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UNIVERSITY OF CALIFORNIA, SAN DIEGO

The Brauer-Siegel Theorem for Fields of Bounded Relative Degree

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Aaron Wong

Committee in charge:

Professor Harold Stark, Chair
Professor Charles Elkan
Professor Ronald Evans
Professor Benjamin Grinstein
Professor Audrey Terras

2007

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The dissertation of Aaron Wong is approved, and
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Chair

University of California, San Diego

2007

To my family and my friends.

He named it Ebenezer, saying, "Thus far has the LORD helped us."

– 1 Samuel 7:12

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ABSTRACT OF THE DISSERTATION

The Brauer-Siegel Theorem for Fields of Bounded Relative Degree

by

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Doctor of Philosophy in Mathematics

University of California San Diego, 2007

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In this dissertation, we undertake the study of the class numbers of fields of bounded relative degree. Fix $B > 1$ and let $\mathcal{B}(B)$ be the set of all number fields M such that M can be reached by a tower of fields, $\mathbb{Q} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ such that $[M_i : M_{i-1}] \leq B$ for $1 \leq i \leq n$. Building on the work of Harold Stark [Sta74] and Andrew Odlyzko [Odl75], we show that there for a fixed B there are only finitely many CM fields M of degree greater than or equal to 387 with a given class number. In the process of proving this, we also obtain lower bounds for the residue of Dedekind zeta functions and $L(1, \chi)$. We also obtain some upper bounds for these functions by mimicking some of Jeffrey Hoffstein's calculations [Hof79].

1

Preliminaries

This chapter is only meant to introduce the main ideas used in this dissertation. Stark’s chapter in [WMLI92] gives a good exposition of these topics as well. For a gentle introduction including many explicit examples, the reader should read [Mar77]. Both [Lan94] and [Neu99] give a more formal presentation.

1.1 Algebraic Theory

The main algebraic structures we’ll be working with in this dissertation are extensions of number fields and their associated invariants.

1.1.1 An Example

Before diving into formal definitions, we will start with an example. We begin with the simplest example of a *number field*, the set of rational numbers,

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\},$$

where

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The common term for \mathbb{Z} is the set of *integers*, but we will be generalizing this word soon so we will call \mathbb{Z} the set of *rational integers* to avoid confusion. There is a special subset of the rational integers, the *primes*. Usually in the context of the rational integers, primes are taken to be positive. However, in the general context we will be considering, we will not have a notion of “positive”, so that in the generalized definition, the rational

primes will be paired off together as being the “same” prime,

$$P = \{\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \dots\}.$$

We are interested in looking at *extensions* of number fields. We will extend the field that we’re looking at by *adjoining* a root of an irreducible polynomial, a process which is described below. For our example, we will adjoin α , a root of $f(x) = x^3 - 2$.

Notice that there are three roots of this polynomial when its roots are viewed over the complex numbers \mathbb{C} . However, the algebraic structure is independent of which root is chosen. The reason is that the only algebraic property we have is that $f(\alpha) = 0$, or $\alpha^3 = 2$. In other words, we cannot distinguish between the real root and the complex pair of roots based on this property alone. We will discuss the implications of this later.

Returning to the example, to adjoin α we will consider the collection of all possible finite sums and products that can be built up from \mathbb{Q} and α . We will write this as $\mathbb{Q}(\alpha)$. It is not hard to see that

$$\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}.$$

We know that the set is at least this big. If we have higher powers of α , we can use the property that $\alpha^3 = 2$ to reduce the power. For example, $\alpha^8 = (\alpha^3)^2\alpha^2 = 2^2\alpha^2 = 4\alpha^2$. Therefore, $\mathbb{Q}(\alpha)$ is exactly this set.

Viewing $\mathbb{Q}(\alpha)$ as a vector space over \mathbb{Q} (with basis vectors 1 , α , and α^2), we see that we have a 3 dimensional vector space. We say that $\mathbb{Q}(\alpha)$ is a degree 3 extension over \mathbb{Q} , and this is written $[\mathbb{Q}(\alpha), \mathbb{Q}] = 3$.

As stated earlier, the algebraic property alone does not distinguish between the various roots of the polynomial. To make the distinction between the roots, we must view them as elements of \mathbb{C} . Since $f(x)$ is a real polynomial, we know that the roots will either be real or come in complex pairs. For the specific example, there is one real root ($\sqrt[3]{2}$, the real cube root of 2) and there is one complex pair of roots, $(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)\sqrt[3]{2}$. For each root, there is an embedding of the number field $\mathbb{Q}(\alpha)$ into \mathbb{C} . The number and types of embeddings are important features, as we will later see. (The embeddings of a particular field are also called its conjugate fields, and we will use these words interchangeably.)

We have briefly looked at $\mathbb{Q}(\alpha)$, but have not answered two basic questions. What are the integers of $\mathbb{Q}(\alpha)$? What are the primes? At this point, it is best to work with the formal definitions.

1.1.2 Field Basics

Keeping the previous example in mind, we will begin to build the formalism of algebraic number theory by discussing general fields and their extensions. Any introductory algebra text will have this information.

Definition 1.1. A field L containing K is said to be an (*algebraic*) *extension* of K if every element of $\beta \in L$ is the root of some non-zero polynomial with coefficients in K . We write L/K to denote that L is an algebraic extension of K . If $f_\beta(\beta) = 0$ and $f_\beta \in K[x]$ is irreducible over K , then we call f_β the *minimal polynomial* of β over K .

We use the word *algebraic* to distinguish this type of extension from a *transcendental* extension. A transcendental extension has elements in it which are not the root of a polynomial with coefficients in the lower field. An example of this is the extension $\mathbb{Q}(\pi)/\mathbb{Q}$. We will not deal with transcendental extensions in this dissertation.

In the example, we looked at the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$. We did not show that every element of $\mathbb{Q}(\alpha)$ is the root of a polynomial with rational coefficients, we only noted that α was the root of such a polynomial. However, it is a theorem that sums and products of algebraic numbers (that is, roots of certain polynomials over a given field) are also algebraic numbers.

Definition 1.2. The *degree* of an extension L/K is the dimension of L over K as a vector space. The degree of L/K is denote $[L : K]$.

We will only be working with finite degree extensions, so we will not discuss infinite extensions here. We were able to come up with an explicit form of $\mathbb{Q}(\alpha)$ in the example that allowed us to quickly see its dimension over \mathbb{Q} as a vector space. The ability to find a single element such that its powers generate the basis for the vector space for the field extension was not unique to this example. For the cases we are considering, it is always true that for a finite degree algebraic extension L/K is of the form $L(\alpha)$ for some α .

Definition 1.3. A *number field* is a finite (algebraic) extension of \mathbb{Q} .

The reason we are interested in number fields is that we want to build on the idea of primes, and the primes in \mathbb{Q} are familiar and fairly well understood.

There are two special type of field extensions that we will be using repeatedly. The first is the *compositum* of number fields.

Definition 1.4. The compositum of number fields K_1, K_2, \dots, K_n , which we denote by $K_1K_2 \cdots K_n$, is the field generated by all the K_i .

In this definition, we are ignoring some technical problems that arise in the general situation since we can view all of our fields to be subsets of \mathbb{C} . If we did not allow ourselves that luxury, we would have to define the compositum of two fields by taking the quotient of a certain tensor product and prove that this extends in a reasonable way to taking the compositum of many fields.

Definition 1.5. A *normal extension* of a number field K is a field N such that

1. $N \supset K$
2. For any $\beta \in K$, if f_β is the minimal polynomial of β over \mathbb{Q} , then all the roots of f_β are contained in K .

If, furthermore, N has the property that it is contained in every other normal extension N' of K , then N is called the *minimal normal extension* of K .

A normal extension of a number field K is a critical idea because the main theorem that we use (Theorem 1.15) requires Galois theory, which can only be applied to normal extensions.

The minimal normal extension of our example field $\mathbb{Q}(\alpha)$ is $N = \mathbb{Q}(\alpha, \omega)$, where ω is a primitive cube root of 1. This is a degree 6 field ($[N : \mathbb{Q}] = 6$) that is degree 2 over $\mathbb{Q}(\alpha)$ ($[N : \mathbb{Q}(\alpha)] = 2$). It is also the compositum of $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\omega)$. Notice that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ and $[N : \mathbb{Q}] = 6 = 3!$. In general, if N is the minimal normal extension of K , then $[N : \mathbb{Q}] \leq [K : \mathbb{Q}]!$.

1.1.3 Algebraic Integers and Unique Factorization

Definition 1.6. We define the set of *algebraic integers* to be the set

$$\mathbb{A} = \{\beta \in \mathbb{C} : \beta \text{ is the root of a monic polynomial with coefficients in } \mathbb{Z}\}.$$

Definition 1.7. The *algebraic integers of a number field K* is the set $K \cap \mathbb{A}$.

We will show that the rational integers are indeed the algebraic integers of \mathbb{Q} . Since every element of \mathbb{Q} is of the form a/b where $a, b \in \mathbb{Z}$ with $b \neq 0$, we see that every element of \mathbb{Q} is the root of a polynomial of the form $f(x) = bx - a$. The algebraic integers

of \mathbb{Q} are those which are the root of a monic polynomial, which means that $b = \pm 1$, and this completes the proof.

In general, computing the algebraic integers of a number field can be complicated task. To demonstrate this, we will compute the integers of $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$. It is easy to check that $a + b\sqrt{5}$ is a solution to

$$x^2 - 2ax + a^2 - 5b^2 = 0.$$

We want to find rational values of a and b so that this is a polynomial with coefficients in \mathbb{Z} .

From the x term, we can see that $a = a'/2$, where a' is a rational integer. We will now consider the constant term. We want

$$\frac{(a')^2}{4} - 5b^2 = \frac{1}{4}((a')^2 - 20b^2) \in \mathbb{Z}.$$

We need the term on the right to be an integer divisible by 4. This means that b can have at most a 2 in the denominator. We let $b = b'/2$ so that our condition is now

$$(a')^2 - 5(b')^2 \in 4\mathbb{Z}.$$

If a' is even, then b' must also be even, and this is the case where a and b are already rational integers. So we now consider the case where a' is odd, which forces b' to be odd. We will write $a' = 2k + 1$ and $b' = 2j + 1$, with $k, j \in \mathbb{Z}$. Then we have

$$\begin{aligned} (a')^2 - 5(b')^2 &= 4k^2 + 4k + 1 - 5(4j^2 + 4j + 1) \\ &= 4k^2 + 4k - 20j^2 - 20j - 4. \end{aligned}$$

This is divisible by 4 for any choice of k and j . Therefore, we have shown that the integers of $\mathbb{Q}(\sqrt{5})$ are numbers of the form $(a + b\sqrt{5})/2$ where a and b are either both even or both odd.

In order to avoid a long discussion about ideals that will not directly be used anywhere else in this dissertation, we will simply have a brief overview of the important ideas, and the reader is advised to pick up a book on algebraic number theory for the details.

The easiest way to view an ideal is that it is the set of all integral linear combinations of a collection of elements. The notation will be understood by writing a couple examples. In \mathbb{Q} , we have

$$(8) = \{8 \cdot x : x \in \mathbb{Q} \cap \mathbb{A}\}$$

and

$$(10, 15) = \{10 \cdot x + 15 \cdot y : x, y \in \mathbb{Q} \cap \mathbb{A}\}.$$

(Note that $\mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$, but we write it this way to illustrate the idea.) In the example of $\mathbb{Q}(\alpha)$, we have

$$(2 + \alpha) = \{(2 + \alpha) \cdot x : x \in \mathbb{Q}(\alpha) \cap \mathbb{A}\}$$

and

$$(1 - \alpha, 3) = \{(1 - \alpha) \cdot x + 3 \cdot y : x, y \in \mathbb{Q}(\alpha) \cap \mathbb{A}\}.$$

Ideals have divisibility based on containment. We say that an ideal \mathfrak{a} divides an ideal \mathfrak{b} if $\mathfrak{a} \supset \mathfrak{b}$. (It is conventional to use a fancy German character to represent ideals in number theory.) In the rational integers, this correspondence is pretty clear upon considering an example. Comparing the ideals (3) and (6) we get

$$(3) = \{3 \cdot x : x \in \mathbb{Z}\} = \{0, \pm 3, \pm 6, \dots\} \supset \{0, \pm 6, \pm 12, \dots\} = (6),$$

and we know that 3 divides 6.

Products of ideals are obtained by multiplying all possible pairs of elements, taking one from each ideal. For example,

$$(2) \cdot (3) = \{2x \cdot 3y : x, y \in \mathbb{Z}\} = \{6z : z \in \mathbb{Z}\} = (6).$$

It turns out that you can simply multiply the generators of the two ideals to get a list of generators for the product ideal, such as

$$(6, 4) \cdot (9, 3) = (54, 18, 36, 12).$$

To verify this, we can use what we know about linear combinations and divisibility in the rational integers, namely that the collection of all integral linear combinations of a set of elements is generated by the greatest common divisor of the elements. In this case, the product reduces back to

$$(2) \cdot (3) = (6).$$

Ideals which can be generated by a single element are called *principal ideals*.

The divisibility property gives us the definition of primes:

Definition 1.8. An ideal $\mathfrak{p} \neq (1)$ is prime if whenever \mathfrak{p} divides $\mathfrak{a}\mathfrak{b}$, then \mathfrak{p} divides \mathfrak{a} or \mathfrak{p} divides \mathfrak{b} .

The algebraic integers of a number field all have unique factorization into prime ideals. The need to generalize from elements to ideals is not immediately obvious. We will work through a brief example to illustrate the problem that arises and how it is resolved.

Consider the number field $\mathbb{Q}(\sqrt{-5})$. The algebraic integers of this field are everything of the form $a+b\sqrt{-5}$ where $a, b \in \mathbb{Z}$. Consider the following two factorizations of 6:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}).$$

It does not take much effort to see that each of the two factorizations cannot be reduced any further as numbers. As numbers, this shows a violation of unique factorization. However, as ideals, the factorization can continue and thus preserve this nice property.

To understand how it works, it's good to look at a case in the rational integers:

$$(60) = (6) \cdot (10) = (4) \cdot (15).$$

Pretend for the moment that you could not tell that you could factor the ideals on the right any further, but you expected unique factorization to hold. For this to work, you would have to surmise that there is a way that (6) splits such that part of it “belongs” to (4), which we will suggestively denote as (6, 4), and the rest of it “belongs” to (15), denoted (6, 15). By doing this to (10) as well, we would get another factorization of (60):

$$(60) = (6, 4) \cdot (6, 15) \cdot (10, 4) \cdot (10, 15). \tag{1.1}$$

Of course, we can see that this is

$$(60) = (2) \cdot (3) \cdot (2) \cdot (5),$$

which confirms our suspicions in this case. Furthermore, by taking appropriate pairs, it is possible to get both of the previous factorizations. For example,

$$(6, 4) \cdot (10, 4) = (60, 24, 40, 16) \implies (2) \cdot (2) = (4).$$

Returning to the $\mathbb{Q}(\sqrt{-5})$ example, following the same process we get

$$(6) = (2, 1 + \sqrt{-5}) \cdot (2, 1 - \sqrt{-5}) \cdot (3, 1 + \sqrt{-5}) \cdot (3, 1 - \sqrt{-5}).$$

This factorization behaves the same way as Equation (1.1), except that there is no single number that generates the ideals on the right. In fact, the term *ideal* originated from the context of saying that there is some *ideal number* which was the greatest common divisor, so that unique factorization of numbers could still be true.

$$\begin{array}{ccc}
(0, 0, 0) & (0, 0, 1) & (0, 0, 2) \\
(0, 1, 0) & (0, 1, 1) & (0, 1, 2) \\
(0, 2, 0) & (0, 2, 1) & (0, 2, 2) \\
(1, 0, 0) & (1, 0, 1) & (1, 0, 2) \\
(1, 1, 0) & (1, 1, 1) & (1, 1, 2) \\
\vdots & \vdots & \vdots
\end{array}$$

Figure 1.1: 18 of the 27 congruence classes of the algebraic integers of $\mathbb{Q}(\alpha)$ modulo (3) .

1.1.4 Field Invariants

The term *field invariant* refers to a collection of values associated to a number field K . These values are intrinsic to the structure of K and do not depend on any particular choices, such as the set of generators. We will not discuss these invariants in detail, but only point out the properties that are relevant to the remainder of the dissertation.

We have already seen in the $\mathbb{Q}(\alpha)$ example that $\alpha^3 = 2$ has three possible choices when viewed over \mathbb{C} . For any number field K , there are always as many embeddings as the degree of the extension over \mathbb{Q} . Furthermore, the complex embeddings (those which are not contained in the real line) always come in complex conjugate pairs. We will express this relationship as $n = r_1 + 2r_2$, where r_1 is the number of real embeddings, r_2 is the number of complex conjugate pairs of embeddings, and $n = [K : \mathbb{Q}]$.

The *norm* of an ideal \mathfrak{a} is the number of congruence classes of the algebraic integers of K modulo \mathfrak{a} and is denoted by $\mathbb{N}_K(\mathfrak{a})$. We say that two algebraic integers x and y are in the same congruence class if $x - y \in \mathfrak{a}$. It is easy to see that for a rational integer n viewed in \mathbb{Q} , $\mathbb{N}_{\mathbb{Q}}((n)) = |n|$. The number field affects how the norm is computed, as we will show in the next example.

We will continue to work with $\mathbb{Q}(\alpha)$, with $\alpha^3 = 2$. We will compute $\mathbb{N}_{\mathbb{Q}(\alpha)}((3))$. Note that

$$(3) = \{3a + 3b\alpha + 3c\alpha^2 : a, b, c \in \mathbb{Z}\}.$$

We can view these as 3-tuples of integers (a, b, c) . Therefore, two algebraic integers are in the same congruence class if each term in the difference between the 3-tuples is divisible by 3. This observation allows one to write out all 27 congruence classes. The list is started in Figure 1.1.

It is also interesting to compute $\mathbb{N}_{\mathbb{Q}(\alpha)}((1 + \alpha))$. Note that

$$\begin{aligned} (1 + \alpha) &= \{(1 + \alpha) \cdot a + (1 + \alpha) \cdot b\alpha + (1 + \alpha) \cdot c\alpha^2 : a, b, c \in \mathbb{Z}\} \\ &= \{(a + 2c) + (a + b)\alpha + (b + c)\alpha^2 : a, b, c \in \mathbb{Z}\}. \end{aligned}$$

The requirement for two algebraic integers to be in the same congruence class is much more complicated. There are other methods to compute the norm besides this brute force method, but we will not introduce them because they will not be used later. Using the idea of 3-tuples again, we see that two algebraic integers are congruent modulo $(1 + \alpha)$ when their difference is in the \mathbb{Z} -span of

$$\{(1, 1, 0), (0, 1, 1), (2, 0, 1)\}.$$

The \mathbb{Z} -span of this set is difficult to discern in this form. However, if we do a \mathbb{Z} -row reduction, we get

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

From this, we can see that any 3-tuple is equivalent to exactly one of

$$\{(0, 0, 0), (0, 0, 1), (0, 0, 2)\},$$

so that $\mathbb{N}_{\mathbb{Q}(\alpha)}((1 + \alpha)) = 3$.

When the norm of an ideal generated by the element x is 1, so $\mathbb{N}((x)) = 1$, we call x a *unit*. In particular, it means that there is some other algebraic integer y such that $xy = 1$. The units of a number field play a role in the computation of the *regulator* of the number field. The regulator is computed by taking the determinant of a matrix which represents the volume of a fundamental mesh in an $(r_1 + r_2)$ -dimensional space (where r_1 and r_2 correspond to the number of real and complex pairs of embeddings of the number field into \mathbb{C}). We will not work through the details of this calculation here even though it will be necessary at one point in this dissertation. The curious reader can read more about the regulator in the references given.

The different \mathcal{D}_K and the discriminant D_K of a number field K are difficult objects to define, but they have nice properties that we will discuss in the next section. The important fact is that the norm of the different is the discriminant, $\mathbb{N}_K(\mathcal{D}_K) = D_K$.

The last invariant that we will discuss is the *class number*, denoted h_K or $h(K)$. The class number is one of the two primary objects of study for this dissertation. Its value reflects the failure of ideals to be principal ideals. There are several methods for computing the class number, but the one we will use will have to wait for a few sections, as there is more formalism that needs to be developed first.

1.1.5 Extensions of Number Fields

Up to this point, we have been focusing on extensions of \mathbb{Q} . However, it is possible to discuss field extensions of any number field. Basically everything we have done over \mathbb{Q} can be repeated starting with a different base field. There are analogous notions of integrality, relative norms, relative differentials, and relative class numbers. The details of these constructions fill several chapters of a book on algebraic number theory, so we will only touch on the essential properties.

We will be building towers of fields throughout this dissertation. If $K \supset L \supset M$ are all fields, with K/L and L/M both finite algebraic extensions, then we have the following properties:

1. The degree is multiplicative: $[K : M] = [K : L][L : M]$.
2. The relative different is multiplicative: $\mathcal{D}(K/M) = \mathcal{D}(K/L)\mathcal{D}(L/M)$.
3. The relative norm respects composition: $\mathbb{N}_{K/M} = \mathbb{N}_{L/K} \circ \mathbb{N}_{M/L}$.

Furthermore, when the base field is \mathbb{Q} , the relative objects reduce appropriately:

1. The relative different is the different: $\mathcal{D}_{K/\mathbb{Q}} = \mathcal{D}_K$.
2. The relative norm is the norm: $\mathbb{N}_{K/\mathbb{Q}} = \mathbb{N}_K$.

1.1.6 CM Fields

There are two concepts to introduce before we can define CM fields:

Definition 1.9. A field is *totally real* if all of its embeddings are contained in the real line. Alternatively, if n is the degree of the extension, then $r_1 = n$ and $r_2 = 0$.

A field is *totally complex* if none of its embeddings are contained in the real line. Alternatively, if n is the degree of the extension, then $r_1 = 0$ and $2r_2 = n$.

Now we can define CM fields:

Definition 1.10. A *CM field* is a totally complex quadratic extension of a totally real field.

The term CM field comes from the theory of complex multiplication. Initially, complex multiplication was almost a geometric theory: A lattice Λ in the complex plane is said to have complex multiplication if there is a complex number $\delta \in \mathbb{C} \setminus \mathbb{R}$ such that $\delta\Lambda \subset \Lambda$. The idea of complex multiplication has been generalized to the theory of elliptic curves. For our purposes, the definition alone is sufficient.

The reason we will want to study CM fields is to control the size of the ratio of the regulators of the totally real field and the totally complex field. The only place this calculation is applied is with Equation (3.4). This is taken directly from [Sta74], so the details of the calculation are not included.

1.2 Analytic Theory

Analytic number theory seeks to gain information out of special complex analytic functions that can be built from number fields. We will begin with the Riemann zeta function, which is the prototype for the Dedekind zeta functions.

1.2.1 The Riemann Zeta Function

There is much that can be said about the Riemann zeta function, but we will confine ourselves to the relevant features.

For $s \in \mathbb{C}$, $s = \sigma + it$, for $\sigma > 1$ we define

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

The unusual choice of symbols was introduced by Bernhard Riemann in his 1859 paper where he discussed the properties of this function and demonstrated how information about the primes can be obtained by understanding the behavior of this function.

