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IRVINE

Stable Day-to-day Departure Time Dynamics at the Corridor and Network Levels: Models, Optimal Pricing, and Applications

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Civil and Environmental Engineering

by

Siwei Hu

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2025

DEDICATION

To the Lord Jesus Christ,
and my parents, Guangqing Liang and Zhizhong Hu

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VITA

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4th IEEE Forum for Innovative Sustainable Transportation Systems, Riverside, CA

Policy and Equity Implications from an Urban Network Assignment Platform with Fair Peer-to-peer Congestion Pricing **Jan 2024**
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Influence of Simulation Timestep Size and Pathfinding Interval Length on Path Patterns and Computational Cost in Dynamic Traffic Assignment **July 2023**
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ABSTRACT OF THE DISSERTATION

Stable Day-to-day Departure Time Dynamics at the Corridor and Network Levels: Models, Optimal Pricing, and Applications

By

Siwei Hu

Doctor of Philosophy in Civil and Environmental Engineering

University of California, Irvine, 2025

Professor R. Jayakrishnan, Chair

Traffic congestion continues to pose serious challenges in metropolitan areas, including travel delays, air pollution, noise, and increased accident risks. These impacts translate to substantial societal costs, with major U.S. cities experiencing billions of dollars in annual congestion-related losses. A prominent feature of congestion is its temporal concentration during peak periods, driven largely by travelers' similar trip timing decisions, such as arriving at work by 9:00 AM. Shifting demand away from peak periods is thus a critical strategy for mitigating congestion. This dissertation addresses this issue by developing and analyzing stable day-to-day dynamical models of travelers' departure time choices and examining how optimal pricing influences these decisions to improve system efficiency.

While traditional transportation analyses focus on equilibrium outcomes, they often overlook the dynamic process by which such equilibria are achieved. This dissertation focuses on how travelers adjust their departure times from day to day and collectively converge to a departure time user equilibrium (DTUE). By capturing these dynamics, we can better design pricing policies that can drive the system toward a system optimal (SO) state. A major emphasis is on the theoretical stability of the dynamical models, which is essential to ensure that they reflect real-world travel behavior that is empirically observed to be convergent.

This dissertation advances the field in four key areas. First, it extends an existing stable single-class day-to-day departure time dynamical model at the corridor level to a multi-class setting, with traveler heterogeneity. When desired arrival times are identical, with different queuing costs relative to unpunctuality costs, the proposed multi-class model is proven to be asymptotically stable, and its stationary state is equivalent to a multi-class DTUE. Second, it applies various pricing schemes to both single- and multi-class dynamical models. It is demonstrated, both theoretically and numerically, that appropriate pricing, such as optimal fine tolling, can drive the system from a DTUE to a stable, stationary SO state.

Third, the dissertation extends departure time modeling from a single corridor to the network level using Vickrey's cordon pricing framework with 48-hour entry data from Manhattan, New York. Results show that Vickrey's marginal cost pricing — combined with day-to-day adjustments — can drive the system to a practically stable optimal state. Finally, the dissertation integrates the network-level departure time dynamics with dynamic traffic assignment (DTA) modeling. Results show that Vickrey's marginal cost pricing significantly improves peak-period congestion, average speeds, vehicle miles traveled (VMT), and vehicle hours traveled (VHT). The work also explores the generalized bathtub model and develops insights on effectively reproducing DTA results at much less computational cost, offering a unique sketch-level dynamic planning method for large networks.

In summary, the dissertation: (1) introduces the first provably stable multi-class day-to-day departure time dynamics; (2) shows that optimal pricing can drive both single- and multi-class systems to a stable SO state; (3) empirically validates Vickrey's pricing in a real-world urban context; and (4) compares the effects of departure-time versus route-choice optimization using DTA. Collectively, these contributions provide a stable modeling framework for analyzing and managing departure time choices at both corridor and network levels, which offers new theoretical insights for designing congestion pricing policies, and provides practical insights on a notably more efficient process for transportation planning at large.

Chapter 1

Introduction

Traffic congestion has become increasingly severe in metropolitan areas, leading to various adverse effects, such as prolonged travel delays across roadway networks (Afrin and Yodo, 2020), deteriorating air pollution (Wen et al., 2020), elevated noise levels from stop-and-go traffic (Kalawapudi et al., 2020), higher risk of traffic accidents (Retallack and Ostendorf, 2019). These issues contribute to substantial social costs. For example, in 2019, the social losses caused by traffic congestion reached remarkably high values in major US cities, such as $11.0 billion in New York, $8.2 billion in Los Angeles (LA), $7.6 billion in Chicago, and $4.5 billion in Philadelphia (McCarthy, 2020). Addressing urban congestion remains an urgent challenge in the context of continued urbanization.

One notable characteristic of traffic congestion is its peaking behavior—delays during peak periods are significantly worse than those during off-peak times. This phenomenon arises because many travelers share similar schedule constraints (e.g., a 9:00 AM work start time). Therefore, understanding and influencing travelers' trip timing—particularly departure time—is essential for mitigating peak congestion. This dissertation focuses on travelers' day-to-day departure time decisions and investigates how pricing mechanisms can encourage travelers shift away from peak periods.

While much of the transportation system analysis emphasizes equilibrium outcomes — such as Wardrop equilibrium for route choice (Wardrop, 1952) or departure time user equilibrium (DTUE) (Vickrey, 1969) — less attention has been paid to how these equilibria are reached. Day-to-day models address this gap by capturing how travelers iteratively adjust their choices over time (Smith, 1984b; Mahmassani et al., 1985; Mahmassani and Chang, 1986; Jin, 2007, 2020b, 2021b). This dissertation develops and analyzes stable day-to-day departure time dynamical models to understand the convergence pathways to a DTUE.

Various policy instruments — such as flexible work hours and congestion pricing — aim to shift travel demand away from peak periods. Among them, pricing is a particularly powerful tool. With a stable dynamical model that captures the day-to-day evolution of travelers' departure time choices, we can evaluate how different pricing schemes influence system performance and drive the system toward a system optimal (SO) state.

Departure time choice modeling is often conducted at either the corridor level or the network level. At the corridor level, Vickrey (1969) introduces the bottleneck model or point queue model to study corridor level departure time choices, by assuming a single origin-destination pair with one route and one mode. Here, the departure time is the sole decision variable. Travelers choose their departure times to balance both queuing costs and unpunctuality costs (penalties of arriving early or arrival late) to minimize the total cost of the trip. The term, unpunctuality cost, was used in Smith (1984b); Daganzo (1985), and it is also called scheduling cost, and schedule delay cost in other literature (Hendrickson and Kocur, 1981; Small, 2015; Jin, 2020b, 2021b). We use the term unpunctuality cost, because it more accurately describes the cost of not adhering to the schedule or violation from a preferred schedule of arrival at the destination.

Unlike the cost formulation, Vickrey (1969) uses the utility formulation with a value of time spent at home, at the office, and in the queue, which is equivalent to the trip cost formulation in other departure time analysis (Arnott et al., 1990, 1993, 1994; Lindsey, 2004). This problem is also called the morning commute problem, as it resembles commuters' departure time decisions from home to work during the morning peak (Li et al., 2020). However, commutes in the evening often exhibit different travel behaviors (De Palma and Lindsey, 2002; Zhang et al., 2005). For example, commuters may perform activities such as grocery shopping on their way home. Thus, the single origin-destination, single-route, and single-mode assumptions may not adequately capture evening commuting behaviors.

For the network level departure time choices, Vickrey (1991, 2020) proposes a framework to consider the entire urban network using the bathtub model to describe the traffic dynamics, capturing hyper-congestion phenomenon with a network fundamental diagram (Godfrey, 1969; Geroliminis and Daganzo, 2008). Here, the trip completion rate is a function of the number of vehicles in the network described by a network fundamental diagram. When the number of vehicles exceeds the critical density specified in the network fundamental diagram, the trip completion rate declines. In contrast, for the bottleneck model, the trip completion rate remains constant to be the bottleneck capacity, when the inflow exceeds capacity. In addition to network traffic dynamics, Vickrey (1991, 2020) also proposes a day-to-day departure time adjustments to iteratively calculate the equilibrium of network-level departure time choices.

Day-to-day travel behavior has been widely studied in both route choice (Smith, 1984b; Peeta and Yang, 2003; Jin, 2007) and departure time choice (Chang, 1985; Mahmassani and Chang, 1986; Guo et al., 2018a; Iryo, 2019; Jin, 2020b, 2021b). These studies can be classified into three categories: (1) empirical studies based on survey or observational data; (2) simulation studies that simulate individual behaviors but may lack mathematical stability guarantees; and (3) theoretical studies that offer mathematical proof of stability. This dissertation contributes to the theoretical domain by developing stable dynamical models of day-to-day departure time choices.

Notably, while dynamical models of day-to-day route choice are often stable (Smith, 1984b; Peeta and Yang, 2003; Jin, 2007), dynamical models of day-to-day departure time choice have historically been unstable (Iryo, 2008; Guo et al., 2018a; Iryo, 2019). The first globally stable day-to-day departure time dynamical model was proposed by Jin (2020b), who followed it with a locally stable model in Jin (2021b). Satsukawa et al. (2024) formulates the departure time choice problem as an atomic game problem, establishes the evolutionary dynamics, and proves its stability. However, these models are limited in three ways: they assume a single traveler class (i.e., homogeneous users), they do not account for tolling mechanisms, and they are restricted to corridor-level departure time decisions.

This dissertation addresses these limitations in three major steps:

1. It extends Jin (2021b)'s locally stable model to a multi-class setting, accounting for user heterogeneity. Besides, it proves that when desired arrival times are identical, with different queuing costs relative to unpunctuality costs, the resulting system converges to a multi-class DTUE and is asymptotically stable.

2. It incorporates pricing mechanisms into both single-class and multi-class models. Specifically, four tolling or incentive schemes are tested, and results show that an optimal fine toll can drive both single-class and multi-class systems from a DTUE to a stable, stationary SO state.

3. It expands the analysis to the network level using the framework of Vickrey (1991, 2020), demonstrating that marginal social cost pricing can reduce peak congestion and improve system performance. Further, the framework is integrated with Dynamic Traffic Assignment (DTA) using realistic inputs (e.g., demand profiles, trip length distributions, and network fundamental diagrams), revealing a reduction in vehicle miles traveled (VMT) and vehicle hours traveled (VHT) under the pricing scheme.

The main contributions of this dissertation are:

1. The development of the first provably stable multi-class day-to-day departure time dynamical model (Chapter 4).

2. The theoretical and numerical demonstration that the optimal fine toll can achieve a stable SO state for both single-class and multi-class travelers in the corridor level (Chapter 5).

3. The integration of all components of Vickrey (1991, 2020) into a single numerical framework using empirical demand from Manhattan, New York, showing the effectiveness of marginal cost pricing in reducing peak congestion (Chapter 6).

4. The application of this framework to a DTA-based route choice scenario, comparing the congestion-reduction benefits of departure time shifts versus SO routing (Chapter 7).

The remainder of this dissertation is structured as follows: Chapter 2 reviews the relevant literature. Chapter 3 introduces the foundational single-class stable local day-to-day departure time dynamical model from Jin (2021b). Chapter 4 extends this model to a multi-class setting and proves its stability. Chapter 5 applies and evaluates various pricing schemes to both single-class and multi-class dynamical models. Chapter 6 explores network-level departure time dynamics under marginal cost pricing. Chapter 7 integrates the framework with DTA to compare benefits from departure time shifting versus route choice optimization. Chapter 8 concludes this dissertation.

Chapter 2

Literature Review

This dissertation is titled “Stable day-to-day departure time dynamics at corridor and network levels: models, optimal tolling and applications”. We review the related literature on the following four aspects: (1) equilibrium analysis of departure time choice in Section 2.1; (2) day-to-day departure time choice models in Section 2.2, including simulation models and dynamical models; (3) stable day-to-day departure time dynamics in Section 2.3; (4) optimal tolling scheme in Section 2.4. This section ends with the identification of research gaps and a summary of the contributions of this dissertation in Section 2.5.

2.1 Equilibrium analysis of departure time choice

The foundation for equilibrium analysis in departure time choice stems from Vickrey (1969), where he considers travelers departing from a single origin, home, to a single downtown destination, the office, with uniformly distributed desired arrival times from 8 AM to 9 AM. Since it resembles the morning commute in the single corridor towards downtown, we call it the departure time decision at the corridor level. Even though Vickrey (1969) does not refer to Wardrop (1952), he extends Wardrop's equilibrium in route choice decisions to departure time choice decisions. Wardrop equilibrium depicts, "The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route." (Wardrop, 1952). It means no traveler can reduce his or her travel time by unilaterally changing his or her route. Vickrey extends Wardrop's equilibrium to the departure time context — the departure time user equilibrium (DTUE), under which all the used departure times result in the same cost, which is less than or equal to what would result from any unused departure time. A traveler's trip cost includes a queuing cost representing the travel time and a unpunctuality cost that is the penalty of arriving early or arriving late (Smith, 1984a; Daganzo, 1985). In this case, travelers choose their departure times that balance queuing and unpunctuality costs and minimize the total trip cost. In DTUE, no traveler can reduce their cost by unilaterally changing their departure times. Vickrey (1969) proposes a bottleneck model with a queue representing traffic delays to study the departure time decisions at the corridor level with a single origin-destination (OD) pair. He derives the equilibrium departure and arrival patterns, the equilibrium cost, and the optimal fine toll to eliminate the queue at the bottleneck.

Subsequent research focused on the mathematical properties of DTUE: building upon Vickrey (1969), Smith (1984a) studies the existence of a DTUE, while Daganzo (1985) studies the uniqueness of such an equilibrium. However, the stability of the departure time user equilibrium has always been a problem troubling researchers, since the convergence process shows oscillatory behavior (de Palma, 2000).

Later studies incorporated heterogeneity in user preferences. Vickrey (1973) conducts an equilibrium analysis for nonidentical travelers with different values for queuing cost relative to unpunctuality cost, introducing travelers' heterogeneity. Newell (1987) considers discrete distributions for preference parameters. Arnott et al. (1994) analyze the welfare effect of congestion tolls for multi-class travelers in a single bottleneck. Lindsey (2004) analyzes the existence and uniqueness of DTUE for multi-class travelers. Researchers are referred to the review article by Li et al. (2020) for more information on the bottleneck model research from the last fifty years.

2.2 Day-to-day departure time choice models

Different from the equilibrium analysis for DTUE, day-to-day departure time choice models aim to capture the convergence process to a DTUE, which can provide more insights on how a DTUE is formed and how to mitigate congestion using pricing schemes in a day-to-day context. This section reviews the related literature on the day-to-day departure time choice models.

Ben-Akiva et al. (1984) propose a dynamic simulation model to capture the evolution of the queues in a single bottleneck from day to day. They use the model to study the impact of capacity variations, demand, flexible work times, and traffic control. Mahmassani et al. (1985) propose a conceptual model incorporating bounded rationality with an indifference band of tolerable scheduling delay (unpunctuality). The results show stable convergence patterns. Even though simulation models provide important insights on travelers' decision-making process and shows stable patterns, they don't have mathematical stability guarantees.

Different from simulation models, researchers have also developed dynamical models to capture travelers' day-to-day departure time adjustment process. Before day-to-day departure time dynamics, Smith (1984b) developed a stable day-to-day dynamical model for route choice, providing mathematical proof for its stability using Lyapunov's second method. Another well-known form of dynamics that comes from evolutionary game theory, replicator dynamics, is also found to be stable in route choice scenarios (Sandholm, 2010). However, Iryo (2008) applies Smith (1984b)'s dynamics from route choice scenarios to departure time choices, but find that the resulting dynamical model is not stable. Iryo (2019) also applies replicator dynamics to departure time choices, and finds it unstable while it is stable in route choice scenarios. Liu et al. (2017) propose a day-to-day evolution framework for both departure time choice and mode choice, considering the impact of user inertia and information provision. Their simulation results show stable convergence to a DTUE, but there is no mathematical proof that yields a stability guarantee. Guo et al. (2018a) considers five different dynamical systems for day-to-day departure time choice, but none of those models is stable. In the end, the authors put forth a general question: "Are we really solving the Dynamic User Equilibrium problem with a departure time choice?"

For day-to-day departure time choice, simulation-based models show stable patterns (Mahmassani et al., 1985), and dynamical models with heuristic approaches can also illustrate stable numerical results (Liu et al., 2017). However, none of these or the above models could be mathematically proven to be stable, which is contradictory to empirically observed convergence in departure time choices (Mahmassani et al., 1986). Such a proof did not emerge until Jin (2020b) proposed the first model of day-to-day departure time dynamics with mathematically established stability. One year later, Jin (2021b) proposed another stable local day-to-day departure time dynamical model, where locality is in relation to the time axis, as explained in the next section. We refer to Jin (2020b) and Jin (2021b) as Jin's first and second stable day-to-day departure time dynamical models. Section 2.3 briefly reviews these two papers as well as another model of stable day-to-day departure time dynamics that appeared after these two papers (Satsukawa et al., 2024).

2.3 Stable day-to-day departure time dynamics

Jin (2020b) proposes the very first stable day-to-day departure time dynamics by considering travelers' decisions in both the planning stage and the execution stage. In the planning stage, travelers make decisions according to the following three principles: (1) they choose their arrival times first and then their departure times; (2) after choosing their arrival times, travelers choose their departure times to balance the total costs; (3) travelers choose arrival times to reduce their unpunctuality cost (scheduling cost) or improve their unpunctuality payoffs (scheduling payoffs). For the execution stage, travelers still choose their departure times.

Jin (2020b) uses a point queue model to describe the within-day dynamics of departure time choice. In terms of day-to-day dynamics, Jin (2020b) first proves the DTUE to be equivalent to arrival time user equilibrium (ATUE), and he further proves the ATUE is equivalent to the scheduling payoff user equilibrium (SPUE). Therefore, the nonlocal departure time choice problem is translated into a local unpunctuality payoff choice problem on an imaginary road, where the capacity of the bottleneck determine the jam density of the imaginary road. When we refer to the departure time choice problem as "nonlocal", we mean that the travelers will shift the departure time "globally", i.e., from the beginning of the peak on the current day to the end of the peak on the next day as long as such shifting reduces the trip cost. When we refer to the unpunctuality payoff choice problem as "local", we mean that the travelers will choose the lowest unpunctuality-cost (highest unpunctuality payoff) option first. When the lower unpunctuality-cost option becomes unavailable, the travelers start to choose higher unpunctuality-cost options. The dynamics of this unpunctuality payoff choice problem is similar to how water fills a tube from bottom to top. This filling process is local in the sense that travelers as a liquid's molecules can only fill out the lower parts first, before the upper parts. Jin (2020b) then uses a model analogous to the well-known LWR model (Lighthill and Whitham, 1955; Richards, 1956) to describe how the imaginary road is filled from bottom to top, which is equivalent to the day-to-day dynamics of travelers choosing unpunctuality payoff.

Jin (2020b) first proves that the stationary state of the system is an SPUE, and then proves that the day-to-day dynamical model is globally stable using Lyapunov's second method, which means that the system will converge to an SPUE from any initial conditions. Since SPUE is equivalent to ATUE and DTUE, the day-to-day dynamical system can lead to a stable stationary DTUE. In his numerical example, the day-to-day dynamical system converges to DTUE after 40 days.

Jin (2021a) introduces another local day-to-day departure time dynamical model inspired by Vickrey (2020), where travelers can only shift their departure times by one time step to their adjacent time steps. Such a time step could be a few minutes long in a practical sense. This departure-time shifting behavior with a one time step constraint is considered as "local". Jin (2021b) first presents the discrete version of the dynamical system, and then derives its continuous version. He proves that the stationary state of the continuous day-to-day dynamical system is equivalent to DTUE using the characteristic wave speed. Note that the characteristic wave speed here should not be confused with wave speeds in traffic. Rather, it refers to an analogous wave of departure rate adjustments in the day-to-day space. With the assumptions on departure rate, rate of change in trip cost at end points, and coefficients for deferral and advancement, Jin (2021b) proves that the stationary state of the dynamical system is asymptotically stable using Lyapunov's second method. These assumptions limit the stability region of this locally stable day-to-day departure time dynamical model. So in his numerical examples, he uses heuristic deferral/advancement coefficients to drive the system to the stability region first. Once the system reaches its stability region, the coefficients are switched to provably stable ones, and the system converges to a DTUE asymptotically.

Compared to Jin (2020b) which requires travelers to have enough information to decide both arrival time and departure time, Jin (2021b) only requires travelers to make departure time decisions, which requires less information. Put simply, if the travelers only make adjustments of departure times to adjacent time steps, say by a few minutes, the model can lead to a stable DTUE. The numerical computations in later sections assume 0.1 hour (6 minutes) for this adjustment, which is in effect an assumption on the collective behavior of the travelers. Both models are stable because of their local behavior. Jin (2020b) uses an LWR-like model, which is a hyperbolic conservation equation to describe the day-to-day dynamics, while Jin (2021b)'s dynamical model is only approximately a hyperbolic conservation equation. The equivalent form of Jin (2021b)'s dynamical model with cumulative departure flow as the variable is exactly a hyperbolic conservation, however. These two studies provide interesting stable examples of hyperbolic conservation laws. Notice that these two dynamical models describe the collective behavior of the population, and do not focus on each traveler's individual behavior in adjusting his or her departure times. This also means that rooting the analytical treatments in this dissertation primarily on Jin (2020b, 2021b), allows the work to be based on simple and logically-acceptable collective behavior assumptions without delving into individual traveler behavior that is certainly much more complex and context-specific.

Different from Jin (2020b, 2021b), which focus on travelers' collective behavior, Satsukawa et al. (2024) formulates the departure time choice problem as an atomic game problem where travelers choose departure times to minimize their trip costs. Under assumptions on the departure adjustment of the first traveler, and travelers' departure times fixed according to the order in DTUE, Satsukawa et al. (2024) establishes better response dynamics and proves its global stability.

Having a stable day-to-day departure time dynamical model lays the foundation for studying various tolling schemes to reduce or even eliminate congestion. Section 2.4 reviews the related literature on the optimal tolling schemes.

2.4 Optimal tolling to manage departure time decisions

Along with the first equilibrium analysis of departure time choice, Vickrey (1969) proposes an optimal fine toll scheme to eliminate bottleneck congestion. Concerned that it might be difficult for travelers to respond to an optimal fine toll, which varies constantly, researchers have proposed other pricing tolling schemes such as single-step optimal coarse tolls (Arnott et al., 1990), multi-step tolls (Laih, 1994), fine rewards, step rewards (Rouwendal et al., 2012), and step feebates (Rouwendal et al., 2012).

Besides analyzing the equilibrium state under tolling schemes, researchers also consider how travelers adjust their departure time choices from day to day after applying toll, e.g., a Walrasian toll charge scheme (Guo et al., 2018b, 2023). However, such day-to-day departure time models with tolls only involve unstable day-to-day departure time dynamics such as Smith's dynamics (Smith, 1984b). None of the existing studies uses stable day-to-day departure time dynamics to study the impacts of optimal tolling.

2.5 Identifying research gaps and thesis contributions

Based on the literature review above, we identify the following research gaps: (1) the stable day-to-day departure time dynamics only consider the simplest case: single-class users, not heterogeneous travelers; (2) all of the existing tolling studies use unstable day-to-day dynamical models. No research has used Jin (2020b, 2021b) or Satsukawa et al. (2024) to study the impact of tolls; (3) all studies with stable day-to-day departure time dynamics focus on the departure time decisions at the corridor level using a single bottleneck model. However, the network level day-to-day departure time decisions have not been fully studied;

(4) models of day-to-day departure time dynamics have not been calibrated or cross-tested using realistic dynamic traffic assignment (DTA) to study the practicality of such dynamics in the presence of route choice behaviors.

To address the research gaps, this dissertation first extends Jin (2021b)'s stable local day-to-day departure time dynamics from single-class to multi-class, capturing travelers' heterogeneity. This dissertation then uses Jin (2021b)'s models of stable single-class departure time dynamics and the proposed stable multi-class departure time dynamics to study various pricing schemes, showing that an optimal fine toll can drive the system to a stable stationary system optimal state in both single-class and multi-class dynamics. This dissertation then studies the day-to-day departure time dynamics at the network level and the impacts of marginal social cost pricing in reducing congestion, based upon Vickrey (1991, 2020). It provides the first numerical example for Vickrey (1991, 2020), validating the effectiveness of Vickrey's marginal social cost pricing in reducing congestion. Finally, this dissertation integrates the network level day-to-day departure time dynamics with a DTA model to compare the congestion reduction effect from route choice and departure time shifting.

Chapter 3

A Stable Day-to-day Departure Time Dynamics for Single-class Travelers

This chapter presents Jin (2021b)'s stable local day-to-day departure time dynamics for single-class travelers at the corridor level, which serves as the foundation of the rest of this dissertation. Inspired by Vickrey (1991, 2020), Jin (2021b)'s dynamics assume that travelers shift their departure times only to the adjacent time steps with lower costs. Section 3.1 presents the notations in this chapter. Section 3.2 presents the definitions of within-day traffic dynamics described by the point queue model, individual trip cost, and departure time user equilibrium. Section 3.3 presents Jin (2021b)'s single-class local day-to-day dynamical system. Section 3.4 presents the stationary state of the dynamical system, while Section 3.5 discusses the stability of the stationary state. Section 3.6 presents the well-definedness condition for numerically solving the dynamical system, and its relationship to the characteristic wave speed along the day-to-day dimension. Section 3.7 shows that the stationary state of the dynamical system is a single-class departure time user equilibrium. This chapter ends with numerical examples in Section 3.8.

3.1 Notations

Table 3.1: Notation in Chapter 3, ordered by appearance

Variable	Meaning	Unit
$f(\tau, t)$	Departure rate at time t on day τ	veh/hr
$F(\tau, t)$	Cumulative departure flow at time t on day τ	veh
$g(\tau, t)$	Arrival rate at time t on day τ	veh/hr
$G(\tau, t)$	Cumulative arrival flow at time t on day τ	veh
$\delta(\tau, t)$	Queue size in the point queue model	veh
$\Upsilon(\tau, t)$	queuing time for travelers departing at time t on day τ	hr
$\Upsilon'(\tau, t)$	queuing time for travelers arriving at time t on day τ	hr
C	Capacity for the point queue model	veh/hr
ϵ	Infinitesimal hyper-real positive number	hr
$f^m(\tau)$	Maximum departure rate for the study period on day τ	veh/hr
$\phi(\tau, t)$	Trip cost for travelers departing at time t from home on day τ	\$
t^*	Desired arrival time for all travelers	hr
λ	Value of travel time	\$/hr
μ	Cost of arriving one extra hour early	\$/hr
ν	Cost of arriving one extra hour late	\$/hr
$\phi_1(\tau, t)$	queuing cost for travelers departing at time t from home on day τ	\$
$\phi_2(\tau, t)$	Unpunctuality cost for travelers departing at time t from home on day τ	\$
$\omega(\tau, t)$	Rate of change in trip cost	\$/hr
$\omega^E(\tau, t)$	Rate of change in trip cost for early arrivals	\$/hr
$\omega^L(\tau, t)$	Rate of change in trip cost for late arrivals	\$/hr
$f_1^m(\tau)$	Maximum departure rate for early arrivals on day τ	veh/hr
$f_2^m(\tau)$	Maximum departure rate for late arrivals on day τ	veh/hr
ϕ^*	Minimum trip cost at the single-class departure time user equilibrium (SC-DTUE)	\$
t_0	Start of departure period at SC-DTUE	hr
t_2	End of departure period at SC-DTUE	hr
T	End of the study period	hr
I	Number of intervals of the study period	1
Δt	Time step size	hr
$\Delta\tau$	Day step size	day
$f_i^d(\tau)$	Deferral rate for travelers departing at time step i on day τ	veh/day
$B_i^d(\tau)$	Deferral coefficient at time step i on day τ	1
$f_i^a(\tau)$	Advance rate for travelers departing at time step i on day τ	veh/day
$B_i^a(\tau)$	Advance coefficient at time step i on day τ	1
$\tilde{g}_i(\tau)$	Net advance flow rate from time step i to time step $i-1$ at time step i on day τ	1
N	Total number of travelers	veh
τ'	The day when the dynamical system reaches SC-DTUE	1
ε	$\omega(\tau, t_2) = 0$ when $\tau \geq \tau' - \varepsilon$	day
$u(\tau, t)$	Deferral or advance coefficient times time step over day step	hr/day
$\alpha(\tau, t)$	Departure rate coefficient of rate of change in trip cost	1
$V(f(\tau, \cdot))$	Lyapunov functional on day τ	1
$A(\tau, t)$	Replacing $\frac{\partial}{\partial\tau}[f(\tau, t) \cdot \omega^2(\tau, t)]$	1
u_0	Positive coefficient for $u(\tau, t)$	1
$f^0(t)$	Initial departure rate on day 0	veh/hr
$f^*(t)$	SC-DTUE departure rate	veh/hr
$\phi^*(t)$	SC-DTUE cost function	\$
$e(j)$	Lebesgue L_1 norm error on day step j	veh/hr

3.2 Definitions

3.2.1 Within-day dynamics: point queue model

For the corridor level departure time choice, we use the bottleneck model or point queue model (Vickrey, 1969; Jin, 2015), which assumes one origin-destination pair (home-work), one route and one mode. The point queue model describes the following scenario: travelers depart from home and then arrive at a queue. After queuing, travelers depart from the queue and arrive at their work destinations. Travelers choose their departure times to balance queuing cost and unpunctuality cost. In some literature, unpunctuality cost is also called scheduling cost.

Let $f(\tau, t)$ be the departure rate from home on day τ at time t (unit: veh/hr), and let $F(\tau, t)$ be the cumulative departure flow from home on day τ at time t (unit: veh). Let $g(\tau, t)$ be the arrival rate at work on day τ at time t (unit: veh/hr), and let $G(\tau, t)$ be the cumulative arrival flow at work on day τ at time t (unit: veh). Notice that the departure rate from home is equivalent to the arrival rate at the queue, and the departure rate from the queue is equivalent to the arrival rate at work. So here we refer departure rate as departure rate from home, and arrival rate as arrival rate at work. Let $\delta(\tau, t)$ be the queue size (unit: veh). We have the following conservation equation for queue size $\delta(\tau, t)$:

$$\delta(\tau, t) = F(\tau, t) - G(\tau, t). \tag{3.1}$$

Let $\Upsilon(\tau, t)$ be the queuing time for travelers departing at time t on day τ (unit: hr), and let $\Upsilon'(\tau, t)$ be the queuing time for travelers arriving at time t on day τ (unit: hr). We have the following equations:

$$F(\tau, t) = G(\tau, t + \Upsilon(\tau, t)) \tag{3.2}$$

$$F(\tau, t - \Upsilon^{'}(\tau, t)) = G(\tau, t). \tag{3.3}$$

Equations (3.1) to (3.3) depict the First-In-First-Out (FIFO) principle of the point queue model (Jin, 2015, 2020b).

If we take the derivative of Equation (3.1) on both sides, we have the following conservation equation in terms of rate of change in queue size and departure/arrival rate:

$$\frac{\partial}{\partial t}\delta(\tau, t) = f(\tau, t) - g(\tau, t). \tag{3.4}$$

Let C be the capacity for the point queue model, which is the maximum service rate of the bottleneck, where $0 \leq g(\tau, t) \leq C$. When the departure rate $f(\tau, t)$ is less than or equal to the capacity C, there is no queue (i.e., $\delta(\tau, t) = 0$). When the departure rate $f(\tau, t)$ is greater than the capacity C, queue develops. We have the following relations for arrival rate $g(\tau, t)$ (Jin, 2015):

$$g(\tau, t) = \min\{f(\tau, t) + \frac{\delta(\tau, t)}{\epsilon}, C\}, \tag{3.5}$$

where $\epsilon = \lim_{\Delta t \to 0^+} \Delta t$ is an indefinitesimal hyper-real number, which is equal to Δt in the discrete version. When $\delta(\tau, t) = 0$, we have $f(\tau, t) \leq C$, so $g(\tau, t) = f(\tau, t)$. When $\delta(\tau, t) > 0$, we have $\frac{\delta(\tau,t)}{\epsilon} \to \infty$, so $g(\tau, t) = C$.

Substituting Equation (3.5) into Equation (3.4), we have the following relation on rate of change in queue size:

$$\frac{\partial}{\partial t}\delta(\tau, t) = \max\{-\frac{\delta(\tau, t)}{\epsilon}, f(\tau, t) - C\}. \tag{3.6}$$

When $\delta(\tau,t)=0$, we have $-\frac{\delta(\tau,t)}{\epsilon}=0$ and $f(\tau,t)-C\leq 0$, so $\frac{\partial}{\partial t}\delta(\tau,t)=0$. When $\delta(\tau,t)>0$, then $-\frac{\delta(\tau,t)}{\epsilon}\rightarrow -\infty$, so $\frac{\partial}{\partial t}\delta(\tau,t)=f(\tau,t)-C$. Let $f^m(\tau)$ denote the maximum departure rate for the study period on day τ, so we have $-C\leq\frac{\partial}{\partial t}\delta(\tau,t)\leq f^m(\tau)-C$.

We use the following equation to update cumulative departure flow, $F(\tau,t)$, from time step t to $t+\Delta t$:

$$F(\tau,t+\Delta t)=F(\tau,t)+f(\tau,t)\Delta t, \tag{3.7}$$

and the following equation to update the cumulative arrival flow, $G(\tau,\cdot)$, from time step t to $t+\Delta t$:

$$G(\tau,t+\Delta t)=G(\tau,t)+g(\tau,t)\Delta t. \tag{3.8}$$

Substituting Equation (3.5) into Equation (3.8), we get the following equation:

$$G(\tau,t+\Delta t)=\min\{F(\tau,t+\Delta t),G(\tau,t)+C\cdot\Delta t\}. \tag{3.9}$$

We have the following discrete version of Equation (3.6) to update the queue size $\delta(\tau,t)$ from t to $t+\Delta t$:

$$\delta(\tau,t+\Delta t)=\max\{0,\delta(\tau,t)+(f(\tau,t)-C)\Delta t\}. \tag{3.10}$$

The queuing time $\Upsilon(\tau,t)$ has the following relation:

$$\Upsilon(\tau,t)=\frac{\delta(\tau,t)}{C}. \tag{3.11}$$

where if there is no queue (i.e., $\delta(\tau,t)=0$), then queuing time is zero (i.e., $\Upsilon(\tau,t)=0$). If queue exists, $\delta(\tau, t) > 0$, the queuing time is as follows: $\Upsilon(\tau, t) = \frac{\delta(\tau,t)}{C}$.

Taking the derivative of both sides of Equation (3.11) and substituting Equation (3.6) into it, we obtain the rate of change in queuing time as follows:

$$\frac{\partial}{\partial t}\Upsilon(\tau, t) = \max\{-\frac{\delta(\tau, t)}{C \cdot \epsilon}, \frac{f(\tau, t)}{C} - 1\}. \tag{3.12}$$

When $\delta(\tau, t) = 0$, $\Upsilon(\tau, t) = 0$, $f(\tau, t) \leq C$ and $\frac{f(\tau,t)}{C} - 1 \leq 0$, so $\frac{\partial}{\partial t}\Upsilon(\tau, t) = 0$. When $\delta(\tau, t) > 0$, $-\frac{\delta(\tau,t)}{C\cdot\epsilon} \to -\infty$, then $\frac{\partial}{\partial t}\Upsilon(\tau, t) = \frac{f(\tau,t)}{C} - 1$. When $f(\tau, t) = 0$, $\frac{\partial}{\partial t}\Upsilon(\tau, t) = -1$. We thus find the following bounds for the rate of change in queuing time:

$$-1 \leq \frac{\partial}{\partial t}\Upsilon(\tau, t) \leq \frac{f^m(\tau)}{C} - 1. \tag{3.13}$$

3.2.2 Trip cost: homogeneous travelers

Let $\phi(\tau, t)$ denote the trip cost for travelers departing at time t from home on day τ (unit: \$). Let t^* be the desired arrival time for all travelers. We have the following trip cost function:

$$\phi(\tau, t) = \lambda \cdot (\Upsilon^0 + \Upsilon(\tau, t)) + \mu \cdot \{t^* - (t + \Upsilon(\tau, t))\}_+ + \nu \cdot \{(t + \Upsilon(\tau, t)) - t^*\}_+, \tag{3.14}$$

where $\{y\}_+ = \max\{0, y\}$, Υ^0 is the travelers' free flow travel time from home to work, while $\Upsilon(\tau, t)$ is the variable queuing time that travelers experience. λ is the value of travel time (unit: \$/hr), while μ, ν are the costs, respectively, of arriving an extra hour early or an extra hour late (unit: \$/hr). The first term in Equation (3.14) refers to the queuing cost, while the second and third terms in Equation (3.14) refer to the unpunctuality cost. Let $\phi_1(\tau, t) = \lambda\cdot(\Upsilon^0+\Upsilon(\tau, t))$ be the queuing cost and $\phi_2(\tau, t) = \mu\cdot\{t^*-(t+\Upsilon(\tau, t))\}_+ + \nu\cdot\{(t+\Upsilon(\tau, t))-t^*\}_+$ be the unpunctuality cost. We assume $\mu < \lambda$ to avoid multiple equilibrium (Arnott et al., 1994). Without loss of generality, we assume $\Upsilon^0 = 0$.

Taking the derivative of Equation (3.14) on both sides, we get the rate of change in trip cost. Let $\omega(\tau, t)$ be the rate of change in trip cost, i.e., $\omega(\tau, t) = \frac{\partial}{\partial t}\phi(\tau, t)$. For early arrivals, we have $\omega^E(\tau, t) = \frac{\partial}{\partial t}\phi^E(\tau, t)$ (E denoting earliness):

$$\omega^E(\tau, t) = \frac{\partial}{\partial t}\phi^E(\tau, t) = (\lambda - \mu) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau, t) - \mu. \tag{3.15}$$

When the bottleneck is uncongested (i.e., $\delta(\tau, t) = 0$), we have $\frac{\partial}{\partial t}\Upsilon^E(\tau, t) = 0$, so $\omega^E(\tau, t) = -\mu$. When the bottleneck is congested (i.e., $\delta(\tau, t) > 0$), we have $\frac{\partial}{\partial t}\Upsilon^E(\tau, t) = \frac{f(\tau,t)}{C} - 1$, so $\omega^E(\tau, t) = (\lambda - \mu) \cdot \frac{f(\tau,t)}{C} - \lambda$.

For late arrivals, we have the rate of change in trip cost as $\omega^L(\tau, t) = \frac{\partial}{\partial t}\phi^L(\tau, t)$ (L denoting lateness):

$$\omega^L(\tau, t) = \frac{\partial}{\partial t}\phi^L(\tau, t) = (\lambda + \nu) \cdot \frac{\partial}{\partial t}\Upsilon^L(\tau, t) + \nu. \tag{3.16}$$

When the bottleneck is uncongested (i.e., $\delta(\tau, t) = 0$), we have $\frac{\partial}{\partial t}\Upsilon^L(\tau, t) = 0$, so $\omega^L(\tau, t) = \nu$. When the bottleneck is congested (i.e., $\delta(\tau, t) > 0$), we have $\frac{\partial}{\partial t}\Upsilon^L(\tau, t) = \frac{f(\tau,t)}{C} - 1$, so $\omega^L(\tau, t) = (\lambda + \nu) \cdot \frac{f(\tau,t)}{C} - \lambda$.

Let $f_1^m(\tau)$ be the maximum departure rate for early arrivals, and $f_2^m(\tau)$ maximum departure rate for late arrivals, so we have $f^m(\tau) = \max\{f_1^m(\tau), f_2^m(\tau)\}$. From Equation (3.13), we have $-1 \leq \frac{\partial}{\partial t}\Upsilon^E(\tau, t) \leq \frac{f_1^m(\tau)}{C} - 1$ for early arrivals, $-1 \leq \frac{\partial}{\partial t}\Upsilon^L(\tau, t) \leq \frac{f_2^m(\tau)}{C} - 1$ for late arrivals, and $-1 \leq \frac{\partial}{\partial t}\Upsilon(\tau, t) \leq \frac{f^m(\tau)}{C} - 1$ for all arrivals. Substituting the bounds of $\frac{\partial}{\partial t}\Upsilon^E(\tau, t)$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau, t)$ into Equations (3.15) and (3.16), we obtain:

$$-\lambda \leq \omega^E(\tau, t) \leq (\lambda - \mu) \cdot \frac{f_1^m(\tau)}{C} - \lambda \tag{3.17}$$

for early arrivals, and

$$-\lambda \leq \omega^L(\tau, t) \leq (\lambda + \nu) \cdot \frac{f_2^m(\tau)}{C} - \lambda \tag{3.18}$$

for late arrivals.

3.2.3 Departure time user equilibrium

Definition 3.1 (Single-class departure time user equilibrium). *A single-class departure time user equilibrium (SC-DTUE) is reached when all chosen departure times share the same trip cost, which is less than or equal to other departure times that are not chosen by travelers. That is, $\phi(\tau, t) = \phi^*$ for $f(\tau, t) > 0$, where $t \in [t_0, t_2]$, while $\phi(\tau, t) \geq \phi^*$ for $f(\tau, t) = 0$ where $t \in (-\infty, t_0) \cup (t_2, +\infty]$. ϕ^* is the minimum trip cost at the SC-DTUE, and t_0 and t_2 are the start and end of the departure period at SC-DTUE.*

From Definition 3.1, we have the following complementarity condition:

$$f(\tau, t) \cdot (\phi(\tau, t) - \phi^*) = 0, \tag{3.19}$$

where if $f(\tau, t) > 0$, $\phi(\tau, t) - \phi^* = 0$, and $\phi(\tau, t) = \phi^*$. If $f(\tau, t) = 0$, then $\phi(\tau, t) - \phi^* \geq 0$, and $\phi(\tau, t) \geq \phi^*$.

3.3 Day-to-day dynamics: Jin (2021)'s dynamics

Inspired by Vickrey (1991, 2020), Jin (2021b) proposes a stable local day-to-day departure time dynamics for single-class travelers. Vickrey (1991, 2020) consider four different travelers' responses to pricing — deferral, advance, suppression and generation. Instead of considering all four responses, which include elastic demand, Jin (2021b) only considers two responses related to fixed demand scenarios — deferral and advance, and only allows travelers to defer and advance by one time step. Jin (2021b) proves that the stationary state of the dynamical system is equivalent to SC-DTUE, and that the stationary state is asymptotically stable. We called it Jin (2021b)'s dynamics, which we will present in Sections 3.3.1, 3.3.2 and 3.3.4.

3.3.1 Discrete version

Let $[0, T]$ be the study time period, which can be, say, a morning peak period. We divide it into I intervals. Then we have $\Delta t = \frac{T}{I}$ as the time step size, where $\Delta t > 0$. The time step size can be a few minutes for practical purposes. Our numerical examples use 0.1 hour (6 minutes) for a time step. Let $\Delta\tau$ be the day step size, where $\Delta\tau > 0$. Here we use the term "day-step" to refer to the discretization of the continuous day-to-day dimension into steps. These steps do not refer to days. In our numerical examples, they are much smaller than a day, the length being determined based on wave characteristics in continuous day and time space (see Section 3.6).

Let $\phi_i(\tau) = \phi(\tau, i\Delta t)$, where $i = 0, 1, 2, ..., I$, which denotes the trip cost for $I + 1$ points in the study period. When $i = 0$, we have $\phi_0(\tau) = \phi(\tau, 0)$, while when $i = I$, we have $\phi_I(\tau) = \phi(\tau, T)$. We have $\omega_i(\tau) = \omega(\tau, (i - \frac{1}{2})\Delta t) = \frac{\phi(\tau, i\Delta t) - \phi(\tau, (i-1)\Delta t)}{\Delta t} = \frac{\phi_i(\tau) - \phi_{i-1}(\tau)}{\Delta t}$, where $i = 1, 2, ..., I$, which denotes the trip cost difference between two consecutive points in the study period. Let $f_i(\tau) = f(\tau, (i - \frac{1}{2})\Delta t)$, where $i = 1, 2, ..., I$, which denotes the departure rate in interval i. Notice that the set of $\{\phi_i(\tau)\}_{i=0}^{I}$ has $I + 1$ elements, while the sets of $\{\omega_i(\tau)\}_{i=1}^{I}$ and $\{f_i(\tau)\}_{i=1}^{I}$ have I elements.

On day τ, if $\omega_{i+1}(\tau) < 0$, that is, $\phi_{i+1}(\tau) < \phi_i(\tau)$, we assume that travelers will defer their departure time by one time step, from i to $i+1$, to reduce their trip costs. Let $f_i^d(\tau)$ be the deferral rate for travelers departing at time step i on day τ (unit: veh/day), so we assume the deferral travelers, $f_i^d(\tau)\Delta\tau$, is determined by the following relation:

$$f_i^d(\tau)\Delta\tau = B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ \cdot f_i(\tau)\Delta t, \tag{3.20}$$

where $B_{i+1}^d(\tau)$ is a positive deferral coefficient, which is to be estimated, and $\{-\omega_{i+1}(\tau)\}_+$ is a scalar whose unit is 1. If we do not consider $\{-\omega_{i+1}(\tau)\}_+$ as a scalar, the units of the left-hand side and the right-hand side of Equation (3.20) do not match. Equation (3.20) depicts the following relation: the number of deferral travelers is equal to the number of travelers departing at time step i on day τ, $f_i(\tau)\Delta t$, times the cost difference between time step $i+1$ and i, $\{-\omega_{i+1}(\tau)\}_+$, and the deferral coefficient, $B_{i+1}^d(\tau)$.

On day τ, if $\omega_i(\tau) > 0$, that is, $\phi_i(\tau) > \phi_{i-1}(\tau)$, we assume that travelers will advance their departure time by one time step, from i to $i-1$, to reduce their trip costs. Let $f_i^a(\tau)$ be the advance rate for travelers departing at time step i on day τ (unit: veh/day), so we assume the advancing travelers, $f_i^a(\tau)\Delta\tau$, is determined by the following relation:

$$f_i^a(\tau)\Delta\tau = B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ \cdot (f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau), \tag{3.21}$$

where $B_i^a(\tau)$ is a positive advance coefficient, which is to be estimated, and $\{\omega_i(\tau)\}_+$ is a scalar whose unit is 1. To avoid the same traveler being deferred and advanced at the same time, we deduct the deferral travelers before we calculate the advancing travelers (i.e., $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau$). Equation (3.21) depicts the following relation: the number of advancing travelers is equal to the remaining non-deferring travelers who depart at time step i on day τ, $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau$, times the cost difference between time step i and $i-1$, $\{\omega_i(\tau)\}_+$, and the advance coefficient, $B_i^a(\tau)$.

After calculating the deferring and advancing travelers, we update the departure rate on day $\tau + \Delta\tau$, $f_i(\tau + \Delta\tau)$, based on the day τ's departure rate at time step i, $f_i(\tau)$, deferral and advance rate at time step i, $f_i^d(\tau)$ and $f_i^a(\tau)$, deferral rate at time step $i-1$, $f_{i-1}^d(\tau)$, and advance rate at time step $i+1$, $f^a_{i+1}(\tau)$. So we have the following relation:

$$f_i(\tau+\Delta\tau)\Delta t = f_i(\tau)\Delta t - f^d_i(\tau)\Delta\tau - f^a_i(\tau)\Delta\tau + f^d_{i-1}(\tau)\Delta\tau + f^a_{i+1}(\tau)\Delta\tau, \tag{3.22}$$

where $f^d_i(\tau)\Delta\tau$ and $f^a_i(\tau)\Delta\tau$ are the travelers flowing out from time step i, while $f^d_{i-1}(\tau)\Delta\tau$ and $f^a_{i+1}(\tau)\Delta\tau$ are the travelers flowing into time step i.

3.3.2 Well-definedness

We present the well-definedness condition of Jin (2021b)'s local dynamical system (Equations (3.20) to (3.22)) in this section.

Definition 3.2 (Well-defined definition)**.** *Jin (2021b)'s local dynamical system (Equations* (3.20) *to* (3.22)*) is well-defined if and only if $f_i(\tau) \geq 0$ for each time step i and each day step τ.*

The departure rate, $f_i(\tau)$, should be non-negative, since we only consider trips departing from home and do not consider trips to travel back home. So we assume $f_i(\tau) \geq 0$. For the local dynamical system (Equations (3.20) to (3.22)) to be well-defined, we need $f_i(\tau+\Delta\tau) \geq 0$ as well. For $f_i(\tau+\Delta\tau) \geq 0$, we need to have $f_i(\tau)\Delta t - f^d_i(\tau)\Delta\tau - f^a_i(\tau)\Delta\tau + f^d_{i-1}(\tau)\Delta\tau + f^a_{i+1}(\tau)\Delta\tau \geq 0$ from Equation (3.22). If the total inflow from time step $i-1$ and $i+1$ into time step i is zero, $f^d_{i-1}(\tau)\Delta\tau + f^a_{i+1}(\tau)\Delta\tau = 0$, then $f_i(\tau)\Delta t - f^d_i(\tau)\Delta\tau - f^a_i(\tau)\Delta\tau \geq 0$. That is, the outflow from time step i to time steps $i-1$ and $i+1$ should not be greater than the number of travelers departing at time step i itself:

$$f^d_i(\tau)\Delta\tau + f^a_i(\tau)\Delta\tau \leq f_i(\tau)\Delta t. \tag{3.23}$$

Substituting $f^a_i(\tau)\Delta\tau$ from Equation (3.21) into Equation (3.23), we have:

$$f^d_i(\tau)\Delta\tau + B^a_i(\tau)\cdot\{\omega_i(\tau)\}_+ \cdot (f_i(\tau)\Delta t - f^d_i(\tau)\Delta\tau) \leq f_i(\tau)\Delta t. \tag{3.24}$$

Arranging $f_i(\tau)\Delta t$ to one side, and $f_i^d(\tau)\Delta\tau$ to the other side, we have:

$$f_i^d(\tau)\Delta\tau(1 - B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+) \leq f_i(\tau)\Delta t(1 - B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+). \tag{3.25}$$

After further simplifying, Equation (3.25) becomes as follows:

$$(1 - B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+)(f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau) \geq 0. \tag{3.26}$$

Here we discuss the following two cases based on $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau$. That is, $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau < 0$ and $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau \geq 0$.

1. If $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau < 0$, then $1 - B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ \leq 0$. Under this condition, we have $f_i(\tau)\Delta t < f_i^d(\tau)\Delta\tau$, which contradicts to $f_i^d(\tau)\Delta\tau + f_i^a(\tau)\Delta\tau \leq f_i(\tau)\Delta t$ from Equation (3.23). Therefore, case 1 is not possible.

2. If $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau \geq 0$, then $1 - B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ \geq 0$ and that is $B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ \leq 1$. From Equation (3.26), $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau \geq 0$ becomes:

$$f_i(\tau)\Delta t \geq f_i^d(\tau)\Delta\tau = B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ \cdot f_i(\tau)\Delta t, \tag{3.27}$$

which becomes $f_i(\tau)\Delta t(1 - B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+) \geq 0$. Since $f_i(\tau)\Delta t \geq 0$, $1 - B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ \geq 0$, and thus $B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ \leq 1$.

From case 2, we have the following well-definedness condition:

$$\begin{aligned} B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ &\leq 1, \\ B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ &\leq 1, \end{aligned} \tag{3.28}$$

where $\{\omega_i(\tau)\}_+$ and $\{-\omega_{i+1}(\tau)\}_+$ are determined by traffic dynamics of the point queue model. We need to further define the bounds for $B_i^a(\tau)$ and $B_{i+1}^d(\tau)$ for Jin (2021b)'s local dynamical system (Equations (3.20) to (3.22)) to be well-defined.

Let $\{\omega_i(\tau)\}_+^{max}$ be the maximum value of $\{\omega_i(\tau)\}_+$, and let $\{-\omega_{i+1}(\tau)\}_+^{max}$ be the maximum value of $\{-\omega_{i+1}(\tau)\}_+$, so from Equation (3.28), we have:

$$\begin{aligned} B_i^a(\tau) &\le \frac{1}{\{\omega_i(\tau)\}_+^{max}}, \\ B_{i+1}^d(\tau) &\le \frac{1}{\{-\omega_{i+1}(\tau)\}_+^{max}}. \end{aligned} \tag{3.29}$$

We need to discuss $\{\omega_i(\tau)\}_+^{max}$ and $\{-\omega_{i+1}(\tau)\}_+^{max}$ to further determine the well-definedness condition.

Let $\omega^E(\tau,t)$ and $\omega^L(\tau,t)$ be the rate of change in trip cost for early and late arrivals. We have $\omega^E(\tau,t) = \frac{\partial}{\partial t}\phi^E(\tau,t) = (\lambda-\mu)\cdot\frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu$ from Equation (3.15), and $\omega^L(\tau,t) = (\lambda+\nu)\cdot\frac{\partial}{\partial t}\Upsilon^L(\tau,t)+\nu$ from Equation (3.16). We discuss the $\{\omega_i(\tau)\}_+^{max}$ and $\{-\omega_{i+1}(\tau)\}_+^{max}$ under the following two cases of $\delta(\tau,t)=0$ and $\delta(\tau,t)>0$:

1. If $\delta(\tau,t)=0$, then $\frac{\partial}{\partial t}\Upsilon^E(\tau,t)=0$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t)=0$, so we have $\omega^E(\tau,t)=-\mu$ and $\omega^L(\tau,t)=\nu$. Since $\{\omega^E(\tau,t)\}_+=0$ for early arrivals and $\{\omega^L(\tau,t)\}_+=\nu$ for late arrivals, we have $\{\omega(\tau,t)\}_+\le\nu$, and $\{\omega_i(\tau)\}_+^{max}=\nu$. Since $\{-\omega^E(\tau,t)\}_+=\mu$ and $\{-\omega^L(\tau,t)\}_+=0$, so we have $\{-\omega(\tau,t)\}_+\le\mu$, and $\{-\omega_{i+1}(\tau)\}_+^{max}=\mu$. So we have:

$$\begin{aligned} B_i^a(\tau) &\le \frac{1}{\{\omega_i(\tau)\}_+^{max}} = \frac{1}{\nu}, \\ B_{i+1}^d(\tau) &\le \frac{1}{\{-\omega_{i+1}(\tau)\}_+^{max}} = \frac{1}{\mu}. \end{aligned} \tag{3.30}$$

2. If $\delta(\tau,t)>0$, then we have $-1\le\frac{\partial}{\partial t}\Upsilon(\tau,t)\le\frac{f^m(\tau)}{C}-1$, so we have $-\lambda\le\omega^E(\tau,t)\le(\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C}-\lambda$ from Equation (3.17) for early arrivals, and $-\lambda\le\omega^L(\tau,t)\le(\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C}-\lambda$ from Equation (3.18) for late arrivals.

   Since $\{\omega^E(\tau,t)\}_+^{max}=(\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C}-\lambda$ and $\{\omega^L(\tau,t)\}_+^{max}=(\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C}-\lambda$, we have $\{\omega_i(\tau)\}_+^{max}=max\{(\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C}-\lambda,(\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C}-\lambda\}$. Since $\{-\omega^E(\tau,t)\}_+^{max}=\lambda$

and $\{-\omega^L(\tau,t)\}_+^{max} = \lambda$, so we have $\{-\omega_{i+1}(\tau)\}_+^{max} = \lambda$. So we have:

$$\begin{aligned} B_i^a(\tau) &\leq \frac{1}{\{\omega_i(\tau)\}_+^{\max}} = \frac{1}{\max\left\{(\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda,\ (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\right\}}, \\ B_{i+1}^d(\tau) &\leq \frac{1}{\{-\omega_{i+1}(\tau)\}_+^{\max}} = \frac{1}{\lambda}. \end{aligned} \tag{3.31}$$

Combining Equation (3.30) from case 1 and Equation (3.31) from case 2, we have the following well-definedness condition:

$$\begin{aligned} 0 < B_i^a(\tau) &\leq \frac{1}{\max\left\{\nu,\ (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda,\ (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\right\}}, \\ 0 < B_{i+1}^d(\tau) &\leq \frac{1}{\max\{\lambda,\mu\}} = \frac{1}{\lambda}. \end{aligned} \tag{3.32}$$

Theorem 3.1 (Well-definedness condition)**.** *Jin (2021b)'s local dynamical system (Equations (3.20) to (3.22)) is well-defined when* $0 < B_i^a(\tau) \leq 1/max\{\nu, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\}$ *and* $0 < B_{i+1}^d(\tau) \leq 1/\lambda$ *for each time step* i *and each day step* τ*.*

3.3.3 Net flows between time steps

Equation (3.22) deals with three time steps: $i-1$, i, and $i+1$. To further simplify the equation and to derive its continuous version, we consider the net flows from time step i to $i-1$, which is the net advancing flows. By net advancing flows, we mean the number of travelers who shift their departure times from time step i on day τ to time step $i-1$ on day $\tau+\Delta\tau$ (i.e., $f_i^a(\tau) > 0$), so travelers are flowing out from time step i into time step $i-1$. At the same time, travelers might also have net deferring flows. By net deferring flows, we mean the number of travelers who shift their departure times from time step $i-1$ on day τ to time step i on day $\tau+\Delta\tau$ (i.e., $f_{i-1}^d(\tau) > 0$), which means travelers are flowing out from time step $i-1$ into time step i.

Since these two kinds of flows exist across the boundary of time steps $i-1$ and i, we only consider the net advance flow from time step i to time step $i-1$ to simplify the analysis.

Let $\tilde{g}_i(\tau)$ to be the net advance flow rate from time step i to $i-1$ from day τ to day $\tau+\Delta\tau$, which is related to both $f_i^a(\tau)$ and $f_{i-1}^d(\tau)$. We first analyze the relationship between $f_i^a(\tau)$ and $f_{i-1}^d(\tau)$, and then present the functional form of $\tilde{g}_i(\tau)$.

We have $f_{i-1}^d(\tau)\Delta\tau = B_i^d(\tau)\cdot\{-\omega_i(\tau)\}_+\cdot f_{i-1}(\tau)\Delta t$ from Equation (3.20), and $f_i^a(\tau)\Delta\tau = B_i^a(\tau)\cdot\{\omega_i(\tau)\}_+\cdot(f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau)$ from Equation (3.21). With the well-definedness condition, we have $f_{i-1}(\tau)\Delta t \geq 0$ and $f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau \geq 0$. If $\omega_i(\tau) > 0$, then $-\omega_i(\tau) < 0$, so we have $\{\omega_i(\tau)\}_+ > 0$ and $\{-\omega_i(\tau)\}_+ = 0$. So we have $f_i^a(\tau) > 0$ and $f_{i-1}^d(\tau) = 0$. If $\omega_i(\tau) < 0$, then we have $-\omega_i(\tau) > 0$, so we have $\{\omega_i(\tau)\}_+ = 0$ and $\{-\omega_i(\tau)\}_+ > 0$. So we have $f_i^a(\tau) = 0$ and $f_{i-1}^d(\tau) > 0$. If $\omega_i(\tau) = 0$, $\{-\omega_i(\tau)\}_+ = \{\omega_i(\tau)\}_+ = 0$, and $f_{i-1}^d(\tau) = f_i^a(\tau) = 0$.

So relationship of $f_{i-1}^d(\tau)$ and $f_i^a(\tau)$ can be summarized as follows:

$$f_{i-1}^d(\tau)\cdot f_i^a(\tau) = 0, \tag{3.33}$$

where if $f_{i-1}^d(\tau) > 0$, then $f_i^a(\tau) = 0$, and vice versa, or they both become zero.

So we have the following functional form of $\tilde{g}_i(\tau)$:

$$\tilde{g}_i(\tau) = sgn(\omega_i(\tau))\cdot\max\{f_{i-1}^d(\tau), f_i^a(\tau)\}, \tag{3.34}$$

where $sgn(y)$ is the sign function, that is:

$$sgn(y) = \begin{cases} 1 & \text{for} \quad y > 0, \\ 0 & \text{for} \quad y = 0, \\ -1 & \text{for} \quad y < 0. \end{cases} \tag{3.35}$$

When $\omega_i(\tau) > 0$ (i.e., $\phi_i(\tau) > \phi_{i-1}(\tau)$), we have $f_i^a(\tau) > 0$ and $f_{i-1}^d(\tau) = 0$, so we have $\tilde{g}_i(\tau) = f_i^a(\tau)$. There is a positive net advance flow from time step i to $i-1$. When $\omega_i(\tau) < 0$ (i.e., $\phi_i(\tau) < \phi_{i-1}(\tau)$), we have $f_i^a(\tau) = 0$ and $f_{i-1}^d(\tau) > 0$, so we have $\tilde{g}_i(\tau) = -f_{i-1}^d(\tau)$. There is a negative net advance flow from time step i to $i-1$, which means there is positive net deferral flow from time step $i-1$ to i. When $\omega_i(\tau) = 0$, there is zero net advance flow from time step i to $i-1$.

So Equation (3.22) can be written as:

$$f_i(\tau + \Delta\tau)\Delta t = f_i(\tau)\Delta t + \tilde{g}_{i+1}(\tau)\Delta\tau - \tilde{g}_i(\tau)\Delta\tau, \tag{3.36}$$

where $\tilde{g}_1(\tau) = \tilde{g}_{I+1}(\tau) = 0$, because time step 1 does not have a net advance flow to time step 0, as time step 0 is out of the study period. Likewise, there is no net advance flow from time step $I+1$ to time step I, because time step $I+1$ is out of the study period.

We have the following property for the dynamical system Equation (3.36):

Theorem 3.2. *The number of trips is conserved on any day step τ in dynamical system (Equation (3.36)).*

Proof. To prove it, we need to show that the total number of trips is N on day $\tau + \Delta\tau$ (i.e., $N = \sum_i f_i(\tau + \Delta\tau)\Delta t$), assuming the total number of trips to be N on day τ (i.e., $N = \sum_i f_i(\tau)\Delta t$) for all τ and $\Delta\tau$.

Let $N = \sum_i f_i(\tau)\Delta t$ on day τ. So on day $\tau+\Delta\tau$, from Equation (3.36), we have the following equation:

$$\sum_{i=1}^{I} f_i(\tau + \Delta\tau)\Delta t = \sum_{i=1}^{I} f_i(\tau)\Delta t + \sum_{i=1}^{I} \tilde{g}_{i+1}(\tau)\Delta\tau - \sum_{i=1}^{I} \tilde{g}_i(\tau)\Delta\tau. \tag{3.37}$$

Arranging the Equation (3.37), we have:

$$\sum_{i=1}^{I} f_i(\tau+\Delta\tau)\Delta t = \sum_{i=1}^{I} f_i(\tau)\Delta t + \tilde{g}_{I+1}(\tau)\Delta\tau - \tilde{g}_1(\tau)\Delta\tau = \sum_{i=1}^{I} f_i(\tau)\Delta t + 0 - 0. \quad (3.38)$$

So we have:

$$\sum_{i=1}^{I} f_i(\tau+\Delta\tau)\Delta t = \sum_{i=1}^{I} f_i(\tau)\Delta t = N. \quad (3.39)$$

It completes the proof that the total number of trips is conserved on any day step τ. □

Corollary 3.2.1. *The number of trips is conserved on any day step τ in Equation (3.22).*

Proof. Since Equation (3.22) is equivalent to Equation (3.36), the number of trips in Equation (3.22) is also conserved on any day step τ. □

3.3.4 Continuous version

Moving $f_i(\tau)\Delta t$ to the left-hand-side, and dividing both sides by $\Delta t \cdot \Delta\tau$, Equation (3.36) becomes:

$$\frac{f_i(\tau+\Delta\tau) - f_i(\tau)}{\Delta\tau} = \frac{\tilde{g}_{i+1}(\tau) - \tilde{g}_i(\tau)}{\Delta t}. \quad (3.40)$$

Let $f(\tau,t) = f(j\Delta\tau, i\Delta t)$, and $\tilde{g}(\tau,t) = \tilde{g}(j\Delta\tau, i\Delta t)$, we can re-write Equation (3.40) as follows:

$$\frac{f((j+1)\Delta\tau, i\Delta t) - f(j\Delta\tau, i\Delta t)}{\Delta\tau} = \frac{\tilde{g}(j\Delta\tau, (i+1)\Delta t) - \tilde{g}(j\Delta\tau, i\Delta t)}{\Delta t}. \quad (3.41)$$

Let $\Delta t \to 0$ and $\Delta\tau \to 0$, Equation (3.41) becomes the following conservation equation in the continuous form:

$$\frac{\partial}{\partial\tau} f(\tau,t) - \frac{\partial}{\partial t}\tilde{g}(\tau,t) = 0, \tag{3.42}$$

where this conservation equation relates the rate of change in departure rate $f(\tau,t)$ from day to day to the rate of change in net advance rate $\tilde{g}(\tau,t)$ from time to time within a given day.

To further simplify Equation (3.42), we analyze the property of $\tilde{g}(\tau,t)$.

We have $\tilde{g}_i(\tau) = sgn(\omega_i(\tau))\cdot\max\{f^d_{i-1}(\tau), f^a_i(\tau)\}$, which includes both $f^d_{i-1}(\tau)$ and $f^a_i(\tau)$. We obtain $f^a_i(\tau)$ from Equation (3.21). To obtain $f^d_{i-1}(\tau)$, we substitute $i-1$ in Equation (3.20) and we have:

$$f^d_{i-1}(\tau) = B^d_i(\tau)\cdot\{-\omega_i(\tau)\}_+ \cdot f_{i-1}(\tau)\cdot\frac{\Delta t}{\Delta\tau}. \tag{3.43}$$

From Equation (3.21), we have $f^a_i(\tau)$ as follows:

$$f^a_i(\tau) = B^a_i(\tau)\cdot\{\omega_i(\tau)\}_+ \cdot \left(f_i(\tau)\cdot\frac{\Delta t}{\Delta\tau} - f^d_i(\tau)\right). \tag{3.44}$$

Substituting $f^d_i(\tau)$ into Equation (3.44), we have $f^a_i(\tau)$ as follows:

$$f^a_i(\tau) = B^a_i(\tau)\cdot\{\omega_i(\tau)\}_+ \cdot \left[(1 - B^d_{i+1}(\tau)\cdot\{-\omega_{i+1}(\tau)\}_+)\cdot f_i(\tau)\cdot\frac{\Delta t}{\Delta\tau}\right]. \tag{3.45}$$

Let $\tau = j\Delta\tau$, and $t = i\Delta t$. We can expand Equation (3.43) as follows:

$$f^d(j\Delta\tau,(i-1)\Delta t) = B^d(j\Delta\tau, i\Delta t)\cdot\{-\omega(j\Delta\tau, i\Delta t)\}_+ \cdot f(j\Delta\tau,(i-1)\Delta t)\cdot\frac{\Delta t}{\Delta\tau}. \tag{3.46}$$

Let $\Delta\tau \to 0$, and $\Delta t \to 0$, Equation (3.46) becomes:

$$f^d(\tau, t^-) = B^d(\tau, t) \cdot \{-\omega(\tau, t)\}_+ \cdot f(\tau, t^-) \cdot \frac{\Delta t}{\Delta\tau}, \tag{3.47}$$

where $t^- = \lim_{\Delta t \to 0}(i-1)\Delta t$ and $\Delta t > 0$.

We can expand Equation (3.45) as follows:

$$\begin{aligned} f^a(j\Delta\tau, i\Delta t) = B^a(j\Delta\tau, i\Delta t) \cdot \{\omega(j\Delta\tau, i\Delta t)\}_+ \\ \cdot \left[(1 - B^d(j\Delta\tau, (i+1)\Delta t) \cdot \{-\omega(j\Delta\tau, (i+1)\Delta t)\}_+) \cdot f(j\Delta\tau, i\Delta t) \cdot \frac{\Delta t}{\Delta\tau}\right]. \end{aligned} \tag{3.48}$$

Let $\Delta\tau \to 0$, and $\Delta t \to 0$, Equation (3.48) becomes:

$$f^a(\tau, t) = B^a(\tau, t) \cdot \{\omega(\tau, t)\}_+ \cdot (1 - B^d(\tau, t^+) \cdot \{-\omega(\tau, t^+)\}_+) \cdot f(\tau, t) \cdot \frac{\Delta t}{\Delta\tau}, \tag{3.49}$$

where $t^+ = \lim_{\Delta t \to 0}(i+1)\Delta t$.

Comparing Equation (3.47) with Equation (3.49), the first two terms in the multiplication, $B^a(\tau, t)$, $B^d(\tau, t)$, and $\omega(\tau, t)$ share the same time variable t and day variable τ. However, the third term in Equation (3.47), $f(\tau, t^-)$, has time variable t^-, while the third term in Equation (3.49), $(1 - B^d(\tau, t^+) \cdot \{-\omega(\tau, t^+)\}_+)$, has the time variable t^+.

Substituting Equation (3.47) and Equation (3.49) into Equation (3.34), and letting $\tau = j\Delta\tau$, and $t = i\Delta t$, we have:

$$\begin{aligned} \tilde{g}(\tau, t) = sgn(\omega(\tau, t)) \cdot \max \Big\{ B^d(\tau, t) \cdot \{-\omega(\tau, t)\}_+ \cdot f(\tau, t^-), \\ B^a(\tau, t) \cdot \{\omega(\tau, t)\}_+ \cdot (1 - B^d(\tau, t^+) \cdot \{-\omega(\tau, t^+)\}_+) \cdot f(\tau, t) \Big\} \cdot \frac{\Delta t}{\Delta\tau}. \end{aligned} \tag{3.50}$$

Equation (3.50) can be written as follows:

$$\tilde{g}(\tau,t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^a(\tau,t)\cdot\omega(\tau,t)\cdot(1 - B^d(\tau,t^+)\cdot\{-\omega(\tau,t^+)\}_+)\cdot f(\tau,t) & \text{for} \quad \omega(\tau,t) \geq 0, \\ B^d(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t^-) & \text{for} \quad \omega(\tau,t) < 0. \end{cases} \tag{3.51}$$

We make the following assumptions:

Assumption 3.3.1. *The departure flow rate function $f(\tau,t)$ is left continuous for all time t. That is, $\lim_{\Delta t\to 0} f(\tau, t-\Delta t) = f(\tau,t^-) = f(\tau,t)$ for $\Delta t > 0$.*

Assumption 3.3.2. *The rate of change in trip cost function $\omega(\tau,t)$ is right continuous for all time t. However, when at SC-DTUE we assume that $\omega(\tau,t)$ is continuous from the right at the start of the departure period, t_0, and continuous from the left at the end of the departure period, t_2, and continuous at other times.*

Suppose the dynamical system reaches SC-DTUE on day τ'. The assumption above can be expressed as follows:

1. *When $\tau < \tau'$, we have $\lim_{\Delta t\to 0}\omega(\tau, t+\Delta t) = \omega(\tau,t^+) = \omega(\tau,t)$ for $\Delta t > 0$ and $t \in [0,T]$.*
2. *When $\tau \geq \tau'$, we have $\lim_{\Delta t\to 0}\omega(\tau,t_0+\Delta t) = \omega(\tau,t_0^+) = \omega(\tau,t_0)$, and $\lim_{\Delta t\to 0}\omega(\tau,t_2-\Delta t) = \omega(\tau,t_2^-) = \omega(\tau,t_2)$. For other times $t \in [0,T]\setminus\{t_0,t_2\}$, $\lim_{\Delta t\to 0}\omega(\tau,t+\Delta t) = \lim_{\Delta t\to 0}\omega(\tau,t-\Delta t) = \omega(\tau,t)$.*

Therefore, when $\tau < \tau'$, given $f(\tau,t^-) = f(\tau,t)$ and $\omega(\tau,t^+) = \omega(\tau,t)$, Equation (3.51) becomes:

$$\tilde{g}(\tau,t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^a(\tau,t)\cdot\omega(\tau,t)\cdot(1 - B^d(\tau,t^+)\cdot\{-\omega(\tau,t)\}_+)\cdot f(\tau,t) & \text{for} \quad \omega(\tau,t) \geq 0, \\ B^d(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t) & \text{for} \quad \omega(\tau,t) < 0. \end{cases}$$

(3.52)

If $\omega(\tau,t) \geq 0$, then $\{-\omega(\tau,t)\}_+ = 0$, so $1 - B^d(\tau,t^+) \cdot \{-\omega(\tau,t)\}_+ = 1$. So we have the following equation:

$$\tilde{g}(\tau,t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^a(\tau,t) \cdot \omega(\tau,t) \cdot f(\tau,t) & \text{for} \quad \omega(\tau,t) \geq 0, \\ B^d(\tau,t) \cdot \omega(\tau,t) \cdot f(\tau,t) & \text{for} \quad \omega(\tau,t) < 0. \end{cases} \tag{3.53}$$

When $\tau \geq \tau'$, the system reaches the SC-DTUE, $\omega(\tau,t) = 0$ for $t \in [t_0, t_2]$. We have $\omega(\tau,t_2^+) = \nu$, so$\{-\omega(\tau,t_2^+)\}_+ = \{-\nu\}_+ = 0$. Notice here $\omega(\tau,t_2^+) \neq \omega(\tau,t_2)$. At other points, we have $\omega(\tau,t^+) = \omega(\tau,t)$. So $1 - B^d(\tau,t^+) \cdot \{-\omega(\tau,t)\}_+ = 1$ when $\omega(\tau,t) \geq 0$. Equation (3.51) still becomes Equation (3.53). Therefore, Equation (3.53) holds from day to day for the entire day period.

Let

$$u(\tau,t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^a(\tau,t) & \text{for} \quad \omega(\tau,t) \geq 0, \\ B^d(\tau,t) & \text{for} \quad \omega(\tau,t) < 0, \end{cases} \tag{3.54}$$

So we have $\tilde{g}(\tau,t) = u(\tau,t) \cdot \omega(\tau,t) \cdot f(\tau,t)$ from Equation (3.53).

Consequently, Equation (3.42) becomes:

$$\frac{\partial}{\partial \tau} f(\tau,t) - \frac{\partial}{\partial t} u(\tau,t) \cdot \omega(\tau,t) \cdot f(\tau,t) = 0. \tag{3.55}$$

Integrating both terms of Equation (3.55) with respect to time t, we have:

$$\frac{\partial}{\partial \tau} \int f(\tau,t)dt - \int \frac{\partial}{\partial t} u(\tau,t) \cdot \omega(\tau,t) \cdot f(\tau,t)dt = 0, \tag{3.56}$$

and Equation (3.56) becomes:

$$\frac{\partial}{\partial \tau} F(\tau, t) - u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0. \tag{3.57}$$

Since $f(\tau, t) = \frac{\partial}{\partial t} F(\tau, t)$, Equation (3.57) becomes:

$$\frac{\partial}{\partial \tau} F(\tau, t) - u(\tau, t) \cdot \omega(\tau, t) \cdot \frac{\partial}{\partial t} F(\tau, t) = 0, \tag{3.58}$$

which is a hyperbolic conservation equation with respective to cumulative departure flow $F(\tau, t)$, and $-u(\tau, t) \cdot \omega(\tau, t)$ is the speed of the characteristic wave for this dynamical system (Equation (3.58)). The unit for $-u(\tau, t) \cdot \omega(\tau, t)$ is hour/day (e.g., t/τ), which describes the speed at which the wave traverses time intervals as the day step increases from τ to $\tau + \Delta\tau$. It resembles the characteristic wave speed in the LWR model (Lighthill and Whitham, 1955; Richards, 1956). However, the characteristic wave speed in the LWR model describes the wave of traffic conditions traveling in the distance-time plane, while the characteristic wave speed in Equation (3.58) describes the wave of travelers' cumulative departure flow adjustments traveling in the day-time plane. This explains why this dissertation mentioned in the earlier sections that our model of day-to-day evolution is LWR-like.

We discuss the property of $u(\tau, t)$ and $\omega(\tau, t)$ in the following paragraphs to further investigate the property of Equations (3.55) and (3.58).

We first discuss $\omega(\tau, t)$. From Equation (3.15), $\omega(\tau, t) = (\lambda - \mu) \cdot \frac{\partial}{\partial t} \Upsilon^E(\tau, t) - \mu$ for early arrivals, and $\omega(\tau, t) = (\lambda + \nu) \cdot \frac{\partial}{\partial t} \Upsilon^L(\tau, t) + \nu$ for late arrivals.

We consider the following two cases for $\delta(\tau, t) = 0$ and $\delta(\tau, t) > 0$.

1. If the bottleneck is uncongested, that is $\delta(\tau, t) = 0$, then $\frac{\partial}{\partial t} \Upsilon(\tau, t) = 0$. Since $\omega(\tau, t) = (\lambda - \mu) \cdot \frac{\partial}{\partial t} \Upsilon^E(\tau, t) - \mu$ for early arrivals and $\omega(\tau, t) = (\lambda + \mu) \cdot \frac{\partial}{\partial t} \Upsilon^L(\tau, t) + \nu$ for late arrivals, we have $\omega(\tau, t) = -\mu$ for early arrivals, and $\omega(\tau, t) = \nu$ for late arrivals.

When $\delta(\tau,t)=0$, we have the following characteristic wave speed for the dynamical system (Equation (3.58)):

$$-u(\tau,t)\cdot\omega(\tau,t)=\begin{cases}\mu\cdot u(\tau,t) & \text{for early arrivals,}\\ -\nu\cdot u(\tau,t) & \text{for late arrivals.}\end{cases} \tag{3.59}$$

2. If the bottleneck is congested, that is $\delta(\tau,t)>0$, then $\frac{\partial}{\partial t}\Upsilon(\tau,t)=\frac{f(\tau,t)}{C}-1$. So we have $\omega(\tau,t)=\frac{\lambda-\mu}{C}\cdot f(\tau,t)-\lambda$ for early arrivals and $\omega(\tau,t)=\frac{\lambda+\nu}{C}\cdot f(\tau,t)-\lambda$ for late arrivals.

   Let

$$\alpha(\tau,t)=\begin{cases}\frac{\lambda-\mu}{C} & \text{for early arrivals,}\\ \frac{\lambda+\nu}{C} & \text{for late arrivals,}\end{cases} \tag{3.60}$$

where we assume that $\lambda+\nu<C$, and thus we have $\alpha(\tau,t)<1$.

So we have $\omega(\tau,t)=\alpha(\tau,t)\cdot f(\tau,t)-\lambda$, when $\delta(\tau,t)>0$.

