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ISBN

9798189270987

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Publication Date

2026-05-01

Peer reviewed|Thesis/dissertation

Scalar Curvature Under Blowup and Positive Scalar Curvature Kähler Surfaces

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Associate Professor Richard Bamler, Co-chair
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Professor John Lott
Professor Maciej Zworski

Spring 2026

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Abstract

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Doctor of Philosophy in Mathematics

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Associate Professor Richard Bamler, Co-chair

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Kähler manifolds exhibit elegant relationships between their curvature and their algebro-geometric structure. Two foundational examples of this phenomenon are that the Ricci curvature of a Kähler metric represents its first Chern class, and Kodaira's embedding theorem that ampleness of a line bundle is characterized by the existence of a positive curvature metric [22]. These two results, together with Yau's classic resolution of the Calabi conjecture [37], characterize Kähler manifolds admitting Kähler metrics with positive Ricci curvature as smooth varieties with ample anti-canonical bundle, that is, smooth Fano varieties.

While the Ricci curvature of Kähler metrics has been very successfully and nearly exhaustively studied, there are many open questions related to the scalar curvature of Kähler metrics. For example, there is not a satisfactory characterization of Kähler manifolds admitting Kähler metrics with positive scalar curvature. An old result of Yau says that a Kähler manifold with positive scalar curvature is such that it has negative Kodaira dimension [36]. However, it has been only a conjecture of Lebrun [24] that the converse should hold in the case of Kähler surfaces, and the story beyond two complex dimensions is still wide open.

In this dissertation, we show that the converse does indeed hold for surfaces. The argument relies on the classification of complex surfaces, which in particular says that all negative Kodaira dimension surfaces are the projective plane, a holomorphic bundle of projective lines over a compact Riemann surface, all of which admit positive scalar curvature Kähler metrics, or are obtained by a sequence of blowups of one of these spaces. It is a result of Hitchin that admitting positive scalar curvature Kähler metrics is a property preserved under point blowups in dimensions three and higher [20]. We therefore resolve Lebrun's conjecture by showing the following much stronger statement: given a Kähler manifold of any dimension, on the blowup of any point there exists a Kähler metric with scalar curvature approximating the scalar curvature of the original metric arbitrarily closely in C^0 .

To my parents, Frederick and Diana.

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Acknowledgments

First, I would like to thank my advisor, Song Sun. What I find most impressive about Song is that, despite his stature in our field, he treats his students as intellectual equals. He has been consistently encouraging, attentive to my progress, and generous with his time and ideas, while fostering independence. I owe the uniquely satisfying experience of contributing something meaningful to the scientific record to him.

I am grateful to Ronan Conlon, Tamás Darvas, Claude LeBrun, Yuji Odaka, Wenhao Ou, Rareş Răşdeaconu, Antoine Song, Ioana Şuvaina, Xiaokui Yang, Ruobing Zhang, and Maciej Zworski, variously for helpful lectures, encouragement, interest in my work, and for being excellent examples to a young mathematician. I especially thank Richard Bamler and John Lott for creating a welcoming environment for young differential geometers at Berkeley, and Gábor Székelyhidi for patiently explaining his work on blowing up extremal Kähler metrics, which helped me complete this thesis.

It was my great fortune that Song’s move to Zhejiang University gave me an unexpected alternate home in Hangzhou, China. Thanks again to Song for making this transition so smooth, and to “the solar system”—Xuemiao Chen, Holly Mandel, Charlie Cifarelli, Hongyi Liu, Junsheng Zhang, Carlos Esparza, Yifan Chen, Mingyang Li, Yueqing Feng, and Keshu Zhou—for being mentors, seminar speakers, translators, guides, travel companions, and friends. I could not have asked for a better academic family.

Berkeley has been the first place outside Michigan that truly felt like home, thanks largely to the people (though the weather helps). There are too many to list, but I especially thank Danny Chupin, Sean Cody, Ethan Dlugie, Swapnil Garg, Michael Heinz, Robin Huang, Daigo Ito, Aidan Kelley, Izak Oltman, Yuji Okitani, Tahsin Saffat, Kenny Schefers, Tobias Shin, Zack Stier, and Vilas Winstein for being great room/building/officemates, mentors, workout partners, and loyal friends. Thank you to everyone who made these past five years so memorable.

I thank my teachers at Southgate Public Schools in Michigan for taking a special interest in me and helping set me on this path. Thank you to my late grandparents, John and Rosemary, for their moral example and their commitment to ensuring that all their grandchildren receive a university education, and also the entire extended Fred family, and my Uncle Rick, for their love and support. Thanks to my brothers, Erik, Jonathon, and Cory, whose examples made pursuing an elite and graduate education feel possible from a young age. Finally, and most of all, thank you to my mom and dad for always placing my needs above their own, for their dedication to supporting my education, and for their unconditional love; I owe all of my successes to you.

Chapter 1

Introduction

Kähler geometry represents what is as close as one can get to an exact intersection between algebraic geometry and differential geometry. This is reflected in the definition: Kähler manifolds are complex manifolds with a Riemannian structure with respect to which the complex structure is orthogonal and parallel. Accordingly, through Kähler metrics we often see a kind of dictionary appearing between the curvature of Kähler manifolds and their vector bundles, and the algebraic geometry of the underlying manifold.

The part of this dictionary concerning the Ricci curvature of Kähler metrics has been very fruitfully investigated. First of all, the Ricci form of a Kähler metric can be viewed as the curvature of the anti-canonical bundle induced by the Kähler metric, and therefore represents (a positive multiple of) the first Chern class. Yau's resolution of the Calabi conjecture gives that any form which represents the first Chern class can be realized as the Ricci form of a unique Kähler metric in each Kähler class [37]. Together with the Kodaira embedding theorem that ampleness of a line bundle corresponds to the existence of a positive Hermitian metric [22], this says the complex manifolds which furnish Kähler metrics with positive Ricci curvature are precisely the smooth Fano varieties, an algebro-geometric characterization. This work by Yau also answered the negative and scalar-flat Kähler-Einstein problems as those manifolds whose canonical bundles are ample and unitary-flat, respectively, and the positive case was settled by Chen, Donaldson, and Sun in terms of a geometric invariant theory-type algebro-geometric stability condition [10] [11] [12].

The scalar curvature of a (in most cases we are restricting to the compact case) Kähler manifold (M, ω) ,

$$S(\omega) := -\omega^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \det(\omega_{p\bar{q}}),$$

certainly has a similarly strong relationship with algebraic geometry as the Ricci curvature. For example, there is the formula

$$\int S(\omega) \frac{\omega^n}{n!} = \frac{2\pi}{(n-1)!} c_1(M) \cup [\omega]^{n-1},$$

which shows that the total scalar curvature (that is, the integral of the scalar curvature) is an invariant of the Kähler class and the homotopy-type of the complex structure, and the

interpretation by Fujiki and Donaldson of the scalar curvature as a moment map on the space of almost-complex structures [17] [15]. However, for some reasons including, technically, that scalar curvature is analytically much more difficult to study due to the prevalence of fourth order equations, and heuristically, that it is simply a much weaker and more flexible measure of curvature than Ricci curvature, there are many remaining open questions. The question of prescribed scalar curvature of Kähler metrics is still completely open, including just the case of constant scalar curvature, though considerable progress on estimates for equations related to constant scalar curvature Kähler metrics has been made in recent years [9].

The main motivation for this dissertation is the question of when a Kähler manifold admits a Kähler metric with scalar curvature positive everywhere. The starting point is an old result of Yau, which shows that even just positive total scalar curvature places quite a heavy restriction on the manifold, though still significantly less so than positive Ricci curvature:

Proposition 1.0.1 (Yau [36]). *Let M be a compact complex manifold for which there exists a Kähler metric of positive total scalar curvature. Then M has Kodaira dimension $-\infty$, that is, the canonical bundle of M and all of its powers have no global holomorphic sections. Moreover, the converse is true in the case of compact Kähler surfaces.*

This is done via a Bochner-type vanishing argument. The converse in the case of surfaces is due to the classification of complex surfaces, the first part of which says that any compact complex surface can be obtained from a minimal one (i.e. a smooth compact complex surface with no embedded $\mathbb{P}^1$ with self-intersection -1) via a finite sequence of blowups, a surgery which produces a new complex manifold by replacing a point with an embedded projective space of codimension 1. Secondly, for compact Kähler surfaces of negative Kodaira dimension, it gives the minimal surfaces as being the projective plane, or a holomorphic $\mathbb{P}^1$ -bundle over a compact Riemann surface. A positive scalar curvature Kähler metric can be constructed explicitly on all such minimal surfaces, and then showing that positive total scalar curvature is preserved under blowup is just a cohomological calculation.

Naturally, the place to get started on the problem of pointwise positive scalar curvature Kähler metrics is Kähler surfaces, where we have this nice classification of negative Kodaira dimension. The minimal surfaces are already taken care of, so we really are pushed onto the questions of whether one can construct a Kähler metric on the blowup whose scalar curvature closely approximates that of the original Kähler metric. Hitchin first investigated this question specifically for positive scalar curvature Kähler metrics, with slight improvements in the surface case made by Sung:

Proposition 1.0.2 (Hitchin [20], Sung [30]). *Let M be a compact complex manifold for which there exists a Kähler metric of positive scalar curvature. Then if M has complex dimension at least 3, the blowup of any point also furnishes such a metric. In the case of complex dimension 2, any negative Kodaira dimension surface that can be obtained by no more than two successive blowups of a minimal surface has a positive scalar curvature Kähler metric.*

Later, LeBrun noted that a minimal Kähler surface need only admit a pointwise positive scalar curvature Riemannian metric to conclude that the surface must have negative Kodaira dimension [24], which was later extended to the non-minimal case by Friedman and Morgan [16]. Due to the gluing theorem of Gromov and Lawson for Riemannian metrics [19], this showed that negative Kodaira dimension surfaces are at least characterized among all Kähler surfaces for admitting positive scalar curvature Riemannian metrics. LeBrun formalized the expectation that all negative Kodaira dimension Kähler surfaces should admit positive scalar curvature Kähler metrics with the conjecture that the existence of a Riemannian positive scalar curvature metric, a Kähler positive scalar curvature metric, and negative Kodaira dimension should all coincide for compact Kähler surfaces. However, the preservation of positive scalar curvature for Kähler metrics under blowup of surfaces was still missing.

By today's standards, the construction of Kähler metrics by Hitchin and Sung is rather naive. In some sense, near the blowup point, the metrics they construct on the blowup behave somewhat like

$$\omega_{euc} + \epsilon p^* \omega_{FS},$$

where $p : Bl_0 \mathbb{C}^n \rightarrow \mathbb{CP}^{n-1}$ is the natural map, ω_{FS} the Fubini-Study metric, and $\epsilon > 0$ small. This metric on $Bl_0 \mathbb{C}^n$ has positive scalar curvature for $n \geq 3$ which allows the argument to go through, but in $n = 2$ this metric is the scalar flat Burns-Simanca metric. Furthermore, their construction in $n = 2$ is heavily dependent on the initial metric.

Luckily, in the past couple decades there have been many advances in constructing Kähler metrics with specified curvature properties on spaces that can be viewed as being composed of “model” pieces, so-called “gluing” methods. This has in particular been applied to the problem of constructing constant scalar curvature, or more generally, extremal Kähler metrics on the blowup of a point across [3] [4] [5] [33] [32] [14]. Among these one can extract a few slightly different approaches for constructing Kähler metrics on a blowup with specified scalar curvature properties, the one we manage to apply to our situation is that of Székelyhidi, especially as described in [31]. We'll state a simple version of one result from these papers here. If M is a complex manifold and $p \in M$, let $\pi : Bl_p M \rightarrow M$ denote the blowdown map, and $[E] \in H^2(Bl_p M, \mathbb{C})$ denote the Poincaré dual of the exceptional divisor.

Proposition 1.0.3. *(Arezzo-Pacard, Székelyhidi) If (M, ω) is a compact Kähler manifold with no holomorphic vector fields, and $S(\omega)$ is constant, then for any ϵ sufficiently small, the class $\pi^*[\omega] - \epsilon^2[E]$ contains a Kähler metric with constant scalar curvature.*

The broad strategy is to first construct an initial family of metrics ω_ϵ on the blowup, and then perturb them by the complex Hessian of a real-valued function to solve the constant scalar curvature equation. Adapting the approach to our setting, we then want to solve some appropriate equation for ϕ of the form

$$S(\omega_\epsilon + \sqrt{-1} \partial \bar{\partial} \phi) - S(\omega) = \text{something shrinking as } \epsilon \rightarrow 0.$$

This naturally leads to studying estimates on the linearized scalar curvature, which is the reason for the no holomorphic vector fields assumption when blowing up constant scalar

curvature Kähler metrics, as the linearized scalar curvature is self-adjoint at a constant scalar curvature Kähler metric and its kernel is identified with a certain subspace of holomorphic vector fields. However, our problem allows for a bit more flexibility, and so we can perturb away any issues with non-invertibility, but for technical reasons we are only able to produce estimates on the adjoint operator. Luckily, the detriment that was the scalar flatness of the Burns-Simanca metric in Hitchin and Sung's approach will instead come to the rescue here. Finally, our approach works in all dimensions, though there are some additional difficulties in complex dimension two as in the case of extremal metrics, coming from the coincidence of the Burns-Simanca metric with the more naive metric on $Bl_0\mathbb{C}^n$ in dimension two. That being said, after identifying an equation that is solvable for small ϵ and fits this paradigm, we successfully arrive at our main result. Note that we can view any function on M as also a function on Bl_pM simply by pulling back along the blowdown map.

Theorem A (Main Result). *Let (M, ω) be a compact Kähler manifold of complex dimension at least 2. For any $p \in M$, there exists a sequence ω_i of Kähler metrics on Bl_pM with*

$$|S(\omega_i) - S(\omega)| \rightarrow 0 \text{ uniformly on } Bl_pM \text{ as } i \rightarrow \infty,$$

whose Kähler classes $[\omega_i] \in H^{1,1}(Bl_pM)$ converge to $\pi^[\omega]$.*

In particular, if (M, ω) has positive scalar curvature, then all its blowups also have Kähler metrics with positive scalar curvature. It's a happy discovery that this is much stronger than we originally set out to do, and we suspect that there are much deeper links between the behavior of the scalar curvature of birationally related manifolds for the community to uncover in the future.

For Kähler surfaces, this finishes the previously mentioned conjecture of LeBrun, completing a classification of compact positive scalar curvature Kähler surfaces and drawing an equivalence with positive scalar curvature Riemannian geometry. We now state it as a theorem.

Theorem B. *Let M be a compact complex surface which admits a Kähler metric. Then the following are equivalent.*

1. *M has a Kähler metric for which the total scalar curvature is positive.*
2. *M has a Kähler metric of positive scalar curvature.*
3. *M has a Riemannian metric of positive scalar curvature.*
4. *M has Kodaira dimension $-\infty$.*
5. *M is obtained from $\mathbb{CP}^2$ or $\mathbb{P}(E)$ by a finite sequence of blowups, where E is a rank 2 holomorphic vector bundle over a compact Riemann surface.*

Disclaimer. Significant parts of this thesis, especially the later parts of the third chapter, and the entirety of the fourth chapter, are strongly based on our paper [8].

Chapter 2

Kähler Geometry

Here we give a quick introduction to Kähler manifolds with an emphasis on the facts and formulas that will be particularly important in this dissertation, and also serves as a vehicle to set some notational conventions. Standard references for the basics of Kähler geometry are [18], [21], and for the specific viewpoint most relevant to this dissertation, the opening of [31].

2.1 Hermitian Geometry of Kähler Manifolds

Kähler geometry concerns triples (M, J, g) where M is a smooth manifold, J is a complex structure on M , and g is a Riemannian metric on M . An almost-complex structure is an endomorphism of the tangent bundle of M such that $J^2 = -\text{Id}_{TM}$, in which case linear algebra forces M to be even dimensional, say having real dimension $2n$. For such an endomorphism it is natural to consider the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$, which decomposes into $\sqrt{-1}$ and $-\sqrt{-1}$ -eigenbundles denoted

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

respectively. $T_{\mathbb{C}}M$ has a natural conjugation operation coming from the complexification, under which $T^{1,0}M$ and $T^{0,1}M$ are complex anti-linearly isomorphic.

An almost-complex structure is upgraded to the status of a complex structure if J is integrable, that is, everywhere locally there are coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ modeled on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ solving the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial}{\partial y_i} &= J \frac{\partial}{\partial x_i}, \\ \frac{\partial}{\partial x_i} &= -J \frac{\partial}{\partial y_i} \end{aligned} \tag{2.1.1}$$

for each $1 \leq i \leq n$. This condition puts a holomorphic structure on M in a way analogous to a smooth structure, since if we take any two coordinate patches satisfying 2.1.1 their

transition will be a biholomorphism under the identifications $z_i = x_i + \sqrt{-1}y_i$. The term integrability comes from the Newlander-Nirenberg theorem stating that J is integrable in the sense described above if and only if $T^{1,0}M$ is integrable in the sense of Frobenius, that is, closed under the Lie bracket [25]. A complex manifold is denoted by the tuple (M, J) to emphasize that complex manifolds may be diffeomorphic but not biholomorphic, the most basic examples given by elliptic curves, that is quotients of $\mathbb{C}$ by different lattices, being diffeomorphic to a torus but having many non-isomorphic induced complex structures. However, we occasionally suppress J when the complex structure is clear.

If (M, J) is a complex manifold, then in a holomorphic coordinate as above the bundles $T^{1,0}M$ and $T^{0,1}M$ are naturally spanned by the holomorphic and anti-holomorphic differentials, respectively:

$$\begin{aligned}\frac{\partial}{\partial z_i} &= \frac{1}{2} \left(\frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right), \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2} \left(\frac{\partial}{\partial x_i} + \sqrt{-1} \frac{\partial}{\partial y_i} \right).\end{aligned}\tag{2.1.2}$$

This naturally makes $T^{1,0}M$ a holomorphic vector bundle on M , the *holomorphic tangent bundle*, the definition of a holomorphic vector bundle naturally being an extension of that of a real (or complex) smooth vector bundle, just having transition maps which are holomorphic. By taking the dual of the complexified tangent bundle we get the bundle of complex 1-forms, which we denote by $\Omega_{\mathbb{C}}^1 M$, having the corresponding decomposition

$$\Omega_{\mathbb{C}}^1 M = \Omega^{1,0} M \oplus \Omega^{0,1} M.$$

Once again these bundles locally have natural spanning sets, given by taking the duals of the vectors in 2.1.2:

$$\begin{aligned}dz_i &= dx_i + \sqrt{-1}dy_i, \\ d\bar{z}_i &= dx_i - \sqrt{-1}dy_i.\end{aligned}$$

We similarly get spanning sets for all exterior powers of the holomorphic tangent and cotangent bundles, and for short we write

$$\Omega^{p,q} M := \bigwedge^p \Omega^{1,0} M \otimes \bigwedge^q \Omega^{0,1} M.$$

Sections (holomorphic or not) of such a bundle are referred to as (p, q) -forms, not to be confused with the terminology around tensors for accounting the indices of covariance and contravariance.

Of course we then have new structures in the exterior calculus, namely that on $\Omega^{p,q} M$ the exterior derivative splits into the operators

$$\partial := \pi^{p+1,q} \circ d, \text{ and } \bar{\partial} := \pi^{p,q+1} \circ d,$$

where $\pi^{p+1,q}$ and $\pi^{p,q+1}$ are the natural projections onto the spaces of $(p+1, q)$ and $(p, q+1)$ -forms, respectively. The integrability of the complex structure guarantees that $d = \partial + \bar{\partial}$.