The clue that information about primes is encoded into this function is found

by rewriting the sum as a product over the (positive) rational primes:

$$\begin{aligned}
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \cdots \\
&= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{(2^2)^s} + \frac{1}{5^s} + \frac{1}{2^s \cdot 3^s} + \frac{1}{7^s} + \frac{1}{(2^3)^s} + \frac{1}{(3^2)^s} + \frac{1}{2^s \cdot 5^s} + \cdots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{(2^2)^s} + \cdots\right) \left(1 + \frac{1}{3^s} + \frac{1}{(3^2)^s} + \cdots\right) \left(1 + \frac{1}{5^s} + \frac{1}{(5^2)^s} + \cdots\right) \cdots \\
&= \left(\frac{1}{1-2^{-s}}\right) \left(\frac{1}{1-3^{-s}}\right) \left(\frac{1}{1-5^{-s}}\right) \cdots \\
&= \prod_p (1-p^{-s})^{-1}.
\end{aligned}$$

The product over primes is known as an *Euler product*. The reason this product works out correctly is that each n^{-s} term can be expressed uniquely as a product of $(p_i^{a_i})^{-s}$ terms, and each combination of such terms appears exactly once in the final product.

The Euler product allows us to quickly compute the logarithmic derivative of $\zeta(s)$:

$$\begin{aligned}
\frac{\zeta'}{\zeta}(s) &= \frac{d}{ds} \log(\zeta(s)) \\
&= \frac{d}{ds} \log\left(\prod_p (1-p^{-s})^{-1}\right) \\
&= -\sum_p \frac{d}{ds} \log(1-p^{-s}) \\
&= -\sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} \\
&= -\sum_p \frac{\log p}{p^s-1}.
\end{aligned}$$

We can also compute the logarithmic derivative a different way, by expanding out the logarithm in a power series:

$$\begin{aligned}
\frac{\zeta'}{\zeta}(s) &= -\sum_p \frac{d}{ds} \log(1-p^{-s}) \\
&= -\sum_p \frac{d}{ds} \left(-\sum_{m=1}^{\infty} p^{-ms}\right) \\
&= -\sum_p \sum_{m=1}^{\infty} \log(p) p^{-ms} \\
&= \sum_{n=1}^{\infty} \Lambda(n) n^{-s},
\end{aligned}$$

where $\Lambda(n)$ is known as the *Von Mangoldt* function and is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^a \text{ for some prime } p \text{ and integer } a > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The sum diverges as $s \rightarrow 1^+$, which shows that there is a pole at $s = 1$. It turns out that the pole is simple and has residue 1. In fact, the Riemann zeta function is analytic everywhere except for this pole. We will see later that the residue of the Dedekind zeta function gives information about the field invariants.

The Riemann zeta function satisfies a functional equation. If we define

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

then $\xi(s)$ is an entire function satisfying $\xi(s) = \xi(1-s)$. The functional equation can be used to compute estimates for the value of $\zeta(s)$ for $\sigma < 0$ by relating them to the values of $\zeta(s)$ for $\sigma > 1$, where we have formulas that allow us to make explicit calculations.

1.2.2 Dedekind Zeta Functions

To understand the transition from the Riemann zeta function to Dedekind zeta functions, we need to reconsider the infinite sum in a different context. Instead of viewing it as a sum over positive rational integers, we treat it as a sum over the ideals of \mathbb{Z} :

$$\zeta(s) = \zeta_{\mathbb{Q}}(s) = \sum_{(n)} \mathbf{N}((n))^{-s}.$$

In the same way the sum over rational integers was expressed as a product over positive prime integers, we can write this as a product over prime ideals:

$$\zeta_{\mathbb{Q}}(s) = \prod_{\mathfrak{p}} (1 - \mathbf{N}(\mathfrak{p})^{-s})^{-1}.$$

Definition 1.11. The *Dedekind zeta function* of a number field K is defined for $\sigma > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathbf{N}(\mathfrak{a})^{-s},$$

where $s = \sigma + it$ and the sum is over all the ideals of K .

By the unique factorization property of ideals, we also have

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \mathbf{N}(\mathfrak{p})^{-s})^{-1}.$$

The Euler product also gives us the logarithmic derivative:

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = \sum_{\mathfrak{p}} \frac{\log \mathbf{N}(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^s - 1}. \quad (1.2)$$

Just like the Riemann zeta function, Dedekind zeta functions are analytic everywhere except at $s = 1$. The residue, κ_K , is given by

$$\kappa_K = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{\omega_K \sqrt{|D_K|}}, \quad (1.3)$$

where r_1 and r_2 are the number of real and complex pairs of embeddings (respectively), h_K is the class number of K , R_K is the regulator of K , ω_K is the number of roots of unity in K , and D_K is the discriminant of K .

Dedekind zeta functions satisfy a functional equation which is similar in form to the one satisfied by the Riemann zeta function. If we set

$$\xi_K(s) = \left(\frac{|D_K|}{2^{2r_2} \pi^n} \right)^{s/2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} s(s-1) \zeta_K(s),$$

where $n = [K : \mathbb{Q}]$, then $\xi_K(s)$ is an entire function satisfying $\xi_K(s) = \xi_K(1-s)$.

1.2.3 Dirichlet L -Functions

Dirichlet L -functions are simpler than Artin L -functions. They can be understood without any background in number theory at all, which cannot be said of their complicated counterparts. A thorough discussion of Dirichlet L -functions will be given as these are the type that arise from quadratic extensions, in particular from the CM field situation.

Definition 1.12. A *character* χ of modulus m is a map from $\mathbb{Z}_{>0}$ into the complex plane such that

1. $\chi(1) = 1$.
2. $\chi(ab) = \chi(a)\chi(b)$.
3. $\chi(a) = 0$ when $(a, m) > 1$.
4. χ is periodic with period m .

It is helpful to look at a few examples of characters to get a sense of how they look. Table 1.1 provides a small sample of characters. The character χ_0 is known as the *trivial character*. Notice that the examples exhibit the following properties (which will not be proven).

The analytic behavior of the L -functions mimics the behavior of the Riemann zeta function. The L -function is analytic everywhere except possibly at $s = 1$ (the L -functions corresponding to the trivial characters have a pole at $s = 1$ and the others are analytic there).

The important feature of Dirichlet L -functions for us is that they arise from CM fields. If K is a totally complex extension of a totally real field k , then then we have

$$\zeta_K(s) = \zeta_k(s)L(s, \chi)$$

for some Dirichlet L -function $L(s, \chi)$. In fact, the factorization holds for any quadratic extension K of any field k once the $L(s, \chi)$ concept is generalized to general number fields. This can be understood in the more complicated context of the formalism of Artin L -functions, but we will refrain from that discussion and present an explicit example.

Consider the CM field $\mathbb{Q}(i)/\mathbb{Q}$, where $i^2 = -1$. It is not hard to show that the algebraic integers of $\mathbb{Q}(i)$ are simply all the numbers of the form $a + bi$ where $a, b \in \mathbb{Z}$. Furthermore, all the ideals of $\mathbb{Q}(i)$ are principal and the norm is given by $\mathbb{N}((a + bi)) = a^2 + b^2$.

We want to compute the expansion for $\zeta_{\mathbb{Q}(i)}(s)$. To do this, we need to know the possible values of $\mathbb{N}((a + bi))$ for all the primes of $\mathbb{Q}(i)$. Fortunately, this is well understood:

1. There is one prime ideal of norm 2, namely $(1 + i)$.
2. If $p \in \mathbb{Z}$ is a prime congruent to 1 modulo 4, then there are two distinct primes \mathfrak{p} such that $\mathbb{N}(\mathfrak{p}) = p$.
3. If $p \in \mathbb{Z}$ is a prime congruent to 3 modulo 4, then (p) itself is a prime ideal and $\mathbb{N}((p)) = p^2$.

Therefore, we have

$$\begin{aligned}
\zeta_{\mathbb{Q}(i)}(s) &= \prod_{\mathfrak{p}} (1 - \mathbb{N}(\mathfrak{p})^{-s})^{-1} \\
&= (1 - 2^{-s})^{-1} \prod_{p \equiv 1 \pmod{4}} (1 - p^{-s})^{-2} \prod_{p \equiv 3 \pmod{4}} (1 - p^{-2s})^{-1} \\
&= (1 - 2^{-s}) \prod_{p \equiv 1 \pmod{4}} (1 - p^{-s})^{-2} \prod_{p \equiv 3 \pmod{4}} \left[(1 - p^{-s})^{-1} (1 + p^{-s})^{-1} \right] \\
&= \prod_p (1 - p^{-s})^{-1} \prod_{p > 2} \left(1 - (-1)^{(p-1)/2} p^{-s} \right)^{-1} \\
&= \zeta_{\mathbb{Q}}(s) L(s, \chi_1),
\end{aligned}$$

where χ_1 is the nontrivial character modulo 4 as shown in Table 1.1.

We can combine the factorization of ζ_K with the functional equation for the Dedekind zeta functions to get a functional equation for $L(s, \chi)$. If we take K to be a quadratic extension field k and $[K : \mathbb{Q}] = 2n$, then we have

$$\xi_{\chi}(s) = \frac{\xi_K(s)}{\xi_k(s)} = \frac{\left(\frac{|D_K|}{2^{2\rho_2} \pi^{2n}} \right)^{s/2} \Gamma(s/2)^{\rho_1} \Gamma(s)^{\rho_2} s(s-1) \zeta_K(s)}{\left(\frac{|D_k|}{2^{2r_2} \pi^n} \right)^{s/2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} s(s-1) \zeta_k(s)},$$

where the number of real and complex pairs of embeddings of k are r_1 and r_2 , the number for K are ρ_1 and ρ_2 .

The complex pairs of embeddings of k all give rise to two complex pairs of embeddings of K . The real embeddings of k may or may not continue to be real when we move to K , but each real embedding will either give rise to two new real embeddings or two complex pairs of embeddings. Let ρ'_2 be the number of complex pairs of embeddings that arose from a real embedding. Then we have the following relationships:

$$\begin{aligned}
r_1 &= \frac{1}{2} \rho_1 + \rho'_2 \\
\rho_2 &= 2r_2 + \rho'_2
\end{aligned}$$

Plugging in these values in the equation above gives

$$\xi_{\chi}(s) = \left(\frac{|D_K|}{|D_k|} \frac{1}{2^{\rho_2 + \rho'_2} \pi^n} \right)^{s/2} \Gamma(s/2)^{\rho_1/2} \Gamma(s)^{(\rho_2 - \rho'_2)/2} \left(\frac{\Gamma(s)}{\Gamma(s/2)} \right)^{\rho'_2} L(s, \chi).$$

It turns out that we can write $|D_K| = D_k^2 f$, where f is the period of χ (which is also the norm of the conductor of χ). Also, by using Legendre's duplication formula

(see [Dav00], for example),

$$\frac{\Gamma(2s)}{\Gamma(s)} = 2^{2s-1} \pi^{-1/2} \Gamma\left(s + \frac{1}{2}\right),$$

we can simplify the expression further:

$$\xi_\chi(s) = \left(\frac{|D_k|f}{2^{\rho_2+\rho'_2}\pi^n}\right)^{s/2} \Gamma(s/2)^{\rho_1/2} \Gamma(s)^{(\rho_2-\rho'_2)/2} \left(2^{(s-1)\rho'_2} \pi^{-\rho'_2/2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)^{\rho'_2}\right) L(s, \chi).$$

After some rearranging, this is

$$\xi_\chi(s) = 2^{-\rho'_2} \pi^{-\rho'_2/2} \left(\frac{|D_k|f}{2^{\rho_2-\rho'_2}\pi^n}\right)^{s/2} \Gamma(s/2)^{\rho_1/2} \Gamma(s)^{(\rho_2-\rho'_2)/2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)^{\rho'_2} L(s, \chi). \quad (1.4)$$

Since $\rho_2 \geq \rho'_2$, we see that all of the exponents on the gamma functions terms are non-negative. This gives the form of the functional equation for L -functions.

In the special case of CM fields, we have $r_1 = n$ and $r_2 = 0$ (since k is totally real) and $\rho_1 = 0$ and $\rho_2 = n$ (since K is totally complex). The relationships force $\rho'_2 = n$ and we get

$$\xi_\chi(s) = 2^{-n} \pi^{-n/2} \left(\frac{|D_k|f}{\pi^n}\right)^{s/2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)^n L(s, \chi).$$

1.2.4 Artin L -Functions

Artin L -functions are a generalization of Dirichlet L -functions. I have chosen to discuss these separately because the level of detail for this section is significantly less than the previous section. This section can be skipped if the reader intends to skip the details of Brauer's Theorem (Theorem 1.15). The statement of that theorem and its applications can be understood without having any knowledge of Artin L -functions.

There is a more general notion of characters than what we have discussed. A *representation* of a group is a map from the group into the group of invertible n -dimensional matrices that preserves the group structure. A representation is called *irreducible* if you cannot block-diagonalize the matrices into more than one block. From the representation, one gets a *character* on the group by taking the trace of the representative matrices. An *irreducible character* is a character derived from an irreducible representation.

For the characters corresponding to the Dirichlet L -functions, we are looking at representations of the cyclic group of order m . The characters are the traces of one-dimensional matrices, which are simply the values of sole matrix entries.

Let H be a subgroup of a group G . Given a character χ on G , it can be viewed as a character on H by restriction, $\psi = \chi|_H$. It turns out that given a character ψ on H , one can define another character on G in a well-defined manner, $\chi = \psi^*$. This is known as the *induced character* of G by χ (of H). A special case of this is the character of the regular representation, χ_{reg} , which is the character induced from the character of the trivial subgroup.

When the group is a Galois group of a field extension, then to each character there is a corresponding Artin L -function. The definition of these L -functions is a bit complicated and can be looked up in the references. It is enough to know that the L -function depends on the character and the field extension (say K/k), and we denote it by $L(s, \chi, K/k)$.

There are four major properties:

1. $L(s, \chi_0, K/k) = \zeta_k(s)$, where χ_0 is the trivial character.

2. If χ_1 and χ_2 are characters of G , then

$$L(s, \chi_1 + \chi_2, K/k) = L(s, \chi_1, K/k)L(s, \chi_2, K/k).$$

3. If $K' \supset K \supset k$ is a bigger Galois extension, and if χ is a character of $G(K/k)$, also viewed as a character of $G(K'/k)$, then

$$L(s, \chi, K/k) = L(s, \chi, K'/k).$$

4. Let $k \subset F \subset K$ be an intermediate field, and let ψ be a character of $G(K/F)$. Let ψ^* be the induced character of $G(K/k)$. Then

$$L(s, \psi, K/F) = L(s, \psi^*, K/k).$$

In the same way that we were able to factor $\zeta_K(s)$ into the product of an Dirichlet L -function and $\zeta_k(s)$ in the CM field situation, there is a corresponding factorization in general:

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi \text{ irreducible}} L(s, \chi, K/k)^{\chi(1)},$$

where $\chi(1)$ is the evaluation of χ on the identity element of the group.

These L -functions are known to be analytic for all $s \neq 1$ for all characters whenever the field extension is normal. *Artin's conjecture* is that these L -functions are analytic for all $s \neq 1$ for any field extension. If Artin's conjecture were true, many of the results of this dissertation could be strengthened.

1.3 Brauer's Theorem

An important key to some of the proofs in this dissertation is the ability to locate a zero of the Dedekind function of one field by knowing the location of a zero of a different Dedekind zeta function. In [Lan94], the main result needed to accomplish this is called *Brauer's Lemma*.

Lemma 1.14. (*Brauer's Lemma*) *Let G be a finite group and χ_{reg} be the character of the regular representation. Then there exist cyclic subgroups $H_j \neq 1$, positive rational numbers λ_j , and one-dimensional characters $\psi_j \neq 1$ of H_j such that*

$$\chi_{\text{reg}} = \chi_0 + \sum_j \lambda_j \psi_j^*.$$

We will use this to prove what Stark refers to as *Brauer's Theorem*:

Theorem 1.15. (*Brauer's Theorem*) *Suppose that K/k is a normal extension and that $\zeta_k(s_0) = 0$. Then $\zeta_K(s_0) = 0$ as well.*

Proof. Let $G = G(K/k)$ be the Galois group of K/k . Then by Brauer's Lemma (Lemma 1.14), we have

$$\chi_{(\text{reg}, G)} = \chi_{(0, G)} + \sum_j \lambda_j \psi_j^*,$$

where $\chi_{(\text{reg}, G)}$ is the character of the regular representation of G , $\chi_{(0, G)}$ is the trivial character on G , λ_j are positive rational numbers, and ψ_j^* are one-dimensional characters induced from cyclic subgroups $H_j \leq G$.

Consider the Artin L -function of K/k given by this character. On the left side, we get

$$L(s, \chi_{(\text{reg}, G)}, K/k) = L(s, \chi_{(0, 1)}^*, K/k) = L(s, \chi_0, K/K) = \zeta_K(s).$$

On the right side, we get

$$\begin{aligned} L\left(s, \chi_{(0, G)} + \sum_j \lambda_j \psi_j^*, K/k\right) &= L(s, \chi_{(0, G)}, K/k) \prod_j L(s, \psi_j^*, K/k)^{\lambda_j} \\ &= \zeta_k(s) \prod_j L(s, \psi_j^*, K/k)^{\lambda_j}. \end{aligned}$$

Notice that $s_0 \neq 1$ and K/k is normal, so that the L -functions are all analytic at s_0 . This implies the result. \square

1.4 The Brauer-Siegel Theorem

In this section, we will briefly discuss the inspirations for the work contained in this dissertation.

1.4.1 The Original Theorem

In [Lan94], Lang devotes a short chapter to this theorem.

Theorem 1.16. (*The Brauer-Siegel Theorem*) *If k_i ranges over a sequence of number fields Galois over \mathbb{Q} , of degree n_i and discriminant D_i , such that $n_i/\log |D_i|$ tends to 0, then we have*

$$\log(h_i R_i) \sim \log |D_i|^{1/2} \text{ as } i \rightarrow \infty,$$

where h_i and R_i are the class number and regulator of k_i , respectively.

The proof of this theorem has two parts. The first is an upper estimate for the residue of the Dedekind zeta function and the second is a lower estimate. The proof for the upper bound is *effective*, which means that one could compute explicit numerical values in the estimates. However, the proof of the lower estimate is *ineffective*, so that the proof relies upon the mere existence of constants without providing a way to compute them.

Notice that in situations where the regulator is well-behaved, this gives a relationship between the size of the class number and the size of the discriminant. We will see this when we work with CM fields.

1.4.2 Some Effective Cases of the Theorem

In [Sta74], Stark provided a method with which one can compute an effective lower estimate of the residue of the Dedekind zeta function in special cases.

Theorem 1.17. (*Stark Theorem 1*) *There is an effectively computable constant $c > 0$ such that*

$$\kappa_k > \frac{c}{ng(n)|D_k|^{1/n}}.$$

Here, we have set $n = n_k$ and $g(n) = 1$ if there is a sequence of fields $\mathbb{Q} = k_0 \subset k_1 \subset \dots \subset k_m = k$ each normal over the preceding field and $g(n) = n!$ otherwise. If k contains no quadratic subfield, we even have

$$\kappa_k > \frac{c}{g(n) \log |D_k|}.$$

This lower bound can be used in place of the one used in the usual proof and makes the entire proof effective.

1.4.3 Fields of Bounded Relative Degree

Stark uses towers of normal fields in the proof of Theorem 1.17. The work in this dissertation is based on a similar idea. Instead of taking fields k where there is a sequence of fields $\mathbb{Q} = k_0 \subset k_1 \subset \cdots \subset k_m = k$ each normal over the preceding field, we focus on fields of bounded relative degree.

Definition 1.18. Let $\mathcal{B}(B)$ be the collection of all fields M for which there is a sequence of fields

$$\mathbb{Q} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_t = M$$

such that for each $1 \leq i \leq t$, $[M_i : M_{i-1}] \leq B$.

One of the primary goals is to prove the following:

Theorem 1.19. *Fix $B > 1$. There exist only finitely many CM fields $M \in \mathcal{B}(B)$ with any given class number.*

We will trace the history of the improvements by Odlyzko ([Odl75]) and Hoffstein ([Hof79]) based on Stark's original work with calculations performed on fields of bounded relative degree. In the end, we will also obtain a number of explicit numerical calculations which give us the values of many of the constants that arise throughout the dissertation.

2

Generalizing Stark's Calculations

The goal of this chapter is to generalize some of the results of [Sta74]. In particular, we will be introducing parameters which will allow us to optimize our final results and give us better values for our numerical computations.

2.1 General Lemmas

These first lemmas are stated in a very general form, but they will specifically be applied to Dedekind zeta functions and L -functions. The first is a straightforward calculation involving the functional equation and the Hadamard product formula and is stated without proof (the proof is found in the original paper). The second is a generalized version of a calculus lemma that was implicit in Stark's paper.

Lemma 2.1. (*Stark Lemma 1*) *Let $f(s)$ be an entire function and let*

$$g(s) = \eta^{s/2} \Gamma\left(\frac{s}{2}\right)^{a_1} \Gamma(s)^{a_2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)^{a_3} f(s),$$

where $\eta > 0$, $a_i \geq 0$ for $i = 1, 2, 3$. Suppose that $f(s)$ is positive for real $s > 1$ and $g(s)$ is an entire function of order 1 whose zeros $\rho = \beta + i\gamma$ all satisfy $0 < \beta < 1$. Suppose further that $g(s) = g(1-s)$. Then

$$\sum_{\rho} \frac{1}{s-\rho} = \frac{1}{2} \log \eta + \frac{a_1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + a_2 \frac{\Gamma'}{\Gamma}(s) + \frac{a_3}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + \frac{1}{2}\right) + \frac{f'}{f}(s),$$

where the ρ and $\bar{\rho}$ terms are grouped together in the sum and are counted according to their multiplicities.

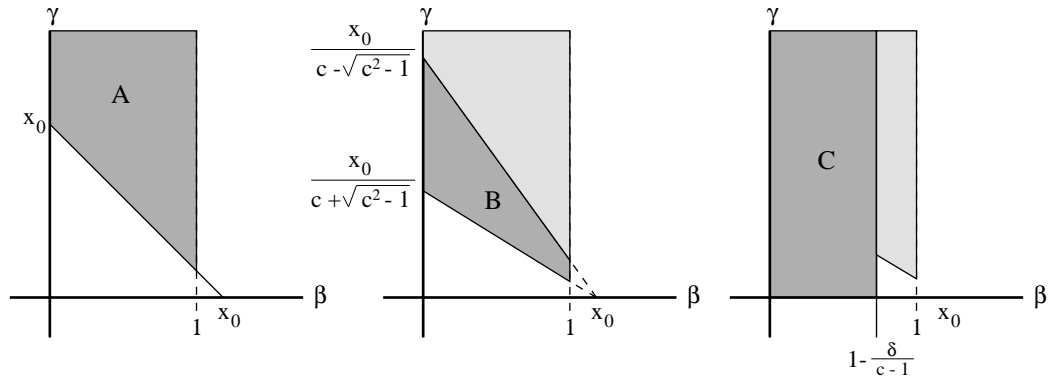


Figure 2.1: Graphs for the proof of Lemma 2.2.

We note that for a number field M of degree n , when $f(s) = s(s-1)\zeta_M(s)$ we have $\eta = |D_M|/(2^{2r_2}\pi^n)$, where $a_1 = r_1$ is the number of real embeddings, $a_2 = r_2$ is the number of complex pairs of embeddings, and $a_3 = 0$.

Lemma 2.2. *Suppose $\delta > 0$, $c > 1$, and let $x_0 = 1 + \delta$. Define*

$$h_{(\beta, \gamma)}(x) = \frac{x - \beta}{(x - \beta)^2 + \gamma^2}$$

for $0 < \beta < 1$, $\gamma \geq 0$, and $1 \leq x \leq x_0$. Then for all (β, γ) except possibly for

$$(\beta, \gamma) \in \left[1 - \frac{\delta}{c-1}, 1\right) \times \left[0, \frac{c\delta}{(c-1)(c + \sqrt{c^2 - 1})}\right],$$

we have $h_{(\beta, \gamma)}(x) \leq ch_{(\beta, \gamma)}(x_0)$.