When $\delta(\tau,t)>0$, we get the following characteristic wave speed for the dynamical system (Equation (3.58)):

$$-u(\tau,t)\cdot\omega(\tau,t)=-u(\tau,t)\cdot(\alpha(\tau,t)\cdot f(\tau,t)-\lambda). \tag{3.61}$$

We now discuss the property of $u(\tau,t)$. From Equation (3.54), we have:

$$u(\tau,t)=\frac{\Delta t}{\Delta\tau}\cdot\begin{cases}B^a(\tau,t) & \text{for}\quad \omega(\tau,t)\geq 0,\\ B^d(\tau,t) & \text{for}\quad \omega(\tau,t)<0.\end{cases}$$

Since $B^a(\tau,t)$ and $B^d(\tau,t)$ are positive advance and deferral coefficients, $u(\tau,t)$ is also positive. From the well-definedness condition (Theorem 3.1), we know that $0<B_i^a(\tau)\leq$

$1/\max\{\nu, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C}-\lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C}-\lambda\}$ and $0 < B_{i+1}^d(\tau) \leq 1/\lambda$. This yields:

$$0 < u(\tau,t) \leq \frac{\Delta t}{\Delta \tau}\cdot\begin{cases} 1/\max\{\nu, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C}-\lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C}-\lambda\} & \text{for} \quad \omega(\tau,t)\geq 0, \\ 1/\lambda & \text{for} \quad \omega(\tau,t)<0. \end{cases} \tag{3.62}$$

3.4 Stationary state

Definition 3.3. *The stationary state of the local dynamical system, Equation (3.55) (discrete version: Equations (3.20) to (3.22)), is reached when $\frac{\partial}{\partial\tau}f(\tau,t)=0$, which means the time-dependent departure rate $f(\tau,t)$ does not change from day to day.*

When $\frac{\partial}{\partial\tau}f(\tau,t) = 0$, we have $\frac{\partial}{\partial\tau}F(\tau,t) = 0$ as well. From Equation (3.57), we have $u(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t) = 0$. Since $u(\tau,t) > 0$ from Equation (3.62), we have the following complementarity condition for $f(\tau,t)$ and $\omega(\tau,t)$:

$$f(\tau,t)\cdot\omega(\tau,t) = 0, \tag{3.63}$$

for all $t\in[0,T]$.

3.5 Stability of the stationary state

We consider the stability in both uncongested and congested cases below.

1. When the bottleneck is uncongested (i.e., $\delta(\tau,t) = 0$), substituting Equation (3.59)

into Equation (3.58), we obtain:

$$\frac{\partial}{\partial \tau}F(\tau,t) + \mu \cdot u(\tau,t) \cdot \frac{\partial}{\partial t}F(\tau,t) = 0, \tag{3.64}$$

for early arrivals, and

$$\frac{\partial}{\partial \tau}F(\tau,t) - \nu \cdot u(\tau,t) \cdot \frac{\partial}{\partial t}F(\tau,t) = 0, \tag{3.65}$$

for late arrivals.

Given $u(\tau,t) > 0$, the characteristic wave speed for early arrival travelers in the dynamical system in Equation (3.64) is $\mu \cdot u(\tau,t) > 0$. This means that, day to day, the early arrival travelers under uncongested conditions will shift their departure times to later time steps.

For late arrival travelers, the characteristic wave speed of the dynamical system in Equation (3.65) is $-\nu \cdot u(\tau,t) < 0$. This means that, day-to-day, the late arrival travelers under uncongested conditions will shift their departure times to early time steps.

So the cumulative departure flow, $F(\tau,t)$, under uncongested conditions will shift to the congested period from both sides, and so does the departure flow rate $f(\tau,t)$. Thus the dynamical system in Equations (3.55) and (3.58) are unstable under uncongested conditions.

2. Here we consider the stability under the congested state (i.e., $\delta(\tau,t) > 0$). We assume that $\delta(\tau,t) > 0$ for $t \in [t_0, t_2]$. Substituting Equation (3.61) into Equation (3.55), we obtain:

$$\frac{\partial}{\partial \tau}f(\tau,t) - \frac{\partial}{\partial t}u(\tau,t) \cdot (\alpha(\tau,t) \cdot f(\tau,t) - \lambda) \cdot f(\tau,t) = 0, \tag{3.66}$$

for $t \in [t_0, t_2]$.

We define a Lyapunov functional as follows:

$$V(f(\tau, \cdot)) = \int_0^T t \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt, \tag{3.67}$$

whose discrete version is as follows:

$$V(f(\tau)) = \sum_1^I (i - \frac{1}{2}) \cdot \Delta t \cdot f_i(\tau) \cdot [\{-\omega_{i+1}(\tau)\}_+^2 + \{\omega_i(\tau)\}_+^2]. \tag{3.68}$$

Assuming that all trips depart within $[t_0, t_2]$, Equation (3.67) becomes:

$$V(f(\tau, \cdot)) = \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt, \tag{3.69}$$

For $V(f(\tau, \cdot))$ to be a Lyapunov functional of the dynamical system in Equation (3.66), we need to show that $V(f(\tau, \cdot))$ satisfies the following three conditions:

(a) $V(f(\tau, \cdot)) \geq 0$, $\forall \tau$.

(b) $V(f(\tau, \cdot)) = 0$ at a stationary state of the dynamical system in Equation (3.66).

(c) $\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) < 0$ at a non-stationary state of the dynamical system in Equation (3.66), while $\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) = 0$ at a stationary state of the dynamical system in Equation (3.66).

Theorem 3.3. *Equation (3.69) is the Lyapunov functional for the dynamical system in Equation (3.66).*

Proof. We show how $V(f(\tau, \cdot))$ from Equation (3.69) will satisfy the three conditions above:

(a) For condition (a), since $t - t_0 \geq 0$, $f(\tau, t) \geq 0$ and $\omega^2(\tau, t) \geq 0$, we have $V(f(\tau, \cdot)) = \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt \geq 0$.

(b) For condition (b), the stationary state (or equilibrium point) for the day-to-day dynamical system is reached when:

$$\frac{\partial}{\partial \tau} f(\tau, t) = 0. \tag{3.70}$$

From Equation (3.63), we have the complementarity condition $f(\tau, t) \cdot \omega(\tau, t) = 0$ at the stationary state. When $f(\tau, t) > 0$, $\omega(\tau, t) = 0$, so we have $\omega^2(\tau, t) = 0$. When $f(\tau, t) = 0$, $\omega(\tau, t) \geq 0$, so we have $\omega^2(\tau, t) \geq 0$. In turn, the following complementarity condition also holds:

$$f(\tau, t) \cdot \omega^2(\tau, t) = 0,$$

for all $t \in [0, T]$.

So $V(f(\tau, \cdot)) = \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt = 0$ at the stationary state.

(c) For condition (c), we take the partial derivative of Equation (3.69) with respect to τ and we get:

$$\begin{aligned} \frac{\partial}{\partial \tau} V(f(\tau, \cdot)) &= \frac{\partial}{\partial \tau} \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt \\ &= \int_{t_0}^{t_2} (t - t_0) \cdot \frac{\partial}{\partial \tau} \left[f(\tau, t) \cdot \omega^2(\tau, t) \right] dt. \end{aligned} \tag{3.71}$$

Let $A(\tau, t) = \frac{\partial}{\partial \tau} f(\tau, t) \cdot \omega^2(\tau, t)$, and Equation (3.71) becomes:

$$\begin{aligned} \frac{\partial}{\partial \tau} V(f(\tau, \cdot)) &= \frac{\partial}{\partial \tau} \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt \\ &= \int_{t_0}^{t_2} (t - t_0) \cdot A(\tau, t) dt. \end{aligned} \tag{3.72}$$

For $A(\tau, t)$, we have:

$$A(\tau, t) = \frac{\partial}{\partial \tau} \left[f(\tau, t) \cdot \omega^2(\tau, t) \right] = \omega^2(\tau, t) \cdot \frac{\partial}{\partial \tau} f(\tau, t) + 2 \cdot f(\tau, t) \cdot \omega(\tau, t) \cdot \frac{\partial}{\partial \tau} \omega(\tau, t).$$

$$(3.73)$$

When $\delta(\tau,t) > 0$, we have $\omega(\tau,t) = \alpha(\tau,t)\cdot f(\tau,t) - \lambda$. From Equation (3.60), we know that $\alpha(\tau,t)$ does not change with respect to the day variable τ, so we can write $\alpha(\tau,t) = \alpha(t)$. Thus we get $\frac{\partial}{\partial\tau}\omega(\tau,t) = \alpha(t)\cdot\frac{\partial}{\partial\tau}f(\tau,t)$. Substituting it into Equation (3.73), we obtain:

$$\begin{aligned} A(\tau,t) &= \omega^2(\tau,t)\cdot\frac{\partial}{\partial\tau}f(\tau,t) + 2\cdot f(\tau,t)\cdot\omega(\tau,t)\cdot\alpha(t)\cdot\frac{\partial}{\partial\tau}f(\tau,t) \\ &= \omega(\tau,t)\cdot[\omega(\tau,t) + 2\cdot f(\tau,t)\cdot\alpha(t)]\cdot\frac{\partial}{\partial\tau}f(\tau,t). \end{aligned} \tag{3.74}$$

Substituting $\omega(\tau,t) = \alpha(t)\cdot f(\tau,t) - \lambda$ into Equation (3.74), we obtain:

$$A(\tau,t) = (\alpha(t)\cdot f(\tau,t) - \lambda)\cdot[3\cdot\alpha(t)\cdot f(\tau,t) - \lambda]\cdot\frac{\partial}{\partial\tau}f(\tau,t). \tag{3.75}$$

From Equation (3.55), we have $\frac{\partial}{\partial\tau}f(\tau,t) = \frac{\partial}{\partial t}u(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t)$. Substituting it into $\frac{\partial}{\partial\tau}f(\tau,t)$ of Equation (3.75), we obtain:

$$A(\tau,t) = (\alpha(t)\cdot f(\tau,t) - \lambda)\cdot[3\cdot\alpha(t)\cdot f(\tau,t) - \lambda]\cdot\frac{\partial}{\partial t}u(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t). \tag{3.76}$$

Now we need to make certain assumptions on $u(\tau,t)$ to be able to move forward with the proof. We know from Equation (3.54) that $u(\tau,t) > 0$. Now we assume that $u(\tau,t)$ has the following form:

$$u(\tau,t) = u_0\cdot\frac{3\omega(\tau,t) + 2\lambda}{f(\tau,t)}, \tag{3.77}$$

where u_0 is a positive coefficient.

When the bottleneck is uncongested, $\omega(\tau,t) = -\mu$ for early arrivals, and $\omega(\tau,t) = \nu$ for late arrivals. So we have $u(\tau,t) = u_0\cdot\frac{-3\mu+2\lambda}{f(\tau,t)}$ for early arrivals, and $u(\tau,t) =$

$u_0 \cdot \frac{3\nu+2\lambda}{f(\tau,t)}$ for late arrivals.

When the bottleneck is congested, we have $\omega(\tau,t) = \alpha(t) \cdot f(\tau,t) - \lambda$, and Equation (3.77) becomes:

$$u(\tau,t) = u_0 \cdot \frac{3\omega(\tau,t) + 2\lambda}{f(\tau,t)} = u_0 \cdot \frac{3\alpha(t) \cdot f(\tau,t) - \lambda}{f(\tau,t)}. \tag{3.78}$$

where $3 \cdot \alpha(t) \cdot f(\tau,t) - \lambda > 0$ and that is, $f(\tau,t) > \frac{\lambda}{3\alpha(t)}$.

Substituting Equation (3.78) into Equation (3.76) with $\omega(\tau,t) = \alpha(t) \cdot f(\tau,t) - \lambda$ under congested condition and crossing out $f(\tau,t)$ in the denominator and nominator, we have:

$$\begin{aligned} A(\tau,t) &= (\alpha(t) \cdot f(\tau,t) - \lambda) \cdot [3 \cdot \alpha(t) \cdot f(\tau,t) - \lambda] \cdot \frac{\partial}{\partial t}\left[u_0 \cdot \frac{3\alpha(t) \cdot f(\tau,t) - \lambda}{f(\tau,t)}\right] \cdot \omega(\tau,t) \cdot f(\tau,t) \\ &= u_0 \cdot (\alpha(t) \cdot f(\tau,t) - \lambda) \cdot [3\alpha(t) \cdot f(\tau,t) - \lambda] \cdot \frac{\partial}{\partial t}(3\alpha(t) \cdot f(\tau,t) - \lambda) \cdot (\alpha(t) \cdot f(\tau,t) - \lambda) \\ &= \frac{u_0}{2} \cdot \frac{\partial}{\partial t}[(3\alpha(t) \cdot f(\tau,t) - \lambda) \cdot (\alpha(t) \cdot f(\tau,t) - \lambda)]^2. \end{aligned} \tag{3.79}$$

Substituting $A(\tau,t)$ from Equation (3.79) back to Equation (3.72), we have:

$$\begin{aligned} \frac{\partial}{\partial \tau} V(f(\tau,\cdot)) &= \frac{\partial}{\partial \tau} \int_{t_0}^{t_2} (t - t_0) \cdot A(\tau,t) dt \\ &= \frac{u_0}{2} \cdot \int_{t_0}^{t_2} (t - t_0) \cdot \frac{\partial}{\partial t}[(3\alpha(t) \cdot f(\tau,t) - \lambda) \cdot (\alpha(t) \cdot f(\tau,t) - \lambda)]^2 dt. \end{aligned} \tag{3.80}$$

Integrating by parts, Equation (3.80) becomes:

$$\begin{aligned}\frac{\partial}{\partial\tau}V(f(\tau,\cdot)) &= \frac{u_0}{2}\cdot\Bigg\{(t-t_0)\cdot[(3\alpha(t)\cdot f(\tau,t)-\lambda)\cdot(\alpha(t)\cdot f(\tau,t)-\lambda)]^2\Big|_{t_0}^{t_2}\\ &\quad -\int_{t_0}^{t_2}[(3\alpha(t)\cdot f(\tau,t)-\lambda)\cdot(\alpha(t)\cdot f(\tau,t)-\lambda)]^2dt\Bigg\}\\ &= \frac{u_0}{2}\cdot\Bigg\{(t_2-t_0)\cdot[(3\alpha(t_2)\cdot f(\tau,t_2)-\lambda)\cdot(\alpha(t_2)\cdot f(\tau,t_2)-\lambda)]^2-0\\ &\quad -\int_{t_0}^{t_2}[(3\alpha(t)\cdot f(\tau,t)-\lambda)\cdot(\alpha(t)\cdot f(\tau,t)-\lambda)]^2dt\Bigg\},\end{aligned} \tag{3.81}$$

where $[(3\alpha(t)\cdot f(\tau,t)-\lambda)\cdot(\alpha(t)\cdot f(\tau,t)-\lambda)]^2 \geq 0$, so we know that the last term of Equation (3.81), $-\int_{t_0}^{t_2}[(3\alpha(t)\cdot f(\tau,t)-\lambda)\cdot(\alpha(t)\cdot f(\tau,t)-\lambda)]^2dt \leq 0$.

Assumption 3.5.1. *At the stationary state, we have $\omega(\tau^{'},t_2)=0$. We assume that before the dynamical system approaches its stationary state on day $\tau^{'}$, rate of change in trip cost at the end of the departure period t_2, $\omega(\tau,t_2)$, already becomes zero. That is, $\exists\varepsilon>0$,*

$$\omega(\tau,t_2)=0, \forall\tau\geq\tau^{'}-\varepsilon. \tag{3.82}$$

In the current Section 3.5, we study the stability of the stationary state. Assumption 3.5.1 means that $\omega(\tau,t_2)=0$ before the dynamical system reaches its stationary state on day $\tau^{'}$.

Notice that $\omega(\tau^{'},t)=0$ for $t\in[t_0,t_2]$ at the stationary state on day $\tau^{'}$, while $\omega(\tau,t_2)=0$ for $\forall\tau\geq\tau^{'}-\varepsilon$ (on some day before day $\tau^{'}$).

With $\omega(\tau,t_2)=0$, we have $\omega(\tau,t_2)=\alpha(t_2)\cdot f(\tau,t_2)-\lambda=0$, and $f(\tau,t_2)=\frac{\lambda}{\lambda+\nu}\cdot C$, and $\alpha(t_2)=\frac{\lambda+\nu}{C}$ from Equation (3.60).

For $\tau\geq\tau^{'}-\varepsilon$, with $\omega(\tau,t_2)=\alpha(t_2)\cdot f(\tau,t_2)-\lambda=0$, the first term of Equation (3.81) becomes zero. So Equation (3.81) becomes:

$$\frac{\partial}{\partial \tau} V(f(\tau,\cdot)) = -\frac{u_0}{2} \cdot \left\{ \int_{t_0}^{t_2} [(3\alpha(t) \cdot f(\tau,t) - \lambda) \cdot (\alpha(t) \cdot f(\tau,t) - \lambda)]^2 dt \right\} \leq 0. \tag{3.83}$$

From Equation (3.77), we have $(3 \cdot \alpha(t) \cdot f(\tau,t) - \lambda) > 0$. For $t \in [t_0, t_2]$, when $\tau^{'} - \varepsilon \leq \tau < \tau^{'}$, we have $\omega^2(\tau,t) = (\alpha(t) \cdot f(\tau,t) - \lambda)^2 > 0$, and when $\tau \geq \tau^{'}$, we have $\omega^2(\tau,t) = (\alpha(t) \cdot f(\tau,t) - \lambda)^2 = 0$.

Thus we find:

$$\frac{\partial}{\partial \tau} V(f(\tau,\cdot)) < 0, \forall \tau^{'} - \varepsilon \leq \tau < \tau^{'},$$

$$\frac{\partial}{\partial \tau} V(f(\tau,\cdot)) = 0, \forall \tau \geq \tau^{'}.$$

That means, when the dynamical system reaches its stability region (i.e., $\tau^{'} - \varepsilon \leq \tau$), we have $\frac{\partial}{\partial \tau} V(f(\tau,\cdot)) < 0$ at the non-stationary states, while $\frac{\partial}{\partial \tau} V(f(\tau,\cdot)) = 0$ at the stationary state, which is SC-DTUE. Notice that the ε describes the size of the asymptotically stable region.

So Equation (3.69) is a Lyapunov functional of the dynamical system in Equation (3.55), and it is asymptotically stable within the stability region during the congested period $[t_0, t_2]$.

□

3.6 CFL condition and characteristic wave speed

Definition 3.4. *For the numerical method of solving the day-to-day dynamical system (i.e., discrete version Equations (3.20) to (3.22)) to be well-defined, $\Delta\tau$ and Δt have to satisfy* *the following condition:*

$$\frac{\Delta\tau}{\Delta t} \leq \frac{1}{\max\limits_{\tau,t} |u(\tau,t) \cdot \omega(\tau,t)|}, \tag{3.84}$$

which resembles the CFL (Courant-Friedrichs-Lewy) condition for the Cell Transmission Model of traffic flow (Jin, 2018).

In numerical traffic modeling, the above condition ensures that traffic flows do not "jump across" more than one spatial segment during computations, which in our analogous case is akin to an assumption that adjustments in departure time are made only to an adjacent time step, and are thus "local" adjustments. Equation (3.84) requires that the day step $\Delta\tau$ is sufficiently small so that the departure rate in one interval only affects its adjacent intervals on the next day step. The unit for $u(\tau,t) \cdot \omega(\tau,t)$ is hour/day (e.g., t/τ), which describes the speed at which the wave traverses time intervals as the day step increases from τ to $\tau + \Delta\tau$. Equation (3.84) depicts that after $\Delta\tau$, the wave still stays in the adjacent time intervals to interval t, $[t - \Delta t, t]$ or $[t, t + \Delta t]$.

So we need to find the $\max\limits_{\tau,t} |u(\tau,t)|$ and $\max\limits_{\tau,t} |\omega(\tau,t)|$ to further determine the well-definedness condition in Definition 3.4.

We discuss $\max\limits_{\tau,t} |u(\tau,t)|$ first. We will discuss it according to the following two scenarios: the bottleneck is congested (i.e., $\delta(\tau,t) > 0$), and the bottleneck is uncongested (i.e., $\delta(\tau, t = 0)$).

From Equation (3.78), when the bottleneck is congested, we have:

$$\begin{aligned} \max_{\tau,t} |u(\tau,t)| &= u_0 \cdot \max\left\{ \left(3 \cdot \frac{\lambda - \mu}{C} - \frac{\lambda}{f(\tau,t)}\right), \left(3 \cdot \frac{\lambda + \nu}{C} - \frac{\lambda}{f(\tau,t)}\right) \right\} \\ &= u_0 \cdot \left(3 \cdot \frac{\lambda + \nu}{C} - \frac{\lambda}{f(\tau,t)}\right). \end{aligned} \tag{3.85}$$

Since $f(\tau,t) > \frac{\lambda}{3\cdot\alpha(t)} > 0$, we have $\dfrac{\lambda}{f(\tau,t)} > 0$, so Equation (3.85) becomes as follows:

$$\max_{\tau,t}|u(\tau,t)| = u_0 \cdot \left(3 \cdot \frac{\lambda+\nu}{C} - \frac{\lambda}{f(\tau,t)}\right) \leq u_0 \cdot \left(3 \cdot \frac{\lambda+\nu}{C}\right), \tag{3.86}$$

where the "=" is reached when there is massive departure at a given time (i.e., $f(\tau,t) \to \infty$), i.e., we have $\frac{\lambda}{f(\tau,t)} \to 0$, and $\max_{\tau,t}|u(\tau,t)| \to u_0 \cdot \left(3 \cdot \dfrac{\lambda+\nu}{C}\right)$. If we set $u_0 = \frac{C}{3\cdot(\lambda+\nu)}$, then Equation (3.86) becomes:

$$\max_{\tau,t}|u(\tau,t)| \leq \frac{C}{3\cdot(\lambda+\nu)} \cdot \left(3 \cdot \frac{\lambda+\nu}{C}\right) = 1, \tag{3.87}$$

where this is the case when $f(\tau,t) > \frac{\lambda}{3\cdot\alpha(t)}$.

However, when $0 \leq f(\tau,t) \leq \frac{\lambda}{3\cdot\alpha(t)}$ and the bottleneck is congested, there exists $3\cdot\alpha(t)\cdot f(\tau,t) - \lambda \leq 0$, $u(\tau,t)$ might be negative. Given $u(\tau,t) = u_0 \cdot \frac{3\omega(\tau,t)+2\lambda}{f(\tau,t)}$, to make sure $u(\tau,t)$ to be positive, we further set $u(\tau,t) = u_0 \cdot \frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}$. As $f(\tau,t) \to 0$, $u(\tau,t) \to \infty$.

So to incorporate both $f(\tau,t) > \frac{\lambda}{3\cdot\alpha(t)}$ and $0 \leq f(\tau,t) \leq \frac{\lambda}{3\cdot\alpha(t)}$ cases, with $u_0 = \frac{C}{3\cdot(\lambda+\nu)}$, we further set $u(\tau,t)$ to be as follows:

$$u(\tau,t) = \min\{u_0 \cdot \frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, \max_{\tau,t}|u(\tau,t)|\} = \min\{u_0 \cdot \frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, 1\}, \tag{3.88}$$

where $f(\tau,t) \geq 0$. So $\max_{\tau,t}|u(\tau,t)| \leq 1$.

We now discuss $\max_{\tau,t}|\omega(\tau,t)|$. In the uncongested case (i.e., $\delta(\tau,t) = 0$), $\max_{\tau,t}|\omega(\tau,t)| = \max\{\mu,\nu\}$ from Equations (3.15) and (3.16) with $\frac{\partial}{\partial t}\Upsilon^E(\tau,t) = 0$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t) = 0$, while in the congested case (i.e., $\delta(\tau,t) > 0$), we have $\max_{\tau,t}|\omega(\tau,t)| = \max\{\lambda, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\}$. Since we assume $\mu < \lambda$, combining both uncongested and congested cases, we find:

$$\max_{\tau,t} |\omega(\tau,t)| = \max\{\nu, \lambda, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\}. \tag{3.89}$$

Theorem 3.4. *For the numerical method of solving the day-to-day dynamical system (i.e., discrete version Equations (3.20) to (3.22)) to be well-defined, $\Delta\tau$ and Δt have to satisfy the following condition:*

$$\frac{\Delta\tau}{\Delta t} \leq \frac{1}{u_0\cdot\left(3\cdot\dfrac{\lambda+\nu}{C}\right)}\cdot\frac{1}{\max\{\nu, \lambda, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\}}. \tag{3.90}$$

If we set $u_0 = \frac{C}{3\cdot(\lambda+\nu)}$, then Equation (3.90) becomes the following condition:

$$\frac{\Delta\tau}{\Delta t} \leq \frac{1}{\max\{\nu, \lambda, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\}}. \tag{3.91}$$

Given the assumption of $u(\tau,t) = \min\{\frac{C}{3\cdot(\lambda+\nu)}\cdot\frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, 1\}$ from Equation (3.88) with $u_0 = \frac{C}{3\cdot(\lambda+\nu)}$ and $\alpha(\tau,t) = \frac{\lambda-\mu}{C}$ for early arrival and $\alpha(\tau,t) = \frac{\lambda+\nu}{C}$ for late arrivals, we further analyze the characteristic wave speed of the dynamical system in Equation (3.58), $-u(\tau,t)\cdot\omega(\tau,t)$. We analyze the following two cases. When $\delta(\tau,t) = 0$, $\omega(\tau,t) = -\mu$ for early arrivals and $\omega(\tau,t) = \nu$ for late arrivals. So we have:

$$-u(\tau,t)\cdot\omega(\tau,t) = \begin{cases} \mu\cdot u(\tau,t) = \mu\cdot\min\{\frac{C}{3\cdot(\lambda+\nu)}\cdot\frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, 1\} & \text{for early arrivals,} \\ -\nu\cdot u(\tau,t) = -\nu\cdot\min\{\frac{C}{3\cdot(\lambda+\nu)}\cdot\frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, 1\} & \text{for late arrivals.} \end{cases} \tag{3.92}$$

Thus, when the bottleneck is uncongested, the characteristic wave speed $-u(\tau,t)\cdot\omega(\tau,t) \in [-\nu,\mu]$, depending on the departure rate $f(\tau,t)$. If the departure rate $f(\tau,t) = 0$, then the characteristic wave speed: $-u(\tau,t)\cdot\omega(\tau,t)=\mu$ for early arrivals which means travelers will shift their departure time to later time steps from day to day, and $-u(\tau,t)\cdot\omega(\tau,t)=-\nu$ for late arrivals, which means travelers will shift their departure time to earlier time steps from day to day.

When $\delta(\tau,t)>0$, we have:

$$
\begin{aligned}
-u(\tau,t)\cdot\omega(\tau,t) &= -u(\tau,t)\cdot(\alpha(t)\cdot f(\tau,t)-\lambda) \\
&= -\min\{\frac{C}{3\cdot(\lambda+\nu)}\cdot\frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)},1\}\cdot(\alpha(t)\cdot f(\tau,t)-\lambda). \quad (3.93)
\end{aligned}
$$

3.7 Stationary state and SC-DTUE

Theorem 3.5. *The stationary state of local dynamical system, Equation* (3.55) *(discrete version: Equations* (3.20) *to* (3.22)*), is equivalent to SC-DTUE.*

Proof. To prove Theorem 3.5, we first prove the SC-DTUE is the stationary state, and we then prove the stationary state is SC-DTUE.

1. Given the system reaches the SC-DTUE, from Definition 3.1 of SC-DTUE, we have $\phi(\tau,t)=\phi^*$ when $f(\tau,t)>0$, and $\phi(\tau,t)\geq\phi^*$ when $f(\tau,t)=0$. Since $\omega(\tau,t)=\frac{\partial}{\partial t}\phi(\tau,t)$, we have $\omega(\tau,t)=0$ for $f(\tau,t)>0$ and $\omega(\tau,t)\geq 0$ for $f(\tau,t)=0$. So we the following complementarity condition:

$$f(\tau,t)\cdot\omega(\tau,t)=0,$$

where if $f(\tau,t)>0$, $\phi(\tau,t)=\phi^*$, so $\omega(\tau,t)=0$. If $f(\tau,t)=0$, then $\phi(\tau,t)\geq\phi^*$, so $\omega(\tau,t)\geq 0$.

Since $f(\tau, t) \cdot \omega(\tau, t) = 0$ at SC-DTUE, then we have $u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$, so $\frac{\partial}{\partial t} u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$. From Equation (3.55), we know $\frac{\partial}{\partial \tau} f(\tau, t) = 0$. This proves the SC-DTUE is in the stationary state.

2. We now prove that the stationary state is SC-DTUE. Given that the system reaches the stationary state, $\frac{\partial}{\partial \tau} f(\tau, t) = 0$, we have $\frac{\partial}{\partial \tau} F(\tau, t) = 0$ as well. According to Equation (3.57), we have $u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$. Since $u(\tau, t) > 0$, we have the following complementarity condition at the stationary state: $f(\tau, t) \cdot \omega(\tau, t) = 0$. When $f(\tau, t) > 0$, we have $\omega(\tau, t) = 0$. When $f(\tau, t) > 0$, we have $\omega(\tau, t) \neq 0$. However, it is possible that there exists an unused departure interval between two used departure intervals. That is, $f(\tau, t) = 0$ for $t \in [T_1, T_2]$, while is $f(\tau, t) > 0$ for $t \in [0, T_1] \cup [T_2, T]$.

   We need to rule out this possibility to prove that SC-DTUE is the only stationary state, meaning that the departure interval is continuous rather than separate.

   We prove it by contradiction. Assume that there exist such a stationary state, in interval $[T_1, T_2]$ where $f(\tau, t) = 0$ and $\delta(\tau, t) = 0$, the characteristic wave speed, $-u(\tau, t) \cdot \omega(\tau, t)$, is μ for early arrivals and $-\nu$ for late arrivals. Let us analyze T_1, which is the boundary between congested interval $[T, T_1]$ and uncongested interval $[T_1, T_2]$. The queue diminishes at T_1. So there is no queuing cost at T_1. The stationary state trip cost at T_1 from the left is: $\phi^*(T_1^-) = \mu \cdot \{t^* - T_1^-\}$, while the trip cost at T_1 from the right is: $\phi^*(T_1^+) = \mu \cdot \{t^* - T_1^+\}$. Since $\{t^* - T_1^+\} < \{t^* - T_1^-\}$, we have $\omega(T_1) < 0$, which means travelers will defer their departure time from T_1^- to T_1^+ from day to day. It leads to $\frac{\partial}{\partial \tau} f(\tau, t) \neq 0$, which contradicts to the stationary state assumption with $\frac{\partial}{\partial \tau} f(\tau, t) = 0$. This rules out the possibility of a stationary state with an unused departure interval inside the used departure interval. We can thus establish that SC-DTUE is the only stationary state, where all the used departure intervals are connected.

By proving that SC-DTUE is the stationary state and that the stationary state is SC-

DTUE as well, we complete the proof that SC-DTUE is equivalent to the stationary state of the day-to-day dynamical system in Equation (3.55) (discrete version: Equations (3.20) to (3.22)). □

From Equation (3.88), we have $u(\tau,t) = \min\{\frac{C}{3\cdot(\lambda+\nu)} \cdot \frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, 1\}$. If $f(\tau,t) = 0$, we have $u(\tau,t) = 1$. From Equation (3.92), at the stationary state (SC-DTUE), we have $f(\tau,t) = 0$ for $t \in [0,t_0) \cup (t_2,T]$, and $f(\tau,t) > 0$ and $\omega(\tau,t) = 0$ for $t \in [t_0,t_2]$. Therefore, the characteristic wave speed is as follows:

$$-u(\tau,t)\cdot\omega(\tau,t) = \begin{cases} \mu & \text{for} \quad t < t_0 \\ 0 & \text{for} \quad t_0 \le t \le t_2 \\ -\nu & \text{for} \quad t > t_0 \end{cases} \tag{3.94}$$

3.8 Numerical examples

This section provides a numerical example of Jin (2021b)'s model of day-to-day departure time dynamics.

3.8.1 Simulation setup

We consider the total number of travelers to be $N = 3,600$ veh, and the capacity of the bottleneck to be $C = 1,800$ veh/hr. We assume that $\lambda = 50$ \$/hr, $\mu = 25$ \$/hr, and $\nu = 100$ \$/hr. We consider the study period to be $[0,6]$ hr, and the desired arrival time for all travelers to be $t^* = 4$ hr. We assume that the initial departure rate on day 0, $f^0(t)$, is as follows:

$$f^0(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4, \\ 1,800 & \text{for} \quad 2.4 < t \leq 4.4, \\ 0 & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{3.95}$$

where the travelers' departure rate is equal to the capacity from 2.4 hr to 4.4 hr, so there is no congestion ($\delta(\tau, t) = 0$) throughout the study period. This is the system optimal state. Under the condition above, the SC-DTUE departure flow rate $f^*(t)$ will be as follows:

$$f^*(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4, \\ 3,600 & \text{for} \quad 2.4 < t \leq 3.2, \\ 600 & \text{for} \quad 3.2 < t \leq 4.4, \\ 0 & \text{for} \quad 4.4 < t \leq 6. \end{cases} \tag{3.96}$$

The SC-DTUE cost function $\phi^*(t)$ will be as follows:

$$\phi^*(t) = \begin{cases} 25 \cdot (4 - t) & \text{for} \quad 0 \leq t \leq 2.4, \\ 40 & \text{for} \quad 2.4 < t \leq 4.4, \\ 100 \cdot (t - 4) & \text{for} \quad 4.4 < t \leq 6. \end{cases} \tag{3.97}$$

With this day-to-day dynamical system, we will show that the initial departure rate $f^0(t)$ can converge to the SC-DTUE departure rate $f^*(t)$. We will also show that all initial departure rates converge to the SC-DTUE departure rate.

3.8.2 Time and day step sizes and advance/deferral coefficients

To simulate the day-to-day dynamical model in Equations (3.20) to (3.22), we need to decide $f_i(\tau)$, $\omega_i(\tau)$, $B_i^a(\tau)$, $B_i^d(\tau)$, Δt and $\Delta \tau$ for all time step i and day τ. We have the initial departure rate on day 0, $f^0(t)$, and $\omega_i(\tau)$ could be determined by the trip cost function from Equation (3.14). We still need to determine $B_i^a(\tau)$, $B_i^d(\tau)$, Δt and $\Delta\tau$.

We first discuss Δt and $\Delta\tau$. The Δt time step size could be set arbitrarily, but with acceptable traveler departure time adjustment ranges in mind (say, as a few minutes), while $\Delta\tau$ is set according to Δt, satisfying Equation (3.84) and as per Equation (3.98) below. Let τ_j be the time (in day units, with the time length of the analysis period considered as a day) at the jth day step, where $j = 0, 1, ..., J$. Let τ_0 be the initial day step, and we have $\tau_j = \tau_{j-1} + \Delta\tau_{j-1}$, for $j = 1, 2, ..., J$. Since we need to satisfy the well-definedness condition for numerical method, $\frac{\Delta\tau}{\Delta t} \leq 1/\max\{\nu, \lambda, (\lambda-\mu)\cdot\frac{f_1^m(\tau)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau)}{C} - \lambda\}$, from Theorem 3.4, we set the jth day step size, $\Delta\tau_j$, to be the upper bound of this condition as follows:

$$\Delta\tau_j = \frac{\Delta t}{\max\{\nu, \lambda, (\lambda-\mu)\cdot\frac{f_1^m(\tau_j)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau_j)}{C} - \lambda\}}. \tag{3.98}$$

In this chapter, we set $\Delta t = 0.1$hr (6 minutes) and total day steps to be $5,001$, starting from day step 0, and ending with day step $5,000$. Please note that the day step sizes are not constant from day to day, as they depend on $f_1^m(\tau_j)$ and $f_2^m(\tau_j)$, which may vary from day to day according to the traffic conditions. Also note that the total number of days is equal to the sum of day step sizes: $\sum_{j=0}^{J}\Delta\tau_j$ (again in day units with the analysis period length on each day considered as a day). For example, $5,001$ day steps in Figure 3.2 of Section 3.8.4 are translated into 3-5 days. However, it does not mean that a real-world system will converge in 3-5 days in reality. If we want to know the actual number of days needed to reach an equilibrium in reality, we need to estimate the deferral and advance coefficients, $B_i^a(\tau)$ and $B_i^d(\tau)$, from travelers' day to day responses to the trip costs (Jin et al., 2020).

Here we consider two sets of $B_i^a(\tau)$, $B_i^d(\tau)$: the heuristic set derived from the well-definedness condition from Theorem 3.1, and the provably stable set derived from the relationship of

$B_i^a(\tau)$, $B_i^d(\tau)$ and $u(\tau, t)$ in Equation (3.54) with the definition of $u(\tau, t)$ as in Equation (3.88).

For the heuristic set, we have the well-definedness condition from Theorem 3.1 as follows: $0 < B_i^a(\tau) \leq 1/\max\{\nu, (\lambda - \mu) \cdot \frac{f_1^m(\tau)}{C} - \lambda, (\lambda + \nu) \cdot \frac{f_2^m(\tau)}{C} - \lambda\}$ and $0 < B_{i+1}^d(\tau) \leq \frac{1}{\lambda}$ for each time step i and each day step τ. So we set $B_i^d(\tau_j)$ to be the upper bound of this well-definedness condition, $\frac{1}{\lambda}$. However, in our testing, setting $B_i^a(\tau_j)$ to be the upper bound of the well-definedness condition (i.e., $1/\max\{\nu, (\lambda - \mu) \cdot \frac{f_1^m(\tau)}{C} - \lambda, (\lambda + \nu) \cdot \frac{f_2^m(\tau)}{C} - \lambda\}$) will not lead to a stable SC-DTUE. So we set $B_i^a(\tau_j)$ to a smaller value (10% of the upper bound) so that the system can converge to a stable stationary SC-DTUE. It is because we are not able to provide a proof on why it leads to a stable state that we call this set of parameter "heuristic". Therefore, we set the heuristic set of $B_i^a(\tau_j)$, $B_i^d(\tau_j)$ is as follows:

$$B_i^d(\tau_j) = \frac{1}{\lambda} \quad \text{and} \quad B_i^a(\tau_j) = \frac{0.1}{\max\{\nu, (\lambda - \mu) \cdot \frac{f_1^m(\tau_j)}{C} - \lambda, (\lambda + \nu) \cdot \frac{f_2^m(\tau_j)}{C} - \lambda\}}, \tag{3.99}$$

where the deferral and advance coefficients $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ are asymmetric.

Notice that Equation (3.99) is not the only heuristic set of parameters that can drive the system to a stable stationary SC-DTUE state. There might be other combinations of $B_i^a(\tau_j)$, $B_i^d(\tau_j)$ that can drive the system to a stable stationary state as well.

For the provably stable set, we have $B_i^a(\tau_j) = \frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$ for $\omega_i(\tau_j) \geq 0$, and $B_i^d(\tau_j) = \frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$ for $\omega_i(\tau_j) < 0$ from Equation (3.54). It means that for Equation (3.54) to hold, we only need $B_i^a(\tau_j) = \frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$ for $\omega_i(\tau_j) \geq 0$, and $B_i^d(\tau_j) = \frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$ for $\omega_i(\tau_j) < 0$, while $B_i^a(\tau_j)$ can be any value for $\omega_i(\tau_j) < 0$, and $B_i^d(\tau_j)$ can be any value for $\omega_i(\tau_j) \geq 0$. So we assume that $B_i^a(\tau_j) = \frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$ under $\omega_i(\tau_j) < 0$, and $B_i^d(\tau_j) = \frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$ under $\omega_i(\tau_j) \geq 0$ as well.

Since we specify as $u(\tau, t) = \min\{u_0 \cdot \frac{\{3\omega(\tau,t)+2\lambda\}_+}{f(\tau,t)}, 1\}$, and both $B_i^a(\tau_j)$ and $B_i^d(\tau_j)$ are determined by the same formula, $\frac{\Delta\tau}{\Delta t} \cdot u_i(\tau_j)$, regardless of the value of $\omega_i(\tau_j)$, we obtain the provably stable set of $B_i^a(\tau_j)$, $B_i^d(\tau_j)$ as follows:

$$B_i^d(\tau_j) = B_i^a(\tau_j) = \frac{\Delta\tau_j}{\Delta t} \cdot \min\{\frac{C}{3 \cdot (\lambda + \nu)} \cdot \frac{\{3\omega(\tau, t) + 2\lambda\}_+}{f(\tau, t)}, 1\}, \tag{3.100}$$

where the deferral and advance coefficients $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ are symmetric.

In numerical simulation testing, we consider the following three scenarios of $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$. First, we show the results with only asymmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ from Equation (3.99) and then with only symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ from Equation (3.100). We then show the results with asymmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ driving the system inside the stability region before we switch to the provably stable set of $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ from Equation (3.100).

3.8.3 Error and stability measurements

We define the following Lebesgue L_1 norm (we use L_1 norm henceforth in this dissertation) as an error to measure the discrepancies between day j's trip cost, $\phi_i^j = \phi(i\Delta t, j\Delta\tau)$, and the SC-DTUE's trip cost ϕ_i^*:

$$e(j) = \sum_{i=0}^{I} |\phi_i^j - \phi_i^*| \cdot \Delta t. \tag{3.101}$$

To measure stability, we calculate the following discrete version of the Lyapunov functional on day τ_j defined in Equation (3.68):

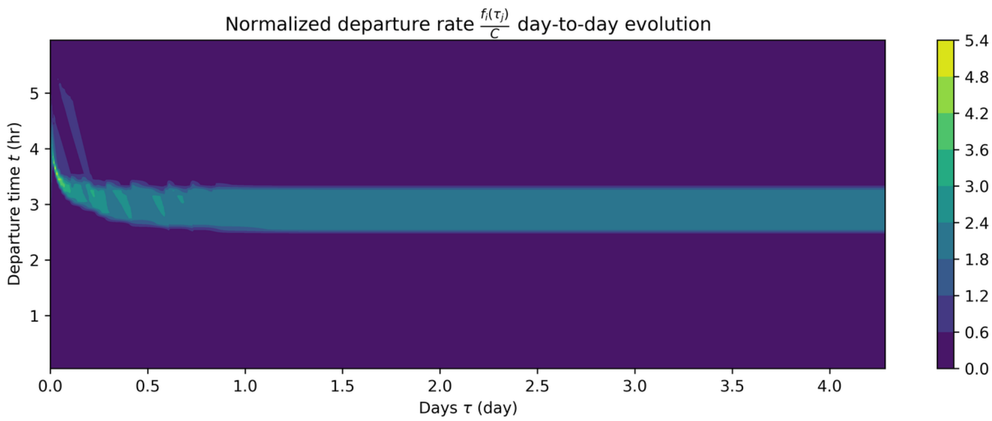
$$V(f(\tau_j)) = \sum_{1}^{I} (i - \frac{1}{2}) \cdot \Delta t \cdot f_i(\tau_j) \cdot [\{-\omega_{i+1}(\tau_j)\}_+^2 + \{\omega_i(\tau_j)\}_+^2],$$

where $V(f(\tau_0)) > 0$. As day step τ_j increases, if $V(f(\tau_j))$ decreases to zero, then it numerically shows that the day-to-day dynamical system is asymptotically stable.

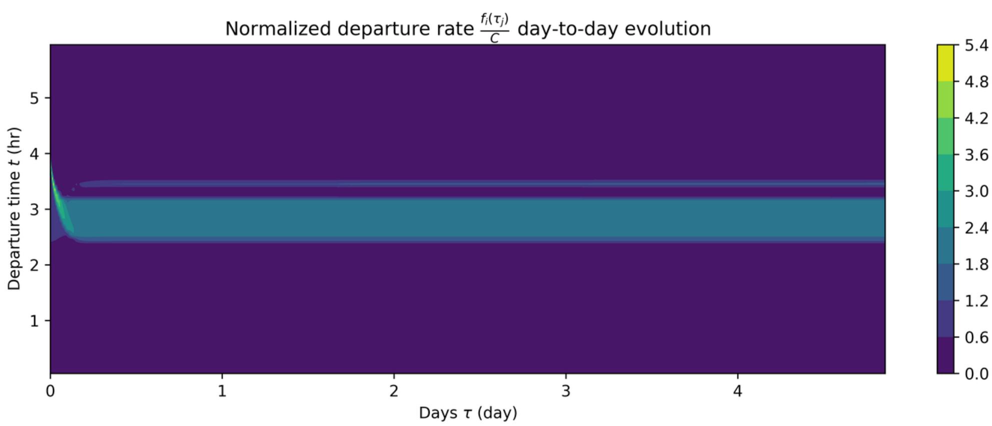
3.8.4 Simulation results

We show the results for the following three cases: (1) asymmetric advance and deferral coefficients only; (2) symmetric advance and deferral coefficients only; and (3) asymmetric advance and deferral coefficients first with a switch to symmetric coefficients from day step $2,500$.

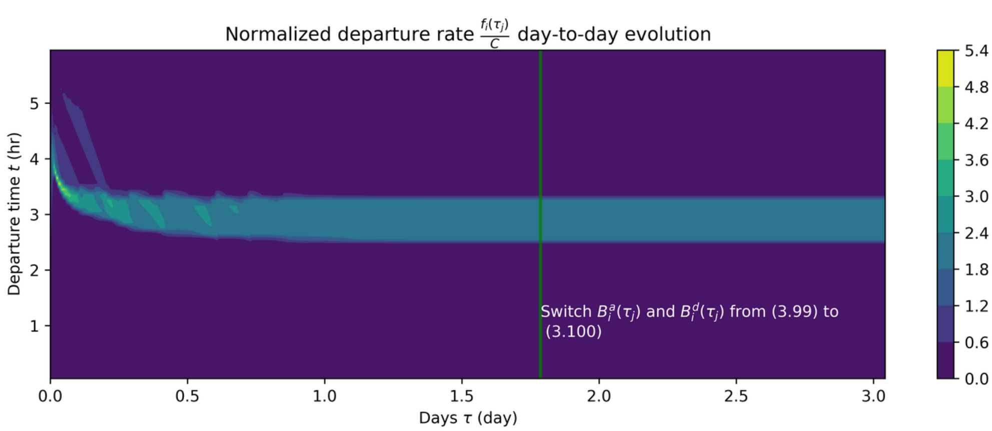
Figure 3.1 shows the day-to-day evolution of the normalized departure rate, $\frac{f_i(\tau_j)}{C}$. Comparing Figures 3.1a and 3.1c and Figure 3.1b, under the system optimal initial condition, the dynamical system converges to a stable stationary SC-DTUE with asymmetric coefficients and switching coefficients, but not with symmetric coefficients.



(a) Asymmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$



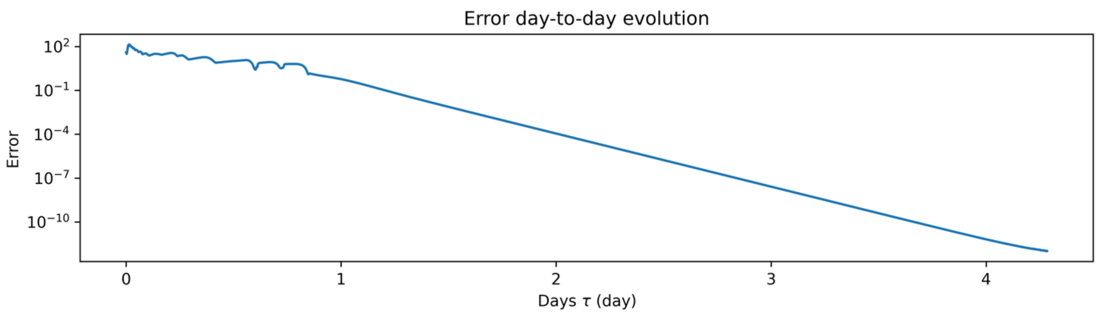
(b) Symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$



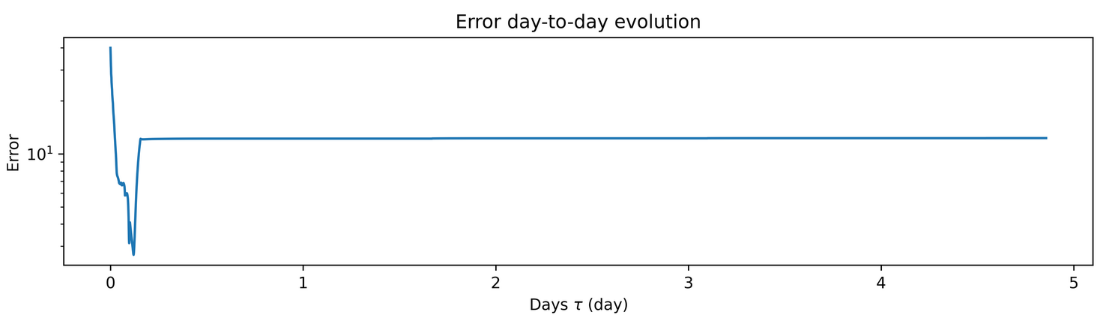
(c) Switch from asymmetric to symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ on day step 2,500

Figure 3.1: Normalized departure rate day-to-day evolution

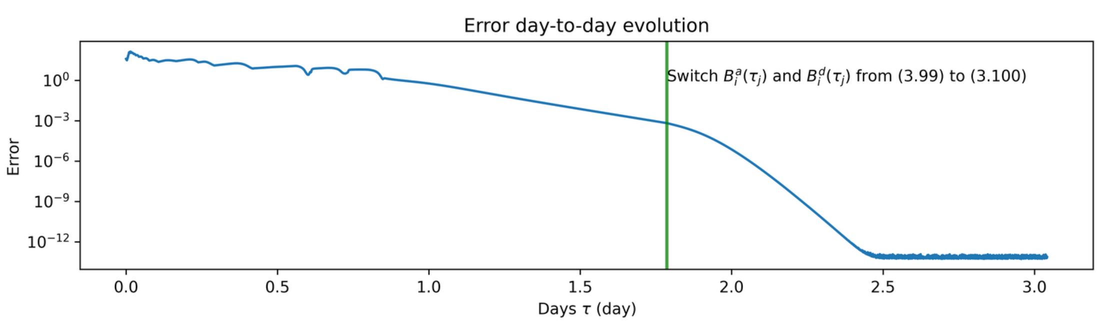
Figure 3.2 shows the day-to-day evolution of the L_1 norm error. The dynamical system cannot converge to SC-DTUE using the symmetric coefficients, while the system can converge to SC-DTUE with both asymmetric coefficients and switching coefficients. Figure 3.2c shows that the error decreases much faster after changing to symmetric coefficients in the switching case. The Lyapunov functional in Figure 3.3 also shows a similar trend.



(a) Asymmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$

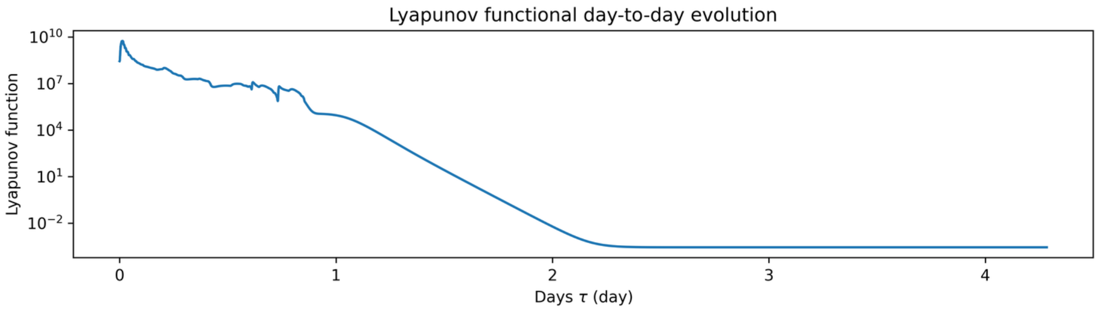


(b) Symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$

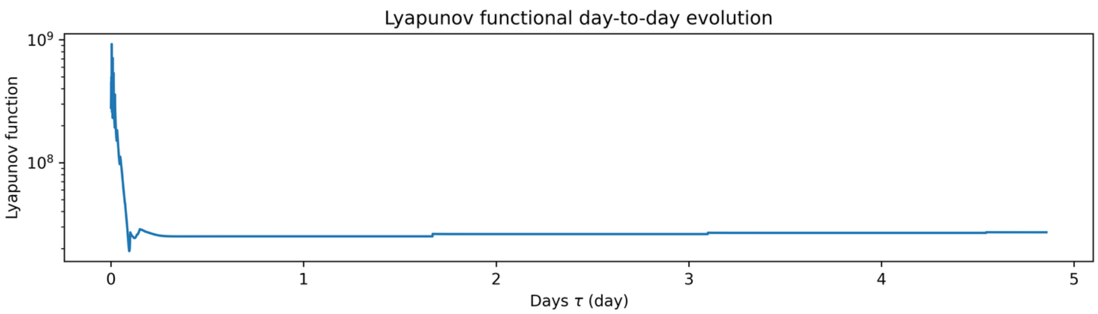


(c) Switch from asymmetric to symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ on day step $2,500$

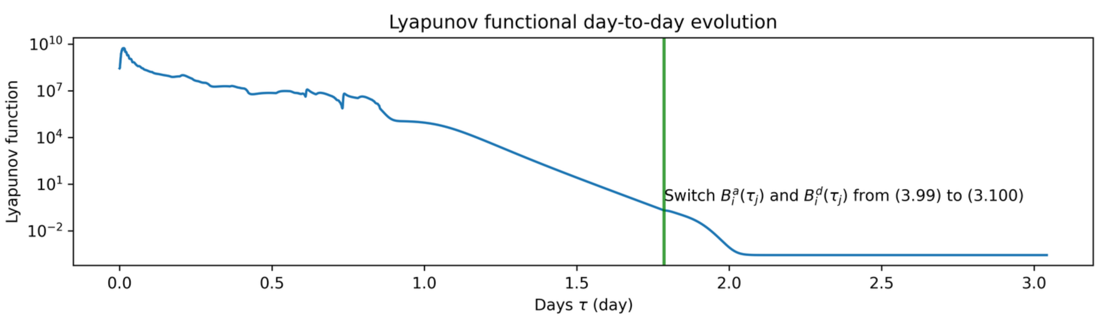
Figure 3.2: Error day-to-day evolution



(a) Asymmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$



(b) Symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$



(c) Switch from asymmetric to symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ on day step 2,500

Figure 3.3: Lyapunov functional day-to-day evolution

Figures 3.4 to 3.6 show the first and last day comparison in departure/arrival rate, cumulative departure/arrival flow, and trip cost. From Figures 3.4 and 3.6, we see that the dynamical system can converge to SC-DTUE with both asymmetric and switching coefficients, while it cannot converge to SC-DTUE with symmetric coefficients, as seen from Figure 3.5.

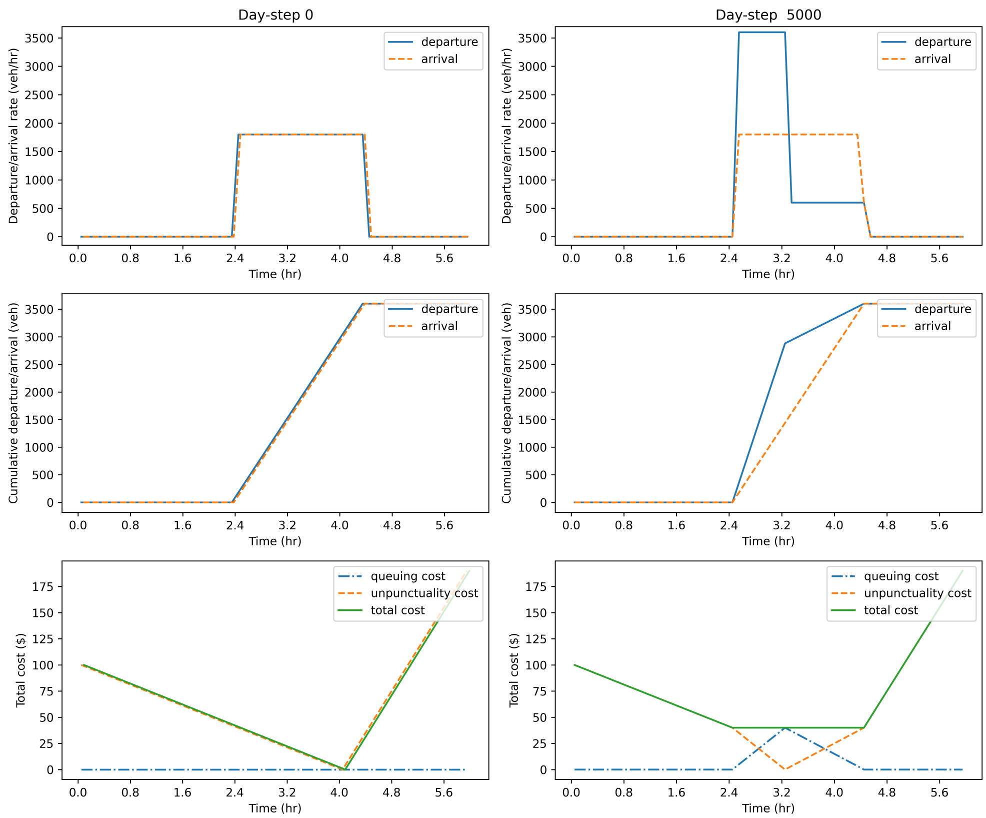


Figure 3.4: First and last day comparison with asymmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$

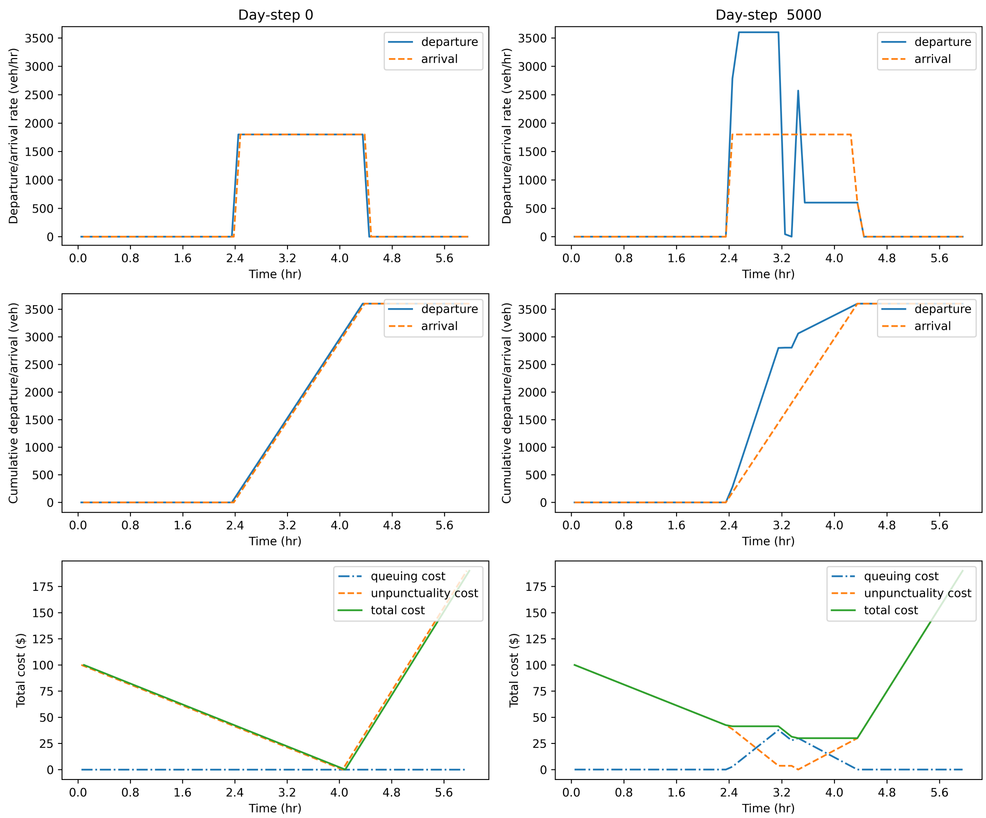


Figure 3.5: First and last day comparison with symmetric $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$

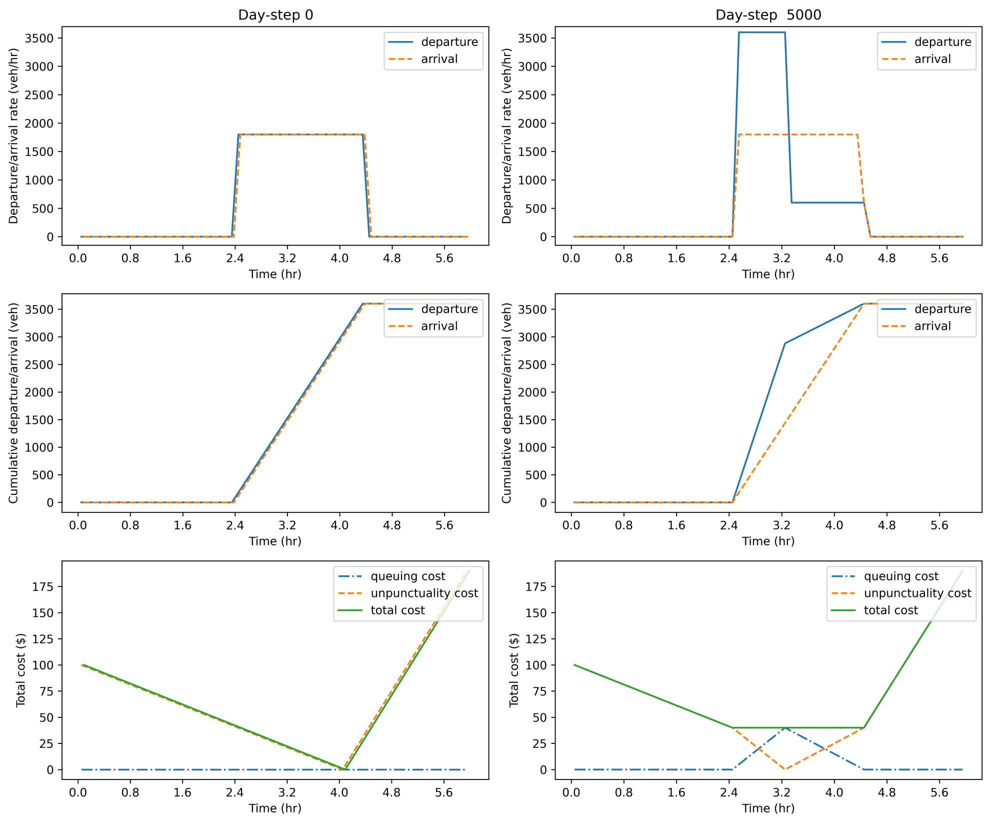


Figure 3.6: First and last day comparison with switching $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ at day step $2,500$

Figures 3.7 to 3.12 show the first and last day comparison and day-to-day evolution under different initial departure patterns. We use the switching coefficients from asymmetric to symmetric at day step 2,500 in all the cases. The results show that the dynamical system converges to a stable SC-DTUE under different initial demand conditions.

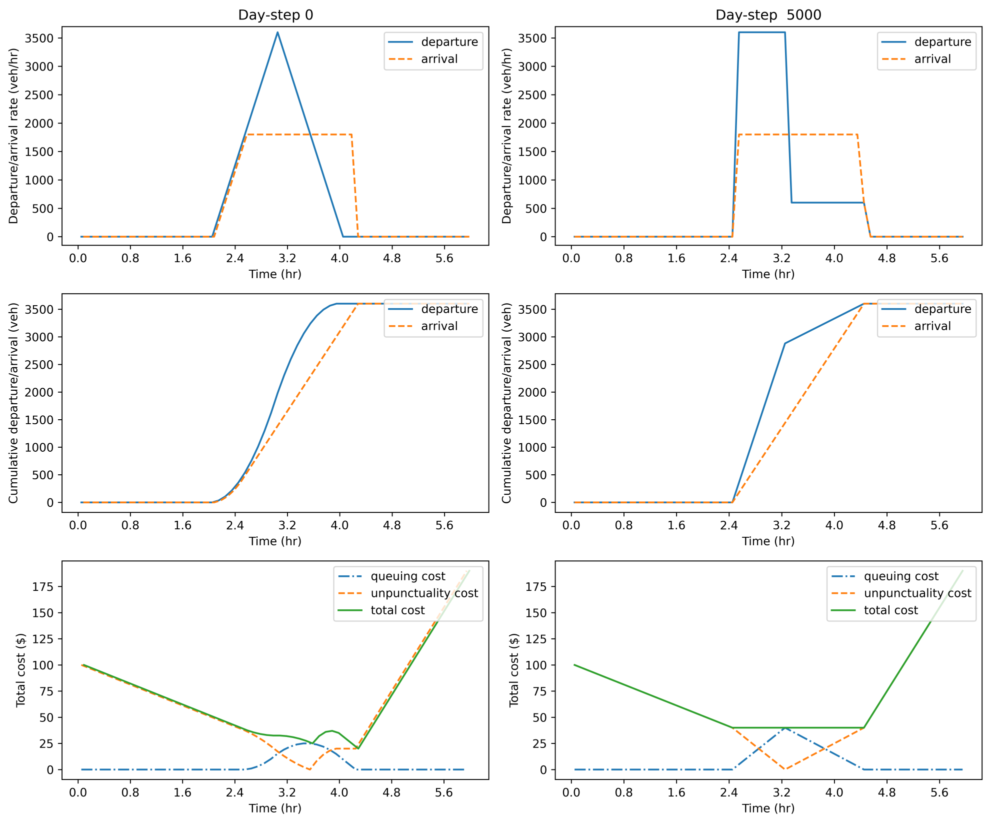


Figure 3.7: First and last day comparison with a triangular initial departure pattern

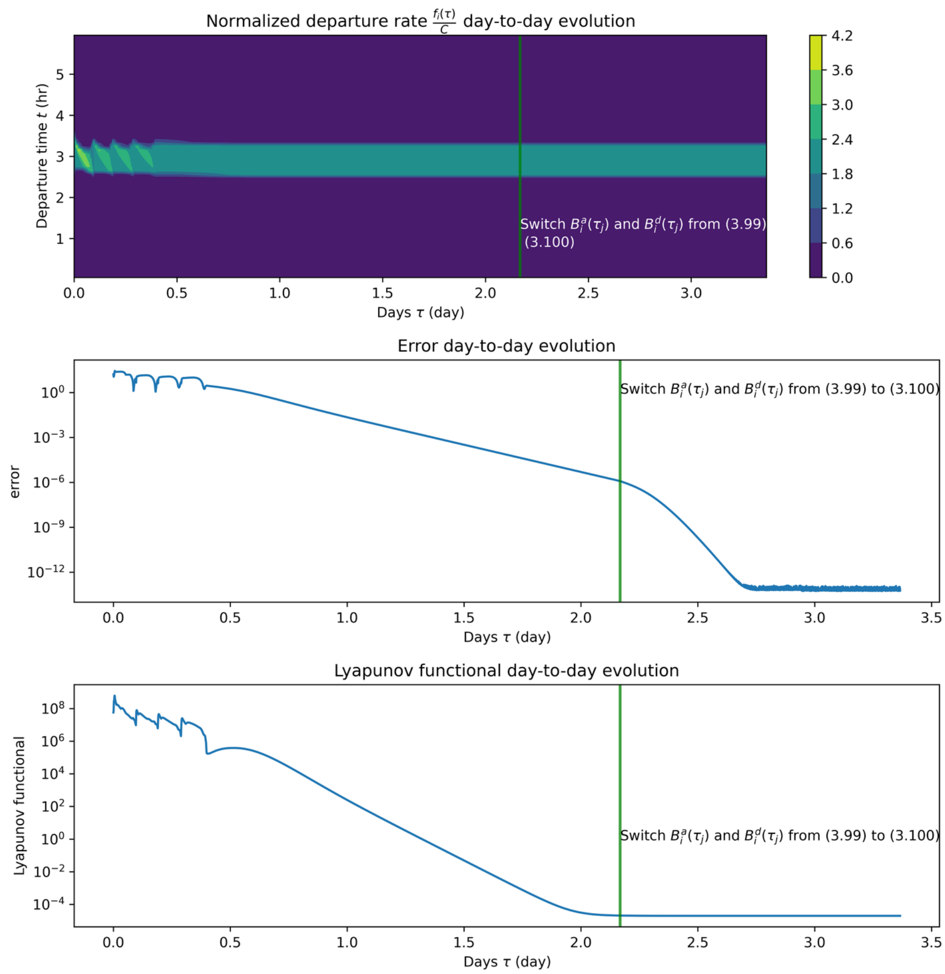


Figure 3.8: Day-to-day evolution with a triangular initial departure pattern

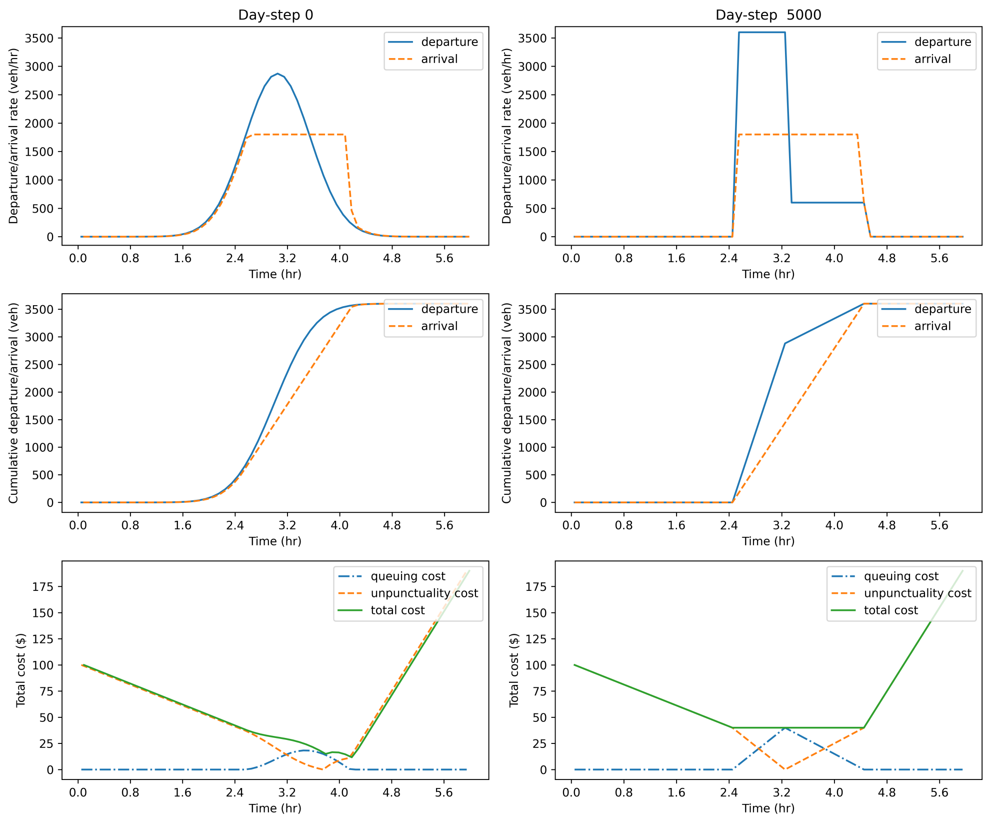


Figure 3.9: First and last day comparison with a Gaussian initial departure pattern

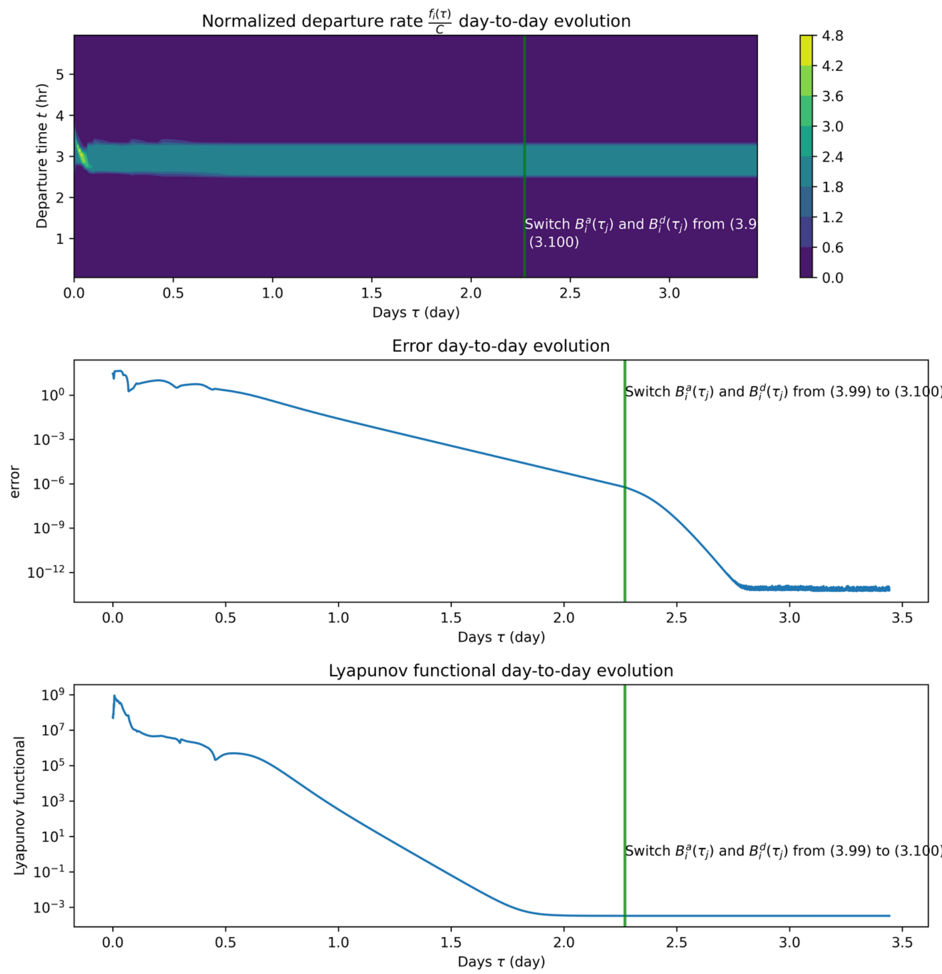


Figure 3.10: Day-to-day evolution with a Gaussian initial departure pattern

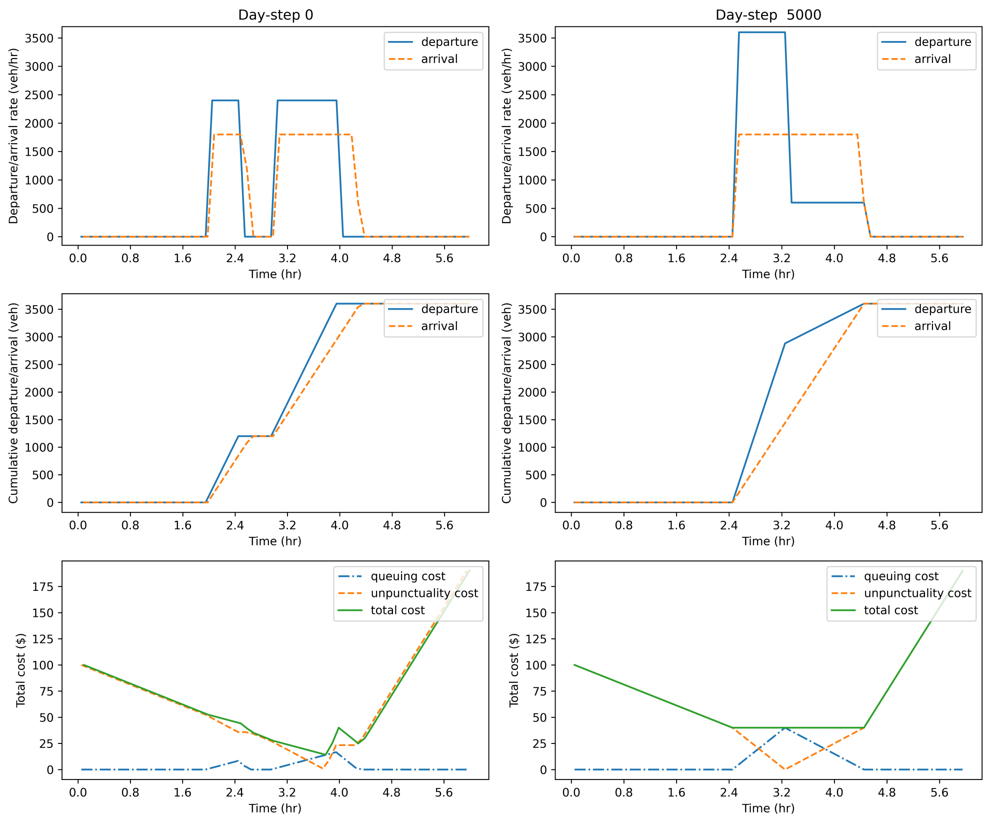


Figure 3.11: First and last day comparison with a 2-peak initial departure pattern

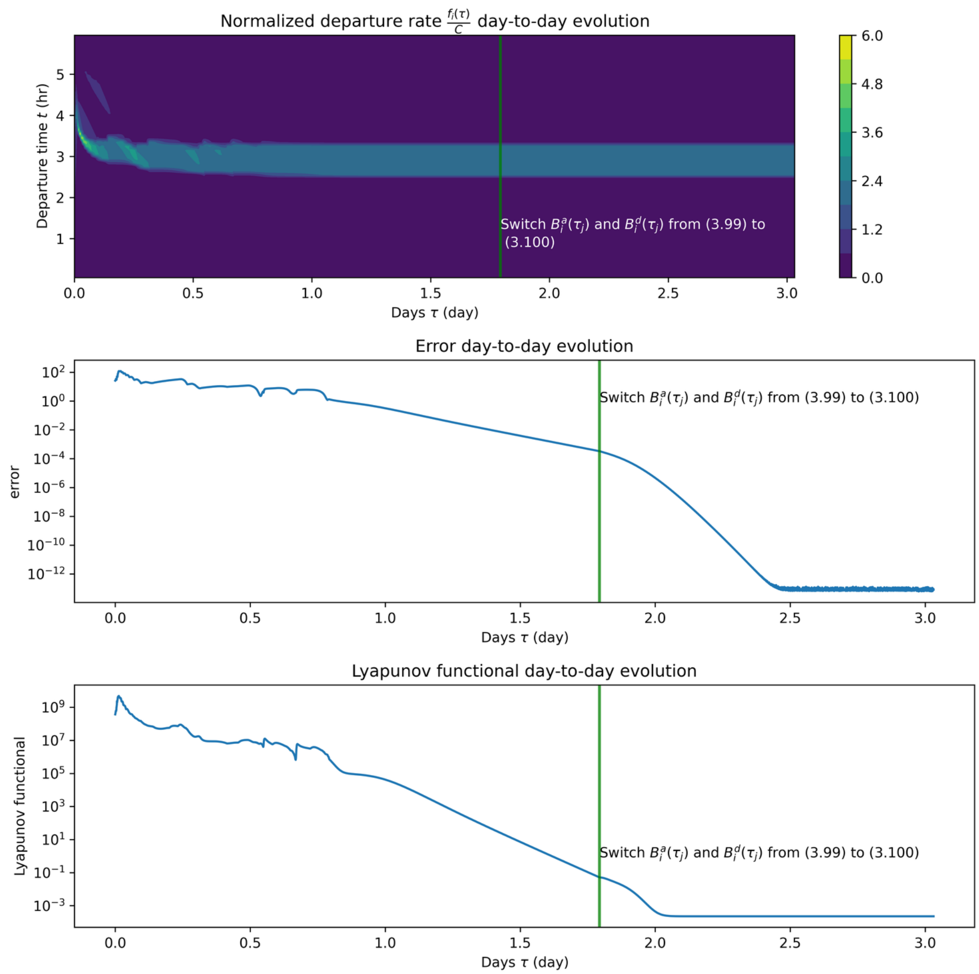


Figure 3.12: Day-to-day evolution with a 2-peak initial departure pattern

3.9 Conclusion

Chapter 3 presents Jin (2021b)'s local stable day-to-day departure time dynamics for a single bottleneck. The results show that within the stability region, the dynamical system is asymptotically stable. With the system optimal as the initial departure pattern, the dynamical system can converge to a stable SC-DTUE with asymmetric coefficients and switching coefficients, but not with symmetric coefficients. With switching coefficients, the dynamical system can converge to SC-DTUE under random initial conditions. Chapter 4 presents the extension of the model of single-class dynamics in this chapter to multi-class dynamics, capturing travelers' heterogeneity.

