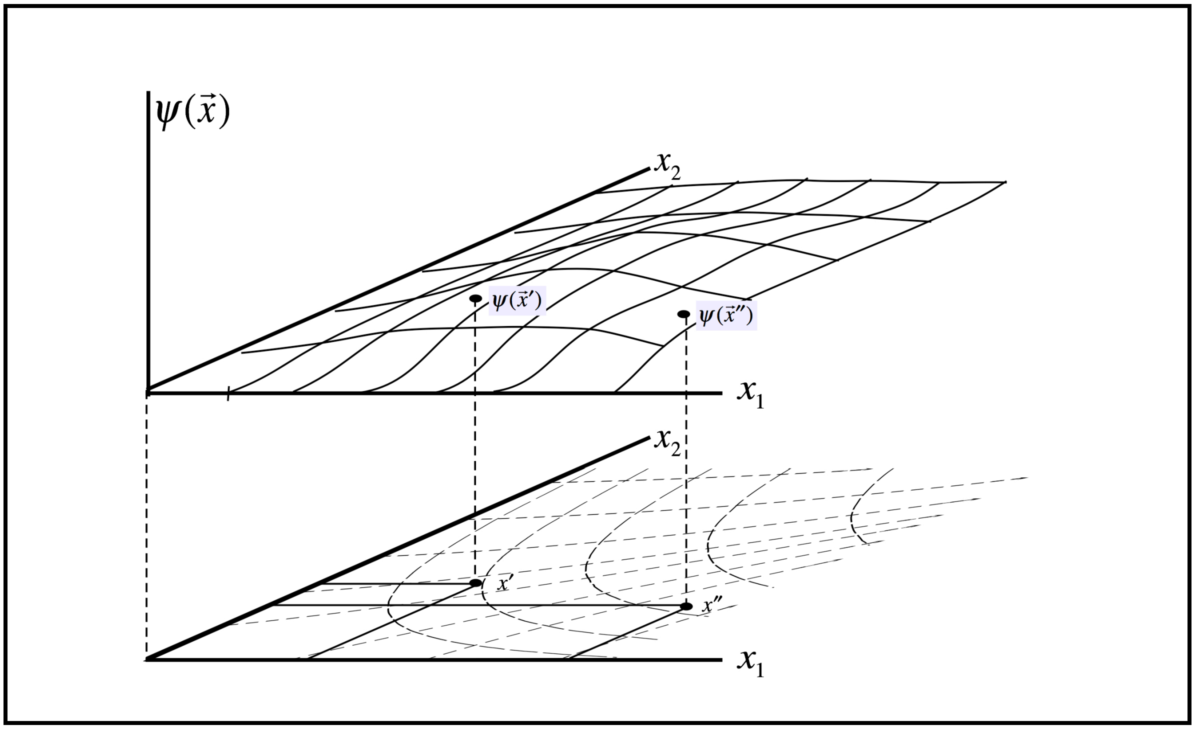

Figure 6.3-1 Demographic Density Function

Solving the above volume integral may be a daunting problem. Fortunately, due to the market mechanism we often will not have to. If the community members have been free to engage in exchange and have come to equilibrium, each member's bundle will reside on the aggregate consumer's income expansion path as shown in Figure 6.3-2. In that case all variables x_i become functions of the consumer's wealth $x_i = x_i[w]$. The integral (Equation 6.3-1) need be taken over the single parameter w.

$$W(\psi) = \int_0^w \psi\left(\vec{x}^*[w,\vec{p}]\right) B\left(\vec{x}^*[w,\vec{p}]\right) dw = \int_0^w \psi(w) B(w) dw \tag{6.3-3}$$

What is assumed in using Equation 6.3-3 is that the distribution is Pareto optimal. Given any consumer k holding bundle $\vec{x}^k$ having market value $\vec{p} \bullet \vec{x}^k = w^k$. If this bundle is not his optimal bundle $\vec{x}^{k*}[w^k]$ per Proposition 4.5-1, then that consumer has the opportunity to make Pareto improving exchanges with other members of the community[8]. There are of course many possible distributions that are Pareto optimal. Such distributions do not necessarily provide the same community welfare, as the following example will show.