Proof. A quick calculation shows that for fixed (β, γ) with $\gamma > 0$, $h_{(\beta, \gamma)}(x)$ has a global maximum at $x = \beta + \gamma$ and $h_{(\beta, \gamma)}(\beta + \gamma) = (2\gamma)^{-1}$. We also find that $h_{(\beta, \gamma)}(x)$ increases on $[\beta, \beta + \gamma]$ and decreases on $[\beta + \gamma, \infty)$. We will consider three cases to prove the result. Figure 2.1 graphically represents the three cases.

The inequality is trivially true when $x_0 \in [\beta, \beta + \gamma]$, for then

$$h_{(\beta, \gamma)}(x) \leq h_{(\beta, \gamma)}(x_0) < ch_{(\beta, \gamma)}(x_0)$$

since the function is increasing there. This translates to the region $x_0 \leq \beta + \gamma$ since $x_0 \geq \beta$ is always true. This is region A in Figure 2.1.

The inequality is also trivially true when $ch_{(\beta,\gamma)}(x_0) \geq (2\gamma)^{-1}$. Computing directly, this happens when

$$\beta + \gamma \left(c - \sqrt{c^2 - 1} \right) \leq x_0 \leq \beta + \gamma \left(c + \sqrt{c^2 - 1} \right).$$

This is region B in Figure 2.1.

We will compute the inequality explicitly for $\beta + \gamma \leq 1$. In this range, the function is decreasing so that the maximum value is attained at $x = 1$.

$$\begin{aligned} \frac{1 - \beta}{(1 - \beta)^2 + \gamma^2} &\leq c \frac{x_0 - \beta}{(x_0 - \beta)^2 + \gamma^2} \\ \implies \gamma^2(-(1 - \beta)(c - 1) - c\delta) &\leq (1 + \delta - \beta)(1 - \beta)((1 - \beta)(c - 1) - \delta). \end{aligned}$$

Note that the left side is always nonpositive and when $\beta \leq 1 - \delta/(c - 1)$ the right side is nonnegative. This is region C in Figure 2.1. \square

Remark 2.3. In the original paper, the inequality was computed for $c = 2$. The allowed region for the potential zero was weaker in γ than our result.

Lemma 2.4. (Stark Lemma 2) *Suppose that $f(s)$ satisfies the conditions of Lemma 2.1 and let $n = a_1 + 2a_2 + a_3$. Suppose further that there exist b , c_1 , c_2 , and d such that*

1. $\frac{f'}{f}(\sigma) \leq \frac{1}{\sigma} + \frac{1}{\sigma-1} + c_1 \log \eta$ for $1 \leq \sigma \leq 1.461$,
2. $d \geq 3$, $d \geq \eta$, and $\log d \geq c_2^{-1}n$,
3. *there is at most one zero of $f(s)$ in the set S defined by*

$$S = \left[1 - \frac{b}{\log d}, 1 \right) \times \left[0, \frac{b}{\log d} \right] \subset \mathbb{C}.$$

If this zero exists, it is real and simple and we denote it as β_0 .

Let $c > 1$ be chosen so that

$$\frac{(c - 1) \left(c + \sqrt{c^2 - 1} \right)}{c} \frac{b}{\log d} \leq 0.461. \quad (2.1)$$

Let $\sigma_0 = \sigma_0(c) = 1 + b(c - 1)(\log d)^{-1} (\leq 1.461)$ and set

$$E = \begin{cases} \frac{1 - \beta_0}{\sigma_0 - \beta_0}, & \text{if } \beta_0 \text{ exists} \\ 1, & \text{otherwise.} \end{cases} \quad (2.2)$$

Then

$$f(1) \geq c_3^{-1} E f(\sigma_0),$$

where $c_3 = c_3(c) > 0$ is computable from c_1 and c_2 .

Proof. We will follow the case where β_0 exists.

$$\sum_{\rho} \frac{1}{\sigma - \rho} = \frac{1}{\sigma - \beta_0} + \sum_{\rho \neq \beta_0} \Re \frac{1}{\sigma - \rho},$$

where we are allowed to take the real part because the sum groups complex conjugate pairs together and for $\rho = \beta + i\gamma$,

$$\frac{1}{\sigma - \rho} + \frac{1}{\sigma - \bar{\rho}} = 2\Re \frac{1}{\sigma - \rho} = 2 \frac{\sigma - \beta}{(\sigma - \beta)^2 + \gamma^2}.$$

We apply Lemma 2.2 with $\delta = b(c-1)(\log d)^{-1}$. Then for zeros outside of the set

$$\left[1 - \frac{b}{\log d}, 1\right) \times \left[0, \frac{c}{c + \sqrt{c^2 - 1}} \frac{b}{\log d}\right] \subset S$$

we have for $1 \leq \sigma \leq \sigma_0 \leq 1.461$,

$$\frac{1}{\sigma - \rho} \leq c \frac{1}{\sigma_0 - \rho}.$$

All of the zeros except β_0 lie outside of S , so we can apply this to all of them. Therefore,

$$\sum_{\rho} \frac{1}{\sigma - \rho} \leq \frac{1}{\sigma - \beta_0} + c \sum_{\rho} \frac{1}{\sigma_0 - \rho}, \quad (2.3)$$

where we have added back the β_0 into the sum (it is positive since $\beta_0 < 1 < \sigma_0$).

By Lemma 2.1,

$$\begin{aligned} \frac{f'}{f}(\sigma) &= \sum_{\rho} \frac{1}{\sigma - \rho} - \frac{1}{2} \log \eta - \frac{a_1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} \right) - a_2 \frac{\Gamma'}{\Gamma}(\sigma) - \frac{a_3}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + \frac{1}{2} \right) \\ &\leq \frac{1}{\sigma - \beta_0} + c \sum_{\rho} \frac{1}{\sigma_0 - \rho} - \frac{1}{2} \log \eta - \frac{a_1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} \right) - a_2 \frac{\Gamma'}{\Gamma}(\sigma) - \frac{a_3}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + \frac{1}{2} \right) \\ &= \frac{1}{\sigma - \beta_0} + \frac{c-1}{2} \log \eta - \frac{a_1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} \right) + \frac{ca_1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma_0}{2} \right) \\ &\quad - a_2 \frac{\Gamma'}{\Gamma}(\sigma) + ca_2 \frac{\Gamma'}{\Gamma}(\sigma_0) - \frac{a_3}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + \frac{1}{2} \right) + \frac{ca_3}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma_0}{2} + \frac{1}{2} \right) + c \frac{f'}{f}(\sigma_0). \end{aligned}$$

Notice that $\Gamma'(x)/\Gamma(x)$ is monotonically increasing for $x > 0$ and is negative for $0 < x < 1.461$. Then for $1 \leq \sigma \leq \sigma_0 \leq 1.461$,

$$\begin{aligned} \frac{f'}{f}(\sigma) &\leq \frac{1}{\sigma - \beta_0} + \frac{c-1}{2} \log \eta - \left(\frac{a_1}{2} + a_2 + \frac{a_3}{2} \right) \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \\ &\quad + c \left(\frac{a_1}{2} + a_2 + \frac{a_3}{2} \right) \frac{\Gamma'}{\Gamma}(\sigma_0) + c \frac{f'}{f}(\sigma_0) \\ &< \frac{1}{\sigma - \beta_0} + \frac{2cc_1 + c - 1}{2} \log \eta - \frac{n}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) + \frac{c}{\sigma_0} + \frac{c}{\sigma_0 - 1}, \end{aligned}$$

where we have used the first hypothesis and dropped the $\Gamma'/\Gamma(\sigma_0)$ term because it is positive.

Integrating this on the interval $1 \leq \sigma \leq \sigma_0$, we get

$$\log f(1) \geq \log f(\sigma_0) - \log \left(\frac{\sigma_0 - \beta_0}{1 - \beta_0} \right) - (\sigma_0 - 1) \left(\frac{2cc_1 + c - 1}{2} \log \eta + \frac{c}{\sigma_0} + \frac{c}{\sigma_0 - 1} - \frac{n \Gamma'}{2 \Gamma} \left(\frac{1}{2} \right) \right).$$

But since $\sigma_0 - 1 = b(c-1)(\log d)^{-1} \leq bc_2(c-1)/n$ and $\eta \leq d$,

$$\begin{aligned} & (\sigma_0 - 1) \left(\frac{2cc_1 + c - 1}{2} \log \eta + \frac{c}{\sigma_0} + \frac{c}{\sigma_0 - 1} - \frac{n \Gamma'}{2 \Gamma} \left(\frac{1}{2} \right) \right) \\ & \leq \frac{b(c-1)}{\log d} \left(\frac{2cc_1 + c - 1}{2} \log \eta \right) + c(\sigma_0 - 1) \left(\frac{1}{\sigma_0} + \frac{1}{\sigma_0 - 1} \right) \\ & \quad - \frac{bc_2(c-1)}{n} \frac{n \Gamma'}{2 \Gamma} \left(\frac{1}{2} \right) \\ & \leq \frac{b(c-1)(2cc_1 + c - 1)}{2} \frac{\log \eta}{\log d} + c \left(2 - \frac{1}{\sigma_0} \right) - \frac{bc_2(c-1)}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \\ & \leq \frac{b(c-1)(2cc_1 + c - 1)}{2} + 1.316c - \frac{bc_2(c-1)}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right). \end{aligned}$$

Exponentiating both sides gives

$$f(1) \geq c_3(c)^{-1} \left(\frac{\sigma_0 - \beta_0}{1 - \beta_0} \right) f(\sigma_0), \quad (2.4)$$

where

$$c_3(c) = c_3 = \exp \left(\frac{b(c-1)(2cc_1 + c - 1)}{2} + 1.316c - \frac{bc_2(c-1)}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right). \quad (2.5)$$

Notice that E is defined to be the middle term of Equation (2.4), so that the proof is complete when β_0 exists.

When β_0 does not exist, the $(\sigma - \beta_0)^{-1}$ term does not arise in Equation (2.3), and so can we take $E = 1$. \square

2.2 Results Near $s = 1$

The lemmas in this section are all calculations near $s = 1$. The first lemma relies on the generality from Lemma 2.2. The next two lemmas are applications of the first lemma, looking first at the residue of a Dedekind zeta function and then at the value of an L -function at $s = 1$.

Table 2.1: Allowed values of μ and ν , where σ is chosen to be $1 + k(\log |D_M|)^{-1}$.

k	0.83	0.83	0.84	0.84	0.90	0.95
μ	2.915	2.915	2.989	3	3.618	4.079
ν	1000	100	10	9.343	3.618	3
k	1.10	1.12	1.22	1.22	1.22	1.22
μ	8.826	10	100	1000	10000	100000
ν	2	1.949	1.688	1.668	1.666	1.666

Lemma 2.5. (Stark Lemma 3a) Let M be an algebraic number field of degree $n = r_1 + 2r_2$ where M has r_1 real conjugate fields and $2r_2$ complex conjugate fields. For $\sigma > 1$,

$$-\frac{\zeta_M}{\zeta_M}(\sigma) < \frac{1}{\sigma} + \frac{1}{\sigma-1} + \frac{1}{2} \log \left(\frac{|D_M|}{2^{2r_2} \pi^n} \right) + \frac{r_1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} \right) + r_2 \frac{\Gamma'}{\Gamma}(\sigma). \quad (2.6)$$

Proof. This is a direct application of Lemma 2.1 with $f(s) = s(s-1)\zeta_M(s)$. \square

Lemma 2.6. (Stark Lemma 3b) Let M be an algebraic number field of degree $n > 1$. Then $\zeta_M(s)$ has at most one zero β in the region

$$1 - (\mu \log |D_M|)^{-1} \leq \beta < 1 \text{ and } |\gamma| \leq (\nu \log |D_M|)^{-1},$$

where μ and ν can be chosen according the Table 2.1. If there is such a zero, then it is also simple.

Proof. We begin with Equation (2.6) and rearrange the terms to get

$$\sum_{\rho} \frac{1}{s-\rho} = \frac{1}{s-1} + \frac{1}{2} \log |D_M| + \left(\frac{1}{s} - \frac{n \log \pi}{2} \right) + \frac{r_1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} \right) + r_2 \left(\frac{\Gamma'}{\Gamma}(s) - \log 2 \right) + \frac{\zeta'_M}{\zeta_M}(s). \quad (2.7)$$

The sum runs over all the zeros ρ of $\zeta_M(s)$ with $0 < \Re(\rho) < 1$. For $s = \sigma > 1$, note that

$$\frac{1}{\sigma-\rho} + \frac{1}{\sigma-\bar{\rho}} = \frac{2(\sigma-\beta)}{(\sigma-\beta)^2 + \gamma^2} > 0,$$

so that

$$\sum'_{\rho} \frac{1}{\sigma-\rho} \leq \sum_{\rho} \frac{1}{\sigma-\rho},$$

where the \sum' is a sum over any subset of the ρ such that if ρ is in the set then so is $\bar{\rho}$.

Notice that for $1 < \sigma < 2.479$ the terms in the second line of Equation (2.7) are all negative, with the r_2 term being the first term to change signs. Therefore,

$$\sum_{\rho}' \frac{1}{\sigma - \rho} < \frac{1}{\sigma - 1} + \frac{1}{2} \log |D_M|.$$

We will first consider the case where $\mu = 3$ and $\nu = 9.343$. Suppose that there is a complex zero in the region. Then its complex conjugate must also be in the region. Taking just these two zeros for \sum' with $\sigma = 1 + 0.84(\log |D_M|)^{-1}$, we get

$$\begin{aligned} \sum_{\rho}' \frac{1}{\sigma - \rho} &\geq \frac{2 \left(\frac{1}{3} + 0.84\right) (\log |D_M|)^{-1}}{\left(\frac{1}{3} + 0.84\right)^2 (\log |D_M|)^{-2} + \frac{1}{9.343^2} (\log |D_M|)^{-2}} \\ &> 1.690 \log |D_M|. \end{aligned}$$

On the other hand,

$$\frac{1}{\sigma - 1} + \frac{1}{2} (\log |D_M|)^{-1} = 1.690 \log |D_M|.$$

This gives a contradiction, so there cannot be a complex pair of zeros in this region. Now suppose that there are two real zeros in the given range. Taking the same value of σ as before, we have

$$\begin{aligned} \sum_{\rho}' \frac{1}{\sigma - \rho} &\geq 2 \cdot \frac{1}{\left(\frac{1}{3} + 0.84\right) (\log |D_M|)^{-1}} \\ &> 1.704 \log |D_M|, \end{aligned}$$

yielding the desired contradiction. Therefore, there is at most one real zero in the region.

The other cases are similar to this one, and the appropriate values of σ are described in Table 2.1. \square

Remark 2.7. Throughout the dissertation, we will take μ and ν to be a fixed pair of values given by this lemma. The specific choice will only be relevant when we attempt to compute numerical values for our bounds. In our applications, the value of b in Lemma 2.4 will be the maximum of μ and ν .

Lemma 2.8. (Stark Lemma 4) *Let $M \neq \mathbb{Q}$ be a number field and let κ_M be the residue of $\zeta_M(s)$ at $s = 1$. Let β_0 be the exceptional zero of $\zeta_M(s)$ if it exists, so that $\beta_0 \geq 1 - (\mu \log |D_M|)^{-1}$. Then for $c > 1$ chosen as in Lemma 2.4*

$$\kappa_M > \begin{cases} c_3^{-1} \frac{c-1}{(c-1)+(b\mu)^{-1}} (1 - \beta_0) & \text{if } \beta_0 \text{ exists} \\ c_3^{-1} b(c-1) (\log |D_M|)^{-1}, & \text{otherwise.} \end{cases}$$

Proof. We want to apply Lemma 2.4 to $f(s) = s(s-1)\zeta_M(s)$ with $\eta = |D_M|$. It can be shown (see [Sta74]) that we can take $c_1 = 0$ and $c_2 = 2/\log 3$. Therefore,

$$f(1) \geq c_3^{-1} E f(\sigma_0) \quad (2.8)$$

where $c_3 = c_3(c)$ is given by Equation (2.5), E is given by Equation (2.2), and $\sigma_0 = \sigma_0(c) = 1 + b(c-1)(\log |D_M|)^{-1}$.

Note that $f(1) = \kappa_M$ and

$$f(\sigma_0) = \sigma_0(\sigma_0 - 1)\zeta_M(\sigma_0) > (\sigma_0 - 1).$$

If β_0 exists then

$$\begin{aligned} E f(\sigma_0) &> \frac{1 - \beta_0}{\sigma_0 - \beta_0} (\sigma_0 - 1) \\ &\geq \frac{b(c-1)(\log |D_M|)^{-1}}{b(c-1)(\log |D_M|)^{-1} + (\mu \log |D_M|)^{-1}} (1 - \beta_0) \\ &= \frac{c-1}{(c-1) + (b\mu)^{-1}} (1 - \beta_0) \end{aligned}$$

Otherwise,

$$E f(\sigma_0) > (\sigma_0 - 1) = b(c-1)(\log |D_M|)^{-1}.$$

Combining these results with Equation (2.8) gives the result. \square

Before we proceed to the next lemma, we will set some notation. Let k be a field of degree $n = r_1 + 2r_2$ where k has r_1 real conjugate fields and $2r_2$ complex conjugate fields. Let K be a quadratic extension of k , so $\zeta_K(s) = \zeta_k(s)L(s, \chi)$. Define $f \geq 1$ by $|D_K| = D_k^2 f$. Notice that $f(s) = L(s, \chi)$ satisfies the hypotheses of Lemma 2.1 with

$$\eta = \frac{|D_k|f}{2^{2r_2}\pi^n}$$

since $\rho_2 = 2r_2 + \rho_2'$ (under the notation of Section 1.2.3).

Lemma 2.9. (Stark Lemma 5) *There is at most one real zero β_0 of $L(s, \chi)$ in the range $1 - (\mu \log D_k^2 f)^{-1} \leq \beta < 1$. Further, for any σ_1 with*

$$1 + b(c-1)(\log D_k^2 f)^{-1} = \sigma_0 \leq \sigma_1 \leq 2,$$

where c is chosen as in Lemma 2.4, we have

$$L(1, \chi) > c_4^{-1} \frac{z(c, \beta_0)}{\sigma_1 - 1} |D_k|^{-\frac{1}{2}(\sigma_1 - 1)} \left(\frac{\sqrt{\pi}}{\Gamma(\sigma_1/2)} \right)^{r_1} \left(\frac{2^{\sigma_1 - 1}}{\Gamma(\sigma_1)} \right)^{r_2} \left(\frac{\pi^{\frac{1}{2}(\sigma_1 - 1)}}{\zeta(\sigma_1)} \right)^n$$

where

$$z(c, \beta_0) = \begin{cases} \frac{c-1}{(c-1)+(b\mu)^{-1}}(1 - \beta_0), & \text{if it exists} \\ b(c-1)(\log D_k^2 f)^{-1}, & \text{otherwise} \end{cases} \quad (2.9)$$

and $c_4 = c_4(c)$ is effectively computable.

Proof. It is shown in [Sta74] that we can apply Lemma 2.4 with $c_1 = 1/2$ and $c_2 = 1/\log 3$, so that

$$L(1, \chi) \geq c_3^{-1} EL(\sigma_0, \chi), \quad (2.10)$$

where E is defined in Equation (2.2).

It is also shown that

$$\begin{aligned} L(\sigma_0, \chi) &> L(\sigma_1, \chi) \frac{\sigma_0(\sigma_0 - 1)}{\sigma_1(\sigma_1 - 1)} \left(\frac{\eta}{f}\right)^{-\frac{1}{2}(\sigma_1 - \sigma_0)} \left(\frac{\Gamma(\sigma_0/2)}{\Gamma(\sigma_1/2)}\right)^{r_1} \left(\frac{\Gamma(\sigma_0)}{\Gamma(\sigma_1)}\right)^{r_2} \\ &> \frac{\sigma_0(\sigma_0 - 1)}{\sigma_1(\sigma_1 - 1)} |D_k|^{-\frac{1}{2}(\sigma_1 - 1)} e^{-(\sigma_0 - 1)n} \left(\frac{\sqrt{\pi}}{\Gamma(\sigma_1/2)}\right)^{r_1} \left(\frac{2^{\sigma_1 - 1}}{\Gamma(\sigma_1)}\right)^{r_2} \left(\frac{\pi^{\frac{1}{2}(\sigma_1 - 1)}}{\zeta(\sigma_1)}\right)^n \end{aligned} \quad (2.11)$$

$$(2.12)$$

Noting that $c_2 \log D_k^2 f \geq n$, by the second hypothesis of Lemma 2.4 we have

$$\exp(-(\sigma_0 - 1)n) \geq \exp(-bc_2(c - 1)).$$

We also have $\sigma_0/\sigma_1 > 1/2$ and $E(\sigma_0 - 1)$ was computed in Lemma 2.8 (we get $E(\sigma_0 - 1) \geq z(c, \beta_0)$). Bringing all these results together we get

$$L(1, \chi) > c_4^{-1} \frac{z(c, \beta_0)}{\sigma_1 - 1} |D_k|^{-\frac{1}{2}(\sigma_1 - 1)} \left(\frac{\sqrt{\pi}}{\Gamma(\sigma_1/2)}\right)^{r_1} \left(\frac{2^{\sigma_1 - 1}}{\Gamma(\sigma_1)}\right)^{r_2} \left(\frac{\pi^{\frac{1}{2}(\sigma_1 - 1)}}{\zeta(\sigma_1)}\right)^n,$$

where $c_4 = c_4(c) = 2c_3 \exp(bc_2(c - 1))$. \square

2.3 Other Results

These results are taken directly from [Sta74] without proof and are given here for reference.

Lemma 2.10. (Stark Lemma 7) *Let K_1, K_2, \dots, K_a be number fields and let $M = K_1 K_2 \cdots K_a$. Then*

$$D_M | \prod_{i=1}^a D_{K_i}^{[M:K_i]}.$$

Lemma 2.11. (Stark Lemma 11) *Let F be a quadratic field. There is an effectively computable constant $c_5 > 0$ such that $\zeta_F(\sigma) \neq 0$ for $\sigma > 1 - (c_5|D_F|^{1/2})^{-1}$. The value of c_5 can be taken to be $\pi/6$.*

Theorem 2.12. (Stark Theorem 3) *Let B be a normal extension of A and suppose β is a real simple zero of $\zeta_B(s)$. Then there is a field F between A and B such that for any field E between A and B , $\zeta_E(\beta) = 0$ if and only if $F \subset E$. Furthermore, $F = A$ or F is quadratic over A .*

3

Application to $\mathcal{B}(B)$

Having generalized the necessary results from [Sta74], we will now apply them to our primary object of study.

3.1 A Lower Bound for κ_M

Our first goal is to obtain a lower bound for the residue κ_M of the Dedekind zeta function $\zeta_M(s)$ associated to the number field M . To do this, we first show that under certain hypotheses the a zero of $\zeta_M(s)$ can be “knocked down” to a quadratic field. Combining this result with Lemma 2.11, we find that we have some information about the location of the zero, which gives us information about the size of κ_M .

Lemma 3.1. *Suppose $M \in \mathcal{B}(B)$ and there is a real β in the range*

$$1 - (\mu(2B - 1)!(2B - 1) \log |D_M|)^{-1} \leq \beta < 1$$

such that $\zeta_M(\beta) = 0$. Then there is a quadratic field F contained in M such that $\zeta_F(\beta) = 0$.

