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INTERSECTING HYPERGRAPHS WITH LARGE COVER NUMBER

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Abstract. In their famous 1974 paper introducing the local lemma, Erdős and Lovász posed a question—later referred to by Erdős as one of his three favorite open problems: What is the minimum number of edges in an r -uniform, intersecting hypergraph with cover number r ? This question was solved up to a constant factor in Kahn’s remarkable 1994 paper. More recently, motivated by applications to Bollobás’ “power of many colours” problem, Alon, Bucić, Christoph, and Krivelevich introduced a natural generalization by imposing a space constraint that limits the hypergraph to use only n vertices. In this note, we settle this question asymptotically, up to a logarithmic factor in n/r in the exponent, for the entire range.

Keywords. Hypergraphs, cover number, intersecting

Mathematics Subject Classifications. 05D05

1. Introduction

The following classical problem was posed by Erdős and Lovász in their influential 1974 paper [EL74], which introduced the local lemma:

Question 1.1. What is the minimum number of edges in an r -uniform, intersecting hypergraph with cover number r ?

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Here, the *cover number* of a hypergraph is the minimum size of a set of vertices that intersects all the edges, and a hypergraph is *intersecting* if any two edges have a non-empty intersection. Following Erdős and Lovász, we will denote the answer to this question by $g(r)$.

In an intersecting r -uniform hypergraph, every edge is a cover, so the cover number is at most r . Therefore, the question asks for the minimum number of edges required to achieve the maximum cover number while maintaining the intersecting property. In addition, if we denote by n the number of vertices, ensuring that the cover number is at least r is equivalent to saying that any vertex subset of size $n - r + 1$ must contain an edge (this makes it an instance of the classical hypergraph Turán problem, see [Kee11, FR85, Sid95, dC91, Sud10, GPY94] for the rich history of this type of questions as well as its connections to covering designs and error correcting codes). One can view this as a (very weak) pseudorandom property, and indeed, random hypergraphs provide very good examples, provided one drops the intersecting requirement. On the other hand, being intersecting is a (very weak) structural assumption, so the question, in a certain sense, seeks sparse hypergraphs which exhibit some random-like properties as well as have some structure.

Erdős and Lovász proved in the original paper that

$$8r/3 - 3 \leq g(r) \leq 4r^{3/2} \log r,$$

where the upper bound is conditional on the existence of a projective plane of order $r - 1$, and r being sufficiently large. They conjectured that the true answer is linear, so that $g(r) = O(r)$. This was one of Erdős' three favorite combinatorial problems (see [Erd81]) alongside Erdős–Faber–Lovász (which has recently been proved in a remarkable paper [KKK⁺23] by Kang, Kelly, Kühn, Methuku, and Osthus) and the Sunflower Conjecture (which, while still open, has seen some remarkable progress recently [ALWZ21, Rao20, Tao20, BCW21]). The Erdős–Lovász conjecture was ultimately resolved by Kahn, first up to a lower order term in [Kah92] and then in full in [Kah94].

In a recent paper [ABCK24], Alon, Bucić, Christoph, and Krivelevich, motivated by applications to a certain “power of many colours” problem (first introduced by Bollobás [BG08]), generalized the question of Erdős and Lovász and asked what happens if one is space constrained.

Question 1.2. Let $n \geq 2r - 1 \geq 3$ be integers. What is the minimum number of edges in an n -vertex r -uniform intersecting hypergraph with cover number equal to r ?

We denote the answer to this question by $f(n, r)$. To get an initial feeling for the problem, we note that if $n \leq 2r - 2$, then any $r - 1$ vertices make a cover, so such hypergraphs do not exist. If $n = 2r - 1$, then one is forced to take a complete r -uniform hypergraph since the complement of any missing edge would be an $r - 1$ cover. This implies that $f(2r - 1, r) = \binom{2r-1}{r}$. On the other hand, when n is quadratic, Kahn's construction provides a tight linear bound. Note that an r -uniform hypergraph with $O(r)$ edges has at most $O(r^2)$ non-isolated vertices, so this follows easily from the result in [Kah94] even without delving deeper into the intricate construction. On the other hand, a more careful examination of the proof shows the construction, still using projective planes as an ingredient, does actually require a quadratic number of vertices. This leads to the natural question of how the behaviour of $f(n, r)$ transitions from being exponential in r when n is linear to being linear in r when n is quadratic.