Chapter 4

A Stable Day-to-day Departure Time Dynamics for Multi-class Travelers

In Chapter 3, we presented a stable model of day-to-day departure time dynamics along the lines of Jin (2021b). In this chapter, we extend the single-class model to multi-class dynamics, capturing travelers' heterogeneity. Here we only consider heterogeneity in the discrete sense with different classes of travelers, rather than the continuous sense, which requires the consideration of the distribution of the values of time, cost of early arrivals, and late arrivals. We will present the definitions for a multi-class heterogeneous bottleneck model, the associated trip costs, and the resulting multi-class departure time user equilibrium in Section 4.2. We present the day-to-day dynamics for multi-class travelers in Section 4.3. We then study the stationary state and its stability in Sections 4.4 and 4.5. We present the CFL condition and the characteristic wave speed in Section 4.6, and use it in Section 4.7 to show that the stationary state of the multi-class dynamical system is equivalent to a multi-class departure time user equilibrium (MC-DTUE) with travelers of the same desired arrival times, different λ/μ, and same μ/ν. We present the numerical examples in Section 4.9, and this chapter concludes in Section 4.10.

4.1 Notations

Table 4.1: Notation in Chapter 4, ordered by appearance

Variable	Meaning	Unit
N	Total number of travelers	veh
G	Set of traveler groups	/
N_g	Number of travelers in group g	veh
$f(\tau,t)$	Total departure rate at time t on day τ	veh/hr
$F(\tau,t)$	Total cumulative departure flow at time t on day τ	veh
$f^g(\tau,t)$	Departure rate for travelers in group g at time t on day τ	veh/hr
$F^g(\tau,t)$	Cumulative departure flow for travelers in group g at time t on day τ	veh
$a^g(\tau,t)$	Group g's portion in the total departure rate at time t on day τ	1
$g(\tau,t)$	Total arrival rate at time t on day τ	veh/hr
$G(\tau,t)$	Total cumulative arrival flow at time t on day τ	veh
$g^g(\tau,t)$	Arrival rate for travelers in group g at time t on day τ	veh/hr
$G^g(\tau,t)$	Cumulative arrival flow for travelers in group g at time t on day τ	veh
$\delta(\tau,t)$	Total queue length in the point queue model	veh
$\delta^g(\tau,t)$	Number of group g's travelers in the queue	veh
$\Upsilon(\tau,t)$	queuing time for travelers departing at time t on day τ	hr
$\Upsilon^{'}(\tau,t)$	queuing time for travelers arriving at time t on day τ	hr
C	Capacity for the point queue model	veh/hr
ϵ	Infinitesimal hyper-real positive number	hr
$\phi^g(\tau,t)$	Trip cost for travelers in group g departing at time t from home on day τ	\$
$t^{g,*}$	Desired arrival time for group g's travelers	hr
λ_g	Value of travel time for group g's travelers	\$/hr
μ_g	Cost of arriving one extra hour early for group g's travelers	\$/hr
ν_g	Cost of arriving one extra hour late for group g's travelers	\$/hr
$\phi_1^g(\tau,t)$	queuing cost for group g's travelers departing at time t from home on day τ	\$
$\phi_2^g(\tau,t)$	Unpunctuality cost for group g's travelers departing at time t from home on day τ	\$
$\omega^g(\tau,t)$	Rate of change in trip cost for group g's travelers	\$/hr
$\omega^{E,g}(\tau,t)$	Rate of change in trip cost for group g's travelers' early arrivals	\$/hr
$\omega^{L,g}(\tau,t)$	Rate of change in trip cost for group g's travelers' late arrivals	\$/hr
$f_1^m(\tau)$	Maximum departure rate for early arrivals on day τ	veh/hr
$f_2^m(\tau)$	Maximum departure rate for late arrivals on day τ	veh/hr
$f^m(\tau)$	Maximum departure rate for the study period on day τ	veh/hr
$\phi^{g,*}$	Minimum trip cost for travelers in group g at the multi-class departure time user equilibrium (MC-DTUE)	\$
t_0	Start of departure period at MC-DTUE	hr
t_2	End of departure period at MC-DTUE	hr
$t_{0,i}^g$	Start of the ith departure interval for group g's travelers at MC-DTUE	hr
$t_{2,i}^g$	End of the ith departure interval for group g's travelers at MC-DTUE	hr
T	End of the study period	hr
I	Number of intervals of the study period	1
Δt	Time step size	hr
$\Delta\tau$	Day step size	day
$f_i^{d,g}(\tau)$	Deferral rate for travelers in group g departing at time step i on day τ	veh/day
$B_i^{d,g}(\tau)$	Deferral coefficient for group g at time step i on day τ	1
$f_i^{a,g}(\tau)$	Advance rate for travelers in group g departing at time step i on day τ	veh/day
$B_i^{a,g}(\tau)$	Advance coefficient for group g at time step i on day τ	1
$\tilde{g}_i^g(\tau)$	Net advance flow rate for group g from time step i to time step $i-1$ at time step i on day τ	1
$\tau^{'}$	The day when the multi-class dynamical system reaches MC-DTUE	1
$u^g(\tau,t)$	Group g's deferral or advance coefficients times time step over day step	hr/day
$\alpha^g(\tau,t)$	Group g's departure rate coefficient of rate of change in trip cost	1
$V(\mathbf{f}(\tau,\cdot))$	Lyapunov functional on day τ	1
$D_i^g(\tau)$	Replacing $\int_{t_{0,i}^g}^{t_{2,i}^g}(t-t_{0,i}^g)\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt$	1
$A_i^g(\tau,t)$	Replacing $\frac{\partial}{\partial\tau}\left\{f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2\right\}$	1
u_0	Positive coefficient for $u(\tau,t)$	1
ε	$\omega(\tau,t_{2,i}^g)=0$ when $\tau\geq\tau^{'}-\varepsilon$	day
$f^*(t)$	MC-DTUE departure rate	veh/hr
$\phi^*(t)$	MC-DTUE cost function	\$

4.2 Definitions

4.2.1 Within-day dynamics: heterogeneous point queue model

Let N be the total number of travelers. Let G be the set of traveler groups. Following Arnott et al. (1994); Lindsey (2004), we use groups (instead of classes) to describe different sets of travelers. Let N_g be the number of travelers in group g, where $g \in G$. So we have $N = \sum_{g \in G} N_g$.

Let $f(\tau, t)$ be the total departure rate at time t on day τ (unit: veh/hr) and $F(\tau, t)$ be the total cumulative departure flow at time t on day τ (unit: veh). For each traveler group g, let $f^g(\tau, t)$ be the departure rate for travelers in group g at time t on day τ, and $F^g(\tau, t)$ be its cumulative departure flow. So we have:

$$F(\tau, t) = \sum_{g \in G} F^g(\tau, t) \text{ and } f(\tau, t) = \sum_{g \in G} f^g(\tau, t). \tag{4.1}$$

We have $N = \int_0^T f(\tau, t)dt$ and $N_g = \int_0^T f^g(\tau, t)dt$.

Let $a^g(\tau, t)$ be group g's portion in the total departure rate at time t on day τ. By definition, we have:

$$a^g(\tau, t) = \frac{f^g(\tau, t)}{f(\tau, t)} \text{ and } a^g(\tau, t) = \frac{F^g(\tau, t)}{F(\tau, t)}, g \in G, \tag{4.2}$$

where $a^g(\tau, t)$ is either pre-defined on the initial day or updated by the multi-class day-to-day dynamical system. Summing over all groups on both sides of Equation (4.2), we have $\sum_g a^g(\tau, t) = \sum_g \frac{f^g(\tau,t)}{f(\tau,t)} = 1$. Therefore, for a given day τ and time t, we have:

$$\sum_{g \in G} a^g(\tau, t) = 1, \forall \tau, t. \tag{4.3}$$

Let $g(\tau, t)$ be the total arrival rate (unit: veh/hr) and $G(\tau, t)$ be the total cumulative arrival flow (unit: vehicles). For each traveler group g, let $g^g(\tau, t)$ be the arrival rate for travelers in group g, and $G^g(\tau, t)$ be its cumulative arrival flow. Let $\delta(\tau, t)$ be the total queue length (unit: vehicles), and $\delta^g(\tau, t)$ be the number of group g's travelers in the queue (unit: vehicles). So for arrival flow, we have:

$$G(\tau, t) = \sum_{g \in G} G^g(\tau, t) \text{ and } g(\tau, t) = \sum_{g \in G} g^g(\tau, t). \tag{4.4}$$

For queues, we have:

$$\delta(\tau, t) = \sum_{g \in G} \delta^g(\tau, t). \tag{4.5}$$

In heterogeneous cases, the dynamics of total departure rate/cumulative departure flow, $f(\tau, t)/F(\tau, t)$, total arrival rate/cumulative arrival flow, $g(\tau, t)/G(\tau, t)$, and total queue length, $\delta(\tau, t)$, are determined by Equations (3.1) to (3.6), and solved numerically by Equations (3.7) to (3.10), as in the homogeneous case.

However, we need to determine group g's cumulative departure/arrival flow, $F^g(\tau, t)/G^g(\tau, t)$, departure/arrival rate, $f^g(\tau, t)/g^g(\tau, t)$, and its queue length $\delta^g(\tau, t)$. Given $a^g(\tau, t)$, from Equation (4.2), we have:

$$F^g(\tau, t) = a^g(\tau, t) \cdot F(\tau, t) \text{ and } f^g(\tau, t) = a^g(\tau, t) \cdot f(\tau, t). \tag{4.6}$$

To determine group g's queue length, $\delta^g(\tau, t)$, we have the following conservation equation as in the homogeneous case (Equation (3.1)):

$$\delta^g(\tau, t) = F^g(\tau, t) - G^g(\tau, t). \tag{4.7}$$

We need to specify $G^g(\tau, t)$ to determine $\delta^g(\tau, t)$. For $G^g(\tau, t)/g^g(\tau, t)$ to be well-defined, they need to satisfy the following conditions: $G(\tau,t)=\sum_g G^g(\tau,t)$ and $g(\tau,t)=\sum_g g^g(\tau,t)$. That is, the sum of each group's cumulative arrival flow/arrival rate is equal to the total cumulative arrival flow/arrival rate. As mentioned above, $G(\tau,t)$ and $g(\tau,t)$ can be determined by the homogeneous point queue model dynamics. From Chapter 3, we have:

$$g(\tau,t)=\min\{f(\tau,t)+\frac{\delta(\tau,t)}{\epsilon},C\},$$

$$G(\tau,t)=\min\{F(\tau,t+\Delta t),G(\tau,t)+C\cdot\Delta t\}.$$

However, for the heterogeneous case, we need to first determine $G^g(\tau,t)$ and then use $g^g(\tau,t)=\frac{\partial}{\partial t}G^g(\tau,t)$ to determine $g^g(\tau,t)$. The first-in-first-out assumption still holds for multi-class travelers. Let $\Upsilon(\tau,t)$ be the queuing time for travelers departing at time t on day τ and $\Upsilon^{'}(\tau,t)$ be the queuing time for travelers arriving at time t on day τ. So we have:

$$F^g(\tau,t)=G^g(\tau,t+\Upsilon(\tau,t)), \tag{4.8}$$

$$F^g(\tau,t-\Upsilon^{'}(\tau,t))=G^g(\tau,t), \tag{4.9}$$

which means that if group g's travelers join the queue at time t, then they will exit the bottleneck at $t+\Upsilon(\tau,t)$, and that if group g's travelers join the queue at time $t-\Upsilon^{'}(\tau,t)$, then they will exit the bottleneck at time t.

For the discrete version, we have the following relationship for cumulative departure flow $F^g(\tau,t)$:

$$F^g(\tau,t+\Delta t)=F^g(\tau,t)+f(\tau,t)\cdot\Delta t. \tag{4.10}$$

For the discrete version, we also have the following relationship for cumulative arrival flow

$G^g(\tau, t)$:

$$G^g(\tau, t + \Delta t) = G^g(\tau, t) + C \cdot a^g(\tau, t - \Upsilon^{'}(\tau, t)) \cdot \Delta t. \tag{4.11}$$

This equation can be interpreted as follows: the portion of group g's travelers among all travelers, $a^g(\tau, t - \Upsilon^{'}(\tau, t))$, remains unchanged from the time they join the queue, $t - \Upsilon^{'}(\tau, t)$, till they exist the bottleneck at time t. So the capacity of the bottleneck, C, is allocated according to the proportions of different traveler groups when they join the queue as $C \cdot a^g(\tau, t - \Upsilon^{'}(\tau, t))$. For $g^g(\tau, t)$, we have:

$$g^g(\tau, t) = \min\{f^g(\tau, t) + \frac{\delta(\tau, t)}{\epsilon}, C \cdot a^g(\tau, t - \Upsilon^{'}(\tau, t))\}, \tag{4.12}$$

When the bottleneck is uncongested (i.e., $\delta(\tau, t) = 0$), we have $\Upsilon^{'}(\tau, t) = 0$, so we get $g^g(\tau, t) = \min\{f^g(\tau, t), C \cdot a^g(\tau, t)\} = f^g(\tau, t)$. When the bottleneck is congested (i.e., $\delta(\tau, t) > 0$), we have $\Upsilon^{'}(\tau, t) > 0$, so we get $g^g(\tau, t) = C \cdot a^g(\tau, t - \Upsilon^{'}(\tau, t))$.

For the discrete version, we have the following relationship:

$$g^g(\tau, t) = \frac{G^g(\tau, t + \Delta t) - G^g(\tau, t)}{\Delta t}. \tag{4.13}$$

queuing time $\Upsilon(\tau, t)$, rate of change in queuing time $\frac{\partial}{\partial t}\Upsilon(\tau, t)$, and the bounds of the queuing time are determined by Equations (3.11) to (3.13) as in Chapter 3.

To implement the multi-class point queue model, we can first model the total cumulative departure/arrival flow $F(\tau, t)/G(\tau, t)$, total departure/arrival rate $f(\tau, t)/g(\tau, t)$, and total queue length $\delta(\tau, t)$. Then we calculate $f^g(\tau, t)$ using Equation (4.6), $F^g(\tau, t)$ using Equation (4.10) and $G^g(\tau, t)$ using Equation (4.11). So we can calculate group g's queue length using Equation (4.7), and $g^g(\tau, t)$ using Equation (4.13).

4.2.2 Trip cost: heterogeneous travelers

To introduce travelers' heterogeneity, we consider the most simple heterogeneous scenario — multi-class heterogeneity, rather than the more complicated cases with continuous distributions.

Let $\phi^g(\tau, t)$ denote the trip cost for group g's travelers departing at time t from home on day τ (unit: \$). Let $t^{g,*}$ be the desired arrival time for group g's travelers. We have the following cost function for group g's travelers:

$$\phi^g(\tau, t) = \lambda_g \cdot (\Upsilon^0 + \Upsilon(\tau, t)) + \mu_g \cdot \{t^{g,*} - (t + \Upsilon(\tau, t))\}_+ + \nu_g \cdot \{(t + \Upsilon(\tau, t)) - t^{g,*}\}_+, \quad (4.14)$$

where the variables related to the congestion of the bottleneck (i.e., free flow travel time $\Upsilon^0(\tau, t)$ and queuing time $\Upsilon(\tau, t)$) are shared by all groups. Without loss of generality, we assume the free flow travel time to be zero (i.e., Υ^0). λ_g is the value of travel time for group g's travelers (unit: \$/hr), while μ_g, ν_g are the costs for arriving early and arriving late for group g's travelers, respectively (unit: \$/hr). The first term in Equation (4.14) refers to queuing cost for group g's travelers, while the second and third terms in Equation (4.14) together refer to the unpunctuality cost for group g's travelers. Let $\phi_1^g(\tau, t) = \lambda_g \cdot (\Upsilon^0 + \Upsilon(\tau, t))$ be the queuing cost and $\phi_2^g(\tau, t) = \mu_g \cdot \{t^{g,*} - (t + \Upsilon(\tau, t))\}_+ + \nu_g \cdot \{(t + \Upsilon(\tau, t)) - t^{g,*}\}_+$ be the unpunctuality cost. We assume $\mu_g < \lambda_g$ for each group to avoid multiple equilibria (Arnott et al., 1994).

Taking the derivative with respect to t on both sides of Equation (4.14), we get the rate of change in trip cost for group g. Let $\omega^g(\tau, t)$ be the rate of change in trip cost for group g's travelers, and that is $\omega^g(\tau, t) = \frac{\partial}{\partial t}\phi^g(\tau, t)$. For early arrivals, we have $\omega^{E,g}(\tau, t) = \frac{\partial}{\partial t}\phi^{E,g}(\tau, t)$ (E for early arrivals):

$$\omega^{E,g}(\tau, t) = \frac{\partial}{\partial t}\phi^{E,g}(\tau, t) = (\lambda_g - \mu_g) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau, t) - \mu_g. \quad (4.15)$$

For late arrivals, we have the following rate of change in trip cost, $\omega^{L,g}(\tau,t) = \frac{\partial}{\partial t}\phi^{L,g}(\tau,t)$ (L for late arrivals):

$$\omega^{L,g}(\tau,t) = \frac{\partial}{\partial t}\phi^{L,g}(\tau,t) = (\lambda_g + \nu_g) \cdot \frac{\partial}{\partial t}\Upsilon^{L}(\tau,t) + \nu_g. \tag{4.16}$$

In Equations (4.15) and (4.16), the queuing time for early and late arrivals, $\Upsilon^E(\tau,t)$ and $\Upsilon^L(\tau,t)$ do not have superscript g, because all groups of travelers share the same queuing time as long as they join the queue at the same time t.

From Equation (3.13), we have $-1 \leq \frac{\partial}{\partial t}\Upsilon^E(\tau,t) \leq \frac{f_1^m(\tau)}{C} - 1$ for early arrivals, $-1 \leq \frac{\partial}{\partial t}\Upsilon^L(\tau,t) \leq \frac{f_2^m(\tau)}{C} - 1$ for late arrivals, and $-1 \leq \frac{\partial}{\partial t}\Upsilon(\tau,t) \leq \frac{f^m(\tau)}{C} - 1$ for all arrivals. Substituting the bounds of $\frac{\partial}{\partial t}\Upsilon^E(\tau,t)$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t)$ into Equations (4.15) and (4.16), we find that:

$$-\lambda_g \leq \omega^{E,g}(\tau,t) \leq (\lambda_g - \mu_g) \cdot \frac{f_1^m(\tau)}{C} - \lambda_g, \tag{4.17}$$

for early arrivals, and

$$-\lambda_g \leq \omega^{L,g}(\tau,t) \leq (\lambda_g + \nu_g) \cdot \frac{f_2^m(\tau)}{C} - \lambda_g, \tag{4.18}$$

for late arrivals.

Let

$$\alpha^g(\tau,t) = \begin{cases} \frac{\lambda_g - \mu_g}{C} & \text{for early arrivals,} \\ \frac{\lambda_g + \nu_g}{C} & \text{for late arrivals,} \end{cases} \tag{4.19}$$

so we have $\lambda_g \leq \omega^g(\tau,t) \leq \alpha^g(\tau,t) f_1^m(\tau) - \lambda_g$ for early arrivals and $\lambda_g \leq \omega^g(\tau,t) \leq \alpha^g(\tau,t) f_2^m(\tau) - \lambda_g$ for late arrivals.

4.2.3 Multi-class departure time user equilibrium

Definition 4.1 (Multi-class departure time user equilibrium)**.** *A multi-class departure time user equlibrium (MC-DTUE) is reached when all used departure times for travelers in a certain group share the same cost, which is smaller than or equal to all unused times for travelers in the same group. We have* $\phi^g(\tau,t) = \phi^{g,*}$ *for* $t \in \bigcup_{i=1}^{\infty}[t^g_{0,i}, t^g_{2,i}]$*, where* $\phi^{g,*}$ *is the minimum trip cost for travelers in group* g*, and* $[t^g_{0,i}, t^g_{2,i}]$ *is the* i*th departure time interval for travelers in group* g*, where* $t^g_{0,i}$ *is the start of the interval and* $t^g_{2,i}$ *is the end of the interval. Notice that in MC-DTUE, travelers from the same group might have two departure intervals — one before the desired arrival time* $t^{g,*}$ *and one after the desired arrival time* $t^{g,*}$*. Thus, we have the subscript* i *in* $[t^g_{0,i}, t^g_{2,i}]$ *to represent the* i*th departure interval, where* $i = 1$ *or* $i = 2$*, representing the first or the second departure interval. Travelers who depart in the middle have only one departure interval. Therefore, for* $t \in \bigcup_{i=1}^{\infty}[t^g_{0,i}, t^g_{2,i}]$*, we have* $f^g(\tau,t) > 0$ *and* $\phi^g(\tau,t) = \phi^{g,*}$*. For* $t \notin \bigcup_{i=1}^{\infty}[t^g_{0,i}, t^g_{2,i}]$*, we have* $f^g(\tau,t) = 0$*, and* $\phi^g(\tau,t) \geq \phi^{g,*}$*.*

From Definition 4.1, we have the following complementarity condition for each traveler group g:

$$f^g(\tau,t)(\phi^g(\tau,t) - \phi^{g,*}) = 0, \tag{4.20}$$

Here, if $f^g(\tau,t) > 0$, $\phi^g(\tau,t) - \phi^{g,*} = 0$, and that is $\phi^g(\tau,t) = \phi^{g,*}$. If $f^g(\tau,t) = 0$, then $\phi^g(\tau,t) - \phi^{g,*} \geq 0$, and that is $\phi^g(\tau,t) \geq \phi^{g,*}$.

4.3 Day-to-day dynamics: extension to multi-class travelers

We presented a stable model of day-to-day dynamics along the lines of Jin (2021b) in Chapter 3. In this chapter, we extend the model from single-class to multi-class scenarios, capturing travelers' heterogeneity. Similar to Jin (2021b), we only consider the fixed demand case where travelers only defer or advance their departure time. We show that with different λ/μ, same μ/ν and same desired arrival time t^* across travelers, the stationary state of the multi-class dynamical system yields an MC-DTUE, and that this state is asymptotically stable.

4.3.1 Discrete version

Same as in Chapter 3, we consider $[0, T]$ as the study period. We divide it into I intervals. Then we have time step size $\Delta t = \frac{T}{I}$. Let $\Delta\tau$ be the day step size. Let $\phi_i^g(\tau) = \phi^g(\tau, i\Delta t)$, where $i = 0, 1, 2, ..., I$, which denotes the trip cost for travelers in group g at $I + 1$ points in the study period. When $i = 0$, we have $\phi_0^g(\tau) = \phi^g(\tau, 0)$, while when $i = I$, we have $\phi_I^g(\tau) = \phi^g(\tau, T)$. We have $\omega_i^g(\tau) = \omega^g(\tau, (i - \frac{1}{2})\Delta t) = \frac{\phi^g(\tau,i\Delta t) - \phi^g(\tau,(i-1)\Delta t)}{\Delta t} = \frac{\phi_i^g(\tau) - \phi_{i-1}^g(\tau)}{\Delta t}$, where $i = 1, 2, ..., I$, which denotes the trip cost difference for group g's travelers between two consecutive points in the study period. Let $f_i^g(\tau) = f^g(\tau, (i - \frac{1}{2})\Delta t)$, where $i = 1, 2, ..., I$, which denotes the group g's departure rate in interval i. Notice that $\{\phi_i^g(\tau)\}_{i=0}^I$ has $I + 1$ elements, while $\{\omega_i^g(\tau)\}_{i=1}^I$ and $\{f_i^g(\tau)\}_{i=1}^I$ have I elements.

We now discuss the day-to-day dynamics for each traveler group g. On day τ, if $\omega_{i+1}^g(\tau) < 0$, that is, $\phi_{i+1}^g(\tau) < \phi_i^g(\tau)$, we assume that travelers from group g will defer their departure times by one time step, from i to $i + 1$, to reduce their trip costs. Let $f_i^{d,g}(\tau)$ be the deferral rate for travelers in group g departing at time step i on day τ (unit: veh/day), so we assume the deferral travelers for group g, $f_i^{d,g}(\tau)\Delta\tau$, is determined by the following relation:

$$f_i^{d,g}(\tau)\Delta\tau = B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+ \cdot f_i^g(\tau)\Delta t, \tag{4.21}$$

where $B_{i+1}^{d,g}(\tau)$ is a positive deferral coefficient for group g, which is to be estimated, and $\{-\omega_{i+1}^g(\tau)\}_+$ is a scalar whose unit is 1. If we do not consider $\{-\omega_{i+1}^g(\tau)\}_+$ as a scalar, the units of the left-hand side and the right-hand side of Equation (4.21) do not match. Equation (4.21) depicts the following relation: the number of deferral travelers in group g is equal to the number of group g's travelers departing at time step i on day τ, $f_i^g(\tau)\Delta t$, times the cost difference of group g's travelers between time step $i+1$ and i, $\{-\omega_{i+1}^g(\tau)\}_+$, and the deferral coefficient, $B_{i+1}^{d,g}(\tau)$.

On day τ, if $\omega_i^g(\tau) > 0$, that is, $\phi_i^g(\tau) > \phi_{i-1}^g(\tau)$, we assume that travelers in group g will advance their departure times by one time step, from i to $i-1$, to reduce their trip costs. Let $f_i^{a,g}(\tau)$ be the advance rate for travelers in group g departing at time step i on day τ (unit: veh/day), so we assume that the number of advancing travelers, $f_i^{a,g}(\tau)\Delta\tau$, is determined by the following relation:

$$f_i^{a,g}(\tau)\Delta\tau = B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \cdot (f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau), \tag{4.22}$$

where $B_i^{a,g}(\tau)$ is a positive advance coefficient for group g, which is to be estimated, and $\{\omega_i^g(\tau)\}_+$ is a scalar whose unit is 1. To avoid the same traveler being deferred and advanced at the same time, we deduct group g's the deferral travelers before we calculate its advancing travelers for the same time step (i.e., $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau$). Equation (4.22) depicts the following relation: the number of advancing travelers in group g is equal to the remaining non-deferring travelers in group g who depart at time step i on day τ, $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau$, times the cost difference of group g's travelers between time step i and $i-1$, $\{\omega_i^g(\tau)\}_+$, and the advance coefficient, $B_i^{a,g}(\tau)$.

After calculating group g's deferral and advance travelers, we update group g's departure rate on day $\tau + \Delta\tau$, $f_i^g(\tau + \Delta\tau)$, based on the day τ's departure rate at time step i, $f_i^g(\tau)$, deferral and advance rate at time step i, $f_i^{d,g}(\tau)$ and $f_i^{a,g}(\tau)$, deferral rate at time step $i-1$, $f_{i-1}^{d,g}(\tau)$, and advance rate at time step $i+1$, $f_{i+1}^{a,g}(\tau)$. We then obtain the following relation:

$$f_i^g(\tau + \Delta\tau)\Delta t = f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau - f_i^{a,g}(\tau)\Delta\tau + f_{i-1}^{d,g}(\tau)\Delta\tau + f_{i+1}^{a,g}(\tau)\Delta\tau, \quad (4.23)$$

where $f_i^{d,g}(\tau)\Delta\tau$ and $f_i^{a,g}(\tau)\Delta\tau$ are group g's travelers flowing out from time step i, while $f_{i-1}^{d,g}(\tau)\Delta\tau$ and $f_{i+1}^{a,g}(\tau)\Delta\tau$ are group g's travelers flowing into time step i.

4.3.2 Well-definedness

We present the well-definedness condition of the multi-class local dynamical system in (Equations (4.21) to (4.23)) in this section.

Definition 4.2 (Well-defined definition)**.** *The multi-class local dynamical system in Equations* (4.21) *to* (4.23) *is well-defined if and only if* $f_i^g(\tau) \geq 0$ *for each group* g*, each time step* i*, and each day step* τ*.*

The departure rate, $f_i^g(\tau)$, should be non-negative, since we only consider trips departing from home and do not consider trips to travel back home. So we assume $f_i^g(\tau) \geq 0$. For the multi-class dynamical system (Equations (4.21) to (4.23)) to be well-defined, we need $f_i^g(\tau + \Delta\tau) \geq 0$ as well. In order for $f_i^g(\tau + \Delta\tau) \geq 0$, we need to have $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau - f_i^{a,g}(\tau)\Delta\tau + f_{i-1}^{d,g}(\tau)\Delta\tau + f_{i+1}^{a,g}(\tau)\Delta\tau \geq 0$ from Equation (4.23). If the total inflow to time step i from time step $i-1$ and $i+1$ is zero, $f_{i-1}^{d,g}(\tau)\Delta\tau + f_{i+1}^{a,g}(\tau)\Delta\tau = 0$, then $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau - f_i^{a,g}(\tau)\Delta\tau \geq 0$. That is, the outflow from time step i to time steps $i-1$ and $i+1$ should not be greater than the number of travelers at time step i iteself:

$$f_i^{d,g}(\tau)\Delta\tau + f_i^{a,g}(\tau)\Delta\tau \leq f_i^g(\tau)\Delta t, \quad (4.24)$$

Substituting $f_i^{a,g}(\tau)\Delta\tau$ from Equation (4.22) into Equation (4.24), we have:

$$f_i^{d,g}(\tau)\Delta\tau + B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \cdot (f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau) \leq f_i^g(\tau)\Delta t. \tag{4.25}$$

Arranging $f_i^g(\tau)\Delta t$ to one side, and $f_i^{d,g}(\tau)\Delta\tau$ to the other side, we have:

$$f_i^{d,g}(\tau)\Delta\tau(1 - B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+) \leq f_i^g(\tau)\Delta t(1 - B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+). \tag{4.26}$$

After further simplifying, Equation (4.26) becomes:

$$(1 - B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+)(f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau) \geq 0. \tag{4.27}$$

Now we can discuss the following two cases based on $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau$. That is $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau < 0$ and $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau \geq 0$.

1. If $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau < 0$, then $1 - B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \leq 0$. Under this condition, we have $f_i^g(\tau)\Delta t < f_i^{d,g}(\tau)\Delta\tau$, which contradicts to $f_i^{d,g}(\tau)\Delta\tau + f_i^{a,g}(\tau)\Delta\tau \leq f_i^g(\tau)\Delta t$ from Equation (4.24). Therefore case 1 is not possible.

2. If $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau \geq 0$, then $1 - B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \geq 0$ and that is $B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \leq 1$. From Equation (4.21), $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau \geq 0$ becomes:

$$f_i^g(\tau)\Delta t \geq f_i^{d,g}(\tau)\Delta\tau = B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+ \cdot f_i^g(\tau)\Delta t, \tag{4.28}$$

which becomes $f_i^g(\tau)\Delta t(1 - B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+) \geq 0$. Since $f_i^g(\tau)\Delta t \geq 0$, $1 - B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+ \geq 0$, and thus $B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+ \leq 1$.

From case 2, we have the following well-definedness condition for the local multi-class dynamical system in (Equations (4.21) to (4.23)):

$$
\begin{aligned}
B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ &\leq 1, \\
B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+ &\leq 1,
\end{aligned} \tag{4.29}
$$

where $\{\omega_i^g(\tau)\}_+$ and $\{-\omega_{i+1}^g(\tau)\}_+$ are determined by traffic dynamics of the point queue model as well as attributes of group g's travelers (e.g., λ_g, μ_g, ν_g, and $t^{g,*}$). To multi-class dynamical system to be well-defined, we need to find the well-definedness conditions for $B_i^{a,g}(\tau)$ and $B_{i+1}^{d,g}(\tau)$ — the deferral and advance coefficient for each group g and time step i on day τ.

Let $\{\omega_i^g(\tau)\}_+^{max}$ be the maximum value of $\{\omega_i^g(\tau)\}_+$, and let $\{-\omega_{i+1}^g(\tau)\}_+^{max}$ be the maximum value of $\{-\omega_{i+1}^g(\tau)\}_+$, so from Equation (4.29), we have:

$$
\begin{aligned}
B_i^{a,g}(\tau) &\leq \frac{1}{\{\omega_i^g(\tau)\}_+^{max}}, \\
B_{i+1}^{d,g}(\tau) &\leq \frac{1}{\{-\omega_{i+1}^g(\tau)\}_+^{max}}.
\end{aligned} \tag{4.30}
$$

We need to discuss $\{\omega_i^g(\tau)\}_+^{max}$ and $\{-\omega_{i+1}^g(\tau)\}_+^{max}$ to further determine the well-definedness condition.

Let $\omega^{E,g}(\tau,t)$ and $\omega^{L,g}(\tau,t)$ be the rate of change in trip cost for early and late arrivals for group g. We have $\omega^{E,g}(\tau,t) = (\lambda_g - \mu_g) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu_g$ from Equation (4.15), and $\omega^{L,g}(\tau,t) = (\lambda_g + \nu_g) \cdot \frac{\partial}{\partial t}\Upsilon^L(\tau,t) + \nu_g$ from Equation (4.16). We discuss the $\{\omega_i^g(\tau)\}_+^{max}$ and $\{-\omega_{i+1}^g(\tau)\}_+^{max}$ under the following two cases of $\delta(\tau,t) = 0$ and $\delta(\tau,t) > 0$:

1. If $\delta(\tau,t) = 0$, then $\frac{\partial}{\partial t}\Upsilon^E(\tau,t) = 0$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t) = 0$, so we have $\omega^{E,g}(\tau,t) = -\mu_g$ and $\omega^{L,g}(\tau,t) = \nu_g$. Since $\{\omega^{E,g}(\tau,t)\}_+ = 0$ for early arrivals and $\{\omega^{L,g}(\tau,t)\}_+ = \nu_g$ for late arrivals, we have $\{\omega^g(\tau,t)\}_+ \leq \nu_g$, and $\{\omega_i^g(\tau)\}_+^{max} = \nu_g$. Since $\{-\omega^{E,g}(\tau,t)\}_+ = \mu_g$ for early arrivals and $\{-\omega^{L,g}(\tau,t)\}_+ = 0$ for late arrivals, so we have $\{-\omega^g(\tau,t)\}_+ \leq \mu_g$,

and $\{-\omega_{i+1}^{g}(\tau)\}_{+}^{max} = \mu_g$. So we have:

$$
\begin{aligned}
B_i^{a,g}(\tau) &\leq \frac{1}{\{\omega_i^g(\tau)\}_{+}^{max}} = \frac{1}{\nu_g}, \\
B_{i+1}^{d,g}(\tau) &\leq \frac{1}{\{-\omega_{i+1}^g(\tau)\}_{+}^{max}} = \frac{1}{\mu_g}.
\end{aligned}
\tag{4.31}
$$

2. If $\delta(\tau,t) > 0$, then we have $-1 \leq \frac{\partial}{\partial t}\Upsilon(\tau,t) \leq \frac{f^m(\tau)}{C} - 1$, so we have $-\lambda_g \leq \omega^{E,g}(\tau,t) \leq (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g$ from Equation (4.17) for early arrivals, and $-\lambda_g \leq \omega^{L,g}(\tau,t) \leq (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g$ from Equation (4.18) for late arrivals.

   Since $\{\omega^{E,g}(\tau,t)\}_{+}^{max} = (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g$ and $\{\omega^{L,g}(\tau,t)\}_{+}^{max} = (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g$, we have $\{\omega_i^g(\tau)\}_{+}^{max} = \max\{(\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\}$. Since $\{-\omega^{E,g}(\tau,t)\}_{+}^{max} = \lambda_g$ and $\{-\omega^{L,g}(\tau,t)\}_{+}^{max} = \lambda_g$, so we have $\{-\omega_{i+1}^g(\tau)\}_{+}^{max} = \lambda_g$. So we have:

$$
\begin{aligned}
B_i^{a,g}(\tau) &\leq \frac{1}{\{\omega_i^g(\tau)\}_{+}^{\max}} = \frac{1}{\max\left\{(\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g,\ (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\right\}}, \\
B_{i+1}^{d,g}(\tau) &\leq \frac{1}{\{-\omega_{i+1}^g(\tau)\}_{+}^{\max}} = \frac{1}{\lambda_g}.
\end{aligned}
\tag{4.32}
$$

Combining Equation (4.31) from case 1 and Equation (4.32) from case 2, we have the following well-definedness condition for $B_i^{a,g}(\tau)$ and $B_{i+1}^{d,g}(\tau)$:

$$
\begin{aligned}
0 < B_i^{a,g}(\tau) &\leq \frac{1}{\max\left\{\nu_g,\ (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g,\ (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\right\}}, \\
0 < B_{i+1}^{d,g}(\tau) &\leq \frac{1}{\max\{\lambda_g, \mu_g\}} = \frac{1}{\lambda_g}.
\end{aligned}
\tag{4.33}
$$

Theorem 4.1 (Well-definedness condition for multi-class dynamical system)**.** *The multi-class local dynamical system in Equations (4.21) to (4.23) is well-defined when* $0 < B_i^{a,g}(\tau) \leq 1/\max\{\nu_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\}$ *and* $0 < B_{i+1}^{d,g}(\tau) \leq \frac{1}{\lambda_g}$ *for each group* g *at time step* i *on day* τ*.*

Comparing Theorem 4.1 for multi-class dynamical system with Theorem 3.1 for single-class dynamical system, we can see that $B_i^{a,g}(\tau)$ and $B_{i+1}^{d,g}(\tau)$ become group specific, but still share the same form as in the single class case in Chapter 3. $B_i^{a,g}(\tau)$ and $B_{i+1}^{d,g}(\tau)$ influence the advance and deferral rate $f_i^{a,g}(\tau)$ and $f_i^{d,g}(\tau)$ only for its own group.

4.3.3 Net flows between time steps

Equation (4.23) deals with three time steps for each group g: $i-1$, i, and $i+1$. There are two two kinds of flows that exist across the boundary of time steps $i-1$ and i: advance flows and deferral flows. By advance flows, we mean travelers in group g who shift their departure times from time step i on day τ to time step $i-1$ on day $\tau+\Delta\tau$ (i.e., $f_i^{a,g}(\tau) > 0$). These travelers flow out from time step i, into time step $i-1$. At the same time, travelers can also have deferral flows, which means they can shift their departure time from time step $i-1$ on day τ to time step i on day $\tau+\Delta\tau$ (i.e., $f_{i-1}^{d,g}(\tau) > 0$). These travelers flow out from time step $i-1$, into time step i. To further simplify the equation and to derive its continuous version, we consider the net advance flows from time step i to $i-1$.

Let $\tilde{g}_i^g(\tau)$ to be the net advance flow rate for group g from time step i to $i-1$, which is related to both $f_i^{a,g}(\tau)$ and $f_{i-1}^{d,g}(\tau)$. We first analyze the relationship between $f_i^{a,g}(\tau)$ and $f_{i-1}^{d,g}(\tau)$, and then present the functional form of $\tilde{g}_i^g(\tau)$.

We have $f_{i-1}^{d,g}(\tau)\Delta\tau = B_i^{d,g}(\tau)\cdot\{-\omega_i^g(\tau)\}_+\cdot f_{i-1}^g(\tau)\Delta t$ from Equation (4.21), and $f_i^{a,g}(\tau)\Delta\tau = B_i^{a,g}(\tau)\cdot\{\omega_i^g(\tau)\}_+\cdot(f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau)$ from Equation (4.22). With the well-definedness condition, we have $f_{i-1}^g(\tau)\Delta t \geq 0$ and $f_i^g(\tau)\Delta t - f_i^{d,g}(\tau)\Delta\tau \geq 0$. If $\omega_i^g(\tau) > 0$, then $-\omega_i^g(\tau) < 0$, so we have $\{\omega_i^g(\tau)\}_+ > 0$ and $\{-\omega_i^g(\tau)\}_+ = 0$. So we have $f_i^{a,g}(\tau) > 0$ and $f_{i-1}^{d,g}(\tau) = 0$. If $\omega_i^g(\tau) < 0$, then we have $-\omega_i^g(\tau) > 0$, so we have $\{\omega_i^g(\tau)\}_+ = 0$ and $\{-\omega_i^g(\tau)\}_+ > 0$. So we have $f_i^{a,g}(\tau) = 0$ and $f_{i-1}^{d,g}(\tau) > 0$. If $\omega_i^g(\tau) = 0$, $\{-\omega_i^g(\tau)\}_+ = \{\omega_i^g(\tau)\}_+ = 0$, and $f_{i-1}^{d,g}(\tau) = f_i^{a,g}(\tau) = 0$.

So relationship of $f_{i-1}^{d,g}(\tau)$ and $f_i^{a,g}(\tau)$ can be summarized as follows:

$$f_{i-1}^{d,g}(\tau) \cdot f_i^{a,g}(\tau) = 0, \tag{4.34}$$

where if $f_{i-1}^{d,g}(\tau) > 0$, then $f_i^{a,g}(\tau) = 0$, and vice versa, or they both become zero. It is the same relationship as single-class in Section 3.3.3, but it is group specific in the multi-class cases.

So we have the following functional form of $\tilde{g}_i^g(\tau)$:

$$\tilde{g}_i^g(\tau) = sgn(\omega_i^g(\tau)) \cdot \max\{f_{i-1}^{d,g}(\tau), f_i^{a,g}(\tau)\}, \tag{4.35}$$

where $sgn(y)$ is the sign function, that is:

$$sgn(y) = \begin{cases} 1 & \text{for} \quad y > 0, \\ 0 & \text{for} \quad y = 0, \\ -1 & \text{for} \quad y < 0, \end{cases} \tag{4.36}$$

when $\omega_i^g(\tau) > 0$ (i.e., $\phi_i^g(\tau) > \phi_{i-1}^g(\tau)$), we have $f_{i-1}^{d,g}(\tau) = 0$ and $f_i^{a,g}(\tau) > 0$, so we have $\tilde{g}_i^g(\tau) = f_i^{a,g}(\tau)$. There is a positive net advance flow for group g's travelers from time step i to $i-1$. When $\omega_i^g(\tau) < 0$ (i.e., $\phi_i^g(\tau) < \phi_{i-1}^g(\tau)$), we have $f_{i-1}^{d,g}(\tau) > 0$ and $f_i^{a,g}(\tau) = 0$, so we have $\tilde{g}_i^g(\tau) = -f_{i-1}^{d,g}(\tau)$, and there is a negative net advance flow for group g's travelers from time step i to $i-1$, which means that there is positive net deferral flow from time step $i-1$ to i. When $\omega_i^g(\tau) = 0$, there is zero net advance flow for group g's travelers from time step i to $i-1$.

Consequently, Equation (4.23) can be written as:

$$f_i^g(\tau + \Delta\tau)\Delta t = f_i^g(\tau)\Delta t + \tilde{g}_{i+1}^g(\tau)\Delta\tau - \tilde{g}_i^g(\tau)\Delta\tau, \tag{4.37}$$

where $\tilde{g}_1^g(\tau) = \tilde{g}_{I+1}^g(\tau) = 0$, because time step 1 does not have any net advance flow to time step 0, with time step 0 being out of the study period. Likewise, there is no net advance flow from time step $I+1$ to time step i, because time step $I+1$ is out of the study period.

We can now state the following property for the multi-class dynamical system in Equation (4.37):

Theorem 4.2. *The number of trips in each group is conserved on any day step τ in the multi-class dynamical system in Equation (4.37), and thereby the total number of trips is conserved on any day step τ as well.*

Proof. To prove it, we need to show total number of trips for group g is N_g on day $\tau + \Delta\tau$ (i.e., $N_g = \sum_i f_i^g(\tau + \Delta\tau)\Delta t$), assuming the total number of trips for group g is N_g on day τ (i.e., $N_g = \sum_i f_i^g(\tau)\Delta t$) for all τ and $\Delta\tau$. So the total number of trips, $N = \sum_g N_g$, is conserved as well.

Let $N_g = \sum_i f_i^g(\tau)\Delta t$ on day τ. So on day $\tau + \Delta\tau$, from Equation (4.37), we have the following equation:

$$\sum_{i=1}^{I} f_i^g(\tau + \Delta\tau)\Delta t = \sum_{i=1}^{I} f_i^g(\tau)\Delta t + \sum_{i=1}^{I} \tilde{g}_{i+1}^g(\tau)\Delta\tau - \sum_{i=1}^{I} \tilde{g}_i^g(\tau)\Delta\tau. \tag{4.38}$$

Arranging the Equation (4.38), we have:

$$\sum_{i=1}^{I} f_i^g(\tau + \Delta\tau)\Delta t = \sum_{i=1}^{I} f_i^g(\tau)\Delta t + \tilde{g}_{I+1}^g(\tau)\Delta\tau - \tilde{g}_1^g(\tau)\Delta\tau = \sum_{i=1}^{I} f_i^g(\tau)\Delta t + 0 - 0. \tag{4.39}$$

So we have:

$$\sum_{i=1}^{I} f_i^g(\tau + \Delta\tau)\Delta t = \sum_{i=1}^{I} f_i^g(\tau)\Delta t = N_g. \tag{4.40}$$

Since N_g is conserved from day τ to day $\tau + \Delta\tau$, we have $N = \sum_g N_g$ conserved from day to day as well. It completes the proof for Theorem 4.2. □

Corollary 4.2.1. *The number of trips for each group is conserved on any day step* τ *in Equation* (4.23).

Proof. Since Equation (4.23) is equivalent to Equation (4.37), the number of trips for each group in Equation (4.23) is also conserved on any day step τ. □

4.3.4 Continuous version

Moving $f_i^g(\tau)\Delta t$ to the left-hand-side, and dividing both sides by $\Delta t \cdot \Delta\tau$, Equation (4.37) becomes:

$$\frac{f_i^g(\tau+\Delta\tau) - f_i^g(\tau)}{\Delta\tau} = \frac{\tilde{g}_{i+1}^g(\tau) - \tilde{g}_i^g(\tau)}{\Delta t}. \tag{4.41}$$

Let $f^g(\tau,t) = f^g(j\Delta\tau, i\Delta t)$, and $\tilde{g}^g(\tau,t) = \tilde{g}^g(j\Delta\tau, i\Delta t)$, we can re-write Equation (4.41) as follows:

$$\frac{f^g((j+1)\Delta\tau, i\Delta t) - f^g(j\Delta\tau, i\Delta t)}{\Delta\tau} = \frac{\tilde{g}^g(j\Delta\tau, (i+1)\Delta t) - \tilde{g}^g(j\Delta\tau, i\Delta t)}{\Delta t}. \tag{4.42}$$

Let $\Delta t \to 0$ and $\Delta\tau \to 0$, Equation (4.42) becomes the following conservation equation for group g in the continuous form:

$$\frac{\partial}{\partial\tau} f^g(\tau,t) - \frac{\partial}{\partial t}\tilde{g}^g(\tau,t) = 0, \tag{4.43}$$

where this conservation equation relates the rate of change in group g's departure rate $f^g(\tau,t)$ from day to day to group g's rate of change in net advance rate $\tilde{g}^g(\tau,t)$ from time to time within a given day.

To further simplify Equation (4.43), we analyze the property of $\tilde{g}^g(\tau, t)$. We have $\tilde{g}_i^g(\tau) = sgn(\omega_i^g(\tau)) \cdot \max\{f_{i-1}^{d,g}(\tau), f_i^{a,g}(\tau)\}$ from Equation (4.35). We obtain $f_i^{a,g}(\tau)$ from Equation (4.22). To obtain $f_{i-1}^{d,g}(\tau)$, we substitute $i-1$ in Equation (4.21) and get:

$$f_{i-1}^{d,g}(\tau) = B_i^{d,g}(\tau) \cdot \{-\omega_i^g(\tau)\}_+ \cdot f_{i-1}^g(\tau) \cdot \frac{\Delta t}{\Delta \tau}. \tag{4.44}$$

From Equation (4.22), we get $f_i^a(\tau)$ as follows:

$$f_i^{a,g}(\tau) = B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \cdot (f_i^g(\tau) \cdot \frac{\Delta t}{\Delta \tau} - f_i^{d,g}(\tau)). \tag{4.45}$$

Substituting $f_i^{d,g}(\tau)$ from Equation (4.21) into Equation (4.45), we get $f_i^{a,g}(\tau)$ as follows:

$$f_i^{a,g}(\tau) = B_i^{a,g}(\tau) \cdot \{\omega_i^g(\tau)\}_+ \cdot \left[(1 - B_{i+1}^{d,g}(\tau) \cdot \{-\omega_{i+1}^g(\tau)\}_+) \cdot f_i^g(\tau) \cdot \frac{\Delta t}{\Delta \tau}\right]. \tag{4.46}$$

Let $\tau = j\Delta\tau$, and $t = i\Delta t$. We can expand Equation (4.44) as follows:

$$f^{d,g}(j\Delta\tau, (i-1)\Delta t) = B^{d,g}(j\Delta\tau, i\Delta t) \cdot \{-\omega^g(j\Delta\tau, i\Delta t)\}_+ \cdot f^g(j\Delta\tau, (i-1)\Delta t) \cdot \frac{\Delta t}{\Delta \tau}. \tag{4.47}$$

Let $\Delta\tau \to 0$, and $\Delta t \to 0$, Equation (4.47) becomes:

$$f^{d,g}(\tau, t^-) = B^{d,g}(\tau, t) \cdot \{-\omega^g(\tau, t)\}_+ \cdot f^g(\tau, t^-) \cdot \frac{\Delta t}{\Delta \tau}, \tag{4.48}$$

where $t^- = \lim_{\Delta t \to 0}(i-1)\Delta t$ and $\Delta t > 0$.

We can expand Equation (4.46) as follows:

$$\begin{aligned} f^{a,g}(j\Delta\tau, i\Delta t) = {} & B^{a,g}(j\Delta\tau, i\Delta t) \cdot \{\omega^g(j\Delta\tau, i\Delta t)\}_+ \\ & \cdot \left[(1 - B^{d,g}(j\Delta\tau, (i+1)\Delta t) \cdot \{-\omega^g(j\Delta\tau, (i+1)\Delta t)\}_+) \cdot f^g(j\Delta\tau, i\Delta t) \cdot \frac{\Delta t}{\Delta \tau}\right]. \end{aligned} \tag{4.49}$$

Let $\Delta\tau \to 0$, and $\Delta t \to 0$, Equation (4.49) becomes:

$$f^{a,g}(\tau,t) = B^{a,g}(\tau,t)\cdot\{\omega^g(\tau,t)\}_+\cdot(1-B^{d,g}(\tau,t^+)\cdot\{-\omega^g(\tau,t^+)\}_+)\cdot f^g(\tau,t)\cdot\frac{\Delta t}{\Delta\tau}, \quad (4.50)$$

where $t^+ = \lim_{\Delta t\to 0}(i+1)\Delta t$ and $\Delta t > 0$.

Comparing Equation (4.48) with Equation (4.50), the first two terms in the multiplication, $B^{a,g}(\tau,t)$, $B^{d,g}(\tau,t)$, and $\omega^g(\tau,t)$ share the same time variable t and day variable τ. However, the third term in Equation (4.48), $f^g(\tau,t^-)$, has time variable t^-, while the third term in Equation (4.50), $(1-B^{d,g}(\tau,t^+)\cdot\{-\omega^g(\tau,t^+)\}_+)$, has the time variable t^+.

Substituting Equation (4.48) and Equation (4.50) into Equation (4.35), and letting $\Delta\tau \to 0$, and $\Delta t \to 0$, we obtain:

$$\begin{aligned}\tilde{g}^g(\tau,t) = sgn(\omega^g(\tau,t))\cdot\max\Big\{&B^{d,g}(\tau,t)\cdot\{-\omega^g(\tau,t)\}_+\cdot f^g(\tau,t^-),\\ &B^{a,g}(\tau,t)\cdot\{\omega^g(\tau,t)\}_+\cdot(1-B^{d,g}(\tau,t^+)\cdot\{-\omega^g(\tau,t^+)\}_+)\cdot f^g(\tau,t)\Big\}\cdot\frac{\Delta t}{\Delta\tau}.\end{aligned} \quad (4.51)$$

Equation (4.51) can be written as follows:

$$\tilde{g}^g(\tau,t) = \frac{\Delta t}{\Delta\tau}\cdot\begin{cases} B^{a,g}(\tau,t)\cdot\omega^g(\tau,t)\cdot(1-B^{d,g}(\tau,t^+)\cdot\{-\omega^g(\tau,t^+)\}_+)\cdot f^g(\tau,t) & \text{for} \quad \omega^g(\tau,t)\geq 0,\\ B^{d,g}(\tau,t)\cdot\omega^g(\tau,t)\cdot f^g(\tau,t^-) & \text{for} \quad \omega^g(\tau,t) < 0.\end{cases} \quad (4.52)$$

We make the following assumptions:

Assumption 4.3.1. *The departure flow rate for group g, $f^g(\tau,t)$, is left continuous for all time t. That is,* $\lim_{\Delta t\to 0} f^g(\tau,t-\Delta t) = f^g(\tau,t^-) = f^g(\tau,t)$ *for* $\Delta t > 0$.

Assumption 4.3.2. *The rate of change in trip cost function $\omega^g(\tau,t)$ is right continuous for all time t. However, when at MC-DTUE we assume that $\omega^g(\tau,t)$ is continuous from the* *right at the start of the departure period, $t^g_{0,i}$, and continuous from the left at the end of the departure period, $t^g_{2,i}$, and continuous in other times.*

Suppose the multi-class dynamical system reaches MC-DTUE on day $\tau^{'}$. The assumption above can be expressed as follows:

1. *When $\tau < \tau^{'}$, we have $\lim_{\Delta t \to 0} \omega^g(\tau, t + \Delta t) = \omega^g(\tau, t^+) = \omega^g(\tau, t)$ for $\Delta t > 0$ and $t \in [0, T]$.*

2. *When $\tau \geq \tau^{'}$, we have $\lim_{\Delta t \to 0} \omega^g(\tau, t^g_{0,i} + \Delta t) = \omega^g(\tau, t^{g,+}_{0,i}) = \omega^g(\tau, t^g_{0,i})$, and $\lim_{\Delta t \to 0} \omega^g(\tau, t^g_{2,i} - \Delta t) = \omega^g(\tau, t^{g,-}_{2,i}) = \omega^g(\tau, t^g_{2,i})$. For other times $t \in [0, T] \setminus \{t^g_{0,i}, t^g_{2,i}\}$, $\lim_{\Delta t \to 0} \omega^g(\tau, t + \Delta t) = \lim_{\Delta t \to 0} \omega^g(\tau, t - \Delta t) = \omega^g(\tau, t)$.*

Therefore, when $\tau < \tau^{'}$, given $f^g(\tau, t^-) = f^g(\tau, t)$ and $\omega^g(\tau, t^+) = \omega^g(\tau, t)$, Equation (4.52) becomes:

$$\tilde{g}^g(\tau, t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^{a,g}(\tau, t) \cdot \omega^g(\tau, t) \cdot (1 - B^{d,g}(\tau, t^+) \cdot \{-\omega^g(\tau, t)\}_+) \cdot f^g(\tau, t) & \text{for} \quad \omega^g(\tau, t) \geq 0 \\ B^{d,g}(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) & \text{for} \quad \omega^g(\tau, t) < 0 \end{cases} \tag{4.53}$$

If $\omega^g(\tau, t) \geq 0$, then $\{-\omega^g(\tau, t)\}_+ = 0$, so $1 - B^{d,g}(\tau, t^+) \cdot \{-\omega^g(\tau, t)\}_+ = 1 - 0 = 1$. So Equation (4.53) becomes the following equation:

$$\tilde{g}^g(\tau, t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^{a,g}(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) & \text{for} \quad \omega^g(\tau, t) \geq 0 \\ B^{d,g}(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) & \text{for} \quad \omega^g(\tau, t) < 0. \end{cases} \tag{4.54}$$

When $\tau \geq \tau^{'}$, the system reaches the MC-DTUE, $\omega^g(\tau, t) = 0$ for $t \in [t^g_{0,i}, t^g_{2,i}]$. As in Assumption 4.3.2, we consider that $\omega^g(\tau, t)$ is continuous from the right at $t = t^g_{0,i}$, continuous from the left at $t = t^g_{2,i}$, and continuous at other times.

When $\tau < \tau^{'}$, we derive Equation (4.54) based on the assumption that $\omega^g(\tau, t)$ is continuous from the right for $t \in [0, T]$. However, when $\tau \geq \tau^{'}$, $\omega^g(\tau, t)$ is continuous from left at $t^g_{2,i}$, and continuous from the right at other times. So we need to discuss $\omega^g(\tau, t^g_{2,i})$ at $t^g_{2,i}$ separately to see if Equation (4.54) still holds at $t^g_{2,i}$ at MC-DTUE.

We have $\omega^g(\tau, t^{g,+}_{2,i}) = \lim_{\Delta t \to 0} \frac{\phi^g(\tau, t^g_{2,i}+\Delta t) - \phi^g(\tau, t^g_{2,i})}{\Delta t} > 0$, because $\phi^g(\tau, t^g_{2,i}) = \phi^{g,*}$ while $\phi^g(\tau, t^{g,+}_{2,i}) > \phi^{g,*}$, and thus $\phi^g(\tau, t^{g,+}_{2,i}) - \phi^g(\tau, t^g_{2,i}) > 0$. So we have $-\omega^g(\tau, t^{g,+}_{2,i}) < 0$ and $\{-\omega^g(\tau, t^{g,+}_2)\}_+ = 0$. At points other than $t^{g,+}_2$ (i.e., $t \in [0, T] \setminus \{t^{g,+}_2\}$), we have $\omega^g(\tau, t^+) = \omega^g(\tau, t)$. So for $\omega^g(\tau, t) \geq 0$, $\{-\omega^g(\tau, t)\}_+ = 0$ and $1 - B^{d,g}(\tau, t^+) \cdot \{-\omega^g(\tau, t)\}_+ = 1 - 0 = 1$. Equation (4.52) still becomes Equation (4.54). Therefore, Equation (4.54) holds for $t^{g,+}_{2,i}$, and thus the entire time and day period.

Let

$$u^g(\tau, t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^{a,g}(\tau, t) & \text{for} \quad \omega^g(\tau, t) \geq 0, \\ B^{d,g}(\tau, t) & \text{for} \quad \omega^g(\tau, t) < 0, \end{cases} \tag{4.55}$$

so we have $\tilde{g}^g(\tau, t) = u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t)$ from Equation (4.54).

So Equation (4.43) becomes:

$$\frac{\partial}{\partial \tau} f^g(\tau, t) - \frac{\partial}{\partial t} u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) = 0. \tag{4.56}$$

Integrating both terms of Equation (4.56) with respect to time t, we have:

$$\frac{\partial}{\partial \tau} \int f^g(\tau, t) dt - \int \frac{\partial}{\partial t} u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) dt = 0, \tag{4.57}$$

and Equation (4.57) becomes:

$$\frac{\partial}{\partial \tau} F^g(\tau, t) - u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) = 0. \tag{4.58}$$

Since $f^g(\tau,t) = \frac{\partial}{\partial t} F^g(\tau,t)$, Equation (4.58) becomes:

$$\frac{\partial}{\partial \tau} F^g(\tau,t) - u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot \frac{\partial}{\partial t} F^g(\tau,t) = 0, \tag{4.59}$$

which is a hyperbolic conservation equation with respect to the cumulative departure flow for group g, $F^g(\tau,t)$, and $-u^g(\tau,t)\cdot\omega^g(\tau,t)$ is the speed of the characteristic wave for group g's cumulative departure flow for the multi-class dynamical system.

We discuss the property of $u^g(\tau,t)$ and $\omega^g(\tau,t)$ in the following paragraphs to further investigate the property of Equations (4.56) and (4.59).

We first discuss $\omega^g(\tau,t)$. From Equation (4.15), we have $\omega^g(\tau,t) = (\lambda_g - \mu_g)\cdot \frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu_g$ for early arrivals, and $\omega^g(\tau,t) = (\lambda_g + \nu_g)\cdot \frac{\partial}{\partial t}\Upsilon^L(\tau,t) + \nu_g$ for late arrivals.

We consider the following two cases for $\delta(\tau,t) = 0$ and $\delta(\tau,t) > 0$.

1. If the bottleneck is uncongested, that is $\delta(\tau,t) = 0$, then $\frac{\partial}{\partial t}\Upsilon(\tau,t) = 0$. Since $\omega^g(\tau,t) = (\lambda_g - \mu_g)\cdot \frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu_g$ for early arrivals and $\omega^g(\tau,t) = (\lambda_g + \mu_g)\cdot \frac{\partial}{\partial t}\Upsilon^L(\tau,t) + \nu_g$ for late arrivals, we have $\omega^g(\tau,t) = -\mu_g$ for early arrivals, and $\omega^g(\tau,t) = \nu_g$ for late arrivals.

   When $\delta(\tau,t) = 0$, we have:

$$-u^g(\tau,t)\cdot\omega^g(\tau,t) = \begin{cases} \mu_g \cdot u^g(\tau,t) & \text{for early arrivals,} \\ -\nu_g \cdot u^g(\tau,t) & \text{for late arrivals.} \end{cases} \tag{4.60}$$

2. If the bottleneck is congested, that is $\delta(\tau,t) > 0$, then $\frac{\partial}{\partial t}\Upsilon(\tau,t) = \frac{f(\tau,t)}{C} - 1$. So we have $\omega^g(\tau,t) = \frac{\lambda_g - \mu_g}{C}\cdot f(\tau,t) - \lambda_g$ for early arrivals and $\omega^g(\tau,t) = \frac{\lambda_g + \nu_g}{C}\cdot f(\tau,t) - \lambda_g$ for late arrivals.

From Equation (4.19), we get

$$\alpha^g(\tau,t) = \begin{cases} \frac{\lambda_g-\mu_g}{C} & \text{for early arrivals,} \\ \frac{\lambda_g+\nu_g}{C} & \text{for late arrivals,} \end{cases}$$

but when we assume that $\lambda_g + \nu_g < C$, we have $\alpha^g(\tau,t) < 1$.

So we have $\omega^g(\tau,t) = \alpha^g(\tau,t) \cdot f(\tau,t) - \lambda_g$, when $\delta(\tau,t) > 0$.

When $\delta(\tau,t) > 0$, we see that:

$$-u^g(\tau,t) \cdot \omega^g(\tau,t) = -u^g(\tau,t) \cdot (\alpha^g(\tau,t) \cdot f(\tau,t) - \lambda_g). \tag{4.61}$$

We now discuss the property of $u^g(\tau,t)$. From Equation (4.55), we have:

$$u^g(\tau,t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^{a,g}(\tau,t) & \text{for} \quad \omega^g(\tau,t) \geq 0, \\ B^{d,g}(\tau,t) & \text{for} \quad \omega^g(\tau,t) < 0. \end{cases}$$

Since $B^{a,g}(\tau,t)$ and $B^{d,g}(\tau,t)$ are positive advance and deferral coefficients, $u^g(\tau,t)$ is also positive. From the well-definedness condition (Theorem 4.1), we know that $0 < B_i^{a,g}(\tau) \leq 1/\max\{\nu_g, (\lambda_g - \mu_g) \cdot \frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g) \cdot \frac{f_2^m(\tau)}{C} - \lambda_g\}$ and $0 < B_{i+1}^{d,g}(\tau) \leq 1/\lambda_g$ for each group g at time step i on day τ. So we have:

$$0 < u^g(\tau,t) \leq \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} 1/\max\left\{\nu_g,\ (\lambda_g - \mu_g) \cdot \frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g) \cdot \frac{f_2^m(\tau)}{C} - \lambda_g\right\} \\ \qquad\qquad\qquad\qquad\qquad \text{for } \omega^g(\tau,t) \geq 0, \\ 1/\lambda_g \\ \qquad\qquad\qquad\qquad\qquad \text{for } \omega^g(\tau,t) < 0 \end{cases} \tag{4.62}$$

4.4 Stationary state

Definition 4.3. *The stationary state of the multi-class dynamical system in Equation (4.56) (discrete version: Equations (4.21) to (4.23)) is reached when* $\frac{\partial}{\partial \tau} f^g(\tau, t) = 0$*, which means that the time-dependent departure rate for each group* $f^g(\tau, t)$ *does not change from day to day.*

When $\frac{\partial}{\partial \tau} f^g(\tau, t) = 0$, we have $\frac{\partial}{\partial \tau} F^g(\tau, t) = 0$ as well. From Equation (4.58), we have $u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) = 0$. Since $u^g(\tau, t) > 0$ from Equation (4.62), we have the following complementarity condition for $f^g(\tau, t)$ and $\omega^g(\tau, t)$ for each group g:

$$f^g(\tau, t) \cdot \omega^g(\tau, t) = 0, \tag{4.63}$$

for all $t \in [0, T]$.

4.5 Stability of the stationary state

We consider the stability in both uncongested and congested cases below.

1. When the system is uncongested (i.e., $\delta(\tau, t) = 0$), substituting Equation (4.60) into Equation (4.59) yields:

   $$\frac{\partial}{\partial \tau} F^g(\tau, t) + \mu_g \cdot u^g(\tau, t) \cdot \frac{\partial}{\partial t} F^g(\tau, t) = 0 \tag{4.64}$$

   for early arrivals, and

   $$\frac{\partial}{\partial \tau} F^g(\tau, t) - \nu_g \cdot u^g(\tau, t) \cdot \frac{\partial}{\partial t} F^g(\tau, t) = 0 \tag{4.65}$$

for late arrivals.

Given $u^g(\tau, t) > 0$, the characteristic wave speed for early arrival travelers in multi-class dynamical system Equation (4.64) is $\mu_g \cdot u^g(\tau, t) > 0$. It means from day-to-day, group g's early arrival travelers in group g under uncongested conditions will shift their departure times towards later time steps.

For late arrival travelers, the characteristic wave for dynamical system Equation (4.65) is $-\nu_g \cdot u^g(\tau, t) < 0$. It means from day-to-day, group g's late arrival travelers under uncongested conditions will shift their departure times towards early time steps.

So the cumulative departure flow, $F^g(\tau, t)$, under uncongested conditions will shift towards the middle of the study period from both sides, and so will the departure flow rate $f^g(\tau, t)$. This renders the dynamical system Equations (4.56) and (4.59) unstable under uncongested conditions.

2. Here we consider the stability under the congested state (i.e., $\delta(\tau, t) > 0$), which is more complicated than the congested state in single-class dynamical system in Chapter 3. Substituting Equation (4.61) into Equation (4.56) yields:

$$\frac{\partial}{\partial \tau} f^g(\tau, t) - \frac{\partial}{\partial t} u^g(\tau, t) \cdot (\alpha^g(\tau, t) \cdot f(\tau, t) - \lambda_g) \cdot f^g(\tau, t) = 0, \tag{4.66}$$

for $t \in [t_0, t_2]$. Here we assume that $\delta(\tau, t) > 0$ for $t \in [t_0, t_2]$, which means the congested period is from t_0 to t_2.

For MC-DTUE, the departure intervals for group g is as follows: $\bigcup_{i=1}^{\infty} [t^g_{0,i}, t^g_{2,i}]$ for $g \in G$, which means $f^g(\tau, t) > 0$ and $\phi^g(\tau, t) = \phi^{g,*}$ for $t \in \bigcup_{i=1}^{\infty} [t^g_{0,i}, t^g_{2,i}]$. We can now state the following theorem:

Theorem 4.3. *At MC-DTUE, there is no unused interval between the used departure intervals of different groups.*

Proof. We prove it by contradiction. We assume that there exist an unused interval between two used departure intervals. Without loss of generality, we assume to have two groups — group 1 and group 2. We have the first departure interval of group 1 as follows: $[t^1_{0,1}, t^1_{2,1}]$ (superscript representing group 1, and subscript $\{0, 1\}$ and $\{2, 1\}$ representing the start and end of the first departure interval), and we have the second departure interval of group 1 as follows: $[t^1_{0,2}, t^1_{2,2}]$. Group 2 only has one departure interval: $[t^2_{0,1}, t^2_{2,1}]$.

Based on the above, there will be an unused interval between group 1's departure interval and group 2's departure interval. For example, if this unused interval exists before the desired arrival time, $\exists[t^1_{2,1}, t^2_{0,1}]$ $(t^1_{2,1} < t^2_{0,1} < t^*)$, then we have $f(\tau, t) = 0$ within the unused interval (i.e., $t \in [t^1_{2,1}, t^2_{0,1}]$), while $f(\tau, t) > 0$ outside the unused interval (i.e., $t \in [t^1_{0,1}, t^1_{2,1}) \cup (t^2_{0,1}, t^2_{2,1}]$).

Since the bottleneck is uncongested in time interval $[t^1_{2,1}, t^2_{0,1}]$, the characteristic (day-to-day) waves in this time interval are described by the uncongested characteristic wave speed shown in Equations (4.64) and (4.65). The early-arrival travelers from group 1 will defer their departure time from day to day at the speed of $\mu_1 \cdot u^1(\tau, t) > 0$. So $t^1_{2,1}$ will eventually move towards $t^2_{0,1}$ and $t^2_{0,1} = t^1_{2,1}$ at MC-DTUE.

If the unused interval exists after the desired arrival time, say $f(\tau, t) = 0$ for $t \in [t^2_{2,1}, t^1_{0,2}]$, then the late-arrival travelers from group 1 will advance their departure times from day to day at the speed of $-\nu_1 \cdot u^1(\tau, t) < 0$. So $t^1_{0,2}$ will eventually move towards $t^2_{2,1}$, and $t^1_{0,2} = t^2_{2,1}$ at MC-DTUE.

Consequently, at MC-DTUE, there is no unused interval between used departure intervals of different groups. □

From Theorem 4.3, there is no unused interval between the used intervals of different groups. For groups g and $g^{'}$, which are adjacent in order of departure, we have one of the following relations: $t^g_{2,i} = t^{g^{'}}_{0,i}$ or $t^g_{0,i} = t^{g^{'}}_{2,i}$. It means that if two groups depart in adjacent order, then the end of the departure of one group is the start of departure of the other group.

Considering that all trips depart within $[t_0, t_2]$, we define a Lyapunov functional as follows:

$$V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \int_{t^g_{0,i}}^{t^g_{2,i}} (t - t^g_{0,i}) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt, \tag{4.67}$$

where $\mathbf{f}(t) = \begin{bmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^G(t) \end{bmatrix}$ is the vector that includes departure rates from all groups.

The discrete version of Equation (4.67) is as follows:

$$V(\mathbf{f}(\tau)) = \sum_g \sum_i \sum_{j\Delta t \in [t^g_{0,i}, t^g_{2,i}]} ((j - \frac{1}{2}) \cdot \Delta t - t^g_{0,i}) \cdot f^g_j(\tau) \cdot [\{-\omega^g_{j+1}(\tau)\}^2_+ + \{\omega^g_j(\tau)\}^2_+] \tag{4.68}$$

Equation (4.67) describes the following calculation: the Lyapunov functional first integrates $(t - t^g_{0,i}) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2$ over each group's departure intervals (e.g., $[t^g_{0,i}, t^g_{2,i}]$) at MC-DTUE, and then sums the integrals over all departure intervals (i.e., $\sum_i$ in Equation (4.67)) and all groups (i.e., $\sum_g$ in Equation (4.67)).

For $V(\mathbf{f}(\tau,\cdot))$ to be a Lyapunov functional of dynamical system Equation (4.66), we need to show $V(\mathbf{f}(\tau,\cdot))$ satisfies the following three conditions:

(a) $V(\mathbf{f}(\tau,\cdot)) \geq 0$, $\forall \tau$.

(b) $V(\mathbf{f}(\tau,\cdot)) = 0$ at the stationary state of the dynamical system Equation (4.66).

(c) $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) < 0$ at the non-stationary states of the dynamical system Equation (4.66), while $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = 0$ at the stationary state of the dynamical system Equation (4.66).

Theorem 4.4. *Equation (4.67) is the Lyapunov functional for the dynamical system in*

Equation (4.66) *with different* λ/μ*, same* μ/ν *and same desired arrival time* t^* *across all travelers (the first type of heterogeneity in Arnott et al. (1994)).*

Proof. We show how $V(\mathbf{f}(\tau,\cdot))$ from Equation (4.67) will satisfy the three conditions above.

Let $D_i^g(\tau) = \int_{t_{0,i}^g}^{t_{2,i}^g}(t - t_{0,i}^g)\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt$ representing the integral over group g's ith departure interval, where i could be the first or second.

(a) For condition (a), since $t - t_{0,i}^g \geq 0$, $f^g(\tau,t) \geq 0$ and $[\omega^g(\tau,t)]^2 \geq 0$, we have $D_i^g(\tau) = \int_{t_{0,i}^g}^{t_{2,i}^g}(t-t_{0,i}^g)\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt \geq 0$. Therefore, $V(\mathbf{f}(\tau,\cdot)) = \sum_g\sum_i D_i^g(\tau) \geq 0$.

(b) For condition (b), the stationary state for the multi-class day-to-day dynamical system is reached when:

$$\frac{\partial}{\partial\tau}f^g(\tau,t) = 0, \text{for} \quad g \in G. \tag{4.69}$$

From Equation (4.63), we have the complementarity condition $f^g(\tau,t)\cdot\omega^g(\tau,t) = 0$ at the stationary state. When $f^g(\tau,t) > 0$, $\omega^g(\tau,t) = 0$, so we have $[\omega^g(\tau,t)]^2 = 0$. When $f^g(\tau,t) = 0$, $\omega^g(\tau,t) \geq 0$, so we have $[\omega^g(\tau,t)]^2 \geq 0$. So the following complementarity condition also holds:

$$f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2 = 0, \tag{4.70}$$

for all $t \in [0,T]$.

So $V(\mathbf{f}(\tau,\cdot)) = \sum_g\sum_i\int_{t_{0,i}^g}^{t_{2,i}^g}(t - t_{0,i}^g)\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt = 0$ at the stationary state.

(c) For condition (c), we take the partial derivative of Equation (4.67) with respect to τ and we have:

$$\begin{aligned}\frac{\partial}{\partial\tau}V(\mathbf{f}(\tau,\cdot)) &= \frac{\partial}{\partial\tau}\sum_g\sum_i\int_{t^g_{0,i}}^{t^g_{2,i}}(t-t^g_{0,i})\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt\\ &= \sum_g\sum_i\frac{\partial}{\partial\tau}\int_{t^g_{0,i}}^{t^g_{2,i}}(t-t^g_{0,i})\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt\\ &= \sum_g\sum_i\int_{t^g_{0,i}}^{t^g_{2,i}}(t-t^g_{0,i})\cdot\frac{\partial}{\partial\tau}\left\{f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2\right\}dt.\end{aligned} \tag{4.71}$$

Let $A^g_i(\tau,t)=\frac{\partial}{\partial\tau}\left\{f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2\right\}$, where g represents group g, and i represents group g's ith departure interval (i.e., $i=1,2$). The group whose departures are closest to the desired arrival time has only one departure interval, while other groups has two departure intervals — one before the desired arrival time and one after the desired arrival time. So Equation (4.71) becomes:

$$\begin{aligned}\frac{\partial}{\partial\tau}V(\mathbf{f}(\tau,\cdot)) &= \sum_g\sum_i\int_{t^g_{0,i}}^{t^g_{2,i}}(t-t^g_{0,i})\cdot\frac{\partial}{\partial\tau}\left\{f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2\right\}dt\\ &= \sum_g\sum_i\int_{t^g_{0,i}}^{t^g_{2,i}}(t-t^g_{0,i})\cdot A^g_i(\tau,t)dt.\end{aligned} \tag{4.72}$$

For $A^g_i(\tau,t)$, we have:

$$\begin{aligned}A^g_i(\tau,t) &= \frac{\partial}{\partial\tau}\left\{f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2\right\}\\ &= \frac{\partial}{\partial\tau}f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2+2\cdot f^g(\tau,t)\cdot\omega^g(\tau,t)\cdot\frac{\partial}{\partial\tau}\omega^g(\tau,t).\end{aligned} \tag{4.73}$$

When $\delta(\tau,t)>0$, we have $\omega^g(\tau,t)=\alpha^g(\tau,t)\cdot f(\tau,t)-\lambda_g$ from Equations (4.15) and (4.16) with $\frac{\partial}{\partial t}\Upsilon(\tau,t)=\frac{f(\tau,t)}{C}-1$. From Equation (4.19), we know that $\alpha^g(\tau,t)$ does not change with respect to day variable τ, so we can write $\alpha^g(\tau,t)=\alpha^g(t)$. So we have $\frac{\partial}{\partial\tau}\omega^g(\tau,t)=\alpha^g(t)\cdot\frac{\partial}{\partial\tau}f(\tau,t)$. Substituting it into Equation (4.73), we have:

$$A_i^g(\tau, t) = \frac{\partial}{\partial \tau} f^g(\tau, t) \cdot [\omega^g(\tau, t)]^2 + 2 \cdot f^g(\tau, t) \cdot \omega^g(\tau, t) \cdot \alpha^g(t) \cdot \frac{\partial}{\partial \tau} f(\tau, t) \quad (4.74)$$

Notice here that for Equation (4.74) we have $\frac{\partial}{\partial \tau} f^g(\tau, t)$ in the first term and $\frac{\partial}{\partial \tau} f(\tau, t)$ in the second term, which is different from its single-class counterpart in Equation (3.74) with only $\frac{\partial}{\partial \tau} f(\tau, t)$.

Since we are interested in the stability of the stationary state, we need to show $f^g(\tau, t) = f(\tau, t)$ for $t \in [t_{0,i}^g, t_{2,i}^g]$ in the stationary state to proceed with the proof. It means that in the departure interval for group g, $[t_{0,i}^g, t_{2,i}^g]$, all departures come from group g, and there is no other groups in group g's departure interval. That is, $f^g(\tau, t) = f(\tau, t)$ for $t \in [t_{0,i}^g, t_{2,i}^g]$, while $f^{g'}(\tau, t) = 0$ for $g' \in G \setminus \{g\}$ $t \in [t_{0,i}^g, t_{2,i}^g]$. So we need to prove the following theorem:

Theorem 4.5. *With different λ/μ, same μ/ν and same desired arrival time t^* across travelers (i.e., the first type of heterogeneity in Arnott et al. (1994)), at the stationary state of the multi-class dynamical system Equation (4.56) (i.e., $\frac{\partial}{\partial \tau} f^g(\tau, t) = 0$ for all groups $g \in G$), group g's departure interval only has one group, and that is group g. That is, $f^g(\tau, t) = f(\tau, t)$ for $t \in [t_{0,i}^g, t_{2,i}^g]$, while $f^{g'}(\tau, t) = 0$ for $g' \in G \setminus \{g\}$ and $t \in [t_{0,i}^g, t_{2,i}^g]$.*

Proof. We prove it by contradiction. We assume that at the stationary state of the multi-class dynamical system, there are two different groups in group g's departure time interval — group g and group g'. That is, for $t \in [t_{0,i}^g, t_{2,i}^g]$, we have $0 < f^g(\tau, t) < f(\tau, t)$ and $0 < f^{g'}(\tau, t) < f(\tau, t)$, where $f^g(\tau, t) + f^{g'}(\tau, t) = f(\tau, t)$. From different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)), we know that the attributes for two groups are different: $\mu_g/\lambda_g \neq \mu_{g'}/\lambda_{g'}$ and $\nu_g/\lambda_g \neq \nu_{g'}/\lambda_{g'}$.