Now we address the presence of g , the Riemannian metric, and how such a metric may interact with the underlying complex geometry. The most obvious thing to demand is that g is *Hermitian*, by which we mean that the complex structure J is orthogonal under g , that is,

$$g(-, -) = g(J-, J-).$$

This makes the metric tensor itself behave nicely with the structure on tensors induced by the complex structure, in particular, if we extend g complex linearly to $T_{\mathbb{C}}M$, then in a local holomorphic coordinate we have

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = 0, \text{ and } g\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right) = 0$$

for all $1 \leq i, j \leq n$, not just when $i \neq j$. Then the metric tensor is just determined by its components

$$g_{i\bar{j}} := g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right).$$

However, this says nothing of how we should take covariant derivatives of tensors in a way that respects the complex structure. Before giving the obvious resolution to this tension, let's take one more aside into Hermitian geometry.

Consider a holomorphic Hermitian vector bundle $(E, h) \rightarrow M$, that is, a holomorphic vector bundle E over M equipped with a Hermitian inner product h . The holomorphic structure of the bundle E is reflected in a naturally induced anti-holomorphic derivative

$$\bar{\partial}_E : \Gamma(E) \rightarrow \Gamma(E \otimes \Omega^{0,1}M).$$

This operator is defined as follows, let $e_1, \dots, e_k$ be a local holomorphic frame for E , then any local section v for E can be written

$$v = \sum_{i=1}^k f_i e_i.$$

Then it is natural to define

$$\bar{\partial}_E v = \sum_{i=1}^k \bar{\partial} f_i \otimes e_i,$$

where the $\bar{\partial}$ appearing in the summation is the one appearing in the exterior calculus. This operator is independent of the choice of holomorphic frame as the transition functions are holomorphic and therefore any additional terms coming from those will be killed by the $\bar{\partial}$ operator when transitioning between frames!

Just as we ask for a canonical covariant derivative in the context of Riemannian geometry respecting both the metric structure (metric tensor is parallel), and the smooth structure

(torsion free), in Hermitian geometry it is natural to ask for a covariant derivative which respects both the metric structure and the complex structure of the bundle. The notion of *metric compatibility* is the same for covariant derivatives

$$\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes \Omega_{\mathbb{C}}^1 M)$$

on Hermitian vector bundles as it is for Riemannian manifolds, namely that for a local vector field v and sections s and t of E we demand

$$v(h(s, t)) = \langle \nabla_v s, t \rangle + \langle s, \nabla_v t \rangle.$$

It turns out that there always exists a unique *Chern connection*, that is, a covariant derivative ∇ on the holomorphic Hermitian vector bundle $(E, h) \rightarrow M$ such that ∇ is compatible with h and

$$\bar{\partial}_E = \pi^{0,1} \circ \nabla,$$

that is, the $(0, 1)$ -part of the covariant derivative agrees with the natural anti-holomorphic derivative on E .

Now, returning to our Hermitian metric g on the complex manifold (M, J) , to ensure that the Levi-Civita connection ∇ induced by g also plays well with the complex structure we further demand that

$$\nabla J \equiv 0.$$

Thus we have arrived at the notion of a *Kähler manifold*, that is, a triple (M, J, g) consisting of a smooth manifold, a complex structure, and a Riemannian metric on M under which J is g -orthogonal and g -parallel. Since J is parallel, we have

$$J \left(\nabla_j \frac{\partial}{\partial z_k} \right) = \nabla_j J \left(\frac{\partial}{\partial z_k} \right) = \sqrt{-1} \nabla_j \frac{\partial}{\partial z_k},$$

where $\nabla_j := \nabla_{\frac{\partial}{\partial z_j}}$. Similarly, $\nabla_{\bar{j}} \frac{\partial}{\partial z_k}$ has type $(1, 0)$, and $\nabla_k \frac{\partial}{\partial \bar{z}_j}$ has type $(0, 1)$. However, by the torsion-freeness of ∇ , we have that

$$\nabla_{\bar{j}} \frac{\partial}{\partial z_k} = \nabla_k \frac{\partial}{\partial \bar{z}_j},$$

and hence they must both vanish. This immediately implies that the holomorphic tangent bundle $T^{1,0}M$ is preserved by ∇ . Furthermore, if we extend g conjugate linearly to $T_{\mathbb{C}}M$, as opposed to complex linearly, then the triple (M, J, g) naturally imposes onto $T^{1,0}M$ the structure of a holomorphic Hermitian vector bundle over M . These calculations then also imply that the Chern connection induced by this holomorphic Hermitian structure agrees with the Levi-Civita connection! Thus, the g -orthogonal and g -parallel conditions of a Kähler metric together serve to perfectly bridge the Riemannian and complex structures on M .

This is one way of justifying Kähler metrics as objects of nature. Another justification of the Kähler condition is as being precisely that which forces the metric to have holonomy (that is, endomorphisms of the tangent spaces induced by parallel translation along closed loops) within $U(n)$, since the complex structure being parallel makes parallel translation J -invariant, and thus Kähler manifolds naturally arise within the subject of special holonomy manifolds.

2.2 Differential Geometry of Kähler Metrics

Now we'll explore more the consequences of the Kähler condition when doing differential geometric computations. To start, while in the author's opinion the definition of a Kähler manifold given in the previous section is the most natural, in practice the one that is most often given, and often used for doing calculations and posing metric problems in Kähler geometry is the following. Given a triple (M, J, g) of a complex manifold with a Hermitian metric, we define a bilinear form

$$\omega(-, -) := g(J-, -),$$

which is easily checked to be alternating and therefore a 2-form, and furthermore it is actually a $(1, 1)$ -form upon complexification (such a form coming from the complexification of a real form is called a *real* $(1, 1)$ -form). It is then a calculation to check that the Kähler condition is equivalent to

$$d\omega = 0,$$

in which case ω is known as the *Kähler form*. Since the Kähler metric g is interchangeable with the Kähler form via twisting with the complex structure, often questions about Kähler metrics are instead phrased entirely as questions about Kähler forms, and we occasionally will abuse language and notation by simply referring to the Kähler form as the metric, and denote a Kähler manifold as (M, ω) , suppressing the complex structure which is assumed to be fixed. We may also simply call a manifold M a Kähler manifold, suppressing the complex structure which we assume to be fixed, meaning there exists a Kähler metric with respect to that complex structure, or we may call a complex manifold (M, J) *Kähler-type* if there exists some Kähler metric with respect to the given complex structure, when we do not want to specify a metric.

Now, being a closed, real 2-form, ω of course defines a de Rham class $[\omega] \in H^2(M, \mathbb{R})$, the *Kähler class*. Often questions in Kähler geometry, as will the fundamental problem of this thesis, ask if given a Kähler class (a class admitting a Kähler form/metric) admits a Kähler metric of specified curvature properties (e.g. constant scalar curvature). Such questions are significantly simplified from a PDE perspective by the following fundamental fact.

Lemma 2.2.1 ($\partial\bar{\partial}$ -lemma). *Let M be a Kähler manifold. Then if η is a closed, real $(1, 1)$ -form, then locally there is a real smooth function $\phi : M \rightarrow \mathbb{R}$ such that*

$$\eta = \sqrt{-1}\partial\bar{\partial}\phi.$$

When we write a Kähler metric (form) locally in this way, ϕ is referred to as a *Kähler potential*.

To give an example of this fact, and give an explicit example of a Kähler manifold (and perhaps the only example that really matters for this thesis) we introduce the Fubini-Study metric on the complex projective space $\mathbb{P}^n$. Now, $\mathbb{P}^n$ has a transitive $U(n+1)$ -symmetry which can be used to define a unique-up-to-scaling natural invariant metric, the Fubini-Study metric, whose symmetry implies that it is Kähler. It is also often useful for explicit computation to note that this metric has an explicit form given as follows: take the projective coordinate

$$Z = [Z_0 : \dots : Z_n],$$

then we have that the Fubini-Study metric is given by

$$\omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log \|Z\|^2.$$

What we mean by this is that if $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ is the quotient map, and $s : U \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ is any local holomorphic section of the quotient map, we have

$$\omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log \|s\|^2.$$

To see that this is well-defined, any two such section differ by $s' = fs$ where f is holomorphic, in which case

$$\sqrt{-1} \partial \bar{\partial} \log \|fs\|^2 = \sqrt{-1} \partial \bar{\partial} \log \|s\|^2 + \sqrt{-1} \partial \bar{\partial} \log f + \sqrt{-1} \partial \bar{\partial} \log \bar{f},$$

and the last two terms vanish when applying $\partial \bar{\partial}$. Since the $SU(n+1) \subset U(n+1)$ symmetry is enough to determine the metric, this metric clearly is the Fubini-Study metric defined earlier, and it has simple and explicit local Kähler potentials for example derived from the standard coordinate patches on $\mathbb{P}^n$.

Now, since the restriction of a Kähler form to a complex submanifold is still a Kähler form, since it remains closed, all projective manifolds are also Kähler, and the most important applications of the main results of this thesis will all be of this type.

Now, one of the most important facts for local computations in Riemannian geometry is the existence of geodesic normal coordinates, where we can choose smooth coordinates locally where the geodesics are the coordinate axes. Another remarkably nice fact about Kähler metrics is that we are still able to do this local approximation by the Euclidean metric but this time in a holomorphic coordinate, however, it is regrettable that we don't have much of any control on the geodesics in a holomorphic coordinate. Still, this will be a particularly crucial fact for the constructions in this dissertation.

Proposition 2.2.2 (Holomorphic Normal Coordinates). *If (M, ω) is a Kähler manifold and $p \in M$, then there exists a local holomorphic chart centered at p in which*

$$\omega = \sqrt{-1}(\delta_{jk} + O(|z|^2))dz^j \wedge d\bar{z}^k.$$

Stated in terms of the metric, in this coordinate the metric is the identity at p and all first derivatives of the metric vanish at p .

The proof is essentially done by Taylor expanding the metric at p and noticing that $d\omega = 0$ forces the linear terms to cancel.

Now, in a holomorphic normal coordinate the Christoffel symbols vanish at the point at which the coordinates are centered, and our earlier observation that

$$\nabla_j \frac{\partial}{\partial z_k}, \nabla_{\bar{j}} \frac{\partial}{\partial z_k}$$

have type $(1, 0)$ and vanish, respectively, implies that the Riemann curvature tensor of a Kähler metric only has components

$$Rm_{i\bar{j}k\bar{l}}.$$

Therefore, the *Ricci curvature* of our metric, which we'll say has Kähler form ω , is defined as a trace of the Riemann curvature tensor

$$\text{Ric}_\omega = R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

It therefore is a tensor of one holomorphic component and one anti-holomorphic component, where $g^{k\bar{l}}$ is the matrix inverse of $g_{k\bar{l}}$. Lastly, of significant importance is the *scalar curvature*

$$S(\omega) := g^{i\bar{j}} R_{i\bar{j}}.$$

We now give useful local formulas for these various notions of curvature.

Lemma 2.2.3. *For a Kähler metric g with Kähler form ω , the Riemann curvature tensor, Ricci curvature, and scalar curvature have local formulas*

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}} (\partial_k g_{i\bar{q}}) (\partial_{\bar{l}} g_{p\bar{j}}), \\ \text{Ric}_\omega &= R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}), \\ S(\omega) &= -g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}). \end{aligned}$$

Proof. For notation, the Christoffel symbols of our metric are defined in a local holomorphic coordinate as

$$\nabla_j \frac{\partial}{\partial z_k} = \Gamma_{jk}^i \frac{\partial}{\partial z_i}.$$

Now, since the metric is covariant constant, we have

$$0 = \nabla_j g_{k\bar{l}} = \partial_j g_{k\bar{l}} - \Gamma_{jk}^p g_{p\bar{l}},$$

and therefore

$$\Gamma_{jk}^i = g^{i\bar{l}} \partial_j g_{k\bar{l}}.$$

Now, the Riemann curvature endomorphism is defined by the equation

$$(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k) \frac{\partial}{\partial z_i} = R_{i\bar{k}\bar{l}}^j \frac{\partial}{\partial z_j}.$$

Now, for example by computing in the center of a normal coordinate we can compute

$$R_{i \ k\bar{l}}^j = -\partial_{\bar{l}} \Gamma_{ki}^j,$$

and therefore combining these two facts we have the first formula. For the Ricci curvature we use the formula for the determinant of a matrix to get

$$-\partial_{\bar{j}} \partial_i \log \det(g_{p\bar{q}}) = -\partial_{\bar{j}}(g^{p\bar{q}} \partial_i g_{p\bar{q}}),$$

and then from above we have

$$= -\partial_{\bar{j}} \Gamma_{ip}^p = R_p^p{}_{i\bar{j}} = R_{i\bar{j}}.$$

□

The last remaining formulas that we will need comes from the perspective hinted at earlier with the introduction of the $\partial\bar{\partial}$ -lemma, that being that when finding a Kähler metric with specified curvature properties in a given Kähler class, we usually start with an initial metric ω in the desired class and then phrase the problem as finding a suitable real function $\phi : M \rightarrow \mathbb{R}$ such that

$$\omega + \sqrt{-1} \partial\bar{\partial}\phi$$

is a Kähler metric and has the curvature properties that we wanted. As such, when studying the prescribed scalar curvature problem, as in our case, one is naturally led to study the *Linearized scalar curvature operator*

$$L_\omega \phi := \frac{d}{dt} \Big|_{t=0} S(\omega + t\sqrt{-1} \partial\bar{\partial}\phi).$$

The thing to note here is that we are varying within the Kähler class by the complex Hessian of a real function, and by the $\partial\bar{\partial}$ -lemma, this indeed exhausts all possible variations within the class and therefore really is a bona fide derivative of the scalar curvature function within the Kähler class. Also, notice that constant functions are always in the kernel of this operator, as the Hessian of a constant function is of course zero.

This operator has a remarkable formula and interaction with the complex geometry of the underlying manifold that we will exploit heavily.

Proposition 2.2.4. *Let (M, ω) be a compact Kähler manifold. Consider the operator*

$$\mathcal{D} : C^\infty(M, \mathbb{C}) \rightarrow C^\infty(\Omega^{0,1} M \otimes \Omega^{0,1} M)$$

$$\phi \mapsto \nabla_{\bar{k}} \nabla_{\bar{q}} \phi.$$

Then we have the following formulas

$$\begin{aligned} L_\omega \phi &= -\Delta_\omega^2 \phi - Ric_\omega \cdot \nabla^2 \phi \\ &= -\mathcal{D}^* \mathcal{D} \phi + \partial S(\omega) \cdot \bar{\partial} \phi, \end{aligned}$$

where $\cdot$ denotes contraction via the metric.

Proof. Denote $\omega_t := \omega + t\sqrt{-1}\partial\bar{\partial}\phi$, and the Ricci form ρ_t . We can alternatively write the determinant of the metric as ω^n , in which case we have

$$\rho_t = -\sqrt{-1}\partial\bar{\partial}\log\omega_t^n.$$

Now, if we expand $\omega_t^n = (\omega + \sqrt{-1}t\partial\bar{\partial}\phi)^n$, the only terms that are linear in t are of the form $\sqrt{-1}t\partial\bar{\partial}\phi \wedge \omega^{n-1}$. Thus,

$$\frac{d}{dt}\bigg|_{t=0} \omega_t^n = n\sqrt{-1}\partial\bar{\partial}\phi \wedge \omega^{n-1},$$

and computing in normal coordinates it is clear to see

$$= \Delta\phi\omega^n.$$

Then we have that

$$\frac{d}{dt}\bigg|_{t=0} \rho_t = -\sqrt{-1}\partial\bar{\partial}\Delta\phi.$$

Now, writing $S(\omega_t) = g_t^{j\bar{k}}R_{t,j\bar{k}}$ where g_t denotes the metric associated to ω_t and R_t the Ricci curvature of ω_t , then by the product rule and the above calculations we have

$$L_\omega\phi = -g^{j\bar{q}}(\partial_p\partial_{\bar{q}}\phi)g^{p\bar{k}}R_{j\bar{k}} - \Delta^2\phi,$$

which written another way is

$$= -\Delta^2\phi - R^{\bar{k}j}\partial_j\partial_{\bar{k}}\phi = -\Delta^2\phi - \text{Ric}_\omega \cdot \nabla^2\phi.$$

Now for the identity involving the operator $\mathcal{D}$. We have

$$\begin{aligned} \Delta^2\phi + \nabla_j(R^{k\bar{j}}\nabla_{\bar{k}}\phi) &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_{\bar{k}}\nabla_p\nabla_{\bar{q}}\phi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\phi) \\ &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_p\nabla_{\bar{k}}\nabla_{\bar{q}}\phi - g^{j\bar{k}}g^{p\bar{q}}\nabla_j(R_{\bar{q}p\bar{k}}^{\bar{m}}\nabla_{\bar{m}}\phi) + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\phi) \\ &= g^{j\bar{k}}g^{p\bar{q}}\nabla_p\nabla_j\nabla_{\bar{k}}\nabla_{\bar{q}}\phi = \mathcal{D}^*\mathcal{D}\phi. \end{aligned}$$

Now by the Bianchi identity $\nabla_k R^{\bar{k}j} = g^{j\bar{k}}\nabla_{\bar{k}}S$ we have

$$\mathcal{D}^*\mathcal{D}\phi = \Delta^2\phi + \nabla_j(R^{k\bar{j}}\nabla_{\bar{k}}\phi) = \Delta^2\phi + R^{\bar{k}j}\nabla_j\nabla_{\bar{k}}\phi + g^{j\bar{k}}\nabla_jS\nabla_{\bar{k}}\phi,$$

as desired. □

The main takeaways are as follows. Firstly, the linearized scalar curvature is the negative of the bilaplacian of the metric to highest order. Secondly, the operator $\mathcal{D}$, up to making the metric identification by raising/lowering indices $\Omega^{0,1}M \cong T^{1,0}M$, is the same as

$$\phi \mapsto \bar{\partial}(\text{grad}^{1,0}\phi).$$

Therefore, ϕ is in the kernel of $-\mathcal{D}^*\mathcal{D}$ exactly when $\text{grad}^{1,0}\phi$ is holomorphic for some function ϕ . From the formula we see the remarkable fact that when the scalar curvature of our Kähler metric is constant, the kernel of the linearized scalar curvature operator is controlled by the holomorphic vector fields on the underlying manifold, and furthermore that the operator is self-adjoint. Flexing these facts to find use in the case of a general, not necessarily constant scalar curvature Kähler metric, will form the crux of the proof of the main results of this thesis.

2.3 Line Bundles on Kähler Manifolds

Line bundles, that is, rank one holomorphic vector bundles, play a crucial role in Kähler geometry for two reasons. The first is due to the complex geometry alone, namely that holomorphic functions are significantly less plentiful than smooth functions, and therefore it is considerably more difficult to construct complex submanifolds. Indeed, a compact complex manifold has no global non-constant holomorphic functions at all, but it may have nontrivial line bundles that do have nontrivial global holomorphic sections. With such a section and possibly allowing some singularities, one can construct complex hypersurfaces, and indeed, up to suitable notions of equivalence, the study of holomorphic line bundles is practically equivalent to the study of the codimension one geometry of the manifold. Secondly, line bundles are an important source of Kähler metrics via their curvature.

The most fundamental example of a line bundle on a n -dimensional complex manifold M is

$$K_M := \bigwedge^n \Omega^{1,0} M,$$

the *canonical bundle*. This is a line bundle we always get for free, and in all but some quite special cases it is nontrivial.

Now, recall that if L is a line bundle with a Hermitian metric h , then it has a unique Chern connection which is metric compatible and whose $(0,1)$ -part is the anti-holomorphic operator $\bar{\partial}_L$ we get for free on a holomorphic vector bundle. The metric compatibility makes the connection form antisymmetric, and the compatibility with the holomorphic operator guarantees that the curvature has no $(0,2)$ -part, and thus combining these two facts ensures that the curvature of a metric on a holomorphic line bundle is a $(1,1)$ -form with values in the endomorphisms of L , which is a trivial bundle, and thus the curvature of a Hermitian holomorphic line bundle can be viewed as a real $(1,1)$ -form. Knowing this, let s be a local holomorphic section of L , then locally the connection is determined by a $(1,0)$ form A given by

$$\nabla_k s = A_k s.$$

Then we have that the curvature is given by

$$-\nabla_{\bar{l}} \nabla_k s = -(\partial_{\bar{l}} A_k) s.$$

Lastly, the properties of the Chern connection plus the holomorphicity of s gives that

$$\partial_k |s|_h^2 = h(\nabla_k s, s) = A_k |s|_h^2,$$

and therefore

$$A_k = (|s|_h^2)^{-1} \partial_k |s|_h^2.$$

Altogether we have the following.