Proof. We will prove the lemma by successively reducing the length of the sequence by one or two until there are less than four fields remaining. Figure 3.1 will guide the proof by graphically keeping track of the fields.

Suppose $K \subset L \subset M$ is a sequence of fields with $[L : K], [M : L] \leq B$ and such that $\zeta_M(\beta) = 0$ where β is in the range given in the statement of the lemma. We want to show that there is a field F between K and M with $\zeta_F(\beta) = 0$ and $[F : K] \leq B$. Then the sequence $K \subset L \subset M$ can be replaced by $K \subset F$ (note that $|D_F| \leq |D_M|$ so that β is still in the required range).

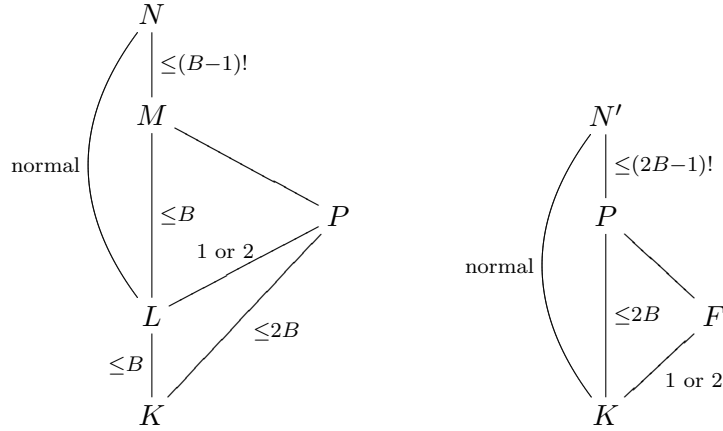


Figure 3.1: Field diagrams for the proof of Lemma 3.1.

Let N be the compositum of all the conjugates of M containing L . This is clearly a normal extension of L . Since $\zeta_M(\beta) = 0$, by Theorem 1.15 we have $\zeta_N(\beta) = 0$. By Lemma 2.10, $|D_N| \leq |D_M|^{([M:L]-1)([M:L]-1)}$, so that

$$|D_N| \leq |D_M|^{([M:L]-1)([M:L]-1)} \leq |D_M|^{(2B-1)(2B-1)}.$$

Note that

$$1 - (\mu \log |D_N|)^{-1} \leq 1 - (\mu(2B-1)(2B-1) \log |D_M|)^{-1} \leq \beta < 1$$

so that by Lemma 2.6 we see that β must be a simple zero of $\zeta_N(s)$.

We can now apply Theorem 2.12 with $A = L$ and $B = N$ to produce a field P contained in N such that $\zeta_P(\beta) = 0$ and either $P = L$ or $[P:L] = 2$. Note that $P \subset M$ since $\zeta_M(\beta) = 0$. If $P = L$, then we can take $F = P$ and we have shortened the sequence by one and the new sequence is $K \subset L$.

In the second case, let N' be the compositum of all the conjugates of P containing K . This is a normal extension of K , so by Theorem 1.15, $\zeta_P(\beta) = 0$ implies that $\zeta_{N'}(\beta) = 0$. We use Lemma 2.10 as before to get $|D_{N'}| \leq |D_P|^{(2B-1)(2B-1)}$. Then we have

$$1 - (\mu \log |D_{N'}|)^{-1} \leq 1 - (\mu(2B-1)(2B-1) \log |D_P|)^{-1} \leq \beta < 1$$

and so by Lemma 2.6, β is a simple zero of $\zeta_{N'}(s)$.

Applying Theorem 2.12 with $A = K$ and $B = N'$, there is a field F contained in N' such that $\zeta_F(\beta) = 0$ and $F = K$ or $[F:K] = 2$. If $F = K$ then we are done, since $K \subset M$ by assumption. In this situation, we have managed to remove two fields from

the sequence (only K remains). Otherwise, we note that $\zeta_P(\beta) = 0$ implies that $F \subset P$ and so $F \subset M$. Clearly, we have $2 = [F : K] \leq B$, so that the new sequence is $K \subset F$.

Suppose the sequence of fields is reduced to three terms, say $\mathbb{Q} \subset K \subset M$. Following the process above we find that there is a field $P \subset M$ such that $\zeta_P(\beta) = 0$ and either $P = K$ or $[P : K] = 2$. In either case, we have a sequence $\mathbb{Q} \subset P$ and $[P : \mathbb{Q}] \leq 2B$.

For a sequence of two terms, the desired quadratic field F is obtained by taking N to be the normal extension of K as described above and following the same procedure. The fact that $\zeta_{\mathbb{Q}}(s)$ has no real zeros forces the field derived from Theorem 2.12 to be quadratic. \square

We will now compute the lower bound for κ_M .

Theorem 3.2. *If $M \in \mathcal{B}(B)$ and $[M : \mathbb{Q}] = n$, then*

$$\kappa_M > c_3^{-1} \frac{c-1}{(c-1) + (b\mu)^{-1}} \min \left[(\mu(2B-1)!(2B-1) \log |D_M|)^{-1}, (c_5 |D_M|^{1/n})^{-1} \right].$$

Furthermore, if M has no quadratic subfields, then

$$\kappa_M > c_3^{-1} b(c-1)(\log |D_M|)^{-1}.$$

Proof. Suppose that $\zeta_M(\beta_0) = 0$ for some β_0 in the range

$$1 - (\mu(2B-1)!(2B-1) \log |D_M|)^{-1} \leq \beta_0 < 1.$$

By Lemma 3.1, there is a quadratic subfield F of M such that $\zeta_F(\beta_0) = 0$.

If M has no quadratic subfields, then there cannot be such a zero. In this case, we apply Lemma 2.8 to get

$$\kappa_M > c_3^{-1} b(c-1)(\log |D_M|)^{-1}.$$

If M has a quadratic subfield, then by Lemma 2.11,

$$\beta_0 < 1 - (c_5 |D_F|^{1/2})^{-1} \leq 1 - (c_5 |D_M|^{1/n})^{-1},$$

where the last inequality follows from the equation $\mathcal{D}(M/\mathbb{Q}) = \mathcal{D}(M/F)\mathcal{D}(F/\mathbb{Q})$, where $\mathcal{D}(A/B)$ is the relative different of A/B (take the norm of the equation). We are assuming that β_0 exists, so this implies that there are no zeros β such that

$$\max \left[1 - (\mu(2B-1)!(2B-1) \log |D_M|)^{-1}, 1 - (c_5 |D_M|^{1/n})^{-1} \right] < \beta < 1. \quad (3.1)$$

Note that this statement is also true if $\zeta_M(s)$ does not have the exceptional zero. Then by Lemma 2.8,

$$\begin{aligned} \kappa_M &> c_3^{-1} \frac{c-1}{(c-1) + (b\mu)^{-1}} (1 - \beta_0) \\ &\geq c_3^{-1} \frac{c-1}{(c-1) + (b\mu)^{-1}} \min \left[(\mu(2B-1)!(2B-1) \log |D_M|)^{-1}, (c_5 |D_M|^{1/n})^{-1} \right]. \end{aligned}$$

□

3.2 Class Number Bounds for CM Fields in $\mathcal{B}(B)$

Let k be a totally real field in $\mathcal{B}(B)$ of degree n and let K be a totally complex quadratic extension of k . Let $\zeta_K(s) = L(s, \chi) \zeta_k(s)$. Define f by $|D_K| = D_k^2 f$. Define

$$g(B) = \begin{cases} 4\mu(B-1)!(B-1), & \text{if } B > 2 \\ \mu, & \text{if } B = 2. \end{cases} \quad (3.2)$$

If we were to apply Theorem 3.1 directly, we would have to take $g(B)$ to be significantly larger. The end result would be $4\mu(2B-1)!(2B-1)$. If K is a CM field of degree $2n$, then $K \in \mathcal{B}(n)$, so that our improved result gives $4\mu(n-1)!(n-1) = 4\mu(1-1/n)n!$. In [Sta74], Stark managed to reduce the corresponding factor down to $16n!$ for CM fields. This shows that the improvement is minimal when k has no subfields, but could be significant when k has many subfields.

This lemma shows how the structure of CM fields allows us to reduce the constant.

Lemma 3.3. *Suppose k is a totally real field with $k \in \mathcal{B}(B)$ and K is a totally complex quadratic extension of k . Suppose further that $\zeta_K(s)$ has a real (simple) zero β in the range*

$$1 - (g(B) \log |D_K|)^{-1} \leq \beta < 1$$

and that $\zeta_k(\beta) \neq 0$. Then there is a complex quadratic field F contained in K such that $\zeta_F(\beta) = 0$.

Proof. As in the proof of Lemma 3.1, it is sufficient to shorten the length of the chain and use induction to complete the argument. In Figures 3.2 and 3.3, the fields in the left column are totally real and the fields in the right column are totally complex.

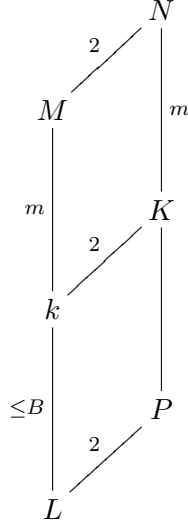


Figure 3.2: Field diagram for the proof of Lemma 3.3.

We will first consider the case where $B > 2$. Suppose that $L \subset k \subset K$ such that $[k : L] \leq B$, k is totally real, K is a totally complex quadratic extension of k , and $\zeta_K(s)$ for has a real simple zero β in the given range. We want to produce a field P which is a totally complex quadratic extension of L such that $\zeta_P(\beta) = 0$ and this is a simple zero.

We will first work with the case that $[k : L] > 2$. Let $k^{(i)}$ denote the conjugates of k over L , where $k^{(1)} = k$. Then let

$$M = k^{(1)}k^{(2)} \dots k^{([k:L]-1)}, N = MK = Kk^{(2)}k^{(3)} \dots k^{([k:L]-1)}.$$

Let $K^{(i)}$ denote the conjugate of K given by sending k to $k^{(i)}$ and let $N^{(i)} = MK^{(i)}$. We want to show that all $N^{(i)}$ are the same, that is N is normal over L .

Note that N is normal over K . Then by Theorem 1.15 we have $\zeta_N(\beta) = 0$. Let $[M : k] = m$. Then $[N : k] = 2m$ since $[N : M][M : k] \geq 2m$ (since $N \neq M$, N being complex and M being real) and $[N : k] = [N : K][K : k] \leq 2m$. By Lemma 2.10, $D_N | D_K^m (D_k^{2m})^{[k:L]-2}$. Since $D_k^2 | D_K$, this implies that $D_N | D_K^{m([k:L]-1)}$. Since M is the a normal extension of L and $[M : L] = m[k : L] \leq [k : L]!$, we have $m \leq ([k : L] - 1)!$. This shows that

$$|D_N| \leq |D_K|^{([k:L]-1)!([k:L]-1)} \leq |D_K|^{(B-1)!(B-1)}. \quad (3.3)$$

Therefore,

$$1 - (\mu \log |D_N|)^{-1} \leq 1 - (\mu(B-1)!(B-1) \log |D_K|)^{-1} \leq \beta < 1,$$

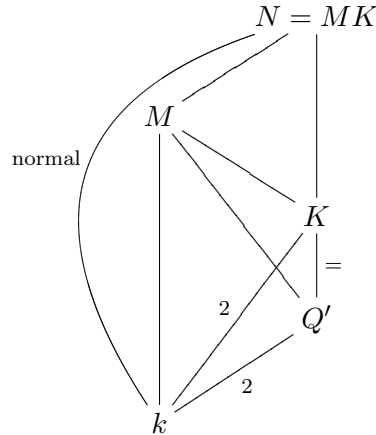


Figure 3.3: A totally real field cannot contain a totally complex field.

which by Lemma 2.6 shows that β is simple zero of $\zeta_N(s)$.

Suppose for a contradiction that not all the $N^{(i)}$ are the same, say $N^{(1)} \neq N^{(2)}$. Let $Q = N^{(1)}N^{(2)}$ and note that this is a biquadratic extension of M . In particular, this allows us to factor the $\zeta_Q(s)$ as

$$\zeta_Q(s) = \zeta_M(s)L(s, \chi^{(1)}, Q/M)L(s, \chi^{(2)}, Q/M)L(s, \chi^{(3)}, Q/M),$$

where the $\chi^{(i)}$ are the irreducible characters of the Galois group of Q/M , with $\zeta_{N^{(1)}}(s) = \zeta_M(s)L(s, \chi^{(1)}, Q/M)$ and $\zeta_{N^{(2)}}(s) = \zeta_M(s)L(s, \chi^{(2)}, Q/M)$, say. Thus, β is a multiple zero of $\zeta_Q(s)$ or β is a zero of $\zeta_M(s)$.

By Lemma 2.10, $D_Q | D_{N^{(1)}}^2 D_{N^{(2)}}^2$. By applying Equation (3.3), this shows that $D_Q | D_K^{4(B-1)!(B-1)}$. Now we see that β satisfies

$$1 - (\mu \log |D_Q|)^{-1} \leq 1 - (4\mu(B-1)!(B-1) \log |D_K|)^{-1} \leq \beta < 1,$$

so that β is a simple zero of $\zeta_Q(s)$. Therefore, it must be the case that β is a zero of $\zeta_M(s)$.

Since $N = MK$ is normal over k and $\zeta_k(\beta) \neq 0$, by Theorem 2.12 with $A = k$ and $B = N$, we see that there is a quadratic extension Q' of k such that $\zeta_{Q'}(\beta) = 0$, $Q' \subset K$, and $Q' \subset M$ (since $\zeta_K(\beta) = \zeta_M(\beta) = 0$). Note that $[K : k] = 2$ and $Q' \subset K$ imply that $K = Q'$. But this gives a contradiction since $K = Q' \subset M$ is impossible as K is complex and M is real. (See Figure 3.3.) Thus, all the $N^{(i)}$ are the same. This shows that N is normal over L .

Finally, we apply Theorem 2.12 with $A = L$ and $B = N$. There is a field P which is degree 1 or 2 over L such that $\zeta_P(\beta) = 0$. Since $L \subset k \subset N$ and $\zeta_k(\beta) \neq 0$,

we cannot have $\zeta_L(\beta) = 0$. Therefore, P must be quadratic over L . If P were real, then $P \subset M$ so that $P \subset k^{(i)}$ for some i and $\zeta_{k^{(i)}}(\beta) = 0$. But the zeta functions of conjugate fields are the same, so this implies that $\zeta_k(\beta) = 0$, giving a contradiction. This shows that P is complex.

Now suppose that $[k : L] = 2$. Then K is biquadratic over L , and in particular it is normal over L . By Lemma 2.6, β is a simple zero of $\zeta_K(s)$. Therefore, we can apply Theorem 2.6 directly with $A = L$ and $B = K$ to obtain a field P of degree 1 or 2 such that $\zeta_P(\beta) = 0$. But since $\zeta_k(\beta) \neq 0$ and we cannot have $P = L$, so P is a quadratic extension of L . Finally, since K is complex and k is real, the remaining two quadratic fields over L are both complex, so P is a complex quadratic extension of L .

When $B = 2$, we can use the argument of the previous paragraph repeatedly to get the result. \square

Before we continue, we will introduce some notation. Define

$$a(s) = 2\Gamma(s/2)\zeta(s)\pi^{(2-s)/2}.$$

Theorem 3.4. *Suppose $k \in \mathcal{B}(B)$ is a totally real field and K is a totally complex quadratic extension of k . Let σ_1 satisfy*

$$1 + b(c-1)(\log(|D_k|^2 f))^{-1} = \sigma_0 \leq \sigma_1 \leq 2,$$

where c is defined as in Lemma 2.4. If there is $\zeta_K(s)$ has a zero in the range

$$1 - (g(B) \log |D_K|)^{-1} \leq \beta < 1,$$

then

$$h(K) > \frac{c-1}{(c-1) + (b\mu)^{-1}} \frac{h(k)}{c_4(\sigma_1-1)a(\sigma_1)^n} \min \left[\frac{|D_k|^{1-\frac{\sigma_1}{2}} f^{\frac{1}{2}}}{g(B) \log D_k^2 f}, c_5^{-1} |D_k|^{1-\frac{\sigma_1}{2}-\frac{1}{n}} f^{\frac{1}{2}-\frac{1}{2n}} \right],$$

where $h(M)$ is the class number of M . If there is no such zero then

$$h(K) > \frac{h(k)b(c-1)}{c_4(\sigma_1-1)a(\sigma_1)^n} \cdot \frac{|D_k|^{1-\frac{\sigma_1}{2}} f^{\frac{1}{2}}}{\log D_k^2 f}.$$

Proof. From Equation (31) of [Sta74],

$$L(1, \chi) \leq \frac{(2\pi)^n h(K)}{|D_k f|^{1/2} h(k)}. \quad (3.4)$$

Let c be defined as in Lemma 2.4 and note that $r_1 = n$ and $r_2 = 0$ since k is a totally real field. Applying Lemma 2.9 we have

$$h(K) > \frac{h(k)z(c, \beta_0)|D_k|^{1-\frac{\sigma_1}{2}}f^{\frac{1}{2}}}{c_4(\sigma_1 - 1)a(\sigma_1)^n}, \quad (3.5)$$

where

$$z(c, \beta_0) = \begin{cases} \frac{c-1}{(c-1)+(b\mu)^{-1}}(1 - \beta_0), & \text{if it exists} \\ b(c-1)(\log D_k^2 f)^{-1}, & \text{otherwise.} \end{cases}$$

Suppose that β_0 exists and can be found in the range

$$1 - (g(B) \log |D_K|)^{-1} \leq \beta_0 < 1.$$

By Lemma 3.3, there is a quadratic field F such that $\zeta_F(\beta_0) = 0$. Then by Lemma 2.11,

$$\beta_0 < 1 - \left(c_5 |D_F|^{1/2}\right)^{-1}.$$

Since $\mathcal{D}(K/\mathbb{Q}) = \mathcal{D}(K/F)\mathcal{D}(F/\mathbb{Q})$ and $[K : \mathbb{Q}] = 2n$, we have $|D_K| \geq |D_F|^n$ and therefore

$$\beta_0 < 1 - \left(c_5 |D_K|^{1/2n}\right)^{-1}.$$

This shows that

$$\beta_0 < \max \left[1 - (g(B) \log |D_K|)^{-1}, 1 - \left(c_5 |D_K|^{1/2n}\right)^{-1} \right]. \quad (3.6)$$

Then by Equation (3.5) combined with this result,

$$\begin{aligned} h(K) &> \frac{c-1}{(c-1)+(b\mu)^{-1}} \frac{h(k)|D_k|^{1-\frac{\sigma_1}{2}}f^{\frac{1}{2}}}{c_4(\sigma_1 - 1)a(\sigma_1)^n} \min \left[(g(B) \log |D_K|)^{-1}, \left(c_5 |D_K|^{1/2n}\right)^{-1} \right] \\ &> \frac{c-1}{(c-1)+(b\mu)^{-1}} \frac{h(k)}{c_4(\sigma_1 - 1)a(\sigma_1)^n} \min \left[\frac{|D_k|^{1-\frac{\sigma_1}{2}}f^{\frac{1}{2}}}{g(B) \log D_k^2 f}, c_5^{-1} |D_k|^{1-\frac{\sigma_1}{2}-\frac{1}{n}}f^{\frac{1}{2}-\frac{1}{2n}} \right]. \end{aligned}$$

If β_0 does not exist, then we immediately get from Equation (3.5) that

$$h(K) > \frac{h(k)b(c-1)}{c_4(\sigma_1 - 1)a(\sigma_1)^n} \cdot \frac{|D_k|^{1-\frac{\sigma_1}{2}}f^{\frac{1}{2}}}{\log D_k^2 f}.$$

□

Remark 3.5. In [Sta74], Stark proved that for any fixed $n > 2$ and $h \geq 1$, there are only finitely many totally complex fields K of degree $2n$ with $h(K) = h$. By taking $B = n$, we can reproduce this result. However, we cannot prove Theorem 1.19 for all n because of the $a(\sigma_1)^n$ factor in the denominator.

4

Odlyzko's Improvement

In [Odl75], Odlyzko improved Stark's result by producing a more explicit growth factor on the class number. We will apply his improvement to Stark's method to prove a partial result for Theorem 1.19.

4.1 Odlyzko's Theorem

Odlyzko showed that by choosing the parameters carefully, it is possible to get a better lower bound on the discriminant. Let K be a CM field with k as its maximal real subfield, $[k : \mathbb{Q}] = n$, and $\zeta_K(s) = L(s, \chi)\zeta_k(s)$. Also, let f be defined by $|D_K| = D_k^2 f$ and define

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s), \quad Z(s) = -\frac{\zeta'_k}{\zeta_k}(s), \quad Z_1(s) = -\frac{d}{ds}Z(s).$$

We will state the main lemma and the main theorem. The proofs are given in full detail in the original paper.

Lemma 4.1. (*Odlyzko Lemma 1*) *Suppose $\sigma \geq 1$ and*

$$\tilde{\sigma} \geq \max \left[\frac{5 + \sqrt{12\sigma^2 - 5}}{6}, 1 + \alpha\sigma \right],$$

where

$$\alpha = \sqrt{\frac{14 - \sqrt{128}}{34}} \approx 0.281.$$

Then

$$\begin{aligned} & \frac{\sigma - x}{(\sigma - x)^2 + y^2} + \frac{\sigma - 1 + x}{(\sigma - 1 + x)^2 + y^2} \\ & \geq \left(\sigma - \frac{1}{2} \right) \left(\frac{y^2 - (\tilde{\sigma} - x)^2}{(y^2 + (\tilde{\sigma} - x)^2)^2} + \frac{y^2 - (\tilde{\sigma} - 1 + x)^2}{(y^2 + (\tilde{\sigma} - 1 + x)^2)^2} \right) \end{aligned}$$

for all $x \in [0, 1]$ and all real y .

Proof. The proof is a long, but straightforward calculation. \square

Theorem 4.2. (Odlyzko Theorem 1) For any $\sigma > 1$ and $\tilde{\sigma}$ as in Lemma 4.1, we have

$$\begin{aligned} \log |D_k| \geq & r_1 \left(\log \pi - \psi \left(\frac{\sigma}{2} \right) \right) + 2r_2 (\log(2\pi) - \psi(\sigma)) + (2\sigma - 1) \left(\frac{r_1}{4} \psi' \left(\frac{\tilde{\sigma}}{2} \right) + r_2 \psi'(\tilde{\sigma}) \right) \\ & + 2Z(\sigma) + (2\sigma - 1)Z_1(\tilde{\sigma}) - \frac{2}{\sigma} - \frac{2}{\sigma - 1} - \frac{2\sigma - 1}{\tilde{\sigma}^2} - \frac{2\sigma - 1}{(\tilde{\sigma} - 1)^2}. \end{aligned}$$

Proof. The proof begins with Equation (2.7) and differentiates both sides with respect to s . The differentiated equation then becomes an estimate for the sum over the zeros in Equation (2.7) through Lemma 4.1. Some of these steps are shown in the proof of Lemma 5.2. \square

Remark 4.3. We can take $\tilde{\sigma} = 1 + \alpha\sigma$ as long as $\sigma - 1$ is sufficiently small. To see that the first condition on $\tilde{\sigma}$ is satisfied, we must have

$$1 + \alpha\sigma \geq \frac{5 + \sqrt{12\sigma^2 - 5}}{6} \implies (36\alpha^2 - 12)\sigma^2 + 12\alpha\sigma + 6 \geq 0.$$

This implies that

$$\sigma \leq \frac{12\alpha + \sqrt{288 - 720\alpha^2}}{24 - 72\alpha^2} = \bar{\alpha} \approx 1.014.$$

4.2 Three Lemmas

The first two lemmas are also taken from Odlyzko's paper. However, he does not compute values for the constants. The proofs presented here give the explicit choices for the constants, which will be used in a later chapter (Chapter 6). The third lemma is actually an implied calculation in the proof of Theorem 2 of the paper. Again, it's included here for the explicit computation of the constants.

