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Journal

Journal of Engineering Mechanics, 152(3)

ISSN

0733-9399

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Publication Date

2026-03-01

DOI

10.1061/jenmdt.emeng-8642

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Peer reviewed

Pyramid-shaped metamaterials with tunable mechanical properties and broad frequency bandgaps

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Kuo, C. H., Lin, Z. Y., Hsu, S. T., Chen, J. H., Chen, J. S., Chen, T., “Pyramid-shaped
metamaterials with tunable mechanical properties and broad frequency bandgaps,” Journal of
Engineering Mechanics, Vol. 152(3), 04025102, 2026.

Abstract

We propose a novel elastic metamaterial offering a number of tunable macroscopic mechanical properties, including effective elastic moduli, overall anisotropy, mass density and frequency band structures. The unit cell of the periodic metastructures is composed of three-dimensional pyramid-shaped inclusions, similar to the geometry of a traditional Chinese food Zongzi. A unique feature of this configuration is that the geometry can be described by a simple algebraic equation, enabling transformation of the geometric configuration into a few variant forms upon simple coordinate scalings. We demonstrate that a number of macroscopic properties can be adjusted by geometric parameters scaling without altering the internal structure or the constituent materials. Numerical simulations are conducted to estimate the effective elastic properties. Specifically, the effective elastic moduli can be reduced by two to three orders of magnitude compared to those of the original constituent material. In addition, the tunable mechanical properties offer flexibilities to control frequency bandgap with adjustable bandwidths. We find that the periodic array of Zongzi metastructure could generate very wide frequency bandgaps, which can be utilized to control wave propagation characteristics. A number of specimens fabricated with 3D printing have been tested for validation of the numerically estimated effective mechanical properties of Zongzi microstructures. The present study provides insightful information towards a new three-dimensional metastructure with tunable mechanical properties and frequency band structures.

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1. Introduction

Metamaterials are engineered composite materials composed of a periodic or non-periodic array of unit cells with artificially designed microstructures. The metamaterial properties can go beyond those of the constituent materials, particularly with tailored functionalities not typically seen in nature. The concept of metamaterials was originated about two decades ago in electromagnetic and optic waves. Specifically, scientists have been able to achieve invisibility cloaking with the use of a novel metamaterial to manipulate light paths and, in the process, an object can be rendered invisible to electromagnetic radiation within a certain frequency range (Pendry et al. 2006). The metamaterial notion was later extended to different physical phenomena, including acoustics, elastodynamics, thermal conductions and some others (see for example, Norris 2015, Milton et al. 2006, Leonhardt 2013). In this study, we focus on the study of elastic metamaterials. Numerous aspects of mechanical metamaterials have been explored in the past decade, such as unit cells with one or more constituent materials, geometry with 1D, 2D and 3D configurations, static and dynamic behavior, isotropic and anisotropic properties, auxetic and non-auxetic behavior, and extremal or pentamode materials. A detailed exposition of mechanical metamaterials can be referred to the articles (Craster et al. 2023, Jiao et al. 2023, Kardic et al. 2019, Yu et al. 2018, Bertoldi et al. 2017, Berger et al. 2017, Kolken and Zadpoor, 2017).

In this study, we propose a new type of microstructure that is composed of a periodic array of three-dimensional pyramid-shaped inclusions (Figure 1a). For convenience, the inclusion will be referred to as Zongzi for its geometric resemblance to a traditional Chinese food, Zongzi, a pyramid-shaped rice dumpling. The present study is motivated by an interesting observation that the geometry of Zongzi is, to some extent, analogous to that of pentamode structures (PMs) (Fig.1) (Milton and Cherkaev 1995, Milton 2002). Pentamode materials are an iconic example of mechanical metamaterials, in which they behave like liquid but with solid structures, so that compression waves and shear waves in PMs could be decoupled (Milton 2002). We will demonstrate that the array of Zongzi structures demonstrates remarkable features that exhibit profound implications for optimal design of metastructures with tunable functionalities. Specifically, we will demonstrate the tunable properties of effective elastic moduli, overall anisotropy, mass density, and frequency band structures. A unique feature of this configuration is that the three-dimensional Zongzi geometry can be described by a simple algebraic equation, which allows simple coordinate scaling to transform the three-dimensional Zongzi configuration into a few variant forms

analytically, yielding tunable macroscopic static and dynamic properties. For convenience of expositions, a schematic diagram and a snapshot of the periodic array of Zongzi is depicted in Figure 2. In view of the geometric configuration, we will show later that the periodic array of Zongzi structure exhibits a face-centered-cubic (fcc) symmetry.

To study the effective elastic properties, we have performed finite element simulations based on static continuum mechanics using the commercial software package, COMSOL Multiphysics with solid mechanics module. As the Zongzi inclusions connect each other at point-like tips, we introduce a ball joint at the connection points to provide stability. We show that the effective constants of the periodic array can be controlled by the size of ball joint, up to a 2 to 3 orders of magnitude. In addition, upon a simple coordinate scaling, the overall symmetry of the periodic array of Zongzi can be made with an orthotropic symmetry consisting of 9 independent elastic constants. Also, we will demonstrate that effective mass density can be adjusted by considering a hollow Zongzi structure. Specifically, a hollow Zongzi with a similar shape of geometry can be precisely characterized with an additional scalar parameter. The effective mass density for a solid Zongzi can be as large as 10 times compared to that of a hollow one. In summary, we can adjust the effective elastic properties, overall anisotropy and mass density of the periodic array of Zongzi simply by incorporating scalar parameters into the algebraic equation of Zongzi inclusion.

In addition to static mechanical properties, we also examine frequency band structures of Zongzi structures. The study of periodic metastructures with a sub-wavelength lattice constant under plane wave excitations is of particular interest in the control of wave propagation through a periodic array of medium, for applications in sound and vibration insulation. These tunable mechanical properties can offer flexibilities to achieve desired frequency bandgaps with tunable bandwidths. We find remarkably that the proposed periodic Zongzi metastructure could generate very wide complete bandgaps in frequency intervals, in which an incident wave can be completely filtered through the periodic medium. To validate the accuracy of our simulations we have implemented a plane wave expansion method (Kushwaha et al. 1994, Dal Poggetto and Serpa 2020) for the periodic array of Zongzi based on Fourier expansions. The study of band structures will offer great value in controlling energy attenuations of wave motions through the periodic medium. Lastly, a number of centimeter-scale specimens of a periodic array of Zongzi were fabricated by 3D printer techniques (Fig.6a) and tested experimentally in the laboratory. We find that our experimental results are in good agreement with the finite element simulations, not only the overall trend but also the simulated data quantitatively.

The outline of this paper is as follows. We start with a description of a periodic array of Zongzi metamaterials with detailed geometrical illustrations given in Section 2. We show that the periodic array of Zongzi exhibits a face-centered cubic symmetry. The theoretical frameworks and numerical simulations of the effective elastic properties, overall anisotropy and mass density are discussed in Section 3. Dynamic behaviors with frequency band structures for the periodic array of Zongzi are discussed in Section 4. Experimental investigation of the effective elastic properties is presented in Section 5. Lastly, some discussions and concluding remarks are addressed in Section 6.

2. A Periodic array of Zongzi structures

The geometry of the Zongzi inclusion can be described by a simple algebraic equation (Zheng and Chen 2001)

$$x^2 + y^2 + z^2 + 2xyz = 1, \quad (1)$$

located within a simple cube, with domain $-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1$ in Cartesian coordinates. The configuration of Zongzi was found previously in a study of admissible Poisson's ratios for an elastically orthotropic material (Zheng and Chen 2001). Here we propose a metamaterial that is composed of a periodic array of unit cell, with a lattice constant a , in which each unit cell contains four Zongzi inclusions arranged in an opposite diagonal manner, as depicted in Fig.1a. In this arrangement, the metamaterial is only constituted by a single ingredient material with void phase outside the Zongzi inclusions. We mention that the three-dimensional configuration of Zongzi can be viewed as an assemblage of rotated ellipses along the three coordinate axes (Fig.1b). For example, the cross-section of the Zongzi geometry along $z = 0$ is simply a circle. For a cross-section along $z = \text{constant} > 0$, the contour is a rotated ellipse with major axis dimension $\sqrt{1+z}$ and minor axis dimension $\sqrt{1-z}$, and with principal axes being along $x' = (x+y)/\sqrt{2}$ and $y' = (-x+y)/\sqrt{2}$. For $z < 0$, the role of the major and the minor axes switches. The present study was motivated by an interesting observation that the geometry of Zongzi is, to some extent, analogous to that of pentamode structures (PMs) (Fig.1c) (Milton and Cherkaev 1995, Milton 2002). Specifically, when the structures of Zongzi and PM structures are aligned together, we see that both geometric configurations share some similarities. First, on the top and bottom surfaces of a unit cell, the two configurations are connected by square diagonals, both being perpendicular to each other; see the red dashed lines

in Fig.1c. That is the top surface of the cube $z = 1$ degenerates to a diagonal edge $x + y = 0$, while the bottom surface $z = -1$ of the closed region becomes another diagonal edge in the opposite direction $x - y = 0$. As Eq. (1) is invariant with the three coordinates, x , y and z , this geometric feature applies also to two other perpendicular directions as well, that is $x = \pm 1$ and $y = \pm 1$.