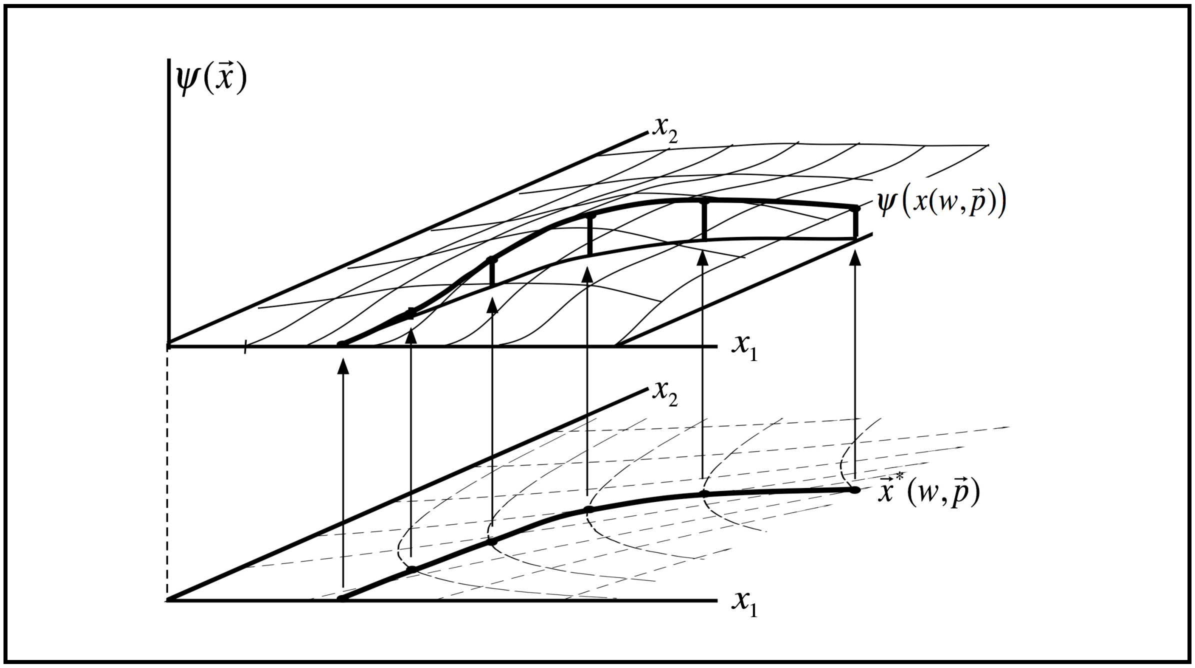

Figure 6.3-2 Demographic Density Along the Aggregate Wealth Expansion Path

Given an economy in which there are n goods to choose from, x_i where $i = (1,2,\ldots n)$. Consider a community of m individuals $k = (1,2,\ldots m)$, possessing bundles $\vec{x}^k$. Each individual is represented by the aggregate marginal price and use-value functions $\vec{R}(\vec{x})$ and $V(\vec{x})$ respectively. Let $\theta = \{\vec{x}^1, \vec{x}^2, \ldots \vec{x}^m\}$ represent the set of bundles $\vec{x}^k$ allocated to the community members. Since these bundles all lie along the common income expansion path $\vec{X}^*(w)$ the allocation set can be written as $\theta = \{\vec{X}^*(w^1), \vec{X}^*(w^2), \ldots, \vec{X}^*(w^m)\}$ or simply $\theta(w^1, w^2, \ldots w^m)$,

[8] This of course will result in a price adjustment, which will not affect the argument made here.