Lemma 2.3.1. *Let (L, h) be a Hermitian holomorphic line bundle. Then the Chern connection has curvature of type $(1, 1)$, and if s is a local nonvanishing holomorphic section of L , the curvature of L is given by*

$$-\partial_k \partial_{\bar{l}} \log |s|_h^2.$$

Let's see an explicit example of this. The complex projective space has a tautological line bundle $\mathcal{O}(-1)$ constructed as follows. For every point $\ell \in \mathbb{P}^n$, there is an associated line in $\mathbb{C}^{n+1}$. By considering pairs (ℓ, p) where p is any point in ℓ , we get a line bundle on $\mathbb{P}^n$. In the coordinate chart

$$z_i = \frac{Z_i}{Z_0}, \quad 1 \leq i \leq n, \quad Z_0 \neq 0,$$

the bundle then has coordinates

$$(z_1, \dots, z_n, k),$$

where the fiber coordinate k corresponds to the point $(k, kz_1, \dots, kz_n) \in \mathbb{C}^{n+1}$. This line bundle then has a natural metric, which is given by just measuring a vector in ℓ with the Euclidean metric in $\mathbb{C}^{n+1}$. In the coordinate patch $Z_0 \neq 0$, we have a nonvanishing holomorphic section simply given by $k = 1$, in which case we can calculate the curvature as

$$-\partial_k \partial_{\bar{l}} \log(1 + |z_1|^2 + \dots + |z_n|^2).$$

Note, this is just the negative of the Fubini-Study metric on $\mathbb{P}^n$. Thus, the natural metric on $\mathcal{O}(-1)$ induces a natural dual metric on $\mathcal{O}(1) := \mathcal{O}(-1)^*$, which therefore has curvature the Fubini-Study metric! Recall that a very ample line bundle L on a complex manifold defines an embedding to a projective space where the pullback of $\mathcal{O}(1)$ along this embedding is isomorphic to L . Thus, holomorphic line bundles are sources of a large class of Kähler metrics, and furthermore by the Kodaira embedding theorem line bundles admitting a metric whose curvature is a Kähler metric on the base manifold are exactly the ample line bundles [22].

Due to the above formula, the curvature of a metric on a line bundle actually defines a closed, real form. If h is the metric, then any other metric on M can be written as $e^{-f}h$ where f is a globally defined function, and defines its own curvature form. Call the two curvature forms ω_h and $\omega_{e^{-f}h}$, respectively. Then

$$\omega_{e^{-f}h} - \omega_h = \partial \bar{\partial} f.$$

Therefore, for normalization reasons, $\frac{1}{2\pi}\omega_h$ is a closed form whose de Rham class does not depend on the choice of metric, and it represents the *first Chern class* of L , $c_1(L)$.

Note the similarity between the formula for the Ricci curvature of a Kähler metric in the previous section. Indeed, the holomorphic tangent bundle $T^{1,0}M$ has a Hermitian metric induced by the Kähler metric, and taking the determinant of that metric we naturally get a metric on the *anti-canonical bundle* K_M^* , whose curvature in a holomorphic coordinate is then given by

$$-\partial_k \partial_{\bar{l}} \log \det(g_{p\bar{q}}),$$

which is precisely the Ricci curvature. Writing this more invariantly, we define the *Ricci form* of a Kähler manifold (M, ω) as

$$\rho(-, -) = \text{Ric}_\omega(J-, -).$$

Combining all of the above, we have the following remarkable fact.

Proposition 2.3.2. *Let (M, ω) be a Kähler manifold, then the Ricci form ρ is a real $(1, 1)$ -form which can be seen as the curvature of K_M^* equipped with the metric induced by ω . ρ is therefore a representative of $2\pi c_1(M) := 2\pi c_1(K_M^*)$.*

Indeed, this means that regardless of the choice of Kähler metric, the Ricci form is always a representative of the same cohomology class. We end this section with the implications of this fact on the scalar curvature. Indeed, since the scalar curvature is given as the trace of the Ricci curvature, in terms of the Ricci form and the Kähler form this can be rephrased as the equation

$$S(\omega) \frac{\omega^n}{n!} = \frac{1}{(n-1)!} \rho \wedge \omega^{n-1}.$$

Therefore, integrating both sides, we see that

$$\int S(\omega) \frac{\omega^n}{n!} = \frac{2\pi}{(n-1)!} c_1(M) \cup [\omega]^{n-1},$$

where $\cup$ refers to the cup product in cohomology. Note that, writing the metric in normal coordinates, it is clearly seen that the volume form of the Kähler metric is $\frac{\omega^n}{n!}$, and therefore the quantity on the left is simply the integral of the scalar curvature on (M, ω) , referred to as the *total scalar curvature* of ω . Thus, of extreme relevance to this thesis, which Kähler classes can admit metrics with scalar curvature everywhere positive (or negative) is restricted by the intersection of the Kähler class with the first Chern class!

Chapter 3

Analysis on Kähler Blowups

Compact algebraic surfaces are classified up to a surgery operation known as *blowing up*. This operation leaves the manifold unchanged everywhere but a point, with that point being replaced by smooth hypersurface isomorphic to projective space. The local and explicit nature of this operation leaves it malleable to an analytic approach. In this chapter we describe the blowup of a complex manifold, how to construct Kähler metrics with nice analytic properties on the blowup of a Kähler manifold, and finally we define function spaces with the geometry of the blowup in mind, so that they will be well suited to the analysis required to perturb the metrics constructed on the blowup to metrics with our desired scalar curvature properties.

3.1 Blowing Up

As we said, the blowup is really a local operation, and so we only really need to understand the blowup of affine space to get a handle on it. Consider the subvariety of $\mathbb{P}^{n-1} \times \mathbb{C}^n$

$$Bl_0\mathbb{C}^n := \{(\ell, p) \in \mathbb{P}^{n-1} \times \mathbb{C}^n \mid p \in \ell\}.$$

Recall, this is actually how we defined the holomorphic line bundle $O(-1)$ over $\mathbb{P}^{n-1}$. When viewed as a line bundle, we focused on the projection onto the projective space

$$p : Bl_0\mathbb{C}^n \rightarrow \mathbb{P}^{n-1}$$

$$(\ell, p) \mapsto \ell,$$

but the interest around the blowup comes from the projection onto affine space

$$\pi : Bl_0\mathbb{C}^n \rightarrow \mathbb{C}^n$$

$$(\ell, p) \mapsto p.$$

Now, every point ℓ in projective space corresponds to a point in affine space going through the origin, but since each line is determined by just two points, as long as $p \neq 0$, $\pi^{-1}(p)$ consists of only one point. Thus, if we define

$$E := \pi^{-1}(0),$$

then π is a biholomorphism $Bl_0 M \setminus E \cong \mathbb{C}^n \setminus \{0\}$, but $E \cong \mathbb{P}^{n-1}$. Thus $Bl_0 \mathbb{C}^n$ is the same as $\mathbb{C}^n$ except the origin gets replaced by a copy of $\mathbb{P}^{n-1}$.

We have described the blowup of $\mathbb{C}^n$ at the origin, so now consider a complex manifold M , and any point $p \in M$. If we take some neighborhood $U \subset M$ of p such that U is biholomorphic to an open subset of $\mathbb{C}^n$ in such a way that p is identified with the origin in affine space, then denote

$$Bl_0 U := \pi^{-1}(U).$$

Then since $Bl_0 U \setminus E \cong U \setminus \{0\}$, simply by replacing U with $Bl_0 U$, we obtain a new complex manifold $Bl_p M$, the *blowup of M at p* . Furthermore, we call the submanifold E the *exceptional divisor*, and we have a natural map $\pi : Bl_p M \rightarrow M$, often referred to as the *blowdown map*, such that when we remove the exceptional divisor we get a biholomorphism $Bl_p M \setminus E \cong M \setminus \{p\}$. Given the structure of the blowup, if we have a local coordinate z on M centered at p , we will often give formulas for objects on $Bl_p M$ in terms of the coordinate z , but we will have to verify that they naturally extend across E . It's not hard to calculate that the normal bundle of E is $\mathcal{O}(-1)$.

Now that we have defined the blowup, we will now turn to some initial ideas on how to construct Kähler metrics on the blowup of a Kähler manifold.

Let's start with the affine space with the Euclidean metric ω_{euc} . First of all, $\pi^* \omega_{euc}$ is not a Kähler metric on $Bl_0 \mathbb{C}^n$ since it vanishes on the tangent space of the exceptional divisor. However, for ϵ sufficiently small

$$\pi^* \omega_{euc} + \epsilon p^* \omega_{FS}$$

is a Kähler metric on the blowup. This can be seen as follows. First of all, this clearly eliminates the issue of the lack of positivity on the tangent of the exceptional divisor. Secondly, by our definition of the Fubini-Study metric, one can see that actually (up to normalization) we can write

$$p^* \omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log |w|,$$

where w is the natural coordinate on $Bl_0 \mathbb{C}^n \setminus E$. Thus,

$$\pi^* \omega_{euc} + \epsilon p^* \omega_{FS} = \sqrt{-1} \partial \bar{\partial} (|w|^2 + \epsilon \log |w|),$$

and since the second derivative of $\log |w|$ is rotationally symmetric and decays at infinity. Thus, choosing ϵ sufficiently small ensures the positivity of the metric everywhere.

Now we construct our most naive Kähler metric on the blowup.

Proposition 3.1.1. *Let (M, ω) be a Kähler manifold, $p \in M$, then if $[E] \in H^2(Bl_p M)$ is the Poincaré dual of E , then for any ϵ sufficiently small, $\pi^*[\omega] - \epsilon[E]$ is a Kähler class on $Bl_p M$.*

Proof. Let U be a holomorphic normal neighborhood with coordinate w centered at p such that

$$\omega = \sqrt{-1}\partial\bar{\partial}(|w|^2 + O(|w|^4)),$$

and so $\pi^*\omega$ has the same formula on $Bl_0U \setminus E$. Now, let χ be a cutoff function that is 1 near p and 0 outside of U . Then, for ϵ sufficiently small,

$$\sqrt{-1}\partial\bar{\partial}(|w|^2 + \epsilon\chi \log |w| + O(|w|^4))$$

by the discussion above extends by $\pi^*\omega$ to a Kähler metric on all of Bl_pM , and the metric restricts to a multiple of the Fubini-Study metric on the exceptional divisor. Since the metric disagrees with $\pi^*\omega$ only on a small neighborhood of E , the degree of the normal bundle of E is negative, the volume of E is positive (as this is a Kähler metric) and the volume of subvarieties are invariants of a Kähler class, the Kähler class must differ from $\pi^*[\omega]$ by a negative multiple of $[E]$. $\square$

Now, the metric constructed in the above proof gives our first window into the difficulty of the problem of preserving the positivity of the scalar curvature of Kähler metrics when blowing up. The reason being that, from the calculations in [20], one can think of the scalar curvature of the metric constructed in the above proof as behaving similarly to that of the metric

$$\pi^*\omega_{euc} + \epsilon p^*\omega_{FS}$$

on $Bl_0\mathbb{C}^n$ as one approaches the blowup point. However, using the formula for the potential of this metric, one can calculate that it has positive scalar curvature everywhere when $n > 2$, but remarkably it is scalar-flat when $n = 2$. Indeed, in [20] it is shown that one can preserve positivity of scalar curvature of Kähler metrics under blowups when $n > 2$, but only in certain cases when $n = 2$, and those cases require special chosen initial metrics, whereas in $n > 2$ one can start with any initial metric.

3.2 The Burns-Simanca Metric and the Initial Metric

ω_ϵ

In constructing our first metrics on the blowup of a Kähler manifold (M, ω) , we discovered that the primary obstruction to the most naive approach is that the most natural metric on $Bl_0\mathbb{C}^n$ is suddenly scalar-flat when $n = 2$, which causes this approach to completely fall apart. However, it turns out this metric actually fits into a family of scalar-flat metrics on $Bl_0\mathbb{C}^n$ in any dimension, which happens to agree with the naive family specifically in $n = 2$. In our approach to constructing Kähler metrics on the blowup in this thesis, we view the problem as a partial differential equation, where one wants to first construct an initial family ω_ϵ of Kähler metrics on Bl_pM , and then write down an equation of the form

$$S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi) - S(\omega) = \text{“something”}$$

which can be solved for in ϕ for small parameter ϵ . Note that we will abuse notation and write $S(\omega)$ in place of $\pi^*S(\omega)$ when obvious due to context.

In our case, we desire to control the scalar curvature on the blowup so it does not differ so much from the original, and so the main issue is constructing the metric ω_ϵ and designing the equation so that it is actually solvable and has right hand side which shrinks as ϵ shrinks. The form of the equation naturally lends itself to studying the linearized scalar curvature operator, and the nice properties of this operator in the presence of a constant-scalar curvature Kähler metric is where we will find we can actually leverage scalar-flatness of metrics on the blowup of affine space, whereas it sunk the previous approach.

Let's now construct the advertised scalar-flat metrics on $Bl_0\mathbb{C}^n$ for $n > 2$, as we have already done the $n = 2$ case. Together with the $n = 2$ case, these metrics are known as the *Burns-Simanca* metrics, and the construction of them in all dimensions is due to [29].

Viewing $Bl_0\mathbb{C}^n$ as the bundle $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$, choose a Hermitian metric on $\mathcal{O}(-1)$ with curvature the negative of the Fubini-Study metric, which we have already seen is realized by a multiple of the natural metric on $\mathcal{O}(-1)$. We will assume the metric to have the form

$$\eta = \sqrt{-1}\partial\bar{\partial}f(s),$$

where f is a strictly convex function, and if z is the coordinates on $\mathbb{P}^{n-1}$ and w the fiberwise coordinate, $s = \log(|w|^2h(z))$. Now, if we choose coordinates at a point such that $dh = 0$, then

$$\eta = f'(s)p^*\omega_{FS} + f''(s)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2},$$

where once again p is the projection to projective space. Then we have

$$\eta^n = \frac{f''(s)}{|w|^2}(f'(s))^{n-1}p^*\omega_{FS}^{n-1} \wedge \sqrt{-1}dw \wedge d\bar{w}.$$

Therefore we have that the Ricci form is

$$\rho = -\sqrt{-1}\partial\bar{\partial}\log(f''(s)(f'(s))^{n-1}) + p^*(n\omega_{FS}),$$

using the fact that the Ricci form of the Fubini-Study metric is $n\omega_{FS}$. Now we take the Legendre transform of f , that is, we rewrite this in terms of the change of coordinates $\phi(\tau) = f''(s)$, where $\tau = f'(s)$, which is a valid change of coordinates since the strictly convex assumption on f implies f' is bijective. Then we have

$$\begin{aligned} \eta &= \tau p^*\omega_{FS} + \phi(\tau)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2}, \\ \rho &= -\sqrt{-1}\partial\bar{\partial}\log(\phi\tau^{n-1}) + np^*\omega_{FS} \\ &= \left(-\phi' - \frac{(n-1)\phi}{\tau} + n\right)p^*\omega_{FS} - \phi\left(\phi' + \frac{(n-1)\phi}{\tau}\right)' \frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2}. \end{aligned}$$

Finally, taking the trace, we have

$$S(\eta) = \frac{-1}{\tau^2} (\tau^2 \phi'' + 2\tau(n-1)\phi' + (n-1)(n-2)\phi - \tau n(n-1)).$$

Now, in accordance with the above calculations, if we want to obtain a metric which extends across the zero section, and on the zero section restricts to the Fubini-Study metric, we need to impose the initial conditions on ϕ (which is defined on $[1, \infty)$) such that

$$\phi(1) = 0, \phi'(1) = 0.$$

In that case the scalar-flat equation is equivalent to

$$\frac{d^2}{d\tau^2} (\tau^{n-1}\phi) = \tau^{n-1}n(n-1).$$

Integrating and applying the boundary conditions we get

$$\phi(\tau) = \tau - (n-1)\tau^{2-n} + (n-2)\tau^{1-n}.$$

Now all we have to do is change coordinates back to s . Note we have that

$$\tau = \phi(\tau) \frac{d}{d\tau} f(s(\tau)),$$

and so

$$\frac{d}{d\tau} f(s(\tau)) = \tau \phi^{-1} = (1 - (n-1)\tau^{1-n} + (n-2)\tau^{-n})^{-1}.$$

Taylor expanding, for large τ we then have

$$\frac{d}{d\tau} f(s(\tau)) = 1 + (n-1)\tau^{1-n} - (n-2)\tau^{-n} + O(\tau^{2-2n}),$$

and so up to a constant we have

$$f(s(\tau)) = \tau - \frac{n-1}{n-2}\tau^{2-n} + \frac{n-2}{n-1}\tau^{1-n} + O(\tau^{3-2n}).$$

As a result we have

$$\frac{ds}{d\tau} = \phi^{-1} = \tau^{-1}(1 + (n-1)\tau^{1-n} - (n-2)\tau^{-n} + O(\tau^{2-2n}))$$

for large τ , and so after integrating, up to a constant we have

$$\log |(z, w)|_h^2 = s = \log \tau + O(\tau^{1-n}).$$

Plugging this into our asymptotic equation for $f(s)$, we then have

$$f(s) = |(z, w)|_h^2 - \frac{n-1}{n-2}|(z, w)|_h^{4-2n} + \frac{n-2}{n-1}|(z, w)|_h^{2-2n} + O(|z|^{6-4n}).$$

As we have already started doing previously, we will from now on use w to denote the natural holomorphic coordinate away from the exceptional divisor/zero section coming from the isomorphism $Bl_0\mathbb{C}^n \setminus E \cong \mathcal{O}(-1) \setminus \mathbb{P}^{n-1} \cong \mathbb{C}^n \setminus \{0\}$. In this coordinate, what we previously wrote as $|(z, w)|_h$ is just a scalar multiple of the squared norm of w . With this notation, the above findings can be summarized as follows.

Proposition 3.2.1. *If $n > 2$, there exists a smooth function $\psi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$ such that $\psi = O(|w|^{4-2n})$ and*

$$\eta = \sqrt{-1}\partial\bar{\partial}(|w|^2 + \psi(w))$$

extends to a scalar-flat Kähler metric on all of $Bl_0\mathbb{C}^n$, whose restriction to the exceptional divisor is a multiple of the Fubini-Study metric. When $n = 2$, the same is true for

$$\eta = \sqrt{-1}\partial\bar{\partial}(|w|^2 + \log |w|).$$

From now on, we refer to η freely as the Burns-Simanca metric. The special thing about these metrics in addition to the fact that they are scalar flat is that the components of the metric decay like $O(|w|^{2-2n})$, the components of the Ricci tensor decay like $O(|w|^{-2n})$, and they are therefore *asymptotically Euclidean*. This property will be used heuristically in our construction, as we will think ourselves as “gluing in” the Burns-Simanca metric to the original metric on $M \setminus \{p\}$, and if we choose holomorphic normal coordinates around p , the ends of the Burns-Simanca metric will “match-up” with the original metric near p .

Now, let (M, ω) be any compact Kähler manifold of dimension at least 2, and p any point in M which is henceforth fixed. We will now construct the initial metrics ω_ϵ on $Bl_p M$.