From Equation (4.61), when $\delta(\tau, t) > 0$, we have the following characteristic wave:

$-u^g(\tau,t)\cdot\omega^g(\tau,t) = -u^g(\tau,t)\cdot(\alpha^g(t)\cdot f(\tau,t)-\lambda_g)$. Since it is a stationary state, the characteristic wave speed is 0. So we have $-u^g(\tau,t)\cdot\omega^g(\tau,t) = -u^g(\tau,t)\cdot(\alpha^g(t)\cdot f(\tau,t)-\lambda_g) = 0$. For group g, since $u^g(\tau,t) > 0$, we have $(\alpha^g(t)\cdot f(\tau,t)-\lambda_g) = 0$, and $f(\tau,t) = \frac{\lambda_g}{\alpha^g(t)}$. So we have the total flow $f(\tau,t) = \frac{\lambda_g}{\lambda_g-\mu_g}\cdot C$ for early arrivals and $f(\tau,t) = \frac{\lambda_g}{\lambda_g+\nu_g}\cdot C$ for late arrivals.

For the other group g', we have $-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t) = -u^{g'}(\tau,t)\cdot(\alpha^{g'}(t)\cdot f(\tau,t)-\lambda_{g'}) = 0$. Since $u^{g'}(\tau,t) > 0$, we have $(\alpha^{g'}(t)\cdot f(\tau,t)-\lambda_{g'}) = 0$, and $f(\tau,t) = \frac{\lambda_{g'}}{\alpha^{g'}(t)}$. So we have the total flow $f(\tau,t) = \frac{\lambda_{g'}}{\lambda_{g'}-\mu_{g'}}\cdot C$ for early arrivals and $f(\tau,t) = \frac{\lambda_{g'}}{\lambda_{g'}+\nu_{g'}}\cdot C$ for late arrivals.

For characteristic waves speed for both groups, $-u^g(\tau,t)\cdot\omega^g(\tau,t)$ and $-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t)$, exist in $[t^g_{0,i}, t^g_{2,i}]$, the total departure rate $f(\tau,t)$ during $[t^g_{0,i}, t^g_{2,i}]$ has to be $f(\tau,t) = \frac{\lambda_g}{\alpha^g(\tau,t)} = \frac{\lambda_{g'}}{\alpha^{g'}(\tau,t)}$ from $\alpha^g(t)\cdot f(\tau,t)-\lambda_g = 0$. Therefore, $\frac{\lambda_g}{\lambda_g-\mu_g} = \frac{\lambda_{g'}}{\lambda_{g'}-\mu_{g'}}$ for early arrivals, and that is $\frac{1}{1-\frac{\mu_g}{\lambda_g}} = \frac{1}{1-\frac{\mu_{g'}}{\lambda_{g'}}}$ for early arrivals, which means $\mu_g/\lambda_g = \mu_{g'}/\lambda_{g'}$. Likewise, we have $\nu_g/\lambda_g = \nu_{g'}/\lambda_{g'}$ for late arrivals, which contradicts our assumption that two groups are different: $\mu_g/\lambda_g \neq \mu_{g'}/\lambda_{g'}$ and $\nu_g/\lambda_g \neq \nu_{g'}/\lambda_{g'}$. So this completes the proof that at the stationary state, group g's departure interval only has one group, and that is group g, with different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)). □

From Theorem 4.5, we have $f^g(\tau,t) = f(\tau,t)$ at the stationary state when $\mu_g/\lambda_g \neq \mu_{g'}/\lambda_{g'}$ and $\nu_g/\lambda_g \neq \nu_{g'}/\lambda_{g'}$. So at the stationary state, Equation (4.74) becomes:

$$\begin{aligned} A^g_i(\tau,t) &= \frac{\partial}{\partial\tau}f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2 + 2\cdot f^g(\tau,t)\cdot\omega^g(\tau,t)\cdot\alpha^g(t)\cdot\frac{\partial}{\partial\tau}f^g(\tau,t) \\ &= \omega^g(\tau,t)\cdot[\omega^g(\tau,t) + 2\cdot f^g(\tau,t)\cdot\alpha^g(t)]\cdot\frac{\partial}{\partial\tau}f^g(\tau,t). \end{aligned} \tag{4.75}$$

At stationary state, we have $\omega^g(\tau,t) = \alpha^g(t)\cdot f(\tau,t)-\lambda_g = \alpha^g(t)\cdot f^g(\tau,t)-\lambda_g$.

Substituting $\omega^g(\tau,t) = \alpha^g(t) \cdot f^g(\tau,t) - \lambda_g$ into Equation (4.75), we have:

$$A_i^g(\tau,t) = (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot [3 \cdot \alpha^g(t) \cdot f^g(\tau,t) - \lambda_g] \cdot \frac{\partial}{\partial \tau} f^g(\tau,t). \tag{4.76}$$

From Equation (4.56), we have $\frac{\partial}{\partial \tau} f^g(\tau,t) = \frac{\partial}{\partial t} u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t)$. Substituting it into $\frac{\partial}{\partial \tau} f^g(\tau,t)$ of Equation (4.76), we have:

$$A_i^g(\tau,t) = (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot [3 \cdot \alpha^g(t) \cdot f^g(\tau,t) - \lambda_g] \cdot \frac{\partial}{\partial t} u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) \tag{4.77}$$

Now we need to make some assumptions on $u^g(\tau,t)$ to be able to move forward with the proof. We know from Equation (4.55) that $u^g(\tau,t) > 0$. Now we assume $u(\tau,t)$ has the following form:

$$u^g(\tau,t) = u_0 \cdot \frac{3\omega^g(\tau,t) + 2\lambda_g}{f^g(\tau,t)}, \tag{4.78}$$

where u_0 is a positive coefficient, which applies to all the groups.

When the bottleneck is uncongested, $\omega^g(\tau,t) = -\mu_g$ for early arrivals, and $\omega^g(\tau,t) = \nu_g$ for late arrivals. So we have $u^g(\tau,t) = u_0 \cdot \frac{-3\mu_g + 2\lambda_g}{f^g(\tau,t)}$ for early arrivals, and $u^g(\tau,t) = u_0 \cdot \frac{3\nu_g + 2\lambda_g}{f^g(\tau,t)}$ for late arrivals.

When the bottleneck is congested, then we have $\omega^g(\tau,t) = \alpha^g(t) \cdot f^g(\tau,t) - \lambda_g$ at stationary state, then Equation (4.78) becomes as follows:

$$u^g(\tau,t) = u_0 \cdot \frac{3\omega^g(\tau,t) + 2\lambda_g}{f^g(\tau,t)} = u_0 \cdot \frac{3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g}{f^g(\tau,t)}. \tag{4.79}$$

where $3 \cdot \alpha^g(t) \cdot f^g(\tau,t) - \lambda_g > 0$ and that is, $f^g(\tau,t) > \frac{\lambda_g}{3\alpha^g(t)}$.

Substituting Equation (4.79) into Equation (4.77) with $\omega^g(\tau,t) = \alpha^g(t) \cdot f^g(\tau,t) -$

λ_g and crossing out $f^g(\tau,t)$ in the denominator and nominator, we have:

$$\begin{aligned}A_i^g(\tau,t) &= u_0 \cdot (\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot[3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g]\\ &\cdot \frac{\partial}{\partial t}\left\{\frac{[3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g]}{f^g(\tau,t)}\cdot(\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot f^g(\tau,t)\right\}\\ &= u_0 \cdot (\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot[3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g]\\ &\cdot \frac{\partial}{\partial t}(3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot(\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\\ &= \frac{u_0}{2}\cdot\frac{\partial}{\partial t}[(3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot(\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)]^2\end{aligned} \tag{4.80}$$

Substituting $A_i^g(\tau,t)$ from Equation (4.80) back to Equation (4.72), we have:

$$\begin{aligned}\frac{\partial}{\partial\tau}V(\mathbf{f}(\tau,\cdot)) &= \sum_g\sum_i\int_{t_{0,i}^g}^{t_{2,i}^g}(t-t_{0,i}^g)\cdot A_i^g(\tau,t)dt\\ &= \sum_g\sum_i\frac{u_0}{2}\cdot\int_{t_{0,i}^g}^{t_{2,i}^g}(t-t_{0,i}^g)\cdot\frac{\partial}{\partial t}[(3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot(\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)]^2dt.\end{aligned} \tag{4.81}$$

We have $D_i^g(\tau)=\int_{t_{0,i}^g}^{t_{2,i}^g}(t-t_{0,i}^g)\cdot f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2dt=\int_{t_{0,i}^g}^{t_{2,i}^g}(t-t_{0,i}^g)\cdot\frac{\partial}{\partial t}[(3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)\cdot(\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g)]^2dt$. So Equation (4.81) becomes:

$$\frac{\partial}{\partial\tau}V(\mathbf{f}(\tau,\cdot))=\sum_g\sum_i\frac{u_0}{2}\cdot D_i^g(\tau). \tag{4.82}$$

Integrating by parts, $D_i^g(\tau)$ becomes:

$$\begin{aligned} D_i^g(\tau) &= \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot \frac{\partial}{\partial t}[(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 dt \\ &= \Bigg\{ (t - t_{0,i}^g) \cdot [(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 \Big|_{t_{0,i}^g}^{t_{2,i}^g} \\ &- \int_{t_{0,i}^g}^{t_{2,i}^g} [(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 dt \Bigg\} \\ &= \Bigg\{ (t_{2,i}^g - t_{0,i}^g) \cdot [(3\alpha^g(t_{2,i}^g) \cdot f^g(\tau,t_{2,i}^g) - \lambda_g) \cdot (\alpha^g(t_{2,i}^g) \cdot f^g(\tau,t_{2,i}^g) - \lambda_g)]^2 - 0 \\ &- \int_{t_{0,i}^g}^{t_{2,i}^g} [(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 dt \Bigg\}, \end{aligned} \tag{4.83}$$

where $[(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 \geq 0$, so we know the last term of Equation (4.83), $-\int_{t_0}^{t_2}[(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 dt \leq 0$.

Assumption 4.5.1. *At the stationary state, we have $\omega^g(\tau', t_{2,i}^g) = 0$. We assume that before the multi-class dynamical system reaches its stationary state on day τ', the rate of change in trip cost at end of group g's ith departure period, $t_{2,i}^g$, $\omega(\tau, t_{2,i}^g)$, already becomes zero. That is, $\exists \varepsilon > 0$,*

$$\omega(\tau, t_{2,i}^g) = 0, g \in G, i = \{1,2\}, \forall \tau \geq \tau' - \varepsilon. \tag{4.84}$$

In the current Section 4.5, we study the stability of the stationary state. Assumption 4.5.1 means that $\omega(\tau, t_{2,i}^g) = 0$ before the multi-class dynamical system reaches its stationary state on day τ'.

Notice that $\omega^g(\tau', t) = 0$ for $t \in [t_{0,i}^g, t_{2,i}^g]$ at the stationary state on day τ', while $\omega^g(\tau, t_{2,i}^g) = 0$ for $\forall \tau \geq \tau' - \varepsilon$ (on some day before day τ').

With $\omega^g(\tau, t_{2,i}^g) = 0$, we have $\omega^g(\tau, t_{2,i}^g) = \alpha(t_{2,i}^g) \cdot f^g(\tau, t_{2,i}^g) - \lambda_g = 0$.

For $\tau \geq \tau^{'} - \varepsilon$, with $\alpha^g(t^g_{2,i}) \cdot f^g(\tau, t^g_{2,i}) - \lambda_g = 0$, the first term of Equation (4.83) becomes zero. So Equation (4.83) becomes as follows:

$$D^g_i(\tau) = -\left\{ \int_{t^g_{0,i}}^{t^g_{2,i}} [(3\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) \cdot (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)]^2 dt \right\} \leq 0. \tag{4.85}$$

From Equation (4.79), we have $(3 \cdot \alpha^g(t) \cdot f^g(\tau,t) - \lambda_g) > 0$. For $t \in [t^g_{0,i}, t^g_{2,i}]$, when $\tau^{'} - \varepsilon \leq \tau < \tau^{'}$, we have $[\omega^g(\tau,t)]^2 = (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)^2 > 0$, and when $\tau \geq \tau^{'}$, we have $[\omega^g(\tau,t)]^2 = (\alpha^g(t) \cdot f^g(\tau,t) - \lambda_g)^2 = 0$. So we have:

$$D^g_i(\tau) < 0, \forall \tau^{'} - \varepsilon \leq \tau < \tau^{'},$$

$$D^g_i(\tau) = 0, \forall \tau \geq \tau^{'}.$$

The Lyapunov functional $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D^g_i(\tau)$, so we have

$$\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D^g_i(\tau) < 0, \forall \tau^{'} - \varepsilon \leq \tau < \tau^{'},$$

$$\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D^g_i(\tau) = 0, \forall \tau \geq \tau^{'}.$$

That means, when the multi-class dynamical system reaches its stability region (i.e., $\tau^{'} - \varepsilon \leq \tau$), we have $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) < 0$ at non-stationary states, while $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = 0$ at stationary state.

So Equation (4.67) is a Lyapunov functional of the multi-class dynamical system in Equation (4.56) with different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)), and it is asymptotically stable at the stationary state.

□

4.6 CFL condition and characteristic wave speed

Definition 4.4. *For the numerical method of solving the multi-class day-to-day dynamical system (i.e., discrete version Equations (4.21) to (4.23)) to be well-defined, $\Delta\tau$ and Δt have to satisfy the following condition:*

$$\frac{\Delta\tau}{\Delta t} \leq \frac{1}{\max\limits_{\tau,t,g} |u^g(\tau,t) \cdot \omega^g(\tau,t)|}, \tag{4.86}$$

which resembles the CFL (Courant-Friedrichs-Lewy) condition for the Cell Transmission Model (Jin, 2018).

Please refer to the discussion in Definition 3.4 of Section 3.6 for the single-class dynamical system. It clarifies the similarities and differences of our analysis on the time-day space and the traditional traffic flow modeling on linear road space. In the multi-class dynamical system, the unit for $u^g(\tau,t) \cdot \omega^g(\tau,t)$ is hour/day (e.g., t/τ), and Equation (4.86) requires that the day step $\Delta\tau$ is sufficiently small, so that after $\Delta\tau$, even the fastest characteristic wave among all groups still stays in the adjacent time intervals to time interval t, $[t-\Delta t, t]$ or $[t, t+\Delta t]$.

So we need to find the $\max\limits_{\tau,t,g} |u^g(\tau,t)|$ and $\max\limits_{\tau,t,g} |\omega^g(\tau,t)|$ to further determine the well-definedness condition in Definition 4.4.

We discuss $\max\limits_{\tau,t,g} |u^g(\tau,t)|$ first. From Equation (4.79), we have $u^g(\tau,t) = u_0 \cdot \frac{3\omega^g(\tau,t)+2\lambda_g}{f^g(\tau,t)} = u_0 \cdot \frac{3\alpha^g(t)\cdot f^g(\tau,t)-\lambda_g}{f^g(\tau,t)} = u_0 \cdot (3\alpha^g(t) - \frac{\lambda_g}{f^g(\tau,t)})$. In order to find the maximum $\max\limits_{\tau,t} |u^g(\tau,t)|$, we need to find the maximum of $\max\limits_{\tau,t} |\omega^g(\tau,t)|$. From Equations (4.17) and (4.18), we know $\max\limits_{\tau,t} |\omega^{E,g}(\tau,t)| = (\lambda_g - \mu_g) \cdot \frac{f_1^m(\tau)}{C} - \lambda_g$ for early arrivals, and $\max\limits_{\tau,t} |\omega^{L,g}(\tau,t)| = (\lambda_g + \nu_g) \cdot \frac{f_2^m(\tau)}{C} - \lambda_g$ for late arrivals.

Therefore, we have:

$$\max_{\tau,t,g} |u^g(\tau,t)| = u_0 \cdot \max_g \left\{ \left(3 \cdot \frac{\lambda_g - \mu_g}{C} - \frac{\lambda_g}{f^g(\tau,t)}\right), \left(3 \cdot \frac{\lambda_g + \nu_g}{C} - \frac{\lambda_g}{f^g(\tau,t)}\right) \right\}. \tag{4.87}$$

Letting $\left(3 \cdot \frac{\lambda_m + \nu_m}{C} - \frac{\lambda_m}{f^m(\tau,t)}\right)$ be the maximum $\left(3 \cdot \frac{\lambda_g + \nu_g}{C} - \frac{\lambda_g}{f^g(\tau,t)}\right)$ among all groups, we have:

$$\begin{aligned} \max_{\tau,t,g} |u^g(\tau,t)| &= \max_g \left\{ u_0 \cdot \left(3 \cdot \frac{\lambda_g - \mu_g}{C} - \frac{\lambda_g}{f^g(\tau,t)}\right), u_0 \cdot \left(3 \cdot \frac{\lambda_g + \nu_g}{C} - \frac{\lambda_g}{f^g(\tau,t)}\right) \right\} \\ &= u_0 \cdot \left(3 \cdot \frac{\lambda_m + \nu_m}{C} - \frac{\lambda_m}{f^m(\tau,t)}\right). \end{aligned} \tag{4.88}$$

Since $f^m(\tau,t) > \frac{\lambda_m}{3 \cdot \alpha^g(t)} > 0$, we have $\frac{\lambda_m}{f^m(\tau,t)} > 0$, so Equation (4.88) becomes:

$$\max_{\tau,t,g} |u^g(\tau,t)| = u_0 \cdot \left(3 \cdot \frac{\lambda_m + \nu_m}{C} - \frac{\lambda_m}{f^m(\tau,t)}\right) \leq u_0 \cdot \left(3 \cdot \frac{\lambda_m + \nu_m}{C}\right), \tag{4.89}$$

where the "=" is reached when there is massive departure at a given time (i.e., $f^m(\tau,t) \to \infty$), and we have $\frac{\lambda_m}{f^m(\tau,t)} \to 0$, and $\max_{\tau,t,g} |u(\tau,t)| \to u_0 \cdot \left(3 \cdot \frac{\lambda_m + \nu_m}{C}\right)$. If we set $u_0 = \frac{C}{3 \cdot (\lambda_m + \nu_m)}$, then Equation (4.89) becomes:

$$\max_{\tau,t,g} |u^g(\tau,t)| \leq \frac{C}{3 \cdot (\lambda_m + \nu_m)} \cdot \left(3 \cdot \frac{\lambda_m + \nu_m}{C}\right) = 1, \tag{4.90}$$

which refers to $f^m(\tau,t) > \frac{\lambda_m}{3 \cdot \alpha^m(t)}$.

However, when $0 \leq f^m(\tau,t) \leq \frac{\lambda_m}{3 \cdot \alpha^m(t)}$ and the bottleneck is congested, $3 \cdot \alpha^m(t) \cdot f^m(\tau,t) - \lambda_m \leq 0$, and $u^m(\tau,t)$ might be negative. Given $u^g(\tau,t) = u_0 \cdot \frac{3\omega^g(\tau,t) + 2\lambda_g}{f^g(\tau,t)}$, to make sure that $u^g(\tau,t)$ is positive, we further set $u^g(\tau,t) = u_0 \cdot \frac{\{3\omega^g(\tau,t) + 2\lambda_g\}_+}{f^g(\tau,t)}$. However, as $f^g(\tau,t) \to 0$, $u^g(\tau,t) \to \infty$, but $u^g(\tau,t)$ cannot be unbounded. Otherwise, the day step $\Delta\tau$ approaches zero (i.e., $\Delta\tau \to 0$). So to make sure that $u^g(\tau,t)$ is bounded from above, we further set

$u^g(\tau,t)$ as follows :

$$u^g(\tau,t) = \min\{u_0 \cdot \frac{\{3\omega^g(\tau,t) + 2\lambda_g\}_+}{f^g(\tau,t)}, \max_{\tau,t,g} |u^g(\tau,t)|\} = \min\{u_0 \cdot \frac{\{3\omega^g(\tau,t) + 2\lambda_g\}_+}{f^g(\tau,t)}, 1\}, \tag{4.91}$$

to incorporate both $f^g(\tau,t) > \frac{\lambda_g}{3\cdot\alpha(t)}$ and $0 \le f(\tau,t) \le \frac{\lambda}{3\cdot\alpha(t)}$ cases, with $u_0 = \frac{C}{3\cdot(\lambda_m+\nu_m)}$. So Equation (4.91) applies to $f^g(\tau,t) \ge 0$, and we have $\max_{\tau,t,g} |u^g(\tau,t)| \le 1$.

We now discuss $\max_{\tau,t,g} |\omega^g(\tau,t)|$. In the uncongested case (i.e., $\delta(\tau,t) = 0$), $\max_{\tau,t,g} |\omega^g(\tau,t)| = \max_g\{\mu_g, \nu_g\} = \nu_m$ from Equations (4.15) and (4.16) with $\frac{\partial}{\partial t}\Upsilon^E(\tau,t) = 0$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t) = 0$, where $\nu_m = \max_g\{\nu_g\}$ is the maximum value for one extra hour late arrivals among all groups. In the congested case (i.e., $\delta(\tau,t) > 0$), we have $\max_{\tau,t,g} |\omega^g(\tau,t)| = \max_g\{\lambda_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\}$. Since we assume $\mu_g < \lambda_g$, combining both uncongested and congested cases, we have:

$$\max_{\tau,t,g} |\omega^g(\tau,t)| = \max_g\{\nu_g, \lambda_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\} \tag{4.92}$$

Theorem 4.6. *For the numerical method of solving the day-to-day dynamical system (i.e., discrete version Equations (4.21) to (4.23)) to be well-defined, $\Delta\tau$ and Δt have to satisfy the following condition:*

$$\begin{aligned}\frac{\Delta\tau}{\Delta t} &\le \frac{1}{\max_{\tau,t,g}|u^g(\tau,t)|} \cdot \frac{1}{\max_g\{\nu_g, \lambda_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\}} \\ &= \frac{1}{1} \cdot \frac{1}{\max_g\{\nu_g, \lambda_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\}}.\end{aligned} \tag{4.93}$$

So Equation (4.93) becomes the following condition:

$$\frac{\Delta\tau}{\Delta t} \le \frac{1}{\max_g\{\nu_g, \lambda_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau)}{C} - \lambda_g\}}, \tag{4.94}$$

where $u_0 = \frac{C}{3\cdot(\lambda_m+\nu_m)}$ *to offset the maximum* $\frac{\{3\omega^g(\tau,t)+2\lambda_g\}_+}{f^g(\tau,t)}$, $3\cdot\frac{\lambda_m+\nu_m}{C}$.

Given the assumption of $u^g(\tau,t) = \min\{\frac{C}{3\cdot(\lambda_g+\nu_g)}\cdot\frac{\{3\omega^g(\tau,t)+2\lambda_g\}_+}{f^g(\tau,t)},1\}$ from Equation (4.91) with $u_0 = \frac{C}{3\cdot(\lambda_m+\nu_m)}$ and $\alpha^g(\tau,t) = \frac{\lambda_g-\mu_g}{C}$ for early arrival and $\alpha^g(\tau,t) = \frac{\lambda_g+\nu_g}{C}$ for late arrivals, we further analyze the characteristic wave speed of the dynamical system in Equation (4.59), $-u^g(\tau,t)\cdot\omega^g(\tau,t)$. We analyze the following two cases: $\delta(\tau,t)=0$ and $\delta(\tau,t)>0$.

When $\delta(\tau,t)=0$, $f(\tau,t)\leq C$, $\omega^g(\tau,t)=-\mu_g$ for early arrivals and $\omega^g(\tau,t)=\nu_g$ for late arrivals. So we have:

$$-u^g(\tau,t)\cdot\omega^g(\tau,t) = \begin{cases} \mu_g\cdot u^g(\tau,t) = \mu_g\cdot\min\{\frac{C}{3\cdot(\lambda_m+\nu_m)}\cdot\frac{\{3\omega^g(\tau,t)+2\lambda_g\}_+}{f^g(\tau,t)},1\} \\ \qquad\qquad\qquad\qquad\text{for early arrivals,} \\ -\nu_g\cdot u^g(\tau,t) = -\nu_g\cdot\min\{\frac{C}{3\cdot(\lambda_m+\nu_m)}\cdot\frac{\{3\omega^g(\tau,t)+2\lambda_g\}_+}{f^g(\tau,t)},1\} \\ \qquad\qquad\qquad\qquad\text{for late arrivals.} \end{cases} \tag{4.95}$$

So when the bottleneck is uncongested, the characteristic wave speed $-u^g(\tau,t)\cdot\omega^g(\tau,t)\in[-\nu_g,\mu_g]$, depending on group g's departure rate $f^g(\tau,t)$. For early arrivals, we have the following characteristic wave speed: $-u^g(\tau,t)\cdot\omega^g(\tau,t)=\mu_g$, which means travelers will shift their departure times to later time steps from day to day. For late arrivals, we have the following characteristic wave speed: $-u^g(\tau,t)\cdot\omega^g(\tau,t)=-\nu_g$, which means travelers will shift their departure times to earlier time steps from day to day. Therefore, when the bottleneck is uncongested, travelers will shift their departure times towards the desired arrival time.

When $\delta(\tau,t)>0$, we have:

$$\begin{aligned} -u^g(\tau,t)\cdot\omega^g(\tau,t) &= -u^g(\tau,t)\cdot(\alpha^g(t)\cdot f(\tau,t)-\lambda_g) \\ &= -\min\{\frac{C}{3\cdot(\lambda_m+\nu_m)}\cdot\frac{\{3\omega^g(\tau,t)+2\lambda_g\}_+}{f^g(\tau,t)},1\}\cdot(\alpha^g(t)\cdot f(\tau,t)-\lambda_g). \end{aligned} \tag{4.96}$$

4.7 Stationary state and MC-DTUE

Theorem 4.7. *The stationary state of the multi-class local dynamical system in Equation (4.56), (discrete version: Equations (4.21) to (4.23)) is equivalent to MC-DTUE under different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)).*

Proof. To prove Theorem 4.7, we first prove that MC-DTUE is the stationary state of the multi-class dynamical system, and we then prove that the stationary state under different λ/μ, same μ/ν and same t^* is MC-DTUE.

1. Given the system reaches the MC-DTUE, from Definition 4.1 of MC-DTUE, we have $\phi^g(\tau,t) = \phi^{g,*}$ when $f^g(\tau,t) > 0$, and $\phi^g(\tau,t) \geq \phi^{g,*}$ when $f^g(\tau,t) = 0$. Since $\omega^g(\tau,t) = \frac{\partial}{\partial t}\phi^g(\tau,t)$, we have $\omega^g(\tau,t) = 0$ for $f^g(\tau,t) > 0$ and $\omega^g(\tau,t) \geq 0$ for $f(\tau,t) = 0$. So we have the following complementarity condition:

   $$f^g(\tau,t) \cdot \omega^g(\tau,t) = 0. \tag{4.97}$$

   Since we have $f^g(\tau,t) \cdot \omega^g(\tau,t) = 0$ at MC-DTUE, then we have $u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) = 0$, so $\frac{\partial}{\partial t}u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) = 0$. From Equation (4.56), we have $\frac{\partial}{\partial \tau}f^g(\tau,t) = 0$. This proves the MC-DTUE is in the stationary state (i.e., $\frac{\partial}{\partial \tau}f^g(\tau,t) = 0$).

2. We then prove the stationary state is MC-DTUE. Given system reaches the stationary state, $\frac{\partial}{\partial \tau}f^g(\tau,t) = 0$, we have $\frac{\partial}{\partial \tau}F^g(\tau,t) = 0$ as well. From Equation (4.58), we have $u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) = 0$. Since $0 < u^g(\tau,t) \leq 1$, we have the following complementarity condition for each group as well: $f^g(\tau,t) \cdot \omega^g(\tau,t) = 0$.

   Now we need to analyze the characteristic wave speed of the stationary state to prove that the stationary state of the multi-class dynamical system under different λ/μ, same μ/ν and same t^* is MC-DTUE. Without loss of generality, we use g and g' to represent two groups.

We will first prove that the stationary state is a single congested period, and we will then show that the stationary state will lead to the departure order of MC-DTUE.

(a) When the bottleneck is uncongested (i.e., $\delta(\tau, t) = 0$), we have $\omega^g(\tau, t) = -\mu_g$ for early arrivals and $\omega^g(\tau, t) = \nu_g$ for late arrivals. So we have the the characteristic wave of group g is $-u^g(\tau, t) \cdot \omega^g(\tau, t) = \mu_g \cdot u^g(\tau, t) > 0$ for early arrivals and $-u^g(\tau, t) \cdot \omega^g(\tau, t) = -\nu_g \cdot u^g(\tau, t) < 0$ for late arrivals. It means that for early arrival travelers of all groups, they will shift their departure times backward towards the desired arrival time ($-u^g(\tau, t) \cdot \omega^g(\tau, t) > 0$), while for late arrival travelers of all groups, they will shift their departure times forward towards the desired arrival time ($-u^g(\tau, t) \cdot \omega^g(\tau, t) < 0$). It will create a single congested period where the characteristic wave speed is zero during the congested period, with positive characteristic wave speed before the congested period and negative characteristic wave speed after the congested period.

(b) Since there is a single congested period, we have $\delta(\tau, t) > 0$ for $t \in [t_0, t_2]$, where t_0 and t_2 are the start and end of the congestion period. Since $\delta(\tau, t) > 0$, we have $f(\tau, t) > 0$. Since the system reaches the stationary state, we have $f^g(\tau, t) = f(\tau, t)$, and $f^g(\tau, t) \cdot \omega^g(\tau, t) = 0$. So we have $\omega^g(\tau, t) = \alpha^g(t) \cdot f^g(\tau, t) - \lambda_g = 0$, and $f^g(\tau, t) = \frac{\lambda_g}{\alpha^g(t)}$. So we have $f^g(\tau, t) = \frac{\lambda_g}{\lambda_g - \mu_g} \cdot C$ for early arrivals, and $f^g(\tau, t) = \frac{\lambda_g}{\lambda_g + \nu_g} \cdot C$ for late arrivals.

Without loss of generality, we assume that $\lambda_{g'} > \lambda_g$, while $\mu_{g'} = \mu_g$ and $\nu_{g'} = \nu_g$, so $\lambda_{g'}/\mu_{g'} > \lambda_g/\mu_g$, and $\mu_{g'}/\nu_{g'} = \mu_g/\nu_g$. Under this assumption, group g' will depart first. We will show it later in this proof.

For the early arrival travelers in group g', the departure rate is: $f^{g',E}(\tau, t) =$

$\frac{\lambda_{g'}}{\lambda_{g'}-\mu_g} \cdot C$. For the early arrival travelers in group g, the departure rate is: $f^{g,E}(\tau,t) = \frac{\lambda_g}{\lambda_g-\mu_g} \cdot C$. Since $\lambda_{g'} > \lambda_g$, $\lambda_g > \mu_g$, and $\lambda_{g'} > \mu_{g'}$, we have $\frac{\lambda_{g'}}{\lambda_{g'}-\mu_g} - \frac{\lambda_g}{\lambda_g-\mu_g} = \frac{\mu_g \cdot (\lambda_g - \lambda_{g'})}{(\lambda_{g'}-\mu_g)\cdot(\lambda_g-\mu_g)} < 0$. Therefore, at the stationary state, we have

$$f^{g',E}(\tau,t) < f^{g,E}(\tau,t). \tag{4.98}$$

For the late arrival travelers in group $g^{'}$, the departure rate is: $f^{g',L}(\tau,t) = \frac{\lambda_{g'}}{\lambda_{g'}+\nu_g} \cdot C$. For the late arrival travelers in group g, the departure rate is: $f^{g,L}(\tau,t) = \frac{\lambda_g}{\lambda_g+\nu_g} \cdot C$. Since $\lambda_{g'} > \lambda_g$, $\lambda_g > \mu_g$, and $\lambda_{g'} > \mu_{g'}$, we have $\frac{\lambda_{g'}}{\lambda_{g'}+\nu_g} - \frac{\lambda_g}{\lambda_g+\nu_g} = \frac{\nu_g \cdot (\lambda_{g'} - \lambda_g)}{(\lambda_{g'}+\mu_g)\cdot(\lambda_g+\mu_g)} > 0$. Therefore, at the stationary state, we have

$$f^{g',L}(\tau,t) > f^{g,L}(\tau,t). \tag{4.99}$$

If group $g^{'}$ will depart first at the stationary state, then we will have uncongested characteristic wave $(-u^g(\tau,t) \cdot \omega^g(\tau,t) > 0)$ before the start of group $g^{'}$'s first departure period $t^{g'}_{0,1}$. So we have:

$$-u^{g'}(\tau,t) \cdot \omega^{g'}(\tau,t) = \mu_{g'} \cdot u^{g'}(\tau,t) > 0. \tag{4.100}$$

Then during group $g^{'}$'s departure period, $[t^{g'}_{0,1}, t^{g'}_{2,1}]$, we will have a characteristic wave speed of zero, so we have:

$$-u^{g'}(\tau,t) \cdot \omega^{g'}(\tau,t) = -u^{g'}(\tau,t) \cdot (\alpha^{g'}(t) \cdot f^{g',E}(\tau,t) - \lambda_{g'}) = 0. \tag{4.101}$$

After the end of group $g^{'}$'s first departure period, $t^{g'}_{2,1}$ and before its second departure period $t^{g'}_{0,2}$, (i.e., $t \in [t^{g'}_{2,1}, t^{g'}_{0,2}]$ or group g's departure period $t \in [t^{g}_{0,1}, t^{g'}_{2,1}]$), we have:

$$-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t) = -u^{g'}(\tau,t)\cdot(\alpha^{g'}(t)\cdot f^{g,E}(\tau,t)-\lambda_{g'}). \tag{4.102}$$

Notice here we have $f^{g,E}(\tau,t)$ in Equation (4.102) instead of $f^{g',E}(\tau,t)$, because group g starts to depart after $t^{g'}_{2,1}$. Since $f^{g',E}(\tau,t) < f^{g,E}(\tau,t)$, we have $0 = \alpha^{g'}(t)\cdot f^{g',E}(\tau,t)-\lambda_{g'} < \alpha^{g'}(t)\cdot f^{g,E}(\tau,t)-\lambda_{g'}$. Since $u^{g'}(\tau,t) > 0$, we have $-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t) = -u^{g'}(\tau,t)\cdot(\alpha^{g'}(t)\cdot f^{g,E}(\tau,t)-\lambda_{g'}) < 0$. So the characteristic wave speed after $t^{g'}_{2,1}$ is negative $(-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t))$.

In summary, for group g''s first departure period $[t^{g'}_{0,1}, t^{g'}_{2,1}]$. There is positive characteristic wave speed before $t^{g'}_{0,1}$, zero characteristic wave speed during $[t^{g'}_{0,1}, t^{g'}_{2,1}]$, and negative characteristic wave speed after $t^{g'}_{2,1}$. So group g's first departure period will be stationary within $[t^{g'}_{0,1}, t^{g'}_{2,1}]$.

For travelers in group g, during $[t^{g}_{0,1}, t^{g}_{2,1}]$, the bottleneck is congested, so we have:

$$-u^{g}(\tau,t)\cdot\omega^{g}(\tau,t) = -u^{g}(\tau,t)\cdot(\alpha^{g}(t)\cdot f^{g,E}(\tau,t)-\lambda_{g}) = 0, \tag{4.103}$$

for early arrivals, and

$$-u^{g}(\tau,t)\cdot\omega^{g}(\tau,t) = -u^{g}(\tau,t)\cdot(\alpha^{g}(t)\cdot f^{g,L}(\tau,t)-\lambda_{g}) = 0, \tag{4.104}$$

for late arrivals.

Before $t^{g}_{0,1}$, the bottleneck is still congested. Since $\alpha^{g}(t)\cdot f^{g,E}(\tau,t)-\lambda_{g} = 0$ and $f^{g',E}(\tau,t) < f^{g,E}(\tau,t)$, we have $\alpha^{g}(t)\cdot f^{g',E}(\tau,t)-\lambda_{g} < 0$, and thus,

$$-u^{g}(\tau,t)\cdot\omega^{g}(\tau,t) = -u^{g}(\tau,t)\cdot(\alpha^{g}(t)\cdot f^{g',E}(\tau,t)-\lambda_{g'}) > 0. \tag{4.105}$$

After $t^g_{2,1}$, the bottleneck is still congested. Since $\alpha^g(t) \cdot f^{g,L}(\tau,t) - \lambda_g = 0$ and $f^{g',L}(\tau,t) > f^{g,L}(\tau,t)$, so we have $\alpha^g(t) \cdot f^{g',L}(\tau,t) - \lambda_g > 0$, and thus,

$$-u^g(\tau,t) \cdot \omega^g(\tau,t) = -u^g(\tau,t) \cdot (\alpha^g(t) \cdot f^{g',L}(\tau,t) - \lambda_{g'}) < 0. \tag{4.106}$$

In summary, for group g's only departure period $[t^g_{0,1}, t^g_{2,1}]$ there is a positive characteristic wave speed before $t^g_{0,1}$, zero characteristic wave speed during $[t^g_{0,1}, t^g_{2,1}]$, and a negative characteristic wave speed after $t^g_{2,1}$. So group g's first departure period will be stationary within $[t^g_{0,1}, t^g_{2,1}]$.

For group g's second departure period, $[t^{g'}_{0,2}, t^{g'}_{2,2}]$, we will have a characteristic wave speed of zero, so we have:

$$-u^{g'}(\tau,t) \cdot \omega^{g'}(\tau,t) = -u^{g'}(\tau,t) \cdot (\alpha^{g'}(t) \cdot f^{g',L}(\tau,t) - \lambda_{g'}) = 0. \tag{4.107}$$

For $t < t^{g'}_{0,2}$, the bottleneck is still congested. Since $\alpha^g(t) \cdot f^{g',L}(\tau,t) - \lambda_g = 0$ and $f^{g,L}(\tau,t) < f^{g',L}(\tau,t)$ from Equation (4.99), we have $\alpha^g(t) \cdot f^{g,L}(\tau,t) - \lambda_g < 0$. So for $t < t^{g'}_{0,2}$, we have the following characteristic wave speed:

$$-u^{g'}(\tau,t) \cdot \omega^{g'}(\tau,t) = -u^{g'}(\tau,t) \cdot (\alpha^{g'}(t) \cdot f^{g,L}(\tau,t) - \lambda_{g'}) > 0. \tag{4.108}$$

For $t > t^{g'}_{2,2}$, the bottleneck is uncongested, and we have the following characteristic wave speed:

$$-u^{g'}(\tau,t) \cdot \omega^{g'}(\tau,t) = -\mu_{g'} \cdot u^{g'}(\tau,t) < 0. \tag{4.109}$$

In summary, for group $g^{'}$'s second departure period $[t^{g'}_{0,2}, t^{g'}_{2,2}]$. There is positive characteristic wave speed before $t^{g'}_{0,2}$, zero characteristic wave speed during $[t^{g'}_{0,2}, t^{g'}_{2,2}]$, and negative characteristic wave speed after $t^{g'}_{2,2}$. So group g's second departure period will be stationary within $[t^{g'}_{0,2}, t^{g'}_{2,2}]$.

So in conclusion, when our multi-class dynamical system reaches its stationary state, the characteristic wave speed around each group's departure period will have the following pattern: positive before the departure period, zero during the departure period, and negative after the departure period. Therefore, each group will stay in their departure period.

This pattern will happen when departure rate for each group is larger than the group departure rate on its left-hand side, and smaller than the group departure rate on its right-hand side, for both early and late departures. In other words, for early arrivals, the group with the smallest $f^{g,E}(\tau, t)$ will depart first, and then the group with the second smallest $f^{g,E}(\tau, t)$, and so on. For late arrivals, the group with the smallest $f^{g,L}(\tau, t)$ will depart first, and then the group with the second smallest $f^{g,L}(\tau, t)$, and so on.

Since $f^{g,E}(\tau, t) = \frac{\lambda_g}{\lambda_g - \mu_g} \cdot C = \frac{1}{1 - \frac{\mu_g}{\lambda_g}} \cdot C$, the smallest $f^{g,E}(\tau, t)$ has the smallest μ_g/λ_g. So the group with the largest λ_g/μ_g will depart first, and then the group with the second largest λ_g/μ_g, and so on. This answers why group g' will depart first for early arrivals with $\lambda_{g'}/\mu_{g'} > \lambda_g/\mu_g$.

Since $f^{g,L}(\tau, t) = \frac{\lambda_g}{\lambda_g + \nu_g} \cdot C = \frac{1}{1 + \frac{\nu_g}{\lambda_g}} \cdot C$, the smallest $f^{g,L}(\tau, t)$ has the largest ν_g/λ_g. So the group with the smallest λ_g/ν_g will depart first, and the group with the second smallest λ_g/ν_g and so on. This answers why group g' will depart later for last arrivals with $\lambda_{g'}/\nu_{g'} > \lambda_g/\nu_g$.

This proves that under different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)), the stationary state can lead to the departure order of MC-DTUE.

We prove that under different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)), MC-DTUE is the stationary state and that the stationary state is MC-DTUE. So this completes the proof of Theorem 4.7. □

4.8 Non-MC-DTUE stationary states

We prove in Section 4.7 that under different λ/μ, same μ/ν and same desired arrival time t^* across travelers (first type of heterogeneity in Arnott et al. (1994)), the stationary state of the multi-class dynamical system is equivalent to MC-DTUE. However, the stationary state under other types of heterogeneity (e.g., different t^*, same λ, μ, and ν across travelers) might not lead to an MC-DTUE. We briefly discuss the following two scenarios based on the distance between $t^{g,*}$ and $t^{g',*}$. We will show that the stationary state does not lead to the departure order of MC-DTUE under $|t^{g,*} - t^{g',*}| \cdot C \leq \frac{\mu}{\mu+\nu} \cdot N_g + \frac{\nu}{\mu+\nu} \cdot N_{g'}$.

1. If we have double peaks, that is $|\frac{\mu}{\mu+\nu} \cdot N_g - \frac{\nu}{\mu+\nu} \cdot N_{g'}| < |t^{g,*} - t^{g',*}| \cdot C \leq \frac{\mu}{\mu+\nu} \cdot N_g + \frac{\nu}{\mu+\nu} \cdot N_{g'}$. Suppose $t^{g',*} < t^{g,*}$, group $g^{'}$ will depart first. Let $[t_0, t_2]$ be the entire congested period. It is possible that group $g^{'}$'s travelers depart at both the beginning and the end of the congestion period, but have different costs for these two departure periods. It is not an MC-DTUE according to its definition in Definition 4.1, and is shown in the departure results in Figure 4.5 and the cost results in Figure 4.6.

2. If we have only one peak, that is $|t^{g,*} - t^{g',*}| \leq |\frac{\mu}{\mu+\nu} \cdot N_g - \frac{\nu}{\mu+\nu} \cdot N_{g'}|$. Suppose $t^{g',*} < t^{g,*}$, group $g^{'}$ will depart first. Let $[t_0, t_2]$ be the entire congested period. It is also possible that group $g^{'}$'s travelers depart at both the beginning and the end of the congestion period, but have different costs for these two departure periods. It is not an MC-DTUE according to its definition in Definition 4.1, and is shown in the departure results inFigure 4.9 and cost results in Figure 4.10.

It shows a limitation of the multi-class dynamical system, in that it cannot capture all kinds of heterogeneity.

4.9 Numerical examples

This section provides a numerical example of the multi-class day-to-day departure time dynamics.

4.9.1 Simulation setup

We consider the total number of travelers, $N = 3,600$ veh. We consider that there are two groups. Both groups have the same number of travelers: $N_1 = N_2 = 1,800$ veh. The capacity of the bottleneck is $C = 1,800$ veh/hr. We consider the following two scenarios: (1) different λ, μ but with the same μ/ν and same desired arrival time t^*, and (2) same λ, μ, ν but with different desired arrival times t^*.

4.9.2 Time and day step sizes and advance/deferral coefficients

To numerically model the day-to-day dynamical model in Equations (4.21) to (4.23), we need to decide $f_i^g(\tau)$, $\omega_i^g(\tau)$, $B_i^{a,g}(\tau)$, $B_i^{d,g}(\tau)$, Δt and $\Delta\tau$ for all time step i and day τ. We have the initial departure rate on day 0, $f^{g,0}(t)$, and $\omega_i^g(\tau)$ could be determined by the trip cost function from Equation (4.14). We still need to determine $B_i^{a,g}(\tau)$, $B_i^{d,g}(\tau)$, Δt and $\Delta\tau$.

We first discuss Δt and $\Delta\tau$. Δt step size could be set randomly, while $\Delta\tau$ is set according to the step size of Δt. Let τ_j be the jth day step, where $j = 0, 1, ..., J$. Let τ_0 be the initial day step, and we have $\tau_j = \tau_{j-1} + \Delta\tau_{j-1}$, for $i = 1, 2, ..., J$. Since we need to satisfy the well-definedness condition for numerical method, from Equation (4.94), we set the jth day step, $\Delta\tau_j$, to be the upper bound of this condition as follows:

$$\Delta\tau_j = \frac{\Delta t}{\max\limits_{g}\{\nu_g, \lambda_g, (\lambda_g - \mu_g)\cdot\frac{f_1^m(\tau_j)}{C} - \lambda_g, (\lambda_g + \nu_g)\cdot\frac{f_2^m(\tau_j)}{C} - \lambda_g\}}. \tag{4.110}$$

In this chapter, we set $\Delta t = 0.1$hr and total day step to be $5,000$, starting from day step 0, and ending with day step $5,000$.

Here we consider two sets of $B_i^{a,g}(\tau)$, $B_i^{d,g}(\tau)$ for each group g: like Chapter 3, the heuristic set derived from the well-definedness condition from Theorem 4.1, and the provably stable set derived from the relationship of $B_i^{a,g}(\tau)$, $B_i^{d,g}(\tau)$ and $u^g(\tau,t)$ in Equation (4.55), and the definition of $u^g(\tau,t)$ in Equation (4.91).

We set the heuristic set of $B_i^{a,g}(\tau_j)$, $B_i^{d,g}(\tau_j)$ for group g is as follows:

$$B_i^{d,g}(\tau_j) = \frac{1}{\lambda_g} \quad \text{and} \quad B_i^{a,g}(\tau_j) = \frac{0.1}{\max\{\nu_g, (\lambda_g - \mu_g) \cdot \frac{f_1^m(\tau_j)}{C} - \lambda_g, (\lambda_g + \nu_g) \cdot \frac{f_2^m(\tau_j)}{C} - \lambda_g\}}, \tag{4.111}$$

where $B_i^{d,g}(\tau_j)$ and $B_i^{a,g}(\tau_j)$ are asymmetric for group g.

We have the provably stable set of $B_i^{a,g}(\tau_j)$, $B_i^{d,g}(\tau_j)$ for group g as follows:

$$B_i^{d,g}(\tau_j) = B_i^{a,g}(\tau_j) = \frac{\Delta\tau_j}{\Delta t} \cdot \min\{\frac{C}{3 \cdot (\lambda_m + \nu_m)} \cdot \frac{\{3\omega^g(\tau,t) + 2\lambda_g\}_+}{f^g(\tau,t)}, 1\}, \tag{4.112}$$

where $B_i^{d,g}(\tau_j)$ and $B_i^{a,g}(\tau_j)$ are symmetric.

With numerical testing described below, we show results with asymmetric $B_i^{d,g}(\tau_j)$ and $B_i^{a,g}(\tau_j)$ from Equation (4.111) to drive the system into the stability region, followed by a switch to the provably stable set of $B_i^{d,g}(\tau_j)$ and $B_i^{a,g}(\tau_j)$ from Equation (4.112). We switch $B_i^{d,g}(\tau_j)$ and $B_i^{a,g}(\tau_j)$ for all groups at day step $2,500$.

4.9.3 Stability measurements

To measure stability, we calculate the following discrete version of Lyapunov function on day τ_j defined in Equation (4.68):

$$V(\mathbf{f}(\tau)) = \sum_g \sum_i \sum_{j\in[t^g_{0,i}, t^g_{2,i}]} ((j-\frac{1}{2})\cdot \Delta t - t^g_{0,i}) \cdot f^g_j(\tau) \cdot [\{-\omega^g_{j+1}(\tau)\}^2_+ + \{\omega^g_j(\tau)\}^2_+]$$

where the $V(\mathbf{f}(\tau_0)) > 0$. If as day step τ_j increases, $V(\mathbf{f}(\tau_j))$ decreases to zero, then it numerically shows that the day-to-day dynamical system is stable.

4.9.4 Results of scenario 1: different λ/μ, same μ/ν and t^*

We assume that $\lambda_1 = 75$ \$/hr and $\lambda_2 = 50$ \$/hr. We have $\frac{\mu_1}{\nu_1} = \frac{\mu_2}{\nu_2} = \frac{1}{4}$. We have $\mu_1 = \mu_2 = 25$ \$/hr, and $\nu_1 = \nu_2 = 100$ \$/hr. So $\frac{\lambda_1}{\mu_1} = \frac{75}{25} = 3$ for group 1, while $\frac{\lambda_2}{\mu_2} = \frac{50}{25} = 2$ for group 2. We consider the study period to be $[0, 6]$ hr, and the desired arrival time for all travelers $t^* = 4$ hr. We assume that the initial departure rate on day 0, $f^0(t)$, is as follows:

$$f^0(t) = \begin{cases} 0 & \text{for} \quad 0 \le t \le 2.4 \\ 1,800 & \text{for} \quad 2.4 < t \le 4.4 \\ 0 & \text{for} \quad 4.4 < t \le 6, \end{cases} \tag{4.113}$$

where group 1 and group 2's departure rate are the same and both equal to half of the total departure rate (e.g., $f^{1,0}(t) = f^{2,0}(t) = \frac{1}{2}\cdot f^0(t) = 900$ veh/hr for $2.4 < t \le 4.4$). Under the condition above, the MC-DTUE departure flow rate, $f^*(t)$, would be as follows:

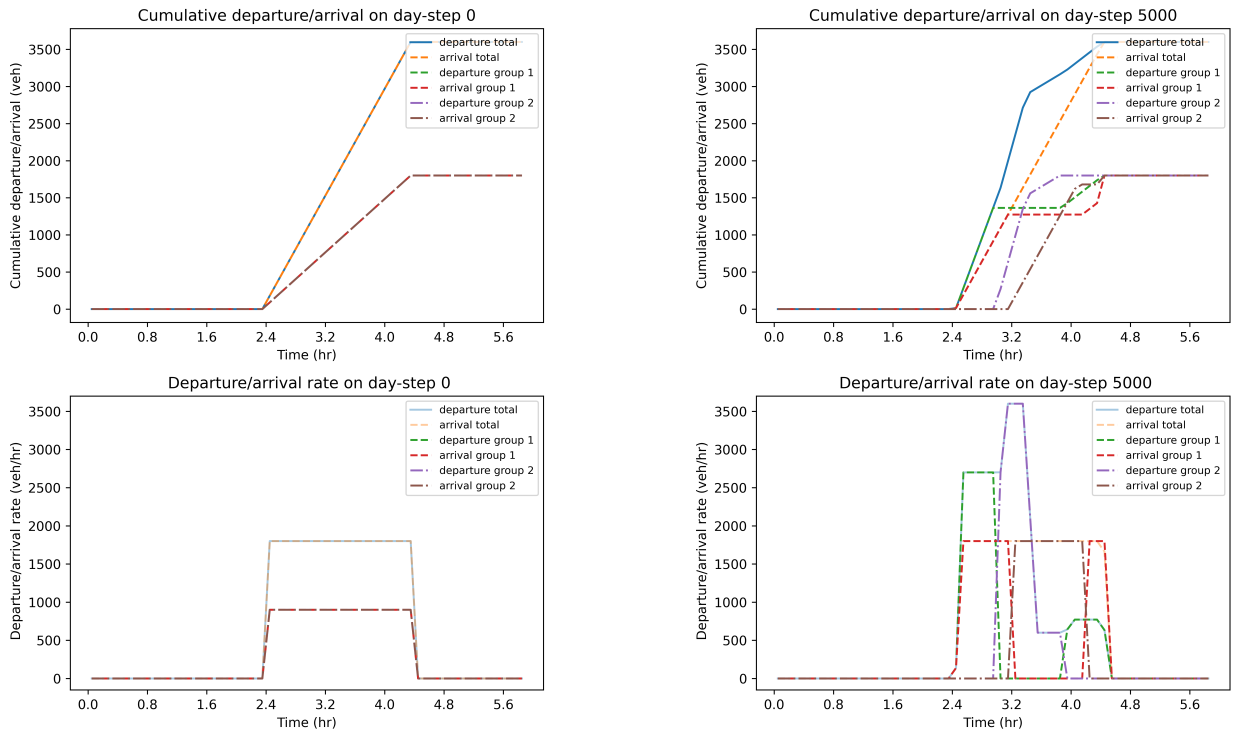
$$f^*(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4 \\ f^{1,E,*}(t) = \frac{\lambda_1}{\lambda_1 - \mu_1} \cdot C = \frac{3}{2} \cdot 1,800 = 2,700 & \text{for} \quad 2.4 < t \leq 2.93 \\ f^{2,E,*}(t) = \frac{\lambda_2}{\lambda_2 - \mu_2} \cdot C = 2 \cdot 1,800 = 3,600 & \text{for} \quad 2.93 < t \leq 3.33 \\ f^{2,L,*}(t) = \frac{\lambda_2}{\lambda_2 + \nu_2} \cdot C = \frac{1}{3} \cdot 1,800 = 600 & \text{for} \quad 3.33 < t \leq 3.93 \\ f^{1,L,*}(t) = \frac{\lambda_1}{\lambda_1 + \nu_1} \cdot C = \frac{3}{7} \cdot 1,800 = 771.4 & \text{for} \quad 3.93 < t \leq 4.4 \\ 0 & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{4.114}$$

where $f^{1,E,*}(t)$, $f^{1,L,*}(t)$, $f^{2,E,*}(t)$, and $f^{2,L,*}(t)$ are calculated from $\omega^g(\tau, t) = \alpha^g(t) \cdot f^g(\tau, t) - \lambda_g = 0$. The analytical solutions for $t^1_{0,1}$, $t^1_{2,1}$, $t^1_{0,2}$, and $t^1_{2,2}$ are complicated, so we just present the final results for these end points in Equation (4.114).

And at MC-DTUE, the cost function $\phi^*(t)$ will be as follows:

$$\phi^*(t) = \begin{cases} 25 \cdot (4 - t) & \text{for} \quad 0 \leq t \leq 2.4 \\ \phi^{1,*}(t) = \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N}{C} = 40 & \text{for} \quad 2.4 < t \leq 2.93 \\ \phi^{2,*}(t) = \frac{\mu_2 \cdot \nu_2}{\mu_2 + \nu_2} \cdot \frac{N_2}{C} + \frac{\lambda_2}{\lambda_1} \cdot \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N_1}{C} = 33.3 & \text{for} \quad 2.93 < t \leq 3.93 \\ \phi^{1,*}(t) = \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N}{C} = 40 & \text{for} \quad 3.93 < t \leq 4.4 \\ 100 \cdot (t - 4) & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{4.115}$$

We will show with the multi-class day-to-day dynamical system, that the initial departure rate $f^0(t)$ can converge to MC-DTUE departure rate $f^*(t)$.



(a) First day-step

(b) Last day-step

Figure 4.1: First and last day-step comparison on cumulative departure/arrival flow and departure/arrival rate (different λ/μ, same μ/ν and same t^*)

Figure 4.1 shows the departure and arrival pattern on the first day and last day-step. On the first day-step from Figure 4.1a, the total departure rate is equal to the total arrival rate, which is both equal to the capacity of the bottleneck. Each group's departure and arrival rate is equal to half of the capacity. On the last day-step from Figure 4.1b, travelers in group 1 shift their departure times to the shoulder of the peak (i.e., 2.4 hr - 2.9hr and 3.9 - 4.4 hr) while travelers in group 2 shift their departure times to the middle of the peak (i.e., 3 hr - 3.9hr). This is because $\frac{\lambda_1}{\mu_1} > \frac{\lambda_2}{\mu_2}$, which means that travelers in group 1 are less willing to endure the queuing cost than the unpunctuality cost compared to travelers in group 2.

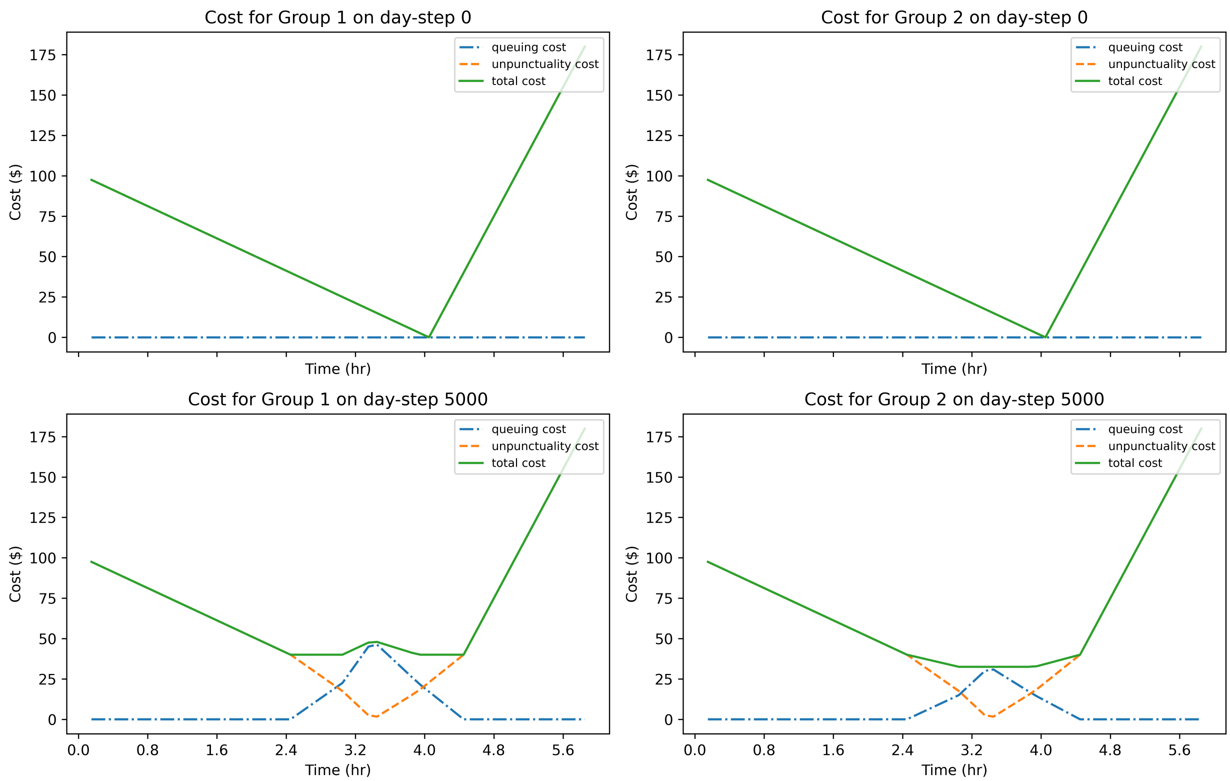


Figure 4.2: First and last day-step comparison for both groups on queuing, unpunctuality, and total costs (different λ/μ, same μ/ν and same t^*)

Figure 4.2 shows the cost comparison on the first day and last day-step. On the first day-step from the upper panel, there is no queue, so the queuing cost is zero, and the total cost is equal to the unpunctuality cost. On the last day-step from the lower panel, travelers in group 1 choose to depart at the shoulder of the peak (i.e., 2.4 hr - 2.9hr and 3.9 - 4.4 hr), which has the lowest total cost for group 1 — \$ 40. Travelers in group 2 choose to depart at the middle of the peak (i.e., 3 hr - 3.9hr), which has the lowest total cost for group 2 — \$ 32.5. The results are aligned with the MC-DTUE cost in Equation (4.115), but not exactly the same, because of the time step in this study is 0.1 hr, while the end points in the analytical solution are two digits after the decimal point (e.g, 2.93 and 3.93).

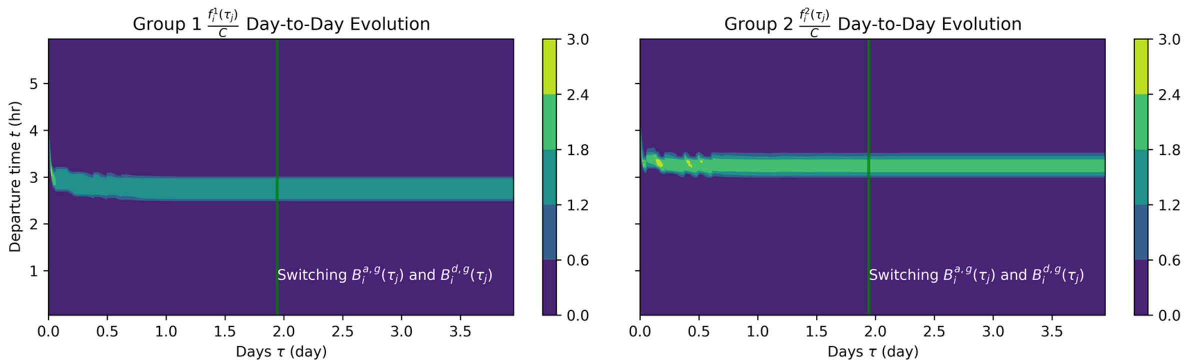


Figure 4.3: Normalized departure rate day-to-day evolution for both groups (different λ/μ, same μ/ν and same t^*)

Figure 4.3 shows the day-to-day evolution of the normalized departure rate for both groups, which shows a stable convergence pattern. Figure 4.4 shows the Lyapunov functional of the multi-class dynamical system, which converges to 10^{-3} as from day to day, especially at a faster speed after switching to provably stable set of $B_i^{a,g}(\tau_j)$, $B_i^{d,g}(\tau_j)$.

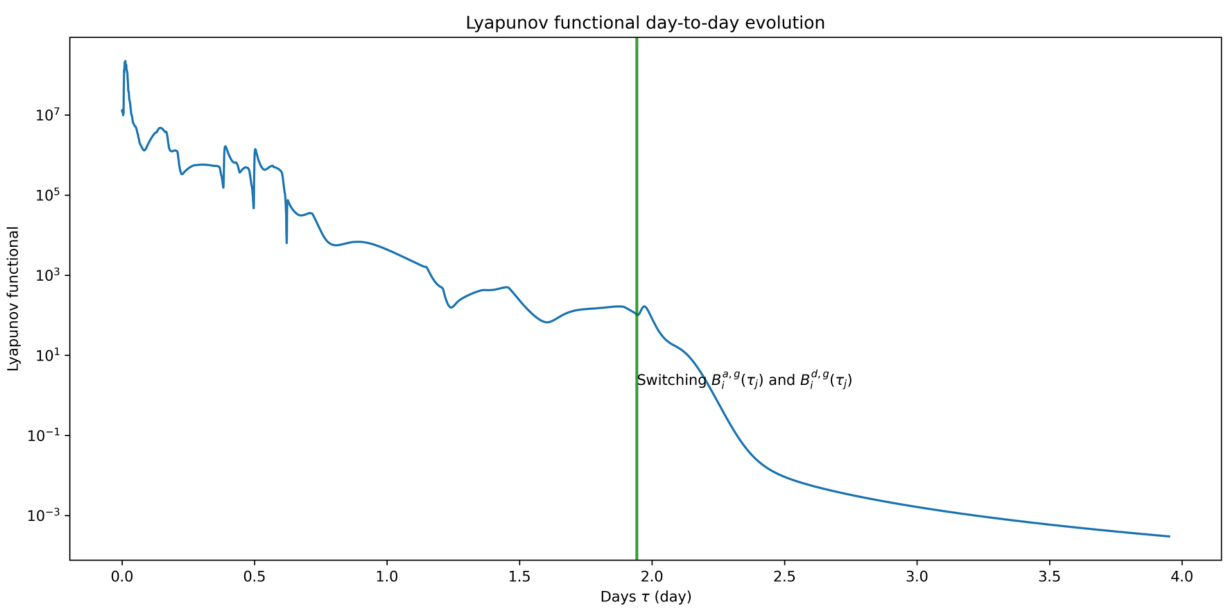


Figure 4.4: Multi-class Lyapunov functional day-to-day evolution (different λ/μ, same μ/ν and same t^*)

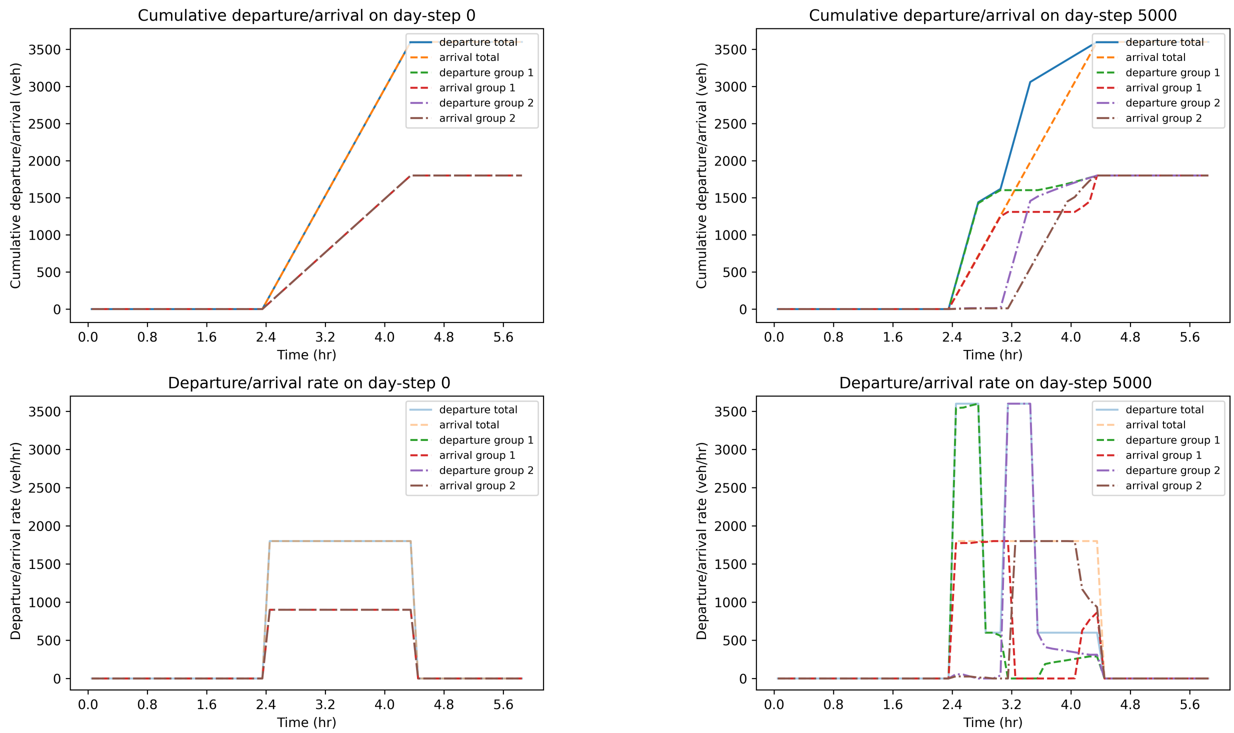
Overall, the results show that the multi-class dynamical system converges to a stable (Lyapunov functional goes to zero) stationary MC-DTUE state (the stationary state departure pattern is MC-DTUE's departure pattern) for different λ/μ, same μ/ν, and same t^*.

4.9.5 Results of scenario 2: different t^*, same λ, μ, and ν

In the following double-peaks and single-peak cases, we switch to provably stable $B_i^{a,g}(\tau_j)$ and $B_{i+1}^{d,g}(\tau_j)$ at day step 4,000.

Double peaks

If we have double peaks, that is $|\frac{\mu}{\mu+\nu} \cdot N_g - \frac{\nu}{\mu+\nu} \cdot N_{g'}| < |t^{g,*} - t^{g',*}| \cdot C \leq \frac{\mu}{\mu+\nu} \cdot N_g + \frac{\nu}{\mu+\nu} \cdot N_{g'}$. In our case, if $0.6 < |t^{1,*} - t^{2,*}| \leq 1$, then we will have double peaks. In this case scenario, we set $t^{1,*} = 3.1$ hr, $t^{2,*} = 4$ hr, while λ, μ, and ν are same for both groups, and that is: $\lambda_1 = \lambda_2 = 50$ \$/hr, $\mu_1 = \mu_2 = 25$ \$/hr, and $\nu_1 = \nu_2 = 100$ \$/hr.



(a) First day-step

(b) Last day-step

Figure 4.5: First and last day-step comparison on cumulative departure/arrival flow and departure/arrival rate (different t^*, same λ, μ, and ν - double peaks)

Figure 4.5 shows the departure and arrival patterns for the first and the last day-step. The cumulative total departure and arrival curve from upper panel of Figure 4.5b shows that there are double peaks. Travelers in group 1 depart from 2.4hr - 3hr and from 3.6 hr - 4.3 hr, while the majority of group 2's travelers depart from 3hr - 4.3 hr, with a few departing from 2.4 hr - 2.6 hr. Comparing the last day's departure pattern in Figure 4.5b with the last day's cost in Figure 4.6, we know that the multi-class dynamical system does not converge to MC-DTUE under different desired arrival times, because travelers' cost in group 1 are not the same, where some departing at the beginning of the congestion period will have lower cost (\$20) than others depart at the end (\$120). For group 2, the majority of travelers who depart from 3hr - 4.3 hr share the same cost at \$30, while those who depart from 2.4 hr - 2.6 hr share the same cost at \$42.5. These results show that the dynamical system converges to the stationary state $(\omega^g(\tau, t) \cdot f^g(\tau, t) = 0)$, but the stationary state is not MC-DTUE.

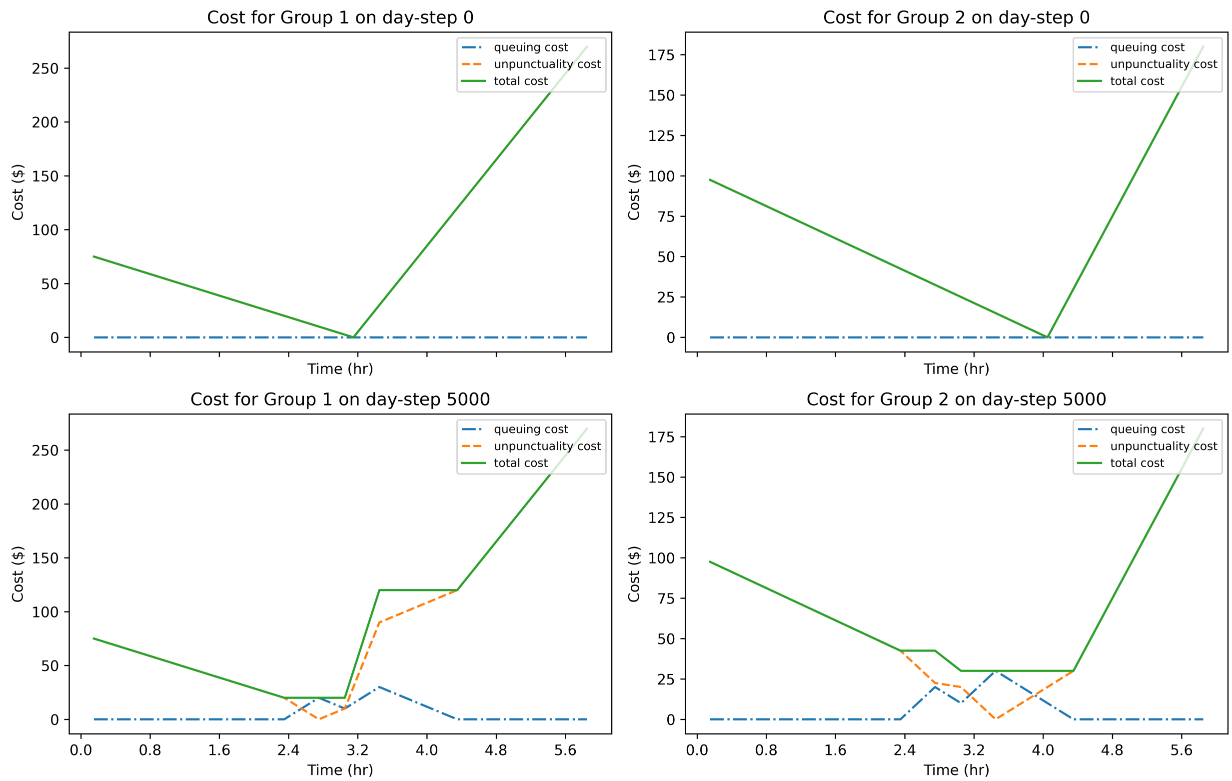


Figure 4.6: First and last day-step comparison for both groups on queuing, unpunctuality, and total costs (different t^*, same λ, μ, and ν - double peaks)

Figure 4.7 shows the day-to-day evolution of the normalized departure rate, which illustrates the stable convergence pattern of the dynamical system. Figure 4.8 shows the Lyapunov functional converges to 10^{-4} from day to day, which shows the system is asymptotically stable in the stability region.

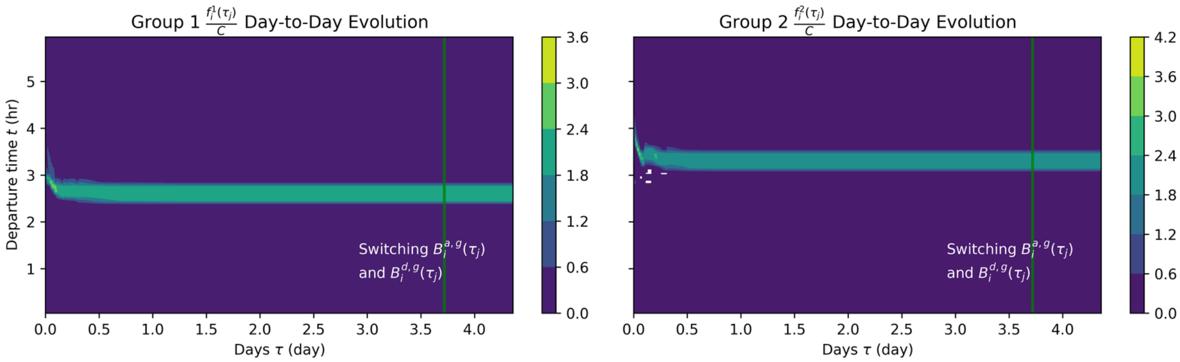


Figure 4.7: Normalized departure rate day-to-day evolution for both groups (different t^*, same λ, μ, and ν - double peaks)

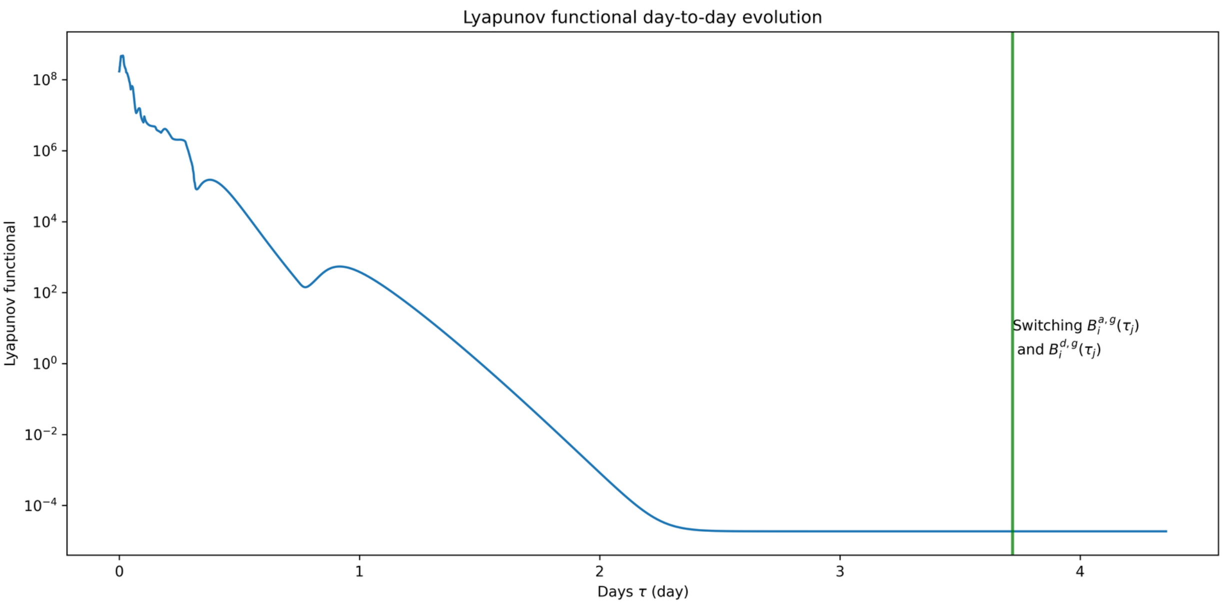


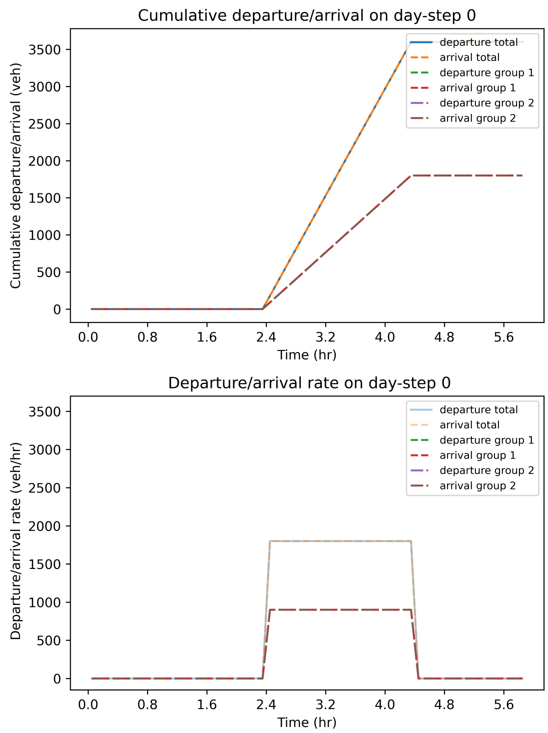
Figure 4.8: Multi-class Lyapunov functional day-to-day evolution (different t^*, same λ, μ, and ν - double peaks)

Overall, the results show that the multi-class dynamical system converges to a stable (Lyapunov functional goes to zero) stationary state for the double-peak scenario with different t^*, same λ, μ, and ν across travelers. However, the stationary state is not MC-DTUE with different t^*, same λ, μ, and ν, because travelers in the same group can have different costs. It is mainly because when $\omega^g(\tau, t) = 0$, travelers will stop shifting their departure time. For group 1's two departure periods, we have $\omega^1(\tau, t) = 0$ for both periods, but the cost in the first departure period is not equal to the cost in the second departure period, indicating the stationary state is not an MC-DTUE.