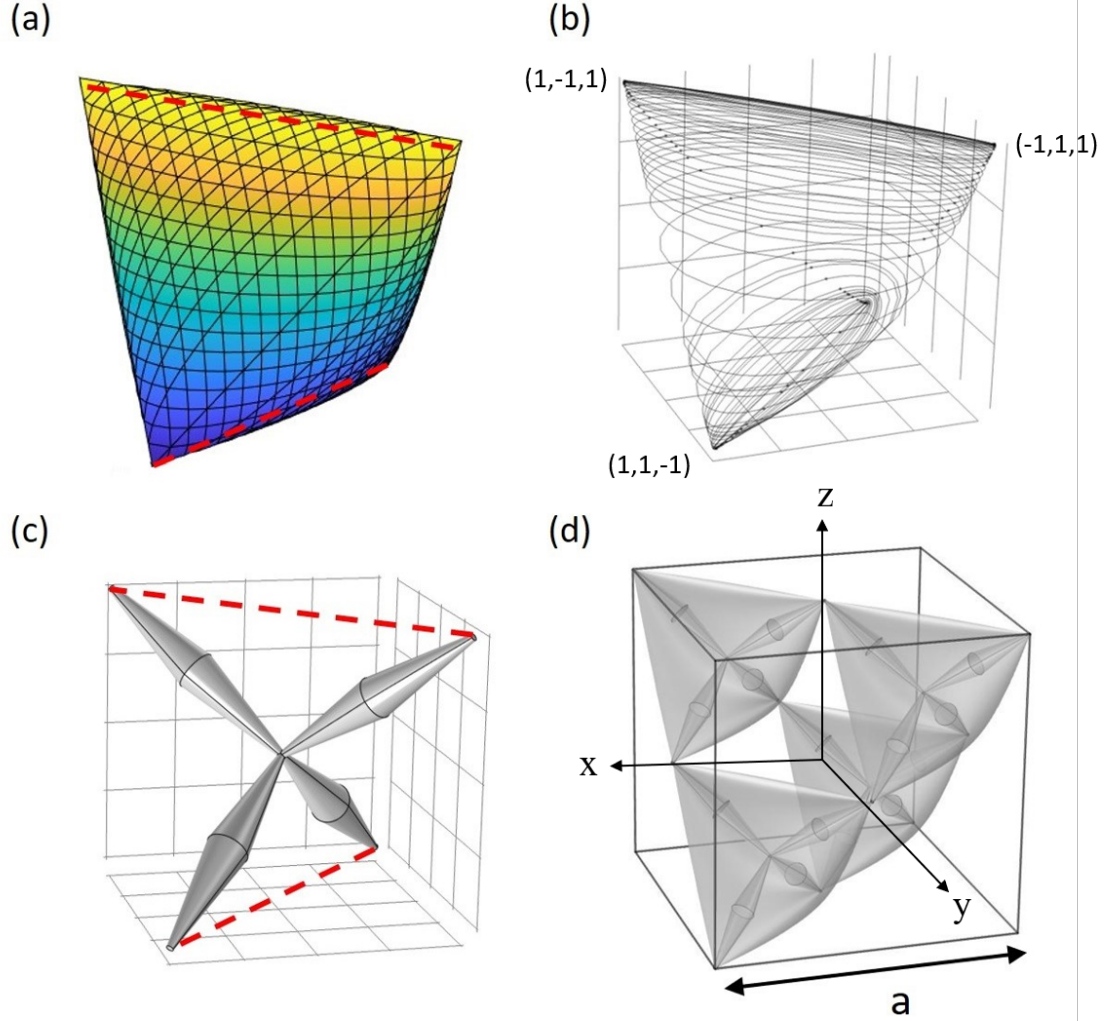


Figure 1. (a) The three-dimensional contour of Zongzi, $x^2 + y^2 + z^2 + 2xyz = 1$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$. Note that the geometry is invariant with respect to, x , y , and z . (b) By setting $z =$ constant, the configuration of Zongzi can be represented as an assembly of ellipses with major axis $\sqrt{1+z}$ and minor axis $\sqrt{1-z}$ for $z > 0$. The role of the major and the minor axis switches for $z < 0$. (c) Pentamode structure, with diamond-type structure consisting of four linkages connecting at a point with symmetric double-cone elements. (d) A unit cell of a periodic array of Zongzi structure, containing 4 Zongzi inclusions, with a lattice constant a . The configuration of Zongzi can be viewed as an outer envelope of pentamode material structure.

Further, the geometry of Zongzi structure can be viewed as an outer envelope of a PM structure (Figure 1d), in which the former structures can be viewed as curved bulk solids, while the latter

is like skeletal structures. We note that a major distinction between the two structures is that the central point is an internal point within a Zongzi inclusion, while it is a point-like connection point for a PM unit cell.

An ideal configuration of Zongzi would be unstable, as each building block of Zongzi is connected with the neighboring elements by a point-like tip with zero contact area, see Figure 2a. To stabilize the structures, we introduce a ball joint with a diameter d at the connection point. The idea of introducing a ball joint is analogous to the notion of finite connection of touching cones in PMs (Kadic et al. 2012). As each connection point will be joined by four different cones, a schematic diagram of the ball joint from different angles is depicted in Figure 2b,c. We will demonstrate later that the diameter of the ball joint plays a crucial role in controlling the effective moduli of the periodic array of Zongzi. A unique feature of the present configuration is that the Zongzi geometry can be described by a simple algebraic equation (1), which allows us to transform the configuration into a few variant forms upon coordinate scalings. This offer a number of flexibilities, through simple analysis, to achieve a number of tunable macroscopic properties, including effective elastic properties, overall anisotropy, and effective mass density. It is observed that altering the coefficients in equation (1) can lead to significant changes in the configuration, which may no longer form a closed contour.

To visualize the periodic array of Zongzi structures, we construct a number of centimeter-scale specimens (Figure 2d) using 3D printing as monolithic pieces without requiring assembly, since the inclusion contour is defined by a simple algebraic equation. were fabricated with 3D printer . In view of the geometric configuration, in Appendix A, we show that the periodic array of Zongzi structures exhibits a face-centered cubic symmetry. In terms of Voigt's notations (Nye 1985, Ting 1996, Dvorak 2013), the elastic constants of cubic solids are described by 3 independent constants, $C_{11} = C_{1111}$, $C_{12} = C_{1122}$, and $C_{44} = C_{2323}$, or by bulk modulus K and two shear moduli μ' and μ'' , with $K = (C_{11} + 2C_{12})/3$, $\mu' = (C_{11} - C_{12})/2$ and $\mu'' = C_{44}$ (Thomas 1966, Walpole 1986, Norris 2006). When the cubic solid exhibits additional symmetries, its overall symmetry becomes isotropic, and the two shear moduli, μ' and μ'' , become equal. Note that for isotropic solids we have $\mu' = \mu''$.

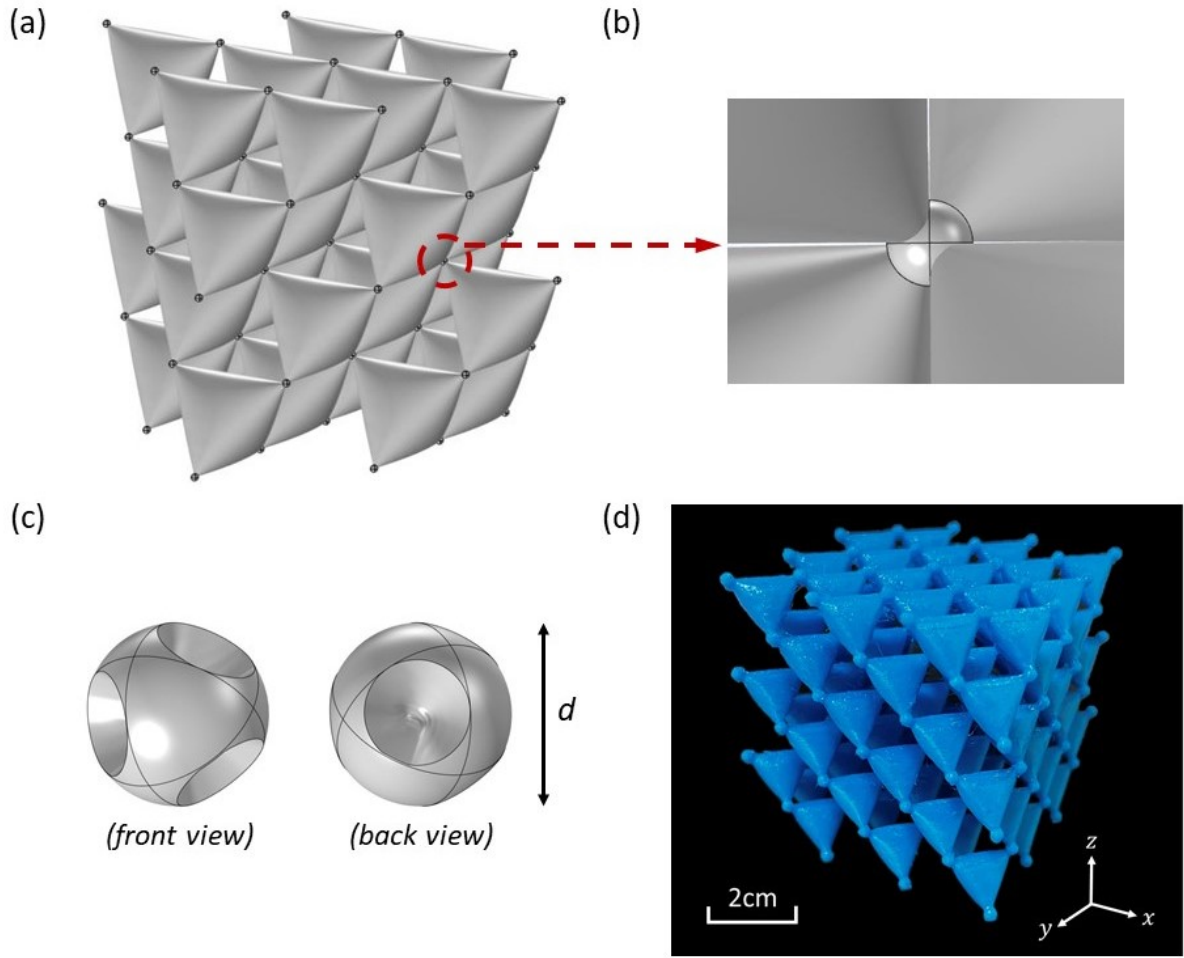


Figure 2. (a) An illustration of a periodic array of Zongzi with ball joints. Figure 2b is an enlarged diagram of a ball joint. Each ball joint will connect 4 end points from different Zongzi inclusions. See the illustration in Fig.2a. Figure 2c is a schematic diagram of a ball joint in two different angles. Figure 2d is a photo of a periodic array of Zongzi structure with ball connectors, manufactured by 3D printers, with a lattice constant $a = 1.86$ cm and $d = 0.26$ cm.