An easy lower bound on $f(n, r)$ can be obtained via a standard double-counting argument, due to de Caen, which also gives the essentially best-known lower bounds for Turán numbers [DC83]. Here, on one hand, for any of the $\binom{n}{r-1}$ vertex subsets of size $r-1$, there must be an edge disjoint from it (or else this subset provides a cover of size less than r). On the other hand, an edge can be disjoint from at most $\binom{n-r}{r-1}$ such subsets showing immediately that

$$f(n, r) \geq \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}} \geq 2^{\Omega(r^2/n)}. \quad (1.1)$$

In this paper, we give an asymptotic answer to the question of Alon, Bucić, Christoph, and Krivelevich by showing that this lower bound is essentially tight.

Theorem 1.3. *For any $3 \leq 2r - 1 \leq n \leq O(r^2)$,*

$$f(n, r) = 2^{\tilde{\Theta}(r^2/n)}.$$

More precisely, we show that there exists an n -vertex, r -uniform hypergraph with cover number r and at most $(n/r)^{O(r^2/n)}$ edges, determining the answer up to a term logarithmic in n/r in the exponent.

For our upper bound on $f(n, r)$, we make use of a hypergraph product construction that preserves both the intersecting property and criticality in terms of cover number and allows us to interpolate between the complete graph construction and the one found by Kahn [Kah94] which are optimal at the respective ends of the regime. We describe this product and establish some of its properties relevant for us in Section 2. One downside of this construction is that the uniformity of the hypergraphs we construct is always composite, and it is surprisingly non-trivial to perform local modifications that preserve both of our desired properties (with one being increasing and one decreasing hypergraph property). Our final ingredient is such a local augmentation lemma which precisely suffices to remove any divisibility requirements on the uniformity. In Section 3, we put these arguments together to prove Theorem 1.3.

Notation. Given a hypergraph H , we denote by $V(H)$ and $E(H)$, its vertex and edge set, respectively. We will denote by $\tau(H)$ the size of a minimum vertex cover of H . All of our logarithms are in base two unless otherwise specified.

2. Wreath product of hypergraphs and its properties

We begin by defining a peculiar type of hypergraph product that preserves intersecting and cover criticality properties. We note that the use of this product in combinatorics dates back at least to the paper of Erdős and Lovász [EL74] and independently the PhD thesis of Frankl [Fra77], but also appears in group theory [Hup67]. A well-known example in combinatorics is the iterated Fano plane construction. See also [FPP24], for a recent application concerned with finding odd sunflowers.

Before giving a formal definition, let us describe the product. We start with two hypergraphs H_1 and H_2 and define their *wreath product* $H_1 \rtimes H_2$ to have vertex set consisting

of $|V(H_1)|$ many vertex-disjoint copies of $V(H_2)$, which we will refer to as *blocks* and identify each block with a vertex of H_1 . Its edges are all constructed as follows. We pick an edge e of H_1 and pick for each $v \in e$ an edge f_v of H_2 , then we make an edge by taking a disjoint union of the edges f_v , where the edge f_v is taken from the block corresponding to v . We now give a formal definition.

Definition 2.1. Given two hypergraphs H_1 and H_2 we define the *wreath product* hypergraph $H_1 \rtimes H_2$ by

$$\begin{aligned} V(H_1 \rtimes H_2) &:= \{(v_1, v_2) \mid v_1 \in V(H_1), v_2 \in V(H_2)\} \text{ and} \\ E(H_1 \rtimes H_2) &:= \{\{(v, u) \mid v \in e, u \in f_v\} \mid e \in E(H_1), \forall v \in e, f_v \in E(H_2)\}. \end{aligned}$$

We note that the operation is not commutative, that is $H_1 \rtimes H_2$ is not in general equal to $H_2 \rtimes H_1$. In the following lemmas, we establish several useful properties of the wreath product of uniform¹ hypergraphs. We begin with some immediate properties to help familiarize the reader with the product.