Lemma 4.4. (Odlyzko Lemma 2) For $\sigma > 1$,

$$Z(\sigma) \geq \log \zeta_k(\sigma) - c_6(\sigma - 1)n,$$

where $c_6 = -2(\log 2)^2 + \log 2 \approx -0.268$.

Proof. If $x \geq 3$ and $\sigma > 1$, then

$$\frac{\log x}{x^\sigma - 1} = \sum_{r \geq 1} (\log x) x^{-r\sigma} \geq \sum_{r \geq 1} \frac{1}{r} x^{-r\sigma} = \log(1 - x^{-\sigma})^{-1}.$$

In the case where $x = 2$, we want to find a bound of a similar form. We note that

$$f(\sigma) = \frac{\log 2}{2^\sigma - 1} - \log(1 - 2^{-\sigma})^{-1}$$

satisfies $f(1) = 0$ and that

$$f''(\sigma) = \frac{2^\sigma (\log 2)^2}{(2^\sigma - 1)^3} ((1 + \log 2) - 2^\sigma (1 - \log 2)).$$

When this is positive, the tangent line approximation will be an underestimate of the actual function. This is true for

$$1 < \sigma \leq \frac{1}{\log 2} \log \left(\frac{1 + \log 2}{1 - \log 2} \right).$$

In fact, since $f'(\sigma) \geq 0$ for

$$\sigma \geq \frac{1}{\log 2} \log \left(\frac{1}{1 - \log 2} \right)$$

and

$$\frac{1}{\log 2} \log \left(\frac{1}{1 - \log 2} \right) < \frac{1}{\log 2} \log \left(\frac{1 + \log 2}{1 - \log 2} \right),$$

the tangent line is an underestimate for all $\sigma > 1$. Therefore,

$$f(\sigma) \geq (\sigma - 1)f'(1),$$

which can be rewritten as

$$\frac{\log 2}{2^\sigma - 1} \geq \log(1 - 2^{-\sigma})^{-1} - c_6(\sigma - 1).$$

Using Equation (1.2),

$$\begin{aligned} Z(\sigma) &= \sum_{\mathfrak{p}} \frac{\log \mathbb{N}(\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^\sigma - 1} \\ &= \sum_{\mathbb{N}(\mathfrak{p}) \neq 2} \frac{\log \mathbb{N}(\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^\sigma - 1} + \sum_{\mathbb{N}(\mathfrak{p})=2} \frac{\log \mathbb{N}(\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^\sigma - 1}. \end{aligned}$$

The worst case scenario is that the rational prime 2 is totally ramified, in which case there are n primes with $\mathbf{N}(\mathfrak{p}) = 2$. Therefore,

$$\begin{aligned} Z(\sigma) &\geq \sum_{\mathfrak{p}} \log(1 - \mathbf{N}(\mathfrak{p})^{-\sigma})^{-1} - c_6(\sigma - 1)n \\ &= \log \zeta_k(\sigma) - c_6(\sigma - 1)n, \end{aligned}$$

where the sum is over all prime ideals and $\mathbf{N}(\mathfrak{p})$ is the norm of the ideal \mathfrak{p} . \square

Lemma 4.5. (Odlyzko Lemma 3) *There exist c_7 and $c_8 = c_8(c_7)$ such that for $1 < \sigma < \sigma' \leq 1 + c_7^{-1}$, we have*

$$Z(\sigma) \geq (1 + (1 + c_8^{-1}))(\sigma' - \sigma)Z(\sigma').$$

Proof. If $x > e$ and $1 < \sigma < \sigma' \leq 2$, then

$$\log \frac{x^{\sigma'} - 1}{x^{\sigma} - 1} = \log x \int_{\sigma}^{\sigma'} \frac{x^u}{x^u - 1} du \geq (\sigma' - \sigma) \log x.$$

Therefore, we have

$$\frac{1}{x^{\sigma} - 1} \geq \frac{x^{\sigma' - \sigma}}{x^{\sigma'} - 1} > \frac{1 + \log x \cdot (\sigma' - \sigma)}{x^{\sigma'} - 1} \geq \frac{1 + (1 + \varepsilon) \cdot (\sigma' - \sigma)}{x^{\sigma'} - 1},$$

for any small $\varepsilon > 0$. Similarly,

$$\log \frac{2^{\sigma'} - 1}{2^{\sigma} - 1} = \log 2 \int_{\sigma}^{\sigma'} \frac{x^u}{x^u - 1} du \geq 2^{\sigma} \log 2 \int_{\sigma}^{\sigma'} \frac{du}{2^u - 1} \geq \frac{2^{\sigma} \log 2}{2^{\sigma'} - 1} (\sigma' - \sigma),$$

so that

$$\frac{1}{2^{\sigma} - 1} \geq \exp\left(\frac{2^{\sigma} \log 2}{2^{\sigma'} - 1} (\sigma' - \sigma)\right) \frac{1}{2^{\sigma'} - 1} > \left(1 + \frac{2^{\sigma} \log 2}{2^{\sigma'} - 1} (\sigma' - \sigma)\right) \frac{1}{2^{\sigma'} - 1}.$$

So if we restrict to $1 < \sigma < \sigma' \leq 1 + c_7^{-1} < \log(1 + \log 4)/\log 2$, then we have

$$\frac{1}{2^{\sigma} - 1} \geq \left(1 + \frac{2 \log 2}{2^{1+c_7^{-1}} - 1} (\sigma' - \sigma)\right) \frac{1}{2^{\sigma'} - 1} \geq \frac{1 + (1 + c_8^{-1})(\sigma' - \sigma)}{2^{\sigma'} - 1},$$

where

$$c_8^{-1} = \frac{2 \log 2}{2^{1+c_7^{-1}} - 1} - 1 > 0. \quad (4.1)$$

Therefore,

$$\frac{1}{\mathbf{N}(\mathfrak{p})^{\sigma}} \geq \frac{(1 + (1 + c_8^{-1}))(\sigma' - \sigma)}{\mathbf{N}(\mathfrak{p})^{\sigma'} - 1},$$

which implies the lemma by multiplying by $\log \mathbf{N}(\mathfrak{p})$ and summing over the prime ideals. \square

Remark 4.6. When c_7 is large, c_8 is small. This shows that if we restrict the values of σ and σ' to a smaller range, then we can take a larger constant in the inequality of the lemma. We can compute an explicit range for c_7 , which will also give an explicit range for c_8 . We have

$$1 < 1 + c_7^{-1} < \frac{\log(1 + \log 4)}{\log 2} \iff \left(\frac{\log(1 + \log 4)}{\log 2} - 1 \right)^{-1} = 3.926 < c_7 < \infty,$$

which implies that

$$(2 \log 2 - 1)^{-1} = 2.589 < c_8 < \infty.$$

Lemma 4.7. *There exist constants c_9 and c_{10} such that when $n \geq c_9(\sigma_1 - 1)^{-1}$, $c_{10}(\sigma - 1) \leq (\sigma_1 - 1)$, and $1 < \sigma < \sigma_1 \leq 1 + c_7^{-1}$ then*

$$\left(1 - (\sigma_1 - 1) - \frac{2}{n} \right) (1 + (1 + c_8^{-1})(\sigma_1 - \sigma)) \geq 1,$$

where c_7 and c_8 are obtained from Lemma 4.5.

Proof. Since $c_8 > 0$, we have

$$1 + (1 + c_8^{-1})(\sigma_1 - \sigma) > 1$$

for $1 < \sigma < \sigma_1 \leq 1 + c_7^{-1}$. Suppose that c_9 and c_{10} are chosen so that

$$(1 + c_8^{-1})(1 - c_{10}^{-1}) > 1 + 2c_9^{-1}. \quad (4.2)$$

This is possible for any fixed value of c_8 if c_9 and c_{10} are taken to be large enough. Notice that

$$n \geq c_9(\sigma_1 - 1)^{-1} \iff -\frac{2}{n} \geq -2c_9^{-1}(\sigma_1 - 1)$$

and

$$c_{10}(\sigma - 1) \leq (\sigma_1 - 1) \implies \sigma_1 - \sigma = (\sigma_1 - 1) - (\sigma - 1) \geq (1 - c_{10}^{-1})(\sigma_1 - 1).$$

Therefore,

$$\begin{aligned} \left(1 - (\sigma_1 - 1) - \frac{2}{n} \right) (1 + (1 + c_8^{-1})(\sigma_1 - \sigma)) &\geq \\ &(1 - (1 + 2c_9^{-1})(\sigma_1 - 1)) (1 + (1 + c_8^{-1})(1 - c_{10}^{-1})(\sigma_1 - 1)). \end{aligned}$$

This expression is of the form $f(x) = (1 - a_1x)(1 + a_2x)$, where $x = \sigma_1 - 1$. A quick calculation shows that $f(x) \geq 1$ with x is between 0 and $(a_2 - a_1)/(a_1a_2)$. Since $\sigma_1 > 1$, we want $f(x) \geq 1$ for some interval with $x > 0$, which requires $a_2 > a_1$. But this is just Equation (4.2), so the proof is complete. \square

Remark 4.8. The right side of Equation (4.2) is always greater than one. For a fixed value of c_8 , we see that we must have $c_{10} > 1 + c_8$. Once c_{10} has been chosen, we can pick c_9 from Equation (4.2),

$$c_9 > \frac{2}{(1 + c_8^{-1})(1 - c_{10}^{-1}) - 1}.$$

4.3 Computing the Lower Discriminant Bound

It is important for our application to obtain a sufficiently strong lower bound on the discriminant. The reason is that we need it to grow fast enough to offset the a $(2\pi)^n$ term in the denominator of our final result.

Theorem 4.9. *For $1 < \sigma < \sigma_1$ sufficiently small and n sufficiently large,*

$$|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}} \geq c_{11}^{-1} \zeta_k(\sigma_1) (2\pi + c_{12}^{-1})^n,$$

where c_{11} and c_{12} are effectively computable.

Proof. We apply Theorem 4.2 with Remark 4.3 so that for $1 < \sigma < \sigma_1 \leq 1 + c_{13}^{-1} = \min(\bar{\alpha}, 1 + c_7^{-1})$ and

$$\tilde{\sigma} \geq \max \left[\frac{5 + \sqrt{12\sigma^2 - 5}}{6}, 1 + \alpha\sigma \right],$$

the following inequality holds:

$$\begin{aligned} \log |D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}} &\geq \left(\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n} \right) \cdot \\ &\cdot \left(\underbrace{n \left(\log \pi - \psi \left(\frac{\sigma}{2} \right) \right)}_A + \underbrace{(2\sigma - 1) \frac{n}{4} \psi' \left(\frac{\tilde{\sigma}}{2} \right)}_B + \underbrace{2Z(\sigma)}_C + \underbrace{(2\sigma - 1) Z_1(\tilde{\sigma})}_D \right. \\ &\quad \left. + \underbrace{\left(-\frac{2}{\sigma} - \frac{2\sigma - 1}{\tilde{\sigma}^2} - \frac{2\sigma - 1}{(\tilde{\sigma} - 1)^2} - \frac{2}{\sigma - 1} \right)}_{E - F} \right). \end{aligned} \quad (4.3)$$

We will work each piece separately and combine them together in the end. Let c_9 and c_{10} satisfy Lemma 4.7. Then for $n \geq c_9(\sigma_1 - 1)^{-1}$ and $c_{10}(\sigma - 1) \leq (\sigma_1 - 1)$,

$$\left(\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n} \right) \geq \frac{1}{2} (1 - (1 + 2c_9^{-1})(\sigma_1 - 1)).$$

For term A , we note that $\psi(\sigma/2)$ is a negative increasing function near $\sigma = 1$ which is concave up. We can calculate a number $c_{14} > 0$ so that for $1 < \sigma < 1 + c_{13}^{-1}$,

$$\psi\left(\frac{\sigma}{2}\right) \leq \psi\left(\frac{1}{2}\right) + c_{14}(\sigma - 1).$$

We can take

$$c_{14} = \frac{\psi\left(\frac{1+c_{13}^{-1}}{2}\right) - \psi\left(\frac{1}{2}\right)}{\frac{1+c_{13}^{-1}}{2} - \frac{1}{2}}.$$

Then we have

$$\begin{aligned} & \frac{1}{2} (1 - (1 + 2c_9^{-1})(\sigma_1 - 1)) n \left(\log \pi - \psi\left(\frac{\sigma}{2}\right) \right) \\ & \geq \frac{n}{2} \left(\log \pi - \psi\left(\frac{1}{2}\right) - c_{14}(\sigma - 1) \right) - \frac{n}{2} (1 + 2c_9^{-1})(\sigma_1 - 1) \left(\log \pi - \psi\left(\frac{1}{2}\right) \right) \\ & \geq \frac{n}{2} \left(\log \pi - \psi\left(\frac{1}{2}\right) \right) - \frac{n}{2} (\sigma_1 - 1) \left(c_{14} + (1 + 2c_9^{-1}) \left(\log \pi - \psi\left(\frac{1}{2}\right) \right) \right). \end{aligned} \quad (4.4)$$

For term B , we use the fact that $\psi'(\tilde{\sigma}/2)$ is a positive, decreasing, concave up function for $\tilde{\sigma} > 0$. Therefore,

$$\psi'\left(\frac{1 + \alpha\sigma}{2}\right) \geq \psi'\left(\frac{1 + \alpha}{2}\right) + \frac{\alpha}{2} \psi''\left(\frac{1 + \alpha}{2}\right) (\sigma - 1).$$

Using this fact, we have

$$\begin{aligned} & \frac{1}{2} (1 - (1 + 2c_9^{-1})(\sigma_1 - 1)) \frac{n}{4} \psi'\left(\frac{\tilde{\sigma}}{2}\right) \\ & \geq \frac{n}{8} \psi'\left(\frac{\tilde{\sigma}}{2}\right) - \frac{n}{8} (1 + 2c_9^{-1})(\sigma_1 - 1) \psi'\left(\frac{\tilde{\sigma}}{2}\right) \\ & \geq \frac{n}{8} \left(\psi'\left(\frac{1 + \alpha}{2}\right) + \frac{\alpha}{2} \psi''\left(\frac{1 + \alpha}{2}\right) (\sigma - 1) \right) - \frac{n}{8} (1 + 2c_9^{-1})(\sigma_1 - 1) \psi'\left(\frac{1 + \alpha}{2}\right) \\ & \geq \frac{n}{8} \psi'\left(\frac{1 + \alpha}{2}\right) - \frac{n}{8} (\sigma_1 - 1) \left((1 + 2c_9^{-1}) \psi'\left(\frac{1 + \alpha}{2}\right) - \frac{\alpha}{2} \psi''\left(\frac{1 + \alpha}{2}\right) \right). \end{aligned} \quad (4.5)$$

Term C is controlled using Lemmas 4.4, 4.5, and 4.7. We get for $n \geq c_9(\sigma_1 - 1)$, $c_{10}(\sigma - 1) \leq (\sigma_1 - 1)$, and $1 < \sigma < \sigma_1 \leq 1 + c_{13}^{-1}$,

$$\begin{aligned} \left(1 - (\sigma_1 - 1) - \frac{2}{n}\right) Z(\sigma) & \geq \left(1 - (\sigma_1 - 1) - \frac{2}{n}\right) (1 + (1 + c_8^{-1})(\sigma_1 - \sigma)) Z(\sigma_1) \\ & \geq \log \zeta_k(\sigma_1) - c_6(\sigma_1 - 1)n. \end{aligned} \quad (4.6)$$

Term E is bounded by a constant,

$$\begin{aligned}
& -\frac{1}{2} (1 - (1 + 2c_9^{-1})(\sigma_1 - 1)) \left(\frac{2}{\sigma} + \frac{2\sigma - 1}{\tilde{\sigma}^2} + \frac{2\sigma - 1}{(\tilde{\sigma} - 1)^2} \right) \\
& \geq -\frac{1}{2} (1 - (1 + 2c_9^{-1})(1 - 1)) \left(\frac{2}{1} + \frac{2(1 + c_{13}^{-1}) - 1}{(1 + \alpha)^2} + \frac{2(1 + c_{13}^{-1}) - 1}{\alpha^2} \right) \\
& \geq -c_{15}.
\end{aligned} \tag{4.7}$$

Term F is easily estimated:

$$-\frac{1}{2} \left(1 - (\sigma_1 - 1) - \frac{2}{n} \right) \frac{2}{\sigma - 1} \geq -\frac{1}{\sigma - 1} + c_{10}(1 + 2c_9^{-1}). \tag{4.8}$$

We note that the D term is positive, so it can be dropped without damaging the inequality. Combining Equations (4.3), (4.4), (4.5), (4.6), (4.7), and (4.8) we get

$$\begin{aligned}
& \log |D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}} \geq \\
& \frac{n}{2} \left(\log \pi - \psi \left(\frac{1}{2} \right) + \frac{1}{4} \psi' \left(\frac{1 + \alpha}{2} \right) \right) + \log \zeta_k(\sigma_1) - c_{15} - \frac{1}{\sigma - 1} + c_{10}(1 + 2c_9^{-1}) \\
& - n(\sigma_1 - 1) \underbrace{\left(\frac{c_{14}}{2} - \frac{\alpha}{16} \psi'' \left(\frac{1 + \alpha}{2} \right) + c_6 + \frac{1 + 2c_9^{-1}}{8} \left(\log \pi^4 - 4\psi \left(\frac{1}{2} \right) + \psi' \left(\frac{1 + \alpha}{2} \right) \right) \right)}_{c_{16}}.
\end{aligned} \tag{4.9}$$

Note that we can explicitly compute that

$$\log \pi - \psi \left(\frac{1}{2} \right) + \frac{1}{4} \psi' \left(\frac{1 + \alpha}{2} \right) \approx 3.925.$$

We can now take the exponential of both sides of Equation (4.9) to get

$$|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}} \geq e^{-c_{15} - (\sigma - 1)^{-1}} \zeta_k(\sigma_1) \left(\frac{7.118}{e^{c_{16}(\sigma_1 - 1)}} \right)^n, \tag{4.10}$$

subject to $n \geq c_9(\sigma_1 - 1)^{-1}$, $c_{10}(\sigma - 1) \leq (\sigma_1 - 1)$, and $1 < \sigma < \sigma_1 \leq 1 + c_{13}^{-1}$. By further restricting σ_1 (say $\sigma_1 \leq 1 + c_{17}^{-1}$ where $c_{17} \geq c_{13}$), it will always be possible to force

$$\frac{7.118}{e^{c_{16}(\sigma_1 - 1)}} > 2\pi.$$

We can then choose σ and σ_1 subject to these constraints to get

$$c_{11} = \exp(c_{15} + (\sigma - 1)^{-1} + c_{10}(1 + 2c_9^{-1}))$$

and

$$c_{12} = \left(\frac{7.118}{\exp(c_{16}(\sigma_1 - 1))} - 2\pi \right)^{-1}.$$

□

4.4 Refining the Estimate

In this section, we will strengthen the lower bounds given on the class number given in Theorem 3.4.

Theorem 4.10. *Suppose $k \in \mathcal{B}(B)$ is a totally real field and K is a totally complex quadratic extension of k . Let $g(B)$ be defined as in Equation (3.2) and $z(c, \beta_0)$ be defined as in Equation (2.9). Let β_0 be the exceptional zero of $L(s, \chi)$ if it exists. For $c > 1$ chosen as in Lemma 2.4, if β_0 exists then*

$$h(K) > h(k) \frac{c_{18}^{-1}}{ng(B)} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} f^{\frac{1}{2} - \frac{1}{2n}},$$

otherwise

$$h(K) \geq h(k) \frac{c_{19}^{-1}}{n} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} f^{\frac{1}{2} - \frac{1}{2n}},$$

where both c_{18} and c_{19} are effectively computable, n is sufficiently large, and $1 < \sigma < \sigma_1$ are sufficiently small.

Proof. In Equation (3.6) of the proof of Theorem 3.4, we proved that there are no zeros β of $L(s, \chi)$ (and of $\zeta_K(s)$) in the range

$$\max \left[1 - (g(B) \log |D_K|)^{-1}, 1 - \left(c_5 |D_K|^{1/2n} \right)^{-1} \right] < \beta < 1.$$

This implies that if β is a zero of $L(s, \chi)$, then

$$\begin{aligned} 1 - \beta &\geq \min \left[(g(B) \log |D_K|)^{-1}, \left(c_5 |D_K|^{1/2n} \right)^{-1} \right] \\ &\geq \frac{1}{g(B)} |D_K|^{-1/2n} \min \left[\frac{|D_K|^{1/2n}}{2n \log |D_K|^{1/2n}}, \frac{18}{\pi} \right] \\ &\geq \frac{1}{ng(B)} |D_K|^{-1/2n} \min \left[\frac{e}{2}, \frac{18n}{\pi} \right] \\ &\geq \frac{c_{20}^{-1}}{ng(B)} |D_k|^{-1/n} f^{-1/2n}, \end{aligned} \tag{4.11}$$

where we have used $e^x/x \geq e$ (with $x = (\log |D_K|)/(2n)$) and $c_{20} = 2/e$. Note that if we had extra information about the size of $|D_k|$ and n , we may be able to choose a better value of c_{20} .

Following Odlyzko, we will start with Equation (3.4),

$$L(1, \chi) \leq \frac{(2\pi)^n h(K)}{|D_k f|^{1/2} h(k)},$$

which can be equivalently written as

$$h(K) \geq h(k)L(1, \chi) \frac{|D_k f|^{1/2}}{(2\pi)^n}. \quad (4.12)$$

We want to find a better bound for $L(1, \chi)$ than what was used in Theorem 3.4. Recall that the proof of Theorem 3.4 was dependent upon $L(s, \chi)$ satisfying the hypotheses of Lemma 2.4 with $c_1 = 1/2$ and $c_2 = 1/\log 3$. We return to Equation (2.11), but instead of using Stark's estimate ($L(\sigma_1, \chi) > \zeta(\sigma_1)^{-n}$), we use $L(\sigma_1, \chi) > \zeta_k(\sigma_1)^{-1}$. This is true since $\zeta_K(\sigma_1) > 1$ and $\zeta_K(s) = L(s, \chi)\zeta_k(s)$. With this as the only change between Equations (2.11) and (2.12), we have for σ_1 satisfying

$$1 + b(c-1)(\log(|D_k|^2 f))^{-1} = \sigma_0 \leq \sigma_1 \leq 2,$$

where c is defined as in Lemma 2.4 that

$$L(\sigma_0, \chi) > \frac{\sigma_0(\sigma_0 - 1)}{\sigma_1(\sigma_1 - 1)} |D_k|^{-\frac{1}{2}(\sigma_1 - 1)} e^{-(\sigma_0 - 1)n} \left(\frac{\sqrt{\pi}}{\Gamma(\sigma_1/2)} \right)^n \left(\frac{\left(\pi^{\frac{1}{2}(\sigma_1 - 1)} \right)^n}{\zeta_k(\sigma_1)} \right).$$

Notice that we have taken $r_1 = n$ and $r_2 = 0$ so that this corresponds to the CM field situation.