First of all, z will always be a holomorphic normal coordinate on M centered at p , so that in this neighborhood

$$\omega = \sqrt{-1}\partial\bar{\partial}(|z|^2 + \varphi(z)), \quad \varphi = O(|z|^4),$$

and without loss of generality we assume the coordinate exists in the ball B_2 , that is $|z| \leq 2$. We glue $Bl_0\mathbb{C}^n$ to $M_p := M \setminus \{p\}$ along the coordinate change $z = \epsilon w$, so $B_1 \subset M$ is identified with the ball of radius ϵ^{-1} in $Bl_0\mathbb{C}^n$. We will denote all balls around the exceptional divisor in $Bl_0\mathbb{C}^n$ with a tilde, so for example B_1 in M is identified with $\tilde{B}_{\epsilon^{-1}}$ in $Bl_0\mathbb{C}^n$. Note that these are not metric balls, rather just balls with respect to the coordinates z and w .

Now let γ be a smooth cutoff function which is 1 outside of B_2 and 0 inside of B_1 on M . Then we set $\gamma_1(z) = \gamma(z/r_\epsilon)$ and $\gamma_2 = 1 - \gamma_1$, where $r_\epsilon = \epsilon^\beta$ for some $\beta \in (1/2, 2/3)$ when $n = 2$, and $\beta = (n-1)/n$ when $n \geq 3$. This choice of β is taken both to ensure that these metrics are Kähler and for further more technical reasons to be seen later. Now we define

$$\begin{aligned} \omega_\epsilon &:= \sqrt{-1}\partial\bar{\partial}(|z|^2 + \gamma_1(z)\varphi(z) + \epsilon^2\gamma_2(z)\psi(\epsilon^{-1}z)) \text{ when } n \geq 3, \text{ and} \\ \omega_\epsilon &:= \sqrt{-1}\partial\bar{\partial}(|z|^2 + \gamma_1(z)\varphi(z) + \epsilon^2\gamma_2(z)\log |\epsilon^{-1}z|), \text{ when } n = 2. \end{aligned} \tag{3.2.2}$$

This is done in such a way so that $\omega_\epsilon \equiv \omega$ on $M \setminus B_{2r_\epsilon}$, and on B_{r_ϵ} we have that $\omega_\epsilon \equiv \epsilon^2\eta$ when we make the identification with $\tilde{B}_{\epsilon^{-1}r_\epsilon}$ in $Bl_0\mathbb{C}^n$. The Kähler class of ω_ϵ is $\pi^*[\omega] - \epsilon^2[E]$.

As written the constant ϵ^2 may not be precisely correct, but in any case it is a scalar multiple of ϵ^2 , and η can be adjusted without loss of generality to make ϵ^2 exact.

One more word on how to think about this gluing: we have not changed the metric outside of a small ball, and we are attaching a very small volume scaling down of the Burns-Simanca metric within that ball. The role of r_ϵ is also that, since it decays slower than ϵ , from the perspective of the Burns metric the region where the interpolation of the metrics ω and $\epsilon^2\eta$ occurs is going to infinity in the coordinate w , where the Burns-Simanca metric is closer to Euclidean. The metric ω_ϵ does not have scalar curvature that approximates ω in the gluing region, in fact it is always scalar-flat near the blowup point. However, this scalar-flat property will make the analysis of L_{ω_ϵ} easier.

3.3 Analysis in Weighted Spaces

The geometry of the blowup naturally breaks up the analysis into $M_p := M \setminus \{p\}$ for when we're away from the blowup point, $Bl_0\mathbb{C}^n$ for when we're near it, and also $\mathbb{C}^n \setminus \{0\}$ which corresponds to the region when we're interpolating between ω and $\epsilon^2\eta$, the “gluing region”. We will want to establish estimates that are uniform in ϵ , and to do so it will be necessary for us to work in function spaces that measure growth and decay of a specified amount at the ends, these are the weighted Hölder spaces. Then on the total space of $Bl_p M$ we will work in a glued version of these function spaces, introduced in [33] for the extremal metrics on blowups problem. For background on weighted Hölder spaces, see, for example, [27]. We start with the case of $\mathbb{R}^n \setminus \{0\}$.

Definition 3.3.1. *Let k a nonnegative integer, $\alpha \in (0, 1)$, and $\delta \in \mathbb{R}$. We define $C^{k,\alpha}(\mathbb{R}^n \setminus \{0\})$ to be the Banach space consisting of locally $C^{k,\alpha}$ functions $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ for which the norm*

$$\|f\|_{C^{k,\alpha}_\delta(\mathbb{R}^n \setminus \{0\})} := \sup_{r>0} \|f_r\|_{C^{k,\alpha}(B_2 \setminus B_1)} < \infty, \text{ where } f_r(x) := r^{-\delta} f(rx).$$

This norm seems opaque at first, but it is really just a slick way of requiring $\nabla^i f = O(|x|^{\delta-i})$ at zero and infinity.

Recall that the primary operator of interest to us is the linearization of the scalar curvature, which we have already seen to be the bilaplacian of the metric to leading order. Thus we require results on the Laplacian in these weighted spaces.

Proposition 3.3.2. *If $\delta \notin \mathbb{Z} \setminus (2 - n, 0)$, the Euclidean Laplacian viewed as a map*

$$\Delta_{euc} : C^{k,\alpha}_\delta(\mathbb{R}^n \setminus \{0\}) \rightarrow C^{k-2,\alpha}_{\delta-2}(\mathbb{R}^n \setminus \{0\})$$

is an isomorphism. Thus, the same is true for the Euclidean bilaplacian $\Delta_{euc}^2 : C^{k,\alpha}_\delta(\mathbb{R}^n \setminus \{0\}) \rightarrow C^{k-4,\alpha}_{\delta-4}(\mathbb{R}^n \setminus \{0\})$ for $\delta \notin \mathbb{Z} \setminus (4 - n, 0)$.

For a proof, see for example [27]. These discrete values for which the operators fail to be isomorphisms are known as *indicial roots*, they arise in the proof by viewing $\mathbb{R}^n \setminus \{0\}$ as a cone with link S^{n-1} , and then the nontrivial kernel at the indicial roots corresponds to the spectrum of the spherical Laplacian.

We now define the weighted spaces on M_p and $Bl_0\mathbb{C}^n$. These are similar to the $\mathbb{R}^n \setminus \{0\}$ case, except these spaces only have one end. Note that given a function $f : M_p \rightarrow \mathbb{R}$, we can view $(1 - \gamma)f$ as a function on $\mathbb{C}^n \setminus \{0\}$ via the coordinate z and extension by zero. Likewise, given a function $f : Bl_0\mathbb{C}^n \rightarrow \mathbb{R}$ we can view γf as a function on $\mathbb{C}^n \setminus \{0\}$. These spaces are also defined in [3].

Definition 3.3.3. *Let k a nonnegative integer, $\alpha \in (0, 1)$, and $\delta \in \mathbb{R}$.*

- Define $C_\delta^{k,\alpha}(M_p)$ to be the Banach space of locally $C^{k,\alpha}$ functions $f : M_p \rightarrow \mathbb{R}$ such that

$$\|f\|_{C_\delta^{k,\alpha}(M_p)} := \|f\|_{C_\omega^{k,\alpha}(M \setminus B_1)} + \|(1 - \gamma)f\|_{C_\delta^{k,\alpha}(\mathbb{C}^n \setminus \{0\})} < \infty.$$

- Define $C_\delta^{k,\alpha}(Bl_0\mathbb{C}^n)$ to be the Banach space of locally $C^{k,\alpha}$ functions $f : Bl_0\mathbb{C}^n \rightarrow \mathbb{R}$ such that

$$\|f\|_{C_\delta^{k,\alpha}(Bl_0\mathbb{C}^n)} := \|f\|_{C_\eta^{k,\alpha}(\tilde{B}_2)} + \|\gamma f\|_{C_\delta^{k,\alpha}(\mathbb{C}^n \setminus \{0\})} < \infty.$$

To give a look at how we will use these spaces, we will prove that L_η has no kernel when viewed as operators between them, if $\delta < 0$. This comes from the correspondence between holomorphic vector fields and the linearized scalar curvature operator of a constant scalar curvature Kähler metric as in Proposition 2.2.4, and Liouville's theorem.

Proposition 3.3.4. *If $\delta < 0$, then the operator*

$$L_\eta : C_\delta^{4,\alpha}(Bl_0\mathbb{C}^n) \rightarrow C_{\delta-4}^{0,\alpha}(Bl_0\mathbb{C}^n)$$

has trivial kernel.

Proof. Recall we have $L_\eta = -\mathcal{D}_\eta^* \mathcal{D}_\eta$. Suppose then we had some $\phi \in C_\delta^{4,\alpha}(Bl_0\mathbb{C}^n)$ in the kernel. The Burns-Simanca metric is asymptotically Euclidean with formula

$$\eta_{j\bar{k}} = \delta_{j\bar{k}} + O(|z|^{2-2n}).$$

If γ is a cutoff supported in $Bl_0\mathbb{C}^n \setminus B_1$ which is equal to 1 outside of B_2 , we can then think of $\gamma\phi$ as a function on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and compare with the Euclidean bilaplacian Δ_{euc}^2 . First, due to the comparison of the Burns-Simanca metric with the Euclidean metric we have

$$\Delta_{euc}^2(\gamma\phi) \in C_{\delta-2-2n}^{0,\alpha}(\mathbb{C}^n \setminus \{0\}).$$

Since the bilaplacian is an isomorphism in these spaces, we have that there exists a $\psi \in C_{\delta+2-2n}^{4,\alpha}(\mathbb{C}^n \setminus \{0\})$ such that

$$\Delta_{euc}^2(\psi) = \Delta^2(\gamma\phi).$$

Thus, $\psi - \gamma\phi$ is biharmonic and decays at infinity. There are no indicial roots in $(4 - 2n, 0)$, and so we have

$$\psi - \gamma\phi \in C_{4-2n}^{4,\alpha}(\mathbb{C}^n \setminus B_1).$$

The decay on ψ then implies that we actually must have had $\gamma\phi \in C_{\delta+2-2n}^{4,\alpha}(\mathbb{C}^n \setminus B_1)$, and thus

$$\phi \in C_{\delta+2-2n}^{4,\alpha}(Bl_0\mathbb{C}^n).$$

This decay is then fast enough to perform the integration by parts

$$0 = \int_{Bl_0\mathbb{C}^n} \phi \mathcal{D}^* \mathcal{D} \phi \eta^n = \int_{Bl_0\mathbb{C}^n} |\mathcal{D}\phi|^2 \eta^n,$$

and thus $\mathcal{D}\phi = 0$. This implies that the $(1, 0)$ -part of the gradient of ϕ is a holomorphic vector field on $Bl_0\mathbb{C}^n$, and since we can extend the resulting holomorphic vector field on $\mathbb{C}^n \setminus B_1$ across the ball by Hartog's extension theorem, it must correspond to a holomorphic vector field on $\mathbb{C}^n$ which decays at infinity. However, by Liouville's theorem, this vector field must vanish, which gives that ϕ is constant. Then, since ϕ decays at infinity, it must be 0. $\square$

Finally, we can define the weighted spaces on $Bl_p M$ as they are defined in [33]. We will provide two equivalent norms. To this end, let $f : Bl_p M \rightarrow \mathbb{R}$, and $\delta \in \mathbb{R}$. For $r \in (\epsilon, 1/2)$ we can define

$$\begin{aligned} f_r : B_2 \setminus B_1 \subset M &\rightarrow \mathbb{R} \\ f_r(z) &= r^{-\delta} f(rz), \end{aligned}$$

and also define

$$\begin{aligned} f_\epsilon : \tilde{B}_1 \subset Bl_0\mathbb{C}^n &\rightarrow \mathbb{R} \\ f_\epsilon(w) &= \epsilon^{-\delta} f(z), \end{aligned}$$

where, again, recall that $\tilde{B}_1 \subset Bl_0\mathbb{C}^n$ is identified with $B_\epsilon \subset Bl_p M$ via $z = \epsilon w$.

Definition 3.3.5. *Let k a nonnegative integer, $\alpha \in (0, 1)$, and $\delta \in \mathbb{R}$. We let $C_\delta^{k,\alpha}(Bl_p M)$ denote the Banach space of locally $C^{k,\alpha}$ functions on $Bl_p M$ with the norm*

$$\|f\|_{C_\delta^{k,\alpha}(Bl_p M)} = \|f\|_{C_\omega^{k,\alpha}(M \setminus B_1)} + \sup_{\epsilon < r < 1/2} r^{-\delta} \|f\|_{C_{r^{-2}\omega_\epsilon}^{k,\alpha}(B_{2r} \setminus B_r)} + \epsilon^{-\delta} \|f\|_{C_{\epsilon^{-2}\omega_\epsilon}^{k,\alpha}(B_\epsilon)}.$$

An equivalent norm in ϵ is given by

$$\|f\|_{C_\delta^{k,\alpha}(Bl_p M)} = \|f\|_{C_\omega^{k,\alpha}(M \setminus B_1)} + \sup_{\epsilon < r < 1/2} \|f_r\|_{C^{k,\alpha}(B_{2r} \setminus B_1)} + \|f_\epsilon\|_{C_\eta^{k,\alpha}(\tilde{B}_1)},$$

where in $B_2 \setminus B_1$ we are measuring the norms with the Euclidean metric.

The first definition can be convenient due to the fact that it is less coordinate dependent. The second is also convenient when working out computations solely in the normal coordinate z in the gluing region. The first definition naturally extends to a weighted norm for tensors on $Bl_p M$. For the second definition, to extend the norm to tensors, by the middle term we will mean to measure the components of the tensor with respect to the normal coordinate z . Despite the notation $C_\delta^{k,\alpha}(Bl_p M)$, the norms also depend on ϵ .

Note that the actual functions in these spaces never change with δ or ϵ , they always just consist of locally $C^{k,\alpha}$ functions on $Bl_p M$, only the norm changes with ϵ and δ . Once again, upon first glance these norms are somewhat complicated, but the idea is that if $\|f\|_{C_\delta^{k,\alpha}(Bl_p M)} \leq c$, then

$$\begin{aligned} |\nabla^i f| &\leq c, |z| \geq 1 \\ |\nabla^i f| &\leq c|z|^{\delta-i}, \epsilon \leq |z| \leq 1 \\ |\nabla^i f| &\leq c\epsilon^{\delta-i}, |z| \leq \epsilon. \end{aligned}$$

We end this section with various estimates on perturbations of ω_ϵ and corresponding geometric quantities in the weighted norms that we will need later. Let g_ϵ denote the metric associated to ω_ϵ . These mostly appear in [33].

Lemma 3.3.6. *For any nonnegative integer k , $\alpha \in (0, 1)$ and $\delta \in \mathbb{R}$, there is a constant c_0 with the following property: there is a constant $C > 0$ independent of ϵ sufficiently small such that for any $\|\phi\|_{C_2^{k+4,\alpha}(Bl_p M)} < c_0$ we have $\omega_\phi := \omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi$ is positive and*

$$\|g_\phi - g_\epsilon\|_{C_{\delta-2}^{k+2,\alpha}(Bl_p M)}, \|g_\phi^{-1} - g_\epsilon^{-1}\|_{C_{\delta-2}^{k+2,\alpha}(Bl_p M)}, \|Rm_{g_\phi} - Rm_{g_\epsilon}\|_{C_{\delta-4}^{k,\alpha}(Bl_p M)} < C\|\phi\|_{C_\delta^{k+4,\alpha}(Bl_p M)},$$

$$\|L_{\omega_\phi} - L_{\omega_\epsilon}\|_{C_\delta^{k+4,\alpha} \rightarrow C_{\delta-4}^{k,\alpha}}, \|L_{\omega_\phi}^* - L_{\omega_\epsilon}^*\|_{C_\delta^{k+4,\alpha} \rightarrow C_{\delta-4}^{k,\alpha}} < C\|\phi\|_{C_2^{k+4,\alpha}(Bl_p M)},$$

where g_ϕ denotes the metric associated to ω_ϕ .

Proof. Note that by the definition of the weighted norms we have

$$\|g_\epsilon\|_{C_0^{k,\alpha}(Bl_p M)} < C \tag{3.3.7}$$

for a constant independent of ϵ . The estimate on $g_\phi - g_\epsilon$ is clear by definition. We then have

$$g_\phi^{-1} - g_\epsilon^{-1} = g_\phi^{-1}(g_\epsilon - g_\phi)g_\epsilon^{-1}.$$

Then the desired estimate follows from

$$\|g_\phi^{-1} - g_\epsilon^{-1}\|_{C_{\delta-2}^{k+2,\alpha}} \leq C\|g_\phi^{-1}\|_{C_0^{k+2,\alpha}}\|\sqrt{-1}\partial\bar{\partial}\phi\|_{C_{\delta-2}^{k+2,\alpha}}\|g_\epsilon^{-1}\|_{C_0^{k+2,\alpha}}.$$

For the curvature estimate, if g is a Kähler metric we have the formula

$$Rm_{g,i\bar{j}k\bar{\ell}} = -\partial_k \partial_{\bar{\ell}} g_{i\bar{j}} + g^{p\bar{q}}(\partial_k g_{i\bar{q}})(\partial_{\bar{\ell}} g_{p\bar{j}}).$$

From this formula we have

$$(Rm_{g_\phi} - Rm_{g_\epsilon})_{i\bar{j}k\bar{\ell}} = -\partial_k \partial_{\bar{\ell}}(g_{\phi,i\bar{j}} - g_{\epsilon,i\bar{j}}) + g_\phi^{p\bar{q}}(\partial_k g_{\phi,i\bar{q}})(\partial_{\bar{\ell}} g_{\phi,p\bar{j}}) - g_\epsilon^{p\bar{q}}(\partial_k g_{\epsilon,i\bar{q}})(\partial_{\bar{\ell}} g_{\epsilon,p\bar{j}}),$$

and then it follows directly from the estimates on the metrics. The estimates on the linearized operator also follow, for example by writing

$$\Delta_{\omega_\phi}^2 \xi - \Delta_{\omega_\epsilon}^2 \xi = g_\phi^{p\bar{q}} \partial_p \partial_{\bar{q}}([g_\phi^{k\bar{\ell}} - g_\epsilon^{k\bar{\ell}}] \partial_k \partial_{\bar{\ell}} \xi) + (g_\phi^{p\bar{q}} - g_\epsilon^{p\bar{q}}) \partial_p \partial_{\bar{q}}(g_\epsilon^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} \xi),$$

and then we have

$$\|\Delta_{\omega_\phi}^2 \xi - \Delta_{\omega_\epsilon}^2 \xi\|_{C_{\delta-4}^{k,\alpha}} \leq C(\|g_\phi^{-1}\|_{C_0^{k,\alpha}} \|g_\phi^{-1} - g_\epsilon^{-1}\|_{C_0^{k+2,\alpha}} \|\xi\|_{C_\delta^{k+4,\alpha}} + \|g_\phi^{-1} - g_\epsilon^{-1}\|_{C_0^{k,\alpha}} \|g_\epsilon^{-1}\|_{C_0^{k+2,\alpha}} \|\xi\|_{C_\delta^{k+4,\alpha}})$$

We can similarly, using the bounds on the curvature tensor and the metric, get bounds on the Ricci and scalar curvatures. $\square$

Chapter 4

Perturbing to Approximate Scalar Curvature

In this chapter we complete the program of perturbing the family of Kähler metrics ω_ϵ constructed on $Bl_p M$, where (M, ω) was a compact Kähler manifold of dimension ≥ 2 , and $p \in M$ some fixed point, to have scalar curvature approximating $S(\omega)$. In the first section, we use the weighted spaces of the previous chapter to study L_{ω_ϵ} , but we will see that we will in reality have to work with a perturbation of the adjoint of this operator. The operator we are able to prove uniform estimates for in the end informs what to put on the right hand side of our toy equation

$$S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi) - S(\omega) = \text{“something decaying in } \epsilon\text{”} \quad (4.0.1)$$

so that it is solvable in ϕ for small ϵ . Due to the lack of decay in the potential for the Burns-Simanca metric in complex dimension 2, we have to slightly refine the metric ω_ϵ to get the proof to go through in the $n = 2$ case. After this alteration the proof goes through with minimal changes, as we will see in the third section, where we apply our estimates to finish the perturbation program. In the final section we discuss the application of our results to positive scalar curvature Kähler surfaces.