Lemma 2.2. *Let H_1 be an r_1 -uniform and H_2 an r_2 uniform hypergraph. Then,*

1. $|V(H_1 \rtimes H_2)| = |V(H_1)| \cdot |V(H_2)|$.
2. $|E(H_1 \rtimes H_2)| = |E(H_1)| \cdot |E(H_2)|^{r_1}$
3. $H_1 \rtimes H_2$ is $r_1 r_2$ -uniform.

Proof. Item 1 is immediate from the definition. Every edge of $H_1 \rtimes H_2$ is a disjoint union of r_1 edges of H_2 , so has size $r_1 r_2$, establishing Item 3. Furthermore, we have $|E(H_1)|$ choices for e and given e , $|E(H_2)|$ choices for each f_v for each of the $|e| = r_1$ of $v \in e$. This gives that the number of edges of $H_1 \rtimes H_2$ equals $|E(H_1)| \cdot |E(H_2)|^{r_1}$, establishing Item 2. $\square$

We establish two key properties of the wreath product in the following two lemmas.

Lemma 2.3. *If H_1 and H_2 are intersecting hypergraphs, then so is $H_1 \rtimes H_2$.*

Proof. Let us take two arbitrary edges $X = \{(v, u) \mid v \in e, u \in f_v\}$ defined by some $e \in E(H_1)$ and $f_v \in E(H_2)$ for all $v \in e$ and $Y = \{(v, u) \mid v \in e', u \in f'_v\}$ defined by some $e' \in E(H_1)$ and $f'_v \in E(H_2)$ for all $v \in e'$. Since H_1 is intersecting, there is some $v \in e \cap e'$. Since H_2 is intersecting, there is some $u \in f_v \cap f'_v$. For this v and u we have $(v, u) \in X \cap Y$. Since X and Y were arbitrary, this shows $H_1 \rtimes H_2$ is indeed intersecting. $\square$

A key property we will use is that the wreath product is multiplicative in terms of the cover number.

Lemma 2.4. *Let H_1 and H_2 be hypergraphs. Then,*

$$\tau(H_1 \rtimes H_2) = \tau(H_1)\tau(H_2).$$

¹We note that we will focus on the uniform case, although all the lemmas have slightly more complicated, general versions as well.

Proof. Let $\tau_i = \tau(H_i)$. Suppose T is a cover of $H_1 \rtimes H_2$ and suppose towards a contradiction that $|T| < \tau_1 \tau_2$. Let $S \subseteq V(H_1)$ consist of vertices v such that T intersects the block corresponding to v in less than τ_2 vertices. Note that $V(H_1) \setminus S$ consists of less than τ_1 vertices, as otherwise, we would have $|T| \geq \tau_1 \cdot \tau_2$. This implies that $V(H_1) \setminus S$ is not a cover for H_1 , and hence there exists an edge $e \in E(H_1)$ such that $e \subseteq S$. This in turn gives that for any $v \in e$ there exists an $f_v \in E(H_2)$ such that $(v, u) \notin T$ for any $u \in f_v$. With this choice of e and f_v 's for $v \in e$, we get an edge $\{(v, u) \mid v \in e, u \in f_v\}$ which is not covered by T . This is a contradiction showing $\tau(H_1 \rtimes H_2) \geq \tau_1 \tau_2$.