In our desired application, we will be using Theorem 4.9, which means we can take a smaller range for σ_1 ,

$$\sigma_0 < \sigma_1 < 1 + c_{17}^{-1}.$$

As before, we have that

$$\exp(-(\sigma_0 - 1)n) \geq \exp(-bc_2(c - 1)),$$

and under the new restrictions on σ_1 , we have

$$\frac{\sigma_0}{\sigma_1} > \frac{1}{1 + c_{17}^{-1}}.$$

Applying these inequalities and rearranging the terms gives

$$L(\sigma_0, \chi) > \frac{1}{(1 + c_{17}^{-1}) \exp(-bc_2(c - 1))} \cdot \frac{\sigma_0 - 1}{\sigma_1 - 1} \left(\frac{\pi^{\sigma_1/2}}{\Gamma(\sigma_1/2)} \right)^n \frac{|D_k|^{-\frac{1}{2}(\sigma_1 - 1)}}{\zeta_k(\sigma_1)}.$$

Combining this result with Equation (2.10),

$$L(1, \chi) \geq c_3^{-1} EL(\sigma_0, \chi),$$

we get

$$\begin{aligned} L(1, \chi) &> \frac{1}{(1 + c_{17}^{-1})c_3 \exp(bc_2(c-1))} \cdot \frac{E(\sigma_0 - 1)}{\sigma_1 - 1} \left(\frac{\pi^{\sigma_1/2}}{\Gamma(\sigma_1/2)} \right)^n \frac{|D_k|^{-\frac{1}{2}(\sigma_1-1)}}{\zeta_k(\sigma_1)} \\ &> c_{21}^{-1} z(c, \beta_0) \frac{|D_k|^{-\frac{1}{2}(\sigma_1-1)}}{\zeta_k(\sigma_1)}, \end{aligned} \quad (4.13)$$

where the computation of $E(\sigma_0 - 1)$ is given in the proof of Lemma 2.8,

$$c_{21} = (1 + c_{17}^{-1})c_3 \exp(bc_2(c-1)), \quad (4.14)$$

and we've used the fact that

$$1 \leq \frac{\pi^{\sigma_1/2}}{\Gamma(\sigma_1/2)} \leq \pi,$$

when $1 \leq \sigma_1 \leq 2$. Combining Equations (4.12) and (4.13), we get

$$h(K) > h(k) c_{21}^{-1} z(c, \beta_0) f^{1/2} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1-1)}}{(2\pi)^n \zeta_k(\sigma_1)}. \quad (4.15)$$

If there is an exceptional zero then by definition,

$$z(c, \beta_0) = \frac{c-1}{(c-1) + (b\mu)^{-1}} (1 - \beta_0).$$

Then from Equation (4.15) combined with Equation (4.11) gives

$$h(K) > h(k) \frac{c_{18}^{-1}}{ng(B)} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1-1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} f^{\frac{1}{2} - \frac{1}{2n}}, \quad (4.16)$$

where

$$c_{18}^{-1} = c_{18}(c)^{-1} = c_{21}^{-1} c_{20}^{-1} \frac{c-1}{(c-1) + (b\mu)^{-1}}.$$

If there is not an exceptional zero, then

$$z(c, \beta_0) = b(c-1) (\log D_k^2 f)^{-1},$$

and we get from this and Equation (4.15)

$$\begin{aligned} h(K) &> h(k) \frac{b(c-1)}{c_{21}} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1-1)} f^{1/2}}{\log D_k^2 f} \frac{1}{(2\pi)^n \zeta_k(\sigma_1)} \\ &= h(k) \frac{b(c-1)}{c_{21}} \frac{|D_k^2 f|^{1/2n}}{2n \log |D_k^2 f|^{1/2n}} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1-1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} f^{\frac{1}{2} - \frac{1}{2n}} \\ &\geq h(k) \frac{c_{19}^{-1}}{n} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1-1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} f^{\frac{1}{2} - \frac{1}{2n}}, \end{aligned} \quad (4.17)$$

where we have once again used $e^x/x \geq 1$ with $x = (\log D_k^2 f)/(2n)$ and

$$c_{19}^{-1} = c_{19}^{-1}(c) = \frac{b(c-1)e}{2c_{21}}.$$

□

Remark 4.11. Notice that in both cases, we have the same general form for the bound but with different constants. What remains is to compute the discriminant bound given by Theorem 4.2.

4.5 Proof of Theorem 1.19

Fix $B > 1$. We will now prove that there are only finitely many CM fields $K \in \mathcal{B}(B)$ of degree with a given class number.

Proof. For fields of fixed degree, Theorem 1.19 follows from [Sta74] (see Remark 3.5). This was effective for degree ≥ 6 and ineffective for degrees 2 and 4. However, [GZ86] makes the result effective for degrees 2 and 4.

In Remark 4.11, we noted that for a CM field $K \in \mathcal{B}(B)$ we had two estimates on the class number, both in the form

$$h(K) \geq h(k) \frac{C}{n} \frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} f^{\frac{1}{2} - \frac{1}{2n}},$$

where C is either $(c_{18}g(B))^{-1}$ or c_{19}^{-1} depending on whether there is an exceptional zero. From Theorem 4.9, we see that for $n \geq 387$,

$$\frac{|D_k|^{\frac{1}{2} - \frac{1}{2}(\sigma_1 - 1) - \frac{1}{n}}}{(2\pi)^n \zeta_k(\sigma_1)} \geq c_{11} \left(1 + \frac{c_{12}^{-1}}{2\pi}\right)^n.$$

Therefore, we have

$$h(K) \geq h(k) C' \frac{(1 + (2c_{12}\pi)^{-1})^n}{n} f^{\frac{1}{2} - \frac{1}{2n}}, \quad (4.18)$$

where

$$C' = \begin{cases} (c_{11}c_{18}g(B))^{-1}, & \text{if the exceptional zero exists} \\ (c_{11}c_{19})^{-1}, & \text{if the exceptional zero does not exist.} \end{cases}$$

This shows that $h(K) \rightarrow \infty$ as $n \rightarrow \infty$, so that given some h , there are no CM fields $K \in \mathcal{B}(B)$ with $h(K) = h$ whose degree is $\geq n_0$ for some n_0 . Therefore, there can only be finitely many such CM fields of a given class number. \square

5

Hoffstein's Upper Bound

Hoffstein ([Hof79]) introduced techniques for both upper and lower bounds for κ_K . We will mimic his upper bound techniques in a way that will allow us to make numerical computations later. The techniques used here are very similar to the work we did in previous chapters.

5.1 A Generalized Lemma

In Lemma 2.4, which was the basis for the calculations of Lemmas 2.8 and 2.9, we computed an effective lower bound of certain functions at $s = 1$. Hoffstein's first lemma is similar in form, except that he finds an upper bound instead. He applies the calculation specifically with $f(s) = s(s - 1)\zeta_k(s)$. He also mimics some of the calculations of [Odl75] to get a stronger constant. We will take a more generic approach to his calculations, using the same level of generality as in Lemma 2.1.

Lemma 5.1. (Hoffstein Lemma 1a) *Let $f(s)$ be an entire function and let*

$$g(s) = \eta^{s/2} \Gamma\left(\frac{s}{2}\right)^{a_1} \Gamma(s)^{a_2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) f(s), \quad (5.1)$$

where $\eta > 0$, $a_i \geq 0$ for $i = 1, 2, 3$. Suppose that $f(s)$ is positive for real $s > 1$ and $g(s)$ is an entire function of order 1 whose zeros $\rho = \beta + i\gamma$ all satisfy $0 < \beta < 1$. Suppose further that $g(s) = g(1 - s)$.

For $\sigma_1 > \sigma \geq 1$, set

$$\bar{E} = \begin{cases} \frac{\sigma - \beta_0}{\sigma_1 - \beta_0}, & \text{if } \beta_0 \text{ is any real zero of } \zeta_k(s) \\ 1, & \text{if no such } \beta \text{ exists.} \end{cases} \quad (5.2)$$

Also, let

$$h(\sigma, \sigma_1) = \eta^{\frac{\sigma_1 - \sigma}{2}} \left(\frac{\Gamma(\sigma_1/2)}{\Gamma(\sigma/2)} \right)^{a_1} \left(\frac{\Gamma(\sigma_1)}{\Gamma(\sigma)} \right)^{a_2} \left(\frac{\Gamma(\frac{\sigma_1}{2} + \frac{1}{2})}{\Gamma(\frac{\sigma}{2} + \frac{1}{2})} \right)^{a_3}. \quad (5.3)$$

Then for $\sigma_1 > \sigma \geq 1$, we have

$$f(\sigma) < f(\sigma_1) \bar{E}h(\sigma, \sigma_1).$$

Proof. We have that

$$g(s) = A \prod_{\rho}' \left(1 - \frac{s}{\rho} \right),$$

where the prime means the product is taken over all zeros of $g(s)$ in complex conjugate pairs. Taking the logarithmic derivative, this becomes

$$\frac{g'}{g}(s) = \sum_{\rho}' \frac{1}{s - \rho} = \sum_{\substack{\gamma \geq 0 \\ \rho \neq \beta_0}}'' \frac{2(s - \beta)}{(s - \beta)^2 + \gamma^2},$$

where the double prime means that the factor of 2 is omitted when $\gamma = 0$.

Suppose that $\zeta(\beta_0) = 0$ for $0 < \beta_0 < 1$. Then for real $s > 1$,

$$\frac{g'}{g}(s) - \frac{1}{s - \beta_0} = \sum_{\substack{\rho \neq \beta_0 \\ \gamma \geq 0}}'' \frac{2(s - \beta)}{(s - \beta)^2 + \gamma^2} > 0. \quad (5.4)$$

We integrate this from σ to σ_1 :

$$\log \frac{g(s)}{s - \beta_0} \Big|_{\sigma}^{\sigma_1} > 0 \implies \frac{g(\sigma_1)}{\sigma_1 - \beta_0} > \frac{g(\sigma)}{\sigma - \beta_0}.$$

Using Equation (5.1), this becomes

$$f(\sigma) < f(\sigma_1) \cdot \frac{\sigma - \beta_0}{\sigma_1 - \beta_0} \cdot \eta^{\frac{\sigma_1 - \sigma}{2}} \left(\frac{\Gamma(\sigma_1/2)}{\Gamma(\sigma/2)} \right)^{a_1} \left(\frac{\Gamma(\sigma_1)}{\Gamma(\sigma)} \right)^{a_2} \left(\frac{\Gamma(\frac{\sigma_1}{2} + \frac{1}{2})}{\Gamma(\frac{\sigma}{2} + \frac{1}{2})} \right)^{a_3},$$

which is the desired expression.

Note that if β_0 does not exist, Equation (5.4) simply reads

$$\frac{g'}{g}(s) > 0,$$

and the result follows by the same process. \square

The next lemma uses a calculation by Odlyzko to obtain a stronger bound when σ and σ_1 are both close to 1.

Lemma 5.2. (Hoffstein Lemma 1b) Let $f(s)$ and $g(s)$ be as in Lemma 5.1 and \bar{E} be defined by Equation (5.2). Suppose that

$$\tilde{\sigma} \geq \max \left[\frac{5 + (12\sigma^2 - 5)^{1/2}}{6}, 1 + \alpha\sigma \right], \quad (5.5)$$

where α is defined as in Theorem 4.2. Then for $\sigma_1 > \sigma \geq 1$, we have

$$f(\sigma) < \frac{f(\sigma_1)\bar{E}h(\sigma, \sigma_1)}{\left(\left(e^{\frac{1}{8}\psi'(\frac{\tilde{\sigma}}{2})} \right)^{a_1} \left(e^{\frac{1}{2}\psi'(\tilde{\sigma})} \right)^{a_2} \left(e^{\frac{1}{8}\psi'(\frac{\tilde{\sigma}}{2} + \frac{1}{2})} \right)^{a_3} e^{\frac{1}{2} \frac{d}{ds} \Big|_{s=\tilde{\sigma}}} \left(\frac{f'(s)}{f(s)} \right) \right)^{(\sigma_1 - \sigma)(\sigma_1 + \sigma - 1)}}. \quad (5.6)$$

Proof. We improve the estimate of Lemma 5.1 by taking a nonzero lower bound in the calculations above. Assuming β_0 exists,

$$\frac{g'}{g}(s) = \sum_{\rho}' \frac{1}{s - \rho} = \frac{1}{2} \sum_{\rho}' \left(\frac{1}{s - \rho} + \frac{1}{s - 1 + \rho} \right) \quad (5.7)$$

(this follows from the functional equation). Pulling out the β_0 term as before, for $s > 1$ we have

$$\frac{g'}{g}(s) - \frac{1}{s - \beta_0} = \frac{1}{2} \sum_{\substack{\rho \neq \beta_0 \\ \gamma \geq 0}}'' 2 \left(\frac{s - \beta}{(s - \beta)^2 + \gamma^2} + \frac{s - 1 + \beta}{(s - 1 + \beta)^2 + \gamma^2} \right),$$

where the double prime means that the factor of 2 is absent if $\gamma = 0$.

We apply Lemma 4.1 with $x = \beta$ and $y = \gamma$. Then we have

$$\frac{g'}{g}(s) - \frac{1}{s - \beta_0} \geq \frac{1}{2} \left(s - \frac{1}{2} \right) \sum_{\substack{\rho \neq \beta_0 \\ \gamma \geq 0}}'' 2 \left(\frac{\gamma^2 - (\tilde{\sigma} - \beta)^2}{(\gamma^2 + (\tilde{\sigma} - \beta)^2)^2} + \frac{\gamma^2 - (\tilde{\sigma} - 1 + \beta)^2}{(\gamma^2 + (\tilde{\sigma} - 1 + \beta)^2)^2} \right).$$

This becomes

$$\frac{g'}{g}(s) - \frac{1}{s - \beta_0} \geq \left(s - \frac{1}{2} \right) \sum_{\rho \neq \beta_0}' \frac{-1}{(\tilde{\sigma} - \rho)^2} > \left(s - \frac{1}{2} \right) \sum_{\rho}' \frac{-1}{(\tilde{\sigma} - \rho)^2}, \quad (5.8)$$

since when $\rho = \beta_0$, the term added back into the sum is negative.

We estimate the sum on the right by differentiating Equation (5.7) with respect to s and take $s = \tilde{\sigma}$:

$$\begin{aligned} \frac{g'}{g}(s) &= \frac{1}{2} \log \eta + \frac{a_1}{2} \psi \left(\frac{s}{2} \right) + a_2 \psi(s) + \frac{a_3}{2} \psi \left(\frac{s}{2} + \frac{1}{2} \right) + \frac{f'}{f}(s) = \sum_{\rho}' \frac{1}{s - \rho} \\ \implies \frac{a_1}{4} \psi' \left(\frac{\tilde{\sigma}}{2} \right) + a_2 \psi'(\tilde{\sigma}) + \frac{a_3}{4} \psi' \left(\frac{\tilde{\sigma}}{2} + \frac{1}{2} \right) + \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{f'}{f}(s) \right) &= \sum_{\rho}' \frac{-1}{(\tilde{\sigma} - \rho)^2}. \end{aligned} \quad (5.9)$$

Combining Equations (5.8) and (5.9), we get

$$\frac{g'}{g}(s) - \frac{1}{s - \beta_0} \geq \left(s - \frac{1}{2}\right) \left(\frac{a_1}{4}\psi' \left(\frac{\tilde{\sigma}}{2}\right) + a_2\psi'(\tilde{\sigma}) + \frac{a_3}{4}\psi' \left(\frac{\tilde{\sigma}}{2} + \frac{1}{2}\right) + \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{f'}{f}(s)\right)\right).$$

We integrate this with respect to s from σ to σ_1 to get

$$\begin{aligned} \log \left(\frac{g(s)}{s - \beta_0}\right) \Big|_{\sigma}^{\sigma_1} &> \left(s^2 - \frac{s}{2}\right) \Big|_{\sigma}^{\sigma_1} \left(\frac{a_1}{4}\psi' \left(\frac{\tilde{\sigma}}{2}\right) + a_2\psi'(\tilde{\sigma}) + \frac{a_3}{4}\psi' \left(\frac{\tilde{\sigma}}{2} + \frac{1}{2}\right) + \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{f'}{f}(s)\right)\right) \\ \implies \frac{g(\sigma_1)}{\sigma_1 - \beta_0} &> \frac{g(\sigma)}{\sigma - \beta_0} \cdot \left(\left(e^{\frac{1}{4}\psi'(\frac{\tilde{\sigma}}{2})}\right)^{a_1} \left(e^{\psi'(\tilde{\sigma})}\right)^{a_2} \left(e^{\frac{1}{4}\psi'(\frac{\tilde{\sigma}}{2} + \frac{1}{2})}\right)^{a_3} e^{\frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{f'}{f}(s)\right)}\right)^{\frac{1}{2}(\sigma_1 - \sigma)(\sigma_1 + \sigma - 1)}. \end{aligned}$$

From Equation (5.1) this becomes

$$f(\sigma) < \frac{f(\sigma_1)\overline{E}h(\sigma, \sigma_1)}{\left(\left(e^{\frac{1}{8}\psi'(\frac{\tilde{\sigma}}{2})}\right)^{a_1} \left(e^{\frac{1}{2}\psi'(\tilde{\sigma})}\right)^{a_2} \left(e^{\frac{1}{8}\psi'(\frac{\tilde{\sigma}}{2} + \frac{1}{2})}\right)^{a_3} e^{\frac{1}{2} \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{f'}{f}(s)\right)}\right)^{(\sigma_1 - \sigma)(\sigma_1 + \sigma - 1)}}.$$

□

5.2 Explicit Upper Bounds

By taking $\sigma = 1$ in Lemma 5.2 and combining the result with Lemma 2.4, we find that under certain restrictions on σ_0 and σ_1 we have

$$Af(\sigma_0) \leq f(1) \leq Bf(\sigma_1),$$

where A and B are computable functions for which we have very explicit forms. We are now going to compute upper bound analogs to Lemmas 2.8 and 2.9.

First, note that for $\sigma = 1$,

$$\max \left[\frac{5 + (12\sigma^2 - 5)^{1/2}}{6}, 1 + \alpha\sigma \right] = 1 + \alpha,$$

so that we can take $\tilde{\sigma} = 1 + \alpha$. This allows us to compute values for the denominator of Equation (5.6):

$$e^{\frac{1}{8}\psi'(\frac{\tilde{\sigma}}{2})} \approx 1.505, e^{\frac{1}{2}\psi'(\tilde{\sigma})} \approx 1.784, \text{ and } e^{\frac{1}{8}\psi'(\frac{\tilde{\sigma}}{2} + \frac{1}{2})} \approx 1.186.$$

5.2.1 Bounding κ_M

As with Lemma 2.8, we will take $f(s) = s(s-1)\zeta_M(s)$. Notice that in order to use Lemma 5.2, we must derive an upper bound on $\zeta_M(\sigma_1)$. Hoffstein accomplishes this bound by building off Odlyzko's main inequality (Theorem 4.2). We will follow Hoffstein's calculations, but give a presentation that allows for easy computations of numerical values. The key idea is to first find a lower bound for the $Z(\sigma)$ and $Z_1(\tilde{\sigma})$ in terms of $\zeta_M(s)$, then relate $\zeta_M(s)$ to the discriminant of M through Odlyzko's formula.

Lemma 5.3. (Hoffstein Lemma 2) *For $1 < \sigma_1 \leq \sigma_m$ and $\tilde{\sigma}_1 \leq \tilde{\sigma}_m$ both chosen to satisfy Equation (5.5), there exists a constant $c_A(\sigma_m)$ such that*

$$c_A(\sigma_m) (2Z(\sigma_1) + (2\sigma_1 - 1)Z_1(\tilde{\sigma}_1)) \geq \log \zeta_k(\sigma_1).$$

Proof. Define $c_B(x, \sigma)$ by

$$c_B(x, \sigma) \left(\frac{2 \log x}{x^\sigma - 1} + \frac{(2\sigma - 1)(\log x)^2 x^{\tilde{\sigma}}}{(x^{\tilde{\sigma}} - 1)^2} \right) = -\log \left(1 - \frac{1}{x^\sigma} \right).$$

Notice that if we can bound $c_B(x, \sigma)$ independent of x , then by setting $x = \mathbb{N}(\mathfrak{p})$ and summing over all prime ideals \mathfrak{p} of M we get a relationship of the desired form.

We write

$$c_B(x, \sigma) = \frac{c_C(x, \sigma)}{1 + c_D(x, \sigma)},$$

where

$$c_C(x, \sigma) = -\frac{x^\sigma - 1}{2 \log x} \log \left(1 - \frac{1}{x^\sigma} \right)$$

and

$$c_D(x, \sigma) = \frac{(2\sigma - 1)x^{\tilde{\sigma}}(\log x)(x^\sigma - 1)}{2(x^{\tilde{\sigma}} - 1)^2}.$$

We want to determine the behavior of $c_C(x, \sigma)$ and $c_D(x, \sigma)$ in order to bound $c_B(x, \sigma)$ independent of x .

Expanding the log term of $c_C(x, \sigma)$ in a power series gives

$$c_C(x, \sigma) = \frac{1}{2 \log x} \left(1 - \sum_{n=1}^{\infty} \frac{x^{-n\sigma}}{n(n+1)} \right).$$

This shows that for $\sigma > 1$,

$$\frac{\partial c_C}{\partial \sigma}(x, \sigma) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^{-n\sigma}}{n+1} > 0.$$

Also, by fixing σ and writing $u = x^\sigma$ we get

$$c_C(u) = -\frac{u-1}{2\sigma \log u} \log\left(1 - \frac{1}{u}\right).$$

Then we have

$$\frac{dc_C}{du}(u) = -\frac{1}{2\sigma \log u} - \frac{\log(u-1)}{2\sigma u (\log u)^2}.$$

Clearly, for $u \geq 2$ this is negative, so that for $x \geq 2$ and $\sigma > 1$,

$$\frac{\partial c_C}{\partial x} = \frac{\partial u}{\partial x} \frac{dc_C}{du} < 0.$$

For $c_D(x, \sigma)$, notice that when $1 < \sigma < \tilde{\sigma}$ are fixed this function first increases and then decreases for $x > 1$. Call the critical point x_0 so that $\partial c_D / \partial x > 0$ when $1 \leq x < x_0$. (x_0 does not have a nice analytic expression.) As σ increases x_0 increases and as $\tilde{\sigma}$ increases x_0 decreases.

For σ_m , pick a valid corresponding $\tilde{\sigma}_m$ according to Equation (5.5). For each pair $\sigma, \tilde{\sigma}$ with $1 < \sigma \leq \sigma_m$ and valid $\tilde{\sigma} \leq \tilde{\sigma}_m$, there is a corresponding $x_0(\sigma, \tilde{\sigma})$. As we range over all values, we have that the minimum x_0 is $x_m = x_0(1, \tilde{\sigma}_m)$, so that $\partial c_D / \partial x > 0$ for $1 \leq x < x_m$ and all $\sigma, \tilde{\sigma}$ in the range specified above.