where w^k is the wealth of the k^{th} individual. We create a second distribution θ' as follows. We identify two individuals; one who is richer, possessing bundle $\vec{x}^r = \vec{X}^*(w^r)$; and one who is poorer, possessing bundle $\vec{x}^r = \vec{X}^*(w^r)$. We redistribute a small amount of wealth Δw, where $\Delta w < w^r - w^p$, from the richer person to the poorer person. This creates a new allocation:

$$\theta' = \{\vec{x}^1{}',\vec{x}^2{}',\ldots\vec{x}^m{}'\} \text{ such that: } \begin{cases} \vec{x}^p{}' = \vec{X}^*\left(w^p + \Delta w\right) \\ \vec{x}^r{}' = \vec{X}^*\left(w^r - \Delta w\right) \\ \vec{x}^k{}' = \vec{x}^k \quad \forall k \neq r \vee p \end{cases} \tag{6.3-4}$$

Since we are treating this as a community of discrete individuals, its welfare is measured using Equation 6.3-1. We compare the welfares provided by the distributions by subtracting them.

$$W(\theta') - W(\theta) = \sum_{k=1}^{m}\left[B\left(\vec{x}^k{}'\right) - B\left(\vec{x}^k\right)\right] \tag{6.3-5}$$

$B(\vec{x}^k) = B(\vec{x}^k{}')$ for all $\vec{x}^k = x^k{}'$ the corresponding terms in the sum cancel, leaving only the use-values of the individuals whose bundles were involved in the redistribution, thus:

$$\begin{aligned} &W(\psi') - W(\psi) \\ &\quad = \left[B\left(\vec{x}^r{}'\right) - B\left(\vec{x}^r\right)\right] + \left[B\left(\vec{x}^p{}'\right) - B\left(\vec{x}^p\right)\right] \\ &\quad = \left[B\left(\vec{X}^*(w^r - \Delta w)\right) - B\left(\vec{X}^*(w^r)\right)\right] + \left[B\left(\vec{X}^*(w^p + \Delta w)\right) - B\left(\vec{X}^*(w^p)\right)\right] \\ &\quad = \left[B\left(w^r - \Delta w\right) - B\left(w^r\right)\right] + \left[B\left(w^p + \Delta w\right) - B\left(w^p\right)\right] \\ &\quad = \left[-\left.\frac{\partial B}{\partial w}\right|_{w^r} \Delta w\right] + \left[\left.\frac{\partial B}{\partial x_i}\right|_{w^p} \Delta w\right] > 0 \end{aligned} \tag{6.3-6}$$

In the last step we have used the fact that d^2B/dw^2 is negative, which follows from the concavity of $B(\vec{x})$, which in turn follows from Assumption 5.

Geometrically, the concavity can be seen from Figure 6.3-3. The figure shows a plot of $B(\vec{x})$ as a concave roof over a plane containing all possible values of $\vec{x}$ (for the two variable case). For bundles $\vec{x}^*(w)$ constrained to lie on the expansion path, the corresponding use value function $B(\vec{x}^*(w))$ is a curve appearing as a projection of the expansion path onto the "roof" formed by $B(\vec{x})$. As result, $B(\vec{x}^*(w))$ or simply $B(w)$ is concave downward towards the wealth expansion path.

The lower part of the figure shows $B(w)$ as seen looking straight down the x_2 axis. We see that $B(w)$ rises faster along this path for smaller values of w than for larger ones. Intuitively, what this means is that the welfare lost by the richer individual is less than the welfare gained by the poorer one, producing a net gain in welfare. Even though θ and θ' represent Pareto optimal distributions. θ' results in higher social welfare than does θ.