To see the upper bound, pick a cover T_i of H_i of size τ_i for $i = 1, 2$. Now let $T := \{(v, u) \mid v \in T_1, u \in T_2\}$. Note that $|T| = \tau_1 \tau_2$ and we claim it is a cover of $H_1 \rtimes H_2$. Consider an edge $X := \{(v, u) \mid v \in e, u \in f_v\}$ defined by some $e \in E(H_1)$ and $f_v \in E(H_2)$ for all $v \in e$. Note that since T_1 is a cover of H_1 , there exists $v \in e \cap T_1$. Since T_2 is a cover of H_2 , there exists $u \in T_2 \cap f_v$. Now $(v, u) \in T \cap X$ showing T covers X . Since X was arbitrary, this shows T is indeed a cover. $\square$

3. Intersecting hypergraphs with high cover number

We say that an r -uniform hypergraph is *critical* if it is intersecting and has cover number equal to r . This definition is motivated by the observation that any intersecting r -uniform hypergraph always has cover number at most r .

Next, we state formally Kahn's celebrated result, mentioned in the introduction, which provides one of the two hypergraphs used in our product construction—the other being the complete hypergraph.

Theorem 3.1 (Kahn, [Kah94]). *There exists $B > 0$ such that for any $r \geq 2$ there exists an r -uniform critical hypergraph with at most Br edges.*

At our precision level, one could use a simpler, albeit weaker, construction such as a projective plane. We now state and prove the product lemma, which allows us to interpolate between the two examples.

Lemma 3.2. *Given r_1 - and r_2 -uniform critical hypergraphs H_1 and H_2 , there exists an $r_1 r_2$ -uniform critical hypergraph with $|V(H_1)| \cdot |V(H_2)|$ vertices and $|E(H_1)| \cdot |E(H_2)|^{r_1}$ edges.*

Proof. Let our new hypergraph be $H_1 \rtimes H_2$. By Lemma 2.2, its number of vertices, edges, and uniformity are as stated. Lemmas 2.3 and 2.4 ensure that it is critical. $\square$

This result suffices to prove our theorem when the uniformity is the product of small primes, since the uniformity of the wreath product graph will be the product of smaller uniformities. However, if our desired uniformity is not of this form—for instance, if it is a prime—Lemma 3.2 cannot yield a hypergraph with the desired uniformity and number of vertices. To address this, we first construct a hypergraph with uniformity close to the target, then adjust using the following augmentation lemma. This lemma permits a small adjustment to the uniformity at the cost of significantly increasing the number of edges. Since we only apply it a few times, the total edge count remains on the same order as that required by the product construction.

Lemma 3.3. *If there exists an r -uniform critical hypergraph H , then there exists an $(r + 1)$ -uniform critical hypergraph with $|V(H)| + r + 1$ vertices and at most $(r + 1) \cdot |E(H)| + 1$ edges.*

Proof. We build our new hypergraph H' by taking $V(H')$ to consist of $V(H)$ together with a set S of $r + 1$ new vertices. For every edge $e \in E(H)$ and every vertex $v \in S$, we create an edge $e \cup v$. Finally, we add S as an edge of H' .

This construction yields an $(r + 1)$ -uniform hypergraph H' with $|V(H)| + r + 1$ vertices and $|E(H)| \cdot |S| + 1 = |E(H)| \cdot (r + 1) + 1$ edges. Additionally, H' is intersecting. To see this, observe that any two edges of the form $e \cup v, f \cup u$ (with $e, f \in E(H)$ and $v, u \in S$) intersect because e and f intersect in H . The edge S intersects every other edge since each edge $e \cup \{v\}$ includes some vertex $v \in S$.

Finally, we show the cover number of H' is $r + 1$. Since H' is intersecting, the cover number is at most $r + 1$. Suppose, for contradiction, that there is a vertex cover T of H' with size r . Because S is an edge of H' , T must contain at least one vertex from S ; thus, $|T \setminus S| \leq r - 1$. However, since $\tau(H) = r$, there exists $e \in E(H)$ not covered by $T \setminus S$. Choose $v \in S \setminus T$ (which exists because $|S| = r + 1 > r = |T|$). Then, the edge $e \cup v$ in H' is not covered by T , a contradiction. $\square$

We are now ready to combine our results and prove the following slightly refined version of Theorem 1.3.