4.1 The Linearized Operator

In this section is that we prove similar estimates on L_{ω_ϵ} in ϵ as appear for example in [33], [31], for a perturbation of the formal adjoint $L_{\omega_\epsilon}^*$. There are many approaches one can take in this step, and one can find several in the literature on blowing up extremal metrics. The one that we manage to make work in our particular problem is the approach that appears in [31].

One encounters some extra analytic difficulties in $n = 2$ when proving estimates on L_{ω_ϵ} in our setting, one of those being, as in the case of extremal metrics, the weaker estimates on L_{ω_ϵ} compared to what we can prove when $n \geq 3$, but the more problematic issue in $n = 2$

is that we are unable to prove the desired estimates on L_{ω_ϵ} at all when the kernel of L_ω^* only consists of functions vanishing at the blow up point. This is the reason why we prove the estimates for L_ω^* instead of L_ω , as $(L_\omega^*)^* = L_\omega$ always at least has constants in its kernel; this is used in step 1.5 of the proof of Proposition 4.1.8. Note that this fact, together with the fact that L_ω is index zero, guarantees that L_ω^* always has a nontrivial kernel.

It is crucial to our argument to notice

$$(L_\omega^* - L_\omega)\phi = -\partial\phi \cdot \bar{\partial}S(\omega) - \partial S(\omega) \cdot \bar{\partial}\phi - \frac{1}{2}\phi\Delta_\omega S(\omega). \quad (4.1.1)$$

In particular, L_ω^* and L_ω only differ by first order terms in ϕ . Note that the highest order terms in L_ω and L_ω^* are the bilaplacian associated to ω . Not only that, but on M_p in the normal coordinate z these terms approximate the usual Euclidean bilaplacian near p , and since η is asymptotically Euclidean and scalar flat, $L_\eta = L_\eta^*$ approximates the Euclidean bilaplacian near infinity.

In general we know little about the kernel of L_ω^* when ω is not cscK. However, fix an L^2 -orthonormal basis $f_1, \dots, f_d$ for the kernel of L_ω .

Lemma 4.1.2. *There exists points $q_1, \dots, q_d \in M_p$ such that the operator*

$$\tilde{L}_\omega^* : C^{4,\alpha}(M) \rightarrow C^{0,\alpha}(M)$$

$$\tilde{L}_\omega^* \phi := L_\omega^* \phi - \sum_{i=1}^d \phi(q_i) f_i$$

is an isomorphism.

Proof. First, regardless of the choice of $q_1, \dots, q_d$, if we have $\tilde{L}_\omega^* \phi = 0$, then we must have that $L_\omega^* \phi = 0$ since the f_i are each orthogonal to the image of L_ω^* .

We claim that we can find $q_1, \dots, q_d \in M_p$, and a basis $\phi_1, \dots, \phi_d$ for $\ker(L_\omega^*)$ such that for $1 \leq i \leq d$ the vectors

$$v_i := (\phi_i(q_1), \dots, \phi_i(q_d))$$

are linearly independent. If this were true, then suppose there was some nonzero ϕ in the kernel of $\tilde{L}_\omega^*$. Then ϕ is also in the kernel of L_ω^* so $\phi = \sum_{j=1}^d c_j \phi_j$, and we would have

$$0 = \sum_{j=1}^d c_j \phi_j(q_i)$$

for each i . This means there is a nonzero vector orthogonal to each v_i^T , which contradicts the linear independence of the v_i , so $\tilde{L}_\omega^*$ is injective. For surjectivity, $\tilde{L}_\omega^*$ differs from the index zero operator L_ω^* by an operator of finite rank, and so is also index zero. Then injectivity implies surjectivity.

Now we prove our claim. We do so inductively; first, if $\phi_1 \in \ker(L_\omega^*)$ is nonzero then of course we can find a point $q_1 \in M_p$ such that $\phi_1(q_1) \neq 0$. Now, suppose we have found

points $q_1, \dots, q_{d-1}$ and linear independent functions $\phi_1, \dots, \phi_{d-1}$ in the kernel of L_ω^* such that for $1 \leq i \leq d-1$ the vectors

$$v'_i := (\phi_i(q_1), \dots, \phi_i(q_{d-1}))$$

are linearly independent. Let $\phi_d \in \ker(L_\omega^*)$ linearly independent of $\phi_1, \dots, \phi_{d-1}$. Suppose there is $q_d \in M_p$ such that $\phi_d(q_d) \neq 0$ and v_d is linearly dependent of $v_1, \dots, v_{d-1}$ defined in the same way as above (they themselves are linearly independent due to the linear independence of the v'_i). Then we have that there exists constant $a_1, \dots, a_{d-1}$ such that

$$\sum_{j=1}^{d-1} a_j \phi_j(q_i) - \phi_d(q_i) = 0 \text{ for } 1 \leq i \leq d, \sum_{j=1}^{d-1} a_j \phi_j - \phi_d \neq 0.$$

Then replacing ϕ_d with $\sum_{j=1}^{d-1} a_j \phi_j - \phi_d$ and replacing q_d with a point where this function is nonzero yields the claim, since in that case v_d would be equal to zero in its first $d-1$ components.

If we could not find such a q_d , then $\phi_d(q_d) \neq 0$ implies that $v_1, \dots, v_d$ are linearly independent as desired. $\square$

From now on we fix $q_1, \dots, q_d \in M_p$ to be a set of points for which the conclusion of the above lemma holds. When we blow up, we are pushed onto studying the operator

$$\tilde{L}_{\omega_\epsilon}^* : C_\delta^{4,\alpha}(Bl_p M) \rightarrow C_\delta^{0,\alpha}(Bl_p M)$$

$$\tilde{L}_{\omega_\epsilon}^* \phi := L_{\omega_\epsilon}^* \phi - \sum_{i=1}^d \phi(q_i) f_i. \quad (4.1.3)$$

Due to the geometry of ω_ϵ , on M_p this operator behaves like $\tilde{L}_\omega^*$ restricted to M_p so we need:

Proposition 4.1.4. *Consider for now $\tilde{L}_\omega^*$ as an operator $\tilde{L}_\omega^* : C_\delta^{4,\alpha}(M_p) \rightarrow C_{\delta-4}^{0,\alpha}(M_p)$.*

- If $\delta > 4 - 2n$, then $\tilde{L}_\omega^*$ has trivial kernel.
- If $n = 2$, fix $\delta \in (-1, 0)$. Then if $\phi \in C_\delta^{4,\alpha}(M_p)$ is in the kernel of $\tilde{L}_\omega^*$, in B_1 we have

$$\phi = a \log |z| + b + \theta$$

for some $a, b \in \mathbb{R}$ and $\theta \in C_{\delta+2}^{4,\alpha}(M_p)$.

Proof. The proof of the first claim is exactly the same as what appears in [3], or [31]. For the second claim, first note that since near p we can use holomorphic normal coordinates $\omega_{j\bar{k}} = \delta_{j\bar{k}} + O(|z|^2)$, if $f \in C_\delta^{k,\alpha}(M_p)$ we have

$$\tilde{L}_\omega^*(f) + \frac{1}{4} \Delta_{\text{euc}}^2(f) \in C_{\delta-2}^{k-4,\alpha}(M_p) \quad (4.1.5)$$

as long as $\delta < 2$ (otherwise we have to account for the zeroth order terms in $\tilde{L}_\omega^*$). We would usually only expect $C_{\delta-4}^{k-4,\alpha}(M_p)$ since these are fourth order operators.

Now consider $\phi \in C_\delta^{4,\alpha}(M_p)$ such that $\tilde{L}_\omega^* \phi = 0$. If we let χ be a cutoff supported in B_1 that is equal to 1 in $B_{1/2}$, then from 4.1.5

$$\Delta_{euc}^2(\chi\phi) \in C_{\delta-2}^{0,\alpha}(\mathbb{C}^2 \setminus \{0\}).$$

From Proposition 3.3.2 there is $\theta \in C_{\delta+2}^{4,\alpha}(\mathbb{C}^2 \setminus \{0\})$ such that $\theta - \chi\phi$ is biharmonic and in $C_\delta^{4,\alpha}(\mathbb{C}^2 \setminus \{0\})$. The only biharmonic functions with $\delta \in (-1, 0)$ are multiples of $\log|z|$ and constants, and thus there are constants $a, b \in \mathbb{R}$ such that

$$\chi\phi = a \log|z| + b + \theta.$$

□

Near the exceptional divisor $L_{\omega_\epsilon}^* = L_{\epsilon^2\eta}^* = L_{\epsilon^2\eta} = \frac{1}{\epsilon^4}L_\eta$, where the last two equalities are due to the fact that η is cscK and the (-2) -homogeneity of the scalar curvature, respectively. There, we are able to use the injectivity of L_η when viewed as an operator on the weighted spaces on $Bl_0\mathbb{C}^n$ when $\delta < 0$.

Finally, to prove the desired estimates on $\tilde{L}_{\omega_\epsilon}^*$ we need Schauder estimates which are independent of ϵ .

Lemma 4.1.6. *Fix $\delta < 0$. There exists a positive constant C independent of all $\epsilon > 0$ small such that for all $\phi \in C_\delta^{4,\alpha}(Bl_p M)$*

$$\|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq C(\|\phi\|_{C_\delta^0(Bl_p M)} + \|\tilde{L}_{\omega_\epsilon}^* \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}).$$

Proof. First, we define a function $\rho : Bl_p M \rightarrow \mathbb{R}$ which describes the weights in the norms on $Bl_p M$,

$$\rho(z) = \begin{cases} 1 & |z| \geq 1 \\ |z| & \epsilon \leq |z| \leq 1 \\ \epsilon & |z| \leq \epsilon \end{cases}.$$

Now, η is asymptotically Euclidean, M is compact, and the sectional curvatures of ω_ϵ satisfy a uniform bound in the $C_{-2}^{k,\alpha}(Bl_p M)$ norm 3.3.7, and thus we have that there exists some $r > 0$ independent of ϵ such that the injectivity radius of ω_ϵ is bounded below by $r\rho(z)$. This implies that for any $\kappa > 0$ one can find fix an r small enough independent of ϵ such that at each point $x \in Bl_p M$, in the normal coordinates of radius $r\rho(x)$ with respect to ω_ϵ centered at x we have

$$\frac{1}{1+\kappa} \|\cdot\|_{C_{(r\rho(x))^{-2}\omega_\epsilon}^{k,\alpha}} \leq \|\cdot\|_{C^{k,\alpha}} \leq (1+\kappa) \|\cdot\|_{C_{(r\rho(x))^{-2}\omega_\epsilon}^{k,\alpha}}, \quad (4.1.7)$$

where the middle norm is measured with respect to the Euclidean metric in the normal coordinates. Since $L_{\omega_{euc}}^* = -\frac{1}{4}\Delta_{euc}^2$, after possibly taking r even smaller we further have for any ϕ

$$\frac{1}{1+\kappa} \|L_{(r\rho(x))^{-2}\omega_\epsilon}^* \phi\|_{C^{k-4,\alpha}} \leq \|\frac{1}{4}\Delta_{euc}^2 \phi\|_{C^{k-4,\alpha}} \leq (1+\kappa) \|L_{(r\rho(x))^{-2}\omega_\epsilon}^* \phi\|_{C^{k-4,\alpha}},$$

again in the normal coordinate ball, and these norms are measured in the Euclidean norm with respect to the normal coordinates. Now, applying the usual Schauder estimates for the Euclidean bilaplacian, there is a constant $D > 0$ such that in any of these normal coordinate balls

$$\|\phi\|_{C^{k,\alpha}} \leq D(\|\phi\|_{C^0} + \|\frac{1}{4}\Delta_{euc}^2\phi\|_{C^{k-4,\alpha}}).$$

Due to the comparisons of the Euclidean Hölder norms with the Hölder norms for ω_ϵ 4.1.7 we then have

$$\|\phi\|_{C_{(r\rho(x))^{-2}\omega_\epsilon}^{k,\alpha}} \leq D(1 + \kappa)^2(\|\phi\|_{C^0} + \|L_{(r\rho(x))^{-2}\omega_\epsilon}^*\phi\|_{C^{k-4,\alpha}}).$$

$D(1 + \kappa)^2$ has no dependence on ϵ . Noting that $L_{(r\rho(x))^{-2}\omega_\epsilon}^* = (r\rho(x))^4 L_{\omega_\epsilon}^*$ and applying this estimate to $\rho(x)^{-\delta}\phi$ over all of these normal coordinate balls we have a $C > 0$ independent of ϵ such that

$$\|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq C(\|\phi\|_{C_\delta^0(Bl_p M)} + \|L_{\omega_\epsilon}^*\phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}).$$

Now to get the estimate for $\tilde{L}_{\omega_\epsilon}^*$, note that there is $c > 0$ independent of ϵ such that

$$\left| \|\tilde{L}_{\omega_\epsilon}^*\phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} - \|L_{\omega_\epsilon}^*\phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \right| \leq c\|\phi\|_{C_\delta^0(Bl_p M)} \max_i \|f_i\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}.$$

The result then follows from the fact that the norms $\|f_i\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}$ are uniformly bounded from above in ϵ when $\delta < 0$. $\square$

We are now ready to prove the main estimate.

Proposition 4.1.8. *Fix $\delta \in (-1, 0)$ sufficiently small. Consider the operator*

$$\tilde{L}_{\omega_\epsilon}^* : C_\delta^{4,\alpha}(Bl_p M) \rightarrow C_{\delta-4}^{0,\alpha}(Bl_p M).$$

- Suppose $n > 2$. For $\epsilon > 0$ small there exists a constant $K > 0$ independent of ϵ such that for all $\phi \in C_\delta^{4,\alpha}(Bl_p M)$,

$$\|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq K\|\tilde{L}_{\omega_\epsilon}^*\phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}.$$

- Suppose $n = 2$. For $\epsilon > 0$ small there exists a constant $K > 0$ independent of ϵ such that for all $\phi \in C_\delta^{4,\alpha}(Bl_p M)$,

$$\|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq K\epsilon^\delta\|\tilde{L}_{\omega_\epsilon}^*\phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}.$$

Proof. We will prove both cases simultaneously and comment where the differences occur. We argue by contradiction: suppose there exists a sequence of functions ϕ_i such that

$$\|\phi_i\|_{C_\delta^{4,\alpha}(Bl_p M)} = 1, \quad \|\tilde{L}_{\omega_{\epsilon_i}}^*\phi_i\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \leq \frac{1}{i}, \quad (4.1.9)$$

in the $n > 2$ case, and in the $n = 2$ case we assume furthermore there is $\epsilon_i \rightarrow 0$ such that

$$\|\phi_i\|_{C_\delta^{4,\alpha}(Bl_p M)} = 1, \|\tilde{L}_{\omega_{\epsilon_i}}^* \phi_i\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \leq \frac{\epsilon_i^{-\delta}}{i}. \quad (4.1.10)$$

From the uniform Schauder estimates Lemma 4.1.6, we have up to choosing a subsequence $\phi_i \rightarrow \phi_\infty$ locally in $C^{4,\alpha}$ on M_p with $\phi_\infty \in C_\delta^{4,\alpha}(M_p)$, where we may have had to replace α which a slightly smaller constant.

Step 1: We show $\phi_\infty \equiv 0$ on M_p . When $n > 2$ this immediately follows from the fact that $\tilde{L}_\omega^* \phi_\infty = 0$ on M_p and our choice of δ due to Proposition 4.1.4. When $n = 2$, our slightly stronger assumption 4.1.10 gives for each $1 \leq j \leq d$

$$\left| \int_{Bl_p M} f_j \tilde{L}_{\omega_{\epsilon_i}}^* (\phi_i) \frac{\omega_{\epsilon_i}^2}{2} \right| = \left| \int_{Bl_p M} f_j L_{\omega_{\epsilon_i}}^* (\phi_i) \frac{\omega_{\epsilon_i}^2}{2} - \sum_{k=1}^d \phi_i(q_k) \int_{Bl_p M} f_j f_k \frac{\omega_{\epsilon_i}^2}{2} \right| < \frac{C}{i}, \quad (4.1.11)$$

where C is a constant independent of ϵ . First note that since $f_1, \dots, f_d$ were assumed to be L_2 -orthonormal that

$$\lim_{i \rightarrow \infty} \int_{Bl_p M} f_j f_k \frac{\omega_{\epsilon_i}^2}{2} = \delta_{jk},$$

and on the other hand we have

$$\int_{Bl_p M} f_j L_{\omega_{\epsilon_i}}^* (\phi_i) \frac{\omega_{\epsilon_i}^2}{2} = \int_{Bl_p M} L_{\omega_{\epsilon_i}} (f_j) \phi_i \frac{\omega_{\epsilon_i}^2}{2}.$$

Now we need to analyze this term as $i \rightarrow \infty$. We break up our manifold into $M \setminus B_{2r_{\epsilon_i}}$, $B_{2r_{\epsilon_i}} \setminus B_{r_{\epsilon_i}}$, and $B_{r_{\epsilon_i}}$, recalling that $r_\epsilon = \epsilon^\beta$ as defined in section 2. For $M \setminus B_{2r_{\epsilon_i}}$, in this region $\omega_{\epsilon_i} = \omega$ so we have

$$\int_{M \setminus B_{2r_{\epsilon_i}}} L_{\omega_{\epsilon_i}} (f_j) \phi_i \frac{\omega_{\epsilon_i}^2}{2} = \int_{M \setminus B_{2r_{\epsilon_i}}} L_\omega (f_j) \phi_i \frac{\omega^2}{2} = 0.$$

Next, in $B_{r_{\epsilon_i}}$ after the coordinate change $z = \epsilon_i w$ we have $\omega_{\epsilon_i} = \epsilon_i^2 \eta$. Note that $L_{\epsilon^2 \eta} = \frac{1}{\epsilon^4} L_\eta$, so from the equation for the linearization of scalar curvature we have

$$\begin{aligned} \int_{B_{r_{\epsilon_i}}} L_{\omega_{\epsilon_i}} (f_j) \phi_i \frac{\omega_{\epsilon_i}^2}{2} &= \int_{\tilde{B}_{r_{\epsilon_i} \epsilon_i^{-1}}} - \left(\frac{1}{4} \Delta_\eta^2 (f_j(\epsilon_i w)) + \frac{1}{2} \text{Ric}_\eta \cdot \nabla_\eta^2 (f_j(\epsilon_i w)) \right) \phi_i(\epsilon_i w) \frac{\eta^2}{2} \\ &= \int_{\tilde{B}_{r_{\epsilon_i} \epsilon_i^{-1}}} - [\eta^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} (\eta^{a\bar{b}} \partial_a \partial_{\bar{b}} (f_j(\epsilon_i w))) + \text{Ric}_\eta^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} (f_j(\epsilon_i w))] \phi_i(\epsilon_i w) \frac{\eta^2}{2}. \end{aligned}$$