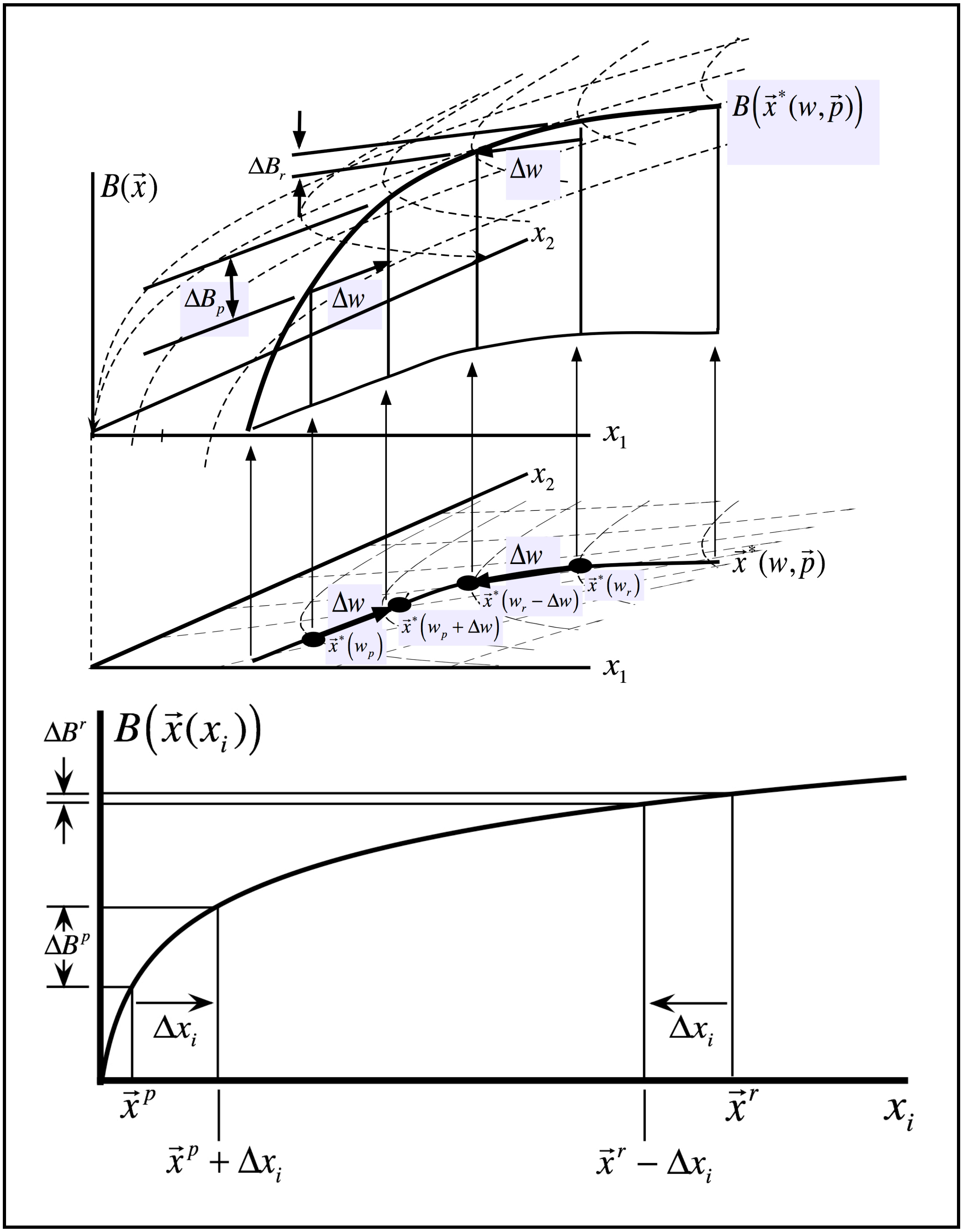

Figure 6.3-3 Impact of Wealth Distribution on Aggregate Well-being

6.4 Community Demand and Community Marginal Demand

Finally we will model a society's demand and marginal demand in a manner analogous to the way we defined aggregate welfare. As before a society's aggregate demand and aggregate marginal demand will be the sums of these quantities of otherwise identical aggregate consumers, holding bundles as they are distributed within the community.

Definition (Demand – for a single good by a discrete community)
Let $\theta = \{\vec{x}^1, \vec{x}^2, \ldots \vec{x}^m\}$ represent the set of bundles $\vec{x}^k$ allocated to a community of m individuals $k = (1,2,\ldots m)$. Each member is described by the demand function $\vec{X}^*(w, \vec{p})$ of the aggregate consumer, and receives a lump sum of wealth w^k. Demand $D_i(\theta, \vec{p})$ for the good x_i by this discrete community is:

$$D_i(\theta, \vec{p}) = \sum_{k=1}^{m} X_i^*\left(w^k, \vec{p}\right) \qquad (6.4\text{-}1)$$