Theorem 3.4. *There exists $C > 0$ such that for any $n \geq 2r - 1 \geq 3$ there exists an n vertex r -uniform critical hypergraph with at most $\max\{(n/r)^{Cr^2/n}, Cr\}$ edges.*

Proof. Let B be the maximum of the constant provided by Theorem 3.1 and the number 4, and choose $C = 32B$. We may assume that $n \leq Br^2$; otherwise, the hypergraph of Theorem 3.1 has at most Br edges and at most $Br^2 \leq n$ vertices, so by padding with isolated vertices if necessary, we obtain the desired hypergraph. Similarly, we may assume $n \geq 8Br$; if not, the hypergraph $K_{2r-1}^{(r)}$ augmented with isolated vertices $n - 2r + 1$ produces a hypergraph with at most $4^r \leq (n/r)^{4r} \leq (n/r)^{32Br^2/n}$ edges. Thus, from now on, we will assume that $8Br \leq n \leq Br^2$.

Let H_1 be the complete r_1 -uniform hypergraph on $2r_1 - 1$ vertices padded with an extra isolated vertex. Note that H_1 is critical, has $2r_1$ vertices, and has less than 4^{r_1} edges. Let H_2 be an r_2 -uniform critical hypergraph with at most Br_2 edges, provided by Theorem 3.1. Note that such a hypergraph has at most Br_2^2 (non-isolated) vertices, and we may remove the isolated ones.

We first construct $H = H_1 \times H_2$, which is $r_1 r_2$ -uniform, has at most $2Br_1 r_2^2$ vertices, and has at most $4^{r_1} \cdot (Br_2)^{r_1} = (4Br_2)^{r_1}$ edges.

In order to keep our augmentation step within the correct order of magnitude, we split our analysis into two cases. In the first case, if $n < 2Br^{3/2}$, we set

$$r_2 := \left\lfloor \frac{n}{4Br} \right\rfloor \geq 2, \quad r_1 := \left\lfloor \frac{r}{r_2} \right\rfloor \geq 4, \quad t := r - r_1 r_2 = r \bmod r_2.$$

Note that $0 \leq t < r_2$.

We now apply Lemma 3.3 a total of t times to increase the uniformity to precisely r . By our choice of r_1, r_2 , the final number of vertices is at most

$$2Br_1r_2^2 + t \cdot r \leq 2Brr_2 + r_2 \cdot r \leq 3Brr_2 < n,$$

which we can pad with isolated vertices if necessary to reach exactly n . The number of edges increases to at most

$$r^{r_2} \cdot (4Br_2)^{r_1} \leq r^{n/(4Br)} \cdot (n/r)^{r/r_2} < (n/r)^{2Br^2/n} \cdot (n/r)^{8Br^2/n} \leq (n/r)^{10Br^2/n},$$

where the first inequality follows from the definitions of r_1, r_2 and the second from the assumption $3 \leq n^2/r^2 \leq 4B^2r$ (which implies $\frac{n^2/r^2}{\log(n^2/r^2)} \leq \frac{4B^2r}{\log(4B^2r)} \leq \frac{4B^2r}{\log r}$ and hence $r^{n/(4Br)} \leq (n^2/r^2)^{Br^2/n}$) along with the fact that $r_2 \geq n/(8Br)$.

If $n \geq 2Br^{3/2}$, we set

$$r_1 := \left\lfloor \frac{8Br^2}{n} \right\rfloor \geq 8, \quad r_2 := \left\lfloor \frac{r}{r_1} \right\rfloor \geq 1, \quad t := r - r_1r_2 = r \bmod r_1.$$

Note that $0 \leq t < r_1$.