3. Tunable effective elastic moduli, overall anisotropy and mass density

We now examine the effective elastic properties of a periodic array of Zongzi structures. The designed microstructure can be generally regarded as a two-phase composite medium, consisting of Zongzi inclusions and a void phase. At the macroscopic level, the representative volume element (RVE) can be homogenized into a continuum medium, characterized by effective material properties. To study the effective elastic properties, we have carried out finite element simulations based on static continuum mechanics using a commercial software package, COMSOL Multiphysics, with solid mechanics module. To characterize the three elastic moduli, three different loading conditions on the boundaries are prescribed (Figure 3). It should be noted that, for an isotropic microstructure, the boundary conditions depicted in the second and third configurations of Fig. 3 are effectively identical. The effective moduli of the periodic array of

Zongzi C_{ijkl} are evaluated based on volume averages over a representative volume element V ,
 $\bar{\sigma}_{ij} = C_{ijkl} \bar{\varepsilon}_{kl}$, where (Hill 1963)

$$\bar{\sigma}_{ij} = \frac{1}{V} \int_V \sigma_{ij} dV, \bar{\varepsilon}_{ij} = \frac{1}{V} \int_V \varepsilon_{ij} dV. \quad (2)$$

Under a homogeneous displacement boundary condition $u_i = \varepsilon_{ij}^0 x_j$, it is known that $\bar{\varepsilon}_{ij} = \varepsilon_{ij}^0$ (Hill 1963). For numerical calculations, we consider a periodic array of Zongzi that is made of a type of polymer with Young's modulus $E_0 = 3$ GPa and Poisson's ratio $\nu_0 = 0.4$ (Kadic et al. 2012). To gain confidence of our computation procedures, we have reconstructed the numerical solutions of the effective elastic properties of PMs presented by Kadic et al. (2012). The finite element calculations for a periodic array of Zongzi structures are performed with $N \times N \times N$ unit cells, totalling $4N^3$ Zongzi inclusions. Our numerical solutions show that $N = 3$, with a total number of 108 Zongzi inclusions, discretized by about $1.7 \sim 4.1 \times 10^6$ tetrahedral elements in finite element calculations, will give converged field solutions.

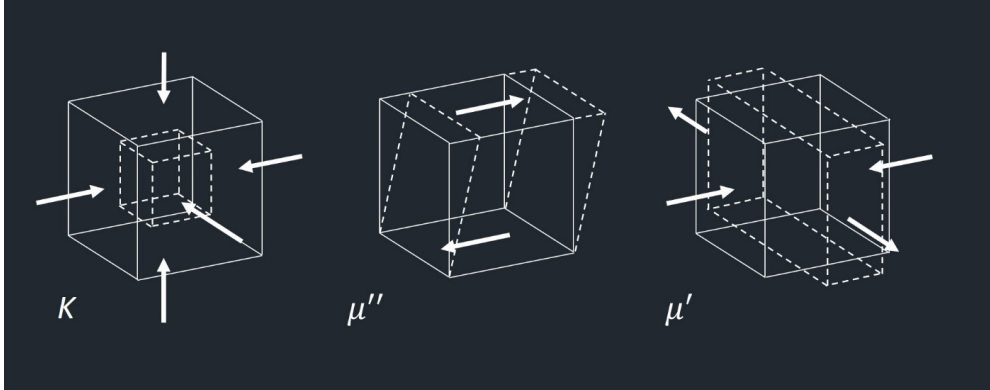


Figure 3. The prescribed boundary loadings for the estimates of K , μ' and μ''

The calculations of effective bulk modulus and two shear moduli are evaluated by prescribing a homogeneous displacement boundary condition along the boundary of a bulk medium (Figure 3) and measure the average stresses (Hill 1963). See also a monograph on computational homogenization on heterogeneous materials (Yvonnet 2019). In Figure 4a, we show the results of the bulk and two shear moduli versus the ball joint diameter d/a . For scalability we emphasize that the calculated effective moduli will be the same for any array with the same d/a . Here we consider the lattice constant $a = 2\text{m}$ for potential applications in seismic metamaterials (Fabbrocino et al. 2016, Brule et al. 2020, Mu et al. 2020), and the diameter d for numerical calculations in the range of $0.01 \leq d/a \leq 0.09$. For different

scalability we have verified the effective elastic properties numerically for different lattice constants. The calculated effective moduli exhibit a linear dependence on d/a , as illustrated in Fig.4a. In comparison of the original bulk material with $K_0 = 5$ GPa and $\mu_0 = 1.07$ GPa, the effective elastic constants can be reduced to 1/100 - 1/1000 in the order of MPa (Fig.4b). We note that since the radius of the ball joint is very small compared to the dimensions of the Zongzi inclusion, its volume contribution is negligible. As a result, the volume fraction of the bulk phase is not a dominant factor in determining the effective material properties. This is of particular interest as we can control the magnitude of effective moduli by up to 2 to 3 orders, simply by varying the joint diameter without changing the internal structures or the constituent material. However, we find that the ratios of K/μ' or K/μ'' for Zongzi structures are reduced by only a factor of 2.5 to 3.5 compared to those in the bulk material, in contrast to PMs in which $\mu' \ll K$ and $\mu'' \ll K$ (Kadic et al. 2012). This could be explained by the intrinsic difference between the two metastructures. Specifically, as we mentioned earlier, the central point of Zongzi structure is an internal point within a bulk medium, while it is a point-like connection for PM structures (see Figure 1). Although Zongzi structure does not provide effective shear modulus as low as that of the PM structures, we emphasize that the present structure offers flexibilities to generate tunable mechanical properties and large frequency bandgaps, which will be addressed later.

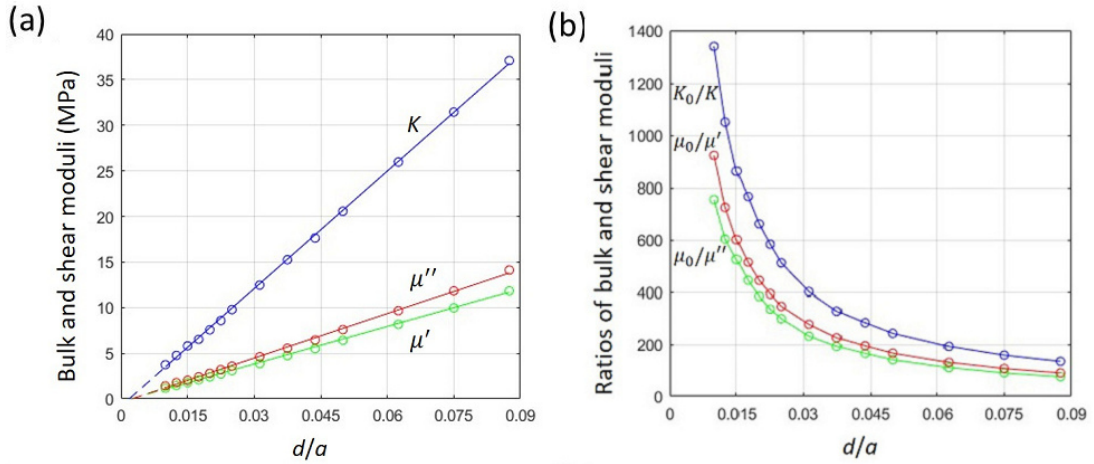


Figure 4. The effective moduli of a periodic array of Zongzi, $N \times N \times N = 3 \times 3 \times 3$, versus the ratio d/a , where the lattice constant $a = 2$ m and d is the ball joint diameter. The bulk material parameters are $K_0 = 5$ GPa and $\mu_0 = 1.07$ GPa. The elastic moduli of K , μ' , and μ'' are presented in Figure 4a and the ratios of K_0/K , μ_0/μ'' , and μ_0/μ' versus d/a are shown in Figure 4b.

Next, we demonstrate that the overall symmetry of the effective medium can be made further anisotropic upon a simple coordinate scaling. To show this, we introduce a change of

length scales along the three coordinate axes. The algebraic equation of Zongzi inclusion in (1) is rewritten as

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 + \left(\frac{z}{\gamma}\right)^2 + 2\left(\frac{x}{\alpha}\right)\left(\frac{y}{\beta}\right)\left(\frac{z}{\gamma}\right) = 1. \quad (3)$$

Here α, β , and γ are positive anisotropic parameters. With this representation, the Zongzi originally placed within a simple cubic is bounded inside a rectangular parallelepiped with domain $|x| \leq \alpha, |y| \leq \beta$, and $|z| \leq \gamma$. The overall symmetry of a periodic array of Zongzi described by (3) is with orthotropic symmetry. For simplicity, we demonstrate in a video in Supplementary Material A showing the geometric configuration for $\alpha = \beta = 1$ with a varying parameter $0.1 \leq \gamma < \infty$.

In Figure 5a-5b, we present the effective elastic moduli of Zongzi with $\alpha = \beta = 1$, and with different values of γ . The effective Young's modulus in three directions versus different values of γ are presented in Figure 5a and the three effective shear moduli are shown in Figure 5b. It is seen that the effective Young's moduli become anisotropic, where the Young's modulus in the z direction increase with the value of γ . This offers a systematic geometric control on the structural anisotropy. We note that, for PMs, Kardic et al. (2013), Cai et al. (2015), and Mohammadi et al. (2020) proposed more complex topological anisotropic lattices to achieve structural anisotropy.

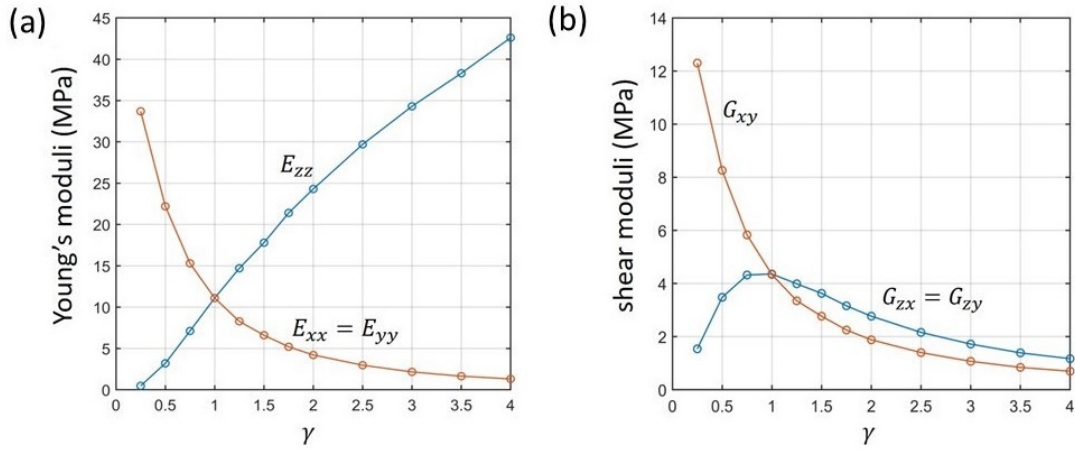


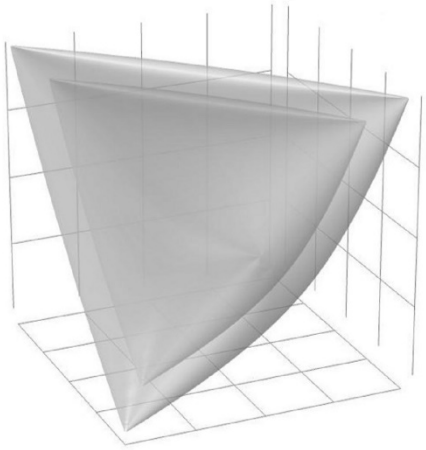
Figure 5. The effective properties of a periodic array of Zongzi structures, Eq. (3), with $\alpha = \beta = 1$ are presented. In Figure 5a, we calculate the effective Young's modulus in three directions versus different value of γ , and in Fig. 5b the three effective shear moduli. Here $a = 2\text{m}$, $d = 0.06\text{ m}$ and $N \times N \times N = 3 \times 3 \times 3$. The parameters of the bulk material are $K_0 = 5\text{ GPa}$ and $\mu_0 = 1.07\text{ GPa}$.