Definition (Demand – for a single good by a large community)
Let ψ represent a large community of consumers, each member described by the demand function $\vec{X}^*(w, \vec{p})$ of the aggregate consumer, and to whom wealth is distributed according to $\psi(w)$. Demand $D_i(\psi, \vec{p})$ for the good x_i by this large community is:

$$D_i(\psi, \vec{p}) = \int_w \psi(w) X^*(w, \vec{p}) dw \qquad (6.4\text{-}2)$$

Definition (Marginal Demand – for a single good by a discrete community)
Let $\theta = \{\vec{x}^1, \vec{x}^2, \ldots \vec{x}^m\}$ represent the set of bundles $\vec{x}^k$ allocated to a community of m individuals $k = (1,2,\ldots m)$. Each member is described by the marginal demand function $\Delta\vec{X}^*(w, I, \vec{p}, \Delta t)$ of the aggregate consumer, possessing a stock of wealth w^k and receiving income I^k. The marginal demand $\Delta D_i(\theta, I, \Delta t)$ for good x_i over the time period Δt by this discrete community is:

$$D_i(\theta, \vec{p}, \Delta t) = \sum_{k=1}^{m} \Delta X_i^*\left(w^k, I^k, \vec{p}, \Delta t\right) \qquad (6.4\text{-}3)$$

Definition (Marginal Demand – for a single good by a large community)
Let ψ represent a large community of consumers, each member described by the marginal demand function $\vec{X}^*(w,\vec{p})$ of the aggregate consumer, and to whom wealth w and income I have been distributed according to $\psi(w,I)$. The marginal demand $\Delta D_i(\psi,\vec{p},\Delta t)$ for good x_i over the time period Δt by this large community is:

$$D_i\left(\psi,\vec{p},\Delta t\right)=\int_w \psi\left(w,I)\right)\Delta X(w,I,\vec{p},\Delta t)dw \qquad (6.4\text{-}4)$$

The behavior of these quantities just defined will depend on the functional form of the marginal price and use value functions. As was mentioned in Chapter 4, the quantities demanded and marginally demanded will be proportional if the wealth expansion paths are straight lines through the origin. If the expansion paths are curved however, these quantities will be quite different. Through an analysis similar to that done at the end of Section 6.3, it can be shown that changes in the distribution of either wealth or income will change the quantities demanded of each of the goods.

6.5) Conclusion

The notion that perfect equality maximizes social welfare might appear to advocate for policies that many, including this author, would find quite troubling. Such could be taken to imply that in a "good" society, everyone must be treated exactly the same. The obvious fact is, that we are all different, and that it is such differences that make specialization in production possible, collaboration beneficial, and creativity appreciable. As was hopefully made abundantly clear in Section 6.2, what we all hold in common are needs, which must be broadly defined. While it is our differences that allow for the diversity in expression that we call art, it is our commonality that allows the art patron to appreciate the artist's intent. Distilling our common needs, from the diverse means we seek to meet those needs is an extremely difficult task, one that, I believe, motivates Sen's Capabilities approach.

With regard to demand vs., marginal demand, rather little has been said beyond the definition of these terms. This is due to the lack of empirical studies that would allow us understand which form of demand best describes consumer behavior. In the comparative static model, with respect to which the usual form of demand is defined, no distinction is drawn between wealth and income. The consumer receives his wealth in a single lump sum, which is usually labeled "income". Unless one truly believes consumers make a lifetime of decisions in a single instant, this model must be taken to presume that the goods the consumer buys are perishable. Such a model can only represent a consumer whose goods are completely consumed before the next pay period. Such consumer, who receives identical "income" payments each period, restocks his supply of goods in the same manner each period, thus maintaining a steady state lifestyle.

While this model may seem unrealistic, the dynamic model, with respect to which marginal demand is defined, also contains a rather extreme assumption. Not all goods that the consumer acquires can be infinitely durable. Some goods (such as food) are clearly perishable and must be replaced each period. While the consumer's stock undoubtedly grows over time. Accurate modeling of such growth must include both perishable and non-perishable goods, thus both demand and marginal demand must be considered. Empirical study will be needed to understand what the correct balance is.

