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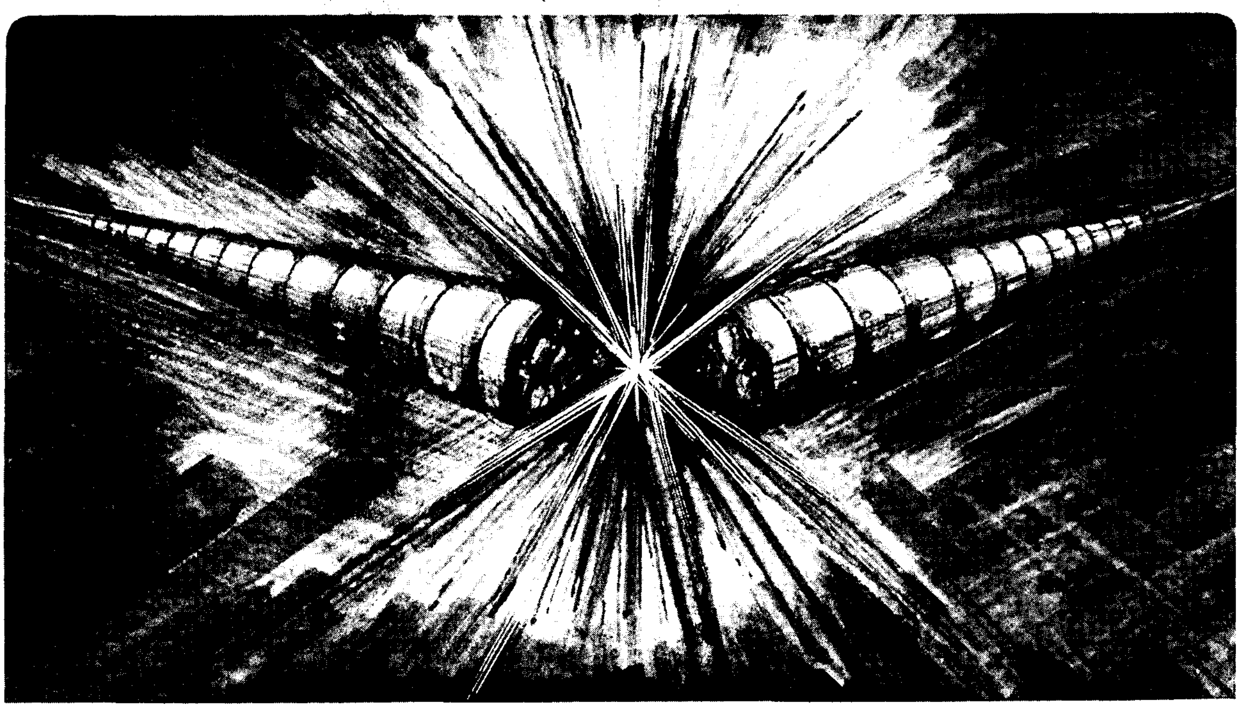

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WAVE ENTROPY: A DERIVATION BY JAYNES' PRINCIPLE*

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Abstract

The Jaynes maximum-entropy principle is used to derive the standard expression for the entropy of a set of weakly interacting waves.

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In discussions of classical wave kinetic equations, use is made of an expression for the wave entropy, as a functional of the mean density of wave action in ray phase space, $\bar{J}(\underline{k},\underline{x})$, and having the property of increasing monotonically in time as $\bar{J}$ evolves. The standard choice[1] is

$$S(\bar{J}) = \int \ln \bar{J}(\underline{k},\underline{x}), \tag{1}$$

with the integration element $d^3x\, d^3k\, /(2\pi)^3$.

It would clearly be desirable to have a classical derivation of this expression. Up to now, one has appealed to the quantum expression,[2] and taken the classical limit of large occupation number. A classical derivation can be based on Jaynes' maximum entropy principle,[3] and provides a striking illustration of its utility. In this note, we review the Jaynes algorithm in performing this derivation.

We begin by recognizing that $(\underline{k},\underline{x})$ space is not a continuum, but rather a set of cells, whose size is determined by Fourier's uncertainty principle. Hence we rewrite Eq. (1) as

$$S(\bar{J}) = \sum \ln \bar{J}_i, \tag{2}$$

where $\bar{J}_i$ is the mean action in the i th cell.

Following Planck, we characterize each cell as an oscillator, possibly nonlinear. With (J_i, Θ_i) as the action-angle variables for the i th oscillator, we introduce the corresponding probability density $\rho_i\,(J_i, \Theta_i)$, and have

$$\bar{J}_i = \int dJ_i \int d\Theta_i \; \rho_i(J_i,\Theta_i) \; J_i$$
$$= \int dJ_i \; \rho_i(J_i) \; J_i. \qquad (3)$$

The Jaynes prescription[3] is to introduce the information-theoretic (Gibbs-Shannon) entropy $S(\rho)$, as a functional of the system distribution function ρ:

$$S(\rho) = - \int \rho \ln \rho, \qquad (4)$$

and maximize it with respect to ρ, subject to the constraint (3). This determines the "best" ρ, as a parametric function of the given data $\{\bar{J}_i\}$. When we carry out this procedure, we obtain, not surprisingly, the Gibbs distribution:

$$\rho(J;\bar{J}) = \Pi_i (\bar{J}_i)^{-1} \exp\,(- J_i/\bar{J}_i). \qquad (5)$$

Since Eq. (4) can be considered as $S = - \langle \ln\rho \rangle$, we form $\langle \ln\rho \rangle$ from (5):

$$\langle \ln\rho \rangle \, (\bar{J}) = \Sigma_i \, (- \ln \bar{J}_i - 1), \qquad (6)$$

and finally obtain the desired expression (2), after discarding the constant term in (6). Thus the entropy associated with the data $\{\bar{J}_i\}$ is the information-theoretic entropy of the best distribution consistent with those data.

I am indebted to Steven M. Omohundro, not only for stressing the importance of Jaynes' principle, but also for providing, in his Ph.D. thesis,[4] the basic mathematical underpinning for that principle.

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