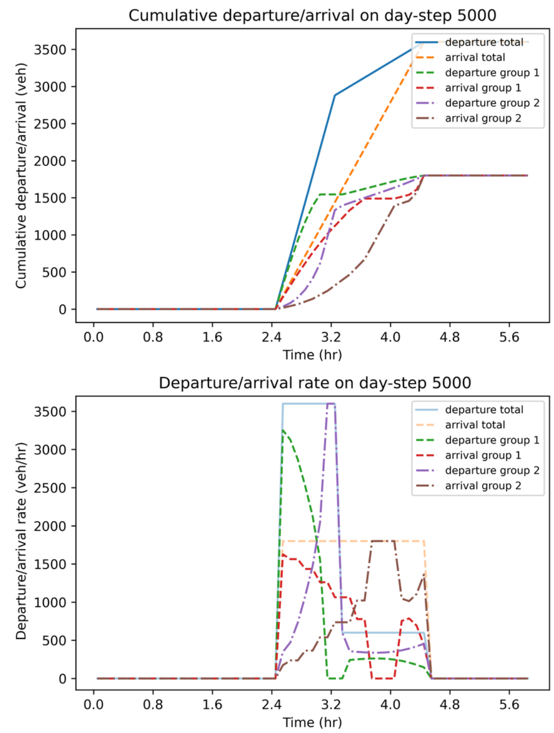
Single peak

If we have only one peak, that is $|t^{g,*} - t^{g',*}| \cdot C \leq |\frac{\mu}{\mu+\nu} \cdot N_g - \frac{\nu}{\mu+\nu} \cdot N_{g'}|$. In our case, if $|t^{1,*} - t^{2,*}| \leq 0.6$, then we have only one peak. In this case scenario, we set $t^{1,*} = 3.7$ hr, $t^{2,*} = 4$ hr, while λ, μ, and ν are the same for both groups as in Section 4.9.5. That is: $\lambda_1 = \lambda_2 = 50$ \$/hr, $\mu_1 = \mu_2 = 25$ \$/hr, and $\nu_1 = \nu_2 = 100$ \$/hr.

Figure 4.9 shows the departure and arrival patterns for the first and the last day-step. The cumulative total departure and arrival curve from the upper panel of Figure 4.9b shows that there is only one peak. Travelers in group 1 departs from 2.5hr - 3hr and from 3.4 hr - 4.4 hr, while all group 2's travelers depart from 2.5hr - 4.4 hr. Comparing the last day's departure pattern in Figure 4.9b with the last day's cost in Figure 4.10, it shows that the multi-class dynamical system does not converge to MC-DTUE under different desired arrival times, because travelers' cost in group 1 are not the same, where some depart at the beginning of the congestion period will have lower cost ($ 32.5) than others depart at the end ($ 70). Group 2's travelers who depart from 2.5hr - 4.4 hr share the same cost at $ 40. These results show that the dynamical system converges to the stationary state ($\omega^g(\tau, t) \cdot f^g(\tau, t) = 0$), but the stationary state is not MC-DTUE.



(a) First day-step



(b) Last day-step

Figure 4.9: First and last day-step comparison on cumulative departure/arrival flow and departure/arrival rate (different t^*, same λ, μ, and ν - single peaks)

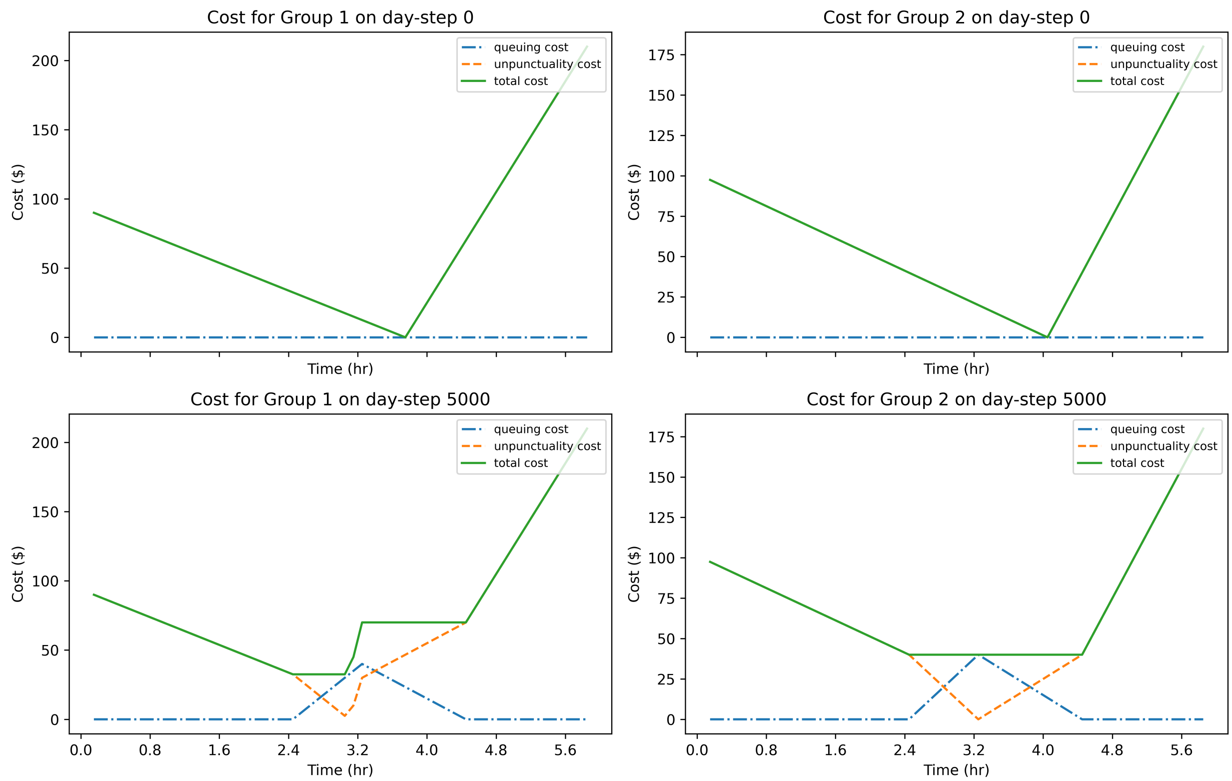


Figure 4.10: First and last day-step comparison for both groups on queuing, unpunctuality, and total costs (different t^*, same λ, μ, and ν - single peaks)

Figure 4.11 shows the day-to-day evolution of the normalized departure rate, which illustrates the stable convergence pattern of the dynamical system. Figure 4.12 shows the Lyapunov functional converges to 10^{-4} from day to day, which shows the system is asymptotically stable in the stability region.

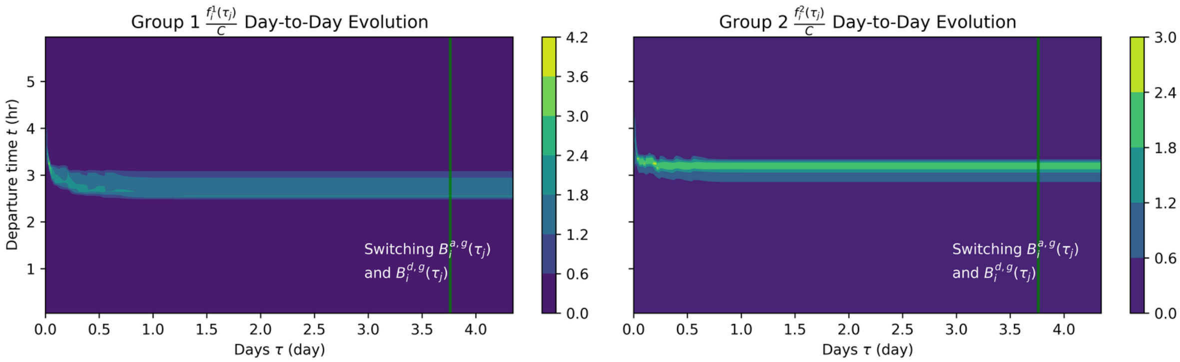


Figure 4.11: Normalized departure rate day-to-day evolution for both groups (different t^*, same λ, μ, and ν - single peaks)

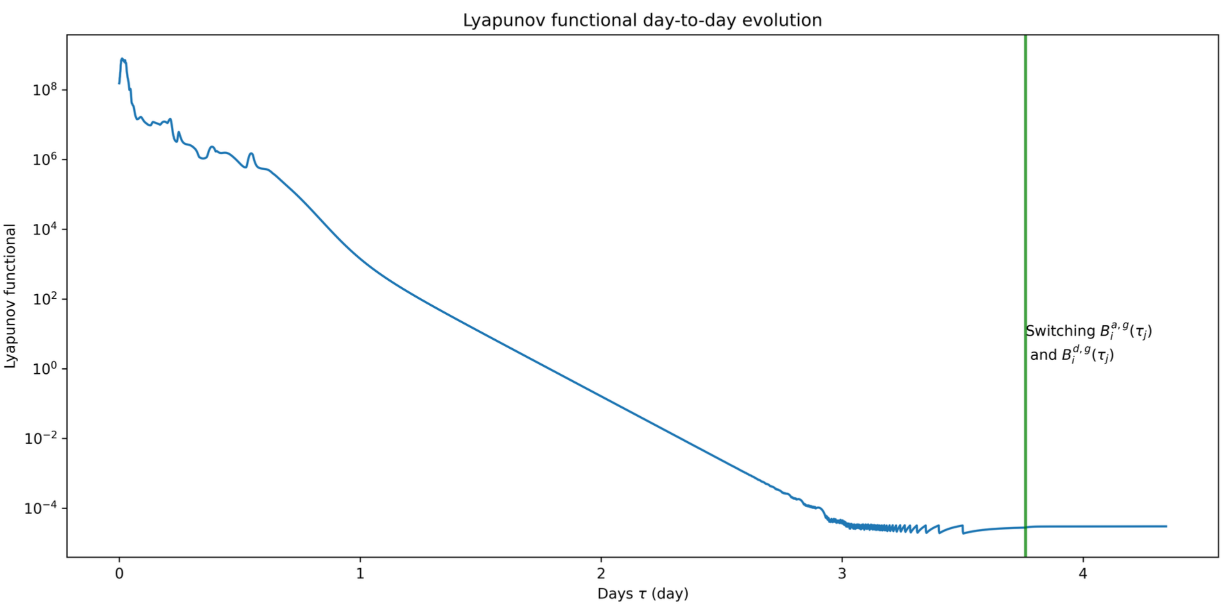


Figure 4.12: Multi-class Lyapunov functional day-to-day evolution (different t^*, same λ, μ, and ν - single peaks)

Overall, the results show that the multi-class dynamical system converges to a stable (Lyapunov functional goes to zero) stationary state for the single peak scenario with different t^*, same λ, μ, and ν across travelers. However, the stationary state is not MC-DTUE, because travelers in the same group can have different costs. It is mainly because when $\omega^g(\tau, t) = 0$, travelers will stop shifting their departure time. For group 1's two departure periods, we have $\omega^1(\tau, t) = 0$ for both periods, but the cost in the first departure period is not equal to the cost in the second departure period, indicating the stationary state is not an MC-DTUE.

4.10 Conclusion

This chapter extended single-class day-to-day dynamical system described in the previous chapter to a multi-class day-to-day dynamical system, capturing travelers' heterogeneity. We proved that the stationary state of the multi-class dynamical system is MC-DTUE under different λ/μ, same μ/ν, and same t^* across travelers, but not under different t^*, same λ, μ, and ν across travelers. The stationary state of the multi-class dynamical system is asymptotically stable within the stable region. This multi-class day-to-day dynamical model can be used to study how travelers from different classes respond to different pricing schemes, which we will examine in the Chapter 5.

Chapter 5

Managing Travelers' Day-to-day Departure Time Decisions with Optimal Pricing at a Single Bottleneck: Homogeneity, Heterogeneity and Stability

We presented a single-class day-to-day dynamical system in Chapter 3 along the lines of Jin (2021b), and extended it to a multi-class day-to-day dynamical system in Chapter 4. In this chapter, we will apply various tolling schemes to both single-class and multi-class dynamical systems, and analyze the resulting state and its stability.

5.1 Notations

Table 5.1 presents the notation in Chapter 5.

Table 5.1: Notation in Chapter 5, ordered by appearance

Variable	Meaning	Unit
$p(\tau, t)$	Toll or incentives	\$
$\phi(\tau, t)$	Trip cost for travelers departing at time t from home on day τ	\$
t^*	Desired arrival time for all travelers	hr
$\phi_1(\tau, t)$	queuing cost for travelers departing at time t from home on day τ	\$
$\phi_2(\tau, t)$	Unpunctuality cost for travelers departing at time t from home on day τ	\$
$\omega(\tau, t)$	Rate of change in trip cost	\$/hr
$\omega^E(\tau, t)$	Rate of change in trip cost for early arrivals	\$/hr
$\omega^L(\tau, t)$	Rate of change in trip cost for late arrivals	\$/hr
N	Total number of travelers	veh
C	Capacity for the point queue model	veh/hr
$P^o(t)$	Optimal fine toll	\$
P^c	Optimal coarse toll	\$
t_q^c	Intermediate variable for optimal coarse toll	\$
t_q^c	Intermediate variable for optimal coarse toll	\$
t^+	Turn-on time for optimal coarse toll	hr
t^-	Turn-off time for optimal coarse toll	hr
$P^r(t)$	Optimal fine reward	\$
$P^f(t)$	Optimal feebate	\$
$V(f(\tau, \cdot))$	Lyapunov functional on day τ	1
$A(\tau, t)$	Replacing $\frac{\partial}{\partial \tau}[f(\tau, t) \cdot \omega^2(\tau, t)]$	1
$\phi^g(\tau, t)$	Trip cost for travelers in group g departing at time t from home on day τ	\$
$t^{g,*}$	Desired arrival time for group g's travelers	hr
$\phi_1^g(\tau, t)$	queuing cost for group g's travelers departing at time t from home on day τ	\$
$\phi_2^g(\tau, t)$	Unpunctuality cost for group g's travelers departing at time t from home on day τ	\$
$\omega^g(\tau, t)$	Rate of change in trip cost for group g's travelers	\$/hr
$\omega^{E,g}(\tau, t)$	Rate of change in trip cost for group g's travelers' early arrivals	\$/hr
$\omega^{L,g}(\tau, t)$	Rate of change in trip cost for group g's travelers' late arrivals	\$/hr
$p^{o,M}(t)$	Multi-class optimal fine toll	\$
$\phi^{g',*}(t)$	Group $g^{'}$'s cost at MC-DTUE	\$
$\alpha^g(\tau, t)$	Group g's departure rate coefficient of rate of change in trip cost	1
$D_i^g(\tau)$	Replacing $\int_{t_{0,i}^g}^{t_{2,i}^g}(t - t_{0,i}^g) \cdot f^g(\tau, t) \cdot [\omega^g(\tau, t)]^2 dt$	1
$A_i^g(\tau, t)$	Replacing $\frac{\partial}{\partial \tau}\left\{ f^g(\tau, t) \cdot [\omega^g(\tau, t)]^2 \right\}$	1
u_0	Positive coefficient for $u(\tau, t)$	1
$\tau^{'}$	The day when the multi-class dynamical system reaches MC-DTUE	1
ε	$\omega(\tau, t_{2,i}^g) = 0$ when $\tau \geq \tau^{'} - \varepsilon$	day
$f^0(t)$	Initial departure rate on day 0	veh/hr
$f^*(t)$	SC-DTUE or MC-DTUE departure rate	veh/hr
$\phi^*(t)$	SC-DTUE or MC-DTUE cost function	\$
$e(j)$	L_1 norm error on day step j	veh/hr

5.2 Tolling on single-class travelers

5.2.1 Definitions

We present the definitions in the same order as in Chapter 3.

Within-day dynamics: single-class point queue model

For within-day traffic dynamics, it is described by the point queue model in Section 3.2.1.

Trip cost: homogeneous travelers

Let $\phi(\tau, t)$ denote the trip cost for travelers departing at time t from home on day τ (unit: \$). Let t^* be the desired arrival time for all travelers. Let $p(\tau, t)$ be the toll or reward. In the rest of this chapter, reward and incentives are used interchangeably. If $p(\tau, t) \geq 0$, then it is a toll, while if $p(\tau, t) < 0$, then it is an reward from the tolling agency or government. We have the following cost function:

$$\phi(\tau, t) = \lambda \cdot (\Upsilon^0 + \Upsilon(\tau, t)) + \mu \cdot \{t^* - (t + \Upsilon(\tau, t))\}_+ + \nu \cdot \{(t + \Upsilon(\tau, t)) - t^*\}_+ + p(\tau, t), \quad (5.1)$$

where the coefficients are described in Section 3.2.2. Without loss of generality, we assume $\Upsilon^0 = 0$. The first term in Equation (5.1) refers to queuing cost, the second and third terms in Equation (5.1) refer together to unpunctuality cost, while the last term in Equation (5.1) refers to the toll or incentives (e.g., time-varying congestion pricing). Let $\phi_1(\tau, t) = \lambda \cdot (\Upsilon(\tau, t))$ be the queuing cost and $\phi_2(\tau, t) = \mu \cdot \{t^* - (t + \Upsilon(\tau, t))\}_+ + \nu \cdot \{(t + \Upsilon(\tau, t)) - t^*\}_+$ be the unpunctuality cost. We assume $\mu < \lambda$ to avoid multiple equilibrium (Arnott et al., 1994).

Taking the derivative of Equation (5.1) on both sides, we have the rate of change in trip cost with pricing. Let $\omega(\tau, t)$ be the rate of change in trip cost, i.e., $\omega(\tau, t) = \frac{\partial}{\partial t}\phi(\tau, t)$. For early arrivals, we have $\omega^E(\tau, t) = \frac{\partial}{\partial t}\phi^E(\tau, t)$ (E denoting earliness):

$$\omega^E(\tau, t) = \frac{\partial}{\partial t}\phi^E(\tau, t) = (\lambda - \mu) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau, t) - \mu + \frac{\partial}{\partial t}p^E(\tau, t). \tag{5.2}$$

For late arrivals, we have the following rate of change in trip cost, $\omega^L(\tau, t) = \frac{\partial}{\partial t}\phi^L(\tau, t)$ (L denoting lateness):

$$\omega^L(\tau, t) = \frac{\partial}{\partial t}\phi^L(\tau, t) = (\lambda + \nu) \cdot \frac{\partial}{\partial t}\Upsilon^L(\tau, t) + \nu + \frac{\partial}{\partial t}p^L(\tau, t). \tag{5.3}$$

Let $f_1^m(\tau)$ be the maximum departure rate for early arrivals, and $f_2^m(\tau)$ maximum departure rate for late arrivals, so we have $f^m(\tau) = \max\{f_1^m(\tau), f_2^m(\tau)\}$. From Equation (3.13), we have $-1 \leq \frac{\partial}{\partial t}\Upsilon^E(\tau, t) \leq \frac{f_1^m(\tau)}{C} - 1$ for early arrivals, $-1 \leq \frac{\partial}{\partial t}\Upsilon^L(\tau, t) \leq \frac{f_2^m(\tau)}{C} - 1$ for late arrivals, and $-1 \leq \frac{\partial}{\partial t}\Upsilon(\tau, t) \leq \frac{f^m(\tau)}{C} - 1$ for all arrivals.

Let $(\frac{\partial}{\partial t}p(\tau, t))^{max}$ be the maximum rate of change of the toll $p(\tau, t)$, and $(\frac{\partial}{\partial t}p(\tau, t))^{min}$ be the minimum rate of change of the pricing $p(\tau, t)$. Substituting the bounds of $\frac{\partial}{\partial t}\Upsilon^E(\tau, t)$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau, t)$ into Equations (5.2) and (5.3), we obtain:

$$-\lambda + (\frac{\partial}{\partial t}p^E(\tau, t))^{min} \leq \omega^E(\tau, t) \leq (\lambda - \mu) \cdot \frac{f_1^m(\tau)}{C} - \lambda + (\frac{\partial}{\partial t}p^E(\tau, t))^{max}, \tag{5.4}$$

for early arrivals, and

$$-\lambda + (\frac{\partial}{\partial t}p^L(\tau, t))^{min} \leq \omega^L(\tau, t) \leq (\lambda + \nu) \cdot \frac{f_2^m(\tau)}{C} - \lambda + (\frac{\partial}{\partial t}p^L(\tau, t))^{max}, \tag{5.5}$$

for late arrivals.

System optimum in a single bottleneck

Definition 5.1 (System optimum)**.** *The system optimum (SO) for a single bottleneck is reached when there is no queue, where the departure rate is less than or equal to the bottleneck capacity for the entire departure interval. That is,* $\delta(\tau, t) = 0$*, and* $f(\tau, t) \leq C$ *for* $t \in [t_0, t_2]$*, where* t_0 *and* t_2 *are the start and end of the departure period at system optimum.*

From Definition 5.1, under SO, we have the following trip cost function without queuing cost and before adding $p(\tau, t)$:

$$\phi(\tau, t) = \mu \cdot \{t^* - t\}_+ + \nu \cdot \{t - t^*\}_+, \tag{5.6}$$

and after adding $p(\tau, t)$:

$$\phi(\tau, t) = \mu \cdot \{t^* - t\}_+ + \nu \cdot \{t - t^*\}_+ + p(\tau, t). \tag{5.7}$$

5.3 Day-to-day dynamics with tolls

We present below the models of day-to-day dynamics explained in the previous chapters, when under applied tolls.

5.3.1 Discrete version

The discrete version is described in Section 3.3.1 in Chapter 3.

5.3.2 Well-definedness

Adding a pricing $p(\tau, t)$ to the trip cost function $\phi(\tau, t)$ will change the shape of $\omega(\tau, t)$, so we discuss the well-definedness condition with tolls here.

Definition 5.2 (Well-defined definition)**.** *Jin (2021b)'s local dynamical system (Equations* (3.20) *to* (3.22)*) with tolls is well-defined if and only if $f_i(\tau) \geq 0$ for each time step i and each day step τ.*

From Section 3.3.2 in Chapter 3, we have the following well-definedness condition:

$$\begin{aligned} B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ &\leq 1 \\ B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ &\leq 1 \end{aligned} \tag{5.8}$$

where $\{\omega_i(\tau)\}_+$ and $\{-\omega_{i+1}(\tau)\}_+$ are determined by traffic dynamics of the point queue model. So we need to find the well-definedness conditions for $B_i^a(\tau)$ and $B_{i+1}^d(\tau)$.

Let $\{\omega_i(\tau)\}_+^{max}$ be the maximum value of $\{\omega_i(\tau)\}_+$, and let $\{-\omega_{i+1}(\tau)\}_+^{max}$ be the maximum value of $\{-\omega_{i+1}(\tau)\}_+$, so Equation (5.8) becomes:

$$\begin{aligned} B_i^a(\tau) &\leq \frac{1}{\{\omega_i(\tau)\}_+^{max}} \\ B_{i+1}^d(\tau) &\leq \frac{1}{\{-\omega_{i+1}(\tau)\}_+^{max}} \end{aligned} \tag{5.9}$$

Since different $p(\tau, t)$ will influence $\omega(\tau, t)$, we need to choose different $B_i^a(\tau)$ and $B_{i+1}^d(\tau)$ to make sure $f_i(\tau) \geq 0$ on a case by case basis. So different pricing can have different well-definedness conditions for $B_i^a(\tau)$ and $B_{i+1}^d(\tau)$.

5.3.3 Continuous version

Chapter 3 presented the details of deriving the continuous version of our day-to-day dynamical system. Here we present the final conservation equation again:

$$\frac{\partial}{\partial \tau} f(\tau, t) - \frac{\partial}{\partial t} u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0, \tag{5.10}$$

where

$$u(\tau, t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^a(\tau, t) & \text{for} \quad \omega(\tau, t) \geq 0, \\ B^d(\tau, t) & \text{for} \quad \omega(\tau, t) < 0. \end{cases} \tag{5.11}$$

Integrating both terms of Equation (5.10) with respect to time t, we have:

$$\frac{\partial}{\partial \tau} F(\tau, t) - u(\tau, t) \cdot \omega(\tau, t) \cdot \frac{\partial}{\partial t} F(\tau, t) = 0, \tag{5.12}$$

which is a hyperbolic conservation equation with respective to cumulative departure flow $F(\tau, t)$, and $-u(\tau, t) \cdot \omega(\tau, t)$ is the speed of the characteristic wave for this system. Adding a toll $p(\tau, t)$ will influence the shape of $\omega(\tau, t)$, and thus the characteristic wave speed, $-u(\tau, t) \cdot \omega(\tau, t)$.

5.4 Tolls/incentives

In this section, we present 4 different tolls/incentives that we test in this chapter.

5.4.1 Optimal fine toll

In the SC-DTUE, the queuing cost, $\phi^1(\tau,t) = \lambda \cdot \Upsilon(\tau,t)$, balances with the unpunctuality cost, $\phi^2(\tau,t) = \mu \cdot \{t^* - (t + \Upsilon(\tau,t))\}_+ + \nu \cdot \{(t + \Upsilon(\tau,t) - t^*)\}_+$, so that the total trip cost, $\phi(\tau,t)$, remains constant during the departure period and is equal to the minimum trip cost (i.e., $\phi(\tau,t) = \phi^*$ for $t \in [t_0, t_2]$).

The trip cost at SC-DTUE is calculated using the following two equations:

$$\begin{aligned} &\mu \cdot (t^* - t_0) = \nu \cdot (t_2 - t^*), \\ &t_2 - t_0 = \frac{N}{C}. \end{aligned} \tag{5.13}$$

Trip cost at SC-DTUE, ϕ^*, is described as follows:

$$\phi^* = \frac{\mu \cdot \nu}{\mu + \nu} \cdot \frac{N}{C}. \tag{5.14}$$

The optimal fine toll is called "optimal" because it can lead to a system optimum state in equilibrium analysis. The optimal fine toll is calculated by assuming that there is no queue ($\delta(\tau,t) = 0$, and $\Upsilon(\tau,t) = 0$), and thus the queuing cost, $\phi^1(\tau,t) = \lambda \cdot \Upsilon(\tau,t)$, is zero. Let $p^o(t)$ be the optimal fine toll, which is only time-dependent. So we have:

$$\phi(\tau,t) = \mu \cdot \{t^* - t\}_+ + \nu \cdot \{t - t^*\}_+ + p^o(t). \tag{5.15}$$

Taking the derivative of both sides of Equation (5.15), for early arrivals, we find:

$$\omega^E(\tau,t) = -\mu + \frac{d}{dt}p^o(t) = 0. \tag{5.16}$$

For late arrivals, we find:

$$\omega^L(\tau,t) = \nu + \frac{d}{dt}p^o(t) = 0. \tag{5.17}$$

So we see that $\frac{d}{dt}p^o(t) = \mu$ for $t \in [t_0, t^*]$ and $\frac{d}{dt}p^o(t) = -\nu$ for $t \in [t_0, t^*]$. Assuming that the toll starts at t_0 and ends at t_2, we derive the following formula for optimal fine toll:

$$p^o(t) = \begin{cases} 0 & \text{for} \quad t < t_0, \\ \phi^* - \mu \cdot (t^* - t) & \text{for} \quad t_0 \le t < t^*, \\ \phi^* + \nu \cdot (t^* - t) & \text{for} \quad t^* \le t \le t_2, \\ 0 & \text{for} \quad t > t_2. \end{cases} \tag{5.18}$$

When $t = t_0$, we have $p^o(t_0) = 0$ from Equation (5.14). When $t = t^*$, we have $p^o(t^*) = \phi^* - \mu \cdot (t^* - t^*) = \phi^*$. When $t = t_2$, we have $p^o(t_2) = 0$.

5.4.2 Optimal coarse toll

Optimal coarse toll is a single-step toll that turns on at t^+ and turns off at t^-, and the toll is constant during $[t^+, t^-]$. Let p^c be the optimal coarse toll, where c represents coarse. From Arnott et al. (1990), we have the optimal coarse toll as follows:

$$p^c = \frac{\mu \cdot \nu}{2 \cdot (\mu + \nu)} \cdot \frac{N}{C}. \tag{5.19}$$

To calculate the t^+ and t^-, Arnott et al. (1990) introduces an intermediate variable t_q^c as follows:

$$t_q^c = t^* - \frac{\nu}{\mu + \nu} \cdot \frac{N}{C} + \frac{(\nu - \lambda) \cdot p^c}{(\mu + \nu) \cdot (\lambda + \nu)}. \tag{5.20}$$

So t^+ and t^- are calculated as follows:

$$\begin{aligned} t^+ &= t_q^c + \frac{p^c}{\mu}, \\ t^- &= t_q^c + \frac{N}{C} - \frac{2 \cdot p^c}{(\lambda + \mu)}. \end{aligned} \tag{5.21}$$

5.4.3 Optimal fine reward

The reasoning behind the optimal fine reward is that instead of charging travelers a toll, the government can give travelers incentives to encourage travelers to shift their departure times to avoid the peak period (Rouwendal et al., 2012).

Similar to optimal fine toll, optimal fine reward also has $\frac{d}{dt}p^o(t) = \mu$ for $t \in [t_0, t^*]$ and $\frac{d}{dt}p^o(t) = -\nu$ for $t \in [t_0, t^*]$, but ϕ^* in Equation (5.18) becomes zero. So the optimal fine reward, $p^r(t)$, is as follows (Rouwendal et al., 2012):

$$p^r(t) = \begin{cases} 0 & \text{for} \quad t < t_0, \\ -\mu \cdot (t^* - t) & \text{for} \quad t_0 \leq t < t^*, \\ \nu \cdot (t^* - t) & \text{for} \quad t^* \leq t \leq t_2, \\ 0 & \text{for} \quad t > t_2. \end{cases} \tag{5.22}$$

5.4.4 Optimal feebate

The reasoning behind the optimal feebate is that instead of charging all travelers a toll or giving all travelers incentives, the government will charge travelers whose departures are closer to their desired arrival times and use the money to compensate travelers who are willing to shift their departure times away from their desired arrival times. It is similar to the trading ideas studied in Lloret Batlle (2017); Nam (2019); Lloret-Batlle and Jayakrishnan (2016); Masoud et al. (2017), but it is not in a peer-to-peer sense, in that individual behavior is not considered. The "trading" behind optimal feebate refers to those who depart closer to their desired arrival time compensating those who depart further from their desired arrival time.

Similar to optimal fine toll and optimal fine reward, optimal feebate also has $\frac{d}{dt}p^o(t) = \mu$ for $t \in [t_0, t^*]$ and $\frac{d}{dt}p^o(t) = -\nu$ for $t \in [t_0, t^*]$, but changes the ϕ^* in Equation (5.18) to $\phi^*/2$.

So the optimal feebate, $p^f(t)$, is as follows (Rouwendal et al., 2012):

$$p^f(t) = \begin{cases} 0 & \text{for} \quad t < t_0 \\ \frac{1}{2}\phi^* - \mu \cdot (t^* - t) & \text{for} \quad t_0 \leq t < t^*, \\ \frac{1}{2}\phi^* + \nu \cdot (t^* - t) & \text{for} \quad t^* \leq t \leq t_2, \\ 0 & \text{for} \quad t > t_2 \end{cases} \tag{5.23}$$

5.5 Stationary state of single-class dynamical system with pricing

Definition 5.3. *With pricing, the stationary state of the local dynamical system in Equation* (3.55)*, (discrete version: Equations* (3.20) *to* (3.22)*), is reached when* $\frac{\partial}{\partial \tau} f(\tau, t) = 0$*, which means the time-dependent departure rate* $f(\tau, t)$ *does not change from day to day.*

When $\frac{\partial}{\partial \tau} f(\tau, t) = 0$, we have $\frac{\partial}{\partial \tau} F(\tau, t) = 0$. So we have $u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$ according to $\frac{\partial}{\partial \tau} F(\tau, t) - u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$. Since $u(\tau, t) > 0$, we have the following complementarity condition:

$$f(\tau, t) \cdot \omega(\tau, t) = 0, \tag{5.24}$$

for all $t \in [0, T]$.

5.6 Stationary state with tolls and system optimum

Theorem 5.1. *With optimal fine toll/optimal fine reward/and optimal feebate, the stationary state of the local dynamical system in Equation* (3.55)*, (discrete version: Equations* (3.20) *to* (3.22)*) is equivalent to system optimum.*

Proof. To prove Theorem 5.1, we proves the system with optimal fine toll as an example. Optimal reward and feebate can follow the same proof. We first prove the system optimum is the stationary state with optimal fine toll, and we then prove the stationary state with optimal fine toll is system optimum.

1. Given the system reaches the system optimum with optimal fine toll, from Definition 5.1 of system optimum and optimal fine toll from Equation (5.18), we have $\phi(\tau, t) = \mu \cdot (t^* - t) + p^o = \mu \cdot (t^* - t) + \phi^* - \mu \cdot (t^* - t) = \phi^*$ for early arrivals and $\phi(\tau, t) = \nu \cdot (t - t^*) + p^o = -\nu \cdot (t^* - t) + \phi^* + \nu \cdot (t^* - t) = \phi^*$ for late arrivals. So we have $\phi(\tau, t) = \phi^*$ for $t \in [t_0, t_2]$ and $\phi(\tau, t) \geq \phi^*$ for $t \in [0, t_0) \cup (t_2, T]$.

   With optimal fine toll, $t_2 - t_0 = N/C$, which means all travelers N can pass through the bottleneck during $[t_0, t_2]$. So system optimum with optimal fine toll can lead to a SC-DTUE, where all used departure times share the same cost, which is less than or equal to the costs of all the unused departure times.

   So we have $f(\tau, t) > 0$ when $\phi(\tau, t) = \phi^*$, and $f(\tau, t) = 0$ when $\phi(\tau, t) \geq \phi^*$. Since $\omega(\tau, t) = \frac{\partial}{\partial t}\phi(\tau, t)$, we have $\omega(\tau, t) = 0$ for $f(\tau, t) > 0$ and $\omega(\tau, t) \neq 0$ for $f(\tau, t) = 0$. So we the following complementarity condition:

$$f(\tau, t) \cdot \omega(\tau, t) = 0. \tag{5.25}$$

   Since $f(\tau, t) \cdot \omega(\tau, t) = 0$ at system optimum with optimal fine toll, then we have $u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$, so $\frac{\partial}{\partial t}u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$. From Equation (3.55), we know $\frac{\partial}{\partial \tau}f(\tau, t) = 0$. This proves that the system optimum with optimal fine toll is in the stationary state.

2. We then prove the stationary state with optimal fine toll is the system optimum. Given system reaches the stationary state, $\frac{\partial}{\partial \tau}f(\tau, t) = 0$, we have $\frac{\partial}{\partial \tau}F(\tau, t) = 0$ as well. So we have $u(\tau, t) \cdot \omega(\tau, t) \cdot f(\tau, t) = 0$ from Equation (3.57). At the stationary state, let the

departure period be $[t_0, t_2$. We analyze the characteristic wave speed of Equation (5.12) $(-u(\tau,t)\cdot\omega(\tau,t))$ to show that stationary state with optimal fine toll is system optimal.

When $t < t_0$, the toll is zero, so we have $-u(\tau,t)\cdot\omega(\tau,t) = \mu$, which is the same as the no toll case. It means travelers will shift their departure times to later times. When $t > t_2$, the toll is zero as well, so we have $-u(\tau,t)\cdot\omega(\tau,t) = -\nu$, which is the same as the no toll case. It means travelers will shift their departure times to early times.

When $t \in [t_0, t_2]$, we have $-u(\tau,t)\cdot\omega(\tau,t) = 0$. This leads to one connected system departure pattern, in stationary state, of the day-to-day dynamical system with optimal fine toll.

Since $\omega(\tau,t) = 0$ for $t \in [t_0, t_2]$ with optimal fine toll, we find from Equation (5.2):

$$\omega^E(\tau,t) = (\lambda-\mu)\cdot\frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu + \frac{d}{dt}p^o(t) = (\lambda-\mu)\cdot\frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu + \mu = 0, \quad (5.26)$$

for early arrivals, and from Equation (5.3):

$$\omega^L(\tau,t) = (\lambda+\nu)\cdot\frac{\partial}{\partial t}\Upsilon^L(\tau,t) + \nu + \frac{d}{dt}p^o(t) = (\lambda+\nu)\cdot\frac{\partial}{\partial t}\Upsilon^L(\tau,t) + \nu - \nu = 0, \quad (5.27)$$

for late arrivals. For $\omega^E(\tau,t)$ and ω^L to be zero, $\frac{\partial}{\partial t}\Upsilon^E(\tau,t)$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t)$ have to be zero. We have $\frac{\partial}{\partial t}\Upsilon(\tau,t) = \max\{-\frac{\delta(\tau,t)}{C\cdot\epsilon}, \frac{f(\tau,t)}{C} - 1\}$ from Equation (3.12). For $\frac{\partial}{\partial t}\Upsilon(\tau,t)$ to be zero, we have $\delta(\tau,t) = 0$ and $f(\tau,t) \leq C$. So there is no queue during $[t_0, t_2]$ in the stationary state with the optimal fine toll, which is the system optimal state.

We thereby prove that the system optimum is the stationary state with optimal fine toll, and also that the stationary state with optimal fine toll is the system optimum. This completes the proof of Theorem 5.1. □

5.7 Stability of the stationary state with tolls

We are interested in knowing what toll can drive the day-to-day dynamical system to a stable stationary system optimal state. More rigorously stated, this means that it reaches a stable system optimum ($\delta(\tau, t) = 0$ for $t \in [t_0, t_2]$) at the stationary state $\frac{\partial}{\partial \tau} f(\tau, t) = 0$.

From Section 5.6, we show that $\delta(\tau, t) = 0$ at the stationary state. However, before reaching the stationary state, we have $\delta(\tau, t) \geq 0$ for $t \in [t_0, t_2]$.

From Section 5.6, in the stationary state, we have $-u(\tau, t) \cdot \omega(\tau, t) = \mu$ before t_0 and $-u(\tau, t) \cdot \omega(\tau, t) = -\nu$ after t_2. It is unstable for $t < t_0$ and $t > t_2$. So we analyze the stability for $t_0 \leq t \leq t_2$, which is congested (i.e., $\delta(\tau, t) > 0$) at the beginning and uncongested (i.e., $\delta(\tau, t) = 0$) at the stationary state.

Here we consider the stability under the congested state (i.e., $\delta(\tau, t) > 0$). For early arrivals, we have $\omega^E(\tau, t) = (\lambda - \mu) \cdot \frac{\partial}{\partial t} \Upsilon^E(\tau, t) - \mu + \mu = (\lambda - \mu) \cdot \frac{\partial}{\partial t} \Upsilon^E(\tau, t)$. When $\delta(\tau, t) > 0$, $\frac{\partial}{\partial t} \Upsilon^E(\tau, t) = \frac{f(\tau, t)}{C} - 1$. So we have $\omega^E(\tau, t) = (\lambda - \mu) \cdot \frac{f(\tau, t)}{C} - (\lambda - \mu)$.

For late arrivals, we have $\omega^L(\tau, t) = (\lambda + \nu) \cdot \frac{\partial}{\partial t} \Upsilon^L(\tau, t) + \nu - \nu = (\lambda + \nu) \cdot \frac{\partial}{\partial t} \Upsilon^L(\tau, t)$. When $\delta(\tau, t) > 0$, $\frac{\partial}{\partial t} \Upsilon^L(\tau, t) = \frac{f(\tau, t)}{C} - 1$. So we have $\omega^L(\tau, t) = (\lambda + \nu) \cdot \frac{f(\tau, t)}{C} - (\lambda + \nu)$.

From Equation (3.60), we find that $\alpha(t) = (\lambda - \mu)/C$ for early arrivals and $\alpha(t) = (\lambda + \nu)/C$ for late arrivals. Combining both early arrivals and late arrivals, we have $\omega(\tau, t) = \alpha(t) \cdot f(\tau, t) - \alpha(t) \cdot C = \alpha(t) \cdot (f(\tau, t) - C)$.

Substituting $\omega(\tau, t) = \alpha(t) \cdot (f(\tau, t) - C)$ into Equation (5.10), we obtain:

$$\frac{\partial}{\partial \tau} f(\tau, t) - \frac{\partial}{\partial t} u(\tau, t) \cdot [\alpha(t) \cdot (f(\tau, t) - C)] \cdot f(\tau, t) = 0, \tag{5.28}$$

for $t \in [t_0, t_2]$.

We define a Lyapunov functional the same as Equation (3.69) in Chapter 3:

$$V(f(\tau,\cdot)) = \int_0^T t \cdot f(\tau,t) \cdot \omega^2(\tau,t)dt, \tag{5.29}$$

whose discrete version is as follows:

$$V(f(\tau)) = \sum_1^I (i - \frac{1}{2}) \cdot \Delta t \cdot f_i(\tau) \cdot [\{-\omega_{i+1}(\tau)\}_+^2 + \{\omega_i(\tau)\}_+^2]. \tag{5.30}$$

When all trips depart within $[t_0, t_2]$, Equation (5.29) becomes:

$$V(f(\tau,\cdot)) = \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau,t) \cdot \omega^2(\tau,t)dt, \tag{5.31}$$

For $V(f(\tau,\cdot))$ to be a Lyapunov functional of dynamical system Equation (5.28) with optimal fine toll, we need to show $V(f(\tau,\cdot))$ satisfies the following three conditions:

1. $V(f(\tau,\cdot)) \geq 0$, $\forall\tau$.
2. $V(f(\tau,\cdot)) = 0$ at stationary state of the dynamical system Equation (5.28).
3. $\frac{\partial}{\partial\tau}V(f(\tau,\cdot)) < 0$ at non-stationary states of the dynamical system Equation (5.28), while $\frac{\partial}{\partial\tau}V(f(\tau,\cdot)) = 0$ at stationary state of the dynamical system Equation (5.28).

Theorem 5.2. *Equation (5.31) is the Lyapunov functional for the dynamical system in Equation (5.28) with optimal fine tolls.*

Proof. We show how $V(f(\tau,\cdot))$ from Equation (5.31) will satisfy the three conditions above:

1. For condition (a), since $t - t_0 \geq 0$, $f(\tau,t) \geq 0$ and $\omega^2(\tau,t) \geq 0$, we have $V(f(\tau,\cdot)) = \int_{t_0}^{t_2}(t - t_0) \cdot f(\tau,t) \cdot \omega^2(\tau,t)dt \geq 0$.

2. For condition (b), the stationary state for the day-to-day dynamical system is reached when:

$$\frac{\partial}{\partial \tau} f(\tau, t) = 0. \tag{5.32}$$

From Equation (5.25), we have the complementarity condition $f(\tau, t) \cdot \omega(\tau, t) = 0$ at the stationary state. When $f(\tau, t) > 0$, $\omega(\tau, t) = 0$, so we have $\omega^2(\tau, t) = 0$. When $f(\tau, t) = 0$, $\omega(\tau, t) \geq 0$, so we have $\omega^2(\tau, t) \geq 0$. Then the following complementarity condition also holds:

$$f(\tau, t) \cdot \omega^2(\tau, t) = 0, \tag{5.33}$$

for all $t \in [0, T]$.

So $V(f(\tau, \cdot)) = \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt = 0$ at the stationary state.

3. For condition (c), we take the partial derivative of Equation (5.31) with respect to τ and we get:

$$\begin{aligned} \frac{\partial}{\partial \tau} V(f(\tau, \cdot)) &= \frac{\partial}{\partial \tau} \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt \\ &= \int_{t_0}^{t_2} (t - t_0) \cdot \frac{\partial}{\partial \tau} [f(\tau, t) \cdot \omega^2(\tau, t)] dt. \end{aligned} \tag{5.34}$$

Letting $A(\tau, t) = \frac{\partial}{\partial \tau} f(\tau, t) \cdot \omega^2(\tau, t)$, Equation (5.34) becomes:

$$\begin{aligned} \frac{\partial}{\partial \tau} V(f(\tau, \cdot)) &= \frac{\partial}{\partial \tau} \int_{t_0}^{t_2} (t - t_0) \cdot f(\tau, t) \cdot \omega^2(\tau, t) dt \\ &= \int_{t_0}^{t_2} (t - t_0) \cdot A(\tau, t) dt. \end{aligned} \tag{5.35}$$

For $A(\tau,t)$, we have:

$$A(\tau,t) = \frac{\partial}{\partial \tau}[f(\tau,t)\cdot\omega^2(\tau,t)] = \frac{\partial}{\partial \tau}f(\tau,t)\cdot\omega^2(\tau,t) + 2\cdot f(\tau,t)\cdot\omega(\tau,t)\cdot\frac{\partial}{\partial \tau}\omega(\tau,t). \tag{5.36}$$

When $\delta(\tau,t) > 0$, we have $\omega(\tau,t) = \alpha(t)\cdot(f(\tau,t) - C)$. So we have $\frac{\partial}{\partial \tau}\omega(\tau,t) = \alpha(t)\cdot\frac{\partial}{\partial \tau}f(\tau,t)$. Substituting it into Equation (5.36), we obtain:

$$\begin{aligned} A(\tau,t) &= \frac{\partial}{\partial \tau}f(\tau,t)\cdot\omega^2(\tau,t) + 2\cdot f(\tau,t)\cdot\omega(\tau,t)\cdot\alpha(t)\cdot\frac{\partial}{\partial \tau}f(\tau,t) \\ &= \omega(\tau,t)\cdot[\omega(\tau,t) + 2\cdot f(\tau,t)\cdot\alpha(t)]\cdot\frac{\partial}{\partial \tau}f(\tau,t) \end{aligned} \tag{5.37}$$

Substituting $\omega(\tau,t) = \alpha(t)\cdot(f(\tau,t) - C)$ into Equation (5.37), we obtain:

$$\begin{aligned} A(\tau,t) &= [\alpha(t)\cdot(f(\tau,t) - C)]\cdot[3\cdot\alpha(t)\cdot f(\tau,t) - \alpha(t)\cdot C]\cdot\frac{\partial}{\partial \tau}f(\tau,t) \\ &= [\alpha(t)\cdot(f(\tau,t) - C)]\cdot[\alpha(t)\cdot(3\cdot f(\tau,t) - C)]\cdot\frac{\partial}{\partial \tau}f(\tau,t) \end{aligned} \tag{5.38}$$

From Equation (5.10), we have $\frac{\partial}{\partial \tau}f(\tau,t) = \frac{\partial}{\partial t}u(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t)$. Substituting it into $\frac{\partial}{\partial \tau}f(\tau,t)$ of Equation (5.38), we obtain:

$$A(\tau,t) = [\alpha(t)\cdot(f(\tau,t) - C)]\cdot[\alpha(t)\cdot(3\cdot f(\tau,t) - C)]\cdot\frac{\partial}{\partial t}u(\tau,t)\cdot\omega(\tau,t)\cdot f(\tau,t). \tag{5.39}$$

Now we need to make certain assumptions on $u(\tau,t)$ to be able to move forward with the proof. We know from Equation (3.54) that $u(\tau,t) > 0$. Now we assume $u(\tau,t)$ to have the following form:

$$u(\tau,t) = u_0\cdot\frac{3\omega(\tau,t) + 2\alpha(t)\cdot C}{f(\tau,t)}, \tag{5.40}$$

where u_0 is a positive coefficient.

When the bottleneck is uncongested, $\omega(\tau,t) = -\mu$ for early arrivals, and $\omega(\tau,t) = \nu$ for late arrivals. So we have $u(\tau,t) = u_0 \cdot \frac{-3\mu+2\alpha(t)\cdot C}{f(\tau,t)} = u_0 \cdot \frac{2\lambda-5\mu}{f(\tau,t)}$ for early arrivals, and $u(\tau,t) = u_0 \cdot \frac{3\nu+2\alpha(t)\cdot C}{f(\tau,t)} = u_0 \cdot \frac{2\lambda+5\nu}{f(\tau,t)}$ for late arrivals.

When the bottleneck is congested, then we have $\omega(\tau,t) = \alpha(t)\cdot(f(\tau,t) - C)$, then Equation (3.77) becomes:

$$u(\tau,t) = u_0 \cdot \frac{3\omega(\tau,t) + 2\alpha(t)\cdot C}{f(\tau,t)} = u_0 \cdot \frac{3\alpha(t)\cdot f(\tau,t) - \alpha(t)\cdot C}{f(\tau,t)}, \tag{5.41}$$

where $\alpha(t)\cdot[3f(\tau,t) - C] > 0$ and that is, $f(\tau,t) > \frac{C}{3}$.

Substituting Equation (5.41) into Equation (5.39) with $\omega(\tau,t) = \alpha(t)\cdot(f(\tau,t) - C)$ and crossing out $f(\tau,t)$ in the denominator and nominator, we derive:

$$\begin{aligned} A(\tau,t) &= [\alpha(t)(f(\tau,t) - C)]\cdot[3\alpha(t)\cdot f(\tau,t) - \alpha(t)C] \\ &\quad \cdot \frac{\partial}{\partial t}\left\{ u_0 \cdot \frac{[3\alpha(t)\cdot f(\tau,t) - \alpha(t)\cdot C]}{f(\tau,t)} \cdot [\alpha(t)(f(\tau,t)-C)]\cdot f(\tau,t) \right\} \\ &= \frac{u_0}{2}\cdot\frac{\partial}{\partial t}\left\{ [\alpha(t)(f(\tau,t)-C)]\cdot[3\alpha(t)\cdot f(\tau,t) - \alpha(t)C] \right\}^2. \end{aligned} \tag{5.42}$$

Substituting $A(\tau,t)$ from Equation (5.42) back to Equation (5.35), we derive:

$$\begin{aligned} \frac{\partial}{\partial\tau} V(f(\tau,\cdot)) &= \frac{\partial}{\partial\tau}\int_{t_0}^{t_2} (t - t_0)\cdot A(\tau,t)dt \\ &= \frac{u_0}{2}\cdot\int_{t_0}^{t_2}(t-t_0)\cdot\frac{\partial}{\partial t}\left\{ [\alpha(t)(f(\tau,t)-C)]\cdot[3\alpha(t)\cdot f(\tau,t) - \alpha(t)C] \right\}^2 dt. \end{aligned} \tag{5.43}$$

Integrating by parts, Equation (5.43) becomes:

$$
\begin{aligned}
\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) = & \frac{u_0}{2} \cdot \Bigg\{ (t - t_0) \left\{ [\alpha(t)(f(\tau,t) - C)] \cdot [3\alpha(t) \cdot f(\tau,t) - \alpha(t)C] \right\}^2 \Big|_{t_0}^{t_2} \\
& - \int_{t_0}^{t_2} \left\{ [\alpha(t)(f(\tau,t) - C)] \cdot [3\alpha(t) \cdot f(\tau,t) - \alpha(t)C] \right\}^2 dt \Bigg\} \\
= & \frac{u_0}{2} \cdot \Bigg\{ (t_2 - t_0) \cdot \left\{ [\alpha(t_2)(f(\tau,t_2) - C)] \cdot [3\alpha(t_2) \cdot f(\tau,t_2) - \alpha(t_2)C] \right\}^2 - 0 \\
& - \int_{t_0}^{t_2} \left\{ [\alpha(t_2)(f(\tau,t_2) - C)] \cdot [3\alpha(t_2) \cdot f(\tau,t_2) - \alpha(t_2)C] \right\}^2 dt \Bigg\},
\end{aligned}
\tag{5.44}
$$

where $\Big\{ [\alpha(t_2)(f(\tau,t_2) - C)] \cdot [3\alpha(t_2) \cdot f(\tau,t_2) - \alpha(t_2)C] \Big\}^2 \geq 0$, so we know the last term of Equation (5.44), $-\int_{t_0}^{t_2} \Big\{ \Big[\alpha(t_2)(f(\tau,t_2) - C)\Big] \cdot [3\alpha(t_2) \cdot f(\tau,t_2) - \alpha(t_2)C] \Big\}^2 dt \leq 0$.

Assumption 5.7.1. *At the stationary state, we have $\omega(\tau^{'}, t_2) = 0$. We assume that before the dynamical system approaches its stationary state on day $\tau^{'}$, and rate of change in trip cost at end of the departure period t_2, $\omega(\tau, t_2)$, becomes zero. That is, $\exists \varepsilon > 0$,*

$$
\omega(\tau, t_2) = 0, \forall \tau \geq \tau^{'} - \varepsilon. \tag{5.45}
$$

In the current Section 5.7, we study the stability of the stationary state. Assumption 5.7.1 means that before the day step approaches the stationary state step $\tau^{'}$, $\omega(\tau, t_2)$ already becomes zero.

With $\omega(\tau, t_2) = 0$, we have $\omega(\tau, t_2) = \alpha(t_2) \cdot (f(\tau, t_2) - C) = 0$, and $f(\tau, t_2) = C$, and $\alpha(t_2) = \frac{\lambda + \nu}{C}$ from Equation (3.60).

For $\tau \geq \tau^{'} - \varepsilon$, with $f(\tau, t_2) - C = 0$, the first term of Equation (5.44) becomes zero.

So Equation (5.44) becomes as follows:

$$\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) = -\frac{u_0}{2} \cdot \left\{ \int_{t_0}^{t_2} \{[\alpha(t)(f(\tau,t) - C)] \cdot [3\alpha(t) \cdot f(\tau,t) - \alpha(t)C]\}^2 \, dt \right\} \leq 0. \tag{5.46}$$

From Equation (5.41), we have $\alpha(t) \cdot (3 \cdot f(\tau,t) - C) > 0$. For $t \in [t_0, t_2]$, when $\tau < \tau^{'}$, we have $\omega^2(\tau,t) = [\alpha(t) \cdot (f(\tau,t) - C)]^2 > 0$, and when $\tau \geq \tau^{'}$, we have $\omega^2(\tau,t) = [\alpha(t) \cdot (f(\tau,t) - C)]^2 = 0$. So we have:

$$\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) < 0, \forall \tau^{'} - \varepsilon \leq \tau < \tau^{'},$$

$$\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) = 0, \forall \tau \geq \tau^{'}.$$

That means, after the system reaches its stability region ($\tau^{'} - \varepsilon \leq \tau$), $\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) < 0$ at non-stationary states, while $\frac{\partial}{\partial \tau} V(f(\tau, \cdot)) = 0$ at stationary state, which is system optimum.

So Equation (5.31) is a Lyapunov functional of dynamical system Equation (5.28) with optimal fine toll, and it is asymptotically stable at the stationary state during the congested period $[t_0, t_2]$. This completes the proof of Theorem 5.2.

□

5.8 Numerical examples: single-class tolling

We test the four tolls/incentives in Section 5.4 by adding a toll in Jin (2021b)'s day-to-day dynamics to test which toll can drive the system to a stable stationary system optimal state.

5.8.1 Simulation setup

We consider the total number of travelers, $N = 3,600$ veh, and the capacity of the bottleneck to be $C = 1,800$ veh/hr. We assume that $\lambda = 50$ \$/hr, $\mu = 25$ \$/hr, and $\nu = 100$ \$/hr. We consider the study period to be $[0, 6]$ hr, and the desired arrival time for all travelers $t^* = 4$ hr. Similar to Chapter 3, we set $\Delta t = 0.1$hr and total day steps to be $5,001$, starting from day step 0, and ending with day step $5,000$. We assume the SO initial departure rate on day 0, $f^0(t)$, as follows:

$$f^0(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4 \\ 1,800 & \text{for} \quad 2.4 < t \leq 4.4 \\ 0 & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{5.47}$$

Under the SO initial condition above, the system converges to the SC-DTUE departure rate, $f^*(t)$, before day-step $2,500$. The equilibrium departure rate is as follows:

$$f^*(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4 \\ 3,600 & \text{for} \quad 2.4 < t \leq 3.2 \\ 600 & \text{for} \quad 3.2 < t \leq 4.4 \\ 0 & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{5.48}$$

We add the four types of tolls/incentives on day-step $2,500$, when the system reaches the SC-DTUE. We aim to find which toll can drive the system from SC-DTUE to a stable stationary SO state.

5.8.2 Time and day step sizes and advance/deferral coefficients

Before adding tolls/incentives on day-step $2,500$, we use the heuristic advance/deferral coefficients, $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ (Equation (3.99)) from the beginning. We switch to provably stable $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ (Equation (3.100)) on day step 2,000.

After adding optimal fine toll on day-step $2,500$, we still use the provably stable $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ (Equation (3.100)) for the rest of the simulation.

However, for optimal coarse toll, optimal fine reward, and optimal feebate, the system cannot converge to the stable stationary state with the provably stable $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ (Equation (3.100)) as in optimal fine toll. Moreover, there will be negative flow if we use the provably stable $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ — it violates the well-definedness condition for $f(\tau, t)$. So we use the following heuristic $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ right after adding these three types of tolls/incentives on day-step $2,500$:

$$B_i^d(\tau_j) = \frac{0.1}{\lambda} \quad \text{and} \quad B_i^a(\tau_j) = \frac{0.1}{\max\{\nu, (\lambda-\mu)\cdot\frac{f_1^m(\tau_j)}{C} - \lambda, (\lambda+\nu)\cdot\frac{f_2^m(\tau_j)}{C} - \lambda\}}, \quad (5.49)$$

where $B_i^d(\tau_j) = \frac{0.1}{\lambda}$ is 1/10 of $B_i^d(\tau_j)$ in Equation (3.99). We switch back to provably stable $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ on day-step $3,500$. It is because the provably stable region is limited, and we need to use the heuristic $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ to drive the system to the stability region, and then use provably stable $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ to have stability guarantee.

5.8.3 Error and stability measurements

Different from error measurements in the differences in costs in Chapter 3, we use error measurements in the difference in departure rate. Before adding tolls/incentives, we use the L_1 norm to measure the differences between current day's departure rate and SC-DTUE's departure rate ($f^*(t)$ in Equation (5.48)). After adding tolls/incentives, we measure the differences between the current day's departure rate $f_i(\tau_j)$ and SO's departure rate ($f^0(t)$ in

Equation (5.47)). The error is as follows:

$$e(j) = \sum_{i=0}^{I} |f_i(\tau_j) - f_i^*| \cdot \Delta t, \text{for} \quad j \leq 2499 \tag{5.50}$$

$$e(j) = \sum_{i=0}^{I} |f_i(\tau_j) - f_i^0| \cdot \Delta t, \text{for} \quad j > 2499 \tag{5.51}$$

To measure stability, we calculate the following discrete version of Lyapunov function on day τ_j defined in Equation (3.68):

$$V(f(\tau_j)) = \sum_{1}^{I} (i - \frac{1}{2}) \cdot \Delta t \cdot f_i(\tau_j) \cdot [\{-\omega_{i+1}(\tau_j)\}_+^2 + \{\omega_i(\tau_j)\}_+^2],$$

where the $V(f(\tau_0)) > 0$. If as day step τ_j increases, $V(f(\tau_j))$ decreases to zero, then it numerically shows that the day-to-day dynamical system is stable.

5.8.4 Simulation results

Optimal fine toll

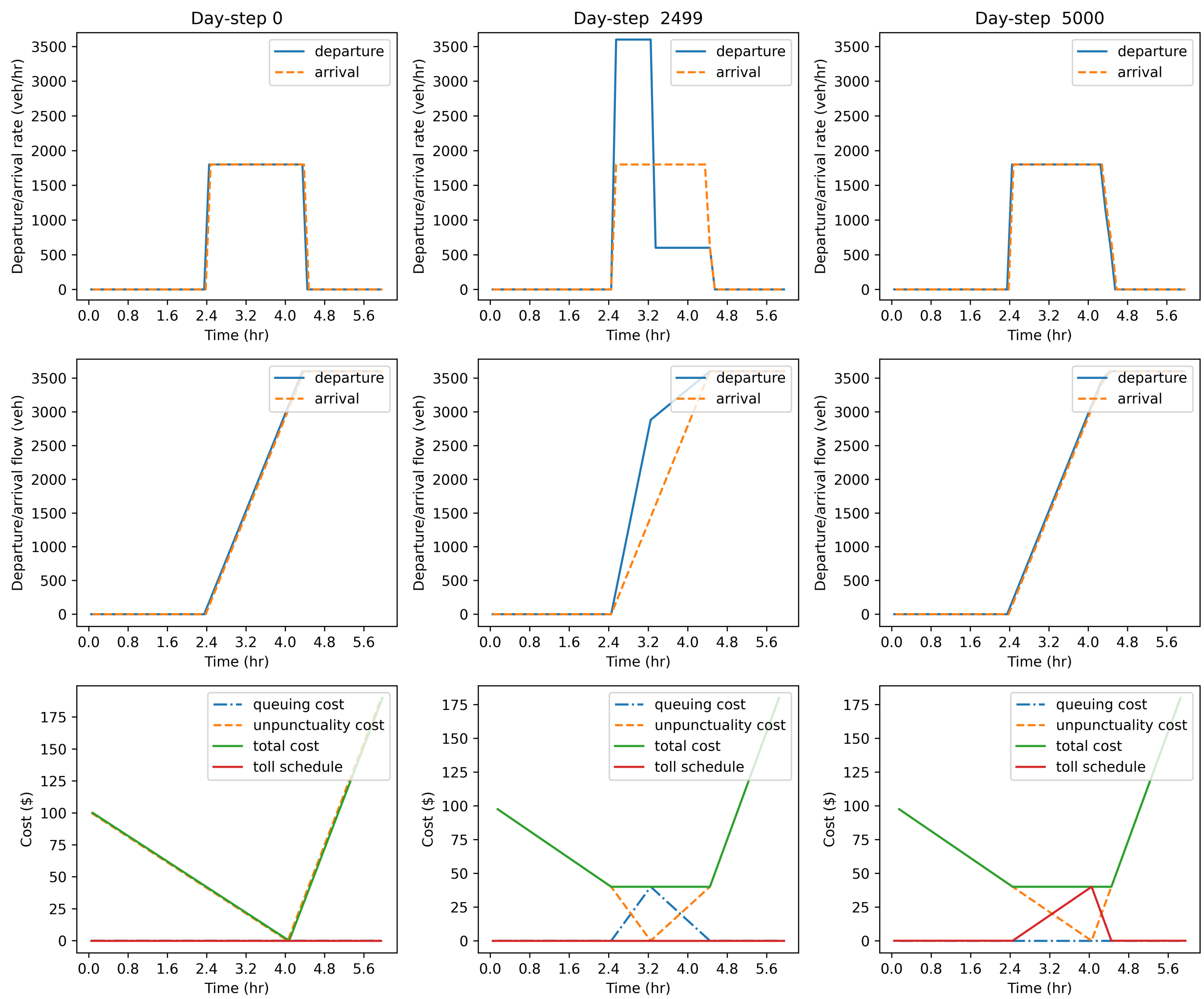


Figure 5.1: First, before adding optimal fine toll, and last day-step comparison

Figure 5.1 shows the departure/arrival rate, the cumulative departure/arrival flow and the trip cost on day steps 0, 2,499, and 5,000. The results show that optimal fine tolls can drive the system from SC-DTUE to the SO state.

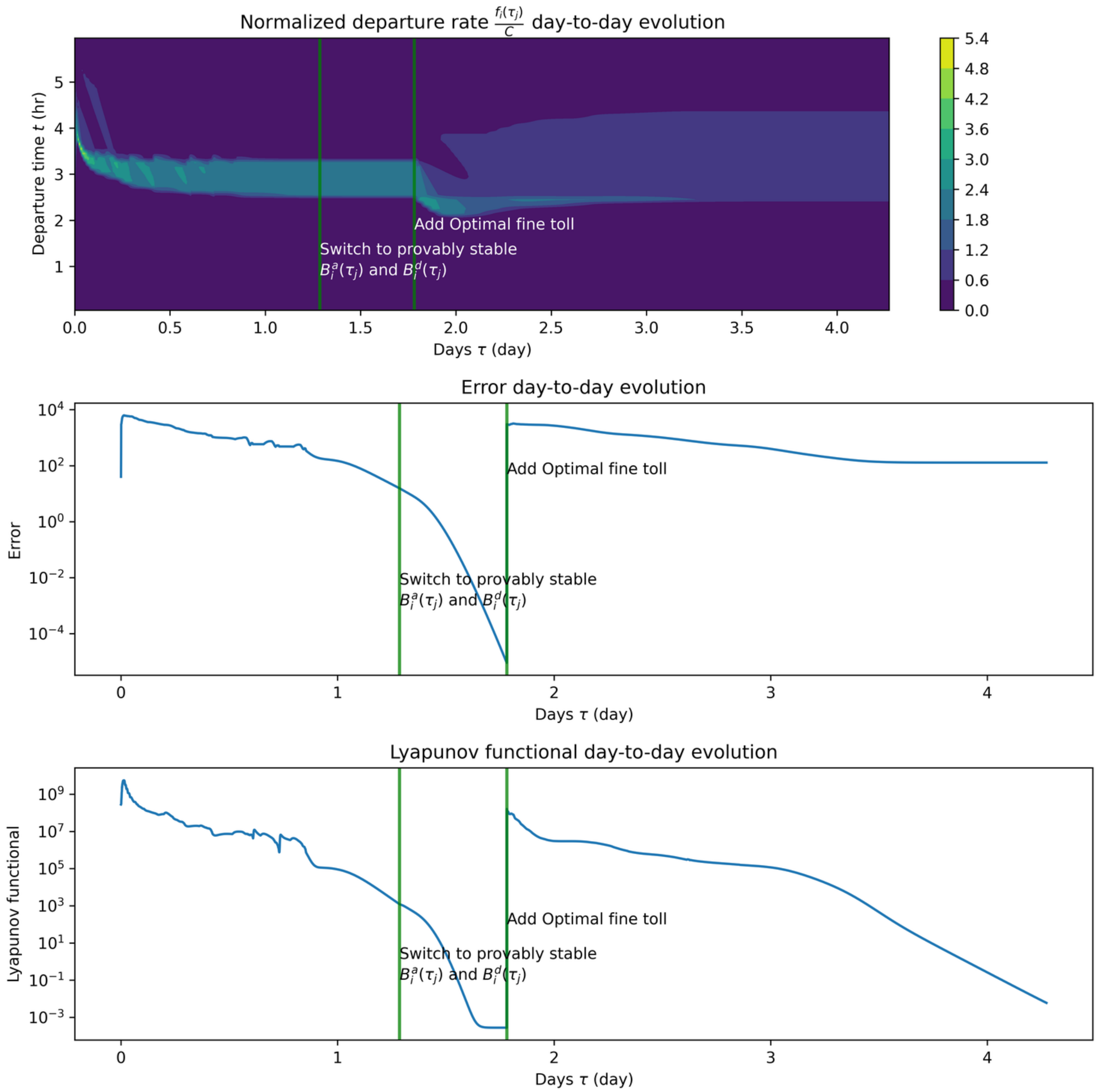


Figure 5.2: Day-to-day evolution of normalized departure rate, error, and Lyapunov functional of adding optimal fine toll

Figure 5.2 shows the day-to-day evolution of normalized departure rate, error, and the Lyapunov functional. The Lyapunov functional result shows that adding the optimal fine toll can drive the system to a stable, stationary SO state. The error does not approach zero, however. The reason is as follows: in the SO condition, there are 20 departure time steps with 1, 800 veh/hr departure rate each. However, after adding the optimal fine toll, there are 21 departure time steps. The first 19 departure time steps have a departure rate of 1, 800 veh / hr, while the last two departure time steps have departure rates of 1, 156 veh / hr and 644 veh / hr. Due to the difference in departure rates, the error does not approach zero. Nonetheless, the final state is still system optimal without any queues.

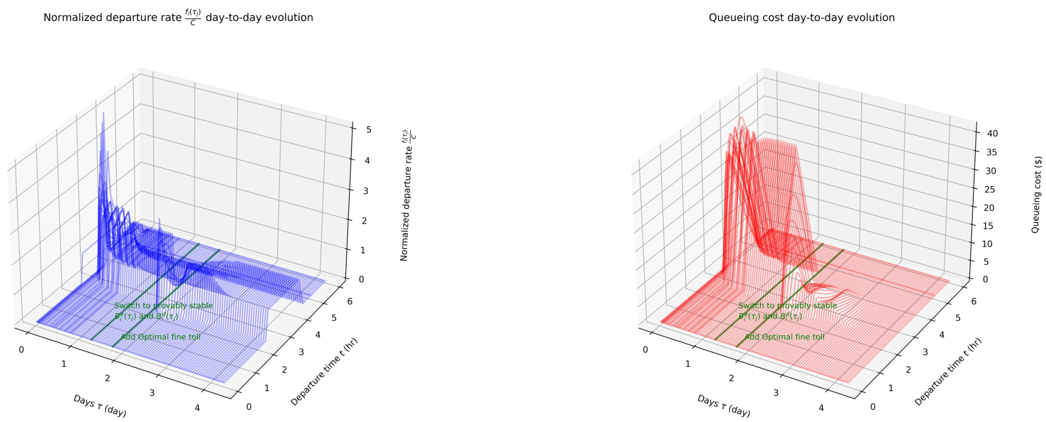


Figure 5.3: Day-to-day evolution of normalized departure rate and queuing cost of adding optimal fine toll

Figure 5.3 shows the day-to-day evolution of the normalized departure rate and the queuing cost. After applying the optimal fine toll, the system converges to SO quickly.

Optimal coarse toll

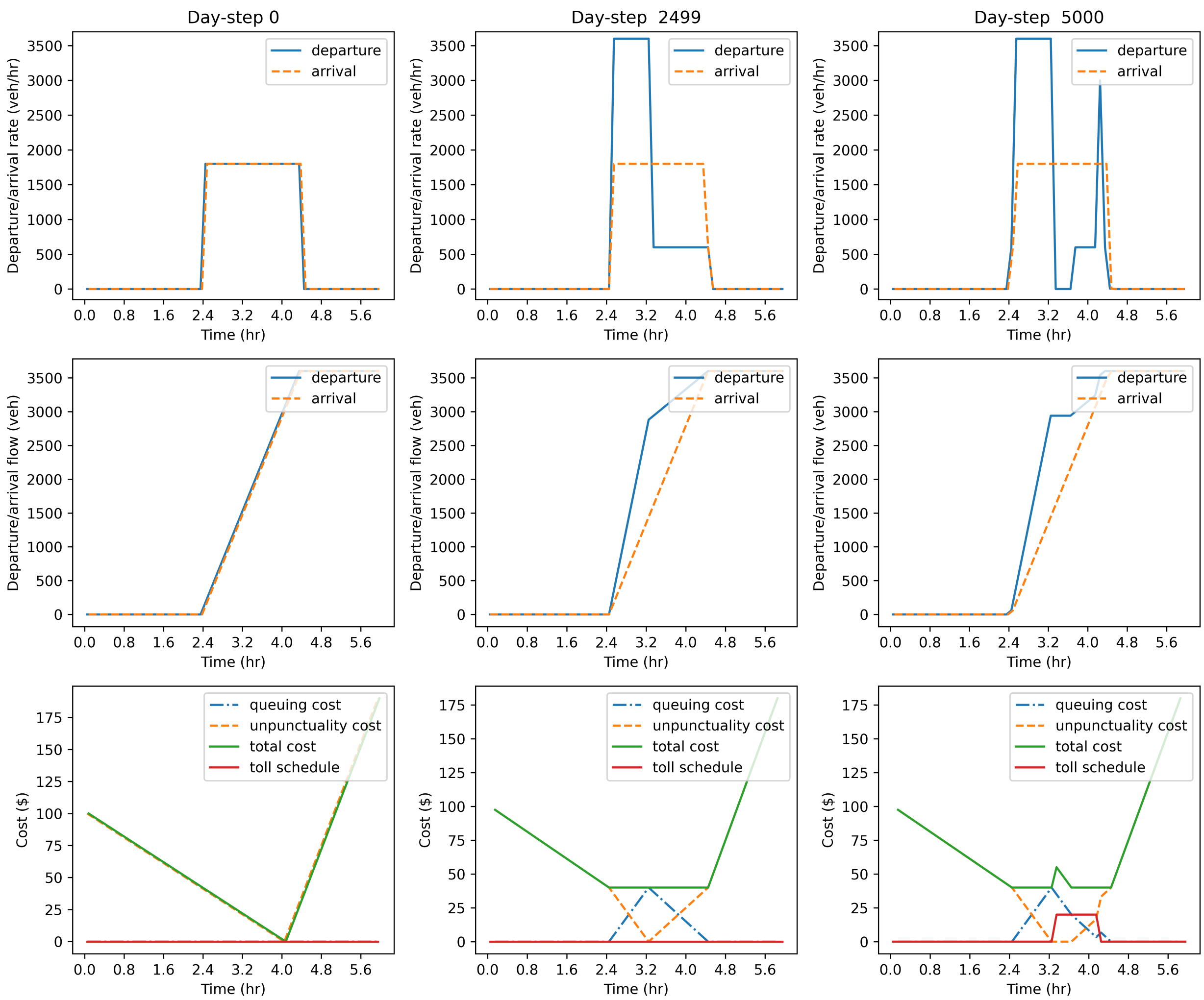


Figure 5.4: First, before adding optimal coarse toll, and last day-step comparison

Figure 5.4 shows the departure/arrival rate, the cumulative departure/arrival flow and the trip cost on day-step 0, 2,499, and 5,000. From the cumulative departure/arrival flow panel, the optimal coarse toll cannot drive the system from SC-DTUE to the SO state — congestion still exist on the last day-step. The results are different from equilibrium analysis in Arnott et al. (1990), where there is no departure for a period of $\frac{p^c}{\lambda} = \frac{20}{50} = 0.4$ hr before the toll is applied. Our results show that there is a a period with no departures of 0.4hr from 3.3 hr to 3.7 hr, after the toll is applied at 3.3 hr. There is a mass departure of 361 vehicles right after the toll is lifted at 4.2 hr, but it is different from the equilibrium analysis result of $2 \cdot C \cdot p^c/(\lambda + \nu) = 2 \cdot 1800 \cdot 20/(50 + 100) = 480$ veh in Arnott et al. (1990). Further research is needed to investigate the reasons for such differences.

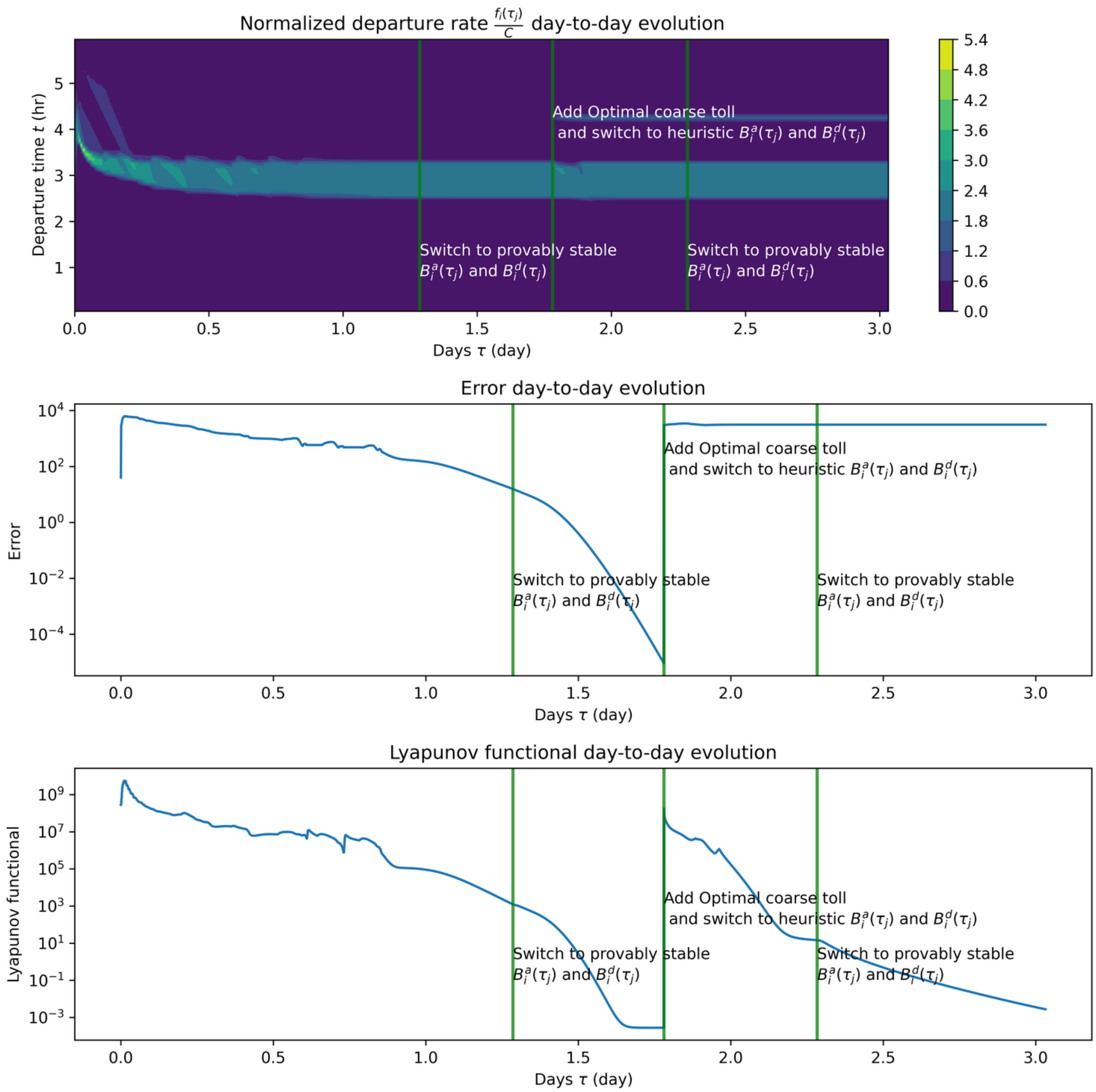


Figure 5.5: Day-to-day evolution of normalized departure rate, error, and Lyapunov functional of adding optimal coarse toll

Figure 5.5 shows the day-to-day evolution of normalized departure rate, error, and Lyapunov functional. The Lyapunov functional result shows that adding the optimal coarse toll with switching $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ can drive the system to a stable, stationary state, albeit not an SO state.

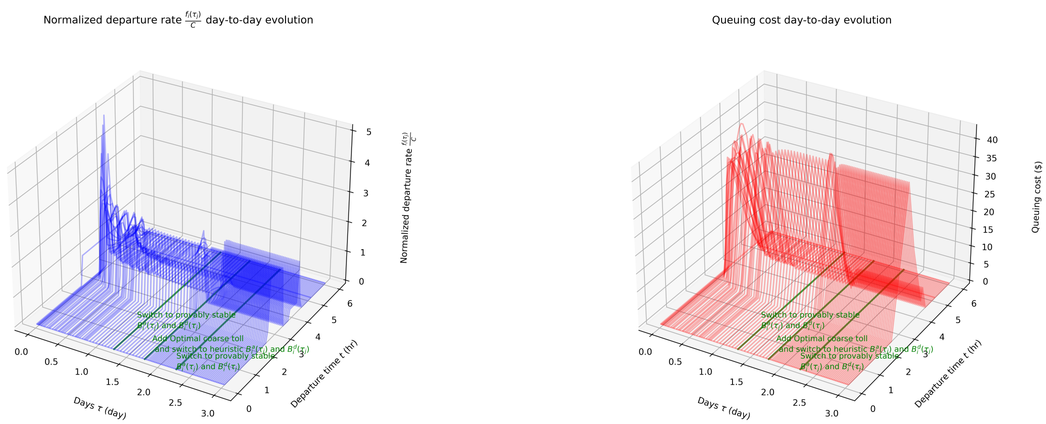


Figure 5.6: Day-to-day evolution of normalized departure rate and queuing cost of adding optimal coarse toll

Figure 5.6 shows the optimal coarse toll eliminates some congestion from day to day, but not all.