Next, we demonstrate that the effective mass density of a periodic array of Zongzi structures can be adjusted by incorporating a thickness parameter δ as shown in Figure 6. To do this, we consider the Zongzi inclusion as hollow, bounded by an outer and an inner surfaces. This allows us to control the volume and the total mass within the unit cell, leading to a tunable mass density. To demonstrate this, we consider the outer surface of Zongzi as the one described in (1). For an inner surface, by letting $x \rightarrow x/(1 - \delta)$, $y \rightarrow y/(1 - \delta)$, $z \rightarrow z/(1 - \delta)$, we can change the algebraic equation of Zongzi (1) to the form

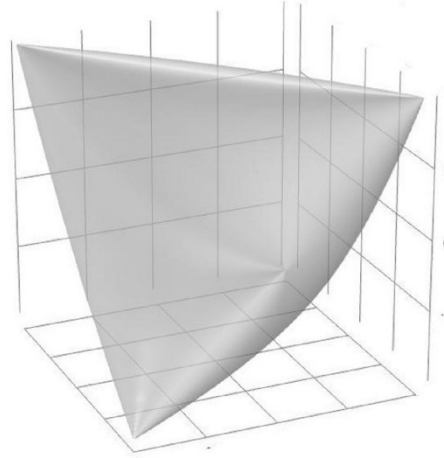
$$x^2 + y^2 + z^2 + 2xyz/(1 - \delta) = (1 - \delta)^2, 0 \leq \delta < 1, \quad (4)$$

where $|x| \leq 1 - \delta$, $|y| \leq 1 - \delta$, and $|z| \leq 1 - \delta$.

(a) $\delta = 0.2$



(b) $\delta = 1$



(c)

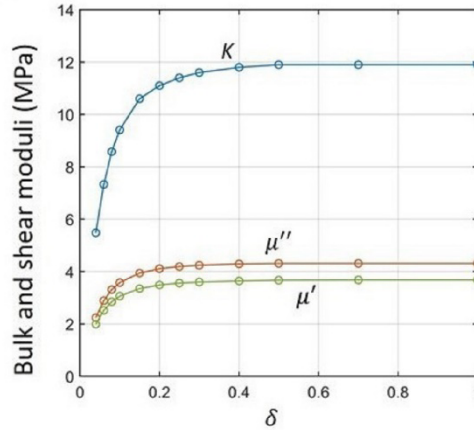


Figure 6. (a) A hollow Zongzi with $\delta = 0.2$, (b) A solid Zongzi with $\delta \rightarrow 1$, (c) the effective elastic properties of a periodic array of Zongzi with $\alpha = \beta = \gamma = 1$ for different values of thickness parameter δ . Here $a = 2\text{m}$, $d = 0.06\text{ m}$ and $N \times N \times N = 3 \times 3 \times 3$.

When $\delta = 0$, Eq.(4) recovers the outer surface of Zongzi. For the inner surface of Zongzi, a certain value of δ between (0,1) describes a similar shape of Zongzi. For example, a value of $\delta = 0.2$ depicts a hollow Zongzi with a small shell thickness, as shown in Figure 6a. When $\delta \rightarrow 1$, the inner surface of Zongzi reduces to a point and thus the Zongzi becomes a solid one, see Figure 6b. For example, the total volume of a solid Zongzi with $\delta \rightarrow 1$ is about 10 times of the hollow one with $\delta = 0.05$. In Figure 6c, we demonstrate the effect of thickness parameter δ on the effective elastic properties. It is seen that the effective elastic moduli of a periodic array of Zongzi structures with a thickness $\delta = 0.05$ can be as low as half of a solid one. We note that the effective mass density of the Zongzi structures is crucial for the frequency bandgap under wave propagation to be addressed later.

4. Frequency band structures

In addition to static continuum mechanics analysis, we examine the bandgap structures of Zongzi metastructures. The study of periodic metastructures with a sub-wavelength lattice constant is of particular interest in the control of wave propagation through a periodic medium.

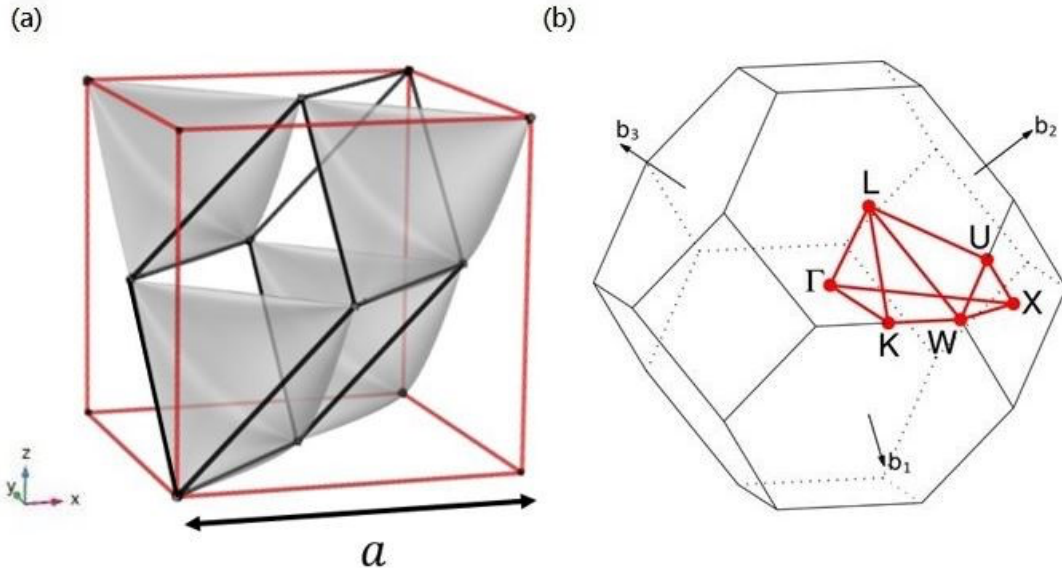


Figure 7. (a) Illustration of Zongzi metamaterial in the extended face-centered-cubic (fcc) unit cell (red) and the primitive cell (black). (b) The corresponding face-centered-cubic Brillouin zone. The irreducible Brillouin zone, $\Gamma \rightarrow X \rightarrow W \rightarrow K \rightarrow \Gamma \rightarrow L \rightarrow U \rightarrow W \rightarrow L \rightarrow K|U \rightarrow X$ (red).

Again we utilize finite-element simulations by a commercial software package COMSOL Multiphysics, with Bloch boundary conditions (Bloch 1929) imposed on the primitive real-space cell in Figure 7a. The corresponding fcc Brillouin zone for a fcc unit cell and the

characteristic points are depicted in Figure 7b (Brillion 1953, Kittle 1996). To verify the accuracy of our finite element simulations, we first reconstruct the band structures for the periodic array of pentamode structures presented in Martin et al. (2012, Fig.1), and good agreement with both results was observed, see Appendix B.

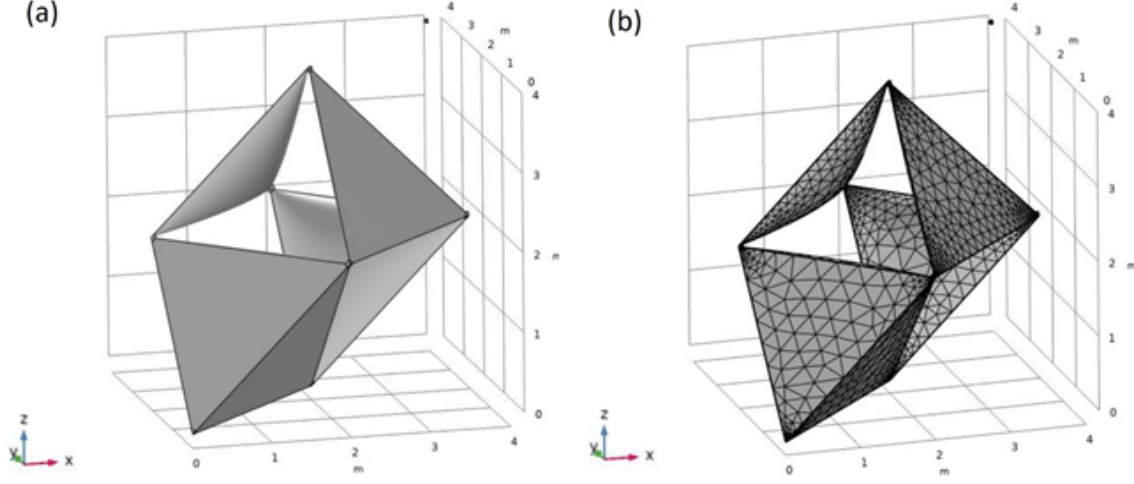
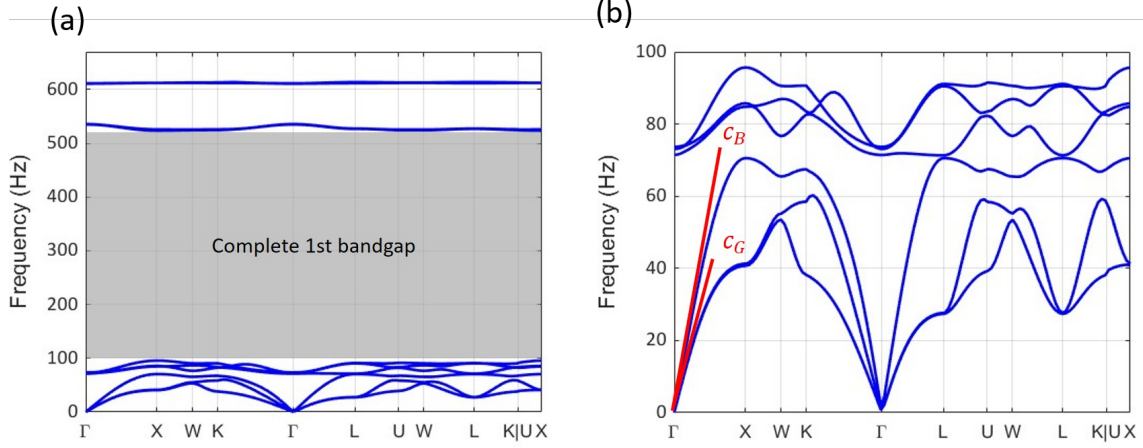


Figure 8. (a) The primitive cell of Zongzi structure. See also the black box in Fig.7a. (b) Finite element mesh with a total of 7350 tetrahedra elements.