CHAPTER 7:
CONCLUSION: SOME BASIC RESULTS

While the notion of redistributions from the wealthy to the poor has been intuitively regarded as welfare improving for centuries, economic science can find no justification for it given its scientific framework. From Aristotle to economists of the early 20th Century, it was commonly believed that if equal quantities of wealth were taken from a rich individual and a poor one, the poorer one would suffer the most. Due to lack of the theoretic tools needed to consider such matters, economists have failed to address questions of distribution, claiming they lie outside the prevue of economic science. The boundaries of such science originate from the assumption that
the value an individual places on goods derives from unobservable forces within a consumers mind. As result, there is no way scientists can compare the value one individual places on a good to that placed on it by another.

As was argued in Chapter 2, application of the scientific by Mill, and later Jevons, led economists to believe that use-value must necessarily be attributed to some physical cause in order to be scientifically useful. As Chapter 2 showed, all attempts to define value in terms of utility or preference have led to notions that cannot necessarily predict either the choices one might make, or the wellbeing he or she might experience. Chapter 2 also showed that ancient thinkers from Aristotle through Aquinas drew economic insights, remarkably as astute as our own, without resort to some observable cause from which value must derive. Furthermore, these ancient thinkers regarded the value one places on goods as shaped by ones discipline and training, not a force to which he passively responds.

Chapter 3 addressed the struggle the marginalist pioneers had with the belief that value must derived from a cause. Our current views flow from Jevons who sought to derive economic principles from the inner workings of the human brain. The French precursors of Walras, who were much more interested in social processes, were skeptical of utility. Their view of value was very similar to that of Aristotle. Operationally, they regarded it as measurable in terms on ones willingness to pay. It may well be that unavailability of the necessary mathematics was the only reason that they fell short of actually defining value in terms of such observable phenomena.

With the conceptual barriers removed, Chapter 4 uses the techniques of vector calculus to formally define use-value in terms of behaviors that can actually be observed. As is the case with energy in physics, use-value is an abstract concept that need not be explained in terms of a single cause. While the intuitive description of use value is as loose as what was given it by Aristotle, Aquinas, or Dupuit, cardinal measurability of the quantities by which it is defined, guarantees that use-value is itself measurable in cardinal terms. There is therefore no reason why use value and marginal prices cannot be compared between individuals.

Since utility and preferences were un-measurable, economists were forced to presume that they vary arbitrarily between individuals, while other social scientists have concluded that an individual's behavior is a product of their socioeconomic condition as evidenced by their demographic profile. As shown in Chapter 5, interpersonal comparability of use-value and marginal prices will allow them to be aggregated empirically among members of demographic groups to determine if such commonality of behavior exists.

Chapter 5 details how aggregation might be accomplished in practice. By aggregating the marginal prices offered by individuals of a common demographic background, existence of any behavior commonalities can be empirically determined. We thus create a model "aggregate" consumer, whose marginal price and use value functions can be presumed to represent all members of the community. Such functions indicate the choices a "typical" individual would make were she to find herself in any given demographic group. This becomes the basis by which the distribution of goods, wealth, or other demographic factors might impact the economic behavior of the community as a whole.

7.1) The Impact of Wealth Distribution on Demand and Marginal Demand – Hobson's Effect

In this demonstration, Engle's law is used to show that redistribution of either wealth or income from a poorer individual to a richer one will reduce community demand for basic goods. This is closely related to the under-consumption problem identified by John Hobson a century ago, and for that reason will be called *Hobson's Effect.*

As before, from a community of individuals, a poorer individual with wealth w^p, and richer individual with wealth w^r are identified. All individuals are given identical incomes $\Delta w = I$. A new allocation is created by redistributing a small amount of income ΔI from the poorer individual to the richer one such that the richer individual's income is $I^r = I + \Delta I$, and the poorer individual's income is $I^p = I - \Delta I$. From Engle's law, it is apparent that a smaller portion of each redistributed dollar will be spent on basic goods *even though the sum of the individual's incomes remains the same.*