Optimal fine reward

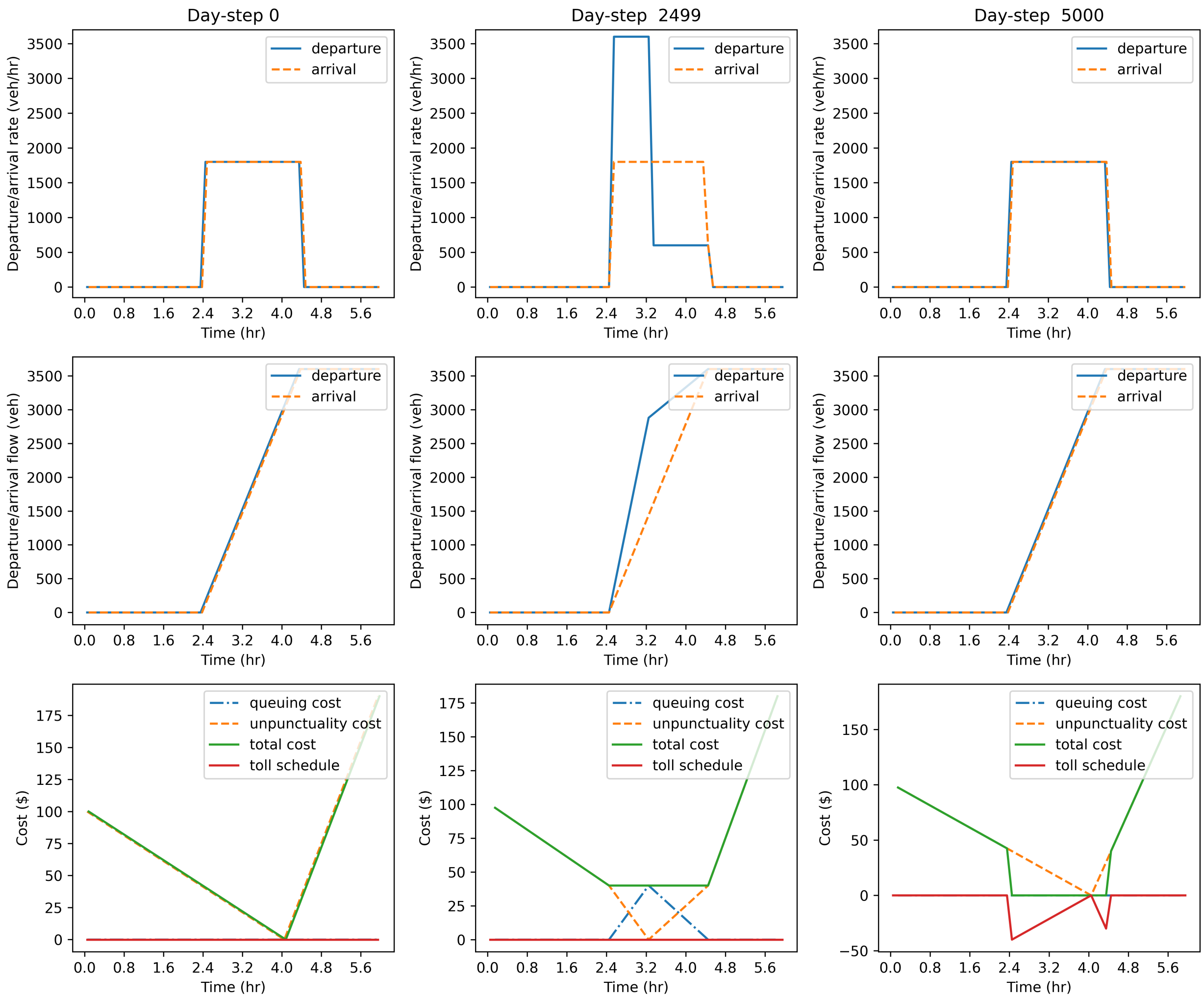


Figure 5.7: First, before adding optimal fine reward, and last day-step comparison

Figure 5.7 shows the departure/arrival rate, the cumulative departure/arrival flow and the trip cost on day-step 0, 2,499, and 5,000. From the cumulative departure/arrival flow panel, the optimal fine reward can drive the system from SC-DTUE to the SO state. The results align with the equilibrium analysis in Rouwendal et al. (2012).

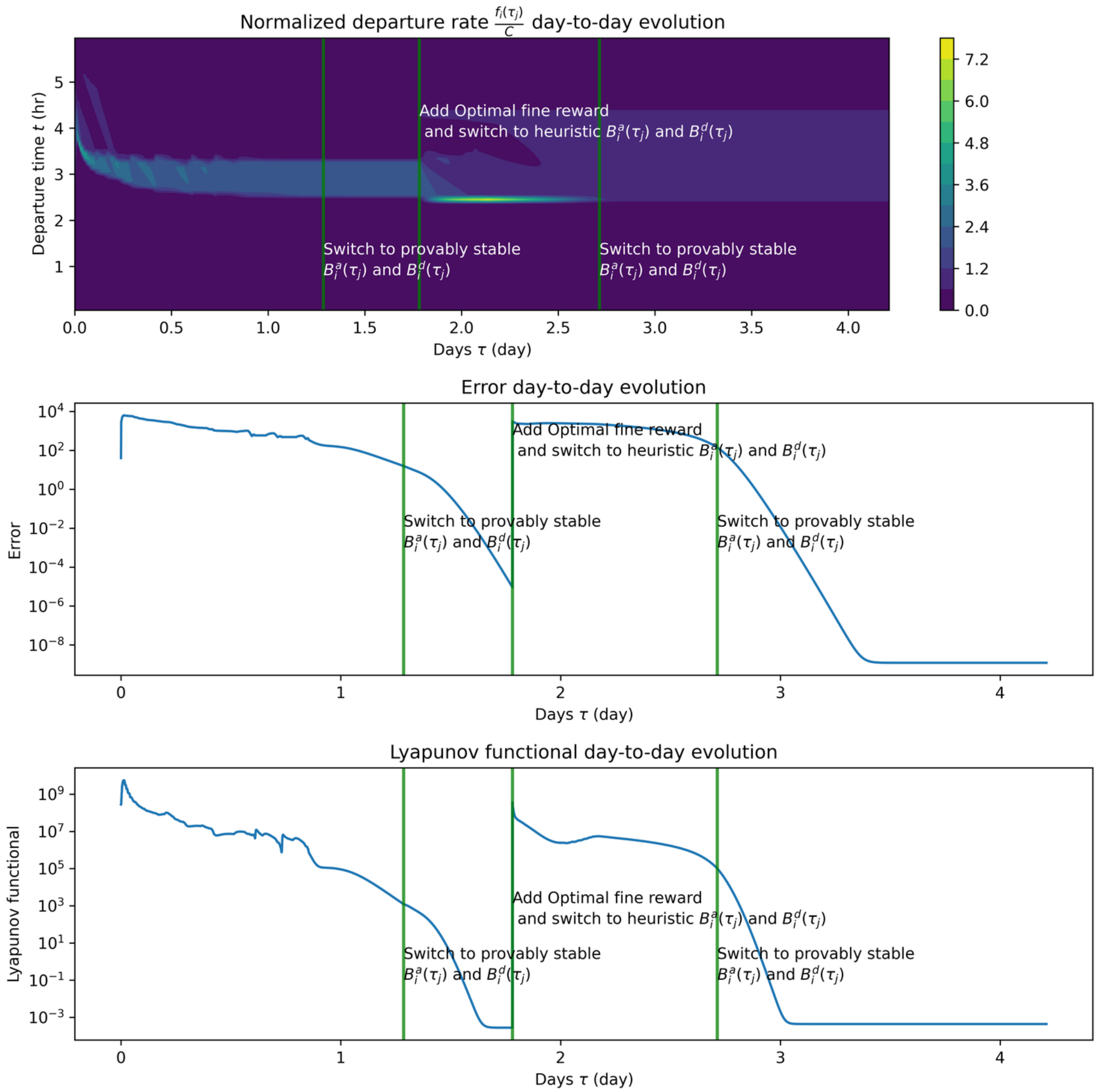


Figure 5.8: Day-to-day evolution of normalized departure rate, error and Lyapunov functional of adding optimal fine reward

Figure 5.8 shows the day-to-day evolution of normalized departure rate, error, and Lyapunov functional. The error and Lyapunov functional result shows that adding the optimal fine reward can drive the system to a stable, stationary SO state.

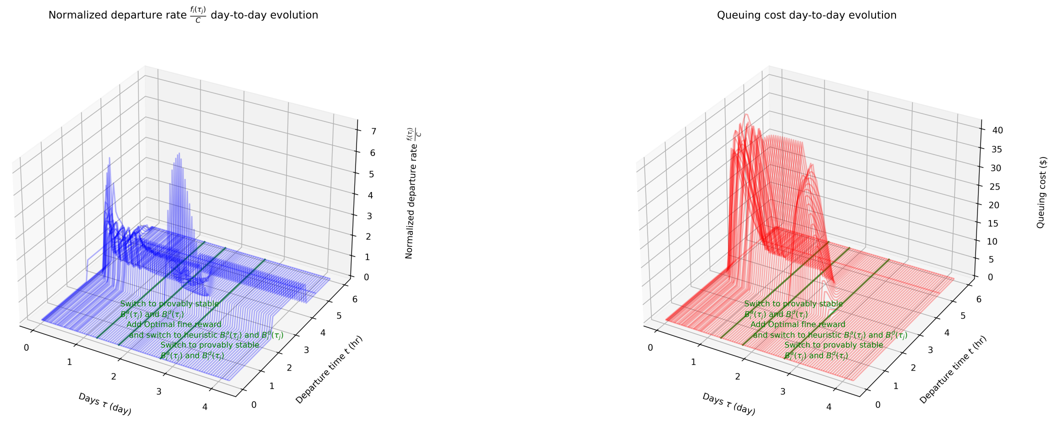


Figure 5.9: Day-to-day evolution of normalized departure rate and queuing cost of adding optimal fine reward

Figure 5.9 shows the optimal fine reward toll can eliminate congestion from day to day, and it takes fewer day steps to reach the SO ($3,583$ day steps) than optimal fine toll ($4,216$ day steps). Further study is needed to investigate the speed of convergence related to $B_i^d(\tau_j)$ and $B_i^a(\tau_j)$ and the tolls/incentives schemes.

Optimal feebate

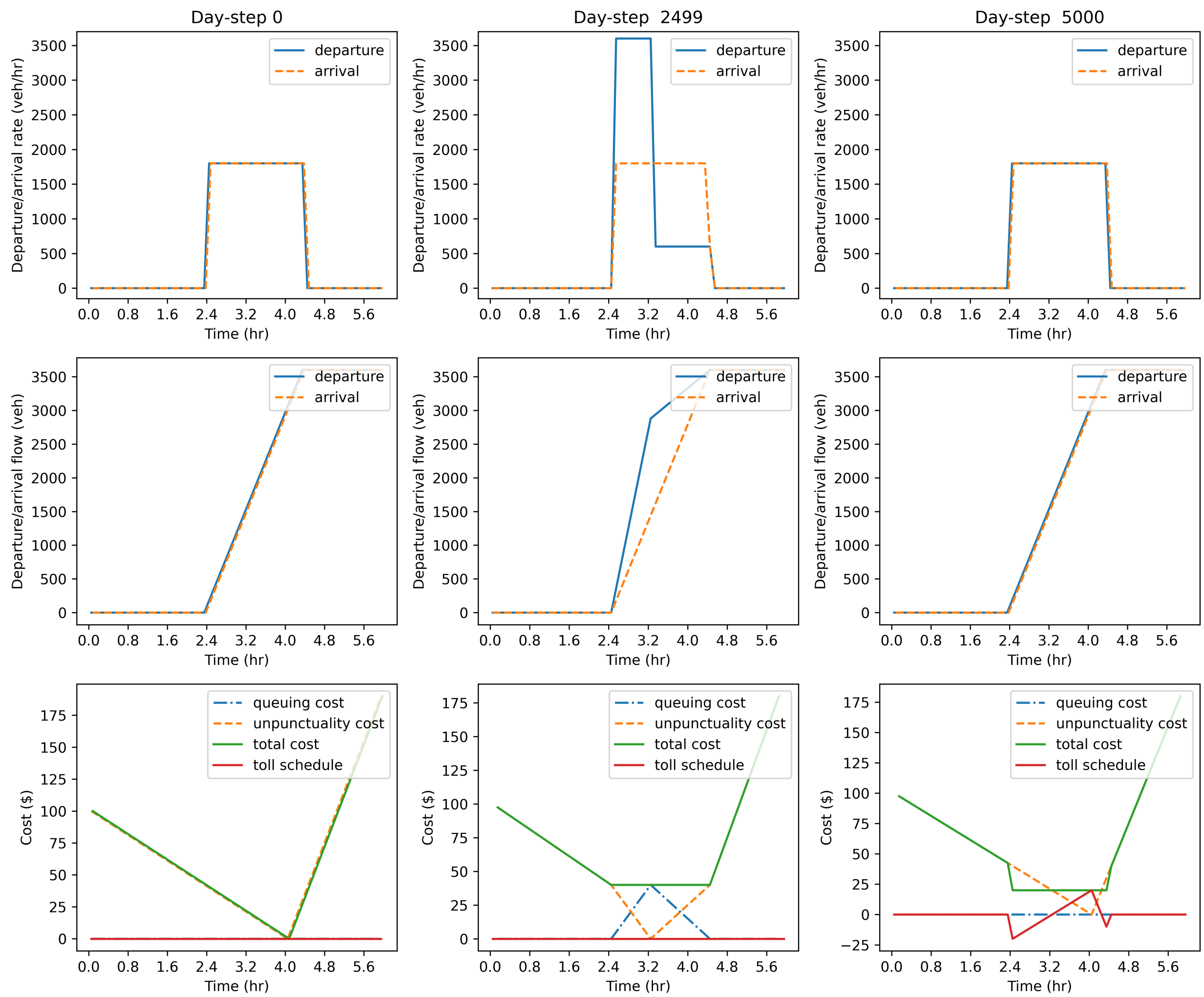


Figure 5.10: First, before adding optimal feebate, and last day-step comparison

Figure 5.10 shows the departure/arrival rate, the cumulative departure/arrival flow and the trip cost on day-step 0, 2,499, and 5,000. From the cumulative departure/arrival flow panel, the optimal feebate can drive the system from SC-DTUE to the SO state. The results align with the equilibrium analysis in Rouwendal et al. (2012).

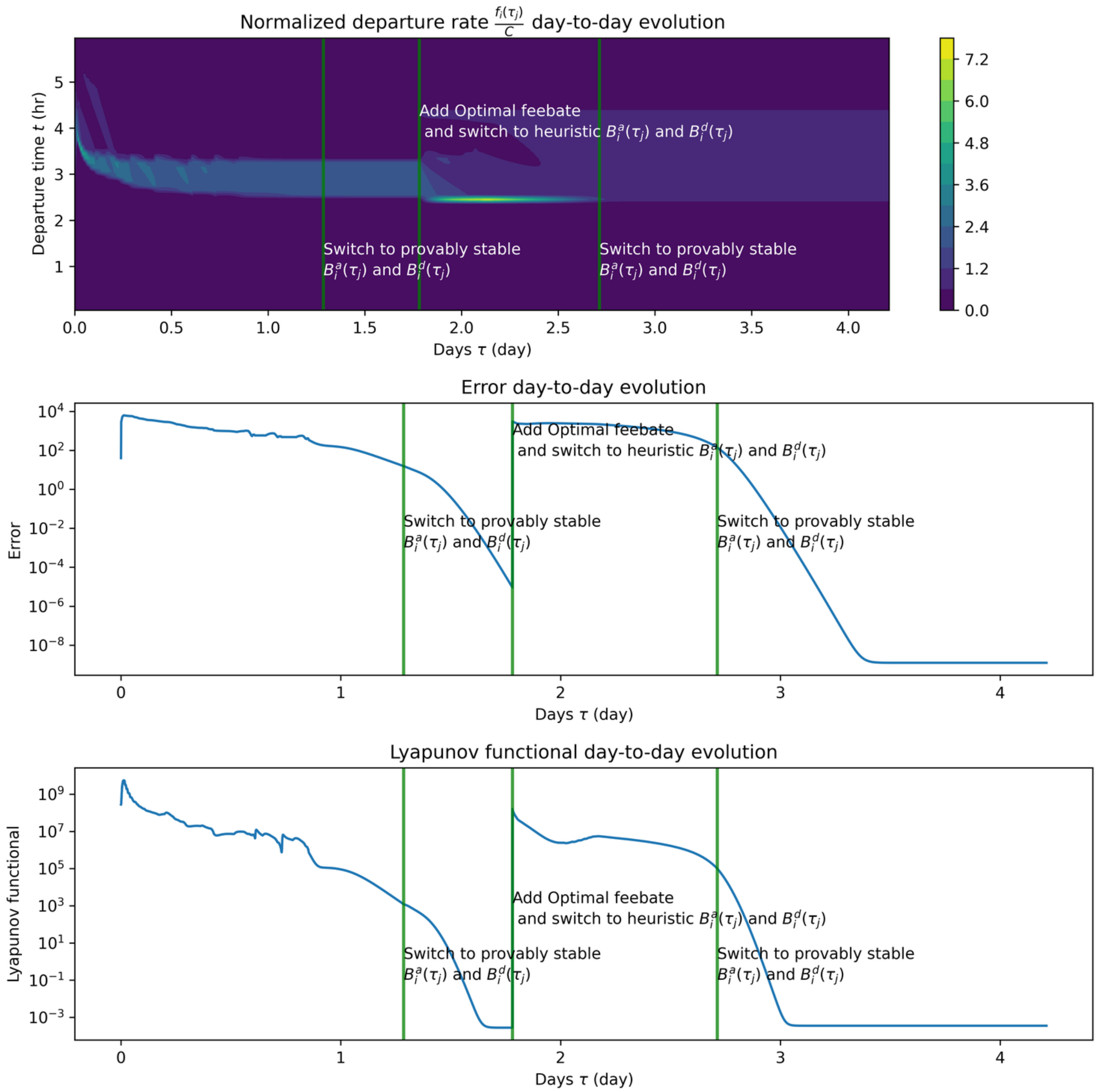


Figure 5.11: Day-to-day evolution of normalized departure rate, error and Lyapunov functional of adding optimal feebate

Figure 5.11 shows the day-to-day evolution of normalized departure rate, error, and Lyapunov functional. The error and Lyapunov functional result shows that adding the optimal feebate can drive the system to a stable, stationary SO state.

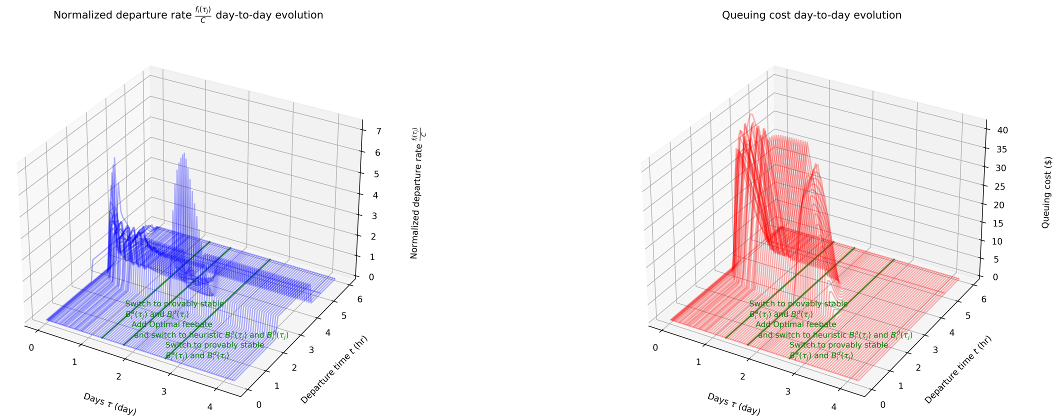


Figure 5.12: Day-to-day evolution of normalized departure rate and queuing cost of adding optimal feebate

Figure 5.12 shows the optimal feebate toll can eliminate congestion from day to day, and it takes the same number of day steps to reach the SO as optimal fine reward (3,583 day steps), which is fewer than optimal fine toll (4,216 day steps).

5.8.5 Conclusion

Among four tolls, the optimal fine toll, optimal fine reward and optimal feebate can drive the system to a stable stationary system optimal state, while optimal coarse toll cannot. We will mathematically prove the stability of the optimal fine toll/reward/and feebate below.

5.9 Tolling on multi-class travelers

We consider applying a toll in multi-class traveler cases in the following sections. Here we only consider the following multi-class scenario: different λ/μ, same μ/ν and same t^* (first type of heterogeneity in Arnott et al. (1994)).

5.9.1 Definitions

We will present the definition in the same order as in Chapter 4.

Within-day dynamics: heterogeneous point queue model

For multi-class within-day traffic dynamics, it is described by the point queue model in Section 4.2.1.

5.9.2 Trip cost: heterogeneous travelers

Let $\phi^g(\tau, t)$ denote the trip cost for group g's travelers departing at time t from home on day τ (unit: \$). Let $t^{g,*}$ be the desired arrival time for group g's travelers. Let $p(t)$ be the toll or incentives, which is only time-dependent. If $p(t) \geq 0$, then the traveler is charged a toll, while $p(t) < 0$, then the traveler is compensated with incentives. We have the following cost function for group g's travelers:

$$\phi^g(\tau, t) = \lambda_g \cdot (\Upsilon^0 + \Upsilon(\tau, t)) + \mu_g \cdot \{t^{g,*} - (t + \Upsilon(\tau, t))\}_+ + \nu_g \cdot \{(t + \Upsilon(\tau, t)) - t^{g,*}\}_+ + p(t), \quad (5.52)$$

Without loss of generality, we assume the free flow travel time to be zero (i.e., $\Upsilon^0 = 0$). The variables are described in Section 4.2.2. Let $\phi_1^g(\tau, t) = \lambda_g \cdot (\Upsilon^0 + \Upsilon(\tau, t))$ be the queuing cost and $\phi_2^g(\tau, t) = \mu_g \cdot \{t^{g,*} - (t + \Upsilon(\tau, t))\}_+ + \nu_g \cdot \{(t + \Upsilon(\tau, t)) - t^{g,*}\}_+$ be the unpunctuality cost. We assume $\mu_g < \lambda_g$ for each group to avoid multiple equilibrium (Arnott et al., 1994).

Let $\omega^g(\tau, t)$ be the rate of change in trip cost for group g's travelers, and that is $\omega^g(\tau, t) = \frac{\partial}{\partial t}\phi^g(\tau, t)$. For early arrivals, we have $\omega^E(\tau, t) = \frac{\partial}{\partial t}\phi^E(\tau, t)$ with tolls (E for early arrivals):

$$\omega^{E,g}(\tau, t) = \frac{\partial}{\partial t}\phi^{E,g}(\tau, t) = (\lambda_g - \mu_g) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau, t) - \mu_g + \frac{d}{dt}p(t) \quad (5.53)$$

For late arrivals, we have the following rate of change in trip cost, $\omega^{L,g}(\tau,t) = \frac{\partial}{\partial t}\phi^{L,g}(\tau,t)$ with tolls (L for late arrivals):

$$\omega^{L,g}(\tau,t) = \frac{\partial}{\partial t}\phi^{L,g}(\tau,t) = (\lambda_g + \nu_g)\cdot\frac{\partial}{\partial t}\Upsilon^{L}(\tau,t) + \nu_g + \frac{d}{dt}p(t), \tag{5.54}$$

where the queuing time for early and late arrivals, $\Upsilon^{E}(\tau,t)$ and $\Upsilon^{L}(\tau,t)$ do not have superscript g, because queuing time is not group specific. All groups of travelers share the same queuing time as long as they join the queue at the same time t.

5.9.3 System optimum for multi-class travelers in a single bottleneck

Definition 5.4 (System optimum for multi-class users)**.** *The system optimum for a single bottleneck is reached when there is no queue, where the departure rate is less than or equal to the bottleneck capacity for the entire departure period. That is, $\delta(\tau,t)=0$, and $f(\tau,t)\leq C$ for $t\in[t_0,t_2]$, where t_0 and t_2 are the start and end of the departure period at system optimum.*

From Definition 5.4, we have the following trip cost function without queuing cost and before adding a tolling:

$$\phi^{g}(\tau,t) = \mu_g\cdot\{t^{g,*}-t\}_+ + \nu_g\cdot\{t-t^{g,*}\}_+, \tag{5.55}$$

and after adding a toll:

$$\phi^{g}(\tau,t) = \mu_g\cdot\{t^{g,*}-t\}_+ + \nu_g\cdot\{t-t^{g,*}\}_+ + p(t). \tag{5.56}$$

5.10 Multi-class day-to-day dynamics with tolls

We present the multi-class day-to-day dynamics with tolls below.

5.10.1 Discrete version

The discrete version is described in Section 4.3.1 in Chapter 4.

5.10.2 Well-definedness

Different tolls can will influence the shape of $\omega(\tau, t)$, which influences the selection of $B_i^{a,g}(\tau)$ and $B_{i+1}^{d,g}(\tau)$ for the dynamical system to be well-defined.

5.10.3 Continuous version

Chapter 4 presents the details of deriving the continuous version of the multi-class day-to-day dynamical system. Here we present the final conservation equation again:

$$\frac{\partial}{\partial \tau} f^g(\tau, t) - \frac{\partial}{\partial t} u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) = 0, \tag{5.57}$$

where

$$u^g(\tau, t) = \frac{\Delta t}{\Delta \tau} \cdot \begin{cases} B^{a,g}(\tau, t) & \text{for} \quad \omega^g(\tau, t) \geq 0 \\ B^{d,g}(\tau, t) & \text{for} \quad \omega^g(\tau, t) < 0, \end{cases} \tag{5.58}$$

Integrating both terms of Equation (5.57) with respect to time t, we have:

$$\frac{\partial}{\partial \tau} F^g(\tau, t) - u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot \frac{\partial}{\partial t} F^g(\tau, t) = 0, \tag{5.59}$$

which is a hyperbolic conservation equation with respective to cumulative departure flow $F^g(\tau, t)$, and $-u^g(\tau, t) \cdot \omega^g(\tau, t)$ is the speed of the characteristic wave for this system. Adding a toll $p(t)$ will influence the shape of $\omega^g(\tau, t)$.

5.11 Tolls for multi-class traveler

In this section, we will only present the optimal fine toll for the following multi-class scenario: different λ/μ, same μ/ν and same t^* (first type of heterogeneity in Arnott et al. (1994)).

5.11.1 Optimal fine toll

The optimal fine toll for multi-class travelers is calculated by assuming that the system optimum is reached (i.e., $\delta(\tau, t) = 0$, and $\Upsilon(\tau, t) = 0$ and Definition 5.4), what kind of pricing can lead to this this system optimum state?

We have $\omega^{E,g}(\tau, t) = \frac{\partial}{\partial t}\phi^{E,g}(\tau, t) = (\lambda_g - \mu_g) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau, t) - \mu_g + \frac{d}{dt}p(t)$ from Equation (5.53) for early arrivals. Assuming the bottleneck reaches system optimal, where there is no congestion (i.e., $\delta(\tau, t) = 0$), we have $\Upsilon(\tau, t) = 0$ and $\frac{\partial}{\partial t}\Upsilon^E(\tau, t) = 0$. For $\omega^{E,g}(\tau, t) = 0$, we need $\frac{d}{dt}p(t) = \mu_g$ for the early-arrival departure interval of group g, $[t^g_{0,1}, t^g_{2,1}]$. Notice that $[t^g_{0,1}, t^g_{2,1}]$ with optimal fine toll could be different from $[t^g_{0,1}, t^g_{2,1}]$ in the MC-DTUE in the no toll cases.

We have $\omega^{L,g}(\tau, t) = \frac{\partial}{\partial t}\phi^{L,g}(\tau, t) = (\lambda_g + \nu_g) \cdot \frac{\partial}{\partial t}\Upsilon^L(\tau, t) + \nu_g + \frac{d}{dt}p(t)$ from Equation (5.54) for late arrivals. Assuming the bottleneck reaches system optimal, where there is no congestion (i.e., $\delta(\tau, t) = 0$), we have $\Upsilon(\tau, t) = 0$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau, t) = 0$. For $\omega^{L,g}(\tau, t) = 0$, we need $\frac{d}{dt}p(t) = -\nu_g$ for the late-arrival departure interval of group g, $[t^g_{0,2}, t^g_{2,2}]$. Notice that $[t^g_{0,2}, t^g_{2,2}]$ with optimal fine toll could be different from $[t^g_{0,2}, t^g_{2,2}]$ in the MC-DTUE in the no toll cases.

We consider there are two groups of travelers, g and g', where we have different λ/μ, different

μ, same μ/ν and same t^*. Assuming $\lambda_{g'}/\mu_{g'} > \lambda_g/\mu_g$, so group $g^{'}$ will depart first from Section 4.7 in Chapter 4. Let t_0 be the start of the congested period and t_2 be the end of the congested period. Let $[t^{g'}_{0,1}, t^{g'}_{2,1}]$ and $[t^{g'}_{0,2}, t^{g'}_{2,2}]$ be the first and the second departure period of group $g^{'}$ under optimal fine toll. Let $[t^{g}_{0,1}, t^{g}_{2,1}]$ be the only departure period of group g under optimal fine toll, and it is in the middle of the congested period. We have the following formula of the optimal fine toll for multi-class travelers:

$$p^{o,M}(t) = \begin{cases} 0 & \text{for} \quad t < t_0 = t^{g'}_{0,1} \\ \mu_{g'} \cdot (t - t^{g'}_{0,1}) & \text{for} \quad t^{g'}_{0,1} \leq t < t^{g'}_{2,1}, \\ \phi^{g',*} + \mu_g \cdot (t - t^{g}_{0,1}) & \text{for} \quad t^{g'}_{2,1} = t^{g}_{0,1} \leq t < t^*, \\ \phi^{g',*} + \nu_g \cdot (t^{g}_{2,1} - t) & \text{for} \quad t^* \leq t < t^{g}_{2,1} = t^{g'}_{0,2}, \\ -\nu_{g'} \cdot (t - t^{g'}_{2,2}) & \text{for} \quad t^{g'}_{0,2} \leq t < t^{g'}_{2,2}, \\ 0 & \text{for} \quad t \geq t_2 = t^{g'}_{2,2}, \end{cases} \tag{5.60}$$

where $\phi^{g',*} = \frac{\mu_{g'} \cdot \nu_{g'}}{2 \cdot (\mu_{g'} + \nu_{g'})} \cdot \frac{N}{C}$.

5.12 Stationary state of multi-class dynamical system with pricing

Definition 5.5. *The stationary state of the multi-class dynamical system in Equation (4.56), (discrete version: Equations (4.21) to (4.23)) with pricing is reached when $\frac{\partial}{\partial \tau} f^g(\tau, t) = 0$, which means that the time-dependent departure rate, $f^g(\tau, t)$ does not change from day to day.*

When $\frac{\partial}{\partial \tau} f^g(\tau, t) = 0$, we have $\frac{\partial}{\partial \tau} F^g(\tau, t) = 0$ as well. According to Equation (4.58), we have $u^g(\tau, t) \cdot \omega^g(\tau, t) \cdot f^g(\tau, t) = 0$ at the stationary state. Since $u^g(\tau, t) > 0$, we have the following complementarity condition:

$$f^g(\tau, t) \cdot \omega^g(\tau, t) = 0, \tag{5.61}$$

for all $t \in [0, T]$.

5.13 Stationary state with tolls and system optimum

Theorem 5.3. *The stationary state of multi-class local dynamical system, Equation (4.56), (discrete version: Equations (4.21) to (4.23)) with optimal fine toll for multi-class travelers is equivalent to system optimum.*

Proof. To prove Theorem 5.3, we first prove the system optimum is the stationary state with optimal fine toll, and we then prove the stationary state with optimal fine toll is system optimum.

1. For group g''s two departure periods $[t^{g'}_{0,1}, t^{g'}_{2,1}]$ and $[t^{g'}_{0,2}, t^{g'}_{2,2}]$, when the system reaches the system optimum (i.e., $\delta(\tau, t) = 0$) with optimal fine toll from Equation (5.60), we have $\phi^{g'}(\tau, t) = \mu_{g'} \cdot (t^* - t) + p^{o,M} = \mu_{g'} \cdot (t^* - t) + \mu_{g'} \cdot (t - t_0) = \mu_{g'} \cdot (t^* - t_0) = \frac{\mu_{g'} \cdot \nu_{g'}}{2 \cdot (\mu_{g'} + \nu_{g'})} \cdot \frac{N}{C}$ for early arrivals and $\phi^{g'}(\tau, t) = \nu_{g'} \cdot (t - t^*) + p^o = \nu_{g'} \cdot (t - t^*) - \nu_{g'}(t - t_2) = \nu_{g'} \cdot (t_2 - t^*) = \frac{\mu_{g'} \cdot \nu_{g'}}{2 \cdot (\mu_{g'} + \nu_{g'})} \cdot \frac{N}{C}$ for late arrivals. Let $\phi^{g',*} = \frac{\mu_{g'} \cdot \nu_{g'}}{2 \cdot (\mu_{g'} + \nu_{g'})} \cdot \frac{N}{C}$.

   For group g's departure period $[t^g_{0,1}, t^g_{2,1}]$, we have we have $\phi^g(\tau, t) = \mu_g \cdot (t^* - t) + p^{o,M} = \mu_g \cdot (t^* - t) + \phi^{g',*} + \mu_g \cdot (t - t^g_{0,1}) = \mu_g \cdot (t^* - t^g_{0,1}) + \phi^{g',*}$ for early arrivals and $\phi^g(\tau, t) = \nu_g \cdot (t - t^*) + p^o = \nu_g \cdot (t - t^*) + \phi^{g',*} + \nu_g(t^g_{2,1} - t) = \nu_g \cdot (t^g_{2,1} - t^*) + \phi^{g',*}$ for late arrivals. We have $\mu_g \cdot (t^* - t^g_{0,1}) = \nu_g \cdot (t^g_{2,1} - t^*)$ from (Arnott et al., 1994). So $\phi^g(\tau, t) = \phi^{g,*}$ for group g's departure period.

Since $\omega^g(\tau,t) = \frac{\partial}{\partial t}\phi^g(\tau,t)$, we have $\omega^g(\tau,t) = 0$ for $f^g(\tau,t) > 0$ and $\omega^g(\tau,t) \neq 0$ for $f^g(\tau,t) = 0$ for both groups.

So we the following complementarity condition:

$$f^g(\tau,t) \cdot \omega^g(\tau,t) = 0. \tag{5.62}$$

Since $f^g(\tau,t) \cdot \omega^g(\tau,t) = 0$ at system optimum with optimal fine toll, then we have $u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) = 0$, so $\frac{\partial}{\partial t} u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) = 0$. From Equation (4.56), we know $\frac{\partial}{\partial \tau} f^g(\tau,t) = 0$. This proves that the system optimum with optimal fine toll is in the stationary state.

2. We now prove that the stationary state with multi-class optimal fine toll is the system optimum. Given system reaches the stationary state, $\frac{\partial}{\partial \tau} f^g(\tau,t) = 0$, we have $\frac{\partial}{\partial \tau} F^g(\tau,t) = 0$ as well. According to Equation (4.58), we have $u^g(\tau,t) \cdot \omega^g(\tau,t) \cdot f^g(\tau,t) = 0$. Since $u(\tau,t) > 0$, we have the following complementary condition: $\omega^g(\tau,t) \cdot f^g(\tau,t) = 0$.

   In the stationary state, the characteristic wave speed of Equation (5.59) is: $-u^g(\tau,t) \cdot \omega^g(\tau,t)$. Let t_0 and t_2 be the start and the end of the departure period.

   When $t < t_0$, the optimal fine toll, $p^{o,M}(t)$, is zero and the bottleneck is uncongested, so we have $-u^g(\tau,t) \cdot \omega^g(\tau,t) = \mu_g$. When $t > t_2$, the optimal fine toll, $p^{o,M}(t)$, is zero as well, so we have $-u^g(\tau,t) \cdot \omega^g(\tau,t) = -\nu_g$. When $t \in [t_0, t_2]$, we have $-u^g(\tau,t) \cdot \omega^g(\tau,t) = 0$. This leads to one connected system departure pattern in stationary state.

   We then consider departure period $t \in [t_0, t_2]$. From Equation (5.53), for group g's early arrivals, we have:

$$\omega^{E,g}(\tau,t) = \frac{\partial}{\partial t}\phi^{E,g}(\tau,t) = (\lambda_g - \mu_g) \cdot \frac{\partial}{\partial t}\Upsilon^E(\tau,t) - \mu_g + \mu_g. \tag{5.63}$$

For group g's late arrivals, from Equation (5.54) we have:

$$\omega^{L,g}(\tau,t) = \frac{\partial}{\partial t}\phi^{L,g}(\tau,t) = (\lambda_g + \nu_g)\cdot\frac{\partial}{\partial t}\Upsilon^L(\tau,t) + \nu_g - \nu_g. \tag{5.64}$$

For $\omega^{E,g}(\tau,t)$ and $\omega^{L,g}$ to be zero, $\frac{\partial}{\partial t}\Upsilon^E(\tau,t)$ and $\frac{\partial}{\partial t}\Upsilon^L(\tau,t)$ have to be zero. We have $\frac{\partial}{\partial t}\Upsilon(\tau,t) = \max\{-\frac{\delta(\tau,t)}{C\cdot\epsilon}, \frac{f(\tau,t)}{C} - 1\}$ from Equation (3.12). For $\frac{\partial}{\partial t}\Upsilon(\tau,t)$ to be zero, we have $\delta(\tau,t) = 0$ and $f(\tau,t) \leq C$. So there is no queue during $[t_0, t_2]$ in the stationary state with the optimal fine toll, which is the system optimal state.

We now prove that the stationary state with multi-class optimal fine toll is system optimal. We will then prove that the departure order with multi-class optimal fine toll is the multi-class system optimal departure order.

Without loss of generality, we consider there to be two groups of travelers, g and $g^{'}$, where $\frac{\lambda_{g'}}{\mu_{g'}} > \frac{\lambda_g}{\mu_g}$, $\frac{\mu_g}{\nu_g} = \frac{\mu_{g'}}{\nu_{g'}}$, $\lambda_{g'} = \lambda_g$, $\mu_g > \mu_{g'}$, $\nu_g > \nu_{g'}$, and $t^{g,*} = t^{g',*}$, so group $g^{'}$ will depart first based on Section 4.7 in Chapter 4. Since the bottleneck is system optimal, we have $\omega^{g,E}(\tau,t) = -\mu_g + \frac{d}{dt}p^{o,M}(t)$ for early arrivals and $\omega^{g,L}(\tau,t) = \nu_g + \frac{d}{dt}p^{o,M}(t)$ for late arrivals.

For group $g^{'}$'s early arrivals, the characteristic wave speed $-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t)$ is as follows:

$$-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t) = \begin{cases} \mu_{g'} > 0 & \text{for} \quad t < t_0 = t_{0,1}^{g'} \\ 0 & \text{for} \quad t_{0,1}^{g'} \leq t \leq t_{2,1}^{g'}, \\ -u^{g'}(\tau,t)\cdot(\mu_g - \mu_{g'}) < 0 & \text{for} \quad t_{2,1}^{g'} < t \leq t^*, \end{cases} \tag{5.65}$$

where rate of change in the optimal fine toll during $[t_{2,1}^{g'}, t^*]$ is $\frac{d}{dt}p^{o,M}(t) = \mu_{g'}$, and $\mu_g - \mu_{g'} > 0$.

For group $g^{'}$'s late arrivals, the characteristic wave speed $-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t)$ is as follows:

$$-u^{g'}(\tau,t)\cdot\omega^{g'}(\tau,t)=\begin{cases}-u^{g'}(\tau,t)\cdot(\nu_{g'}-\nu_g)>0 & \text{for} \quad t^*<t<t_{0,2}^{g'}\\ 0 & \text{for} \quad t_{0,2}^{g'}\le t\le t_{2,2}^{g'},\\ -\nu_{g'}<0 & \text{for} \quad t>t_2=t_{0,2}^{g'},\end{cases} \tag{5.66}$$

where rate of change in the optimal fine toll during $[t^*, t_{0,2}^{g'}]$ is $\frac{d}{dt}p^{o,M}(t) = -\nu_g$, and $\nu_{g'} - \nu_g < 0$.

For group g's arrivals, the characteristic wave speed $-u^g(\tau,t)\cdot\omega^g(\tau,t)$ is as follows:

$$-u^{g}(\tau,t)\cdot\omega^{g}(\tau,t)=\begin{cases}-u^{g}(\tau,t)\cdot(-\mu_g+\mu_{g'})>0 & \text{for} \quad t<t_{0,1}^{g}\\ 0 & \text{for} \quad t_{0,1}^{g}\le t\le t_{2,1}^{g},\\ -u^{g}(\tau,t)\cdot(\nu_g-\nu_{g'})<0 & \text{for} \quad t>t_{2,1}^{g}.\end{cases} \tag{5.67}$$

From the characteristic wave speed, we know that group g' departs from the shoulder of the peak, and group g departs in the middle of the peak. This is the order of departure of multi-class system optimum.

We prove that the system optimum is the stationary state with multi-class optimal fine toll, and that the stationary state with multi-class optimal fine toll is the system optimum. So this completes the proof of Theorem 5.3. □

5.14 Stability of the stationary state with multi-class optimal fine toll

We are interested in knowing whether the multi-class optimal fine toll can drive the multi-class day-to-day dynamical system to a stable stationary system optimal state. More rigorously, this means that it reaches a stable system optimum ($\delta(\tau,t)=0$ for $t\in[t_0,t_2]$) at the stationary state $\frac{\partial}{\partial\tau}f^g(\tau,t)=0$.

From Section 5.13, we have $\delta(\tau,t)=0$ at the stationary state. However, before reaching the stationary state, we have $\delta(\tau,t)\geq 0$ for $t\in[t_0,t_2]$.

From Section 5.13, in the stationary state, we have $-u^g(\tau,t)\cdot\omega^g(\tau,t)=\mu_g$ before t_0 and $-u^g(\tau,t)\cdot\omega^g(\tau,t)=-\nu_g$ after t_2. It is unstable for $t<t_0$ and $t>t_2$. So we analyze the stability for the period $t_0\leq t\leq t_2$, which is congested (i.e., $\delta(\tau,t)>0$) at the beginning and uncongested (i.e., $\delta(\tau,t)=0$) at the stationary state.

Here we consider the stability under the congested state (i.e., $\delta(\tau,t)>0$). For early arrivals, we have $\omega^{E,g}(\tau,t)=(\lambda_g-\mu_g)\cdot\frac{\partial}{\partial t}\Upsilon^E(\tau,t)-\mu_g+\mu_g=(\lambda_g-\mu_g)\cdot\frac{\partial}{\partial t}\Upsilon^E(\tau,t)$. When $\delta(\tau,t)>0$, $\frac{\partial}{\partial t}\Upsilon^E(\tau,t)=\frac{f(\tau,t)}{C}-1$. So we have $\omega^{E,g}(\tau,t)=(\lambda_g-\mu_g)\cdot\frac{f(\tau,t)}{C}-(\lambda_g-\mu_g)$.

For late arrivals, we have $\omega^{L,g}(\tau,t)=(\lambda_g+\nu_g)\cdot\frac{\partial}{\partial t}\Upsilon^L(\tau,t)+\nu_g-\nu_g=(\lambda_g+\nu_g)\cdot\frac{\partial}{\partial t}\Upsilon^L(\tau,t)$. When $\delta(\tau,t)>0$, $\frac{\partial}{\partial t}\Upsilon^L(\tau,t)=\frac{f(\tau,t)}{C}-1$. So we have $\omega^L(\tau,t)=(\lambda_g+\nu_g)\cdot\frac{f(\tau,t)}{C}-(\lambda_g+\nu_g)$.

With $\alpha^g(\tau,t)=(\lambda_g-\mu_g)/C$ for early arrivals, and $\alpha^g(\tau,t)=(\lambda_g+\nu_g)/C$ for late arrivals, we have

$$\omega^g(\tau,t)=\alpha^g(t)\cdot f(\tau,t)-\alpha^g(t)\cdot C=\alpha^g(t)\cdot(f(\tau,t)-C). \tag{5.68}$$

Substituting $\omega^g(\tau,t)=\alpha^g(t)\cdot(f(\tau,t)-C)$ into Equation (5.10), we derive:

$$\frac{\partial}{\partial\tau}f^g(\tau,t)-\frac{\partial}{\partial t}u^g(\tau,t)\cdot[\alpha^g(t)\cdot(f(\tau,t)-C)]\cdot f^g(\tau,t)=0, \tag{5.69}$$

for $t\in[t_0,t_2]$.

Let $[t^g_{0,i},t^g_{2,i}]$ be group g's ith departure interval in multi-class system optimal. We define a

Lyapunov functional the same as Equation (4.67) in Chapter 4:

$$V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt, \tag{5.70}$$

where $\mathbf{f}(t) = \begin{bmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^G(t) \end{bmatrix}$ is the vector including departure rate from all groups.

The discrete version of Equation (5.70) is as follows:

$$V(\mathbf{f}(\tau)) = \sum_g \sum_i \sum_{j\Delta t \in [t_{0,i}^g, t_{2,i}^g]} ((j - \frac{1}{2}) \cdot \Delta t - t_{0,i}^g) \cdot f_j^g(\tau) \cdot [\{-\omega_{j+1}^g(\tau)\}_+^2 + \{\omega_j^g(\tau)\}_+^2] \tag{5.71}$$

Equation (5.70) described the following calculation: the Lyapunov functional first integrates $(t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2$ over each group's departure intervals (e.g., $[t_{0,i}^g, t_{2,i}^g]$) at system optimum, and sums the integrals over all departure intervals (i.e., $\sum_i$ in Equation (5.70)) and all groups (i.e., $\sum_g$ in Equation (5.70)).

For $V(\mathbf{f}(\tau,\cdot))$ to be a Lyapunov functional of dynamical system Equation (4.66) with multi-class optimal fine toll, we need to show $V(\mathbf{f}(\tau,\cdot))$ satisfies the following three conditions:

1. $V(\mathbf{f}(\tau,\cdot)) \geq 0$, $\forall \tau$.
2. $V(\mathbf{f}(\tau,\cdot)) = 0$ at the stationary state of the dynamical system Equation (5.69) with multi-class optimal fine toll.
3. $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) < 0$ at non-stationary states of the dynamical system Equation (5.69) with multi-class optimal fine toll, while $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = 0$ at the stationary state of the dynamical system Equation (5.69) with multi-class optimal fine toll.

Theorem 5.4. *Equation (5.70) is the Lyapunov functional for multi-class dynamical system in Equation (5.69) with multi-class optimal fine tolls.*

Proof. We show how $V(\mathbf{f}(\tau,\cdot))$ from Equation (5.69) will satisfy the three conditions above. Let $D_i^g(\tau) = \int_{t_{0,i}^g}^{t_{2,i}^g}(t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2$ representing the integral over group g's ith departure interval, where i could be first or second.

1. For condition (a), since $t - t_{0,i}^g \geq 0$, $f^g(\tau,t) \geq 0$ and $[\omega^g(\tau,t)]^2 \geq 0$, we have $D_i^g(\tau) = \int_{t_{0,i}^g}^{t_{2,i}^g}(t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt \geq 0$. Therefore, $V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i D_i^g(\tau) \geq 0$.

2. For condition (b), the stationary state for the multi-class day-to-day dynamical system is reached when:

$$\frac{\partial}{\partial \tau} f^g(\tau,t) = 0, \text{for} \quad g \in G. \tag{5.72}$$

From Equation (5.62), we have the complementarity condition $f^g(\tau,t) \cdot \omega^g(\tau,t) = 0$ at the stationary state. When $f^g(\tau,t) > 0$, $\omega^g(\tau,t) = 0$, so we have $[\omega^g(\tau,t)]^2 = 0$. When $f^g(\tau,t) = 0$, $\omega^g(\tau,t) \geq 0$, so we have $[\omega^g(\tau,t)]^2 \geq 0$. So the following complementarity condition also holds:

$$f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 = 0, \tag{5.73}$$

for all $t \in [0,T]$.

So $V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g}(t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt = 0$ at stationary state.

3. For condition (c), we take the partial derivative of Equation (5.70) with respect to τ

and we have:

$$\begin{aligned}
\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) &= \frac{\partial}{\partial \tau} \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt \\
&= \sum_g \sum_i \frac{\partial}{\partial \tau} \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt \\
&= \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot \frac{\partial}{\partial \tau} \Bigg\{ f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 \Bigg\} dt.
\end{aligned} \tag{5.74}$$

Let $A_i^g(\tau,t) = \frac{\partial}{\partial \tau} \Bigg\{ f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 \Bigg\}$, where g represents group g, and i represents group g's ith departure interval (i.e., $i = 1, 2$). The group who departure are closest to the desired arrival time has only one departure interval, while other groups has two departure intervals — one before the desired arrival time and one after the desired arrival time. So Equation (5.74) becomes:

$$\begin{aligned}
\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) &= \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot \frac{\partial}{\partial \tau} \Bigg\{ f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 \Bigg\} dt \\
&= \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot A_i^g(\tau,t) dt.
\end{aligned} \tag{5.75}$$

For $A_i^g(\tau,t)$, we derive:

$$\begin{aligned}
A_i^g(\tau,t) &= \frac{\partial}{\partial \tau} \Bigg\{ f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 \Bigg\} \\
&= \frac{\partial}{\partial \tau} f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 + 2 \cdot f^g(\tau,t) \cdot \omega^g(\tau,t) \cdot \frac{\partial}{\partial \tau} \omega^g(\tau,t).
\end{aligned} \tag{5.76}$$

When $\delta(\tau,t) > 0$, we have $\omega^g(\tau,t) = \alpha^g(t) \cdot (f(\tau,t) - C)$ from Equation (5.68), so we have $\frac{\partial}{\partial \tau} \omega^g(\tau,t) = \alpha^g(t) \cdot \frac{\partial}{\partial \tau} f(\tau,t)$. Substituting it into Equation (5.76), we get:

$$A_i^g(\tau,t) = \frac{\partial}{\partial \tau} f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 + 2 \cdot f^g(\tau,t) \cdot \omega^g(\tau,t) \cdot \alpha^g(t) \cdot \frac{\partial}{\partial \tau} f(\tau,t). \tag{5.77}$$

Notice here that for Equation (5.77) we have $\frac{\partial}{\partial\tau}f^g(\tau,t)$ in the first term and $\frac{\partial}{\partial\tau}f(\tau,t)$ in the second term, which is different from its single-class counterpart in Equation (5.37) with only $\frac{\partial}{\partial\tau}f(\tau,t)$.

From Theorem 4.5 in Chapter 4, we have $f^g(\tau,t) = f(\tau,t)$ at the stationary state. So at the stationary state, Equation (5.77) becomes:

$$\begin{aligned} A_i^g(\tau,t) &= \frac{\partial}{\partial\tau}f^g(\tau,t)\cdot[\omega^g(\tau,t)]^2 + 2\cdot f^g(\tau,t)\cdot\omega^g(\tau,t)\cdot\alpha^g(t)\cdot\frac{\partial}{\partial\tau}f^g(\tau,t) \\ &= \omega^g(\tau,t)\cdot[\omega^g(\tau,t) + 2\cdot f^g(\tau,t)\cdot\alpha^g(t)]\cdot\frac{\partial}{\partial\tau}f^g(\tau,t) \end{aligned} \tag{5.78}$$

At stationary state, we have $\omega^g(\tau,t) = \alpha^g(t)\cdot(f(\tau,t)-C) = \alpha^g(t)\cdot(f^g(\tau,t)-C)$. Substituting $\omega^g(\tau,t) = \alpha^g(t)\cdot(f^g(\tau,t)-C)$ into Equation (5.78), we have:

$$A_i^g(\tau,t) = [\alpha^g(t)\cdot(f^g(\tau,t)-C)]\cdot[\alpha^g(t)\cdot(3\cdot f^g(\tau,t)-C)]\cdot\frac{\partial}{\partial\tau}f^g(\tau,t) \tag{5.79}$$

From Equation (4.56), we have $\frac{\partial}{\partial\tau}f^g(\tau,t) = \frac{\partial}{\partial t}u^g(\tau,t)\cdot\omega^g(\tau,t)\cdot f^g(\tau,t)$. Substituting it into $\frac{\partial}{\partial\tau}f^g(\tau,t)$ of Equation (5.79), we have:

$$A_i^g(\tau,t) = [\alpha^g(t)\cdot(f^g(\tau,t)-C)]\cdot[\alpha^g(t)\cdot(3\cdot f^g(\tau,t)-C)]\cdot\frac{\partial}{\partial t}u^g(\tau,t)\cdot\omega^g(\tau,t)\cdot f^g(\tau,t) \tag{5.80}$$

Now we need to make certain assumptions on $u^g(\tau,t)$ to be able to move forward with the proof. We know from Equation (4.55) that $u^g(\tau,t) > 0$. Now we assume $u^g(\tau,t)$ has the following form:

$$u^g(\tau,t) = u_0\cdot\frac{3\omega^g(\tau,t) + 2\alpha^g(t)\cdot C}{f^g(\tau,t)}, \tag{5.81}$$

where u_0 is a positive coefficient, which applies to all the groups.

When the bottleneck is uncongested, $\omega^g(\tau,t) = -\mu_g$ for early arrivals, and $\omega^g(\tau,t) = \nu_g$

for late arrivals. So we have $u^g(\tau,t) = u_0 \cdot \frac{2\lambda_g - 5\mu_g}{f^g(\tau,t)}$ for early arrivals, and $u^g(\tau,t) = u_0 \cdot \frac{2\lambda_g + 5\nu_g}{f^g(\tau,t)}$ for late arrivals.

When the bottleneck is congested, then we have $\omega^g(\tau,t) = \alpha^g(t) \cdot (f^g(\tau,t) - C)$ at stationary state, then Equation (5.81) becomes as follows:

$$u^g(\tau,t) = u_0 \cdot \frac{3\omega^g(\tau,t) + 2\alpha^g(t) \cdot C}{f^g(\tau,t)} = u_0 \cdot \frac{\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)}{f^g(\tau,t)}. \tag{5.82}$$

where $\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C) > 0$ and that is, $f^g(\tau,t) > \frac{C}{3}$.

Substituting Equation (5.82) into Equation (5.80) with $\omega^g(\tau,t) = \alpha^g(t) \cdot (f^g(\tau,t) - C)$ and crossing out $f^g(\tau,t)$ in the denominator and nominator, we have:

$$\begin{aligned} A_i^g(\tau,t) &= \cdot[\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \\ &\quad \cdot \frac{\partial}{\partial t}\left\{ u_0 \cdot \frac{\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)}{f^g(\tau,t)} \cdot [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot f^g(\tau,t) \right\} \\ &= \frac{u_0}{2} \cdot \frac{\partial}{\partial t}\left\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \right\}^2. \end{aligned} \tag{5.83}$$

Substituting $A_i^g(\tau,t)$ from Equation (5.83) back to Equation (5.75), we have:

$$\begin{aligned} \frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) &= \sum_g \sum_i \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot A_i^g(\tau,t) dt \\ &= \sum_g \sum_i \frac{u_0}{2} \cdot \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot \frac{\partial}{\partial t}\left\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \right\}^2 dt. \end{aligned} \tag{5.84}$$

We have $D_i^g(\tau) = \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot f^g(\tau,t) \cdot [\omega^g(\tau,t)]^2 dt = \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot \frac{\partial}{\partial t} \Big\{ [\alpha^g(t) \cdot (f^g(\tau,t) -$

$C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)]\Bigg\}^2 dt$. So Equation (5.84) becomes:

$$\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau,\cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D_i^g(\tau) \tag{5.85}$$

Integrating by parts, $D_i^g(\tau)$ becomes:

$$\begin{aligned}
D_i^g(\tau) &= \int_{t_{0,i}^g}^{t_{2,i}^g} (t - t_{0,i}^g) \cdot \frac{\partial}{\partial t} \Bigg\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \Bigg\}^2 dt \\
&= \Bigg\{ (t - t_{0,i}^g) \cdot \Big\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \Big\}^2 \Big|_{t_{0,i}^g}^{t_{2,i}^g} \\
&\quad - \int_{t_{0,i}^g}^{t_{2,i}^g} \Big\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \Big\}^2 dt \Bigg\} \\
&= \Bigg\{ (t_{2,i}^g - t_{0,i}^g) \cdot \Big\{ [\alpha^g(t_{2,i}^g) \cdot (f^g(\tau,t_{2,i}^g) - C)] \cdot [\alpha^g(t_{2,i}^g) \cdot (3 \cdot f^g(\tau,t_{2,i}^g) - C)] \Big\}^2 - 0 \\
&\quad - \int_{t_{0,i}^g}^{t_{2,i}^g} \Big\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \Big\}^2 dt \Bigg\},
\end{aligned} \tag{5.86}$$

where $\Bigg\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \Bigg\}^2 \geq 0$, so we know the last term of Equation (5.86), $-\int_{t_{0,i}^g}^{t_{2,i}^g} \Bigg\{ [\alpha^g(t) \cdot (f^g(\tau,t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau,t) - C)] \Bigg\}^2 dt \leq 0$.

Assumption 5.14.1. *At the stationary state, we have $\omega^g(\tau', t_{2,i}^g) = 0$. We assume that before the multi-class dynamical system reaches its stationary state on day τ', and rate of change in trip cost at end of group g's ith departure period, $t_{2,i}^g$, $\omega(\tau, t_{2,i}^g)$, already becomes zero. That is, $\exists \varepsilon > 0$,*

$$\omega(\tau, t_{2,i}^g) = 0, \forall \tau \geq \tau' - \varepsilon, g \in G, i = \{1, 2\}. \tag{5.87}$$

In the current Section 5.14, we study the stability of the stationary state. Assumption 5.14.1 means that before day step reaches the stationary state step $\tau^{'}$, $\omega(\tau, t_{2,i}^g)$ becomes zero already. With $\omega^g(\tau, t_{2,i}^g) = 0$, we have $\omega^g(\tau, t_{2,i}^g) = \alpha(t_{2,i}^g) \cdot f^g(\tau, t_{2,i}^g) - \lambda_g = 0$. For $\tau \geq \tau^{'} - \varepsilon$, with $\alpha^g(t_{2,i}^g) \cdot (f^g(\tau, t_{2,i}^g) - C) = 0$, the first term of Equation (5.86) becomes zero. So Equation (5.86) becomes as follows:

$$D_i^g(\tau) = -\left\{ \int_{t_{0,i}^g}^{t_{2,i}^g} \left\{ [\alpha^g(t) \cdot (f^g(\tau, t) - C)] \cdot [\alpha^g(t) \cdot (3 \cdot f^g(\tau, t) - C)] \right\}^2 dt \right\} \leq 0. \tag{5.88}$$

From Equation (5.82), we have $\alpha^g(t) \cdot (3 \cdot f^g(\tau, t) - C) > 0$. For $t \in [t_{0,i}^g, t_{2,i}^g]$, when $\tau^{'} - \varepsilon \leq \tau < \tau^{'}$, we have $[\omega^g(\tau, t)]^2 = [\alpha^g(t) \cdot (3 \cdot f^g(\tau, t) - C)]^2 > 0$, and when $\tau \geq \tau^{'}$, we have $[\omega^g(\tau, t)]^2 = [\alpha^g(t) \cdot (3 \cdot f^g(\tau, t) - C)]^2 = 0$. So we have:

$$D_i^g(\tau) < 0, \forall \tau^{'} - \varepsilon \leq \tau < \tau^{'}$$

$$D_i^g(\tau) = 0, \forall \tau \geq \tau^{'}$$

The Lyapunov functional $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau, \cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D_i^g(\tau)$, so we have

$$\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau, \cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D_i^g(\tau) < 0, \forall \tau^{'} - \varepsilon \leq \tau < \tau^{'}$$

$$\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau, \cdot)) = \sum_g \sum_i \frac{u_0}{2} \cdot D_i^g(\tau) = 0, \forall \tau \geq \tau^{'}$$

That means, after the system reaches its stability region ($\tau^{'} - \varepsilon \leq \tau$), we have $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau, \cdot)) < 0$ at non-stationary states, while $\frac{\partial}{\partial \tau} V(\mathbf{f}(\tau, \cdot)) = 0$ at stationary state. When Lyapunov functional becomes zero, $f^g(\tau, t) = C$ for each group g from $\omega^g(\tau, t) = \alpha^g(t) \cdot (f^g(\tau, t) - C) = 0$.

So Equation (5.70) is a Lyapunov functional of multi-class dynamical system Equation (5.69) with multi-class optimal fine toll, and it is asymptotically stable at the stationary state during the entire the congested period $[t_0, t_2]$.

This completes theh proof of Theorem 5.4. $\square$

5.15 Numerical examples: multi-class tolling

We test the optimal fine toll in Section 5.11.1 in the multi-class day-to-day dynamics in Chapter 4 to study the impacts of congestion pricing.

5.15.1 Simulation set up

We consider the total number of travelers, $N = 3,600$ veh. We consider two groups. Both groups share the same number of travelers: $N_1 = N_2 = 1,800$ veh. The capacity of the bottleneck is $C = 1,800$ veh/hr. We only consider the following scenarios: different λ, μ but with the same $\frac{\mu}{\nu}$ and same desired arrival time t^*. Similar to Chapter 4, we set $\Delta t = 0.1$hr and total day steps to be $5,001$, starting from day step 0, and ending with day step $5,000$.

We consider the following two scenarios with different sets of λ, μ, and ν: (1) $\lambda_1 = 75$ \$/hr and $\lambda_2 = 50$ \$/hr, $\mu_1 = \mu_2 = 25$ \$/hr, and $\nu_1 = \nu_2 = 100$ \$/hr; (2) $\lambda_1 = \lambda_2 = 50$ \$/hr, $\mu_1 = 15$, $\mu_2 = 25$ \$/hr, and $\nu_1 = 60$, $\nu_2 = 100$ \$/hr. We consider the study period to be $[0, 6]$ hr, and the desired arrival time for all travelers $t^* = 4$ hr. We assume that the initial departure rate on day 0, $f^0(t)$, is as follows:

$$f^0(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4 \\ 1,800 & \text{for} \quad 2.4 < t \leq 4.4 \\ 0 & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{5.89}$$

where group 1 and group 2's departure rate are the same and both equal to half of the total departure rate (e.g., $f^{1,0}(t) = f^{2,0}(t) = \frac{1}{2} \cdot f^0(t) = 900$ veh/hr for $2.4 < t \leq 4.4$). For the scenario (1), the MC-DTUE departure flow rate, $f^*(t)$, would be as follows:

$$f^*(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 2.4 \\ f^{1,E,*}(t) = \frac{\lambda_1}{\lambda_1 - \mu_1} \cdot C = \frac{3}{2} \cdot 1,800 = 2,700 & \text{for} \quad 2.4 < t \leq 2.93 \\ f^{2,E,*}(t) = \frac{\lambda_2}{\lambda_2 - \mu_2} \cdot C = 2 \cdot 1,800 = 3,600 & \text{for} \quad 2.93 < t \leq 3.33 \\ f^{2,L,*}(t) = \frac{\lambda_2}{\lambda_2 + \nu_2} \cdot C = \frac{1}{3} \cdot 1,800 = 600 & \text{for} \quad 3.33 < t \leq 3.93 \\ f^{1,L,*}(t) = \frac{\lambda_1}{\lambda_1 + \nu_1} \cdot C = \frac{3}{7} \cdot 1,800 = 771.4 & \text{for} \quad 3.93 < t \leq 4.4 \\ 0 & \text{for} \quad 4.4 < t \leq 6. \end{cases} \tag{5.90}$$

which is the same as Equation (4.114) in Chapter 4. And at MC-DTUE, the cost function $\phi^*(t)$ will be as follows:

$$\phi^*(t) = \begin{cases} 25 \cdot (4 - t) & \text{for} \quad 0 \leq t \leq 2.4 \\ \phi^{1,*}(t) = \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N}{C} = 40 & \text{for} \quad 2.4 < t \leq 2.93 \\ \phi^{2,*}(t) = \frac{\mu_2 \cdot \nu_2}{\mu_2 + \nu_2} \cdot \frac{N_2}{C} + \frac{\lambda_2}{\lambda_1} \cdot \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N_1}{C} = 33.3 & \text{for} \quad 2.93 < t \leq 3.93 \\ \phi^{1,*}(t) = \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N}{C} = 40 & \text{for} \quad 3.93 < t \leq 4.4 \\ 100 \cdot (t - 4) & \text{for} \quad 4.4 < t \leq 6, \end{cases} \tag{5.91}$$

Since $\mu_1 = \mu_2 = 25$ \$/hr, and $\nu_1 = \nu_2 = 100$, the multi-class optimal fine toll becomes single class optimal fine toll as follows:

$$p^o(t) = \begin{cases} 0 & \text{for} \quad t < 2.4 \\ 40 - 25 \cdot (4 - t) & \text{for} \quad 2.4 \leq t < 4, \\ 40 + 100 \cdot (4 - t) & \text{for} \quad t^* \leq t \leq 4.4, \\ 0 & \text{for} \quad t > 4.4. \end{cases} \tag{5.92}$$

For the scenario (2), the MC-DTUE departure flow rate, $f^*(t)$, would be as follows:

$$f^*(t) = \begin{cases} 0 & \text{for} \quad 0 \le t \le 2.4 \\ f^{1,E,*}(t) = \frac{\lambda_1}{\lambda_1 - \mu_1} \cdot C = \frac{10}{7} \cdot 1,800 = 2,571.4 & \text{for} \quad 2.4 < t \le 2.96 \\ f^{2,E,*}(t) = \frac{\lambda_2}{\lambda_2 - \mu_2} \cdot C = 2 \cdot 1,800 = 3,600 & \text{for} \quad 2.96 < t \le 3.36 \\ f^{2,L,*}(t) = \frac{\lambda_2}{\lambda_2 + \nu_2} \cdot C = \frac{1}{3} \cdot 1,800 = 600 & \text{for} \quad 3.36 < t \le 3.96 \\ f^{1,L,*}(t) = \frac{\lambda_1}{\lambda_1 + \nu_1} \cdot C = \frac{5}{11} \cdot 1,800 = 818.2 & \text{for} \quad 3.96 < t \le 4.4 \\ 0 & \text{for} \quad 4.4 < t \le 6. \end{cases} \tag{5.93}$$

And at MC-DTUE, the cost function $\phi^*(t)$ will be as follows:

$$\phi^*(t) = \begin{cases} 25 \cdot (4 - t) & \text{for} \quad 0 \le t \le 2.4 \\ \phi^{1,*}(t) = \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N}{C} = 24 & \text{for} \quad 2.4 < t \le 2.96 \\ \phi^{2,*}(t) = \frac{\mu_2 \cdot \nu_2}{\mu_2 + \nu_2} \cdot \frac{N_2}{C} + \frac{\lambda_2}{\lambda_1} \cdot \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N_1}{C} = 32 & \text{for} \quad 2.96 < t \le 3.96 \\ \phi^{1,*}(t) = \frac{\mu_1 \cdot \nu_1}{\mu_1 + \nu_1} \cdot \frac{N}{C} = 24 & \text{for} \quad 3.96 < t \le 4.4 \\ 100 \cdot (t - 4) & \text{for} \quad 4.4 < t \le 6, \end{cases} \tag{5.94}$$

We have the following formula of the optimal fine toll for multi-class travelers:

$$p^{o,M}(t) = \begin{cases} 0 & \text{for} \quad t < 2.4 \\ 15 \cdot (t - 2.4) & \text{for} \quad 2.4 \le t < 3.2, \\ \phi^{1,*} + 25 \cdot (t - 3.2) & \text{for} \quad 3.2 \le t < 4, \\ \phi^{1,*} + 100 \cdot (4.3 - t) & \text{for} \quad 4 \le t < 4.3, \\ -60 \cdot (t - 4.5) & \text{for} \quad 4.3 \le t < 4.5, \\ 0 & \text{for} \quad t \ge 4.5, \end{cases} \tag{5.95}$$

where $\phi^{1,*} = \frac{\mu_1 \cdot \nu_1}{2 \cdot (\mu_1 + \nu_1)} \cdot \frac{N}{C} = 12$.

5.15.2 Time and day step sizes and advance/deferral coefficients

We select $B_i^{a,g}(\tau)$, $B_i^{d,g}(\tau)$, Δt and $\Delta\tau$ according to Section 5.15.2.

5.15.3 Stability measurements

To measure stability, we calculate the following discrete version of Lyapunov function on day τ_j defined in Equation (4.68):

$$V(\mathbf{f}(\tau)) = \sum_g \sum_i \sum_{j \in [t_{0,i}^g, t_{2,i}^g]} ((j - \frac{1}{2}) \cdot \Delta t - t_{0,i}^g) \cdot f_j^g(\tau) \cdot [\{-\omega_{j+1}^g(\tau)\}_+^2 + \{\omega_j^g(\tau)\}_+^2]$$

where the $V(\mathbf{f}(\tau_0)) > 0$. If as day step τ_j increases, $V(\mathbf{f}(\tau_j))$ decreases to zero, then it numerically shows that the day-to-day dynamical system is stable.

5.15.4 Simulation results

5.15.5 Scenario 1: different λ, same μ, ν, and t^*

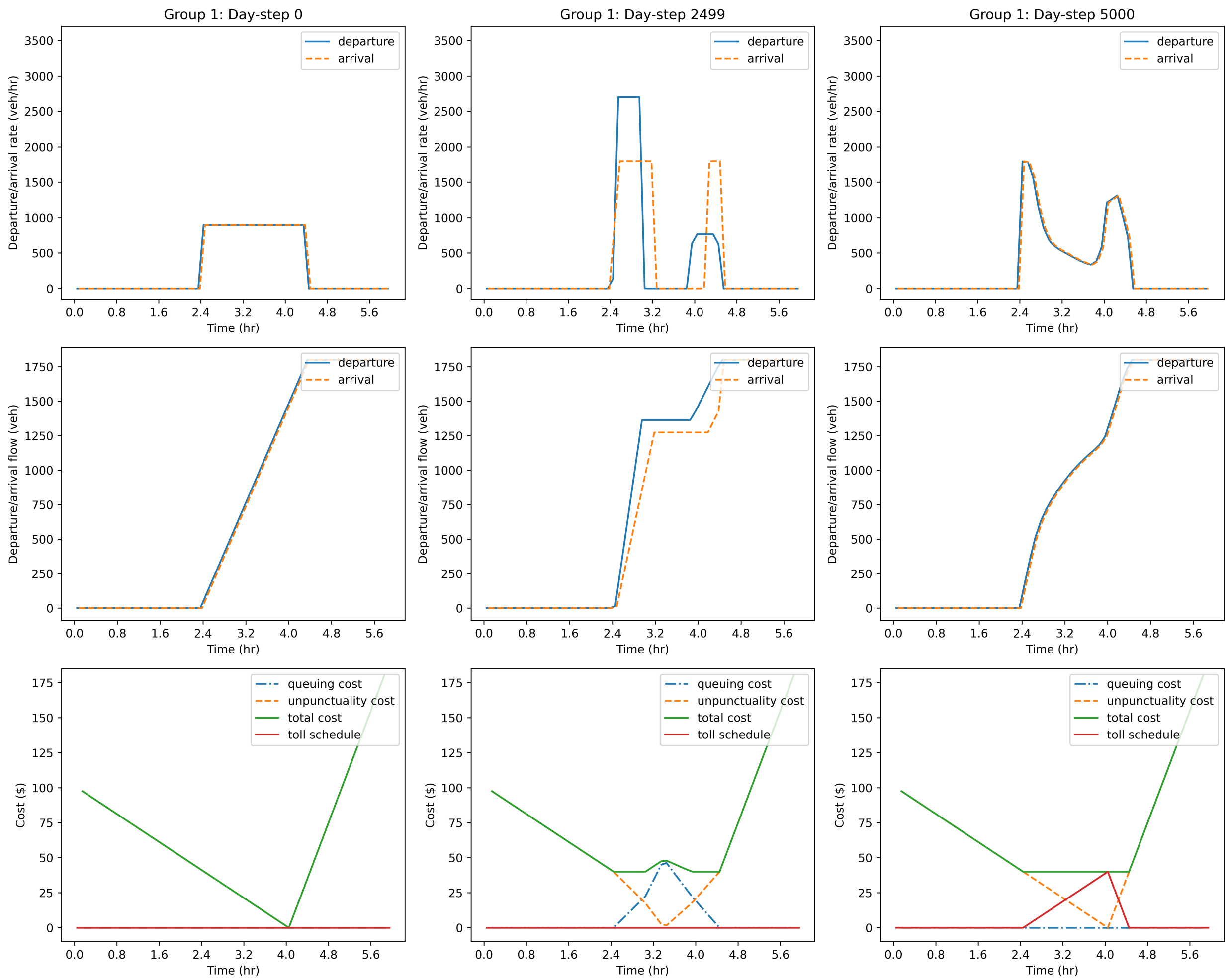


Figure 5.13: First, before adding optimal fine toll, and last day-step comparison (group 1)

Figure 5.13 shows group 1 departure / arrival rate, cumulative departure / arrival flow, and costs on first, 2,499 and 5,000 day step, while Figure 5.14 shows the same metrics for group 2. At the MC-DTUE on day-step 2,499, group 1 departs during the shoulder of the peak period, where the minimum cost for group 1 occurs, while group 2 departs in the middle of the peak, where the minimum cost for group 2 occurs. After adding the optimal fine toll on day-step 2,499, group 1 travelers start to shift their departure time to the middle of the peak, while group 2's travelers start to shift to the shoulder of the peak. The shifting continues until the queue is eliminated, and the system becomes uncongested. Once the system becomes uncongested, the $\omega_i(\tau_j)$ becomes zero, so the two groups of travelers are not able to fully separate. As a result, there will be mixed departure between groups at the stationary SO state, also shown in Figure 5.15.

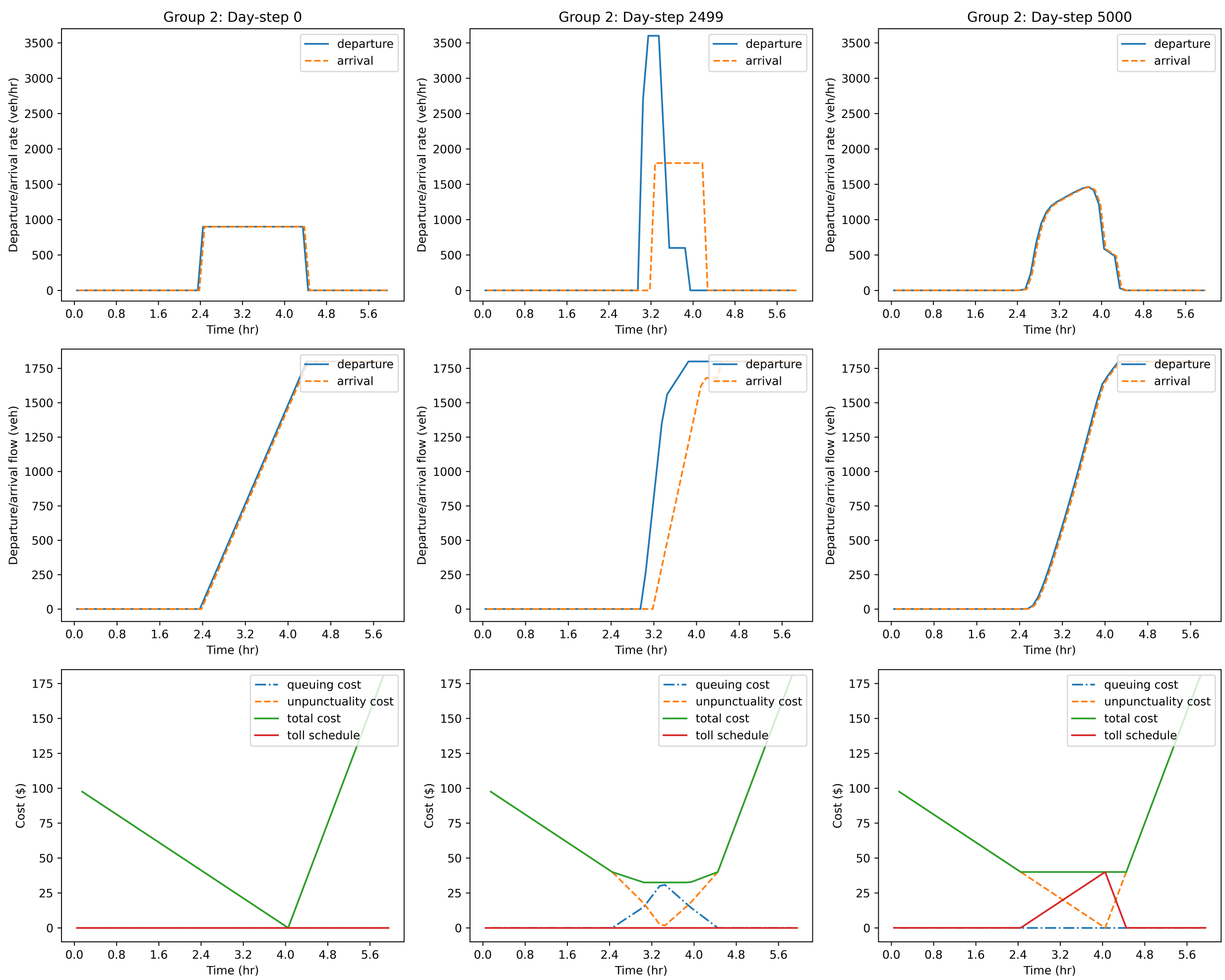


Figure 5.14: First, before adding optimal fine toll, and last day-step comparison (group 2)

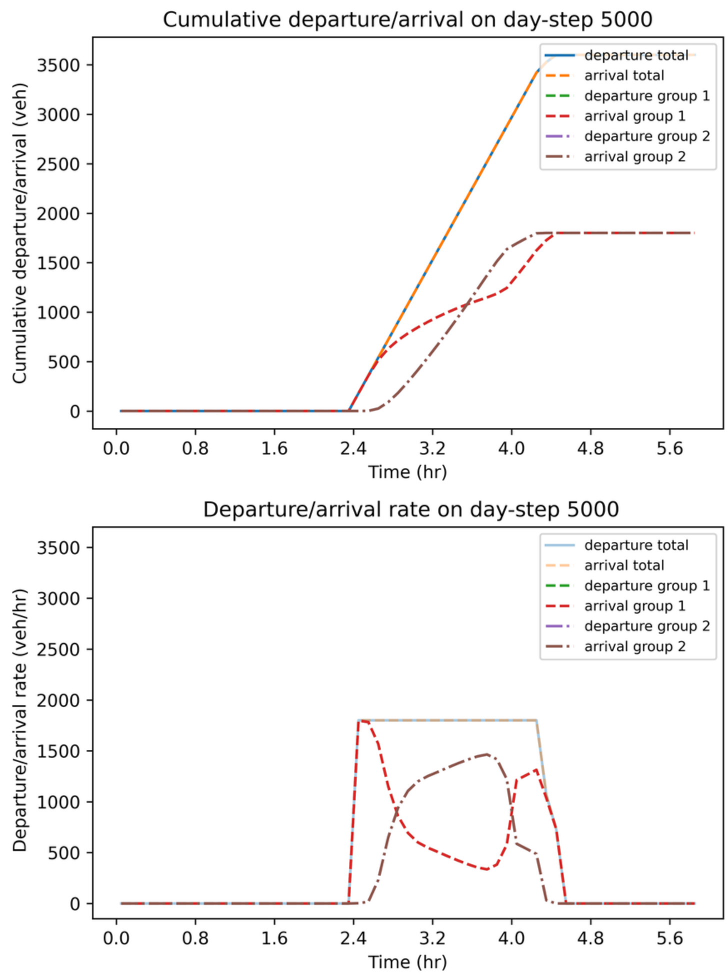


Figure 5.15: Cumulative departure/arrival flow and departure/arrival rate on the last day-step

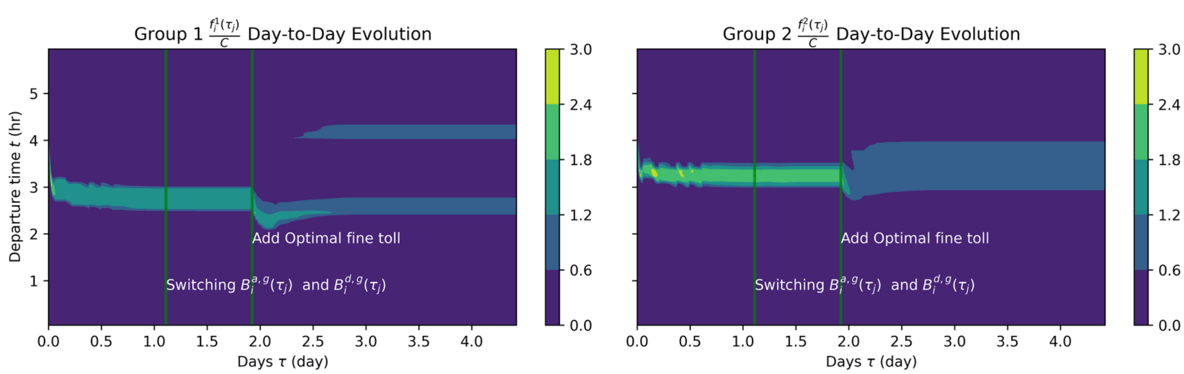


Figure 5.16: Day-to-day evolution of normalized departure rate of two groups of adding optimal fine toll (scenario 1)

Figure 5.16 shows the stable convergence pattern of the normalized departure rate of two groups of adding optimal fine toll. Figure 5.17 shows the Lyapunov functional of the multi-class dynamical system drops to 10^{-20} after adding the optimal fine toll. It means the optimal fine toll can drive the system to a stable stationary SO state.