To examine the bandgap structures for the periodic Zongzi structures, we consider a primitive cell, depicted as in Figure 8a with a mesh diagram shown in Figure 8b. A converged solution with finite element simulations is achieved by a total of 7350 tetrahedra elements within the primitive cell. To connect with our previous static analysis, we use the same material parameters $E_0 = 3\text{GPa}$, $\nu_0 = 0.4$ and $\rho_0 = 1190\text{kg/m}^3$ for numerical illustrations. To examine the wave behavior, we consider a periodic array of Zongzi structure with lattice constant $a = 2\text{m}$ and the ball joint diameter $d = 0.06\text{m}$. We show that different length scales, namely different lattice constants with ball joint diameters, can be adjusted to achieve different frequency ranges. In Figure 9a, we present the frequency bandgap structure of the Zongzi structures. The baseline geometry of Zongzi is solid with homogeneous material as mentioned before. Unlike the pentamode structures (Martin et al. 2012, Huang and Zhang 2021), the Zongzi structures do not have any single-mode regions for the low frequency regime, see also the left column in Figure B1. This is due to the fact that our effective bulk and shear moduli for Zongzi structures do not differ much as shown in Fig.4, contrasting to that of pentamode structures. However, we find that, remarkably, the periodic Zongzi metastructure composed of a single material could exhibit a very wide complete frequency bandgap (Figure 9a), in the range of 98-528 Hz. This means that, within the bandgap frequency interval, an incident wave can be completely filtered through the periodic medium. To further validate the accuracy of our

finite-element calculations for Zongzi structures, in a separate study (Lin, 2023), we have implemented a plane wave expansion (PWE) method (Kushwaha et al. 1994, Dal Poggetto and



Serpa 2020) based on Fourier expansions for a comparison. It shows that the bandwidth and the frequency interval are in reasonable agreement between these two different approaches. The formulation algorithm of plane wave expansion methods for Fourier coefficients corresponding to the primitive region of Zongzi structures is rather complicated, and the details of PWE method will be reported elsewhere.

Figure 9. Calculated Band structures for the periodic array of Zongzi structure, with $a = 2\text{m}$ and $d = 0.06\text{m}$. The geometry of Zongzi is solid made of a homogeneous material with parameters $E_0 = 3\text{GPa}$, $\nu_0 = 0.4$ and $\rho_0 = 1190\text{kg/m}^3$. (a) The calculated first complete band structures of Zongzi metastructure. (b) The first six bands, in which the first two bands correspond to the lateral motion of shear wave along the Γ X direction, and the third band can be attributed to the compression wave motion along the wave direction. The red straight lines are fitted to dispersion branches, in which c_B corresponds to phase velocity of compression wave and c_G is the phase velocity of shear wave.

In Figure 9a, we demonstrate the first frequency bandgap, where the lower limit of the full bandgap is the 6th mode, while the upper limit of the bandgap is the 7th mode. To perceive further the underlying mechanism of the bandgap, we examine the wave modes in the dispersion relation. These bandgaps arise from the interaction between the neighboring structural unit and the incoming waves, leading to interference effects that attenuate the wave energy. Within the bandgap frequency range, local resonance behavior can be observed between the neighboring Zongzi inclusions, characterized by out-of-phase motion between different inclusions. A video file, Audio S2.gif, was included in supplementary materials, which shows the out-of-phase motion between the neighboring Zongzi inclusions. For illustrations, we generate a video in Supplementary Materials B that describes the wave motions for the 6th mode and the 7th mode. Interestingly, it appears that, within this excitation frequency, the Zongzi inclusions between neighboring blocks move in opposite directions. This suggests a mechanism of local resonances

(Liu et al. 2000), often characterized in three-phase composite systems containing a hard inclusion with a soft coating within a hard matrix medium. See for instance the relevant references, in which both continuum models and discrete models are simulated (Liu et al. 2005, Huang et al. 2009, Huang and Sun 2009). Note that our present periodic array could be considered as two-phase systems consisting of the constituent material of Zongzi inclusions with the surrounding void phase. Further examining the band structures in Figure 9b for the array of Zongzi, it appears that, for the wave vector along the ΓX direction, the first two bands correspond to the lateral displacement due to shear wave, while the third band can be attributed to the compression waves as the displacement vector is along the wave direction. In Figure 10, we illustrate the first three bands corresponding to Figure 9b, where the yellow ones indicate the undeformed structure and the red ones show the deformed structure, with the wave vector $\mathbf{k}$ pointing in the ΓX direction (the y axis).

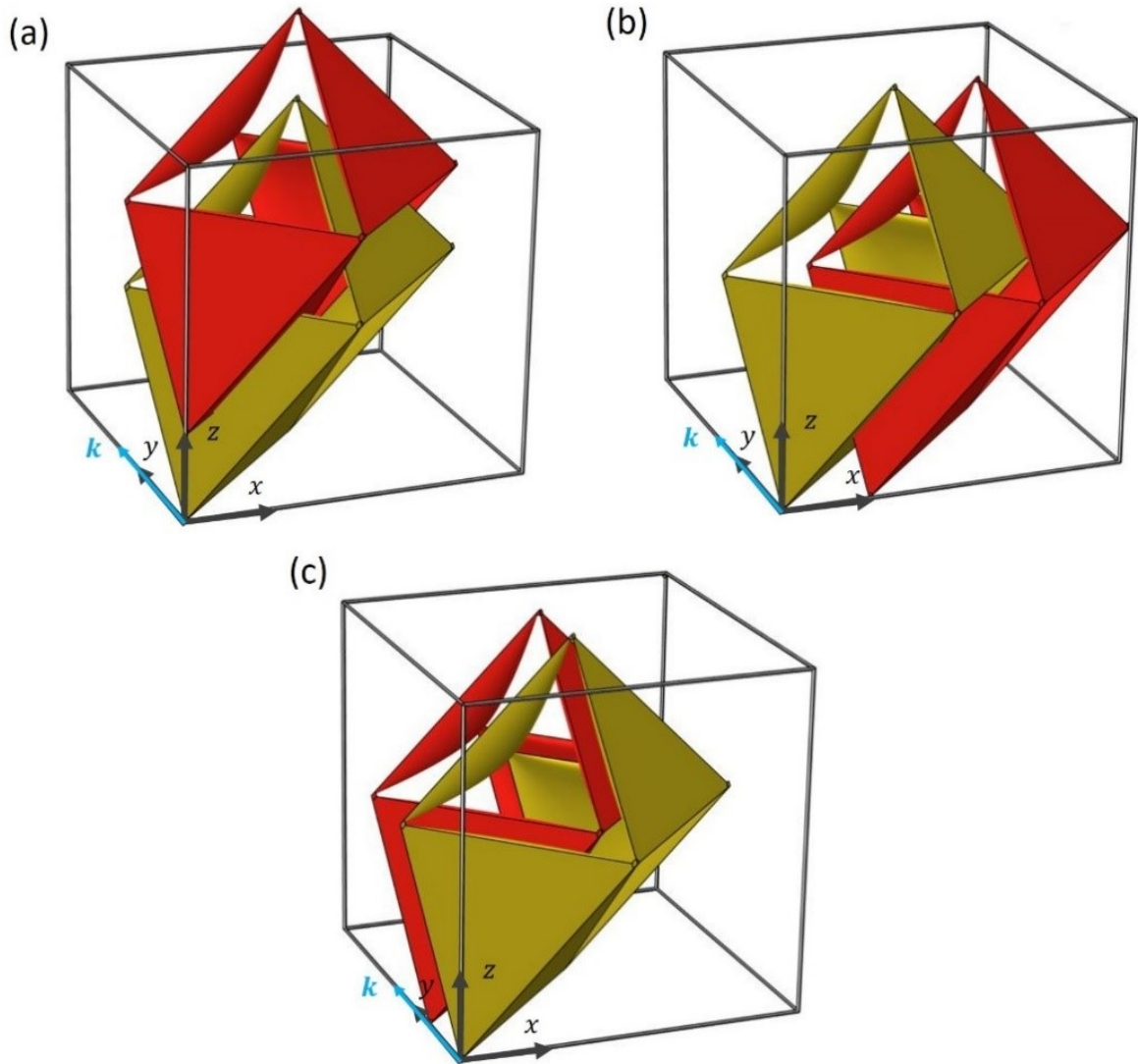


Figure 10. Illustrations of the first three modes for the band structure calculated with the parameters given in Fig.9. Fig.10 (a), (b) and (c) correspond respectively to the first, second and the third band. The wave vector k is along the ΓX direction, i.e., the y axis. We display the undeformed structure in yellow and the deformed structure in red (enlarged snapshot).

To examine the influence of the constituent material properties on the frequency bandgap, in Figure 11 we change the material parameters from $E_0 = 800$ MPa to $E_0 = 3$ MPa, with the same Poisson's ratio. It turns out that the upper and lower limits of the first bandgap vary substantially. Specifically, for $E_0 = 3$ MPa, the first bandgap is only around 3.5 to 16 Hz, which is of particular interest for applications in seismic waves with relatively low frequencies and long wavelengths. Our numerical results demonstrated that, with the same mass density, if the Young's modulus is increased by 100 times, the frequency is increased by the square root, i.e., by factor of 10. The scalability between the lattice constants and the bandgap frequencies is demonstrated in Table 1 for different lattice constants a and joint diameters d . The results show that, for a same ratio of d/a , when the lattice constant a is increased by a factor of n , the upper and lower frequency limits are decreased by a factor of n .