We also have that from the asymptotics of the Burns-Simanca metric, $\eta^{k\bar{\ell}} = \delta^{k\bar{\ell}} + O(|w|^{-2})$, $\text{Ric}_\eta^{k\bar{\ell}} = O(|w|^{-4})$, $|\nabla^k (f_j(\epsilon w))| \leq C \epsilon^k$, and $|\phi_i(\epsilon w)| = |\phi_i(z)| \leq C \epsilon^\delta$, where C represents possibly different constants but nevertheless is independent of ϵ . Thus,

$$\left| \int_{B_{r_{\epsilon_i}}} L_{\omega_{\epsilon_i}} (f_j) \phi_i \frac{\omega_{\epsilon_i}^2}{2} \right| \leq C (\epsilon_i^4 + \epsilon_i^2) \epsilon^\delta \text{Vol}_\eta(\tilde{B}_{r_{\epsilon_i} \epsilon_i^{-1}}) \leq C \epsilon_i^2 \epsilon_i^\delta r_{\epsilon_i}^4 \epsilon_i^{-4} = C \epsilon_i^{4\beta-2+\delta},$$

so this goes to zero as $i \rightarrow \infty$ as long as $\beta > 1/2$ and δ is taken small enough. Lastly, we look at the integral over $B_{2r_{\epsilon_i}} \setminus B_{r_{\epsilon_i}}$. Here we have

$$\int_{B_{2r_{\epsilon_i}} \setminus B_{r_{\epsilon_i}}} L_{\omega_{\epsilon_i}}(f_j) \phi_i \frac{\omega_{\epsilon_i}^2}{2} = \int_{B_{2r_{\epsilon_i}} \setminus B_{r_{\epsilon_i}}} - \left(\frac{1}{4} \Delta_{\omega_{\epsilon_i}}^2 f_j + \frac{1}{2} \text{Ric}_{\omega_{\epsilon_i}} \cdot \nabla_{\omega_{\epsilon_i}}^2 f_j \right) \phi_i \frac{\omega_{\epsilon_i}^2}{2}.$$

From the explicit formula for ω_{ϵ_i} 3.2.2, in the region $r_{\epsilon_i} \leq |z| \leq 2r_{\epsilon_i}$ we have

$$\omega_{\epsilon_i}^{k\bar{\ell}} = \delta^{k\bar{\ell}} + O\left(\frac{\epsilon_i^2}{r_{\epsilon_i}^2} \log |\epsilon_i|\right), \quad \text{Ric}_{\omega_{\epsilon_i}}^{k\bar{\ell}} = O\left(\frac{\epsilon_i^2}{r_{\epsilon_i}^4} \log |\epsilon_i|\right),$$

and thus

$$\begin{aligned} \left| \int_{B_{2r_{\epsilon_i}} \setminus B_{r_{\epsilon_i}}} L_{\omega_{\epsilon_i}}(f_j) \phi_i \frac{\omega_{\epsilon_i}^2}{2} \right| &\leq C \left(1 + \frac{\epsilon_i^2}{r_{\epsilon_i}^4} \log |\epsilon_i| \right) \epsilon^\delta \text{Vol}_{\omega_{\epsilon_i}}(B_{2r_{\epsilon_i}} \setminus B_{r_{\epsilon_i}}) \\ &\leq C \epsilon_i^2 r_{\epsilon_i}^{-4} \log |\epsilon_i| \epsilon_i^\delta r_{\epsilon_i}^4 = C \epsilon_i^{2+\delta} \log |\epsilon_i|. \end{aligned}$$

This goes to zero as $i \rightarrow \infty$ for δ small. The upshot is that from 4.1.11 we then get

$$\lim_{i \rightarrow \infty} \left| \int_{B_{l_p M}} f_j \tilde{L}_{\omega_{\epsilon_i}}^*(\phi_i) \frac{\omega_{\epsilon_i}^2}{2} \right| = |\phi_\infty(q_j)| = 0,$$

and thus on M_p , we not only have $\tilde{L}_\omega^* \phi_\infty = 0$ but

$$L_\omega^* \phi_\infty = 0.$$

We then have that in B_1 there is $a, b \in \mathbb{R}$ and $\theta \in C_{\delta+2}^{4,\alpha}(M_p)$ such that $\phi_\infty = a \log |z| + b + \theta$ from Proposition 4.1.4.

Step 1.5: Continuing our focus on the $n = 2$ case, we now show that there is a constant $\kappa \neq 0$ such that for any $\xi \in C^\infty(M)$,

$$\int_M L_\omega(\xi) \phi_\infty \frac{\omega^2}{2} = -a \kappa \xi(p).$$

Thus, testing this equation with $\xi \equiv 1$, we must have that $a = 0$. Therefore ϕ_∞ extends smoothly to all of M , being a bounded weak solution to an elliptic equation on M . Then $\phi_\infty \equiv 0$ by the invertibility of $\tilde{L}_\omega^*$ on M (Lemma 4.1.2), as desired. This is the key step that necessitates us proving estimates on $L_{\omega_\epsilon}^*$ instead of L_{ω_ϵ} . Anyway, we write

$$\int_M L_\omega(\xi) \phi_\infty \frac{\omega^2}{2} = \lim_{r \rightarrow 0} \int_{M \setminus B_r} - \left(\frac{1}{4} \Delta_\omega^2 \xi + \frac{1}{2} \text{Ric}_\omega \cdot \nabla_\omega^2 \xi \right) \phi_\infty \frac{\omega^2}{2}.$$

Now we apply integration by parts to each term in the integrand. Denoting by N the unit normal to the boundary sphere ∂B_r , we first look at

$$\begin{aligned} \int_{M \setminus B_r} \Delta_\omega^2(\xi) \phi_\infty \frac{\omega^2}{2} &= \int_{M \setminus B_r} \xi \Delta_\omega^2(\phi_\infty) \frac{\omega^2}{2} - \int_{\partial B_r} \xi \nabla_N(\Delta_\omega \phi_\infty) \text{Vol}_{\partial B_r} + \int_{\partial B_r} \Delta_\omega(\phi_\infty) \nabla_N(\xi) \text{Vol}_{\partial B_r} \\ &\quad - \int_{\partial B_r} \Delta_\omega(\xi) \nabla_N(\phi_\infty) \text{Vol}_{\partial B_r} + \int_{\partial B_r} \phi_\infty \nabla_N(\Delta_\omega \xi) \text{Vol}_{\partial B_r}. \end{aligned}$$

We investigate the boundary terms as $r \rightarrow 0$. Since ϕ_∞ behaves like $a \log |z|$ near zero, we have that the last three terms all go to zero in r since the terms involving the derivatives of ϕ_∞ blow up at most like $O(r^{-2})$, whereas the volume of ∂B_r decays like $O(r^3)$. The only issue is the first boundary term. When r is small we have

$$\xi \nabla_N(\Delta_\omega \phi_\infty) = a \xi \nabla_N[(\delta^{k\bar{\ell}} + O(r^2)) \partial_k \partial_{\bar{\ell}} \log |z|] + \xi \nabla_N(\Delta_\omega \theta).$$

Due to the estimate on θ , the last term decays at worst like $O(r^{\delta-1})$, and so integrating and taking the limit we have

$$\begin{aligned} \lim_{r \rightarrow 0} \int_{M \setminus B_r} -\frac{1}{4} \Delta_\omega^2(\xi) \phi_\infty \frac{\omega^2}{2} &= \int_M -\frac{1}{4} \Delta_\omega^2(\phi_\infty) \xi \frac{\omega^2}{2} + \lim_{r \rightarrow 0} \frac{a}{4} \int_{\partial B_r} \xi \nabla_N(\Delta_{euc} \log |z|) \text{Vol}_{\partial B_r} \\ &= \int_M -\frac{1}{4} \Delta_\omega^2(\phi_\infty) \xi \frac{\omega^2}{2} + a \kappa \xi(p). \end{aligned}$$

For the other integration by parts, we have

$$\begin{aligned} &\int_{M \setminus B_r} (\text{Ric}_\omega \cdot \nabla_\omega^2 \xi) \phi_\infty \frac{\omega^2}{2} \\ &= \int_{M \setminus B_r} \xi \text{tr}_\omega \nabla_\omega (\text{tr}_\omega \nabla_\omega (\phi_\infty \text{Ric}_\omega)) \frac{\omega^2}{2} + \int_{\partial B_r} \langle \nabla \xi \otimes N^b, \phi_\infty \text{Ric}_\omega \rangle \text{Vol}_{\partial B_r} \\ &\quad - \int_{\partial B_r} \xi (\text{tr}_\omega \nabla (\phi_\infty \text{Ric}_\omega))(N) \text{Vol}_{\partial B_r}. \end{aligned}$$

Both of the boundary terms have integrands which blow up no worse than $O(r^{-1})$, so both boundary terms go to zero as $r \rightarrow 0$. Thus, we finally have that

$$\int_M L_\omega(\xi) \phi_\infty \frac{\omega^2}{2} = \int_M \xi L_\omega^*(\phi_\infty) \frac{\omega^2}{2} - a \kappa \xi(p) = -a \kappa \xi(p),$$

since ϕ_∞ is in the kernel of L_ω^* on M_p .

Step 2: The rest of the proof is now the same in any dimension and proceeds as in [31]. To recount, we have as in 4.1.9 ϕ_i such that

$$\|\phi_i\|_{C_\delta^{4,\alpha}(Bl_p M)} = 1, \|\tilde{L}_{\omega_{\epsilon_i}}^*(\phi_i)\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} < \frac{1}{i},$$

(the second bound is actually slightly stronger in $n = 2$) and $\phi_i \rightarrow 0$ locally in $C^{4,\alpha}$, where again we may have to replace α with a slightly smaller constant. We then have the uniform Schauder estimate of Lemma 4.1.6

$$\|\phi_i\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq C(\|\phi_i\|_{C_\delta^0(Bl_p M)} + \|\tilde{L}_{\omega_{\epsilon_i}}^*(\phi_i)\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}),$$

so then we have

$$1 \leq C\|\phi_i\|_{C_\delta^0(Bl_p M)} + \frac{C}{i} \iff \frac{1}{C} - \frac{1}{i} \leq \|\phi_i\|_{C_\delta^0(Bl_p M)} \leq 1,$$

and so $\|\phi_i\|_{C_\delta^0(Bl_p M)}$ is uniformly bounded below by a positive constant, say C' . Applying the uniform Schauder estimates to $\psi_i := \frac{\phi_i}{\|\phi_i\|_{C_\delta^0(Bl_p M)}}$ we get

$$\|\psi_i\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq C + \frac{C}{i\|\phi_i\|_{C_\delta^0(Bl_p M)}} \leq C + \frac{C}{iC'}.$$

Since the last term goes to zero we have a family of functions such that

$$\|\psi_i\|_{C_\delta^0(Bl_p M)} = 1, \|\psi_i\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq C, \quad (4.1.12)$$

$$\|L_{\omega_{\epsilon_i}}^*(\psi_i)\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \rightarrow 0, \psi_i \rightarrow 0 \text{ locally in } C^{4,\alpha} \text{ on } M_p. \quad (4.1.13)$$

Now, we want to look at the point $x_i \in Bl_p M$ where $\rho_i^{-\delta}\psi_i$ achieves its maximum, where ρ_i is as defined in the proof of Lemma 4.1.6 but with respect to ϵ_i , which due to the above assumptions is when

$$\rho_i^{-\delta}(x_i)\psi_i(x_i) = 1.$$

Since $\psi_i \rightarrow 0$ locally on M_p , $\rho_i(x_i) \rightarrow 0$. Then $\epsilon_i^{-1}\rho_i(x_i)$ is either bounded or unbounded; one can think of the former case as the ψ_i concentrating close to the exceptional divisor as $i \rightarrow \infty$, the latter case is when ψ_i concentrates in the Euclidean neck region.

Step 3: We first suppose the former. In this case we have that $\epsilon_i^{-1}\rho_i(x_i) < R$ for some R and for all i . Then under the identification of $B_1 \subset Bl_p M$ with the ball $\tilde{B}_{\epsilon^{-1}} \subset Bl_0 \mathbb{C}^n$, we have that all the x_i are inside $\tilde{B}_R$. Thus, up to taking a subsequence we have $x_i \rightarrow x_\infty \in \tilde{B}_R$. Rescaling by ϵ_i , $\epsilon_i^{-\delta}\psi_i$ can be viewed as functions on larger subsets of $Bl_0 \mathbb{C}^n$ still with a uniform $C_\delta^{4,\alpha}(Bl_0 \mathbb{C}^n)$ bound from 4.1.12. Thus we have that up to a subsequence and replacing α , $\epsilon_i^{-\delta}\psi_i \rightarrow \psi_\infty$ locally in $C^{4,\alpha}$ on $Bl_0 \mathbb{C}^n$.

Now note that on $Bl_0 \mathbb{C}^n$ we have that $\epsilon_i^{-2}\omega_{\epsilon_i}$ locally converges to the Burns-Simanca metric η and thus since

$$\epsilon_i^{-\delta}\|L_{\omega_{\epsilon_i}}^*\psi_i\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \rightarrow 0 \text{ from 4.1.13, and } L_{\epsilon_i^{-2}\omega_{\epsilon_i}} = \epsilon_i^4 L_{\omega_{\epsilon_i}},$$

we have $L_\eta \psi_\infty = 0$ on $Bl_0 \mathbb{C}^2$. On the other hand by assumption

$$\epsilon_i^{-\delta}\rho_i(x_i)^\delta > R^\delta,$$

and by definition of x_i

$$\psi_i(x_i) = \rho_i(x_i)^\delta,$$

so

$$\psi_\infty(x_\infty) \geq R^\delta.$$

Thus, we have a contradiction of Proposition 3.3.4 since ψ_∞ decays at infinity but is nonzero.

Step 4: Lastly, we need to deal with the case when $\epsilon_i^{-1}\rho_i(x_i)$ is unbounded. Note again that $\overline{\rho_i(x_i)} \rightarrow 0$, and thus in this case we have $\rho_i(x_i) = |x_i|$. Choose $r_i \rightarrow 0$ and $R_i \rightarrow \infty$ such that

$$R_i|x_i| \rightarrow 0, \quad r_i|x_i|\epsilon_i^{-1} \rightarrow \infty.$$

We consider the annulus $A_i := B_{R_i|x_i|} \setminus B_{r_i|x_i|}$. By rescaling by $|x_i|$, we identify A_i with $B_{R_i} \setminus B_{r_i} \subset \mathbb{C}^n \setminus \{0\}$. On $B_{R_i} \setminus B_{r_i}$ we consider the metric $|x_i|^{-2}\omega_{\epsilon_i}$ (here ω_{ϵ_i} on $B_{R_i} \setminus B_{r_i}$ is ω_{ϵ_i} on A_i pulled back along the rescaling by $|x_i|$). The point of these annuli is that on M_p they are shrinking to p , where ω is becoming euclidean, and on $Bl_0\mathbb{C}^n$ they are going to infinity where the Burns-Simanca metric is Euclidean. In other words, these metrics converge locally in any C^k norm to the standard Euclidean metric on $\mathbb{C}^n \setminus \{0\}$.

Now, under this identification all of the points x_i lie on the unit sphere in $\mathbb{C}^n \setminus \{0\}$, and thus up to a subsequence $x_i \rightarrow x_\infty$. Further, the functions $|x_i|^{-\delta}\psi_i$, when viewed as functions on larger annuli in $\mathbb{C}^n \setminus \{0\}$, inherit uniform $C_\delta^{4,\alpha}$ bounds from 4.1.12. Thus, up to a sequence $|x_i|^{-\delta}\psi_i \rightarrow \psi_\infty$ locally in $C^{4,\alpha}$ for possibly smaller α . We then have $\psi_\infty \in C_\delta^{4,\alpha}$ and

$$\psi_\infty(x_\infty) = 1, \quad \Delta_{euc}^2 \psi_\infty = 0,$$

a contradiction of Proposition 3.3.2 based on our choice of δ . □

The above proposition can be summarized as follows.

Corollary 4.1.14. *Let $\delta \in (-1, 0)$ sufficiently small.*

- Suppose $n > 2$. For ϵ small there exists a constant $K > 0$ independent of ϵ such that $\tilde{L}_{\omega_\epsilon}^*$ is an isomorphism and

$$\|(\tilde{L}_{\omega_\epsilon}^*)^{-1}\|_{C_\delta^{4,\alpha} \rightarrow C_{\delta-4}^{0,\alpha}} \leq K.$$

- Suppose $n = 2$. For ϵ small there exists a constant $K > 0$ independent of ϵ such that $\tilde{L}_{\omega_\epsilon}^*$ is an isomorphism and

$$\|(\tilde{L}_{\omega_\epsilon}^*)^{-1}\|_{C_\delta^{4,\alpha} \rightarrow C_{\delta-4}^{0,\alpha}} \leq K\epsilon^\delta.$$

Proof. The above proposition shows that $\tilde{L}_{\omega_\epsilon}^*$ is injective. The operator $\tilde{L}_{\omega_\epsilon}^*$ differs from the index zero operator $L_{\omega_\epsilon}^*$ by a finite rank operator, and thus must also have index zero. Thus we also have surjectivity, and the statement about the norms follows directly. □

We wish to draw attention to the fact that we could have proven the exact same estimates but for the operator L_{ω_ϵ} in $n \geq 3$. This makes the remainder of the proof of our main results slightly cleaner, we only insist on $L_{\omega_\epsilon}^*$ to uniformize the approach across dimensions. The precise reason that we are unable to prove the estimates for L_{ω_ϵ} in $n = 2$ is due to the added analytic difficulty necessitating the existence of a function on M in the kernel of L_ω^* which is nonvanishing at the blowup point. Since our problem is open in nature (in the topological sense), one could ask if this condition is satisfied for Kähler metrics near our initial metric ω , perhaps even within its own Kähler class. Similar generic aspects of eigenfunctions have been studied for example by Uhlenbeck [34] in the obviously less restrictive Riemannian case, it would be interesting to study similar questions in the Kähler case. So, we end this section with a question.

Question 4.1.15. *If (M, ω) is a compact Kähler manifold, then does any open set containing ω in the Kähler class $[\omega]$ contain a Kähler metric ω' such that $L_{\omega'}$ has only constants in its kernel?*

4.2 Alteration of ω_ϵ and the $n = 2$ Case

For this section we assume $n = 2$ exclusively. We will construct instead a metric $\tilde{\omega}_\epsilon$ on $Bl_p M$ which is a slight alteration of ω_ϵ but still is a scaling down of the Burns-Simanca metric in B_{r_ϵ} . The main advantage will be that the gluing annulus $B_{2r_\epsilon} \setminus B_{r_\epsilon}$ will have better behavior, namely we will no longer have to multiply the log term in the definition of ω_ϵ by a cutoff. This better approximation is necessitated by the need to prove some estimate on the difference $S(\omega_\epsilon) - S(\omega)$. What appears here is only a minor alteration to the construction in [31].

The main point is to argue that there exists a function $g \in \text{Ker}(L_\omega^*)$ on all of M , and a function Γ on M_p such that

$$L_\omega \Gamma = g \tag{4.2.1}$$

on M_p , and $\Gamma(z) = \log |z| + \psi(z)$ near p , where $\psi \in C_1^{4,\alpha}(M_p)$. Note that by elliptic regularity g must be smooth on M , and Γ is also smooth on M_p .

To this end, let χ once again be a cutoff supported in B_2 and equal to 1 in B_1 . Let $\chi_r := \chi(z/r)$ for any $r > 0$. Then we will define the metric

$$\omega'_r := \sqrt{-1} \partial \bar{\partial} (|z|^2 + \chi_r(z) \varphi(z))$$

on $\mathbb{C}^2 \setminus \{0\}$. This metric is flat outside B_{2r} and equal to ω in B_r . We need the following estimates.

Lemma 4.2.2. • *For any $\delta \in \mathbb{R}$ and sufficiently small $r > 0$ there is a positive constant C independent of r such that*

$$\|(L_{\omega'_r} + \frac{1}{4} \Delta_{euc}^2) f\|_{C_{\delta-4}^{0,\alpha}(\mathbb{C}^2 \setminus \{0\})} \leq Cr^2 \|f\|_{C_\delta^{4,\alpha}(\mathbb{C}^2 \setminus \{0\})}$$

for any $f : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{R}$. Then for r sufficiently small $L_{\omega'_r}$ is an isomorphism whenever δ is not an indicial root of Δ_{euc}^2 .

- For any $\delta \leq -2$ we have $L_{\omega'_r}(\log |z|) \in C_\delta^{0,\alpha}(B_1)$.