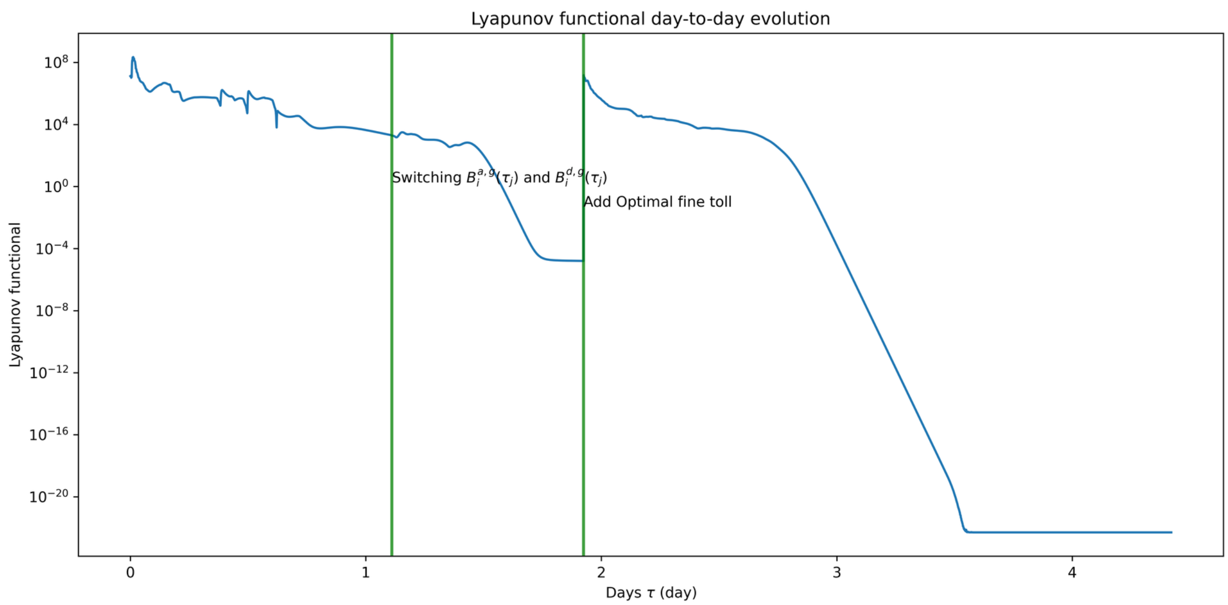


Figure 5.17: Lyapunov functional day-to-day evolution of two groups of adding optimal fine toll (scenario 1)

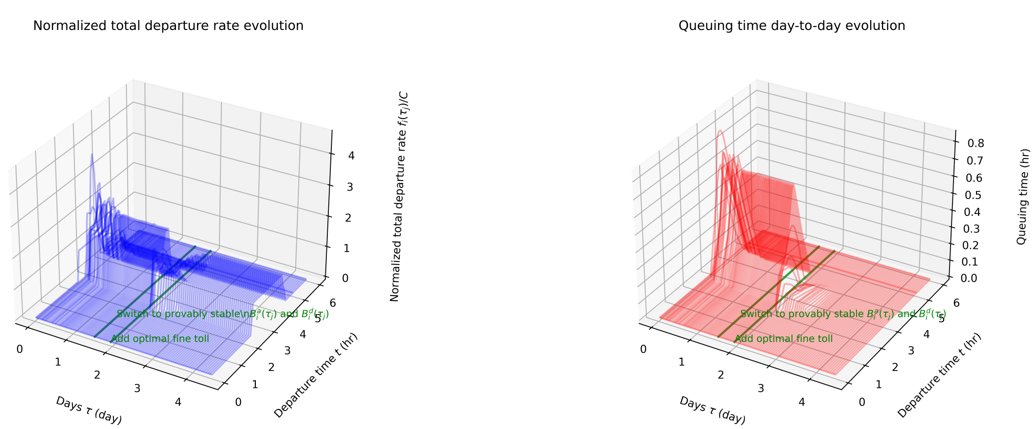


Figure 5.18: Day-to-day evolution of normalized departure rate and queuing time of adding optimal fine toll (scenario 1)

Figure 5.18 shows that the system converges to SO stable fairly quickly after adding optimal fine toll. The queuing times for all time steps reach below $1e-5$ on day-step $3,559$.

5.15.6 Scenario 2: different μ, same λ, μ/ν, and t^*

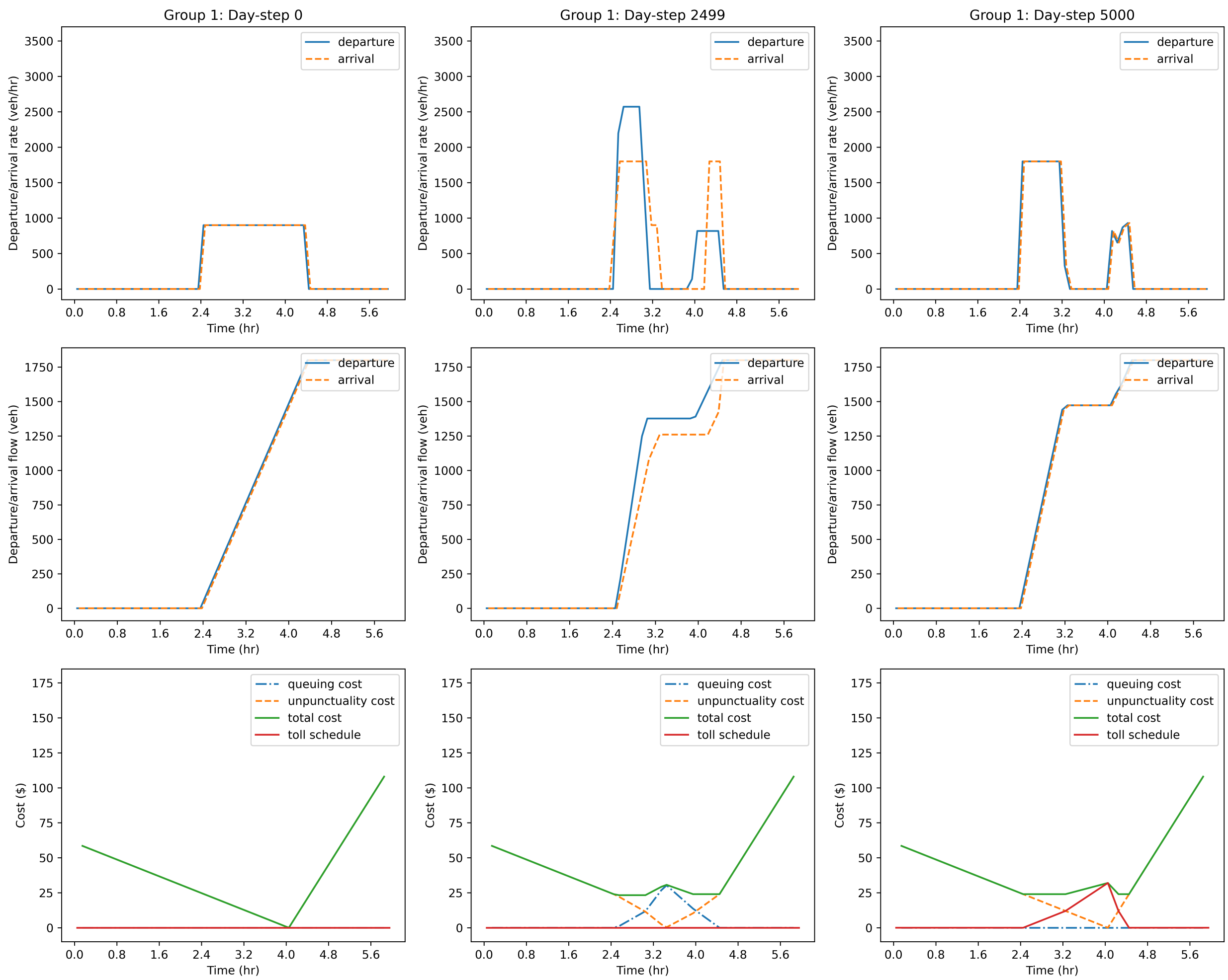


Figure 5.19: First, before adding optimal fine toll, and last day-step comparison (group 1)

Figure 5.19 shows group 1 departure / arrival rate, cumulative departure / arrival flow, and costs on first, $2,499$ and $5,000$ day step, while Figure 5.20 shows the same metrics for group 2. At the MC-DTUE on day-step $2,499$, group 1 departs during the shoulder of the peak period, where the minimum cost for group 1 occurs, while group 2 departs in the middle of the peak, where the minimum cost for group 2 occurs. After adding the optimal fine toll on day-step 2,499, group 1 and group 2's travelers start to shift their departure times according to their μ and ν, which aligns with Arnott and Small (1994). Group 1's travelers still depart on the shoulder but more early arrivals and fewer late arrivals compared with MC-DTUE. On the last day-step, groups of travelers are organized by their unpunctuality cost, where travelers with the larger μ and ν are close to the desired arrival time, also shown in Figure 5.21.

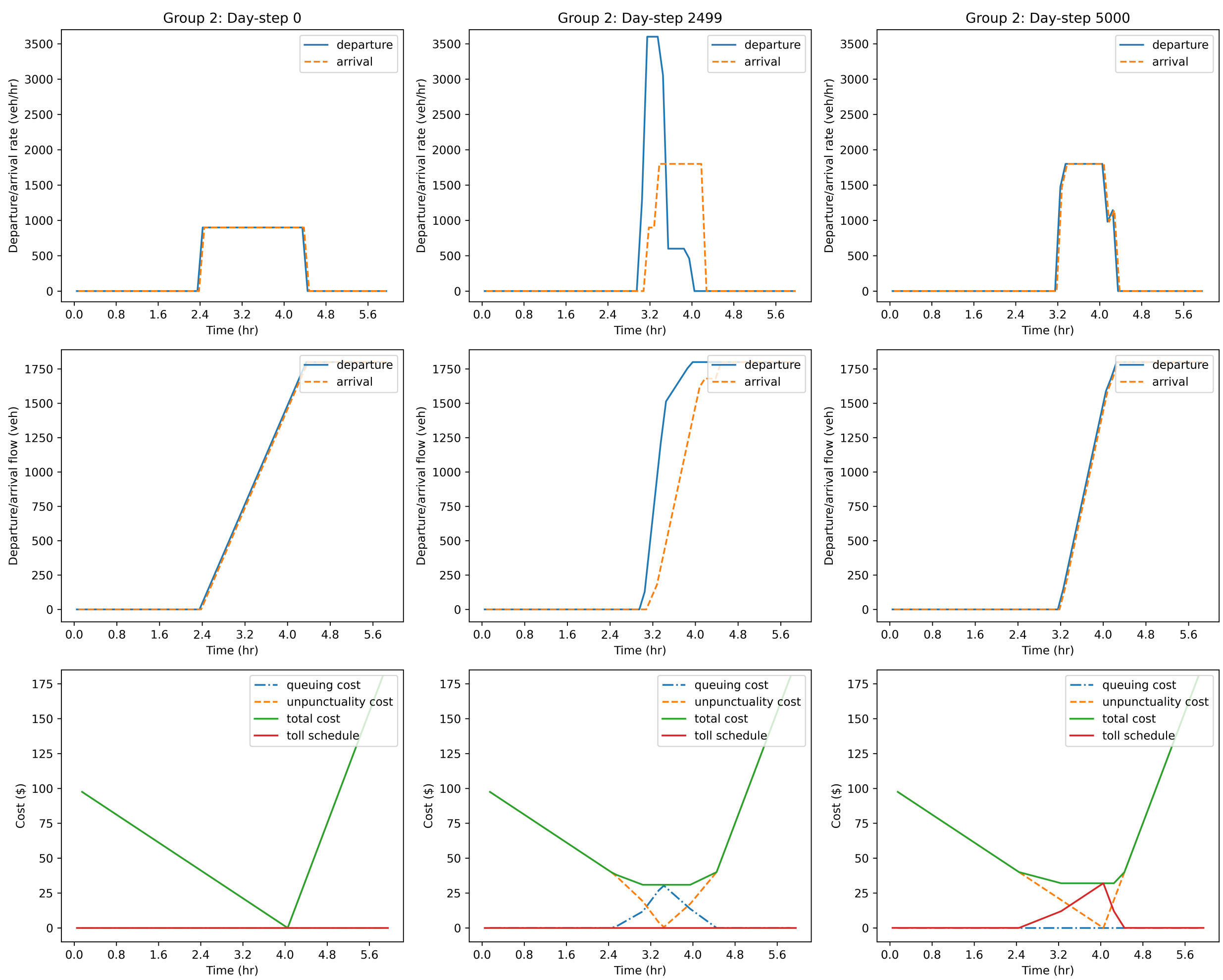


Figure 5.20: First, before adding optimal fine toll, and last day-step comparison (group 2)

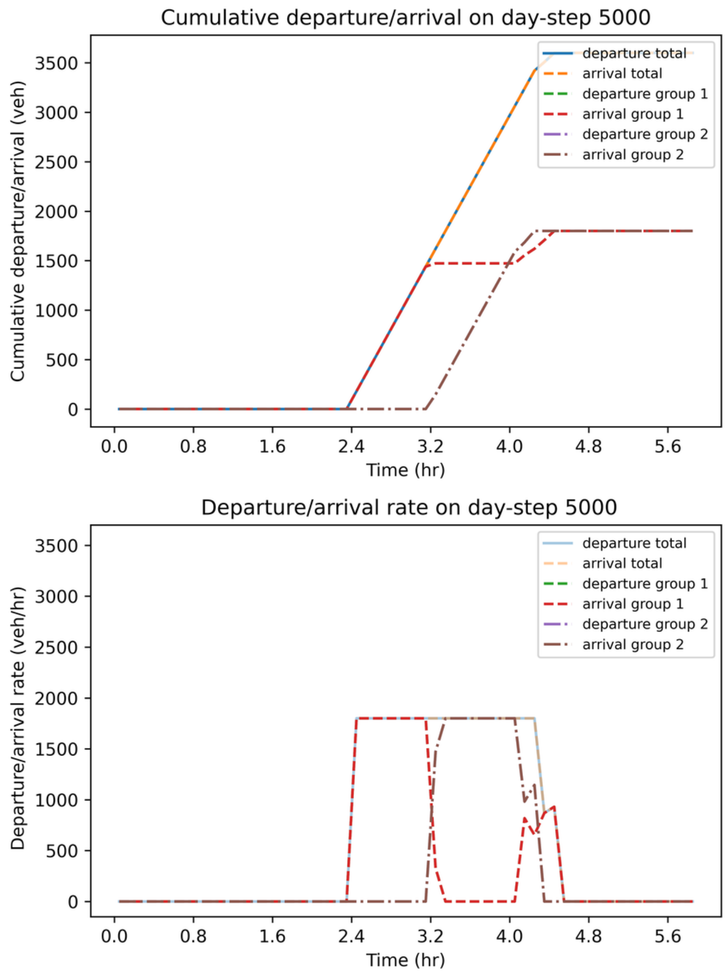


Figure 5.21: Cumulative departure/arrival flow and departure/arrival rate on the last day-step

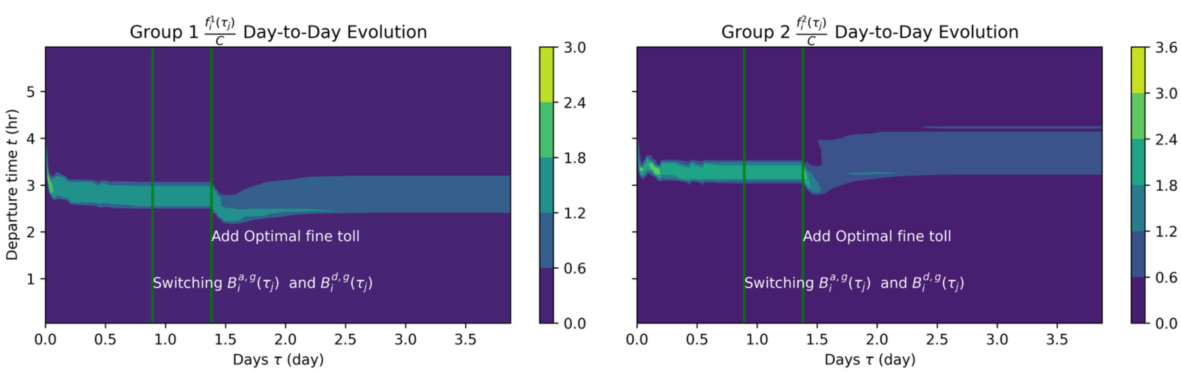


Figure 5.22: Departure time day-to-day evolution of two groups of adding optimal fine toll (scenario 2)

Figure 5.22 shows the stable convergence pattern of the normalized departure rate of two groups of adding optimal fine toll. Figure 5.23 shows the Lyapunov functional of the multi-class dynamical system reaches 10^{-7} after adding the optimal fine toll. It means the optimal fine toll can drive the system to a stable stationary SO state.

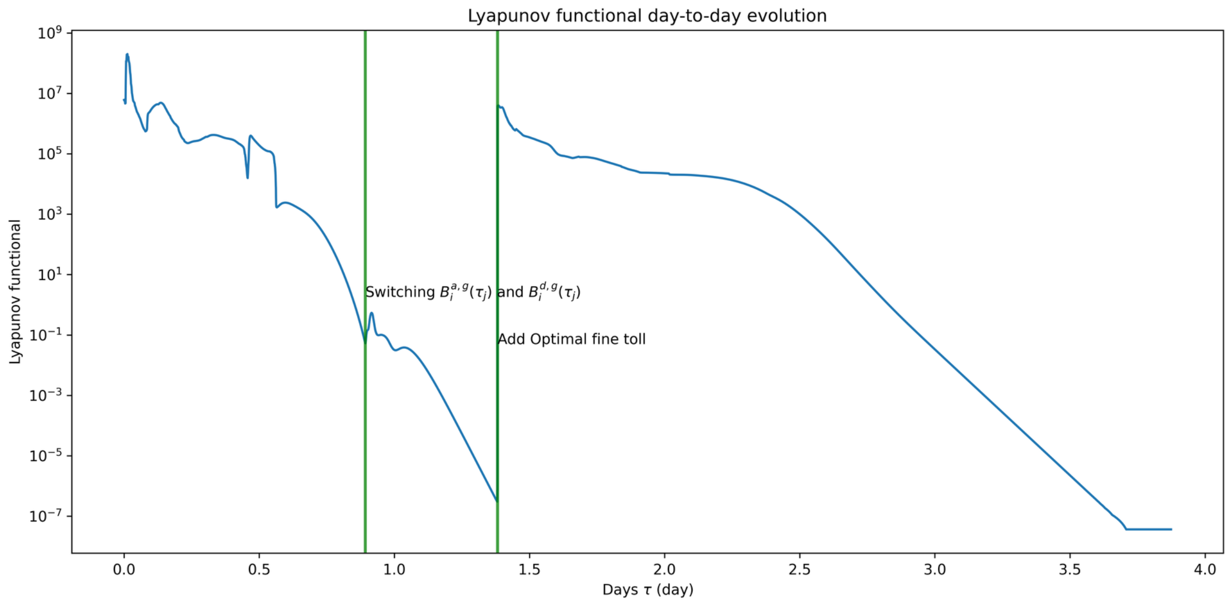


Figure 5.23: Lyapunov functional day-to-day evolution of two groups of adding optimal fine toll (scenario 2)

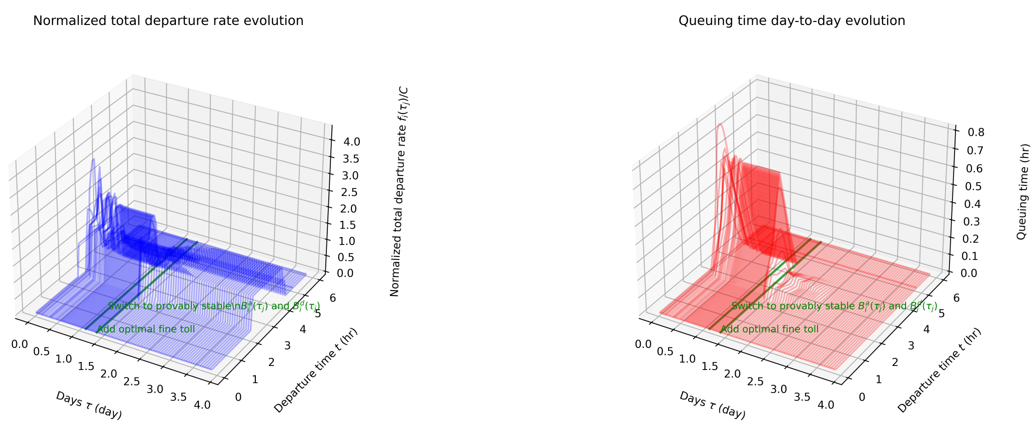


Figure 5.24: Day-to-day evolution of normalized departure rate and queuing time of adding optimal fine toll (scenario 2)

Figure 5.24 shows that the system converges to SO stable fairly quickly after adding optimal fine toll. The queuing times for all time steps reach below $1e-5$ on day-step $4,089$.

5.16 Conclusion

In this chapter, we study different tolls/incentives using both the single-class day-to-day departure time dynamics from Jin (2021b) in Chapter 3, and the multi-class day-to-day departure time dynamics in Chapter 4. The results show that for single-class dynamical system, the optimal fine toll, optimal fine reward, and optimal feebate can drive the system to a stable stationary SO state, because these three tolls/incentives have the same slope in tolls ($\frac{d}{dt}p(t) = \mu$ for early arrivals, and $\frac{d}{dt}p(t) = -\nu$ for late arrivals). On the other hand, the optimal coarse toll cannot drive the system to such an SO state. When adding optimal coarse toll, optimal fine reward, and optimal feebate, the cost difference $\omega_i(\tau_j)$ might become so large that the previous $B_i^a(\tau_j)$ and $B_i^d(\tau_j)$ might lead to negative departure rates. So we switch to heuristic $B_i^a(\tau_j)$ and $B_i^d(\tau_j)$ after adding these three tolls/incentives, and switch back to provably stable $B_i^a(\tau_j)$ and $B_i^d(\tau_j)$ on day-step $3,500$.

For multi-class dynamical system, we apply the optimal fine toll in two different scenarios: (1) different λ, same μ, ν, and t^* across travelers; (2) different λ/μ, same λ, μ/ν, and t^* across travelers. The results show that the optimal fine toll can drive the system to a stable stationary SO state. When μ and ν are the same for two groups, the system will have mixed departures for both groups throughout the departure period. When μ and ν are different for two groups, they will be organized according to μ and ν with the largest μ and ν being closer to the middle of the peak.

Chapters 3 to 5 study the day-to-day departure time dynamics in the corridor level using the point queue model in single-class, multi-class and tolling scenarios. Chapter 6 will examine the day-to-day departure time dynamics in the network level with Vickrey (1991, 2020)'s improvement-based dynamics.

Chapter 6

Managing Travelers' Departure Time Decisions in a Congested Network with Marginal Social Cost Pricing

Chapters 3 to 5 considered the day-to-day departure time dynamics in the corridor level with a single bottleneck. Chapter 3 presented the single-class day-to-day dynamics proposed by Jin (2021b), while Chapter 4 presented the multi-class extension of it, capturing travelers' heterogeneity. Chapter 5 considered adding tolling into both single-class and multi-class day-to-day departure time dynamics and proved that the optimal fine toll can drive the system to a stable stationary system optimal state in both single-class and multi-class cases. In this chapter, we will present the day-to-day departure time dynamics at the network level with Vickrey's bathtub model (Vickrey, 1991, 2020, 1994, 2019), as well as managing travelers' departure time decisions using marginal social cost pricing.

6.1 Introduction

Cordon congestion pricing has gained increasing attention, particularly as New York City became the sixth city to implement such a policy on January 5, 2025, joining Singapore, London, Stockholm, Milan, and Gothenburg (Ley et al., 2025). Since the introduction of a \$9 congestion toll in New York City, positive effects have been observed, such as a 7.5% decrease in weekday entry to the congestion zone (Ley et al., 2025) and a 10% to 30% average decrease in trip times (Khalifeh and Nessen, 2025). Meanwhile, mixed opinions have also emerged (Hu et al., 2025). However, implementation approaches vary widely across cities, and the design of congestion pricing remains largely ad hoc (Lehe, 2019).

Over the years, researchers have developed various models to design cordon congestion pricing. Some studies employ static congestion models, such as the Bureau of Public Roads (BPR) function (Zhang and Yang, 2004; Liu et al., 2014), which assume that traffic conditions within a given time interval are independent of other intervals. However, static models do not capture within-day traffic dynamics. To address this limitation, other researchers use dynamic models, like the bathtub model. However, they often focus on equilibrium states (Arnott, 2013; Arnott et al., 2016), where the marginal social cost depends solely on the number of vehicles in the network rather than the time-dependent marginal effect of an additional traffic at one time at subsequent times (Vickrey, 1994, 2019).

Unlike previous studies focusing on equilibrium states, Vickrey (1991, 2020) proposes a marginal social cost pricing scheme that considers the marginal effect in a transient state until the end of congestion. Vickrey proposes this approach for cordon-level congestion pricing to regulate inflowing traffic into Midtown Manhattan. Vickrey (1991, 2020) includes four sections. The first section presents the bathtub model to describe the within-day traffic dynamics. The second section describes a day-to-day entry traffic adjustment mechanism to capture four kinds of travelers' responses to congestion pricing — deferral, advance, suppression, and generation. The third section presents an iterative method to estimate the marginal social cost backward from the end of the congestion. The fourth section presents a method to numerically solve the bathtub model, incorporating the curvature of the network fundamental diagram.

However, Vickrey (1991, 2020) only provides a theoretical framework for marginal cost pricing, without numerical verification. To bridge this gap, this chapter presents the first attempt to verify Vickrey's scheme using 48-hour traffic entry data for Manhattan, New York (Gutman et al., 2016). Among the four traveler responses to congestion pricing, we focus on two that pertain to fixed demand —deferral and advance. The results show that after applying the marginal social cost pricing, the time-dependent traffic entry rate converges to a practically stable state, with travelers shifting their entry times from peak to off-peak. After the day-to-day adjustment process, the marginal social cost decreases to less than 0.6 hour-equivalents. These numerical findings provide valuable validation of Vickrey (1991, 2020)'s theoretical contributions to congestion pricing.

The rest of the chapter is organized as follows: Section 6.2 outlines the methodology of this chapter, including notation, system workflow, description of four sections of Vickrey (2020), and the convergence criteria of system. Section 6.3 presents the simulation setup and results, and Section 6.4 summarizes the main findings and future research directions.

6.2 Methodology

6.2.1 Notation

Table 6.1 presents the notation in Chapter 6.

Table 6.1: Notation in Chapter 6, ordered by appearance

Variable	Meaning	Unit
	Variables from Section 6.2.3	
t	Time variable	hr
$\delta(t)$	Number of vehicles attempting to move in a network at time t	veh
$f(t)$	Influx of vehicles to network at time t	veh/hr
$g(t)$	Outflux of vehicles from the network at time t	veh/hr
B	Average trip length	mile
L	Total lane miles of the network	lane-mile
$V(\rho)$	Network speed density relationship (network fundamental diagram)	mile/hr
	Variables from Section 6.2.4	
T	Time variable for an hour	hr
q	Extra trips added to the network at time T	veh
$M(T)$	Marginal social cost at time T	hr
dq	Small number of extra trips added to the network at time T	veh
h	Time variable ($h = t - T$, $h \in [0, 1]$)	hr
$y(T, h, dq)$	Remaining active trips at time h after dq active trips are added to the network at time T	veh
$Z(T, h)$	Rate of change in remaining active trips at time h after dq active trips ($dq \to 0$) are added to the network at time T	1
u	Free flow speed in the network fundamental diagram	mile/hr
ρ_j	Jam density in the network fundamental diagram	veh/(lane-mile)
$\tilde{A}$	First term of Equation (6.24)	1
$\tilde{B}$	Coefficient in front of h in the second term of Equation (6.24)	1
C_0	Constant	1
$P(T)$	Toll schedule at T	hr
$v(t)$	Network average speed of the bathtub model at time t	mile/hr
	Variables from Section 6.2.5	
$f(\tau, t)$	Influx of vehicles to network at time t on day τ	veh/hr
$\phi(\tau, t)$	Travel cost at time t on day τ	hr
$P(\tau, t)$	Toll schedule at time t on day τ	hr
σ_i^j	Cost increment after adding the marginal social cost toll at time step i on day step j compared with day step 0	hr
ω_i^j	Difference in cost increment in time step i and $i - 1$ on day step j	hr
$f_i^d(\tau)$	Deferral rate at time step i on day τ	veh/day
$f_i^a(\tau)$	Advance rate at time step i on day τ	veh/day
$B_i^d(\tau)$	Deferral coefficient at time step i on day τ	1
$B_i^a(\tau)$	Advance coefficient at time step i on day τ	1
	Variables from Section 6.2.6	
$\tilde{C}$	Constant	1
$\hat{A}$	Coefficient of Equation (6.58)	1
$\hat{B}$	Coefficient of Equation (6.58)	1
$\hat{C}$	Coefficient of Equation (6.58)	1
$\hat{D}$	Coefficient of Equation (6.58)	1
C^1	Constant	1
	Variables from Section 6.2.7	
$RMSPE_j$	Root Mean Squared Percentage Error	1

6.2.2 System workflow

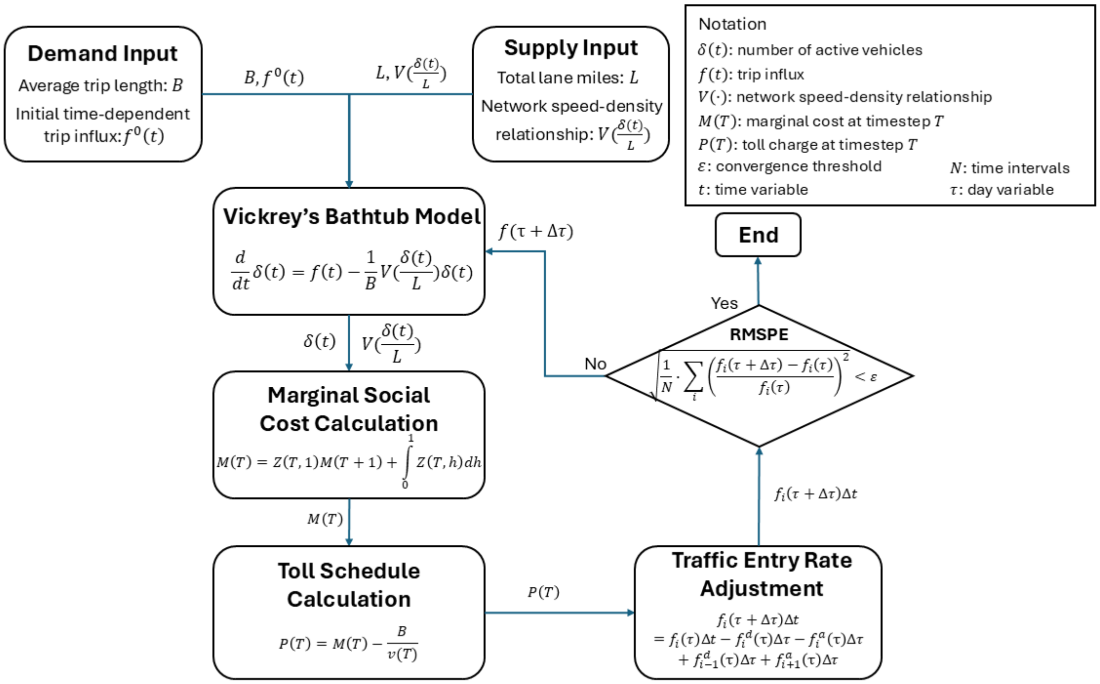


Figure 6.1: System workflow

Figure 6.1 shows the workflow of the system. The model takes demand inputs, including the average trip distance and initial traffic entry pattern, and supply inputs, such as total lane miles and network speed-density relationships (Godfrey, 1969; Geroliminis and Daganzo, 2008). Using these inputs, we use the numerical method in Vickrey (2020)'s fourth section (presented in Section 6.2.6) to solve Vickrey's bathtub model to obtain the within-day dynamics of the number of active vehicles in the network. We then use the iterative method in Vickrey (2020)'s third section (presented in Section 6.2.4) to estimate the marginal social cost and calculate the toll schedule accordingly for the next day. The system updates the traffic entry pattern for the next day according to the traffic adjustment mechanism in Vickrey (2020)'s second section (presented in Section 6.2.5). If the relative difference in the traffic entry pattern is less than the convergence threshold, then the model stops and returns the equilibrium result. Otherwise, the model will numerically solve Vickrey's bathtub model again with next day's traffic entry pattern.

The rest of Section 6.2 is organized as follows: Section 6.2.3 presents Vickrey's bathtub model, Section 6.2.4 presents Vickrey's marginal cost pricing calculation for a transient state, Section 6.2.5 presents the day-to-day dynamics of travelers' network entry time choices, Section 6.2.6 presents the numerical method to solve Vickrey's bathtub model, and Section 6.2.7 presents the convergence criteria.

6.2.3 Within-day dynamics: Vickrey's bathtub model

We use Vickrey's bathtub model (Vickrey, 1991, 2020) to model the within-day traffic dynamics in a network.

Let $\delta(t)$ be the number of vehicles attempting to move in a network at time t, $f(t)$ be the influx of vehicles entering the network at time t (unit: veh/hr), and $g(t)$ be the outflux of vehicles exiting the network at time t (unit: veh/hr). We have the following conservation equation for within-day dynamics of the number of active vehicles in the network:

$$\frac{d}{dt}\delta(t) = f(t) - g(t) \tag{6.1}$$

Under the following three assumptions: (1) undifferentiated streets; (2) the existence of a network fundamental diagram; and (3) negative exponential remaining trip distance, Vickrey (1991, 2020) derives the outflux, $g(t)$, as follows:

$$g(t) = \frac{1}{B} \cdot \delta(t) \cdot V(\frac{\delta(t)}{L}), \tag{6.2}$$

where B is the average trip length, L is the network total lane-miles, and $V(\cdot)$ is the network fundamental diagram, which related the network vehicle density (unit: veh/lane-mile) to the network space mean speed (unit: mile/hr). Equation (6.2) captures the congestion mechanism in a network.

Substiting Equation (6.2) into Equation (6.1), we have Vickrey's bathtub model as follows:

$$\frac{d}{dt}\delta(t) = f(t) - \frac{1}{B} \cdot \delta(t) \cdot V(\frac{\delta(t)}{L}) \tag{6.3}$$

We present the marginal social cost calculation of Vickrey's bathtub model in section 6.2.4.

6.2.4 Marginal social cost pricing in Vickrey's bathtub model

Let T be the time variable for an hour, where $T = \{0, 1, ...\}$. Let q be the extra active trips added to the network at time T, and $M(T)$ be the marginal social cost (unit: hour) at time T. So the marginal social cost of a small number of extra trips, dq, added to the network, $dq \cdot M(T)$, is equal to the marginal social cost of the remaining active trips at $T+1$ time step, times the marginal social cost at time step $T+1$, $M(T+1)$, plus the additional travel time experienced by the vehicles in the network from T to $T+1$ (Vickrey, 1991, 2020). Let $y(T, h, dq)$ be the remaining active trips at time t after dq active trips are added to the network at time T, where $t \in [T, T+1]$, and $h = t - T$, so $h \in [0, 1]$. Therefore, we have:

$$y(T, h, dq) = \delta(T, h, dq) - \delta(T, h, 0) \tag{6.4}$$

The marginal social cost satisfies the following equation:

$$dq \cdot M(T) = y(T, 1, dq) \cdot M(T+1) + \int_0^1 y(T, h, dq)dh \tag{6.5}$$

where $y(T, 1, dq)$ is the remaining active trips at time $T+1$ after dq trips are added to the network at time T.

Expanding $\int_0^1 y(T,h,dq)dh$ in Equation (6.5) according to Equation (6.4), we find:

$$\int_0^1 y(T,h,dq)dh = \int_0^1 \delta(T,h,dq)dh - \int_0^1 \delta(T,h,0)dh \tag{6.6}$$

where $\int_0^1 \delta(T,h,dq)dh$ is the total travel time experienced by travelers from time T to time $T+1$ due to dq trips added into the network at time T, while $\int_0^1 \delta(T,h,0)dh$ is the original total experienced travel time without extra traffic. So $\int_0^1 \delta(T,h,dq)dh - \int_0^1 \delta(T,h,0)dh$ describes the extra travel time experienced by travelers from time T to $T+1$, due to dq trips added to into the network at time T.

Dividing both sides of Equation (6.5) by dq, we get the following equation:

$$M(T) = \frac{y(T,1,dq)}{dq} \cdot M(T+1) + \int_0^1 \frac{y(T,h,dq)}{dq} dh \tag{6.7}$$

When $dq \to 0$, we get the following equation:

$$M(T) = \frac{\partial y(T,1,q)}{\partial q} \cdot M(T+1) + \int_0^1 \frac{\partial y(T,h,q)}{\partial q} dh \tag{6.8}$$

Taking the partial derivative of Equation (6.4) with respect to q, we obtain:

$$\frac{\partial y(T,h,dq)}{\partial q} = \frac{\partial \delta(T,h,dq)}{\partial q} - \frac{\partial \delta(T,h,0)}{\partial q} = \frac{\partial \delta(T,h,dq)}{\partial q} \tag{6.9}$$

where $\delta(T,h,0)$ is a constant, so $\frac{\partial \delta(T,h,0)}{\partial q} = 0$.

Since we evaluate marginal social cost of a very small amount of extra traffic dq, which is close to zero, we calculate $\frac{\partial y(T,h,dq)}{\partial q}$ at $dq = 0$. Let $Z(T,h)$ denote $\frac{\partial y(T,h,dq)}{\partial q}$ evaluated at

$dq = 0$, so we have the following equation:

$$Z(T,h) = \frac{\partial y(T,h,0)}{\partial q} = \frac{\partial \delta(T,h,0)}{\partial q} \tag{6.10}$$

Substituting Equation (6.10) into Equation (6.8), we derive the marginal cost equation evaluated at $dq = 0$:

$$M(T) = Z(T,1) \cdot M(T+1) + \int_0^1 Z(T,h)dh \tag{6.11}$$

We now estimate $Z(T,h)$ below. We take the partial derivative with respect to q (extra added trips to the network) in both rides of Vickrey's bathtub model (Equation (6.3)), and we have the following equation:

$$\frac{\partial}{\partial q}\frac{\partial \delta(t)}{\partial t} = \frac{\partial f(t)}{\partial q} - \frac{\partial}{\partial q}(\frac{1}{B} \cdot \delta(t) \cdot V(\frac{\delta(t)}{L})) \tag{6.12}$$

where the influx $f(t)$ in exogenous and is not dependent on q, so the first term in Equation (6.12) becomes zero (i.e., $\frac{\partial f(t)}{\partial q} = 0$). So Equation (6.12) becomes:

$$\frac{\partial}{\partial t}\frac{\partial \delta(t)}{\partial q} = -\frac{1}{B} \cdot (\frac{\partial \delta(t)}{\partial q} \cdot V(\frac{\delta(t)}{L}) + \delta(t) \cdot \frac{\partial}{\partial q}V(\frac{\delta(t)}{L}))) \tag{6.13}$$

Expanding $\frac{\partial}{\partial q}V(\frac{\delta(t)}{L})$ in the second term of Equation (6.13), we can derive:

$$\frac{\partial}{\partial q}V(\frac{\delta(t)}{L}) = \frac{\partial V}{\partial(\delta(t)/L)} \cdot (\frac{\partial(\delta(t)/L)}{\partial q}) = \frac{\partial V}{\partial(\rho(t))} \cdot \frac{1}{L}(\frac{\partial \delta(t)}{\partial q}), \tag{6.14}$$

where $\rho(t)$ is the density of the network fundamental diagram (unit: veh/lane-mile). Substituting Equation (6.14) into Equation (6.13), we have the following equation:

$$\frac{\partial}{\partial t}\frac{\partial \delta(t)}{\partial q} = -\frac{1}{B} \cdot (\frac{\partial \delta(t)}{\partial q} \cdot V(\frac{\delta(t)}{L}) + \delta(t) \cdot \frac{\partial V}{\partial(\rho(t))} \cdot \frac{1}{L} \cdot (\frac{\partial \delta(t)}{\partial q})) \tag{6.15}$$

Arraging Equation (6.15), we obtain:

$$\frac{\partial}{\partial t}\frac{\partial \delta(t)}{\partial q} = -\frac{1}{B}\cdot(V(\frac{\delta(t)}{L}) + \delta(t)\cdot\frac{\partial V}{\partial(\rho(t))}\cdot\frac{1}{L})\cdot\frac{\partial \delta(t)}{\partial q} \tag{6.16}$$

If the speed-density relationship is linear (Greenshields network fundamental diagram), we have:

$$V(\rho) = u\cdot(1-\frac{\rho}{\rho_j}) \tag{6.17}$$

where u is the free flow speed while ρ_j is the jam density in the network fundamental diagram. So $\frac{\partial V}{\partial \rho(t)} = -\frac{u}{\rho_j}$.

Substituting $\frac{\partial V}{\partial \rho(t)} = -\frac{u}{\rho_j}$ and Equation (6.17) into Equation (6.16), we derive:

$$\frac{\partial}{\partial t}\frac{\partial \delta(t)}{\partial q} = -\frac{1}{B}\cdot(u\cdot(1-\frac{\delta(t)/L}{\rho_j}) + \delta(t)\cdot(-\frac{u}{\rho_j})\cdot\frac{1}{L})\cdot\frac{\partial \delta(t)}{\partial q} \tag{6.18}$$

Arranging the terms of Equation (6.18), we derive:

$$\frac{\partial}{\partial t}\frac{\partial \delta(t)}{\partial q} = -\frac{1}{B}\cdot u\cdot(1-2\cdot\frac{\delta(t)}{L\cdot\rho_j})\cdot\frac{\partial \delta(t)}{\partial q} \tag{6.19}$$

Replacing $\frac{\partial \delta(t)}{\partial q}$ in Equation (6.19) with $Z(T,h) = \frac{\partial \delta(T,h,0)}{\partial q}$ and changing time variable from t to h, Equation (6.19) becomes:

$$\frac{\partial}{\partial h}Z(T,h) = -\frac{1}{B}\cdot u\cdot(1-2\cdot\frac{\delta(h)}{L\cdot\rho_j})\cdot Z(T,h), \tag{6.20}$$

where $h\in[0,1]$.

Dividing both sides of Equation (6.20) by $Z(T,h)$, yields:

$$\frac{\frac{\partial Z(T,h)}{\partial h}}{Z(T,h)} = -\frac{1}{B} \cdot u \cdot (1 - \frac{2}{L \cdot \rho_j} \cdot \delta(h)) \tag{6.21}$$

Since the variable in $\delta(h)$ is h ($h = t - T$) instead of t, by assuming $\delta(t)$ varying linearly, we can re-write $\delta(h)$ as follows:

$$\delta(h) = \delta(T) + h \cdot \frac{\delta(T+1) - \delta(T)}{(T+1) - T} = \delta(T) + h \cdot (\delta(T+1) - \delta(T)) \tag{6.22}$$

Substituting Equation (6.22) into $\delta(h)$ in Equation (6.21), we have:

$$\frac{\frac{\partial Z(T,h)}{\partial h}}{Z(T,h)} = -\frac{1}{B} \cdot u \cdot [1 - \frac{2}{L \cdot \rho_j} \cdot (\delta(T) + h \cdot (\delta(T+1) - \delta(T)))] \tag{6.23}$$

Arranging the terms with h and terms without h of Equation (6.23), we have:

$$\frac{\frac{\partial Z(T,h)}{\partial h}}{Z(T,h)} = -\frac{1}{B} \cdot u \cdot [(1 - \frac{2}{L \cdot \rho_j} \cdot \delta(T)) - \frac{2}{L \cdot \rho_j} \cdot (\delta(T+1) - \delta(T)) \cdot h] \tag{6.24}$$

We set $\tilde{A}$ to be the first term of Equation (6.24) and $\tilde{B}$ to be coefficient in front of h in the second term in Equation (6.24), so we have:

$$\tilde{A} = -\frac{1}{B} \cdot u \cdot (1 - \frac{2}{L \cdot \rho_j} \cdot \delta(T)); \tilde{B} = \frac{1}{B} \cdot u \cdot (\frac{2}{L \cdot \rho_j} \cdot (\delta(T+1) - \delta(T))) \tag{6.25}$$

Equation (6.24) then becomes:

$$\frac{\frac{\partial Z(T,h)}{\partial h}}{Z(T,h)} = \tilde{A} + \tilde{B} \cdot h \tag{6.26}$$

Integrating both sides of the Equation (6.26) with respect to h, we obtain:

$$\ln Z(T,h) = \tilde{A} \cdot h + \tilde{B} \cdot \frac{1}{2} \cdot h^2 + C_0 \tag{6.27}$$

where C_0 is a constant to be determined by the initial condition when $h = 0$.

When $h = 0$, $\delta(T,0,q) = \delta(T,0,0) + q$, and $\delta(T,0,0)$ is independent of q, so we find:

$$\frac{\partial \delta(T,0,q)}{\partial q} = \frac{\partial}{\partial q}(\delta(T,0,0) + q) = 0 + 1 = 1, \tag{6.28}$$

$Z(T,0) = \frac{\partial \delta(T,0,q)}{\partial q}$ evaluated at $q = 0$, so we have $Z(T,0) = 1$.

When $h = 0$, Equation (6.27) becomes:

$$\ln Z(T,0) = \ln 1 = \tilde{A} \cdot 0 + \tilde{B} \cdot \frac{1}{2} \cdot 0 + C_0 = C_0 = 0, \tag{6.29}$$

Substituting $C_0 = 0$ back to Equation (6.27), we get:

$$\ln Z(T,h) = \tilde{A} \cdot h + \tilde{B} \cdot \frac{1}{2} h^2 \tag{6.30}$$

We thus obtain:

$$Z(T,h) = exp(\tilde{A} \cdot h + \tilde{B} \cdot \frac{1}{2} h^2) \tag{6.31}$$

When $h = 1$, $Z(T,1) = exp(\tilde{A} + \frac{1}{2}\tilde{B})$, so Equation (6.11) becomes:

$$M(T) = exp(\tilde{A} + \frac{1}{2}\tilde{B}) \cdot M(T+1) + \int_0^1 exp(\tilde{A} \cdot h + \tilde{B} \cdot \frac{1}{2} h^2) dh \tag{6.32}$$

The second term of Equation (6.32) integrates a quadratic exponent. Let $H(h) = \tilde{A} \cdot h + \tilde{B} \cdot \frac{1}{2} h^2$, and we further approximate the quadratic function $H(h)$ with a linear function

$\tilde{H}(h) = (\tilde{A} + \frac{1}{2}\tilde{B}) \cdot h$ with the same end points of $(0, 0)$ and $(1, \tilde{A} + \frac{1}{2}\tilde{B})$, so we have:

$$H(h) = \tilde{A} \cdot h + \tilde{B} \cdot \frac{1}{2}h^2 \approx (\tilde{A} + \frac{1}{2}\tilde{B}) \cdot h = \tilde{H}(h) \tag{6.33}$$

Consequently, the second term of Equation (6.32) becomes:

$$\int_0^1 exp(H(h))dh \approx \int_0^1 exp(\tilde{H}(h))dh = \frac{exp[(\tilde{A} + \frac{1}{2}\tilde{B}) \cdot h]}{\tilde{A} + \frac{1}{2}\tilde{B}}\Big|_0^1 = \frac{exp(\tilde{A} + \frac{1}{2}\tilde{B}) - exp(0)}{\tilde{A} + \frac{1}{2}\tilde{B}} \tag{6.34}$$

Substituting Equation (6.34) in Equation (6.32), we find that:

$$M(T) = exp(\tilde{A} + \frac{1}{2}\tilde{B}) \cdot M(T + 1) + \frac{exp(\tilde{A} + \frac{1}{2}\tilde{B}) - 1}{\tilde{A} + \frac{1}{2}\tilde{B}} \tag{6.35}$$

where $\tilde{A}$ and $\tilde{B}$ can be calculated using $\delta(T)$ and $\delta(T + 1)$ from Equation (6.25).

We use Equation (6.35) to estimate the marginal social cost backward from an assumed marginal social cost where the marginal social cost is equal to the average cost when the network has minimum number of vehicles in the early morning (e.g., 3am). So the marginal social cost should be estimated from 3am on the next day (27th hour) backwards to 3am on the first day with an assumed marginal cost on 27th hour. If the marginal cost at 3am, $M(3)$, is significantly different from the marginal cost at 27th hour, $M(27)$, then we can set $M(27) = M(3)$ and estimate the marginal social cost again from Equation (6.35) using the new $M(27)$.

We can calculate the toll schedule, $P(T)$, as follows:

$$P(T) = M(T) - \frac{B}{v(T)} \tag{6.36}$$

where the toll, $P(T)$, is equal to marginal social cost $M(T)$ at time T, minus average cost,

$\frac{B}{v(T)}$, which is the average trip length, B, divided by network speed at time T, $v(T)$.

6.2.5 Day-to-day dynamics in travelers' network entry time choices

We describe how travelers change their times to enter the network from day to day in response to the marginal social cost toll, $P(T)$. We can also consider the network entry time choices as travelers' departure time choices at the network level.

In Section 6.2.3, we use $f(t)$ to denote the influx to the network, and we only consider the within day dynamics. In this section, we consider day-to-day dynamics, so we add the day variable τ to $f(t)$, so let $f(\tau, t)$ denote the influx to the network at time t on day τ. Let $\phi(\tau, t)$ denote the travelers trip cost at time t on day τ, $v(\tau, t)$ denote the network speed at time t on day τ, and $P(\tau, t)$ denote the toll (unit: hour) at time t on day τ. We assume that there is no toll on day 0, and toll starts on day $0 + \Delta\tau$. So on day 0, the average cost $\phi(0, t)$ experienced by travelers is as follows:

$$\phi(0, t) = \frac{B}{v(0, t)}. \tag{6.37}$$

On day τ ($\tau > 0$), travelers' experience average cost with toll, $P(\tau, t)$, is as follows:

$$\phi(\tau, t) = P(\tau, t) + \frac{B}{v(\tau, t)}. \tag{6.38}$$

We use the discrete version of $\phi(\tau, t)$ to for the rest of this section. Let I be the study time interval, J be the study day period. We divide I into N intervals, and J into M periods. So we have $\Delta t = \frac{I}{N}$ and $\Delta\tau = \frac{J}{M}$. Let $\phi(j\Delta\tau, i\Delta t) = \phi_i^j$, $P(j\Delta\tau, i\Delta t) = P_i^j$, and $v(j\Delta\tau, i\Delta t) = v_i^j$, where $i \in \{0, 1, ..., N\}$ and $j \in \{0, 1, ..., M\}$. So we have the following equation for the cost on time step i and day step j, ϕ_i^j:

$$\phi_i^j = P_i^j + \frac{B}{v_i^j}, \quad j > 0. \tag{6.39}$$

When $j = 0$ on day 0, we have $\phi_i^0 = \frac{B}{v_i^0}$.

Let $\sigma(\tau, t)$ denote the cost increment on the current day τ compared with day 0. We have the discrete version of $\sigma(j\Delta\tau, i\Delta t) = \sigma_i^j$. so we have:

$$\sigma_i^j = \phi_i^j - \phi_i^0 = P_i^j + \frac{B}{v_i^j} - \frac{B}{v_i^0} \tag{6.40}$$

Let $\omega(j\Delta\tau, (i - \frac{1}{2})\Delta t) = \omega_i^j$ denote the difference in cost increment at time step i and $i-1$, where $i \in \{1, 2, ..., N\}$. Notice here $\omega(\tau, t)$ refers to the difference in cost increment between two consecutive time steps, not the difference in the cost. Therefore, we have:

$$\omega_i^j = \sigma_i^j - \sigma_{i-1}^j. \tag{6.41}$$

Vickrey (2020) considers four kinds of travelers' responses to congestion pricing — deferral, advance, suppression, and generation, where deferral and advance relate to the fixed demand case, and suppression, and generation relate to the elastic demand case. Here we only consider two kinds of responses related to fixed demand. We assume that travelers will advance their network entry time or defer their network entry time in response to the differences in two consecutive cost increments, ω_i^j. For travelers departing at time step i on day τ, on day $\tau + \Delta\tau$, we have the following situations:

1. If $\omega_{i+1}^j < 0$, $\sigma_{i+1}^j < \sigma_i^j$, the cost increment at time step $i+1$ is less than the cost increment at time step i, travelers will defer their network entry time from time step i to $i+1$. Let $f_i^d(\tau)$ be the deferral rate at time step i on day τ (unit: veh/day). We

have the following network entry deferral relationship:

$$f_i^d(\tau)\Delta\tau = B_{i+1}^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ \cdot f_i(\tau)\Delta t \tag{6.42}$$

where $B_{i+1}^d(\tau)$ is a positive deferral coefficient, which is to be estimated. $\{-\omega_{i+1}(\tau)\}_+ = \max\{0, -\omega_{i+1}(\tau)\}$, and $f_i(\tau)$ is the influx at time step i on day τ.

2. If $\omega_i^j \geq 0$, $\sigma_i^j \geq \sigma_{i-1}^j$, the cost increment at time step i is greater than or equal to the cost increment at time step $i-1$, travelers will advance their network entry time from time step i to $i-1$. Let $f_i^a(\tau)$ be the advance rate at time step i on day τ (unit: veh/day). Since the same traveler cannot both defer and advance, we exclude the deferral travelers (i.e., $f_i^d(\tau)\Delta\tau$) when we calculate the advance travelers. We have the following network entry advancing relationship:

$$f_i^a(\tau)\Delta\tau = B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ \cdot (f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau) \tag{6.43}$$

where $B_i^a(\tau)$ is a positive advance coefficient, which is to be estimated.

We have the following relationship to update the influx in time step i on day $\tau + \Delta\tau$ as follows:

$$f_i(\tau + \Delta\tau)\Delta t = f_i(\tau)\Delta t - f_i^d(\tau)\Delta\tau - f_i^a(\tau)\Delta\tau + f_{i-1}^d(\tau)\Delta\tau + f_{i+1}^a(\tau)\Delta\tau \tag{6.44}$$

The well-definedness condition of Equation (6.42) and Equation (6.43) is $f_i(\tau) \geq 0$ for $i \in I$ and $\tau \in J$. That is, there is no negative influx in any time step on any given day. This condition could be written as follows equivalently (Jin, 2021b):

$$B_i^d(\tau) \cdot \{-\omega_{i+1}(\tau)\}_+ \leq 1; B_i^a(\tau) \cdot \{\omega_i(\tau)\}_+ \leq 1. \tag{6.45}$$

The condition in Equation (6.45) can be written as follows:

$$B_i^d(\tau) \leq \frac{1}{\{-\omega_{i+1}(\tau)\}_+^{max}}; B_i^a(\tau) \leq \frac{1}{\{\omega_i(\tau)\}_+^{max}}. \tag{6.46}$$

where $\{-\omega_{i+1}(\tau)\}_+^{max}$ is the maximum value of $\{-\omega_{i+1}(\tau)\}_+$, while $\{\omega_i(\tau)\}_+^{max}$ is the maximum value of $\{\omega_i(\tau)\}_+$.

$$\omega_i^j = \sigma_i^j - \sigma_{i-1}^j = (P_i^j + \frac{B}{v_i^j} - \frac{B}{v_i^0}) - (P_{i-1}^j + \frac{B}{v_{i-1}^j} - \frac{B}{v_{i-1}^0}) \tag{6.47}$$

From Equation (6.36), we have $M(T) = P(T) + \frac{B}{v(T)}$. Let M_i^j denote the marginal social cost for time step i on day step j. Equation (6.47) becomes:

$$\omega_i^j = (M_i^j - \frac{B}{v_i^0}) - (M_{i-1}^j - \frac{B}{v_{i-1}^0}) = (M_i^j - M_{i-1}^j) + (\frac{B}{v_{i-1}^0} - \frac{B}{v_i^0}) \leq (M_{max}^j - M_{min}^j) + (\frac{B}{v_{min}^0} - \frac{B}{v_{max}^0}) \tag{6.48}$$

where M_{max}^j is the maximum value of marginal social cost on day j, and M_{min}^j is the minimum value of marginal social cost on day j, which is equal to the average cost at free flow speed. v_{min}^0 is the minimum speed on day 0, while v_{max}^0 is the maximum speed on day 0. When network speed reaches zero (i.e., network gridlock), $\frac{B}{v_{min}^0}$ and M_{max} reach infinity. So $\{-\omega_{i+1}^j\}_+^{max} \to \infty$ and $\{\omega_i^j\}_+^{max} \to \infty$. From Equation (6.46), if $\{-\omega_{i+1}^j\}_+^{max} \to \infty$ and $\{\omega_i^j\}_+^{max} \to \infty$, we have $B_i^d(\tau) \to 0$ and $B_i^a(\tau) \to 0$, when network speed reaches 0.

So there are no $B_i^d(\tau)$ and $B_i^a(\tau)$ that can make the system well-defined under all conditions, but we can choose $B_i^d(\tau)$ and $B_i^a(\tau)$ in the ad-hoc sense when modeling day-to-day network entry times (or network level departure time) adjustment mechanism in Vickrey's bathtub model. In this study, $B_i^d(\tau)$ and $B_i^a(\tau)$ are chosen according to Equation (6.65).

6.2.6 A numerical method to solve Vickrey's bathtub model

Vickrey (2020) proposes a numerical method to solve Vickrey's bathtub model, considering the curvature of the network fundamental diagram.

We assume that we know all the variables at time T, and we want to solve for $T+h$, where $h \in [0,1]$. From Equation (6.3), we have:

$$\frac{d}{dt}\delta(t) = f(t) - \frac{1}{B}\cdot\delta(t)\cdot V(\frac{\delta(t)}{L})$$

Changing the variable from t to $T+h$, Equation (6.3) becomes the following:

$$\frac{d\delta(T+h)}{dh} = f(T+h) - \frac{1}{B}\cdot\delta(T+h)\cdot V(\frac{\delta(T+h)}{L}) \tag{6.49}$$

We have $g(\delta) = \frac{1}{B}\cdot\delta\cdot V(\frac{\delta}{L})$, so $\frac{d}{d\delta}g(\delta) = \frac{1}{B}\cdot(V(\frac{\delta}{L}) + \delta\cdot\frac{dV}{d\delta}\cdot\frac{1}{L}) = \frac{1}{B}\cdot(u - 2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})$. The linear approximation of $g(T+h)$ based on the tangent of $g(T)$ at T is:

$$g(T+h) = g(T) + \frac{1}{B}\cdot(u - 2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})\cdot(\delta(T+h)-\delta(T)), \tag{6.50}$$

which uses the slope of the network fundamental diagram at $\delta(T)$, $\frac{1}{B}\cdot(u - 2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})$, and the distance between the number of vehicles at $T+h$ and T, $\delta(T+h)-\delta(T)$.

Since the network fundamental diagram is in the quadratic form, we add another second-order term to Equation (6.50), $\tilde{C}h^2$, to capture the curvature of the network fundamental diagram. So Equation (6.50) becomes:

$$g(T+h) = g(T) + \frac{1}{B}\cdot(u - 2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})\cdot(\delta(T+h)-\delta(T)) + \tilde{C}\cdot h^2, \tag{6.51}$$

We will compare the exact form of $\frac{d^2}{dt^2}g(t)$ with the approximation $\frac{d^2}{dh^2}g(T+h)$ to determine the constant $\tilde{C}$.

For $g(t)$, we have:

$$\begin{aligned}
&\frac{d}{dt}g(t) = \frac{dg}{d\delta}\cdot\frac{d\delta(t)}{dt} = \frac{1}{B}\cdot(u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(t)}{L})\cdot\frac{d}{dt}\delta(t) \\
&\frac{d^2}{dt^2}g(t) = \frac{1}{B}\cdot\left\{-2\cdot\frac{u}{\rho_j}\cdot\frac{1}{L}\cdot(\frac{d}{dt}\delta(t))^2 + (u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(t)}{L})\cdot\frac{d^2}{dt^2}\delta(t)\right\}
\end{aligned} \tag{6.52}$$

For $g(T+h)$, we have:

$$\begin{aligned}
&\frac{d}{dh}g(T+h) = \frac{1}{B}\cdot(u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})\cdot\frac{d}{dh}\delta(T+h) + 2\cdot\tilde{C}\cdot h \\
&\frac{d^2}{dh^2}g(T+h) = \frac{1}{B}\cdot(u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})\cdot\frac{d^2}{dh^2}\delta(T+h) + 2\cdot\tilde{C}
\end{aligned} \tag{6.53}$$

Equating Equation (6.52) and Equation (6.53), we have:

$$2\cdot\tilde{C} = -2\cdot\frac{1}{B}\cdot\frac{u}{\rho_j}\cdot\frac{1}{L}\cdot(\frac{d}{dt}\delta(t))^2 \tag{6.54}$$

Therefore, we have $\tilde{C} = -\frac{1}{B}\cdot\frac{u}{\rho_j}\cdot\frac{1}{L}\cdot(\frac{d}{dt}\delta(t))^2$. We assume that $f(T+h) = f(T)+(f(T+1)-f(T))\cdot h$. Substituting Equation (6.51) into Equation (6.49), we have the following equation:

$$\frac{d\delta(T+h)}{dh} = \left\{f(T)+(f(T+1)-f(T))\cdot h\right\} - \left\{g(T)+\frac{1}{B}\cdot(u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})\cdot(\delta(T+h)-\delta(T))+\tilde{C}\cdot h^2\right\}, \tag{6.55}$$

which is in the form of $\frac{d\delta(T+h)}{dh} = \hat{A}\delta(T+h)+\hat{B}h^2+\hat{C}h+\hat{D}$, where $\hat{A} = -\frac{1}{B}\cdot(u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})$, $\hat{B} = -\tilde{C} = \frac{1}{B}\cdot\frac{u}{\rho_j}\cdot\frac{1}{L}\cdot(\frac{d}{dt}\delta(t))^2$, $\hat{C} = (f(T+1)-f(T))$, and $\hat{D} = f(T)-g(T)+\frac{1}{B}\cdot(u-2\cdot\frac{u}{\rho_j}\cdot\frac{\delta(T)}{L})\cdot\delta(T)$. Arranging $\frac{d\delta(T+h)}{dh} = \hat{A}\delta(T+h)+\hat{B}h^2+\hat{C}h+\hat{D}$, we have:

$$\frac{d\delta(T+h)}{dh} - \hat{A}\delta(T+h) = \hat{B}h^2+\hat{C}h+\hat{D} \tag{6.56}$$

Multiply both sides with $e^{-\hat{A}\cdot h}$, we have:

$$e^{-\hat{A}\cdot h} \cdot \frac{d\delta(T+h)}{dh} - \hat{A} \cdot e^{-\hat{A}\cdot h} \cdot \delta(T+h) = e^{-\hat{A}\cdot h} \cdot (\hat{B}h^2 + \hat{C}h + \hat{D}), \tag{6.57}$$

and it becomes:

$$(e^{-\hat{A}\cdot h} \cdot \delta(T+h))' = e^{-\hat{A}\cdot h} \cdot (\hat{B}h^2 + \hat{C}h + \hat{D}) \tag{6.58}$$

Integrating both sides of the equation with respect to h, we have:

$$e^{-\hat{A}\cdot h} \cdot \delta(T+h) = \int e^{-\hat{A}\cdot h} \cdot (\hat{B}h^2 + \hat{C}h + \hat{D})dh \tag{6.59}$$

For the right hand side, integrating by parts, we have

$$\int e^{-\hat{A}h} \cdot (\hat{B}h^2 + \hat{C}h + \hat{D})\, dh = e^{-\hat{A}h} \left[\frac{\hat{B}}{\hat{A}^3}(\hat{A}^2h^2 - 2\hat{A}h + 2) + \frac{\hat{C}}{\hat{A}^2}(\hat{A}h - 1) - \frac{\hat{D}}{\hat{A}} \right] + C_1 \tag{6.60}$$

When $h = 0$, then

$$\int e^{-\hat{A}h} \cdot (\hat{B}h^2 + \hat{C}h + \hat{D})\, dh = \left[\frac{\hat{B}}{\hat{A}^3} \cdot (2) + \frac{\hat{C}}{\hat{A}^2}(-1) - \frac{\hat{D}}{\hat{A}} \right] + C_1 = \delta(T) \tag{6.61}$$

So we have:

$$C_1 = \delta(T) - \frac{2\hat{B}}{\hat{A}^3} + \frac{\hat{C}}{\hat{A}^2} + \frac{\hat{D}}{\hat{A}} \tag{6.62}$$

Substituting Equation (6.62) back to Equation (6.60), and substituting Equation (6.60) back to Equation (6.59), dividing both sides by $e^{-\hat{A}\cdot h}$, we have:

$$\delta(T+h) = \left[\frac{\hat{B}}{\hat{A}^3}(\hat{A}^2h^2 - 2\hat{A}h + 2) + \frac{\hat{C}}{\hat{A}^2}(\hat{A}h - 1) - \frac{\hat{D}}{\hat{A}} \right] + e^{\hat{A}h} \cdot \left[\delta(T) - \frac{2\hat{B}}{\hat{A}^3} + \frac{\hat{C}}{\hat{A}^2} + \frac{\hat{D}}{\hat{A}} \right], \tag{6.63}$$

where $\hat{A} = -\frac{1}{B} \cdot (u - 2 \cdot \frac{u}{\rho_j} \cdot \frac{\delta(T)}{L})$, $\hat{B} = \frac{1}{B} \cdot \frac{u}{\rho_j} \cdot \frac{1}{L} \cdot (\frac{d}{dt}\delta(t))^2$, $\hat{C} = (f(T+1) - f(T))$, and $\hat{D} = f(T) - g(T) + \frac{1}{B} \cdot (u - 2 \cdot \frac{u}{\rho_j} \cdot \frac{\delta(T)}{L}) \cdot \delta(T)$. Replacing h in Equation (6.63) with Δt, we use Equation (6.63) to simulate Vickrey's bathtub model from t to $t + \Delta t$.

6.2.7 Convergence criteria

We use Root Mean Squared Percentage Error (RMSPE) on day step j as the convergence criteria:

$$RMSPE_j = \sqrt{\frac{1}{N} \cdot \sum_i (\frac{f_i^j - f_i^{j-1}}{f_i^{j-1}})^2}, j \in \{1, 2, 3, ...M\} \tag{6.64}$$

where f_i^j represents the influx in time step i on day step j. If $RMSPE_j < \varepsilon$, then we consider the system converges to an optimal network level departure time choice equilibrium driven by marginal social cost. We set $\varepsilon = 0.001$ in this chapter.

6.3 Simulation study

6.3.1 Simulation setup

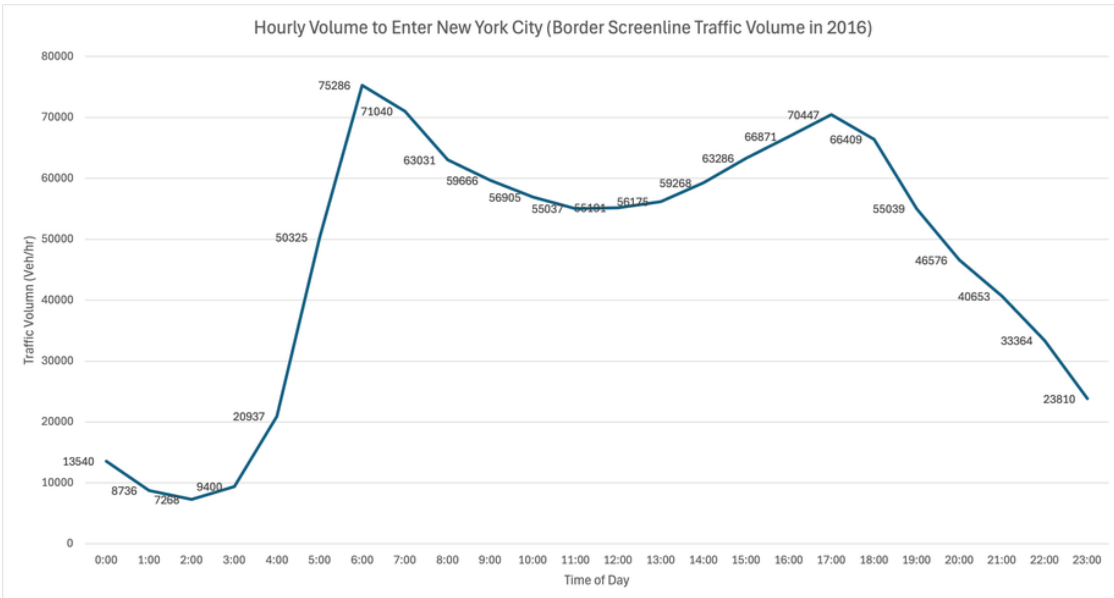


Figure 6.2: 24-hour traffic entry pattern in New York city

We use 48-hour Manhattan traffic entry data with 2.2 million trips as demand input to Vickrey's bathtub model, which is duplicated from 24-hour traffic entry data from (Gutman et al., 2016), as the marginal cost calculation requires starting from 3 a.m. on the next day. Figure 6.2 shows the entry pattern. We have the following settings: $L = 225$ miles, $u = 30$ mile/hr, $\rho_j = 200$ veh/lane· mile, and $B = 5$ miles. We set $B_i^d(\tau)$ and $B_i^a(\tau)$ as follows:

$$B_i^d(\tau) = B_i^a(\tau) = \frac{1}{2} \cdot \frac{1}{P_{max}^1 - P_{min}^1 + 2 \cdot (B/v^{min} - B/u)} \tag{6.65}$$

where P_{max}^1 is the maximum toll on day step 1 derived from the no-toll pattern on day step 0, while P_{min}^1 is the minimum toll on day step 1 derived from the no-toll pattern on day step 0. Theoretically, v^{min} can reach zero, but we select v^{min} to be 1 mile/hr. So $B/v^{min} = B/1 = 5$.

In this study, the departure time study period is a two-day 48-hour period, $I = 48$. We divide it into 48 periods, so $N = 48$, $dt = \frac{I}{N} = 1$ hr. We set the study day steps to $6,000$ steps, so $M = 6,000$, and we set the day step size to be 1 (i.e., $\Delta\tau = 1$). Section 6.3.2 presents the simulation results.

6.3.2 Simulation results

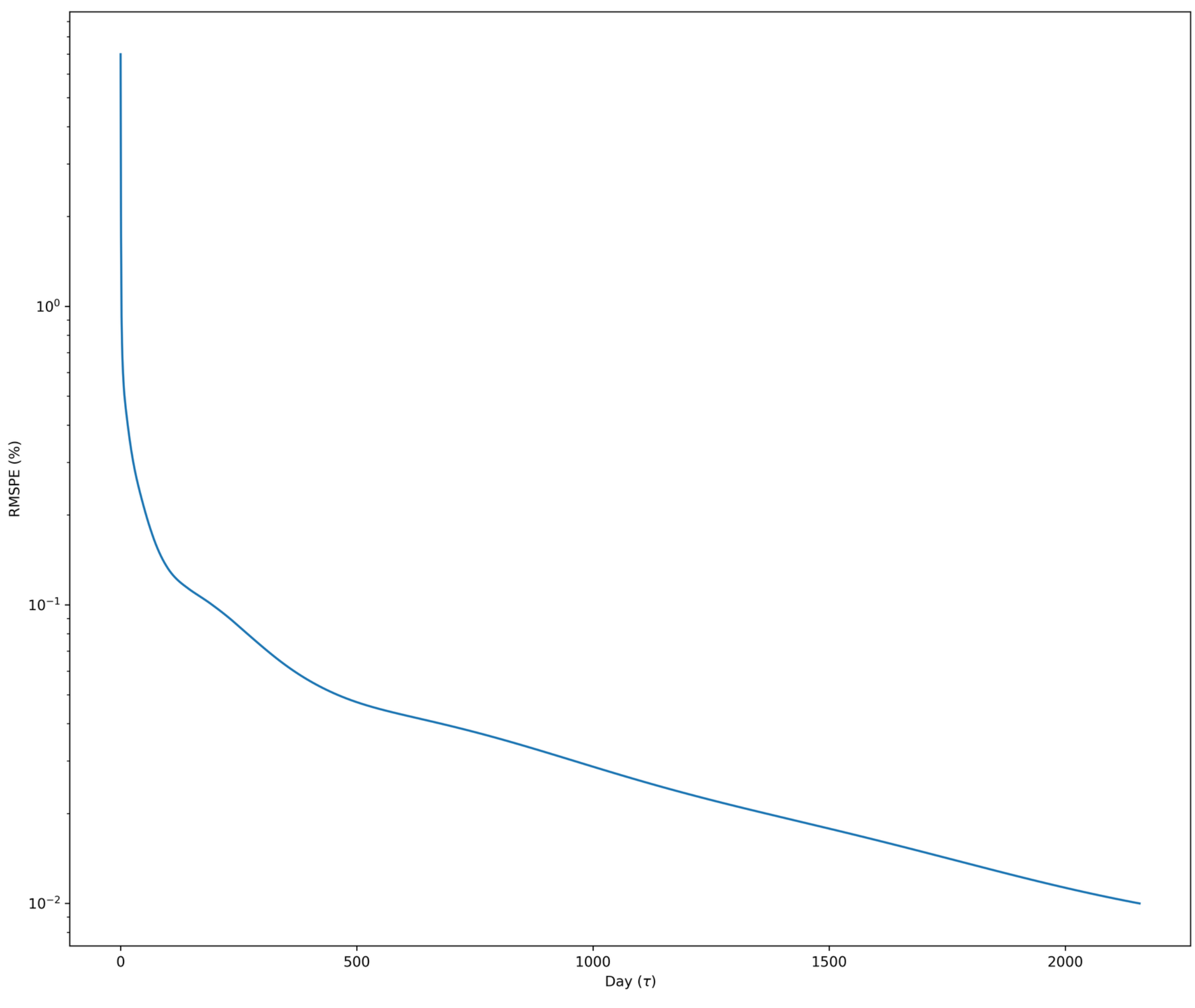
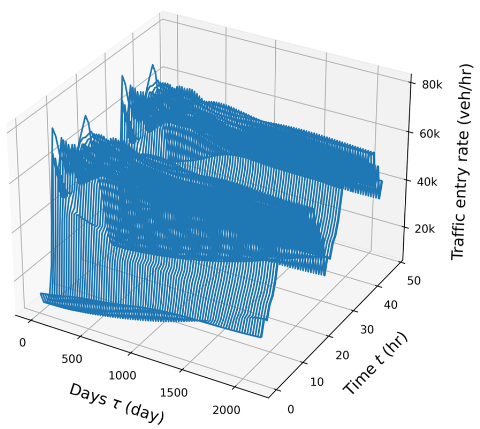
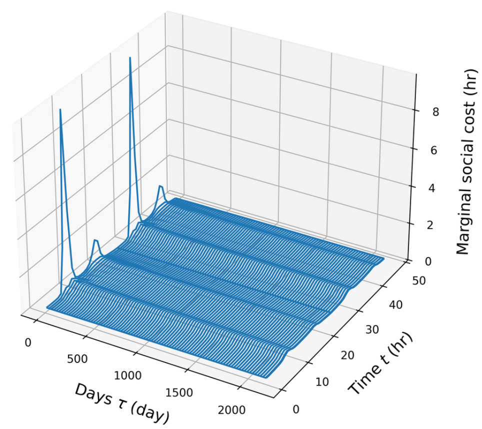


Figure 6.3: Day-to-day evolution of RMSPE error

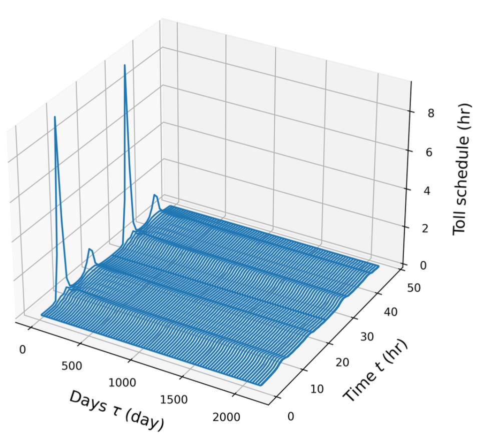
Figure 6.3 shows that the system converges on day step $2,158$, and the RMSPE error converges to 10^{-2} eventually. This means that the model converges to a practically stable state where the time-dependent network influx converges to a stable pattern from day to day. Since this system is too complicated to find a Lyapunov function to theoretically prove its stability, we call the system reaches a “practically stable” state, where the RMSPE decreases below the threshold.