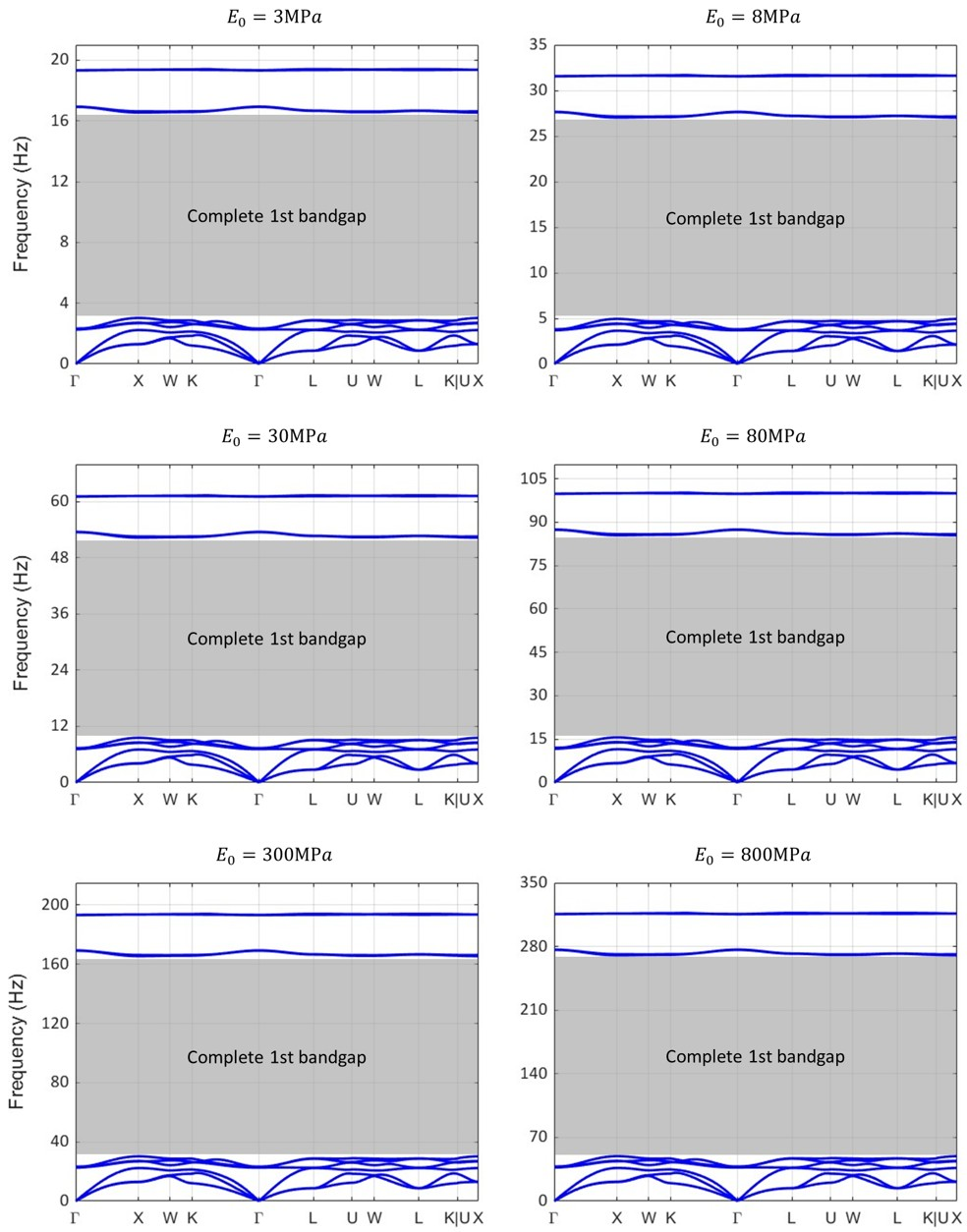


Figure 11. Calculated band structures for a periodic array of Zongzi structures, with different Young's modulus from $E_0 = 3 \text{ MPa}$ to $E_0 = 800 \text{ MPa}$, and $a = 2 \text{ m}$ and $d = 0.06 \text{ m}$. Other material parameters are $\nu_0 = 0.4$ and $\rho_0 = 1190 \text{ kg/m}^3$.

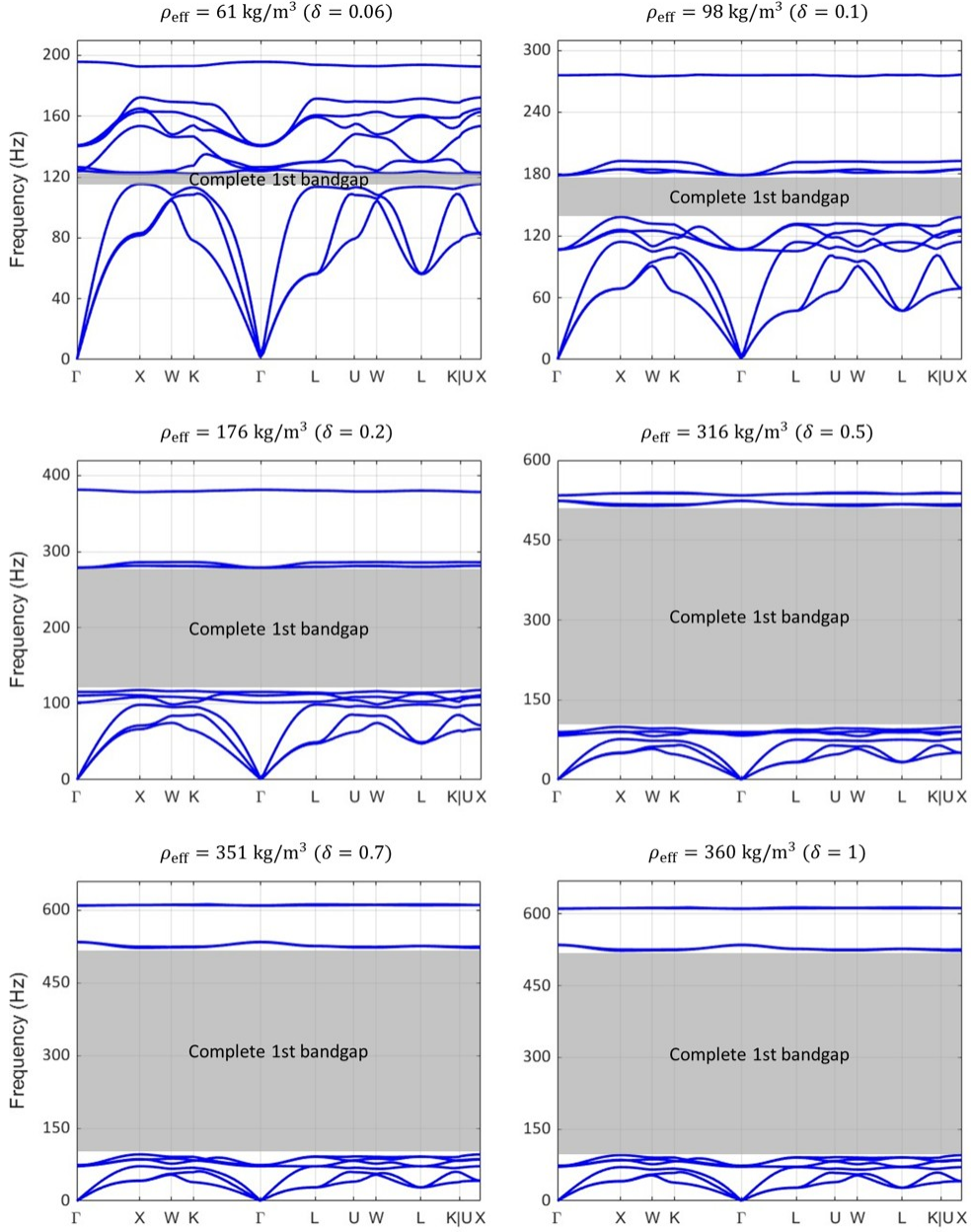


Figure 12. Calculated band structures for a periodic array of Zongzi structures, with different thickness parameter δ within the range $0.06 \leq \delta \leq 1$. As in Figure 11, $a = 2\text{m}$ and $d = 0.06\text{m}$. Other material parameters are $\nu_0 = 0.4$ and $\rho_0 = 1190\text{kg/m}^3$.

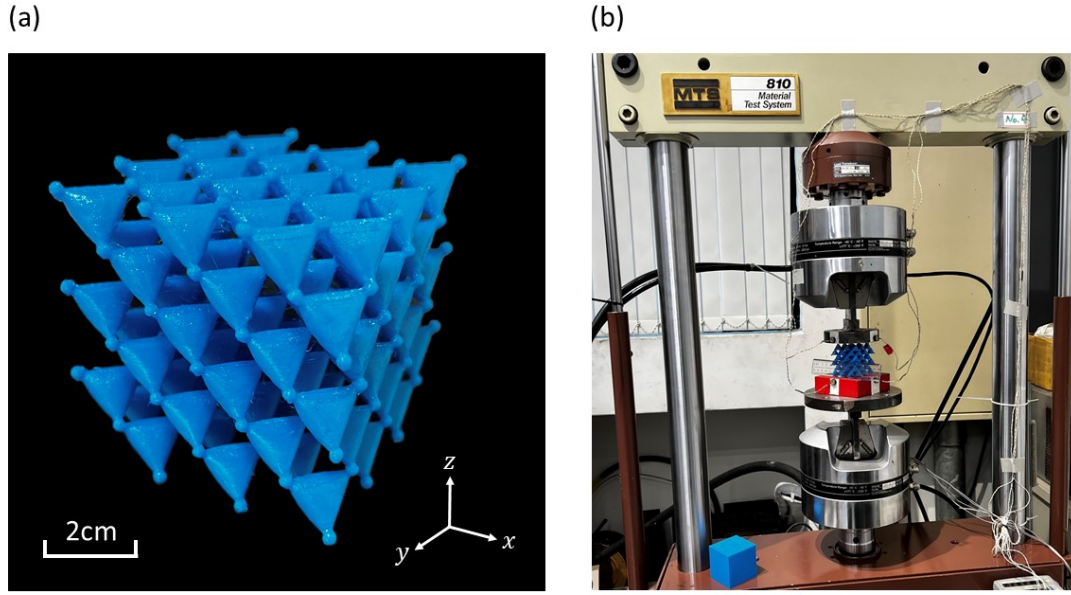
Lastly, we also examine the effect of thickness parameter δ on the frequency bandgap. Different value of δ changes the volume of inclusion and thus the effective mass density. In Figure 12, we present the band structures of Zongzi metastructures with different thickness parameters δ , within the range $0.06 \leq \delta \leq 1$. When the value of δ is getting close to 1, the Zongzi inclusion is like a solid inclusion, while a small value of $\delta = 0.06$ corresponds to a hollow inclusion with thin thickness. The finite element meshes within the primitive cell increase from 7350 for the meshes presented in Figure 8b to 10000-60000 tetrahedra elements

for our numerical simulations in Figure 12. Our numerical results indicate that, for the first bandgap, the upper limit of the bandgap increases substantially from 61 Hz for a thin hollow structure to 264 Hz for a solid Zongzi structure, while the lower limit does not influence much by the thickness parameter. This suggest that the thickness paramater has a great influence on the band width of the first band gap.