Proof. For the first claim, outside of B_{2r} we already have that $L_{\tilde{\omega}_r} = -\frac{1}{4}\Delta_{euc}^2$, so we only focus within B_{2r} . Note that within B_{2r} we have

$$\nabla^i(\chi_r \varphi) = O(|z|^{4-i}),$$

and the metric is given by

$$\omega'_{r,j\bar{k}} = \delta_{j\bar{k}} + \partial_j \partial_{\bar{k}}(\chi_r \varphi).$$

As long as r is sufficiently small, in B_{2r} we have

$$\omega_r'^{j\bar{k}} = \delta^{j\bar{k}} - \partial_j \partial_{\bar{k}}(\chi_r \varphi) + O(|z|^4).$$

For the Ricci curvature in B_{2r} we have

$$R_{j\bar{k}} = \partial_j \partial_{\bar{k}}(\log \det[\delta_{\ell\bar{m}} + \partial_\ell \partial_{\bar{m}}(\chi_r \varphi)]) = \partial_j \partial_{\bar{k}}(\partial_\ell \partial_{\bar{\ell}}(\chi_r \varphi)) + O(|z|^2),$$

so then we have that in B_{2r}

$$L_{\omega'_r} f = -\frac{1}{4}\Delta_{euc}^2 f + \delta^{\ell\bar{m}} \partial_\ell \partial_{\bar{m}}[\partial_j \partial_{\bar{k}}(\chi_r \varphi) \partial_j \partial_{\bar{k}} f] + \partial_j \partial_{\bar{k}}(\partial_\ell \partial_{\bar{\ell}}(\chi_r \varphi)) \partial_j \partial_{\bar{k}} f + O(|z|).$$

Thus we have in B_{2r} there is some constant C such that

$$\begin{aligned} |z|^2 \left(\frac{1}{4}\Delta_{euc}^2 f + L_{\omega'_r} f \right) &= |z|^2 (\delta^{\ell\bar{m}} \partial_\ell \partial_{\bar{m}}[\partial_j \partial_{\bar{k}}(\chi_r \varphi) \partial_j \partial_{\bar{k}} f] + \partial_j \partial_{\bar{k}}(\partial_\ell \partial_{\bar{\ell}}(\chi_r \varphi)) \partial_j \partial_{\bar{k}} f + O(|z|)) \\ &\leq Cr^2 \partial_j \partial_{\bar{k}} f \end{aligned}$$

for r small, which proves the desired estimate.

For the second claim we have

$$\begin{aligned} L_{\omega'_r} \log |z| &= -\frac{1}{4}\Delta_{euc}^2 \log |z| + \delta^{\ell\bar{m}} \partial_\ell \partial_{\bar{m}}[\partial_j \partial_{\bar{k}}(\chi_r \varphi) \partial_j \partial_{\bar{k}} \log |z|] \\ &\quad + \partial_j \partial_{\bar{k}}(\partial_\ell \partial_{\bar{\ell}}(\chi_r \varphi)) \partial_j \partial_{\bar{k}} \log |z| + O(|z|^{-1}). \end{aligned}$$

The claim then follows since $\log |z|$ is biharmonic away from zero. $\square$

From this lemma we have that for any r sufficiently small there exists $\xi_1 \in C_{2-\tau}^{4,\alpha}$ for any $\tau > 0$ such that

$$L_{\omega'_r}(\xi_1) = L_{\omega'_r}(\log |z|),$$

which implies that near p

$$L_\omega(\chi_r(z) \log |z| - \chi_r(z) \xi_1(z)) = 0,$$

and the left side is a function on all of M . Arguing in the exact same fashion as in Lemma 4.1.2, if $g_1, \dots, g_d$ is an L^2 -orthonormal basis for $\text{Ker}(L_\omega^*)$ then there exists points $q'_1, \dots, q'_d \in M_p$ such that there is a function $\xi_2 \in C^{4,\alpha}(M)$ satisfying

$$L_\omega(\chi_r \log |z| - \chi_r \xi_1 - \xi_2) = - \sum_{i=1}^d \xi_2(q'_i) g_i$$

on M_p . We can subtract a constant from ξ_2 to ensure that $\xi_2(p) = 0$, and then we set

$$\Gamma = \chi_r(\log |z| - \xi_1) - \xi_2,$$

which then has the desired properties.

Now we define

$$\tilde{\omega}_\epsilon := \omega_\epsilon + \epsilon^2 \sqrt{-1} \partial \bar{\partial} (\gamma_1(\Gamma - \log \epsilon)). \quad (4.2.3)$$

We have that near p , $\Gamma(z) = \log |z| + \psi$ with $\psi \in C_1^{4,\alpha}(M_p)$, so we can alternatively write

$$\tilde{\omega}_\epsilon = \sqrt{-1} \partial \bar{\partial} (|z|^2 + \gamma_1(z) \varphi(z) + \epsilon^2 \log |z| + \epsilon^2 \gamma_1(z) \psi(z)). \quad (4.2.4)$$

Note that in this presentation it is clear that we have preserved the property that the metric is equal to $\epsilon^2 \eta$ in B_{r_ϵ} after changing coordinates.

Lastly, we will need to have the same bounds on $(\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}$ as we did for $(\tilde{L}_{\omega_\epsilon}^*)^{-1}$. We have them because $\|\gamma_1 \Gamma\|_{C_2^{4,\alpha}(Bl_p M)} \leq C \frac{\log |r_\epsilon|}{r_\epsilon^2}$, and using Lemma 3.3.6 in the last inequality we have

$$\begin{aligned} \|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} &\leq K \epsilon^\delta \|\tilde{L}_{\omega_\epsilon}^* \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \leq K \epsilon^\delta (\|\tilde{L}_{\tilde{\omega}_\epsilon}^* \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} + \|(L_{\omega_\epsilon}^* - L_{\tilde{\omega}_\epsilon}^*) \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}) \\ &\leq K \epsilon^\delta (\|\tilde{L}_{\tilde{\omega}_\epsilon}^* \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} + C(\epsilon^{2(1-\beta)} \beta \log |\epsilon|) \|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)}), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} &\leq \frac{K \epsilon^\delta}{1 - KC \epsilon^{2(1-\beta)+\delta} \beta \log |\epsilon|} \|\tilde{L}_{\tilde{\omega}_\epsilon}^* \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)} \\ &\leq (K+1) \epsilon^\delta \|\tilde{L}_{\tilde{\omega}_\epsilon}^* \phi\|_{C_{\delta-4}^{0,\alpha}(Bl_p M)}, \end{aligned} \quad (4.2.5)$$

for small ϵ , given that $\delta \in (-1, 0)$. We will just replace $K+1$ with K when discussing this bound later.

4.3 Further Estimates and Completion of the Perturbation

Per 4.0.1, we now use the estimates on $(\tilde{L}_{\omega_\epsilon}^*)^{-1}$ and $(\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}$ to find and solve an equation with right hand side going to zero as $\epsilon \rightarrow 0$, providing the main result. We deal with the $n > 2$ case first.

The Equation for $n > 2$:

We would like to find ϕ solving the equation

$$S(\omega) - S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi) = -\partial\phi \cdot \bar{\partial}S(\omega_\epsilon) - \partial S(\omega_\epsilon) \cdot \bar{\partial}\phi - \frac{1}{2}\phi\Delta_{\omega_\epsilon}S(\omega_\epsilon) - \sum_{i=1}^d \phi(q_i)f_i, \quad (*_1)$$

for each ϵ sufficiently small, where the f_i and q_i are as in Lemma 4.1.2. Linearizing, we can instead write $S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi)$ as

$$S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi) = S(\omega_\epsilon) + L_{\omega_\epsilon}(\phi) + Q_{\omega_\epsilon}(\phi),$$

where $Q_{\omega_\epsilon}(\phi)$ are the higher order terms. Then the equation $(*_1)$ is equivalent to

$$S(\omega) - S(\omega_\epsilon) - Q_{\omega_\epsilon}(\phi) = L_{\omega_\epsilon}\phi - \partial\phi \cdot \bar{\partial}S(\omega_\epsilon) - \partial S(\omega_\epsilon) \cdot \bar{\partial}\phi - \frac{1}{2}\phi\Delta_{\omega_\epsilon}S(\omega_\epsilon) - \sum_{i=1}^d \phi(q_i)f_i.$$

From the formula 4.1.1 and the definition 4.1.3 we recognize the right side of the equation now as $\tilde{L}_{\omega_\epsilon}^*\phi$, multiplying both sides by the inverse we have that $(*_1)$ is equivalent to

$$(\tilde{L}_{\omega_\epsilon}^*)^{-1}(S(\omega) - S(\omega_\epsilon) - Q_{\omega_\epsilon}(\phi)) = \phi.$$

Now, if we consider the map

$$\mathcal{N}_\epsilon : C_\delta^{4,\alpha}(Bl_p M) \rightarrow C_\delta^{4,\alpha}(Bl_p M)$$

$$\phi \mapsto (\tilde{L}_{\omega_\epsilon}^*)^{-1}(S(\omega) - S(\omega_\epsilon) - Q_{\omega_\epsilon}(\phi)),$$

$(*_1)$ can be rephrased as a fixed point problem for the nonlinear operator $\mathcal{N}_\epsilon$. We will now work to apply the Banach fixed point theorem. We already have bounds on $(\tilde{L}_{\omega_\epsilon}^*)^{-1}$ from Corollary 4.1.14, we also need bounds on $S(\omega_\epsilon) - S(\omega)$ and $Q_{\omega_\epsilon}(\phi)$.

Lemma 4.3.1. *Let k be a nonnegative integer, $\alpha \in (0, 1)$, and $\delta \in \mathbb{R}$. Then if c_0 is as in Lemma 3.3.6 and*

$$\|\phi_1\|_{C_2^{k+4,\alpha}(Bl_p M)}, \|\phi_2\|_{C_2^{k+4,\alpha}(Bl_p M)} \leq c_0,$$

then there is some $C > 0$ independent of ϵ such that

$$\|Q_{\omega_\epsilon}(\phi_2) - Q_{\omega_\epsilon}(\phi_1)\|_{C_\delta^{k,\alpha}(Bl_p M)} \leq C(\|\phi_1\|_{C_2^{k+4,\alpha}(Bl_p M)} + \|\phi_2\|_{C_2^{k+4,\alpha}(Bl_p M)})\|\phi_1 - \phi_2\|_{C_\delta^{k+4,\alpha}(Bl_p M)},$$

for all ϵ small.

Proof. By the mean value inequality there is a $t \in [0, 1]$ such that $\chi = t\phi_1 + (1-t)\phi_2$ satisfies

$$\|Q_{\omega_\epsilon}(\phi_2) - Q_{\omega_\epsilon}(\phi_1)\|_{C_\delta^{k,\alpha}(Bl_p M)} \leq \|DQ_{\omega_\epsilon, \chi}(\phi_2 - \phi_1)\|_{C_\delta^{k,\alpha}(Bl_p M)}.$$

On the other hand we have

$$DQ_{\omega_\epsilon, \chi} = L_{\omega_\chi} - L_{\omega_\epsilon},$$

where $\omega_\chi := \omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\chi$. Then since $\|\chi\|_{C_2^{k+4, \alpha}(Bl_p M)} \leq c_0$ by Lemma 3.3.6 we have

$$\begin{aligned} \|Q_{\omega_\epsilon}(\phi_2) - Q_{\omega_\epsilon}(\phi_1)\|_{C_\delta^{k, \alpha}(Bl_p M)} &\leq C\|\chi\|_{C_2^{k+4, \alpha}(Bl_p M)}\|\phi_1 - \phi_2\|_{C_\delta^{k+4, \alpha}(Bl_p M)} \\ &\leq C(\|\phi_1\|_{C_2^{k+4, \alpha}(Bl_p M)} + \|\phi_2\|_{C_2^{k+4, \alpha}(Bl_p M)})\|\phi_1 - \phi_2\|_{C_\delta^{k+4, \alpha}(Bl_p M)}. \end{aligned}$$

□

Lemma 4.3.2. *Let $\delta < 0$ be chosen sufficiently small. There is a constant $c_1 > 0$ independent of ϵ such that if*

$$\|\phi_1\|_{C_2^{4, \alpha}(Bl_p M)}, \|\phi_2\|_{C_2^{4, \alpha}(Bl_p M)} \leq c_1,$$

then

$$\|\mathcal{N}_\epsilon(\phi_1) - \mathcal{N}_\epsilon(\phi_2)\|_{C_\delta^{4, \alpha}(Bl_p M)} \leq \frac{1}{2}\|\phi_1 - \phi_2\|_{C_\delta^{4, \alpha}(Bl_p M)},$$

for all ϵ small.

Proof. First, we have that

$$\mathcal{N}_\epsilon(\phi_1) - \mathcal{N}_\epsilon(\phi_2) = (\tilde{L}_{\omega_\epsilon}^*)^{-1}(Q_{\omega_\epsilon}(\phi_2) - Q_{\omega_\epsilon}(\phi_1)).$$

From the bound on $(\tilde{L}_{\omega_\epsilon}^*)^{-1}$ in Corollary 4.1.14 and the bounds in Lemma 4.3.1, the result follows once c_1 is chosen small enough. □

Lemma 4.3.3. *For ϵ sufficiently small we have*

$$\|S(\omega_\epsilon) - S(\omega)\|_{C_{\delta-4}^{2, \alpha}(Bl_p M)} \leq Cr_\epsilon^{4-\delta}$$

for some constant C independent of ϵ .

Proof. $S(\omega_\epsilon) \equiv S(\omega)$ on $M \setminus B_{2r_\epsilon}$ and $S(\omega_\epsilon) \equiv 0$ on B_{r_ϵ} , so the lemma really only needs to be checked in $B_{2r_\epsilon} \setminus B_{r_\epsilon}$. Rewriting 3.2.2 in $n \geq 3$, in this region the key is that

$$\omega_\epsilon = \omega_{euc} + \sqrt{-1}\partial\bar{\partial}(\gamma_1(z)\varphi(z) + \epsilon^2\gamma_2(z)\psi(\epsilon^{-1}z)),$$

and for any δ

$$\|\gamma_1(z)\varphi(z) + \epsilon^2\gamma_2(z)\psi(\epsilon^{-1}z)\|_{C_\delta^{6, \alpha}(B_{2r_\epsilon} \setminus B_{r_\epsilon})} \leq Cr_\epsilon^{4-\delta}.$$

This estimate is because when ϵ is small, in the annulus $B_{2r_\epsilon} \setminus B_{r_\epsilon}$ we have

$$\epsilon^2\psi(\epsilon^{-1}z) = O\left(\epsilon^2 \frac{r_\epsilon^{4-2n}}{\epsilon^{4-2n}}\right) = O(r_\epsilon^4),$$

specifically due to our choice of β . With this estimate we can conclude

$$\|S(\omega_\epsilon)\|_{C_{\delta-4}^{2, \alpha}(B_{2r_\epsilon} \setminus B_{r_\epsilon})} \leq Cr_\epsilon^{4-\delta}.$$

□

Combining all the estimates, we can solve $(*_1)$.

Proposition 4.3.4. *Assume $n > 2$ and $\delta < 0$ sufficiently small. Then for all $\epsilon > 0$ sufficiently small, $(*_1)$ admits a solution with $\|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq c_1 \epsilon^{2-\delta}$.*

Proof. Let $\mathcal{B}$ denote the ball of radius $c_1 \epsilon^{2-\delta}$ in $C_\delta^{4,\alpha}(Bl_p M)$. If $\phi \in \mathcal{B}$, then $\|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq c_1$. Then the above estimates give that $\mathcal{N}_\epsilon$ is a contraction on $\mathcal{B}$ for ϵ small. We show in addition that $\mathcal{N}_\epsilon(\mathcal{B}) \subset \mathcal{B}$ for ϵ small, from which the proposition follows immediately by the Banach fixed point theorem.

First, we have from Lemma 4.3.2

$$\begin{aligned} \|\mathcal{N}_\epsilon(\phi)\|_{C_\delta^{4,\alpha}(Bl_p M)} &\leq \|\mathcal{N}_\epsilon(\phi) - \mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(Bl_p M)} + \|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(Bl_p M)} \\ &\leq \frac{1}{2} \|\phi\|_{C_\delta^{4,\alpha}(Bl_p M)} + \|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(Bl_p M)}. \end{aligned}$$

Recall $\mathcal{N}_\epsilon(0) = (\tilde{L}_{\omega_\epsilon}^*)^{-1}(S(\omega) - S(\omega_\epsilon))$, so from the uniform estimates on $(\tilde{L}_{\omega_\epsilon}^*)^{-1}$ in Corollary 4.1.14 and the estimate on $S(\omega) - S(\omega_\epsilon)$ in Lemma 4.3.3 we have independent of ϵ the bound

$$\|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq C r_\epsilon^{4-\delta}.$$

Now, $(4-\delta)\beta > 2-\delta$ when $n > 2$ and δ is chosen small enough, so then when ϵ is sufficiently small

$$\|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq \frac{1}{2} c_1 \epsilon^{2-\delta},$$

which completes the proof. $\square$

The Equation for $n = 2$:

The $n = 2$ case is only very slightly different. We instead solve the equation

$$S(\omega) - S(\tilde{\omega}_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi) = -\partial\phi \cdot \bar{\partial}S(\tilde{\omega}_\epsilon) - \partial S(\tilde{\omega}_\epsilon) \cdot \bar{\partial}\phi - \frac{1}{2}\phi\Delta_{\tilde{\omega}_\epsilon}S(\tilde{\omega}_\epsilon) - \sum_{i=1}^d \phi(q_i)f_i - \epsilon^2 g, \quad (*_2)$$

where we have the addition of g from 4.2.1. By an entirely similar calculation as above, $(*_2)$ is equivalent to

$$(\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}(S(\omega) - S(\tilde{\omega}_\epsilon) + \epsilon^2 g - Q_{\tilde{\omega}_\epsilon}(\phi)) = \phi,$$

so we instead consider the fixed point problem for the operator

$$\mathcal{N}_\epsilon : C_\delta^{4,\alpha}(Bl_p M) \rightarrow C_\delta^{4,\alpha}(Bl_p M)$$

$$\phi \mapsto (\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}(S(\omega) - S(\tilde{\omega}_\epsilon) + \epsilon^2 g - Q_{\tilde{\omega}_\epsilon}(\phi)).$$

Once again we just need to prove similar estimates as in the $n > 2$ case.

Lemma 4.3.5. *Let $\delta < 0$ be chosen sufficiently small. There is a constant $c_1 > 0$ independent of ϵ such that if*

$$\|\phi_1\|_{C_2^{4,\alpha}(Bl_p M)}, \|\phi_2\|_{C_2^{4,\alpha}(Bl_p M)} \leq c_1 \epsilon^{-\delta},$$

then

$$\|\mathcal{N}_\epsilon(\phi_1) - \mathcal{N}_\epsilon(\phi_2)\|_{C_\delta^{4,\alpha}(Bl_p M)} \leq \frac{1}{2} \|\phi_1 - \phi_2\|_{C_\delta^{4,\alpha}(Bl_p M)},$$

for all ϵ small.

Proof. The proof is identical to Lemma 4.3.2, we just need $\epsilon^{-\delta}$ to account for the slightly weaker bound on $(\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}$. $\square$

Lemma 4.3.6. *Let $\delta < 0$ be chosen sufficiently small. For ϵ sufficiently small we have*

$$\|S(\tilde{\omega}_\epsilon) - \epsilon^2 g - S(\omega)\|_{C_{\delta-4}^{2,\alpha}} \leq C r_\epsilon^{4-\delta}$$

for some constant C independent of ϵ .