Formally, we proceed as before: Given an economy in which there are n goods to choose from, where one good x_B is a normal good satisfying a basic need. Consider a community of m individuals $k = (1,2,\ldots m)$, possessing bundles $\vec{x}^k$ which includes some quantity of the basic good x_B. Each individual is represented by the aggregate marginal price and use-value functions $\vec{R}(\vec{x})$ and $V(\vec{x})$ respectively. Let $\psi = \{\vec{x}^1, \vec{x}^2, \ldots \vec{x}^m\}$ represent the set of bundles $\vec{x}^k$ allocated to the community members. The community's marginal demand for x_B is given by:

$$\Delta D_B(\psi) = \sum_{k=1}^{m} \Delta x_B^k = \sum_{k=1}^{m} \Delta X_B^*\left(w^k, I^k\right) \tag{7.1-1}$$

Since these bundles all lie along the common income expansion path $\vec{X}^*(w)$ the allocation set can be written as $\psi = \{\vec{X}^*(w^1, I^1), \vec{X}^*(w^2, I^2), \ldots, \vec{X}^*(w^m, I^m)\}$, where w^k and I^k are the wealth and income of the k^{th} individual. We create a new allocation set:

$$\psi' = \{\vec{x}^1{}', \vec{x}^2{}', \ldots \vec{x}^m{}'\} \text{ such that: } \begin{cases} \vec{x}_B^p{}' = \vec{X}_B^*\left(w^p + I - \Delta I\right) \\ \vec{x}^r{}' = \vec{X}^*\left(w^r + I + \Delta I\right) \\ \vec{x}^k{}' = \vec{x}^k \quad \forall k \neq r \vee p \end{cases} \tag{7.1-2}$$

The change in marginal demand for the basic good x_B is:

$$\begin{aligned}
&\Delta D_B(\psi') - \Delta D_B(\psi) \\
&\quad = \left[\Delta D_B\left(\vec{x}^r{}'\right) - \Delta D_B\left(\vec{x}^r\right)\right] \qquad\qquad + \left[\Delta D_B\left(\vec{x}^p{}'\right) - \Delta D_B\left(\vec{x}^p\right)\right] \\
&\quad = \left[\Delta X_B\left(w^r, I + \Delta I\right) - \Delta X_B\left(w^r, I\right)\right] + \left[\Delta X_B\left(w^p, I - \Delta I\right) - \Delta X_B\left(w^p, I\right)\right] \\
&\quad = \left[\frac{\partial}{\partial I}\left[\Delta X_B\left(w^r, I\right)\right]\Delta I\right] \qquad\qquad - \left[\frac{\partial}{\partial I}\left[\Delta X_B\left(w^p, I\right)\right]\Delta I\right] < 0
\end{aligned}$$