5. Experimental validation

For experimental tests, we have constructed a number of specimens of the periodic array of Zongzi at a centimeter scale (Figure 2d, Figure 13a) using a 3D printer, KINGSSSEL K-2327, with blue PLA plastic filament with a diameter of 1.75 mm. The specimens were 3D printed in one piece without the need for assembly, as the contour of the Zongzi inclusion is defined by a simple algebraic equation as discussed in Equations (3) and (4). To validate the effective elastic properties computed by the finite element simulations, a number of samples were tested experimentally. Specifically, a number of centimeter-scale specimen of the periodic array of Zongzi, containing $N = 3 \times 3 \times 3$ unit cells, were fabricated with 3D printer (Figure 13a). The lattice constant is $a = 1.86$ cm with different sizes of ball joint diameter d . A testing instrument, MTS 810 Material Test System, was utilized for experimental tests (Figure 13b). The mechanical properties of the base material, printed as a solid cube, were measured as $E_0 = 2.17$ GPa and $\nu_0 = 0.4$. In Figure 14, we compare the data by the finite element calculations with the experimental results. We see that the experimental results are in reasonable agreement with the finite element calculations, not only the overall trend but also the simulated data quantitatively. It appears that larger, yet marginal, discrepancies occur for samples with smaller values of d . This could be due to the loss of precision control in printing smaller specimen. Also, a drastic difference in finite element mesh dimensions for the ball joint and for the bulk phase of Zongzi could lead to certain computational errors.

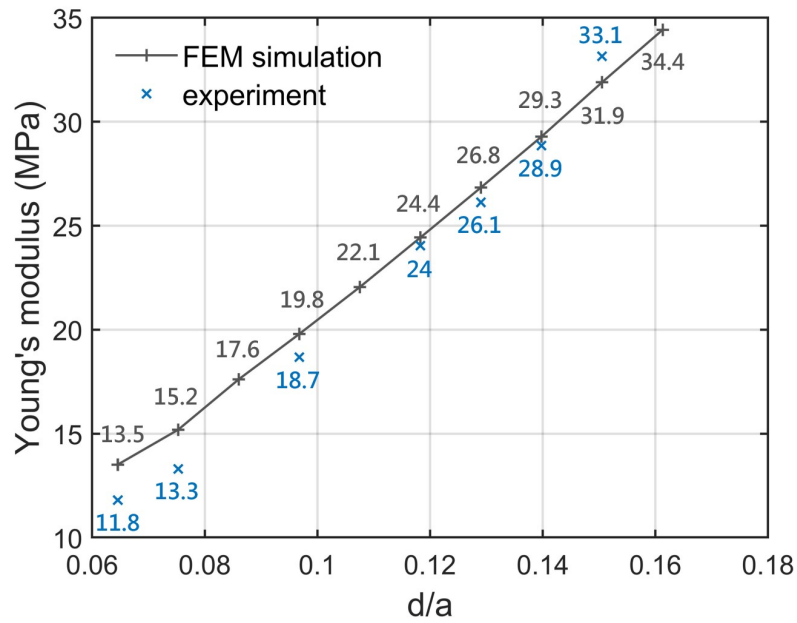
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523 Figure 13. (a) A test specimen of a periodic array of Zongzi with ball connectors, with
 524 lattice constant $a = 1.86\text{cm}$, ball joint diameter $d = 0.26\text{cm}$ and $N = 3 \times 3 \times 3$. (b) A testing
 525 instrument, MTS 810 Material Test System.
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529 Figure 14. A comparison of the effective Young's modulus based on COMSOL finite element
 530 simulation and experimental tests is shown with different values of d . A periodic array of Zongzi with
 531 a lattice constant $a = 1.86\text{ cm}$ was fabricated by 3D printer. The printing source material is blue PLA
 532 plastics. A testing instrument, MTS 810 Material Test System, is utilized for the tests. The material
 533 parameters of the source material were measured as $E_0 = 2.17\text{ GPa}$ and $\nu_0 = 0.4$.
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6. Discussions and Concluding Remarks

We present a new type of elastic metastructures composed of a periodic array of pyramid-shaped inclusions, referred to as Zongzi. Each Zongzi inclusion is connected with neighboring inclusions by solid ball joints. We demonstrate that the effective properties of the periodic array of Zongzi can be controlled by the size of ball joints within a range of $0.01 \leq d/a \leq 0.09$.

The frequency bandgap structures of Zongzi metastructures have also been examined. The tunable mechanical properties also provide degrees of flexibilities in the manipulation of wave characteristics, in which the frequency bandgaps can be systematically adjusted. Particularly, we find that, with proper geometric parameter scaling, the periodic array of Zongzi structures could generate a very wide bandgap, with tunable frequency ranges. This is due to local resonance mechanism, leading to an opposite motion between neighboring Zongzi inclusions. The local resonant behavior could be controlled by the ratio between ball-joint radius and the lattice constant of Zongzi microstructure. The investigated Zongzi metamaterial offers unique features in tunable mechanical properties and frequency bandgap, which have tremendous applications benefiting from the control of wave propagation.

Although the ratio of bulk and shear moduli in the Zongzi microstructure is lower than that of the PM materials, its flexibility to generate large bandgaps and the potential to tune its effective properties without modifying the internal structure or material composition as shown in this study make it particularly attractive for vibration isolation in earthquake engineering applications.

Appendix A

In this Appendix, we will show that the periodic array of Zongzi structures exhibits a cubic symmetry. To show this, we refer to the geometric configuration and coordinate systems as in Figure 1d. We will demonstrate that the Zongzi and PM structures are both cubic symmetries. To identify the symmetry planes of the periodic array, we place the origin of the Cartesian coordinates at the central point within the simple cube, as shown in Figure 1d. It can be seen that there exists 6 symmetry planes within the unit cell of Zongzi structures. The unit normals of the symmetry planes are indicated in Table A1. The presence of symmetry in the periodic array of Zongzi structures will reduce the number of independent elastic constants. For doing this, we introduce a transformation from one set of orthogonal axes x_i to another set of new axes x'_i given by $x'_i = a_{ij}x_j$, so that $C'_{ijkl} = a_{ip}a_{jq}a_{kr}a_{ls}C_{pqrs}$. For example, for symmetry plane a , the unit normal of symmetry plane is given by $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and the transformation matrix of the reflectional symmetry will be

$$a_{ij} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A1})$$

By direct expansion of the tensor transformation, it can be seen that the elastic constants can be reduced as $c'_{1111} = c_{2222}$, $c'_{1133} = c_{2233}$, $c'_{2233} = c_{1313}$, $c'_{1123} = c_{2213}$, $c'_{2223} = c_{1113}$, $c'_{3323} = c_{3313}$, $c'_{2312} = c_{1321}$,

For symmetry plane b , the non-zero component of the transformation matrix will become $a_{12} = a_{21} = -1$, $a_{33} = 1$. And the elastic constants can be reduced as $c_{1123} = c_{2213} = c_{2223} = c_{1113} = c_{2312} = c_{1321} = 0$. In a similar way, one can repeat the derivations for other symmetry planes and show that final expression for the elastic stiffness matrix can be simplified as

$$\mathbf{C} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1122} & 0 & 0 & 0 \\ c_{1122} & c_{1111} & c_{1122} & 0 & 0 & 0 \\ c_{1122} & c_{1122} & c_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{2323} \end{bmatrix}. \quad (\text{A2})$$

Appendix B

To validate the accuracy of our numerical simulations of band structures, in Appendix B we demonstrate the results of our calculations of the band structures of three-dimensional pentamode metamaterials studied by Martin et al. (2012, Figure 1). The material and geometric parameters follow exactly the data from Martin et al. (2012). The results of the right column in Figure B1 are our calculations, while the ones from the left column of Figure B1 was the original plots from Martin et al. (2012). In the calculation, we utilized COMSOL commercial finite element program with about by a total of 10000 tetrahedra elements within the primitive cell. The results show good agreement between the two calculations.