Proof. In B_{r_ϵ} there is nothing to prove since there $\tilde{\omega}_\epsilon$ is scalar flat. In $B_{2r_\epsilon} \setminus B_{r_\epsilon}$ we have from rewriting 4.2.4

$$\tilde{\omega}_\epsilon = \omega_{euc} + \sqrt{-1} \partial \bar{\partial} (\epsilon^2 \log |z| + \gamma_1(\varphi + \epsilon^2 \psi)).$$

Since $L_{\omega_{euc}} = -\frac{1}{4} \Delta_{euc}^2$ 2.2.4 we have

$$S(\tilde{\omega}_\epsilon) = -\frac{1}{4} \Delta_{euc}^2 (\gamma_1(\varphi + \epsilon^2 \psi)) + Q_{\omega_{euc}} (\epsilon^2 \log |z| + \gamma_1(\varphi + \epsilon^2 \psi)).$$

Now, for the $Q_{\omega_{euc}}$ term we have, recalling that $\psi \in C_1^{4,\alpha}(M_p)$

$$\begin{aligned} & \|Q_{\omega_{euc}} (\epsilon^2 \log |z| + \gamma_1(\varphi + \epsilon^2 \psi))\|_{C_{\delta-4}^{2,\alpha}(B_{2r_\epsilon} \setminus B_{r_\epsilon})} \\ & \leq C \|\epsilon^2 \log |z| + \gamma_1(\varphi + \epsilon^2 \psi)\|_{C_2^{6,\alpha}(B_{2r_\epsilon} \setminus B_{r_\epsilon})} \|\epsilon^2 \log |z| + \gamma_1(\varphi + \epsilon^2 \psi)\|_{C_\delta^{6,\alpha}(B_{2r_\epsilon} \setminus B_{r_\epsilon})} \\ & \leq C (\epsilon^{2-2\beta} \log |\epsilon| + \epsilon^{2\beta}) (\epsilon^{2-\delta} \log |\epsilon| + \epsilon^{(4-\delta)\beta}) \leq r_\epsilon^{4-\delta}, \end{aligned}$$

where we have used Lemma 4.3.1, and thus

$$\|S(\tilde{\omega}_\epsilon)\|_{C_{\delta-4}^{2,\alpha}(B_{2r_\epsilon} \setminus B_{r_\epsilon})} \leq C r_\epsilon^{4-\delta}.$$

On $M \setminus B_{2r_\epsilon}$ we have from 4.2.3 $\tilde{\omega}_\epsilon = \omega + \epsilon^2 \sqrt{-1} \partial \bar{\partial} \Gamma$ and thus from 4.2.1

$$S(\tilde{\omega}_\epsilon) = S(\omega) + \epsilon^2 L_\omega \Gamma + Q_\omega (\epsilon^2 \Gamma) = S(\omega) + \epsilon^2 g + Q_{\omega_\epsilon} (\epsilon^2 \Gamma).$$

We can estimate the last term using Lemma 4.3.1 once again to get

$$\|Q_{\omega_\epsilon} (\epsilon^2 \Gamma)\|_{C_{\delta-4}^{2,\alpha}(M \setminus B_{2r_\epsilon})} \leq C \|\epsilon^2 \Gamma\|_{C_2^{6,\alpha}(M \setminus B_{2r_\epsilon})} \|\epsilon^2 \Gamma\|_{C_\delta^{6,\alpha}(M \setminus B_{2r_\epsilon})} \leq C \frac{\epsilon^2}{r_\epsilon^2} \log |\epsilon| \frac{\epsilon^2}{r_\epsilon^\delta} \log |\epsilon| \leq C r_\epsilon^{4-\delta}$$

due to our range of β . Putting these regions together gives the desired estimate. $\square$

Combining all the estimates, we can solve $(*_2)$.

Proposition 4.3.7. *Assume $n = 2$ and $\delta < 0$ sufficiently small. Then for all $\epsilon > 0$ sufficiently small, $(*_2)$ admits a solution with $\|\phi\|_{C_\delta^{4,\alpha}(B_{l_p}M)} \leq c_1\epsilon^{2-2\delta}$.*

Proof. Let $\mathcal{B}$ denote the ball of radius $c_1\epsilon^{2-2\delta}$ in $C_\delta^{4,\alpha}(B_{l_p}M)$. If $\phi \in \mathcal{B}$, then $\|\phi\|_{C_\delta^{4,\alpha}(B_{l_p}M)} \leq c_1\epsilon^{-\delta}$. Then the above estimates give that $\mathcal{N}_\epsilon$ is a contraction on $\mathcal{B}$ for ϵ small. Again, we show that $\mathcal{N}_\epsilon(\mathcal{B}) \subset \mathcal{B}$ for ϵ small, and we conclude by the Banach fixed point theorem.

First, we have using Lemma 4.3.5

$$\begin{aligned} \|\mathcal{N}_\epsilon(\phi)\|_{C_\delta^{4,\alpha}(B_{l_p}M)} &\leq \|\mathcal{N}_\epsilon(\phi) - \mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(B_{l_p}M)} + \|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(B_{l_p}M)} \\ &\leq \frac{1}{2}\|\phi\|_{C_\delta^{4,\alpha}(B_{l_p}M)} + \|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(B_{l_p}M)}. \end{aligned}$$

Recall $\mathcal{N}_\epsilon(0) = (\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}(S(\omega) - S(\tilde{\omega}_\epsilon) + \epsilon^2g)$, so from the uniform estimates on $(\tilde{L}_{\tilde{\omega}_\epsilon}^*)^{-1}$ from 4.2.5 and the estimates in Lemma 4.3.6 on $S(\omega) - S(\tilde{\omega}_\epsilon) + \epsilon^2g$ we have independent of ϵ the bound

$$\|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(B_{l_p}M)} \leq C\epsilon^\delta r_\epsilon^{4-\delta}.$$

Now, $\delta + (4 - \delta)\beta > 2 - 2\delta$ for our range of β if δ is chosen small enough, so then when ϵ is sufficiently small

$$\|\mathcal{N}_\epsilon(0)\|_{C_\delta^{4,\alpha}(B_{l_p}M)} \leq \frac{1}{2}c_1\epsilon^{2-2\delta},$$

which completes the proof. $\square$

Proof of the Main Result:

Once again, the point is that both $(*_1)$ and $(*_2)$ are equations for the difference $S(\omega) - S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi)$, and we can solve them with sufficiently good estimates on ϕ and $S(\omega_\epsilon)$ and their derivatives, so that the right hand side of the equation becomes small in ϵ . This gives us our main result.

Proof. (of the Main Result) Let's do the $n = 2$ case, the $n > 2$ case is entirely similar, just without the g term. We can get that $(*_2)$ has a solution with $\|\phi\|_{C_\delta^{4,\alpha}(B_{l_p}M)} \leq c_1\epsilon^{2-2\delta}$ for all ϵ sufficiently small (Proposition 4.3.7). If we can show that

$$\max_M | -\partial\phi \cdot \bar{\partial}S(\tilde{\omega}_\epsilon) - \partial S(\tilde{\omega}_\epsilon) \cdot \bar{\partial}\phi - \frac{1}{2}\phi\Delta_{\tilde{\omega}_\epsilon}S(\tilde{\omega}_\epsilon) - \sum_{i=1}^d \phi(q_i)f_i - \epsilon^2g | \rightarrow 0, \text{ as } \epsilon \rightarrow 0,$$

then $S(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi)$ will have the same sign everywhere as $S(\omega)$. Since M is compact, g and the f_i are bounded, and $|\phi| \leq c_1\epsilon^2$ on all of M , so we do not need to be concerned about

the last two terms. For the first two terms, note first of all that $S(\tilde{\omega}_\epsilon) \equiv 0$ in B_{r_ϵ} , so in that region we are fine. In $M \setminus B_{r_\epsilon}$ we have

$$|\nabla \phi| \leq c_1 r_\epsilon^{\delta-1} \epsilon^{2-2\delta} = c_1 \epsilon^{2-\beta+(\beta-2)\delta},$$

$$|\nabla[S(\omega_\epsilon) - \epsilon^2 g - S(\omega)]| \leq C r_\epsilon^{\delta-5} r_\epsilon^{4-\delta} = C \epsilon^{-\beta},$$

and $\epsilon^{2(1-\beta)+(\beta-2)\delta} \rightarrow 0$ as $\epsilon \rightarrow 0$ for our range of β , so the terms $-\partial\phi \cdot \bar{\partial}S(\tilde{\omega}_\epsilon) - \partial S(\tilde{\omega}_\epsilon) \cdot \bar{\partial}\phi$ are fine. What remains is the term $-\frac{1}{2}\phi\Delta_{\tilde{\omega}_\epsilon}S(\tilde{\omega}_\epsilon)$. For this note once again that in $M \setminus B_{r_\epsilon}$ we have

$$|\phi| \leq c_1 r_\epsilon^\delta \epsilon^{2-2\delta} = c_1 \epsilon^{2+(\beta-2)\delta},$$

$$|\nabla^2[S(\omega_\epsilon) - \epsilon^2 g - S(\omega)]| \leq C r_\epsilon^{\delta-6} r_\epsilon^{4-\delta} = C \epsilon^{-2\beta},$$

and $\epsilon^{2(1-\beta)+(\beta-2)\delta} \rightarrow 0$ as $\epsilon \rightarrow 0$ for our range of β .

Lastly, we need to check that any solution ϕ to $(*_1)$ or $(*_2)$ will be smooth, so that the metric we construct on $Bl_p M$ with our desired scalar curvature properties is indeed smooth. This is a routine elliptic bootstrapping argument but we will write the details anyway. Let $\phi \in C^{4,\alpha}(Bl_p M)$ be a solution to $(*_1)$ or $(*_2)$. Then in particular, in either case for some $h \in C^\infty(M)$,

$$S(\tilde{\omega}_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi) = \partial\phi \cdot \bar{\partial}S(\tilde{\omega}_\epsilon) + \partial S(\tilde{\omega}_\epsilon) \cdot \bar{\partial}\phi + \frac{1}{2}\phi\Delta_{\tilde{\omega}_\epsilon}S(\tilde{\omega}_\epsilon) + h.$$

We can also split this into a system of equations

$$\frac{1}{2}\Delta_{\tilde{\omega}_\epsilon}F = \partial\phi \cdot \bar{\partial}S(\tilde{\omega}_\epsilon) + \partial S(\tilde{\omega}_\epsilon) \cdot \bar{\partial}\phi + \frac{1}{2}\phi\Delta_{\tilde{\omega}_\epsilon}S(\tilde{\omega}_\epsilon) + h,$$

$$\log \det(\tilde{g}_{\phi,j\bar{k}}) = F,$$

where for convenience we have written $\tilde{\omega}_\phi := \tilde{\omega}_\epsilon + \sqrt{-1}\partial\bar{\partial}\phi$, and for the associated metric $\tilde{g}_\phi$. For the first equation, we see that F solves a linear second order elliptic PDE with coefficients in $C^{2,\alpha}(Bl_p M)$, and the right hand side is in $C^{3,\alpha}(Bl_p M)$, so we have that $F \in C^{4,\alpha}(Bl_p M)$. Differentiating both sides of the second equation we get

$$(\tilde{g}_\phi)^{j\bar{k}}\partial_j\partial_{\bar{k}}(\partial_i\phi) = -(\tilde{g}_\phi)^{j\bar{k}}(\partial_i\tilde{g}_{\epsilon,j\bar{k}}) + \partial_i F,$$

so $\partial_i\phi$ can be thought of as solving a linear second order elliptic PDE with coefficients in $C^{2,\alpha}(Bl_p M)$ and right hand side in $C^{2,\alpha}(Bl_p M)$. Since we already had that $\partial_i\phi \in C^{3,\alpha}(Bl_p M)$ this yields that $\partial_i\phi \in C^{4,\alpha}(Bl_p M)$, and finally $\phi \in C^{5,\alpha}(Bl_p M)$. Continuing this argument inductively gives that ϕ is actually smooth. $\square$

4.4 Positive Scalar Curvature Kähler Surfaces and Further Discussion

Positive Scalar Curvature Kähler Surfaces

The most interesting consequences of the main result come in the case of positive scalar curvature in complex dimension 2. If (M, ω) is a Kähler manifold, and ρ is its Ricci form, recall we have the following equation for the total scalar curvature,

$$\int S(\omega) \frac{\omega^n}{n!} = \frac{2\pi c_1(M) \cup [\omega]^{n-1}}{(n-1)!}, \quad (4.4.1)$$

showing that the total scalar curvature of a Kähler metric actually only depends on the Kähler class. This formula can be used to show that the positivity of total scalar curvature has a dramatic effect on the holomorphic sections of the canonical bundle.

Proposition 4.4.2 (Yau [36]). *Let M be a compact complex manifold for which there exists a Kähler metric of positive total scalar curvature. Then M has Kodaira dimension $-\infty$, that is, the canonical bundle of M and all of its powers have no global holomorphic sections. Moreover, the converse is true in the case of compact Kähler surfaces.*

We will now list some basic facts about the Kodaira-Enriques classification of complex surfaces, for which a reference is [6]. Kodaira dimension is invariant under blowup, and the Kodaira-Enriques classification classifies the Kähler surfaces of Kodaira dimension $-\infty$ up to blowups at points. First of all, a complex surface is called *minimal* if it has no complex submanifolds biholomorphic to $\mathbb{CP}^1$ with self-intersection -1 . Equivalently, a complex surface is minimal if it cannot be obtained from another via blowing up at a point. Furthermore, every complex surface is obtained from a minimal one via a finite sequence of blowups at points. The Kodaira-Enriques classification gives the minimal Kähler surfaces of Kodaira dimension $-\infty$ as follows.

Proposition 4.4.3. *Any compact Kähler surface of Kodaira dimension $-\infty$ is obtained from $\mathbb{CP}^2$ or $\mathbb{P}(E)$ by a finite sequence of blowups at points, where $E \rightarrow \Sigma$ is a rank two holomorphic vector bundle over a compact Riemann surface.*

A surface which is birationally equivalent to $\mathbb{CP}^2$ is called *rational*. Among the surfaces of Kodaira dimension $-\infty$, these are precisely those with zero first betti number [6]. Any surface of the form $\mathbb{P}(E)$ as above is known as a *ruled surface*. Of course, the Fubini-Study metric has positive scalar curvature, and it was shown in [36] that ruled surfaces also admit positive scalar curvature Kähler metrics. Thus, our main result shows we can replace positive total scalar curvature in Yau's result with pointwise positive scalar curvature in the case of Kähler surfaces.

Due to the gluing construction for positive scalar curvature Riemannian metrics of Gromov and Lawson, it was already known that these surfaces at least had Riemannian metrics

of positive scalar curvature [19]. In fact, one can say even more. A complex surface is said to be of *Kähler-type* if there exists a Riemannian metric which is Kähler with respect to the complex structure. This is equivalent to having even first betti number [6]. It was shown first in the minimal case in [24] and then in the general case in [16] using Seiberg-Witten invariants that the existence of a positive scalar curvature Riemannian metric, not necessarily Kähler, on a compact complex surface of Kähler type, is enough to conclude the complex structure has Kodaira dimension $-\infty$. Thus, we have verified Theorem B.

The contribution of this dissertation is validating the presence of (2) in Theorem B. One can phrase this result as drawing the following equivalences. First, positive scalar curvature Riemannian geometry and positive scalar curvature Kähler geometry are in some sense the same on compact real 4-manifolds for which a Kähler structure exists (ignoring polarizations). Second, the algebro-geometric condition of Kodaira dimension $-\infty$ for a compact complex algebraic surface can be differential geometrically restated as the existence of a Kähler metric with positive scalar curvature.

Higher Dimensions and Further Questions

The natural question to ask is if Theorem B extends to higher dimensions. The equivalence with Riemannian positive scalar curvature does not persist in higher dimensions. In [23] the counterexample of a smooth hypersurface of degree $n + 3$ in $\mathbb{CP}^{n+1}$ where $n \geq 3$ is provided. This variety is simply connected and non-spin with real dimension ≥ 6 , and thus it has a Riemannian metric of positive scalar curvature by the results of [19]. However, it has Kodaira dimension n and thus no positive scalar curvature Kähler metric.

On the other hand, there exists many conjectures in the literature on the relationship between positive (total) scalar curvature of Kähler metrics (and non-Kähler generalizations) and algebraic geometry in arbitrary dimension; a discussion of these conjectures can be found for example in [35]. In higher dimensions, one replaces (5) in Theorem B with the following notion: a complex manifold M is said to be *uniruled* if every point in M is contained in a *rational curve*, that is, a smooth complex curve biholomorphic to $\mathbb{CP}^1$. We state here a Kähler version of the conjecture.

Conjecture 4.4.4. *Let (M, ω) be a compact Kähler manifold of arbitrary dimension. Then the following are equivalent.*

1. *M has a Kähler metric for which the total scalar curvature is positive.*
2. *M has a Kähler metric of positive scalar curvature.*
3. *M has Kodaira dimension $-\infty$.*
4. *M is uniruled.*

It is reasonable to contend that this may be optimistic. In dimension $n \geq 3$, Kähler manifolds, and in fact projective manifolds, are not closed under bimeromorphic transformation, leading to the definition of the Fujiki and Moishezon classes, or manifolds birational to

Kähler and projective manifolds, respectively. The proof of our main result makes heavy use of this unique property for surfaces. However, *balanced manifolds*, Hermitian manifolds with co-closed associated $(1,1)$ -form, are closed under bimeromorphic transformation [1] [2]. Then the approach of [36] can be mirrored to show (1) and (4) are equivalent for balanced metrics on Moishezon manifolds [13], where one must consider instead the Chern scalar curvature in the case of a non-Kähler balanced metric. Still, this is at most a warning sign for one particular approach to the above conjecture.

Recently, Ou [26] extended the results of [7] to Kähler manifolds, which together with the results of [35] implies that (1) and (4) are equivalent for Kähler manifolds if one weakens the condition on the metric to being *Gauduchon*. Namely, uniruled is equivalent to non-pseudoeffective canonical bundle (meaning it does not admit a singular metric with weakly positive curvature) [26], which is equivalent to the existence of a Gauduchon metric with positive total Chern scalar curvature [35]. A Gauduchon metric is a Hermitian metric whose associated $(1,1)$ -form's $(n-1)$ th exterior power is $\partial\bar{\partial}$ -closed.

The equivalence of (3) and (4) in general is already a deep conjecture in birational geometry and so we will not address it here. The proposal of the equivalence of (1) and (2) is perhaps the most optimistic. Philosophically, we interpret this as a question of whether having pointwise positive scalar curvature is truly an algebraic concept for Kähler metrics, or just a vestige of the differential geometric point of view, unlike the total scalar curvature which certainly is algebraic. Some hope for the algebraic nature of pointwise positive scalar curvature is provided by the recent work of Sha [28] on a systolic inequality for the *holomorphic* 2-systole of a positive scalar curvature Kähler surface. In particular, the minimum volume of a holomorphic curve in a positive scalar curvature Kähler surface is controlled by the inverse of the minimum of its scalar curvature.

In this vein, one can ask if pointwise positive scalar curvature, like total scalar curvature, is really a property of the Kähler class, that is, the *polarization*. We summarize the above in the following questions.

Question 4.4.5. *If (M, ω) is a compact Kähler manifold which admits a Gauduchon metric with positive (resp. negative) total Chern scalar curvature, then must it also admit a positive (resp. negative) total scalar curvature Kähler metric?*

Question 4.4.6. *If (M, ω) is a compact Kähler manifold with positive (resp. negative) total scalar curvature, then does the Kähler class $[\omega]$ also admit a Kähler metric with pointwise positive (resp. negative) scalar curvature?*

The positive case of Question 4.4.5 is simply the missing piece of the equivalence of (1) and (4) in Conjecture 4.4.4 after the work of [26] and [35]. An affirmative answer in the negative case would also provide a Kähler version of the relationship between the image of total scalar curvature and pseudoeffectivity of the (anti-)canonical bundle in [35]. An affirmative answer to Question 4.4.6 would show that the existence of a positive (or negative) scalar curvature Kähler metric is a property of the polarization.

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