We have

$$c_B(x, \sigma) = \frac{c_C(x, \sigma)}{1 + c_D(x, \sigma)},$$

and since $c_D(x, \sigma) > 0$, $\partial c_C / \partial x < 0$, and $\partial c_C / \partial \sigma > 0$, for $x \geq x_m$,

$$c_B(x, \sigma) \leq c_C(x, \sigma) \leq c_C(x_m, \sigma_m).$$

For $2 \leq x < x_m$, $c_D(x, \sigma)$ increases and $c_C(x, \sigma)$ decreases, so $c_B(x, \sigma)$ is a decreasing function and

$$c_B(x, \sigma) \leq c_B(2, \sigma).$$

Then for all $x \geq 2$,

$$c_B(x, \sigma) \leq c_A(\sigma_m)$$

where

$$c_A(\sigma_m) = \max_{1 \leq \sigma \leq \sigma_m} (c_B(2, \sigma), c_C(x_m, \sigma_m)).$$

So we have for $1 < \sigma \leq \sigma_m$ and valid $\tilde{\sigma} \leq \tilde{\sigma}_m$,

$$c_A(\sigma_m) \left(\frac{2 \log x}{x^\sigma - 1} + \frac{(2\sigma - 1)(\log x)^2 x^{\tilde{\sigma}}}{(x^{\tilde{\sigma}} - 1)^2} \right) \geq -\log\left(1 - \frac{1}{x^\sigma}\right).$$

We let $x = \mathbb{N}(\mathfrak{p})$ and sum over all prime ideals \mathfrak{p} of M to get

$$c_A(\sigma_m) (2Z(\sigma) + (2\sigma - 1)Z_1(\tilde{\sigma})) \geq \log \zeta_M(\sigma).$$

□

Lemma 5.4. (Hoffstein Lemma 3) For $1 < \sigma_1 \leq \sigma_m$ and $\tilde{\sigma}_1$ chosen as in Equation (5.5),

$$\zeta_M(\sigma_1) < \left(\frac{|D_M|}{c_E(\sigma_1)^n} \right)^{c_A(\sigma_m)}$$

where, in the notation of Theorem 4.2,

$$\begin{aligned} \log c_E(\sigma_1) = & \frac{r_1}{n} \left(\log \pi - \psi \left(\frac{\sigma_1}{2} \right) - \frac{2\sigma_1 - 1}{4} \psi' \left(\frac{\tilde{\sigma}_1}{2} \right) \right) \\ & + \frac{2r_2}{n} \left(\log 2\pi - \psi(\sigma_1) + \frac{2\sigma_1 - 1}{2} \psi'(\tilde{\sigma}_1) \right) \\ & - \frac{1}{n} \left(\frac{2}{\sigma_1} + \frac{2}{\sigma_1 - 1} + \frac{2\sigma_1 - 1}{\tilde{\sigma}_1^2} + \frac{2\sigma_1 - 1}{(\tilde{\sigma}_1 - 1)^2} \right). \end{aligned}$$

Proof. Notice that from Theorem 4.2 we can write

$$\log |D_M| \geq n \log c_E(\sigma_1) + 2Z(\sigma_1) + 2(\sigma_1 - 1)Z_1(\tilde{\sigma}_1).$$

We now use Lemma 5.3 to get

$$\log |D_M| \geq n \log c_E(\sigma_1) + \frac{\log \zeta_M(\sigma_1)}{c_A(\sigma_m)},$$

which is equivalent to the desired result. □

We bring these results together to get a result which mimics the form of Hoffstein's Theorem 1'.

Theorem 5.5. Let M be a number field with all of the usual notation. Then

$$\kappa_M < \frac{|D_M|^{c_A(\sigma_m) + \frac{1-\sigma_1}{2}}}{c_F(\sigma_1, \sigma_m)^n} \cdot \bar{E},$$

where

$$\begin{aligned} \log c_F(\sigma_1, \sigma_m) = & c_A(\sigma_m) \log c_E(\sigma_1) \\ & + \sigma_1(\sigma_1 - 1) \left(\frac{-6.633}{n} + \frac{r_1}{n}(0.409) + \frac{r_2}{n}(0.579) \right) + \frac{r_1}{n}(0.572) \\ & + (\sigma_1 - 1) \left(\frac{r_2}{n}(0.693) + 0.572 \right) - \frac{1}{n} \log(\sigma_1(\sigma_1 - 1)) \\ & - \frac{r_1}{n} \log \left(\Gamma \left(\frac{\sigma_1}{2} \right) \right) - \frac{r_2}{n} \log(\Gamma(\sigma_1)), \quad (5.10) \end{aligned}$$

and \bar{E} is defined as in Lemma 5.1.

Proof. We apply Lemma 5.2 with $f(s) = s(s-1)\zeta_M(s)$. This gives

$$\sigma(\sigma-1)\zeta_M(\sigma) < \frac{\sigma_1(\sigma_1-1)\zeta_M(\sigma_1)\overline{E}h(\sigma, \sigma_1)}{\left(\left(e^{\frac{1}{8}\psi'(\frac{\sigma}{2})} \right)^{r_1} \left(e^{\frac{1}{2}\psi'(\tilde{\sigma})} \right)^{r_2} e^{\frac{1}{2}\frac{d}{ds}\Big|_{s=\tilde{\sigma}}\left(\frac{\zeta'_M(s)+\frac{1}{s}+\frac{1}{s-1}}{\zeta_M(s)}\right)} \right)^{(\sigma_1-\sigma)(\sigma_1+\sigma-1)}}.$$

We let $\sigma \rightarrow 1$ and $\tilde{\sigma} = 1 + \alpha$ to get

$$\kappa_M < \frac{\sigma_1(\sigma_1-1)\zeta_M(\sigma_1)\overline{E}h(1, \sigma_1)}{\left((1.505)^{r_1} (1.784)^{r_2} e^{-\frac{1}{2\tilde{\sigma}^2} - \frac{1}{2(\tilde{\sigma}-1)^2} + \frac{1}{2}\frac{d}{ds}\Big|_{s=\tilde{\sigma}}\left(\frac{\zeta'_M(s)}{\zeta_M(s)}\right)} \right)^{(\sigma_1-1)\sigma_1}}.$$

Fix σ_m and pick $1 < \sigma_1 < \sigma_m$. From the Dirichlet series, we see that

$$\frac{d}{ds}\Big|_{s=\tilde{\sigma}}\left(\frac{\zeta'_M(s)}{\zeta_M(s)}\right) > 0,$$

we can drop this term from the denominator. Use Equation (5.3) to replace $h(1, \sigma_1)$ and apply Lemma 5.4:

$$\kappa_M < \frac{\sigma_1(\sigma_1-1)\left(\frac{|D_M|}{c_E(\sigma_1)^n}\right)^{c_A(\sigma_m)}\overline{E}\left(\frac{|D_M|}{2^{2r_2}\pi^n}\right)^{\frac{\sigma_1-1}{2}}\left(\frac{\Gamma(\frac{\sigma_1}{2})}{\sqrt{\pi}}\right)^{r_1}\Gamma(\sigma_1)^{r_2}}{\left((1.505)^{r_1}(1.784)^{r_2}(1.361 \cdot 10^{-3})\right)^{(\sigma_1-1)\sigma_1}}$$

After rearranging the terms, this can be written as

$$\kappa_M < \frac{|D_M|^{c_A(\sigma_m)+\frac{\sigma_1-1}{2}}}{c_F(\sigma_1, \sigma_m)^n} \cdot \overline{E},$$

where $c_F(\sigma_1, \sigma_m)$ is defined in Equation (5.10). \square

In Chapter 6, we will pick numerical values for σ_m and σ_1 and obtain an explicit numerical upper bound for the residue.

5.2.2 Bounding $L(1, \chi)$

We will now consider the case of a generic quadratic extension K/k . We take χ to be the character satisfying $\zeta_K(s) = L(s, \chi)\zeta_k(s)$, $[k : \mathbb{Q}] = n$, and define f so that $|D_K| = D_k^2 f$.

Theorem 5.6. *Let K/k be described as above. Then*

$$L(1, \chi) < \left(\frac{D_k^2 |f|}{c_E(\sigma_m)^n} \right)^{c_A(\sigma_1)} \frac{\overline{E}h(1, \sigma_1)}{\left((1.505)^{a_1} (1.784)^{a_2} (1.186)^{a_3} (1.935 \cdot 10^{-3}) \right)^{(\sigma_1-1)(\sigma_1)}},$$

where a_1 , a_2 , and a_3 are chosen to make the functional equation for $\xi_\chi(s)$ (Equation (1.4)) satisfy the hypotheses of Lemma 5.2.

Proof. We apply Lemma 5.2 with $f(s) = L(s, \chi)$. Taking $\sigma = 1$ and $\tilde{\sigma} = 1 + \alpha$, we get

$$L(1, \chi) < \frac{L(\sigma_1, \chi) \overline{Eh}(1, \sigma_1)}{\left((1.505)^{a_1} (1.784)^{a_2} (1.186)^{a_3} e^{\frac{1}{2} \frac{d}{ds} \Big|_{s=\tilde{\sigma}}} \left(\frac{L'}{L}(s, \chi) \right) \right)^{(\sigma_1-1)(\sigma_1)}}.$$

To bound $L(\sigma_1, \chi)$ we will use a crude bound by combining Lemma 5.4 and the fact that $\zeta_k(\sigma_1) > 1$:

$$L(\sigma_1, \chi) = \frac{\zeta_K(\sigma_1)}{\zeta_k(\sigma_1)} < \left(\frac{D_k^2 |f|}{c_E(\sigma_m)^n} \right)^{c_A(\sigma_1)}.$$

To bound the $(L'/L)(s, \chi)$ term we note that,

$$\begin{aligned} \left| \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{L'}{L}(s, \chi) \right) \right| &= \left| \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\sum_{n=1}^{\infty} \chi(n) \Lambda(n) n^{-s} \right) \right| \\ &= \left| - \sum_{n=1}^{\infty} \chi(n) \log(n) \Lambda(n) n^{-\tilde{\sigma}} \right| \\ &\geq - \sum_{n=1}^{\infty} \log(n) \Lambda(n) n^{-\tilde{\sigma}} \\ &= - \frac{d}{ds} \Big|_{s=\tilde{\sigma}} \left(\frac{\zeta'}{\zeta}(s) \right) \\ &\approx -12.495. \end{aligned}$$

Bringing these calculations together gives

$$L(1, \chi) < \left(\frac{D_k^2 |f|}{c_E(\sigma_m)^n} \right)^{c_A(\sigma_1)} \frac{\overline{Eh}(1, \sigma_1)}{\left((1.505)^{a_1} (1.784)^{a_2} (1.186)^{a_3} (1.935 \cdot 10^{-3}) \right)^{(\sigma_1-1)(\sigma_1)}}.$$

□

The following corollary follows immediately.

Corollary 5.7. *If K is a CM field whose maximal subfield is k , then*

$$L(1, \chi) < \left(\frac{D_k^2 |f|}{c_E(\sigma_m)^n} \right)^{c_A(\sigma_1)} \frac{\overline{Eh}(1, \sigma_1)}{\left((1.186)^n (1.935 \cdot 10^{-3}) \right)^{(\sigma_1-1)(\sigma_1)}}.$$

6

Explicit Numerical Computations

This chapter is dedicated to computing specific values of the constants found in the previous chapters. As best as we can, we will find the optimal values of the constants and the most explicit expressions. We will proceed through the calculations in the same order they are presented in this dissertation.

6.1 General Information

Before we begin into a long succession of calculations, it will be useful to review all of the major constants that were introduced and describe their dependencies.

We began with a generalization of the lemmas found in [Sta74]. In Lemma 2.4, we introduced four constants. The numbers c_1 and c_2 were values taken directly from [Sta74] and had two different choices, depending on the situation. When working with the residue of the Dedekind zeta function of an arbitrary field M , we have

$$c_1 = 0 \text{ and } c_2 = 2/\log 3. \tag{6.1}$$

When working with quadratic field extensions, we have

$$c_1 = 1/2 \text{ and } c_2 = 1/\log 3. \tag{6.2}$$

The value of b was related to the size of the exceptional box in which the functions would have at most one zero. As noted in Remark 2.7, we take $b = \min[\mu, \nu]$, where μ and ν are taken from Table 2.1 and control the vertical and horizontal dimensions of the exceptional box, namely that there is at most one zero of the given function in the region

$$1 - (\mu \log d)^{-1} \leq \beta < 1 \text{ and } |\gamma| \leq (\nu \log d)^{-1},$$

for an appropriately chosen d . The value of c was a constant chosen in Equation (2.1) and $d > 1$ ends up being the discriminant of some field (according to the specific application). In the conclusion of Lemma 2.4, we obtain an effective constant c_3 in Equation (2.5) and we find c_4 in Lemma 2.9, where it is related to the lower bound of $L(1, \chi)$. The value of c_5 is fixed, $c_5 = \pi/6$, and is given by Lemma 2.11.

The next series of constants came from Odlyzko's work. The value of c_6 is fixed and is part of the bound of the logarithmic derivative of $\zeta_k(s)$ for $s > 1$ (Lemma 4.4). We find c_7 and c_8 in Lemma 4.5. The choice of c_7 is made so that we can restrict the value of the parameters σ and σ' . Once c_7 is chosen, we immediately get c_8 from Equation (4.1). In Remark 4.8, we even computed the allowed range of values for these two constants. Given these values, Lemma 4.7 introduces the two more constants as constraints. The value of c_9 affects the degrees of the fields that we are considering and c_{10} is a bound that further affects our choice of σ and σ_1 (in particular, it keeps them from getting too close to each other). The result of Theorem 4.9 is the calculation of c_{11} and c_{12} . Also, in Theorem 4.10, we compute c_{18} and c_{19} .

Finally, Hoffstein's work is not so much the calculation of constants, but evaluating functions at specific values. The choice of σ_m gives the upper bound for the parameter σ_1 , to which there is a corresponding $\widetilde{\sigma}_1$ given by Equation (5.5). The calculation of the functions $c_A(\sigma_m)$, $c_E(\sigma_1)$, and $c_F(\sigma_1, \sigma_m)$ are all given explicitly in the text and are too involved to discuss here.

6.2 The Results of Stark's Method

We will begin with the work from Chapter 2. These results are very straightforward, but serve as a good way to ease into the more complicated calculations that follow.

6.2.1 The Lower Bound for κ_M

In Lemma 2.8, we showed that for an arbitrary number field M and $c > 1$ chosen such that

$$\frac{(c-1)(c+\sqrt{c^2-1})}{c} \frac{b}{\log |D_M|} \leq 0.461,$$

that we have

$$\kappa_M > \begin{cases} c_3^{-1} \frac{c-1}{(c-1)+(b\mu)^{-1}} (1 - \beta_0), & \text{if } \beta_0 \text{ exists} \\ c_3^{-1} b(c-1)(\log |D_M|)^{-1}, & \text{otherwise,} \end{cases}$$

where c_3 is taken from Equation (2.5),

$$c_3 = \exp \left(\frac{b(c-1)(2cc_1 + c - 1)}{2} + 1.316c - \frac{bc_2(c-1)}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right).$$

Notice that we are in the situation of Equation (6.1), so that

$$c_3 = \exp \left(\frac{b(c-1)^2}{2} + 1.316c - \frac{b(c-1)}{\log 3} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right).$$

In a completely generic setting, we can do no better than to take $M \neq \mathbb{Q}$ so that $|D_M| \geq 3$. Once we pick the values for μ and ν from Table 2.1, we immediately have b , which leaves us with c as the only free variable.

Suppose that the exceptional zero exists. In this case, we have

$$\kappa_M > \frac{1}{\exp \left(\frac{b(c-1)^2}{2} + 1.316c - \frac{b(c-1)}{\log 3} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right)} \frac{c-1}{(c-1) + (b\mu)^{-1}} (1 - \beta_0) = A_1(1 - \beta_0),$$

where c must satisfy

$$\frac{(c-1) \left(c + \sqrt{c^2 - 1} \right)}{c} \leq \frac{0.461 \cdot \log 3}{b}. \quad (6.3)$$

When there is no exceptional zero, we have

$$\kappa_M > \frac{b(c-1)}{\exp \left(\frac{b(c-1)^2}{2} + 1.316c - \frac{b(c-1)}{\log 3} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right)} (\log |D_M|)^{-1} = A_2(\log |D_M|)^{-1},$$

where c must satisfy the same condition as above. In either situation, we can use a computer to choose c so that the value of A_i is maximal, giving us the best lower bound. The results of these calculations are given in Table 6.1.

Remark 6.1. Although the values for A_1 appear to be significantly better than even the optimal value of A_2 , notice that the exceptional zero must be within $(\mu \log |D_M|)^{-1}$ of 1, which makes the $1 - \beta_0$ term very small. This will be a recurring theme in the rest of these calculations.

6.2.2 The First Lower Bound for $L(1, \chi)$

Suppose that K is a quadratic extension of k and define f by $|D_K| = D_k^2 f$. Suppose that k is a degree n field with r_1 real conjugate fields and $2r_2$ complex conjugate

Table 6.1: Values for A_1 and A_2 with the optimal choice of c .

A_2^* represents the optimal value of A_2 over all c , not subject to Equation (6.3).

μ	2.915	2.915	2.989	3	3.618	4.079
ν	1000	100	10	9.343	3.618	3
c	1.120	1.085	1.083	1.083	1.067	1.075
A_1	0.0639	0.0639	0.0650	0.652	0.0738	0.0771
c	1.120	1.120	1.117	1.117	1.099	1.117
A_2	0.0420	0.0420	0.0422	0.0422	0.0437	0.422
A_2^*	0.0427	0.0427	0.0429	0.0429	0.0446	0.0429
μ	8.826	10	100	1000	10000	100000
ν	2	1.949	1.688	1.668	1.666	1.666
c	1.081	1.079	1.034	1.011	1.004	1.001
A_1	0.101	0.111	0.197	0.243	0.260	0.266
c	1.167	1.171	1.194	1.194	1.196	1.196
A_2	0.0385	0.0382	0.0367	0.0365	0.0366	0.0366
A_2^*	0.0388	0.0385	0.0369	0.0367	0.0367	0.0367

fields. Let $L(s, \chi)$ be defined by this extension, $L(s, \chi) = \zeta_K(s)/\zeta_k(s)$. In Lemma 2.9, we showed that

$$L(1, \chi) > c_4^{-1} \frac{z(c, \beta_0)}{\sigma_1 - 1} |D_k|^{-\frac{1}{2}(\sigma_1 - 1)} \left(\frac{\sqrt{\pi}}{\Gamma(\sigma_1/2)} \right)^{r_1} \left(\frac{2^{\sigma_1 - 1}}{\Gamma(\sigma_1)} \right)^{r_2} \left(\frac{\pi^{\frac{1}{2}(\sigma_1 - 1)}}{\zeta(\sigma_1)} \right)^n,$$

where

$$1 + b(c - 1)(\log D_k^2 f)^{-1} \leq \sigma_1 \leq 2,$$

c is chosen so that

$$\frac{(c - 1) \left(c + \sqrt{c^2 - 1} \right)}{c} \frac{b}{\log D_k^2 f} \leq 0.461,$$

and

$$z(c, \beta_0) = \begin{cases} \frac{c-1}{(c-1)+(b\mu)^{-1}}(1 - \beta_0), & \text{if } \beta_0 \text{ exists} \\ b(c - 1)(\log D_k^2 f)^{-1}, & \text{otherwise.} \end{cases}$$

The value of c_4 is given explicitly in the proof,

$$\begin{aligned} c_4 &= 2c_3 \exp(bc_2(c - 1)) \\ &= 2 \exp \left(\frac{b(c - 1)(2cc_1 + c - 1)}{2} + 1.316c - \frac{bc_2(c - 1)}{2} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) - 2 \right) \right). \end{aligned}$$

In this case, there are two parameters to choose, c and σ_1 . The choice of each is independent of the other, with c affecting the c_4 and $z(c, \beta_0)$ terms and σ_1 affecting the rest. We can optimize c as before, however the choice of σ_1 may vary with the application. For example, if k is of small degree, it would be better to choose σ_1 small so that the $(\sigma_1 - 1)^{-1}$ term dominates the expression. However, if the field has a large degree then by taking $\sigma_1 = 2$ the last term is greater than one and this suggests that $L(1, \chi)$ is very large.

We will start by choosing c following the pattern prescribed in the previous section. We will take $D_k \geq 3$ and $f \geq 1$. We are in the situation where Equation (6.2) applies. Therefore,

$$c_4 = 2 \exp \left(\frac{b(c-1)(2c-1)}{2} + 1.316c - \frac{b(c-1)}{2 \log 3} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) - 2 \right) \right).$$

If the zero exists, then

$$\begin{aligned} L(1, \chi) &> \frac{\frac{c-1}{(c-1)+(b\mu)^{-1}}}{2 \exp \left(\frac{b(c-1)(2c-1)}{2} + 1.316c - \frac{b(c-1)}{2 \log 3} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) - 2 \right) \right)} f(\sigma_1)(1 - \beta_0) \\ &= A_3 f(\sigma_1)(1 - \beta_0), \end{aligned}$$

where $f(\sigma_1)$ consists of all the remaining terms in the product and c is subject to

$$\frac{(c-1) \left(c + \sqrt{c^2 - 1} \right)}{c} \leq \frac{0.461 \cdot \log 3}{b}.$$

If β_0 does not exist, then we have

$$\begin{aligned} L(1, \chi) &> \frac{b(c-1)}{2 \exp \left(\frac{b(c-1)(2c-1)}{2} + 1.316c - \frac{b(c-1)}{2 \log 3} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) - 2 \right) \right)} f(\sigma_1)(\log D_k^2 f)^{-1} \\ &= A_4 f(\sigma_1)(\log D_k^2 f)^{-1}, \end{aligned}$$

where c is subject to the same constraint. Table 6.2 gives the results of the numerical computations.

Since there is no uniformly ideal manner in which to choose σ_1 , we will simply take a few values and compute the results. But before we do this, we will rearrange the

Table 6.2: Values for A_3 and A_4 with the optimal choice of c .

μ	2.915	2.915	2.989	3	3.618	4.079
ν	1000	100	10	9.343	3.618	3
c	1.073	1.073	1.071	1.071	1.057	1.065
A_3	0.0281	0.0281	0.0286	0.0287	0.0327	0.0344
c	1.115	1.115	1.113	1.112	1.097	1.112
A_4	0.0172	0.0172	0.0173	0.0173	0.0178	0.0173
μ	8.826	10	100	1000	10000	100000
ν	2	1.949	1.688	1.668	1.666	1.666
c	1.071	1.070	1.031	1.010	1.003	1.001
A_3	0.0485	0.0510	0.0957	0.120	0.129	0.133
c	1.153	1.156	1.173	1.174	1.174	1.174
A_4	0.0158	0.0157	0.0151	0.0151	0.0151	0.0151

Table 6.3: Values of A_5 , A_6 , A_7 , and A_8 corresponding to choices of σ_1 .

σ_1	1.01	1.05	1.10	1.25	1.50	2.00
A_5	100	20	10	4	2	1
A_6	0.005	0.0250	0.0500	0.125	0.250	0.500
A_7	0.0101	0.0524	0.110	0.310	0.737	1.910
A_8	0.0102	0.0547	0.119	0.380	1.083	3.820

terms:

$$\begin{aligned}
f(\sigma_1) &= \frac{|D_k|^{-\frac{1}{2}(\sigma_1-1)}}{\sigma_1-1} \left(\frac{\sqrt{\pi}}{\Gamma(\sigma_1/2)} \right)^{r_1} \left(\frac{2^{\sigma_1-1}}{\Gamma(\sigma_1)} \right)^{r_2} \left(\frac{\pi^{\frac{1}{2}(\sigma_1-1)}}{\zeta(\sigma_1)} \right)^{r_1+2r_2} \\
&= \frac{|D_k|^{-\frac{1}{2}(\sigma_1-1)}}{\sigma_1-1} \left(\frac{\pi^{\frac{\sigma_1}{2}}}{\Gamma(\sigma_1/2)\zeta(\sigma_1)} \right)^{r_1} \left(\frac{(2\pi)^{\sigma_1-1}}{\Gamma(\sigma_1)\zeta(\sigma_1)} \right)^{r_2} \\
&= A_5 |D_k|^{-A_6} A_7^{r_1} A_8^{r_2}.
\end{aligned}$$

These results can be found in Table 6.3. Notice these values confirm the suggestion that larger values of σ_1 are better for fields of large degree (so that r_1 and r_2 are large) and smaller values of σ_1 are better for fields of small degree.