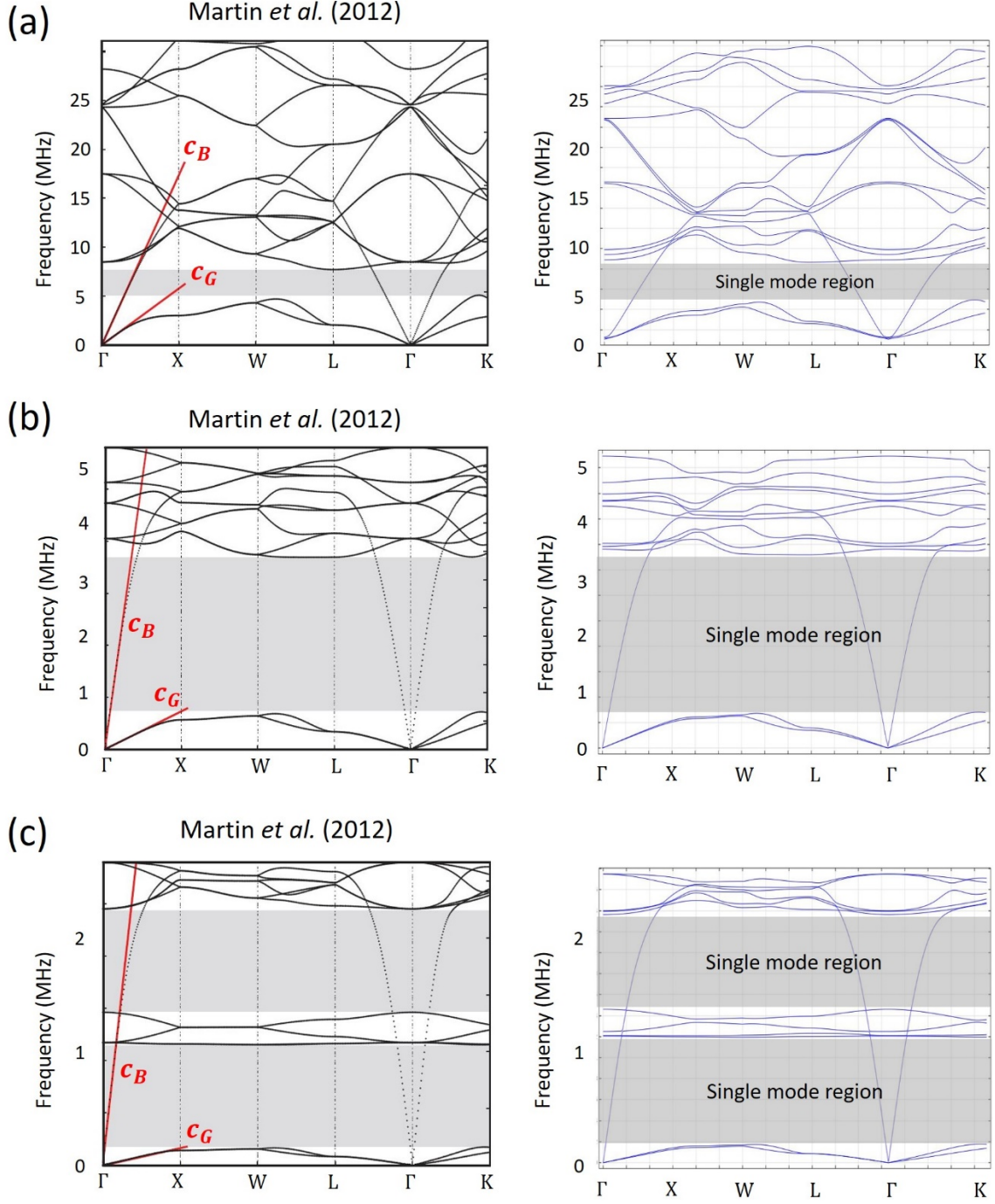


Figure B1. Reconstruction of band structures of pentamode metamaterials presented in Martin *et al.* (2012, Figure 1). All the geometric and material parameters follow the paper (Martin *et al.* 2012). Left panel: the results of Figure 1 in Martin *et al.* (2012). Right panel: the present numerical simulations. The fixed parameters for the pentamode structures are $a = 37.3\mu\text{m}$, $D = 3\mu\text{m}$. The diameter of the connection region d . (a) $d = 3\mu\text{m}$, (b) $d = 0.55\mu\text{m}$, (c) $d = 0.2\mu\text{m}$.

Data Availability: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Acknowledgements:

This work was supported by the National Science and Technology Council, Taiwan, under the project NSTC 113-2221-E-006-063-MY3. J.S Chen was supported by the Yushan scholar program sponsored by the Ministry of Education, Taiwan, under contract MOE-113-YSFEE-0005-001-P1. We thank to the National Research Center of Earthquake Engineering (NCEE), Taiwan, for the assistance of experimental tests of our specimens, and also to the National Center for High-performance Computing (NCHC), Taiwan, for providing computational and storage resources.

Supplementary Materials A: Audio S1

Supplementary Materials B: Audio S2

Figure Captions

Figure 1. (a) The three-dimensional contour of Zongzi, $x^2 + y^2 + z^2 + 2xyz = 1$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$. Note that the geometry is invariant with respect to, x , y , and z . (b) By setting $z = \text{constant}$, the configuration of Zongzi can be represented as an assembly of ellipses with major axis $\sqrt{1+z}$ and minor axis $\sqrt{1-z}$ for $z > 0$. The role of the major and the minor axis switches for $z < 0$. (c) Pentamode structure, with diamond-type structure consisting of four linkages connecting at a point with symmetric double-cone elements. (d) A unit cell of a periodic array of Zongzi structure, containing 4 Zongzi inclusions, with a lattice constant a . The configuration of Zongzi can be viewed as an outer envelope of pentamode material structure.

Figure 2. (a) An illustration of a periodic array of Zongzi with ball joints. Figure 2b is an enlarged diagram of a ball joint. Each ball joint will connect 4 end points from different Zongzi inclusions. See the illustration in Fig.2a. Figure 2c is a schematic diagram of a ball joint in two different angles. Figure 2d is a photo of a periodic array of Zongzi structure with ball connectors, manufactured by 3D printers, with a lattice constant $a = 1.86 \text{ cm}$ and $d = 0.26 \text{ cm}$.

Figure 3. The prescribed boundary loadings for the estimates of K , μ' and μ'' .

Figure 4. The effective moduli of a periodic array of Zongzi, $N \times N \times N = 3 \times 3 \times 3$, versus the ratio d/a , where the lattice constant $a = 2\text{m}$ and d is the ball joint diameter. The bulk material parameters are $K_0 = 5 \text{ GPa}$ and $\mu_0 = 1.07 \text{ GPa}$. The elastic moduli of K , μ' , and μ'' are presented in Figure 4a and the ratios of K_0/K , μ_0/μ'' , and μ_0/μ' versus d/a are shown in Figure 4b.

Figure 5. The effective properties of a periodic array of Zongzi structures, Eq. (3), with $\alpha = \beta = 1$ are presented. In Figure 5a, we calculate the effective Young's modulus in three directions versus different value of γ , and in Fig. 5b the three effective shear moduli. Here $a = 2\text{m}$, $d = 0.06 \text{ m}$ and $N \times N \times N = 3 \times 3 \times 3$. The parameters of the bulk material are $K_0 = 5 \text{ GPa}$ and $\mu_0 = 1.07 \text{ GPa}$.

Figure 6. (a) A hollow Zongzi with $\delta = 0.2$, (b) A solid Zongzi with $\delta \rightarrow 1$, (c) the effective elastic properties of a periodic array of Zongzi with $\alpha = \beta = \gamma = 1$ for different values of thickness parameter δ . Here $a = 2\text{m}$, $d = 0.06 \text{ m}$ and $N \times N \times N = 3 \times 3 \times 3$.

Figure 7. (a) Illustration of Zongzi metamaterial in the extended face-centered-cubic (fcc) unit cell (red) and the primitive cell (black). (b) The corresponding face-centered-cubic Brillouin zone. The irreducible Brillouin zone, $\Gamma \rightarrow X \rightarrow W \rightarrow K \rightarrow \Gamma \rightarrow L \rightarrow U \rightarrow W \rightarrow L \rightarrow K|U \rightarrow X$ (red).

Figure 8. (a) The primitive cell of Zongzi structure. See also the black box in Fig.7a. (b) Finite element mesh with a total of 7350 tetrahedra elements.

Figure 9. Calculated Band structures for the periodic array of Zongzi structure, with $a = 2\text{m}$ and $d = 0.06\text{m}$. The geometry of Zongzi is solid made of a homogeneous material with parameters $E_0 = 3\text{GPa}$, $\nu_0 = 0.4$ and $\rho_0 = 1190\text{kg/m}^3$. (a) The calculated first complete band structures of Zongzi metastructure. (b) The first six bands, in which the first two bands correspond to the lateral motion of shear wave along the ΓX direction, and the third band can be attributed to the compression wave motion along the wave direction. The red straight lines are fitted to dispersion branches, in which c_B corresponds to phase velocity of compression wave and c_G is the phase velocity of shear wave.

Figure 10. Illustrations of the first three modes for the band structure calculated with the parameters given in Fig.9. Fig.10 (a), (b) and (c) correspond respectively to the first, second and the third band. The

wave vector k is along the ΓX direction, i.e., the y axis. We display the undeformed structure in yellow and the deformed structure in red (enlarged snapshot).

Figure 11. Calculated band structures for a periodic array of Zongzi structures, with different Young's modulus from $E_0 = 3$ MPa to $E_0 = 800$ MPa, and $a = 2$ m and $d = 0.06$ m. Other material parameters are $\nu_0 = 0.4$ and $\rho_0 = 1190$ kg/m³.

Figure 12. Calculated band structures for a periodic array of Zongzi structures, with different thickness parameter δ within the range $0.06 \leq \delta \leq 1$. As in Figure 11, $a = 2$ m and $d = 0.06$ m. Other material parameters are $\nu_0 = 0.4$ and $\rho_0 = 1190$ kg/m³.

Figure 13. (a) A test specimen of a periodic array of Zongzi with ball connectors, with lattice constant $a = 1.86$ cm, ball joint diameter $d = 0.26$ cm and $N = 3 \times 3 \times 3$. (b) A testing instrument, MTS 810 Material Test System.

Figure 14. A comparison of the effective Young's modulus based on COMSOL finite element simulation and experimental tests is shown with different values of d . A periodic array of Zongzi with a lattice constant $a = 1.86$ cm was fabricated by 3D printer. The printing source material is blue PLA plastics. A testing instrument, MTS 810 Material Test System, is utilized for the tests. The material parameters of the source material were measured as $E_0 = 2.17$ GPa and $\nu_0 = 0.4$.

Figure B1. Reconstruction of band structures of pentamode metamaterials presented in Martin et al. (2012, Figure 1). All the geometric and material parameters follow the paper (Martin et al. 2012). Left panel: the results of Figure 1 in Martin et al. (2012). Right panel: the present numerical simulations. The fixed parameters for the pentamode structures are $a = 37.3 \mu\text{m}$, $D = 3 \mu\text{m}$. The diameter of the connection region d . (a) $d = 3 \mu\text{m}$, (b) $d = 0.55 \mu\text{m}$, (c) $d = 0.2 \mu\text{m}$.

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Table 1. The range of first bandgap for different lattice constants a and d with fixed a/d

a	d	Lower limit (Hz)	Upper limit (Hz)	Bandwidth (Hz)
40 cm	1.2 cm	490	2640	2150
2 m	0.06 m	98	528	430
4 m	0.12 m	49	264	215
8 m	0.24 m	25	132	107

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Table A1 The unit normal of symmetry planes

Symmetry plane	unit normal (x_1, x_2, x_3)
a	$1/\sqrt{2} (1, -1, 0)$
b	$1/\sqrt{2} (1, 1, 0)$
c	$1/\sqrt{2} (1, 0, -1)$
d	$1/\sqrt{2} (1, 0, 1)$
e	$1/\sqrt{2} (0, 1, -1)$
g	$1/\sqrt{2} (0, 1, 1)$

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