We now apply Lemma 3.3 a total of t times to increase the uniformity to precisely r . The final vertex count is at most

$$2Br_1r_2^2 + t \cdot r \leq 2Brr_2 + r_1 \cdot r \leq \frac{2Br^2}{r_1} + \frac{8Br^3}{n} \leq \frac{2Br^2}{4Br^2/n} + \frac{2n}{B} = \frac{n}{2} + \frac{2n}{B} \leq n,$$

where we used $r_1 \geq 4Br^2/n$ in the third inequality and $B \geq 4$ in the final inequality. The number of edges increases to at most

$$r^{r_1} \cdot (4Br_2)^{r_1} = (4Brr_2)^{r_1} \leq (4Br^2)^{8Br^2/n} \leq (n/r)^{32Br^2/n},$$

using our case assumption (which implies $n/r \geq 2B\sqrt{r}$) in the final inequality. $\square$

4. Concluding remarks

In this paper, we answer a vertex-constrained version of the original Erdős–Lovász problem, concerned with finding the minimum number of edges in an r -uniform, intersecting hypergraph on n vertices with cover number r —posed by Alon, Bucić, Christoph, and Krivelevich. We resolve the problem up to a $\log(n/r)$ factor in the exponent by showing that $f(n, r) \leq (n/r)^{O(r^2/n)}$, for $n \leq O(r^2)$.

It is natural to wonder whether the simple lower bound (1.1) telling us $f(n, r) \geq \binom{n}{r-1} / \binom{n-r}{r-1}$ might be tight. This turns out not to be the case.

Proposition 4.1. *Let $2r - 1 \leq n < r^2$. Let H be an n -vertex, r -uniform hypergraph with cover number at least r . Then,*

$$|E(H)| > \frac{n}{5r} \cdot \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}} \geq \frac{n}{r} \cdot e^{\Omega(r^2/n)}$$

Proof. The stated inequality holds by (1.1) (which does not make use of the intersecting property) if $n < 5r$, so we may assume $n \geq 5r$. Combined with the assumed r^2 upper bound, we may also assume $r > 5$.

Let $t = \lfloor \frac{n}{r} \rfloor \leq r$. Pick any set of $r - t$ vertices of H . There must exist at least t edges outside it: if there were at most $t - 1$ edges outside this set, we can cover them with $t - 1$ vertices and obtain a cover of the whole hypergraph of size $r - t + t - 1 = r - 1$.

There are $\binom{n-r}{r-t}$ sets of size $r - t$ living outside any edge, so each edge is counted at most $\binom{n-r}{r-t}$ times. Therefore, it follows that

$$\begin{aligned} |E(H)| &\geq t \cdot \frac{\binom{n}{r-t}}{\binom{n-r}{r-t}} = t \cdot \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}} \cdot \frac{(n-2r+t) \cdots (n-2r+2)}{(n-r+t) \cdots (n-r+2)} \\ &\geq t \cdot \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}} \cdot \left(1 - \frac{r}{n-r+2}\right)^{t-1} \\ &\geq t \cdot \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}} \cdot e^{\frac{-7(t-1)r}{6(n-r+2)}} \geq \frac{t}{e^{7/6}} \cdot \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}} > \frac{n}{5r} \cdot \frac{\binom{n}{r-1}}{\binom{n-r}{r-1}}, \end{aligned}$$

where in the third inequality we used that $1 - x \geq e^{-7x/6}$ with $x = \frac{r}{n-r+2} \leq \frac{1}{4}$, in the fourth inequality that $t - 1 \leq \frac{n-r+2}{r}$, and in the final one that $t \geq \frac{5n}{6r}$. $\square$

We made no particular effort to optimize the constant in the above bound. A more careful analysis of the argument shows that for $n \geq 3r$, we get a (slightly) stronger bound compared to (1.1). Moreover, when $n \geq r^2$, the minimum number of edges is clearly r (e.g. a hypergraph consisting of r vertex-disjoint edges). Finally, if H is known to be intersecting, the above bound can be improved by roughly a factor of two since in an intersecting hypergraph one can cover any set of up to $2t - 2$ edges with $t - 1$ vertices. It would be interesting to obtain a more substantial improvement from the intersecting assumption.