6.2.3 First Calculations on $\mathcal{B}(B)$

In Chapter 3, we computed some results for lower bounds on the residue κ_M of arbitrary fields in $\mathcal{B}(B)$ and the class number $h(k)$ of the maximal real subfield of a CM

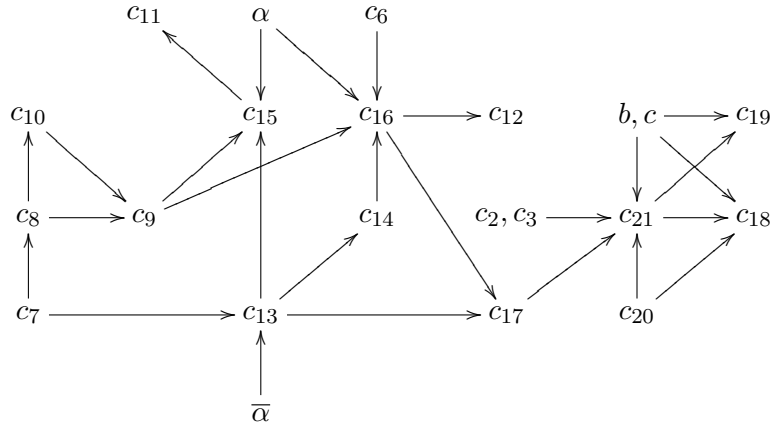


Figure 6.1: The web of dependencies for the constants in Chapter 4.

field K . However, an inspection of Theorems 3.2 and 3.4 shows that the major constants in these results are exactly the same as those in the previous sections.

6.3 The Results of Odlyzko's Method

As seen in Figure 6.1, there are about twenty constants and parameters introduced in Chapter 4, so before we begin we will review the definitions and constraints for everything. However, instead of presenting the information in the order it was presented in the proofs, we will present them in the order that makes sense for performing numerical calculations.

6.3.1 An Overview of the Constants

The first constant introduced is

$$\alpha = \sqrt{\frac{14 - \sqrt{128}}{34}} \approx 0.281,$$

which is a value that arises from the proof of Lemma 4.1 (which was not included in this dissertation). From this, we get

$$\bar{\alpha} = \frac{12\alpha + \sqrt{288 - 720\alpha^2}}{24 - 72\alpha^2} \approx 1.014,$$

which is an upper bound for the parameter σ that allows us to fix the value of another parameter, $\tilde{\sigma} = 1 + \alpha\sigma$.

We get

$$c_6 = -2(\log 2)^2 + \log 2 \approx -0.268$$

from the proof of Lemma 4.4, which is a part of an estimate for the function $Z(\sigma)$, which is defined at the beginning of Chapter 4. From Lemma 4.5 and Remark 4.6, we get

$$\left(\frac{\log(1 + \log 4)}{\log 2} - 1 \right)^{-1} = 3.926 < c_7 < \infty$$

and

$$c_8 = \left(\frac{2 \log 2}{2^{1+c_7^{-1}} - 1} - 1 \right)^{-1},$$

where c_7 is used as an upper bound and c_8 contributes to another lower estimate of $Z(\sigma)$.

By choosing c_9 and c_{10} such that

$$(1 + c_8^{-1})(1 - c_{10}^{-1}) > 1 + 2c_9^{-1},$$

then for $1 < \sigma < \sigma_1 \leq 1 + c_7^{-1}$, $n \geq c_9(\sigma_1 - 1)^{-1}$, and $c_{10}(\sigma - 1) \leq (\sigma_1 - 1)$, we can apply the result of Lemma 4.7.

The process of computing c_{11} and c_{12} in Theorem 4.9 goes through 5 different constants (c_{13} through c_{17}). Instead of presenting them in the order presented in the text, we will review them in an order that makes sense computationally. We start with

$$c_{13} \geq \max \left(\frac{1}{\alpha - 1}, c_7 \right),$$

which is related to the restrictions on σ and σ_1 . In fact, we will simply take this to be an equality for c_{13} . From this we immediately obtain

$$c_{14} = \frac{\psi \left(\frac{1+c_{13}^{-1}}{2} \right) - \psi \left(\frac{1}{2} \right)}{\frac{1+c_{13}^{-1}}{2} - \frac{1}{2}}.$$

This value comes out of bounding term A in the proof. Once c_{14} is computed, we get

$$\begin{aligned} c_{16} &= \frac{c_{14}}{2} - \frac{\alpha}{16} \psi'' \left(\frac{1+\alpha}{2} \right) + c_6 + \frac{1+2c_9^{-1}}{8} \left(\log \pi^4 - 4\psi \left(\frac{1}{2} \right) + \psi' \left(\frac{1+\alpha}{2} \right) \right) \\ &= \frac{c_{14}}{2} + \frac{3.925}{c_9} + 1.840, \end{aligned}$$

which arises from combining several terms together. We then pick c_{17} so that when $\sigma_1 \leq 1 + c_{17}^{-1}$ we have

$$\frac{7.118}{e^{c_{16}(\sigma_1-1)}} > 2\pi,$$

which is equivalent to

$$(\sigma_1 - 1) < c_{16}^{-1} \log \left(\frac{7.118}{2\pi} \right).$$

This implies that we want to take

$$c_{17} > c_{16} \left(\log \left(\frac{7.118}{2\pi} \right) \right)^{-1}.$$

We also want to have $c_{17} \geq c_{13}$, so that we can take

$$c_{17} > \max \left(c_{16} \left(\log \left(\frac{7.118}{2\pi} \right) \right)^{-1}, c_{13} \right).$$

The value of c_{15} can be computed from previously obtained values:

$$c_{15} = \frac{1}{2} \left(2 + \frac{2(1 + c_{13}^{-1}) - 1}{(1 + \alpha)^2} + \frac{2(1 + c_{13}^{-1}) - 1}{\alpha^2} \right).$$

In the end, we get that for $1 < \sigma < \sigma_1 < 1 + c_{17}^{-1}$, $n \geq c_9(\sigma_1 - 1)^{-1}$, and $c_{10}(\sigma - 1) \leq (\sigma_1 - 1)$,

$$c_{11} = \exp(c_{15} + (\sigma - 1)^{-1} - c_{10}(1 + 2c_9^{-1}))$$

and

$$c_{12} = \left(\frac{7.118}{\exp(c_{16}(\sigma_1 - 1))} - 2\pi \right)^{-1}.$$

Moving on to the calculation of c_{18} and c_{19} from Theorem 4.10, we find that these are much more straight forward values to compute. We have $c_{20} = 2/e$ from Equation (4.11). From Equation (4.14),

$$c_{21} = (1 + c_{17}^{-1})c_3 \exp(bc_2(c - 1)),$$

where c_1 and c_2 are taken as in Equation (6.2). From these two values, we get

$$c_{18} = c_{21}c_{20} \left(\frac{c - 1}{(c - 1) + (b\mu)^{-1}} \right)^{-1}$$

and

$$c_{19} = \frac{2c_{21}}{b(c - 1)e}.$$

It is not necessary to compute each constant explicitly. For example, the value of c_8 is given directly from knowing c_7 . Therefore, the results can be recomputed given only a few of the many constants and parameters. Table 6.4 gives a few examples of valid choices.

Table 6.4: Examples of valid choices of the constants and parameters.

c_7	3.927	70	70	1000	69.346
c_9	40992	5.8	103	5.21	5.572
c_{10}	10000	100	4	10000	25781
c_{17}	70	70	70	1000	69.346
$\sigma - 1$	$1.41 \cdot 10^{-6}$	$1.41 \cdot 10^{-4}$	$3.54 \cdot 10^{-3}$	$8.9 \cdot 10^{-8}$	$5.58 \cdot 10^{-7}$
$\sigma_1 - 1$	$1.42 \cdot 10^{-2}$	$1.42 \cdot 10^{-2}$	$1.42 \cdot 10^{-2}$	$9 \cdot 10^{-4}$	$1.44 \cdot 10^{-2}$
$n \geq$	$2.887 \cdot 10^6$	408.5	7253.52	5788.9	386.9

6.3.2 Another Lower Bound for $L(1, \chi)$

Equation (4.13) in the middle of the proof of Theorem 4.10 gives another lower bound for $L(1, \chi)$ which is an improvement over Lemma 2.9. For

$$1 + b(c - 1)(\log(|D_k|^2 f))^{-1} = \sigma_0 \leq \sigma_1 \leq 1 + c_{17}^{-1},$$

where c is chosen so that

$$\frac{(c - 1) \left(c + \sqrt{c^2 - 1} \right)}{c} \frac{b}{\log D_k^2 f} \leq 0.461,$$

we have

$$L(1, \chi) > c_{21}^{-1} z(c, \beta_0) \frac{|D_k|^{-\frac{1}{2}(\sigma_1 - 1)}}{\zeta_k(\sigma_1)}.$$

Notice that the exponent of $|D_k|$ is of the same form, but instead of an explicit dependence on the number and types of field embeddings, it depends on the value of the Dedekind zeta function near $s = 1$.

If we plug in the formula for c_{21} we get

$$L(1, \chi) > \frac{2}{(1 + c_{17}^{-1})c_4} z(c, \beta_0) \frac{|D_k|^{-\frac{1}{2}(\sigma_1 - 1)}}{\zeta_k(\sigma_1)},$$

and we already computed values for $c_4^{-1} z(c, \beta_0)$ in Table 6.2. We cannot obtain good numerical bounds for the remaining terms because we do not have a bound for $\zeta_k(\sigma_1)$ except for $\zeta_k(\sigma_1) < \zeta(\sigma_1)^n$. Since we cannot find choices of the parameters to allow us to compute bounds for fields of very small degree, this bound is unhelpful to us and we will not compute numerical values for them.

Table 6.5: Possible values of B_1 and B_2 using the parameters given by Table 6.4.

$B_1 - 1$	0.066	0.056	0.066	0.128	0.054
B_2	$10^{-303671}$	10^{-3025}	10^{-124}	$10^{-4873707}$	$10^{-763094}$
$n \geq$	$2.887 \cdot 10^6$	408.5	7253.5	5788.9	386.9

6.3.3 A Lower Bound for $h(K)$ for CM Fields

In the proof of Theorem 1.19, where we show that

$$h(K) \geq h(k)C' \frac{(1 + (2c_{12}\pi)^{-1})^n}{n} f^{\frac{1}{2} - \frac{1}{2n}},$$

where

$$C' = \begin{cases} (c_{11}c_{18}g(B))^{-1}, & \text{if the exceptional zero exists} \\ (c_{11}c_{19})^{-1}, & \text{if the exceptional zero does not exist.} \end{cases}$$

Notice that

$$c_{18}^{-1} = \frac{2c_{20}^{-1}}{1 + c_{17}^{-1}}A_3 = \frac{e}{1 + c_{17}^{-1}}A_3$$

and

$$c_{19}^{-1} = \frac{e}{1 + c_{17}^{-1}}A_4,$$

so we can use the values from Table 6.2. This indicates that we only need to compute

$$B_1 = 1 + (2c_{12}\pi)^{-1}$$

and

$$B_2 = \frac{e \cdot c_{11}^{-1}}{1 + c_{17}^{-1}}.$$

The results of these calculations are given in Table 6.5.

Remark 6.2. The first observation from this table is how poor these bounds are, especially for B_2 . The reason that they are so bad is that we are attempting to obtain uniform bounds for fields of both high and low degree.

In the last four columns we can also begin to see the various trade-offs that are made between the degree of the field, the size of B_1 , and the size of B_2 .

6.3.4 Application to Theorem 1.19

Our numerical calculations have shown that we can apply our main results to fields whose degree is at least 387. We will analyze this lower bound on the degree and explain the major obstacles to obtaining a result for smaller fields.

From Lemma 4.7, the condition on n is

$$n \geq c_9(\sigma_1 - 1)^{-1}.$$

We have that c_9 is bounded below,

$$c_9 > \frac{2}{(1 + c_8^{-1})(1 - c_{10}^{-1}) - 1},$$

and by Remark 4.6 we know that $c_8 > (2 \log 2 - 1)^{-1}$, so that as we let $c_{10} \rightarrow \infty$ we get $c_9 > 2(2 \log 2 - 1)^{-1} \approx 5.177$. The restriction

$$1 < \sigma < \sigma_1 < 1 + c_{17}^{-1}$$

gives us an lower bound on $(\sigma_1 - 1)^{-1}$, namely that this is greater than c_{17} . By tracing the inequalities, we find that

$$c_{17} \geq c_{13} > \frac{1}{\bar{\alpha} - 1} \approx 69.345.$$

Therefore, the techniques we have developed can at best prove the result for fields with

$$n > \frac{2}{(2 \log 2 - 1)(\bar{\alpha} - 1)} \approx 359.02.$$

However, we are unable to even attain this ideal because to minimize c_9 as above, we must take c_7 very large, which affects the choice of c_{17} . In fact, if we take $c_7 = c_{17}$ and let c_{17} approach $(\bar{\alpha} - 1)^{-1}$ (still taking $c_{10} \rightarrow \infty$), we get

$$n > \frac{2(2^{\bar{\alpha}} - 1)}{(2 \log 2 - 2^{\bar{\alpha}} + 1)(\bar{\alpha} - 1)} \approx 386.34.$$

We can see from the last column of Table 6.4 that we have found explicit values of these parameters to obtain this optimal value (since n is always an integer).

It is not a surprise that our efforts fail to prove get results for fields of small degree. In [Sta74], Stark observed that finding effective values for the lower bound of the residue of Dedekind zeta functions, and hence class numbers of CM fields, is very difficult for fields of small degree.

6.4 The Results of Hoffstein's Method

6.4.1 An Overview of the Calculations

Starting from Lemma 5.3, we pick σ_m to be an upper bound for σ_1 , so that $1 < \sigma_1 \leq \sigma_m$. From the choice of σ_m , we take $\widetilde{\sigma}_m$ such that

$$\widetilde{\sigma}_m \geq \max \left[\frac{5 + (12\sigma_m^2 - 5)^{1/2}}{6}, 1 + \alpha\sigma_m \right].$$

In fact, we will take this to be an equality. For each σ_1 , we consider all possible choices of $\widetilde{\sigma}_1 < \widetilde{\sigma}_m$ subject to the corresponding restriction. We are seeking a value $c_A(\sigma_m)$ that will give us the inequality.

In the proof of the lemma, there are three functions introduced,

$$c_C(x, \sigma) = -\frac{x^\sigma - 1}{2 \log x} \log \left(1 - \frac{1}{x^\sigma} \right)$$

and

$$c_D(x, \sigma) = \frac{(2\sigma - 1)x^{\widetilde{\sigma}}(\log x)(x^\sigma - 1)}{2(x^{\widetilde{\sigma}} - 1)^2},$$

which together give

$$c_B(x, \sigma) = \frac{c_C(x, \sigma)}{1 + c_D(x, \sigma)}.$$

We defined $x_0(\sigma, \widetilde{\sigma})$ to be the critical point of $c_D(x, \sigma)$ when σ and $\widetilde{\sigma}$ were fixed. After analyzing the function, we determined that the smallest value of $x_0(\sigma, \widetilde{\sigma})$ over the ranges $1 < \sigma \leq \sigma_m$ and valid $\widetilde{\sigma} \leq \widetilde{\sigma}_m$ was $x_m = x_0(1, \widetilde{\sigma}_m)$. Although there is no closed form expression for this value, it can easily be obtained numerically. This value then immediately gives $c_C(x_m, \sigma_m)$, which is one part of the bound for $c_A(\sigma_m)$. The second part of the bound for $c_A(\sigma_m)$ is another calculation. We $1 \leq \sigma \leq \sigma_m$, we want to maximize the function $c_B(2, \sigma)$. Once again, we do not have a closed form for this, but we can do it numerically. We then take

$$c_A(\sigma_m) = \max_{1 \leq \sigma \leq \sigma_m} (c_B(2, \sigma), c_C(x_m, \sigma_m)).$$

Once this value is established, everything else is a matter of choosing σ_1 such that $1 < \sigma_1 \leq \sigma_m$ and $\widetilde{\sigma}_1$ such that

$$\max \left[\frac{5 + (12\sigma_1^2 - 5)^{1/2}}{6}, 1 + \alpha\sigma_1 \right] \leq \widetilde{\sigma}_1 \leq \widetilde{\sigma}_m,$$

then using a computer to calculate various values. Table 6.6 lists some possible choices of σ_m and the values of the corresponding constants.

Table 6.6: The values of $\widetilde{\sigma}_m$, x_m , and $c_A(\sigma_m)$ for a given choice of σ_m

σ_m	2	1.5	1.3	1.1	1.01
$\widetilde{\sigma}_m$	1.926	1.615	1.485	1.348	1.284
x_m	2.710	4.938	7.864	18.217	34.889
$c_A(\sigma_m)$	0.466	0.400	0.385	0.365	0.354

6.4.2 An Upper Bound for κ_M

In Theorem 5.5, we showed that

$$\kappa_M < \frac{|D_M|^{c_A(\sigma_m) + \frac{\sigma_1 - 1}{2}}}{c_F(\sigma_1, \sigma_m)^n} \cdot \bar{E},$$

where \bar{E} is defined as in Lemma 5.1, $c_A(\sigma_m)$ as described above,

$$\begin{aligned} \log c_F(\sigma_1, \sigma_m) &= c_A(\sigma_m) \log c_E(\sigma_1) \\ &+ \sigma_1(\sigma_1 - 1) \left(\frac{-6.633}{n} + \frac{r_1}{n}(0.409) + \frac{r_2}{n}(0.579) \right) + \frac{r_1}{n}(0.572) \\ &+ (\sigma_1 - 1) \left(\frac{r_2}{n}(0.693) + 0.572 \right) - \frac{1}{n} \log(\sigma_1(\sigma_1 - 1)) \\ &\quad - \frac{r_1}{n} \log \left(\Gamma \left(\frac{\sigma_1}{2} \right) \right) - \frac{r_2}{n} \log(\Gamma(\sigma_1)), \end{aligned}$$

and

$$\begin{aligned} \log c_E(\sigma_1) &= \frac{r_1}{n} \left(\log \pi - \psi \left(\frac{\sigma_1}{2} \right) - \frac{2\sigma_1 - 1}{4} \psi' \left(\frac{\widetilde{\sigma}_1}{2} \right) \right) \\ &+ \frac{2r_2}{n} \left(\log 2\pi - \psi(\sigma_1) + \frac{2\sigma_1 - 1}{2} \psi'(\widetilde{\sigma}_1) \right) \\ &\quad - \frac{1}{n} \left(\frac{2}{\sigma_1} + \frac{2}{\sigma_1 - 1} + \frac{2\sigma_1 - 1}{\widetilde{\sigma}_1^2} + \frac{2\sigma_1 - 1}{(\widetilde{\sigma}_1 - 1)^2} \right). \end{aligned}$$

While this form is very cumbersome, it turns out that after rearranging the terms, this can be more simply expressed as

$$\kappa_M < \frac{C_1 |D_M|^{C_2}}{C_3^{r_1} \cdot C_4^{r_2}} \cdot \bar{E},$$

Table 6.7: The values of the constants C_1 through C_5 for the choices of σ_m , σ_1 , and $\widetilde{\sigma}_1$.

σ_m	2	2	2	1.1	1.1
σ_1	1.347	1.001	1.001	1.05	1.001
$\widetilde{\sigma}_1$	1.926	1.926	1.282	1.312	1.282
C_1	166129	10^{867}	10^{872}	$1.801 \cdot 10^{22}$	10^{872}
C_2	0.639	0.466	0.466	0.390	0.366
C_3	4.315	3.474	2.908	2.395	2.310
C_4	29.293	13.017	16.284	10.018	8.919

with

$$C_1 = \sigma_1 \cdot (\sigma_1 - 1) \cdot \exp \left[(6.633) \cdot \sigma_1 \cdot (\sigma_1 - 1) + \frac{2}{\sigma_1} + \frac{2}{\sigma_1 - 1} + \frac{2\sigma_1 - 1}{\widetilde{\sigma}_1^2} + \frac{2\sigma_1 - 1}{(\widetilde{\sigma}_1 - 1)^2} \right]$$

$$C_2 = c_A(\sigma_m) + \frac{\sigma_1 - 1}{2}$$

$$C_3 = \exp \left[c_A(\sigma_m) \left(\log \pi - \psi \left(\frac{\sigma_1}{2} \right) - \frac{2\sigma_1 - 1}{4} \psi' \left(\frac{\widetilde{\sigma}_1}{2} \right) \right) \right. \\ \left. + (0.409) \cdot \sigma_1 \cdot (\sigma_1 - 1) + (0.572) - \log \Gamma \left(\frac{\sigma_1}{2} \right) + (0.572) \cdot (\sigma_1 - 1) \right]$$

$$C_4 = \exp \left[2c_A(\sigma_m) \left(\log 2\pi - \psi(\sigma_1) + \frac{2\sigma_1 - 1}{2} \psi'(\widetilde{\sigma}_1) \right) \right. \\ \left. + (0.579) \cdot \sigma_1 \cdot (\sigma_1 - 1) + (0.693) \cdot (\sigma_1 - 1) - \log \Gamma(\sigma_1) + (1.157) \cdot (\sigma_1 - 1) \right]$$

Table 6.7 has the results of some these constants with the explicit choices of the parameters.

7

Future Work

This dissertation leaves the door open for many improvements and future investigations. There is plenty of room to improve this result and lots of places to begin searching for inspiration. There was not enough time to attempt to replicate Hoffstein's lower bound methods and extend them to class number calculations and to $\mathcal{B}(B)$. This would be the next logical step.

Stéphane Louboutin has done a large amount of work with ideas related to the Brauer-Siegel Theorem and CM fields (see [Lou05], [Lou03], [Lou06] and their references), and there are certainly many ideas and techniques that can be gleaned from those papers and applied to our situation.

Bibliography

- [Dav00] Harold Davenport. *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2000. Revised and with a preface by Hugh L. Montgomery.
- [GZ86] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L -series. *Invent. Math.*, 84(2):225–320, 1986.
- [Hof79] Jeffrey Hoffstein. Some analytic bounds for zeta functions and class numbers. *Invent. Math.*, 55(1):37–47, 1979.
- [Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994.
- [Lou03] Stéphane Louboutin. Explicit lower bounds for residues at $s = 1$ of Dedekind zeta functions and relative class numbers of CM-fields. *Trans. Amer. Math. Soc.*, 355(8):3079–3098 (electronic), 2003.
- [Lou05] Stéphane R. Louboutin. The Brauer-Siegel theorem. *J. London Math. Soc. (2)*, 72(1):40–52, 2005.
- [Lou06] Stéphane R. Louboutin. Lower bounds for relative class numbers of imaginary abelian number fields and CM-fields. *Acta Arith.*, 121(3):199–220, 2006.
- [Mar77] Daniel A. Marcus. *Number fields*. Springer-Verlag, New York, 1977. Universitext.
- [Neu99] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.
- [Odl75] A. M. Odlyzko. Some analytic estimates of class numbers and discriminants. *Invent. Math.*, 29(3):275–286, 1975.
- [Sta74] H. M. Stark. Some effective cases of the Brauer-Siegel theorem. *Invent. Math.*, 23:135–152, 1974.

- [WMLI92] M. Waldschmidt, P. Moussa, J. M. Luck, and C. Itzykson, editors. *From number theory to physics*. Springer-Verlag, Berlin, 1992. Papers from the Meeting on Number Theory and Physics held in Les Houches, March 7–16, 1989.