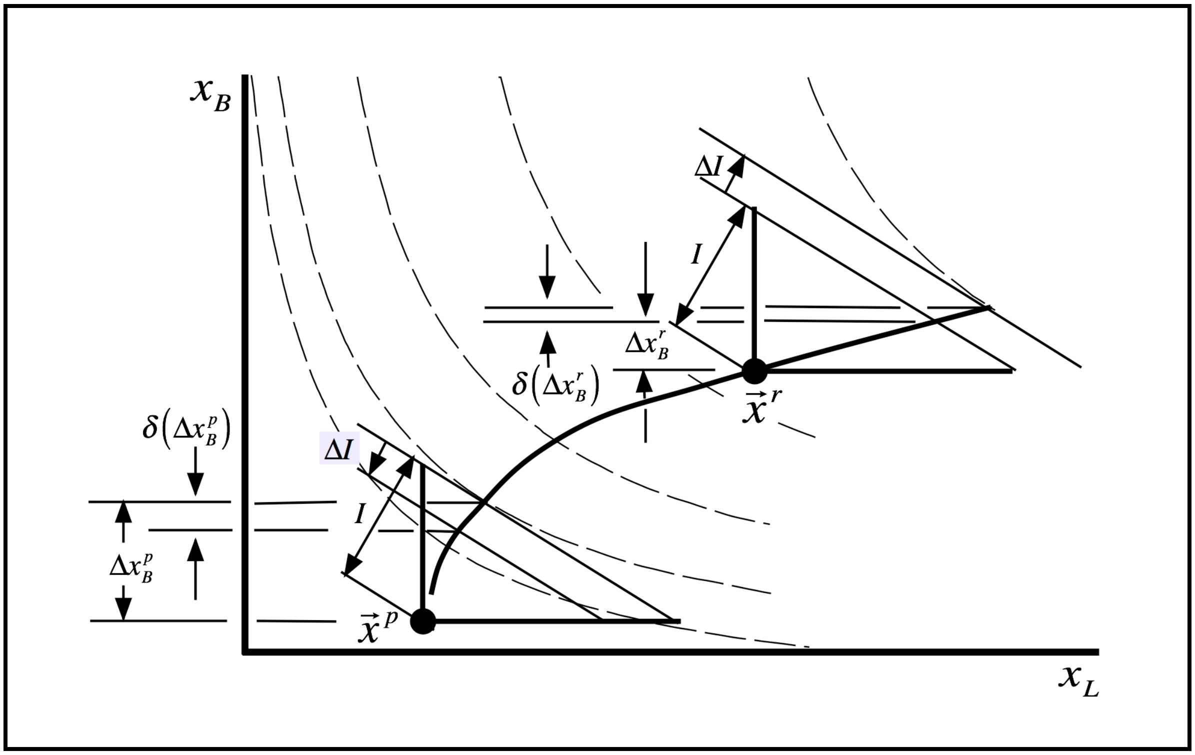

Figure 7.1-1 Impact of Wealth Distribution on Demand and Marginal Demand

Two other things are to be noted: If a redistribute wealth by taking a small bundle $\Delta\vec{x}$ from the poorer individual's bundle $\vec{x}_p$ and add it to the wealthier person's bundle $\vec{x}_r$ the portion of the poorer individual's budget spent on basic goods will i*ncrease* while the portion of the wealthier person's budget spent on such goods will decrease further. The poor person will not likely demand more basic goods due to his decreased budget, though he will be more motivated to use the resources at his disposal more productively. The wealthier person on the other hand will clearly demand less and be *less* motivated to use his resources efficiently. As result, if the redistributed wealth is in the form of property, such is likely to be used less efficiently causing a decrease in output. This may well explain the inverse relationship between the size of land holdings in Latin America, and the productivity with which such holdings are used. This is the *Latifundio – Minifundio* pattern of land use mentioned in Chapter 1.

A final observation is that wealth inequality may be self-propelling. As the demand for basic goods (which are perishable) decreases with increasing wealth, an increasing share of the income of the wealthy is dedicated to purchase durable goods. This causes the wealthy to progress along the wealth expansion path at a faster rate than the poor, which may cause fixed resources to be increasingly redistributed from poor to rich. This in turn may cause such resources to be used with decreasing efficiency.

7.2) Closing Comments

As was said at the very beginning, markets are the means by which the human species organizes itself to collectively meet its survival needs through production and distribution of goods and services. As such, markets are a force of nature, powerful, yet blind. As was shown in Chapter 5, markets function under most circumstances to obtain for us what we collectively seek. They make no guarantee that the end sought will actually enhance our well-being. Markets are not oracles. As a force of nature the market mechanism is the wind that fills our sails. Like sailors, maritime engineers, and aircraft designers, it is up to us to harness the power of the wind to our own ends. To be effective in that, we need to understand ourselves and to address the nominative questions as to what ends we should seek.

The tools of Chapter 4 allow us to banish romantic assumptions regarding the efficacy of human rationality. Rather than assuming that consumers are infinitely wise calculators of their own self-interest, we need assume merely that humans assign value to things according to reasons of their own choosing. If we are serious about understanding how markets behave, we need to base our models on a scientifically gained understanding of human behavior.

A key goal of this work was to facilitate incorporation of the results of our sister social sciences into economics. Time will tell if that goal has been met.

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