As mentioned in the introduction, the version of the problem without the intersecting assumption leads us to Turán numbers. The Turán number $T(n, k, r)$ is defined as the maximum number of edges in an r -uniform hypergraph H on n vertices that does not contain $K_k^{(r)}$. Equivalently, every k -subset of vertices of H omits some r -subset as an edge. This condition is equivalent to the complementary hypergraph $\bar{H} := \{S \in \binom{[n]}{r} \mid S \notin H\}$ having the (k, r) -covering property—that is, every set of k vertices should contain an edge of $\bar{H}$. Let $U(n, k, r)$ denote the minimum number of edges in an r -uniform hypergraph on n vertices with the (k, r) -covering property. Then, $U(n, k, r) = \binom{n}{r} - T(n, n - r + 1, r)$. Since having (k, r) -covering property is equivalent to having cover number at least $n - k + 1$, determining our function $f(n, r)$ is equivalent to determining the minimum number of edges in an intersecting r -uniform hypergraph satisfying the $(n - r + 1, r)$ -covering property. This implies

$$f(n, r) \geq U(n, n - r + 1, r) = \binom{n}{r} - T(n, n - r + 1, r).$$

We note that Proposition 4.1 gives a lower bound on $U(n, n - r + 1, r)$.

Both the Turán and the covering questions have been extensively studied over the years², although most of the attention was afforded to instances with r and k being fixed. The instance relevant for us is when $k = n - r + 1$ and $2r - 1 \leq n \leq O(r^2)$. In this regime, both r and k are growing with n (with k very close to n). In particular, we are interested in Turán problem for almost spanning hypergraphs. As mentioned above, it is easy to see that $U(n, n - r + 1, r) = r$ for $n \geq r^2$. For our regime, just below this threshold, Sidorenko [Sid82] shows that

$$T(n, n - r + 1, r) = \begin{cases} 3r - \left\lfloor \frac{2n}{r} \right\rfloor & \text{if } r \text{ even } \quad \frac{3r^2}{4} \leq n \leq r^2 \\ 3r - \left\lfloor \frac{2(n-r)}{r-1} \right\rfloor & \text{if } r \text{ odd } \quad \frac{3r^2+r}{4} \leq n \leq r^2. \end{cases}$$

The best upper bound we are aware of comes from repeatedly taking an edge belonging to the most $n - r + 1$ vertex sets not containing any edge already. Since the probability that a random edge belongs to an $n - r + 1$ vertex set is $p = \binom{n-r+1}{r} / \binom{n}{r}$, we can always pick out a new edge that belongs to at least this proportion of the $n - r + 1$ sets not yet containing one. Hence, after i iterations we are left with at most $(1 - p)^i \binom{n}{n-r+1} \leq e^{-ip} \binom{n}{n-r+1}$ sets of $n - r + 1$ vertices not containing an edge. This implies we can hit them all with $p^{-1} \log \binom{n}{n-r+1}$ edges. Putting together this upper bound with the lower bound from Proposition 4.1, we get

$$\frac{n}{5r} \leq \frac{U(n, n - r + 1, r)}{\binom{n}{r-1} / \binom{n-r}{r-1}} < 2r \log \frac{n}{r}.$$

It would be interesting to close this gap. Returning to our question of separating $f(n, r)$ and $U(n, n - r + 1, r)$, with this upper bound in mind, it would suffice to improve Proposition 4.1 by making use of the intersecting property to improve the gain factor by only $2r \log \frac{n}{r}$.

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References

- [ABCK24] Noga Alon, Matija Bucić, Micha Christoph, and Michael Krivelevich. The power of many colours. In *Forum of Mathematics, Sigma*, volume 12, pages e118 1–23. Cambridge University Press, 2024. doi:10.1017/fms.2024.120.
- [ALWZ21] Ryan Alweiss, Shachar Lovett, Kewen Wu, and Jiapeng Zhang. Improved bounds for the sunflower lemma. *Annals of Mathematics*, 194(3):795–815, 2021. doi:10.4007/annals.2021.194.3.5.
- [BCW21] Tolson Bell, Suchakree Chueluecha, and Lutz Warnke. Note on sunflowers. *Discrete Mathematics*, 344(7):Paper No. 112367, 3, 2021. doi:10.1016/j.disc.2021.112367.

²See for example the survey [Sid95] where all the different perspectives we discuss here are explored.

- [BG08] Béla Bollobás and András Gyárfás. Highly connected monochromatic subgraphs. *Discrete Mathematics*, 308(9):1722–1725, 2008. doi:10.1016/j.disc.2006.01.030.
- [DC83] Dominique De Caen. Extension of a theorem of Moon and Moser on complete subgraphs. *Ars Combinatoria*, 16:5–10, 1983.
- [dC91] Dominic de Caen. The current status of Turán’s problem on hypergraphs. *Extremal problems for finite sets*, 3:187–197, 1991.
- [EL74] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Colloquia Mathematica Societatis János Bolyai*, 10:609–627, 1974.
- [Erd81] P. Erdős. On the combinatorial problems which I would most like to see solved. *Combinatorica*, 1(1):25–42, 1981. doi:10.1007/BF02579174.
- [FPP24] Peter Frankl, János Pach, and Dömötör Pálvölgyi. Odd-sunflowers. *J. Combin. Theory Ser. A*, 206:Paper No. 105889, 14, 2024. doi:10.1016/j.jcta.2024.105889.
- [FR85] Peter Frankl and Vojtěch Rödl. Lower bounds for Turán’s problem. *Graphs and Combinatorics*, 1:213–216, 1985. doi:10.1007/BF02582949.
- [Fra77] P. Frankl. *Extremalis halmazrendszerek (in Hungarian)*. PhD thesis, MTA, 1977.
- [GPY94] Dan Greene, Michal Parnas, and Frances Yao. Multi-index hashing for information retrieval. In *Proceedings 35th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 722–731. IEEE, 1994. doi:10.1109/SFCS.1994.365720.
- [Hup67] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, Heidelberg, 1967. doi:10.1007/978-3-642-64981-3.
- [Kah92] Jeff Kahn. On a problem of Erdős and Lovász: random lines in a projective plane. *Combinatorica*, 12(4):417–423, 1992. doi:10.1007/BF01305234.
- [Kah94] Jeff Kahn. On a problem of Erdős and Lovász. II. $n(r) = O(r)$. *Journal of the American Mathematical Society*, 7(1):125–143, 1994. doi:10.2307/2152722.
- [Kee11] Peter Keevash. Hypergraph Turán problems. *Surveys in Combinatorics*, 392:83–140, 2011. doi:10.1017/cbo9781139004114.004.
- [KKK⁺23] Dong Kang, Tom Kelly, Daniela Kühn, Abhishek Methuku, and Deryk Osthus. A proof of the Erdős–Faber–Lovász conjecture. *Annals of Mathematics*, 198(2):537–618, 2023. doi:10.4007/annals.2023.198.2.2.
- [Rao20] Anup Rao. Coding for sunflowers. *Discrete Analysis*, pages Paper No. 2, 8, 2020. doi:10.19086/da.11887.
- [Sid82] AF Sidorenko. *Extremal constants and inequalities for distributions of sums of random vectors*. PhD thesis, Moscow State University, 1982.
- [Sid95] Alexander Sidorenko. What we know and what we do not know about Turán numbers. *Graphs and Combinatorics*, 11(2):179–199, 1995. doi:10.1007/BF01929486.

- [Sud10] Benny Sudakov. Recent developments in extremal combinatorics: Ramsey and Turán type problems. In *Proceedings of the International Congress of Mathematicians*, pages 2579–2606. World Scientific, 2010.
- [Tao20] Terence Tao. The Sunflower lemma via Shannon entropy, 2020. What’s New [Terence Tao Blog]. URL: <https://terrytao.wordpress.com/2020/07/20/the-sunflower-lemma-via-shannon-entropy>.