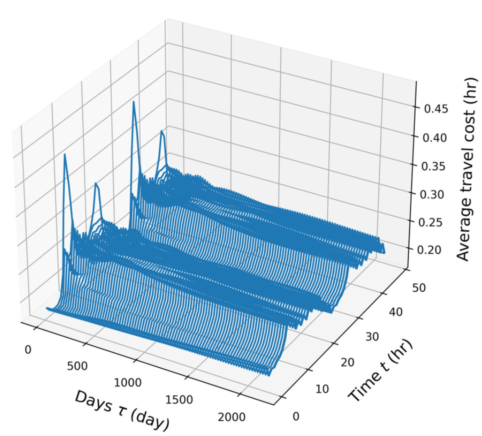
(a) Traffic entry rate (veh/hr)



(b) Marginal social cost (hr)



(c) Toll schedule (hr)



(d) Average travel cost (hr)

Figure 6.4: Day-to-day evolution of traffic entry rate, marginal social cost, toll schedule, and average travel cost

Figure 6.4 shows the day-to-day evolution of the traffic entry rate, marginal social cost, toll schedule, and average travel cost. After applying the marginal social cost pricing along with the traffic entry adjustment process, travelers shift their entry times from peak to off-peak. From day to day, the marginal social cost decreases very quickly at the beginning and remains relatively stable afterwards. Comparing the last day step with day step 0, the maximum marginal social cost decreases by 94.7% from 9.63 to 0.51 hour-equivalents. At the same time, the maximum toll decreases from 9.30 to 0.26 hour-equivalents. Additionally, maximum average travel cost decreases by 40.5% from 0.42 to 0.25 hours. With the day-to-day inflowing traffic adjustment mechanism, the marginal social cost pricing can lead to a practically stable optimal equilibrium, shifting travelers' entry times from peak to off-peak periods, reducing the average cost in the network.

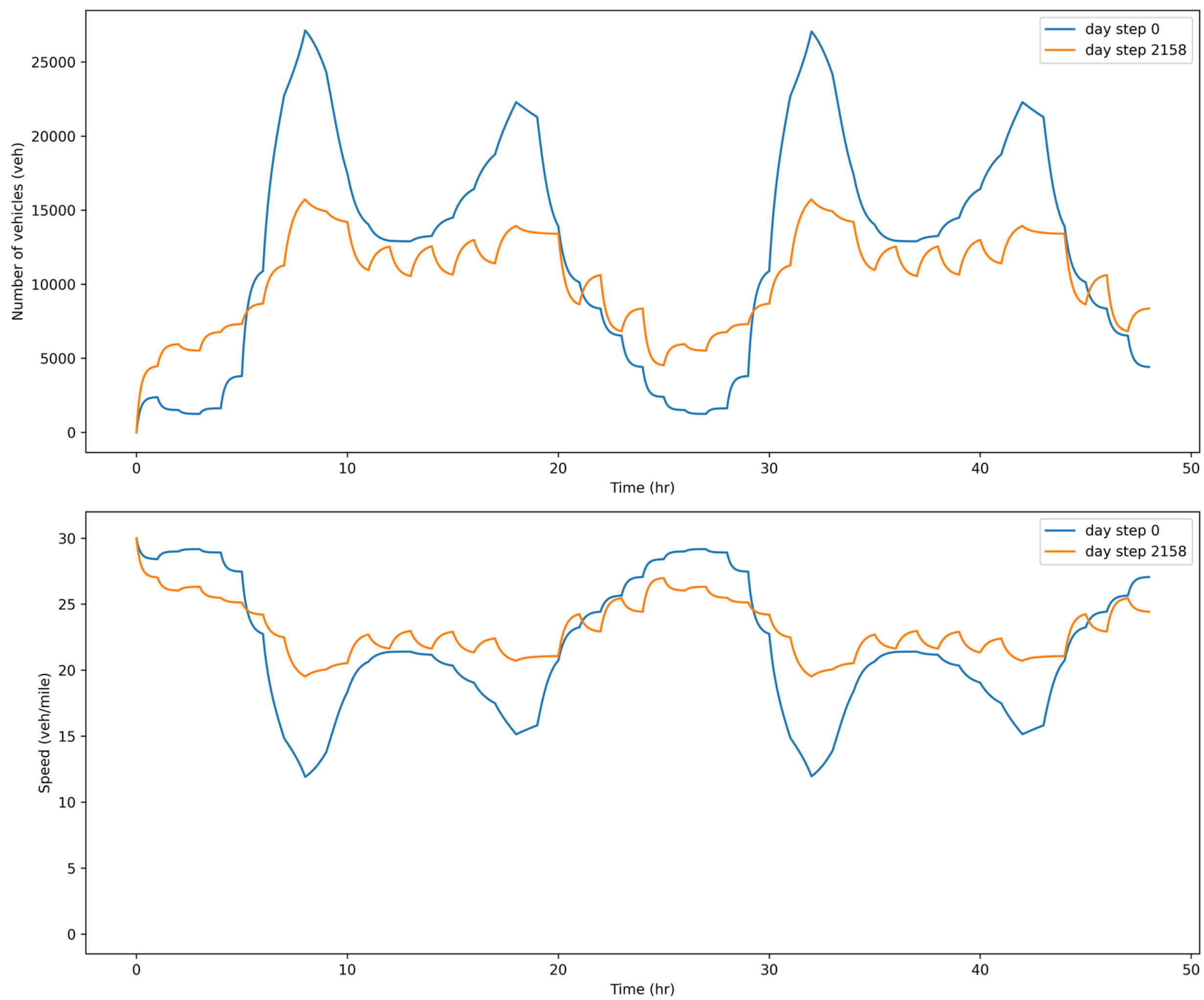


Figure 6.5: Time-dependent number of vehicles and network speed on day-step 0 and day-step 2,158

Figure 6.5 compares the number of vehicles (network queue) and network speed in Vickrey's bathtub model on day step 0 and day step 2,158. The maximum number of vehicles in the network decreases by 42%, from 27,125 on day step 0 to 15,725 on day step 2,158. The minimum speed in the network increases by 64%, from 11.9 mph to 19.5 mph. The results show that Vickrey's marginal cost pricing can effectively reduce the severity of the congestion and improve the network space mean speed from day to day.

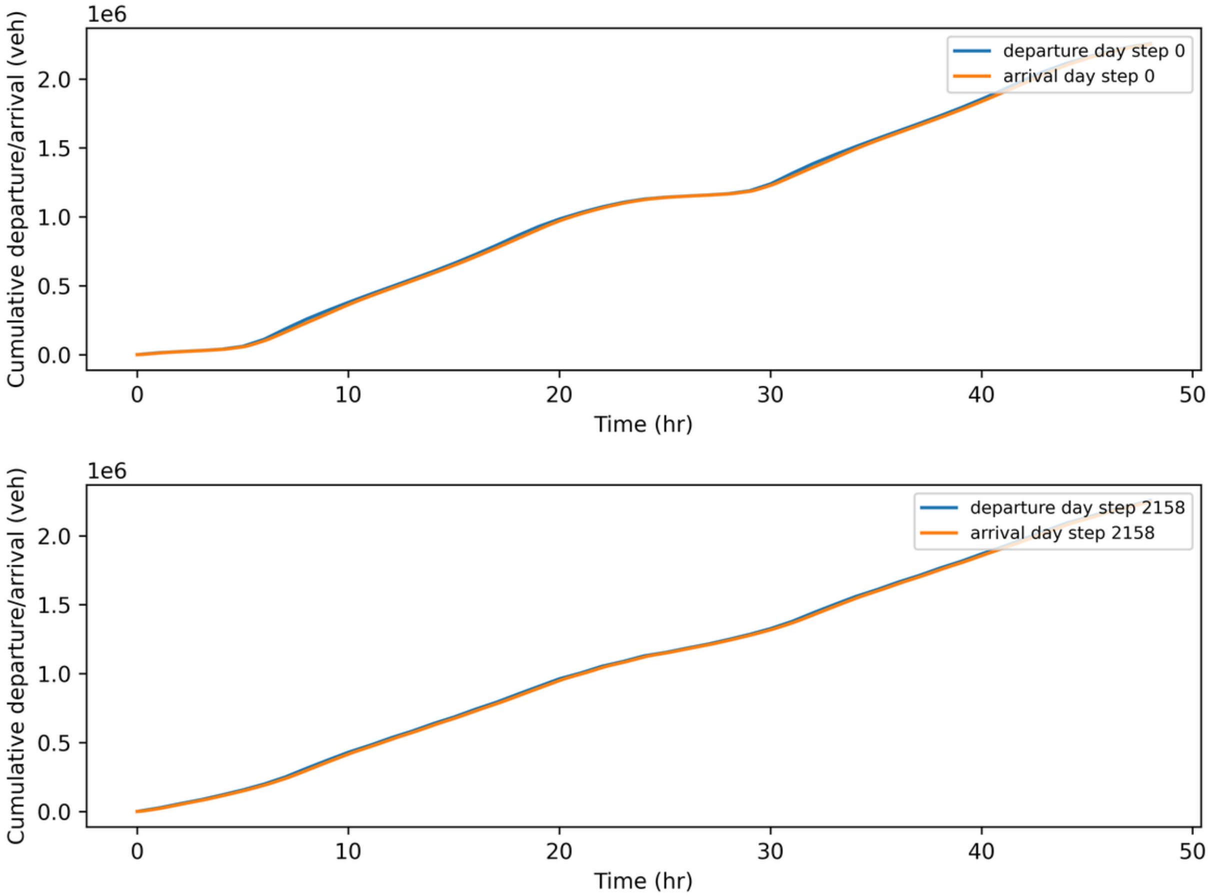


Figure 6.6: Cumulative departure and arrival flow on day-step 0 and day-step 2, 158

Figure 6.6 shows the cumulative departure and arrival flow of Vickrey's bathtub model on the first and the last day step. The results show that the area between the cumulative departure and arrival curves on the last day step becomes smaller than that on the first day step. From the cumulative arrival flow and the cumulative departure flow, we calculate the queuing time for each arrival time step shown in Figure 6.7. The results show that compared with the first day step, queuing time decreases significantly on the last day step. The total vehicle hours traveled (VHT) for all travelers decreases by 17%, from 603, 825 hours on the first day step to 501, 281 hours on the last day step.

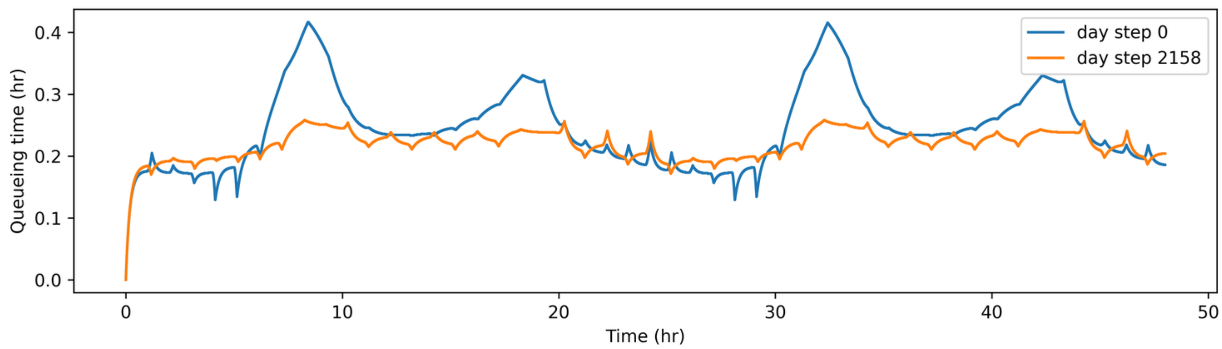


Figure 6.7: queuing time for each arrival time step on day-step 0 and day-step 2,158

6.4 Conclusion

This chapter presents the first numerical implementation of all four sections of Vickrey (1991, 2020) using 48-hour traffic entry data for Manhattan, New York City. The results show that after applying the dynamic marginal social cost pricing and day-to-day deferral and advance mechanisms, the system converges to a practically stable optimal equilibrium. Vickrey's marginal social cost pricing scheme produces meaningful, nonnegative tolls in hour-equivalents, which can be converted to monetary values using the average value of time. For instance, on the final day, a maximum 0.26 hour-equivalents can be translated to a $4.42 toll, assuming a $17/hour value of time in New York City (Ulak et al., 2020). These numerical results provide a practical reference for determining the magnitude of cordon congestion pricing.

Compared with the initial conditions, travelers shift their entry times from peak to off-peak. The marginal social cost, together with the toll schedule, decreases very quickly from day to day due to the adjustments in travelers' entry time decisions. At the same time, the average travel cost also decreases from day to day. However, an oscillatory pattern emerges in the within-day traffic entry dynamics, likely due to the one-hour time step, which we plan to refine in future research.

This research opens several avenues for future work. First, the different time steps for marginal social cost calculations can be examined. Second, further investigation is needed to understand how different initial traffic entry conditions influence equilibrium patterns. Third, extending the model to incorporate alternative network fundamental diagram shapes (Cassidy et al., 2011; Jin and Yu, 2015) would enhance its applicability across diverse urban environments. Lastly, incorporating all four traveler responses to congestion pricing—including suppression and generation—would provide a more comprehensive analysis of demand elasticity and behavioral adjustments.

Chapter 7

Managing Travelers' Departure Time Decisions at a Realistic Dynamic Network with Marginal Social Cost Pricing

Chapters 3 to 5 consider the day-to-day departure time dynamics at the corridor level with a single bottleneck, while Chapters 6 and 7 consider the day-to-day departure time dynamics at the network level. Chapters 3 and 4 only consider the day-to-day departure time dynamics without tolls, while Chapter 5 considers the single-class and multi-class day-to-day dynamics in Chapters 3 and 4 in the presence of tolls. Chapter 6 considers day-to-day departure time dynamics at the network level with marginal social cost pricing. In this chapter, we use the data from a dynamic traffic assignment model to validate the supply and demand framework in Chapter 6 and to examine the practicality of marginal cost pricing in a realistic network with route choice behavior.

7.1 Introduction

In this chapter, we will apply Vickrey's marginal social cost pricing in Chapter 6 to a realistic dynamic Los Angeles (LA) I-10 expressway network with 1-hour demand in a DTA (Dynamic Traffic Assignment) model with embedded traffic simulation developed by Nam (2019), which is originated from Jayakrishnan et al. (1995). Readers are referred to Nam (2019) for details of the DTA simulator. We use the DTA model results to examine three types of network fundamental diagrams — Greenshields, triangular, and trapezoidal. We then use the resulting network fundamental diagrams as inputs to Vickrey's bathtub model and the generalized bathtub model (Jin, 2020a) with the same one-hour demand to find out what combination of network fundamental diagrams and a bathtub model can best replicate the DTA model results. We find that the generalized bathtub model with Greenshields network fundamental diagram can best replicate the DTA results.

To better capture the network dynamics and to avoid heavy computation involved in DTA modeling, we then use the generalized bathtub model with Greenshields network fundamental diagram to capture the within-day dynamics, instead of Vickrey's bathtub model as in Chapter 6.

The system converges on day-step $4,001$. We then use the resulting demand pattern to run the 1-hour DTA simulation again, and we find that the total travel time is reduced by 8.16% compared with the first day-step.

The rest of the chapter is organized as follows: Section 7.2 presents the methodology in this chapter, including calibration of supply and demand inputs in Sections 7.2.2 and 7.2.3, generalized bathtub model in Section 7.2.4, comparison among different bathtub models under different network fundamental diagram to decide which combination can best replicate the DTA simulation results in Section 7.2.5, marginal social cost calculation with 15 second time step in Section 7.2.6, day-to-day dynamics in Section 7.2.7, numerical methods in

Section 7.2.8 and convergence criteria in Section 7.2.9. Section 7.3 presents the simulation study and the results. This chapter concludes with major findings in Section 7.4.

7.2 Methodology

7.2.1 System workflow

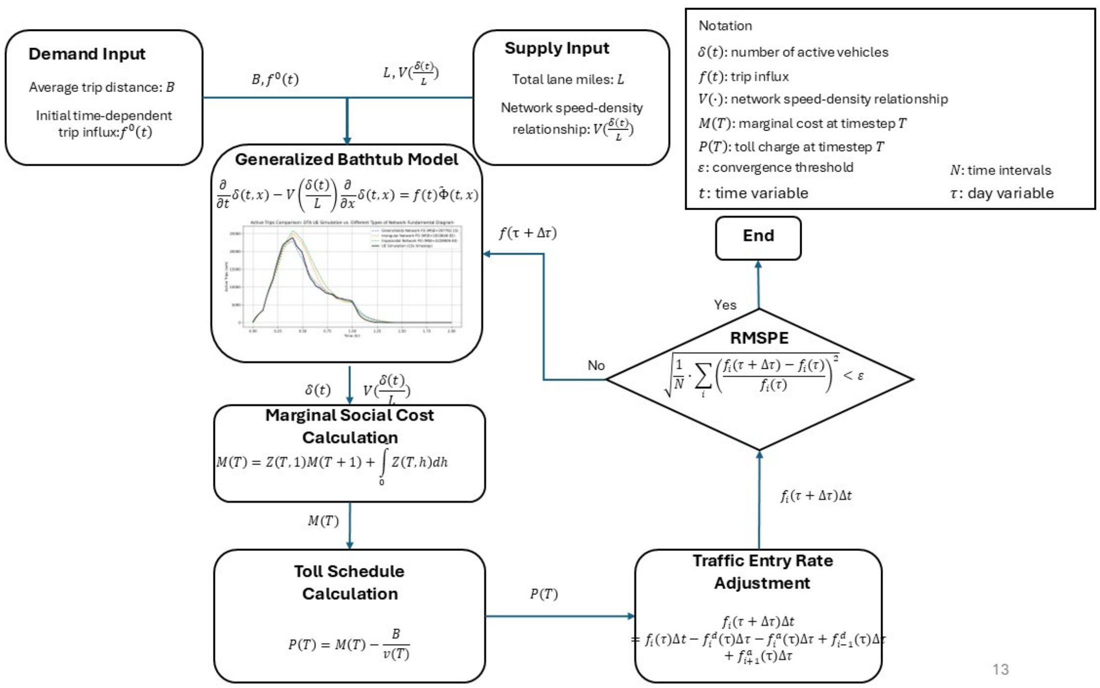
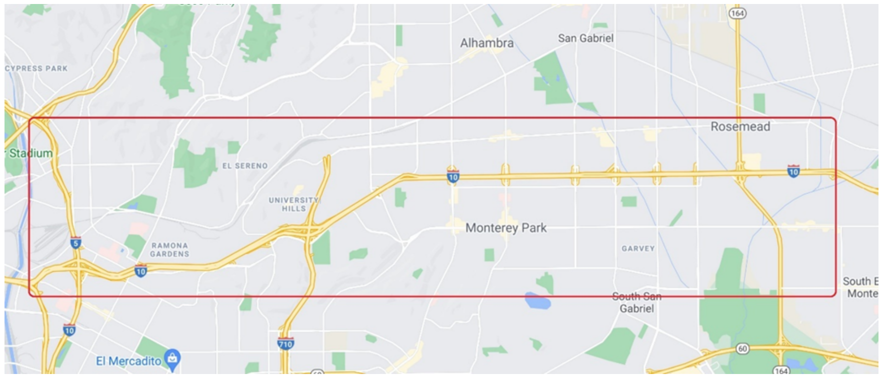


Figure 7.1: System workflow of Chapter 7

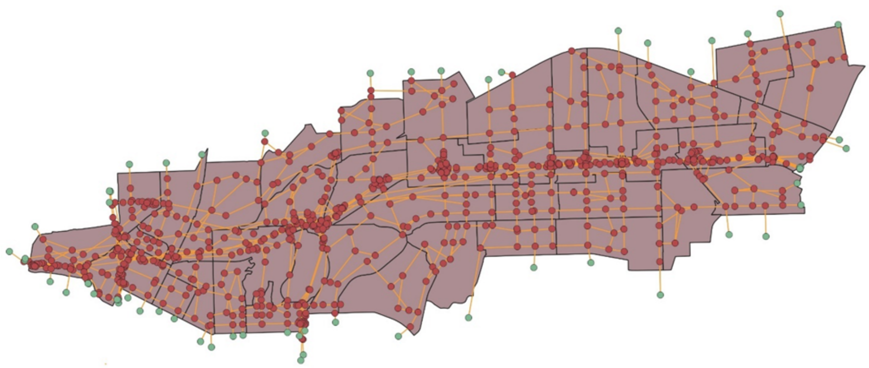
Figure 7.1 shows the workflow of Chapter 7, which uses the generalized bathtub model (Jin, 2020a) to replace Vickrey's bathtub model in Chapter 6 for within-day dynamics. Chapter 6 uses numerical data, while Chapter 7 uses embedded-simulation DTA data for the demand and supply inputs. The demand input includes the empirical trip length distribution and initial traffic entry pattern. The empirical trip length distribution is estimated from the origin-destination (OD) demand pattern of the DTA model. The initial traffic entry pattern is estimated from the time-dependent demand profile from the DTA. The supply input includes the total lane miles and the network speed-density relationships (Godfrey, 1969; Geroliminis and Daganzo, 2008). The total lane miles are estimated from the total lane miles of the LA-I10 expressway network from the DTA simulation, while the network speed-density relationships are estimated from the network space mean speed and the vehicle density in the network by running the DTA model with 100%, 150% and 200% demand.

Using these inputs, we use the integral numerical method in Jin (2020a) to solve the generalized bathtub model to obtain the within-day dynamics of the number of active vehicles in the network. We then use the iterative method in Vickrey (2020)'s third section with a smaller 15-second time step (instead of 1-hour time step in Chapter 6) to estimate the marginal social cost and calculate the toll schedule accordingly for the next day. The system updates the traffic entry pattern for the next day according to the traffic adjustment mechanism in Vickrey (2020)'s second section as in Chapter 6. If the relative difference in the traffic entry pattern is less than the convergence threshold, then the model stops and returns the equilibrium result. Otherwise, the model will numerically solve the generalized bathtub model again with next day's traffic entry pattern. After the model converges, we use the resulting demand pattern to run the DTA simulation again to examine how much congestion could be reduced. in a realistic network with route choices using Vickrey's marginal social cost pricing.

7.2.2 Calibration of supply input



(a) Google map



(b) QGIS

Figure 7.2: LA I-10 expressway network

The supply inputs of the system includes total lane miles and the network fundamental diagram. We generate those inputs with the LA I-10 expressway network using the DTA model shown in Figure 7.2. The LA I-10 network has 799 nodes and 1,927 links. The total lane miles of the network is 604 $lane-miles$. To calibrate the network fundamental diagram, we run the dynamic traffic assignment with 100% (78,196.6 trips), 150% (117,294.9 trips) and 200% (156,393.2 trips) of the 1-hour demand to obtain space mean speed-density data points. We rule out the outliers and keep the points close to the main trend.

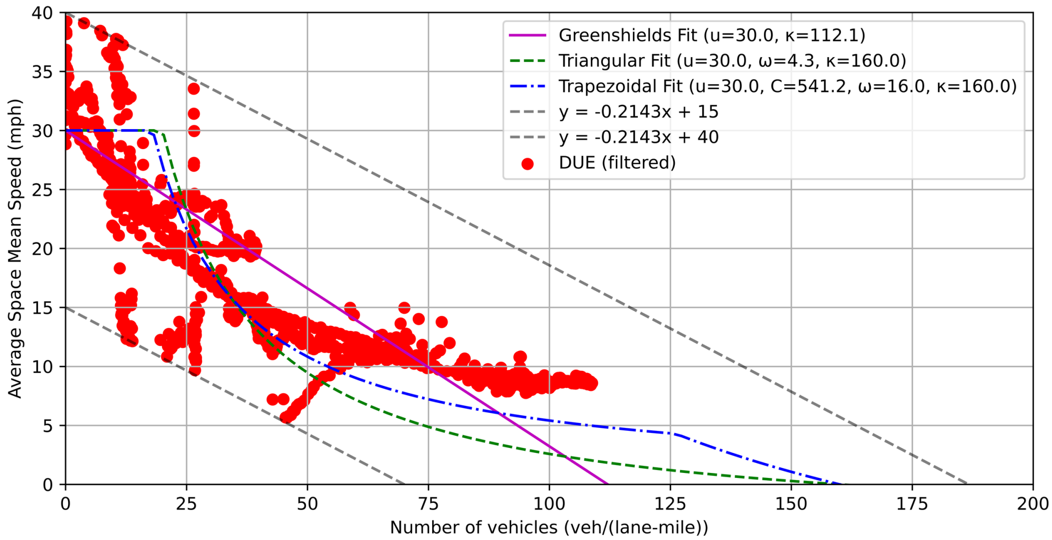


Figure 7.3: Calibrated network fundamental diagram

The red dots in Figure 7.3 present the resulting data points. We use the data to fit three types of network fundamental diagrams: Greenshields, triangular (used for freeway networks), and trapezoidal (used for signalized intersections). Let ρ be the network vehicle density (veh/lane-mile), and ρ_j be the jam density. Let u be the free-flow speed of the network fundamental diagram. The Greenshields fundamental diagram has the following form:

$$V(\rho) = u \cdot (1 - \frac{\rho}{\rho_j}). \tag{7.1}$$

The triangular fundamental diagram has the following form:

$$V(\rho) = \min\{u, \omega \cdot (\frac{\kappa}{\rho} - 1)\}. \tag{7.2}$$

The trapezoidal fundamental diagram has the following form:

$$V(\rho) = \min\{u, \frac{C}{\rho}, \omega \cdot (\frac{\kappa}{\rho} - 1)\}. \tag{7.3}$$

To ensure that the three different network fundamental diagrams produce results that align with the DTA, we set the free flow speed for three types of network fundamental diagrams to be the same as 30 mph ($u = 30$). The jam density for both triangular and trapezoidal fundamental diagram is set to be 160 veh/lane-mile ($\rho_j = 160$). So the maximum likelihood algorithm will determine the rest of the parameters for the fundamental diagrams. The purple solid curve, green dashed curve, and blue dash-dotted curve in Figure 7.3 present the resulting Greenshields, triangular and trapezoidal network fundamental diagrams.

7.2.3 Calibration of demand input

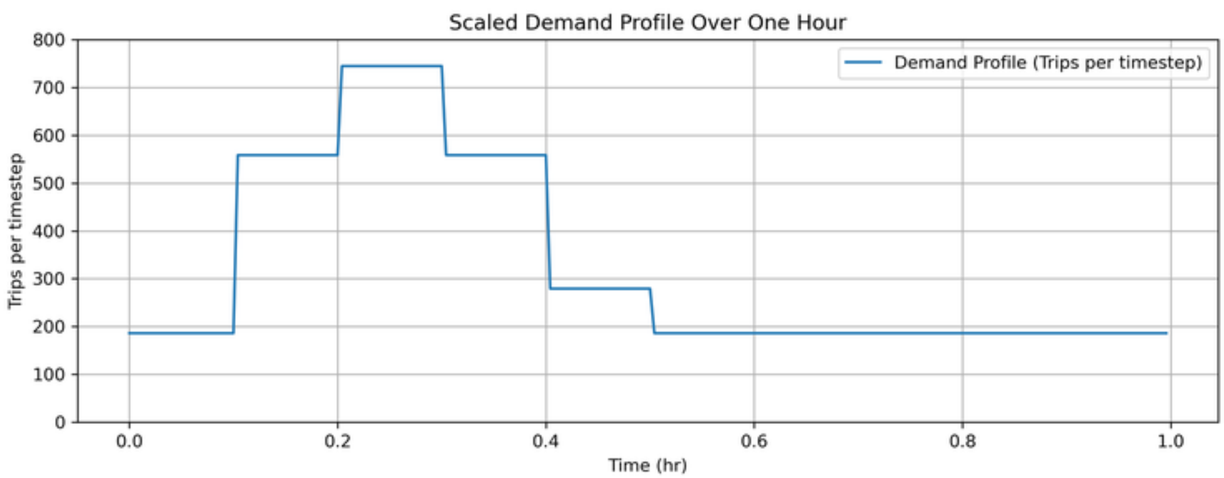


Figure 7.4: Initial trip entry pattern $f^0(t)$

The demand input includes the time-dependent initial trip entry pattern $f^0(t)$ and the trip length distribution of the network. Figure Figure 7.4 shows the initial trip entry profile, which has the maximum entry rate of 744 trips/15 seconds from 0.2 - 0.3 hour. Figure 7.5 shows the trip length distribution from the DTA simulation based of the OD demand pattern. The average trip length is 3.75 miles, so we also plot the theoretical exponential distribution.We will use the average trip length as B in Vickrey's bathtub model and the empirical distribution for the generalized bathtub model.

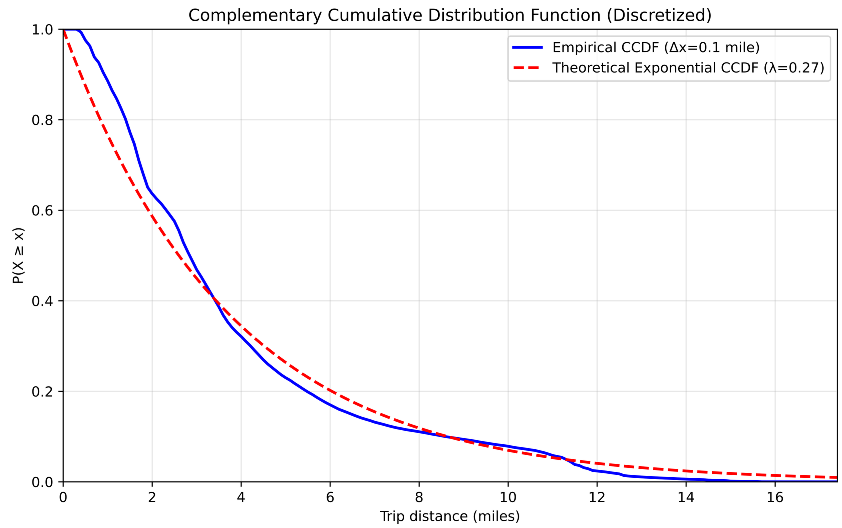


Figure 7.5: Trip length distribution

7.2.4 Generalized bathtub model

Let x be the remaining distance to the travelers' destinations. Let $\delta(t,x)$ be the number of vehicles at time t with remaining distance no less than x. Let $\delta(t)$ be the number of vehicles in the network at time t, where $\delta(t) = \delta(t,0)$. Let $V(\frac{\delta(t)}{L})$ be the network fundamental diagram, outputting the network space mean speed v from the network vehicle density $\rho(t) = \frac{\delta(t)}{L}$. Let $f(t)$ be the influx into the network at time t (unit: veh/hr). Let $\tilde{\Phi}(t,x)$ be the cumulative distribution function of entering trips with distance no less than x, where $\tilde{\Phi}(t,0) = 1$. The generalized bathtub model has the following form (Jin, 2020a):

$$\frac{\partial}{\partial t}\delta(t,x) - V(\frac{\delta(t)}{L}) \cdot \frac{\partial}{\partial x}\delta(t,x) = f(t) \cdot \tilde{\Phi}(t,x), \tag{7.4}$$

where $f(t)$ is from the demand input from initial entry pattern or from the update by day-to-day dynamics, while $\tilde{\Phi}(t,x)$ is estimated from the empirical trip length distribution, and $V(\frac{\delta(t)}{L})$ is the network fundamental diagram.

7.2.5 Comparison between Vickrey's bathtub model and generalized bathtub model

We model both Vickrey's bathtub model and generalized bathtub model under different network fundamental diagrams with the 1-hour demand from the embedded-simulation DTA. Figure 7.6 presents the results on the number of active trips in the network by time. The results show that the generalized bathtub model with the Greenshields network fundamental diagram (dashed blue curve) produces results that are closest to the DTA simulation. Therefore, we use generalized bathtub model with the Greenshields network fundamental diagram to describe the within-day dynamics of the framework shown in Figure 7.1.

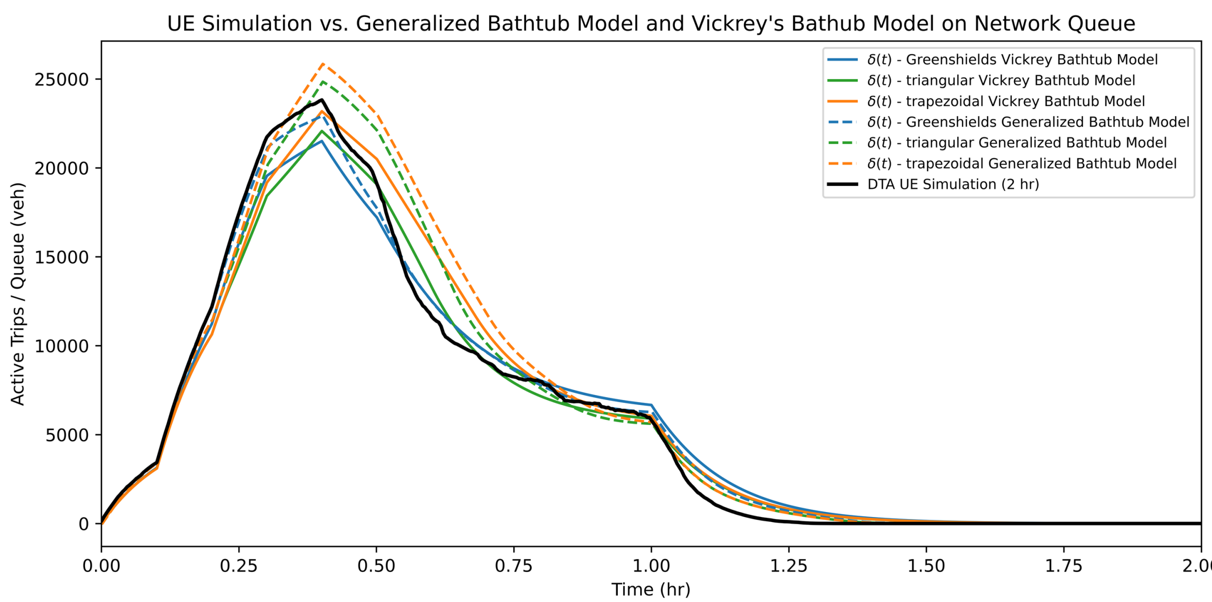


Figure 7.6: Comparison on Vickrey's bathtub model and generalized bathtub model under different network fundamental diagrams

7.2.6 Marginal social cost pricing calculation with 15-second time step

Different from the 1-hour time step in Chapter 6, in the DTA simulation, trips enter the network every 15 seconds. So we need to extend the Equation (6.35) to capture the marginal social cost in a 15-second time step. The derivation for 15 second time step marginal social cost follows the derivation in Section 6.2.4 for 1 hour time step marginal social cost. The resulting marginal cost estimation is as follows:

$$M(t) = EXP(H) * M(t+dt) + \frac{1}{\tilde{A} + \frac{1}{2} \cdot \tilde{B} * dt} * [EXP(H) - 1] \tag{7.5}$$

where $\tilde{A} = -\frac{1}{B} \cdot u \cdot (1 - \frac{2}{L \cdot \rho_j} \cdot \delta(T))$, $\tilde{B} = \frac{1}{B} \cdot u \cdot (\frac{2}{L \cdot \rho_j} \cdot (\delta(T+1) - \delta(T)))$, $H = \tilde{A} \cdot dt + \frac{1}{2} \cdot \tilde{B} \cdot dt^2$, and $dt = 15$ seconds. We use Equation (7.5) to estimate the marginal social cost backward from the last time step of the DTA simulation where the marginal social cost is equal to the average cost (average trip length/free-flow speed). Notice that the the marginal social cost estimation still uses Vickrey's bathtub model with Greenshields fundamental diagram, like B, ρ_j, and u in $\tilde{A}$ and $\tilde{B}$. It is an approximation of marginal social cost for the generalized bathtub model and the DTA simulation.

We can calculate the toll schedule, $P(t)$, as follows:

$$P(t) = M(t) - \frac{B}{v(t)} \tag{7.6}$$

where the toll, $P(t)$, is equal to marginal social cost $M(t)$ at time t, minus average cost $\frac{B}{v(t)}$, which is the average trip length, B, divided by network speed at time t, $v(t)$.

7.2.7 Day-to-day dynamics in travelers' departure time choices

We use the same day-to-day dynamics, which focuses on the cost increment compared with day-step 0, to capture travelers' departure time choices as in Chapter 6, but the time step changes from 1 hour to 15 seconds. Here, we allow travelers shift their departure times from fist time step to the last time step and vice versa.

7.2.8 A numerical method to solve generalized bathtub model

We use the difference-integration method to solve the genralized bathtub model numerically (Jin, 2020a). We divide the range of trip distance $[0, X]$ into I intervals with $\Delta x = \frac{X}{I}$, where X is the maximum trip distance. We discretize the study period $[0, T]$ into J time steps with $\Delta t = \frac{T}{J}$. At $j\Delta t (j = 0, 1, ..., J)$, the number of active trips in the network is δ^j, the travel speed by v^j, and the characteristic travel distance by z^j. The characteristic travel distance describes how far the vehicles can travel based on the network speed, and has the following equation:

$$z(t) = \int_0^t v(b)db \tag{7.7}$$

So we have the following equations to update the network space mean speed of the generalized bathtub model v^j at time step j, the characteristic travel distance z^{j+1}, and number of active tripsδ^{j+1} at time step $j+1$:

$$v^j = V(\frac{\delta^j}{L}) \tag{7.8}$$

$$z^{j+1} = z^j + v^j \cdot \Delta t \tag{7.9}$$

$$\delta^{j+1} = \delta(0, z^{j+1}) + \sum_{m=0}^{j} f(m\Delta t) \cdot \tilde{\Phi}(m\Delta t, z^{j+1} - z^m) \cdot \Delta t, \tag{7.10}$$

where the second term is the cumulative network inflow above the characteristic travel distance curve from time step 0 to time step j. Then for $j = 0, ..., J-1$ and $i = 0, ..., I$, we have:

$$\delta((j+1)\Delta t, i\Delta x) = \delta(0, i\Delta x + z^{j+1}) + \sum_{m=0}^{j} f(m\Delta t) \cdot \tilde{\Phi}(m\Delta t, i\Delta x + z^{j+1} - z^m) \cdot \Delta t, \tag{7.11}$$

which could be used to update the number of active trips with remaining distance $i\Delta x$ above 0.

7.2.9 Convergence criteria

We use Root Mean Squared Percentage Error (RMSPE) on day step j as the convergence criteria:

$$RMSPE_j = \sqrt{\frac{1}{N} \cdot \sum_i (\frac{f_i^j - f_i^{j-1}}{f_i^{j-1}})^2}, j \in \{1, 2, 3, ...M\} \tag{7.12}$$

where f_i^j represents the influx in time step i on day step j. If $RMSPE_j < \varepsilon$, then we consider the system converges to an optimal departure time choice equilibrium driven by marginal social cost. We set $\varepsilon = 0.001$ in this chapter.

7.3 Simulation study

7.3.1 Simulation setup

After calibrating the demand and supply input, we have the following settings: $L = 225$ miles, $u = 30$ mile/hr, $\rho_j = 112.6$ veh/lane· mile, and $B = 3.7523$ miles for marginal social cost calculation. We set $B_i^d(\tau)$ and $B_i^a(\tau)$ as follows:

$$B_i^d(\tau) = B_i^a(\tau) = \gamma(\tau) \cdot \frac{1}{P_{max}^1 - P_{min}^1 + 2 \cdot (B/v^{min} - B/u)} \tag{7.13}$$

where P_{max}^1 is the maximum toll on day step 1 derived from the no-toll pattern on day step 0, while P_{min}^1 is the minimum toll on day step 1 derived from the no-toll pattern on day step 0. Theoretically, v^{min} can reach zero, but we select v^{min} to be 1 mile/hr. So $B/v^{min} = B/1 = 3.7523$. $\gamma(\tau_j)$ is a day-dependent scaling factor that scales $B_i^d(\tau)$ and $B_i^a(\tau)$ to adjust the convergence speed of the system. We set the scaling factor as follows:

$$\gamma(\tau_j) = \begin{cases} 100 & \text{for} \quad 0 \leq j \leq 500 \\ 50 & \text{for} \quad 500 < j \leq 1,000 \\ 20 & \text{for} \quad 1,000 < j \leq 2,000 \\ 10 & \text{for} \quad 2,000 < j \leq 3,000 \\ 5 & \text{for} \quad 3,000 < j \leq 4,000 \\ 1 & \text{for} \quad 4,000 < j \leq 5,000. \end{cases} \tag{7.14}$$

In this study, the departure time study period is a one-hour period, $I = 1$. We divide it into 240 periods, so $N = 240$, and $dt = \frac{I}{N} = 15$ sec. We set the study day steps to 5,000 steps, so $M = 5,000$, and we set the day step size to be 1 (i.e., $\Delta\tau = 1$). Section 7.3.2 presents the simulations results.

7.3.2 Simulation results

In this section, we present the results from day-to-day departure time adjustments with generalized bathtub model, and its resulting optimal demand pattern. We call it *Vickrey's optimal demand*. We then use Vickrey's optimal demand to run the DTA simulation with dynamic user equilibrium (DUE) and dynamic system optimal (DSO) at the route choice level, and compare the results from DTA simulation with the initial demand pattern on day 0 with DUE and DSO.

Generalized bathtub model results

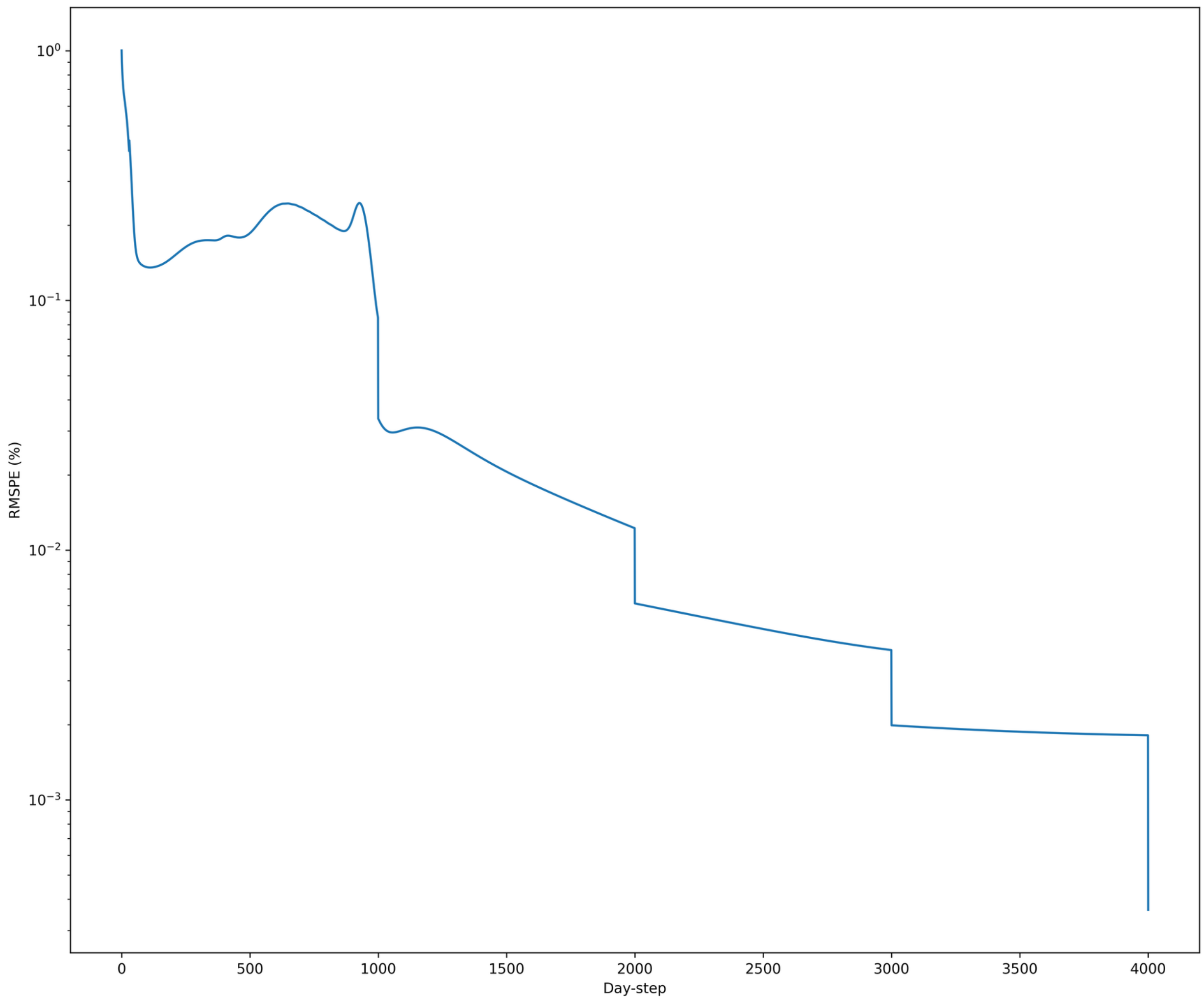
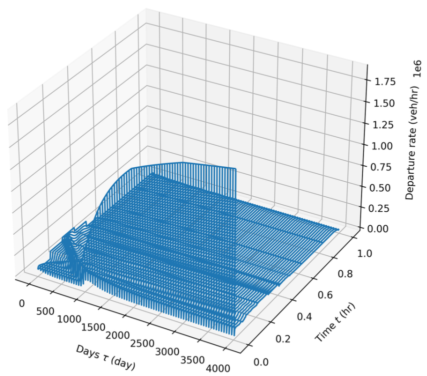
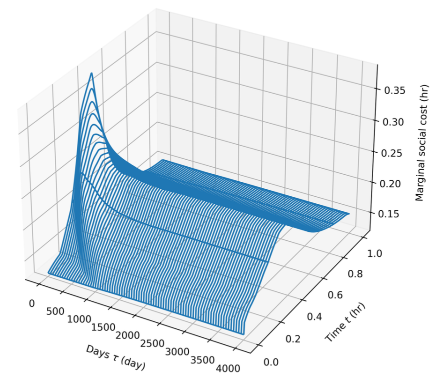


Figure 7.7: Day-to-day evolution of RMSPE error

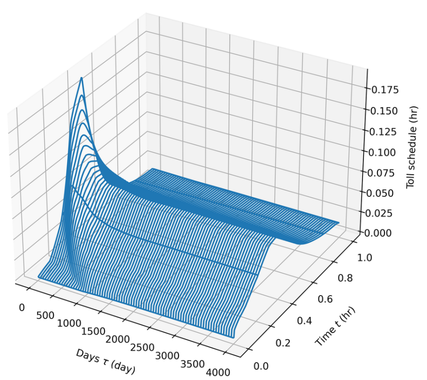
Figure 7.7 shows that the system converges on day step 4, 001, and the RMSPE error converges to 0.001 eventually. This system is too complicated to theoretically prove its stability by finding a Lyapunov function, we consider the system to be reaching a "practically stable" state, where the RMSPE decreases below the threshold.



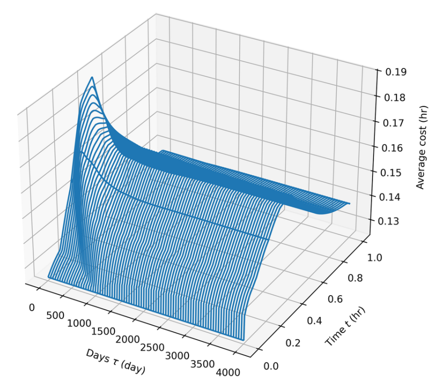
(a) Departure rate (veh/hr)



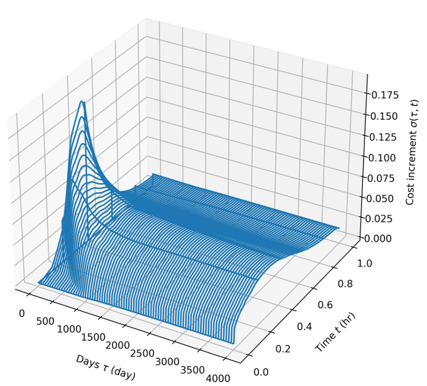
(b) Marginal social cost (hr)



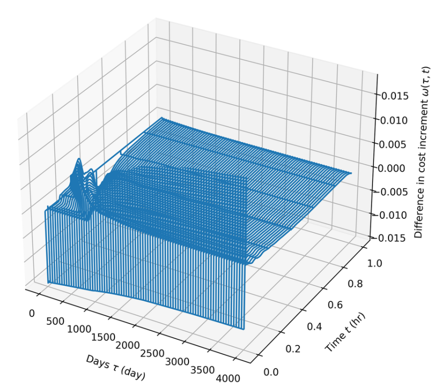
(c) Toll schedule (hr)



(d) Average travel cost (hr)



(e) Cost increment $\sigma(\tau, t)$ (hr)



(f) Difference in cost increment $\omega(\tau, t)$ (hr)

Figure 7.8: Day-to-day evolution of departure rate, marginal social cost, toll schedule, average travel cost, cost increment, and difference in cost increment

Figure 7.8 shows the day-to-day evolution of the departure rate (at the network level), marginal social cost, toll schedule, average travel cost, cost increment and difference in cost increment. After applying the marginal social cost pricing along with the departure time adjustment process, travelers shift their departure times to the beginning of the 1 hour period, as shown in Figure 7.8a. This is because travelers in this day-to-day dynamical system will always shift their departure times to the time step with the least cost increment. time step 0 has the minimum cost increment of 0.003 hr, so travelers will shift their departure times to the first time step, creating a mass departure at the beginning. Even the last time step has the higher increment than the first time step — 0.014 hr. So allowing travelers to shift their departure time from the first step to the last step won't eliminate the mass departure at the beginning.

Figures 7.8b to 7.8e show that marginal social cost, toll schedule, average cost, and cost increment decrease from day-to-day as travelers start to adjust their departure times. When $j = 4,001$, the difference in cost increment has a negative value at time step 0, because $\omega_0(\tau_j) = \sigma_0(\tau_j) - \sigma_I(\tau_j) = 0.003 - 0.014 < 0$, and has a high positive value at time step 1, because $\omega_1(\tau_j) = \sigma_1(\tau_j) - \sigma_0(\tau_j) = 0.021 - 0.003 > 0$. At other time steps, the $\omega(\tau, t)$'s are close to zero.

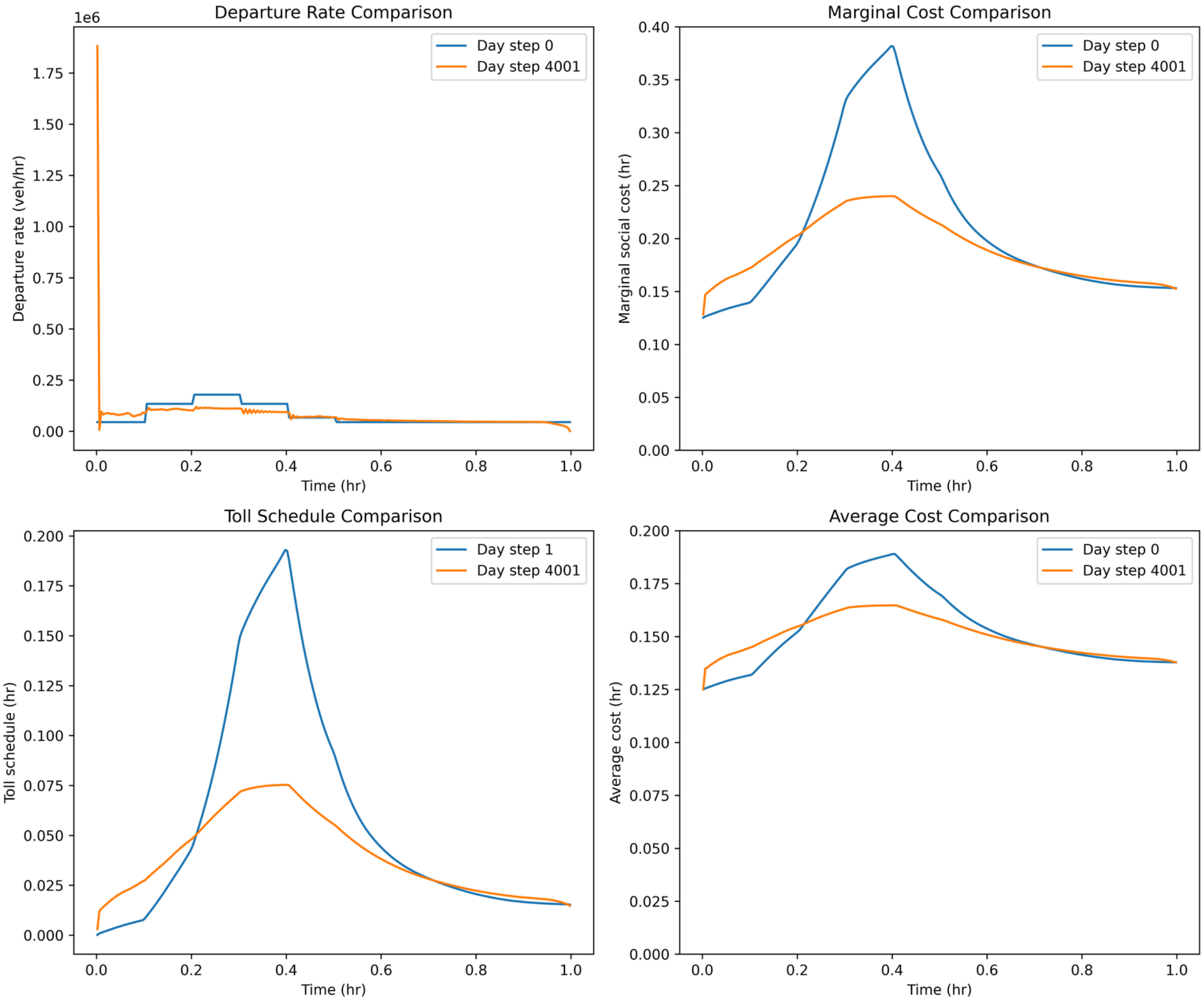


Figure 7.9: Comparison of departure rate, marginal social cost, toll schedule, average travel cost on day step 0 and day step 4,001

Figure 7.9 compares departure rate, marginal social cost, toll schedule, average travel cost on day step 0 and day step 4,001. Comparing the last day step with day step 0, the maximum marginal social cost decreases by 37.1% from 0.38 to 0.24 hour-equivalents.

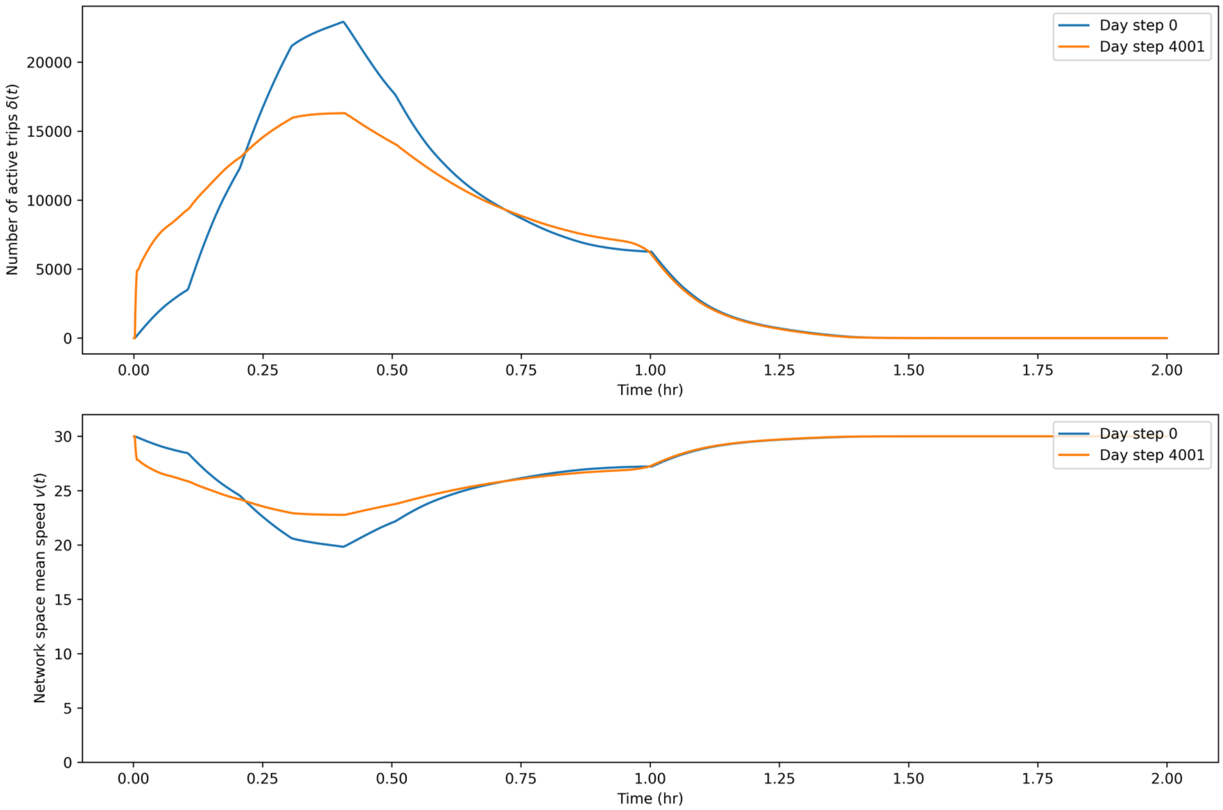
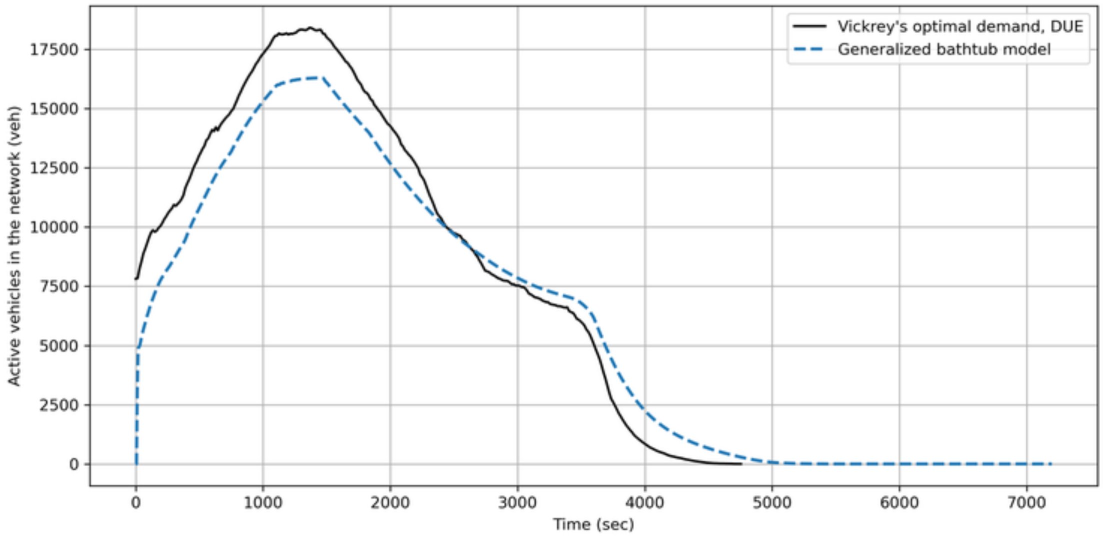


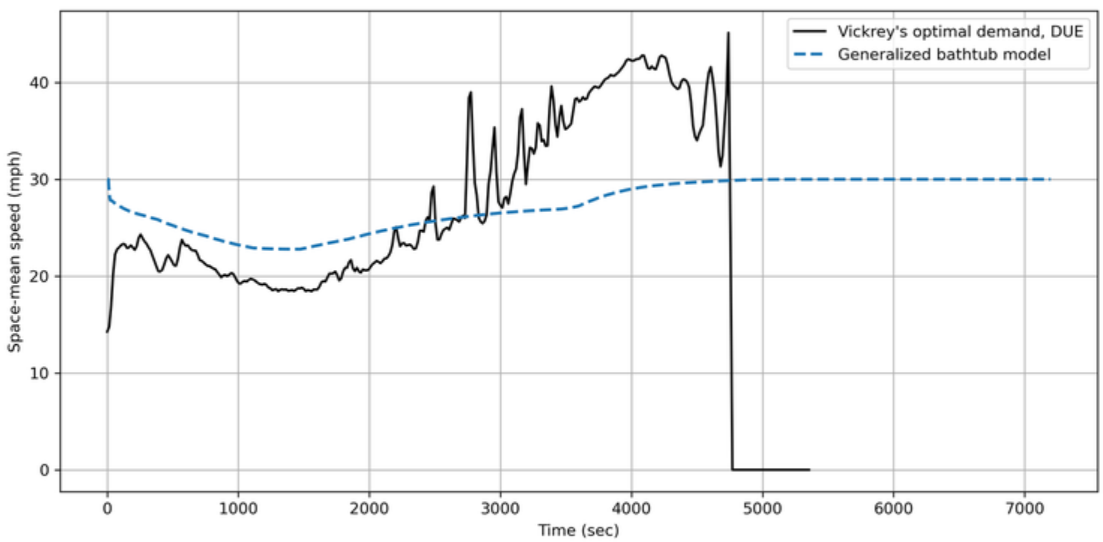
Figure 7.10: Time-dependent number of vehicles and network speed on day-step 0 and day-step $4,001$

Figure 7.10 compares the number of vehicles (network queue) and network speed in Vickrey's bathtub model on day step 0 and day step $4,001$. The maximum number of vehicles in the network decreases by 29%, from $22,917$ on day step 0 to $16,302$ on day step $4,001$. The minimum speed in the network increases by 15%, from 19.8 mph to 22.8 mph. The results show that Vickrey's marginal cost pricing can effectively reduce the severity of the congestion and improve the network space mean speed from day to day.

Results from DTA with embedded simulation



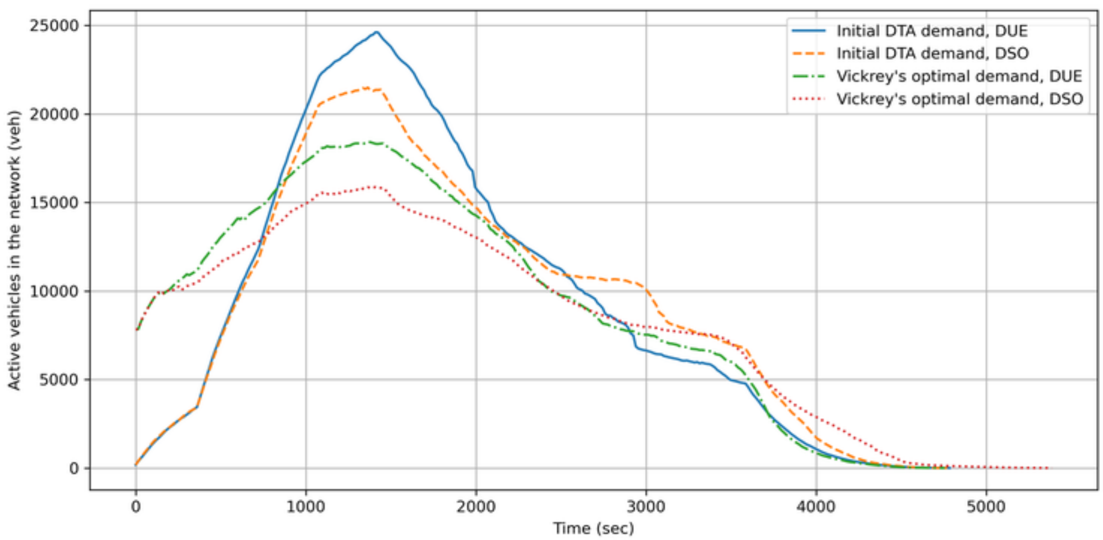
(a) Active vehicles in the network



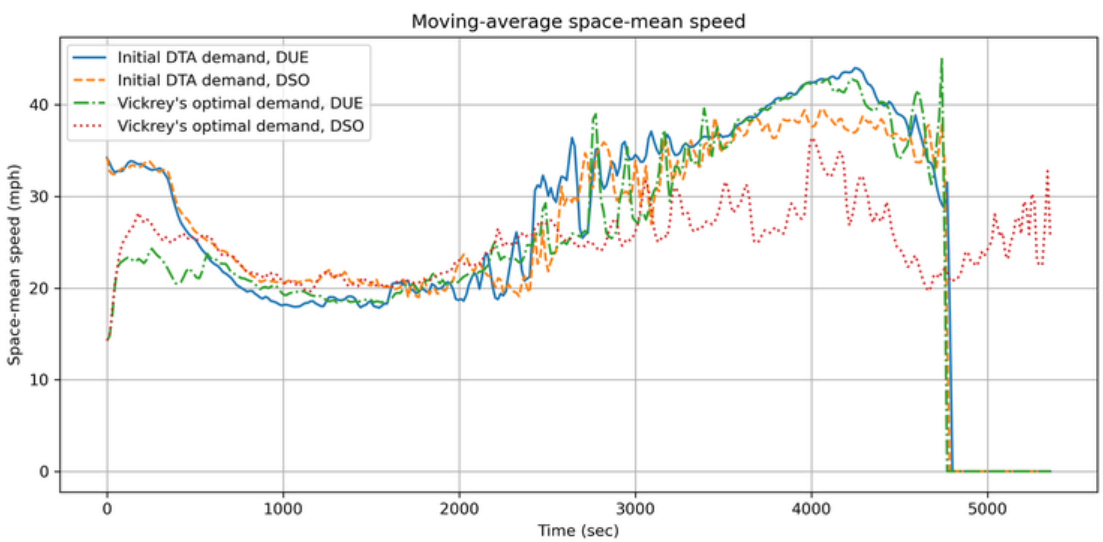
(b) Space mean speed

Figure 7.11: Comparison of generalized bathtub model and DTA simulation with Vickrey's optimal demand on the last day-step

Figure 7.11 compares the results from the generalized bathtub model on the last day-step with the DTA simulation results under a DUE assignment scenario. The results show that generalized bathtub model can replicate the trend of active number of vehicles, even though not 100% accurately. However, the generalized bathtub model is not able to replicate the speed profile accurately.



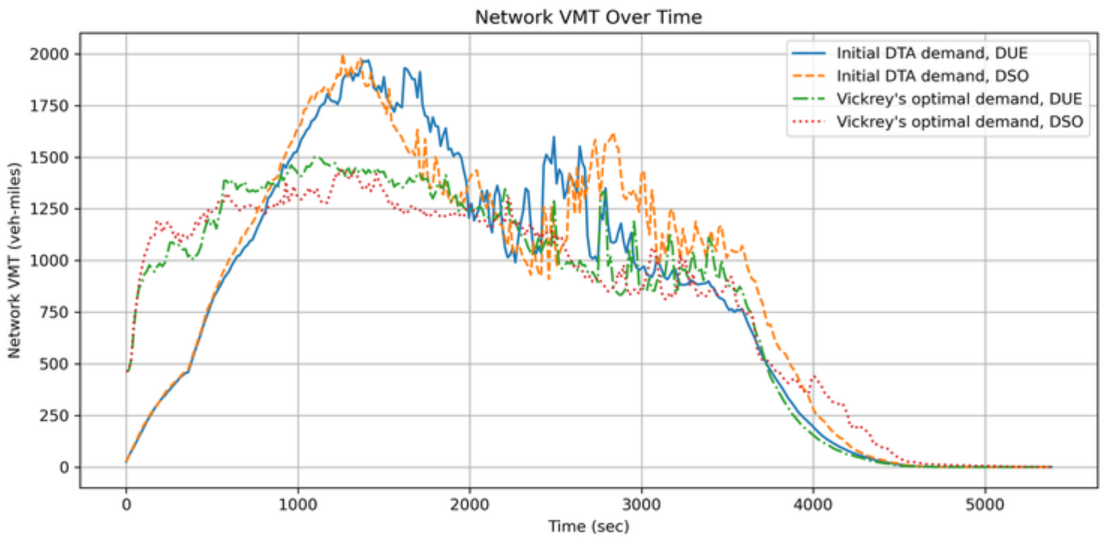
(a) Active vehicles in the network



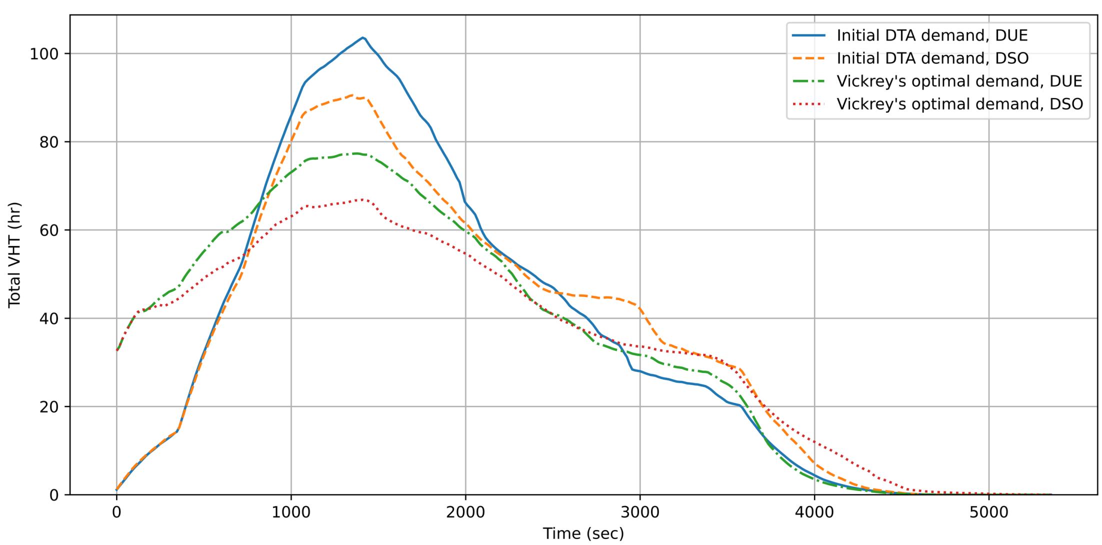
(b) Space mean speed

Figure 7.12: Comparison of initial demand and Vickrey's optimal demand under UE and SO in DTA simulation

Figure 7.12 compares the active vehicles and network space mean speed under DUE and DSO assignments with both initial demand and Vickrey's optimal demand. Among all four scenarios, Vickrey's optimal demand with DSO assignment leads to the least maximum number of vehicles in the network, but at the same time increases the active vehicles in the shoulder of the peak. For space mean speed, DSO has the higher speed from $1,000$ to $2,000$ time steps with both initial demand and Vickrey's optimal demand. In terms of improving speed during the heavily congested period, route level DSO decision seems to have a more significant impact than shifting departure time decisions. But shifting departure time decisions has a significant impact on reducing maximum number of vehicles in the network.



(a) VMT



(b) VHT

Figure 7.13: Comparison of VMT and VHT for initial demand and Vickrey's optimal demand under UE and SO in DTA simulation

Figure 7.13 compares vehicle miles traveled (VMT) and vehicle hours traveled (VHT) under these four scenarios. With Vickrey's optimal demand pattern, network VMT is more evenly distributed throughout the simulation. The VHT trend in Figure 7.13b is simular to the active vehicle trend in Figure 7.12a. Table 7.1 presents the total VMT, VHT and average space mean speed from the DTA simulation. Comparing Vickrey's optimal demand with initial DTA demand on DUE assignment type, the VHT decreases by 0.9%. Comparing DSO with DUE assignment type with initial DTA demand, the VHT decreases by 1%. However, with both Vickrey's optimal demand and DSO, the VHT decreses by 4.7%. At the same time Vickrey's optimal demand can also decrease VMT. It is because the short distance routes become less congested, and have a lower travel time with Vickrey's optimal demand. Therefore, for travelers, the short distance routes are the short travel time routes. So they will travel on the short distance routes, thereby reducing the VMT.

Table 7.1: Network-level performance comparison across scenarios

Scenario	**VMT (veh-miles)**	**VHT (veh-hours)**	**Average space mean speed (mph)**
Initial DTA demand, DUE	297,550.74	12,854.43	23.15
Initial DTA demand, DSO	312,423.37	12,710.33	24.58
Vickrey's optimal demand, DUE	293,191.98	12,718.42	23.05
Vickrey's optimal demand, DSO	291,282.84	12,233.26	23.81

7.4 Conclusion

This chapter integrates the day-to-day departure time adjustment mechanism from Chapter 6 with a dynamic traffic assignment (DTA) simulation to examine the effects of marginal social cost pricing in a realistic, dynamic network. We begin by calibrating the model's demand and supply inputs using simulation data from the I-10 Expressway network in Los Angeles. To avoid the high computational cost of running DTA simulations at every iteration, we replace the DTA simulation with a generalized bathtub model incorporating the Greenshields fundamental diagram to approximate within-day network dynamics.

The results show that with marginal social cost pricing and the departure time adjustment mechanism, travelers shift their departure times toward the beginning of the 1-hour period.

We refer to this resulting demand pattern as *Vickrey's optimal demand*. We then compare DTA simulation outcomes under both dynamic user equilibrium (DUE) and dynamic system optimal (DSO) conditions, using both the initial and Vickrey's optimal demand profiles. The findings indicate that shifting to Vickrey's optimal demand achieves a similar reduction in vehicle-hours traveled (VHT) as the DSO scenario, while also significantly mitigating congestion severity and reducing vehicle-miles traveled (VMT). This chapter offers new insights into the distinct impacts of congestion pricing on departure time decisions versus route choice decisions.

Chapter 8

Conclusion and Discussion

Traffic congestion imposes significant costs on cities, largely due to peak-period demand driven by synchronized traveler schedules. This dissertation, on the one hand, develops stable day-to-day dynamical models of travelers' departure time choices and, on the other hand, examines how optimal pricing can shift travel demand away from the peak to improve system efficiency using the proposed stable dynamical models.

This dissertation first presents the stable local day-to-day departure time dynamics from Jin (2021b) in Chapter 3, which lays a theoretical foundation for the rest of this dissertation. Stable dynamics here considers a fixed demand scenario, where there is no induced or suppressed demand. It assumes that travelers will only defer or advance their departure times by one time step of a reasonable length (conceivably a few minutes) from day to day, making the travelers' departure time shifting behavior local. This dissertation presents the proof, along the lines of Jin (2021b) that the stationary state of dynamical system is equivalent to SC-DTUE, and that the stationary state is asymptotically stable using Lyapunov's second method. However, due to the size of the stability region, we need to first use heuristic deferral and advance coefficients to drive the system to its stability region, and then switch to the provably stable deferral and advance coefficients. After switching coefficients, the system will converge to SC-DTUE asymptotically.

In Chapter 4, this dissertation extends the stable day-to-day departure time dynamics from single-class to a multi-class setting, capturing travelers' heterogeneity. We consider the simplest case of heterogeneity — groups of travelers — in the fixed demand scenario. As in single-class dynamics, we assume that travelers in each group will only defer or advance their departure times by one time step from day to day, and that the trip cost function is group specific. We prove that, under the first type of heterogeneity in Arnott et al. (1994) (different ratios for queuing cost over unpunctuality cost, λ/μ, across travelers, but the same ratio for arriving early and arrival late penalties, μ/ν, and the same desired arrival time, t^*), the stationary state of the multi-class dynamical system is equivalent to MC-DTUE, and that the stationary state is asymptotically stable based on Lyapunov's second method. We provide numerical examples for two different types of traveler heterogeneity: (1) different λ/μ, same μ/ν and same t^*, and (2) different desired arrival time t^*, but same λ, μ and ν. The simulation results also show that with the first type of heterogeneity, the multi-class system converges to a stable stationary MC-DTUE state, which aligns with our analytical results. However, with second type of heterogeneity, the multi-class dynamical system does not converge to an MC-DTUE state.

The stable single-class and multi-class dynamics are then used in this dissertation to study various optimal pricing schemes to drive the system from DTUE to SO in Chapter 5. We apply four different tolls and incentives schemes to the case of single-class dynamics: optimal fine toll, optimal coarse toll, optimal fine reward, and optimal feebate. We prove that the optimal fine toll can drive the system to a stable stationary SO state, as optimal fine reward and optimal feebate can also be proven stable following the same proof. The simulation results align with our proof, and show that optimal fine toll, optimal fine reward, and optimal feebate can drive the system from SC-DTUE to a stable SO. However, optimal coarse toll can only reduce the queuing time, but cannot drive the system to an SO state. For multi-class cases, we study only one toll — optimal fine toll. We prove that the optimal fine toll in a multi-class scenario can also drive the system to a stable stationary SO state. In the numerical example, we study two scenarios of two groups' heterogeneity with respect to tolling: (1) different λ/μ, but same μ, ν, and t^*; (2) different μ and ν, but same λ, μ/ν, and t^*. The results show that the optimal fine toll can drive the system to a stable stationary SO state in both scenarios. However, for the order of departure in SO, scenario (1) allows for mixed departure within the departure period because of the same μ and ν, while scenario (2) will order the travelers according to their μ's, which aligns with the equlibrium results in Arnott et al. (1994).

Building upon the corridor-level dynamics in Chapters 3 and 4, this dissertation then studies the day-to-day departure time dynamics at the network level, regarding travelers' network entry times. Chapter 6 is built upon the seminal work by Vickrey (1991, 2020). Vickrey (1991, 2020) presents a theoretical framework on modeling network traffic dynamics with a bathtub model, travelers' responses to congestion pricing, marginal social cost pricing estimation, and solving the bathtub model considering the curvature of the network fundamental diagram. However, Vickrey's work is a purely theoretical study without any numerical examples verifying its effectiveness in reducing congestion. This dissertation provides the first numerical example for it, applying it to data on 48 hours of traffic entry to Manhattan, NY, and showing that the system converges to a practically stable optimal equilibrium with Vickrey's marginal social cost pricing and day-to-day departure time adjustment mechanism. The results also show that the network congestion can be significantly reduced, including a 17% decrease in VHT, and a 64% increase in minimum speed. Different from Jin (2021b)'s dynamics which focuses on the difference in trip cost in two consecutive time steps, Vickrey (1991, 2020)' dynamics focuses the difference in cost increment in two consecutive time steps. As a result, Jin (2021b)'s dynamics will lead to a stationary state where all departure time steps share the same cost, while Vickrey (1991, 2020)'s dynamics will lead to a stationary state where all departure time steps share the same cost increment.

Building upon Chapter 6's study on network level departure time choice, this dissertation lastly examines the departure time decisions in a realistic dynamic network using a DTA model with embedded simulation in Chapter 7. We first use the data from the DTA to calibrate the demand and supply input of the model framework in Chapter 6. Since the DTA modeling is computationally expensive (2 hours per computer run), we use the generalized bathtub model calibrated by the DTA modeling outputs for within-day traffic dynamics (accomplished in seconds per run). After travelers adjust their departure time decisions and the model converges, we use the resulting demand pattern to run the DTA model under DUE and DSO assignment scenarios. The results show that the reduction in VHT due to departure time shifting is comparable to the reduction due to route shifting to a DSO. Moreover, shifting departure time decisions will reduce the VMT as well, while shifting travelers from DUE to DSO routes will usually increase the network VMT.

In summary, this dissertation studies the departure time dynamics at both corridor level and network levels. For corridor level, we use the point queue model to describe the within-day traffic dynamics, and Jin (2021b)'s cost-based dynamics to describe the day-to-day departure time decision dynamics. We are able to prove that the dynamical system can converge to a stable stationary DTUE state without pricing, and to a stable stationary SO state with optimal pricing. For network level, we use Vickrey's bathtub model in Chapter 6 and generalized bathtub model in (Jin, 2020a) in Chapter 7 to describe the within-day dynamics, and Vickrey (2020)'s cost increment-based dynamics to describe the day-to-day departure time decision dynamics. We are able to numerically show that the system converges to a practically stable optimal equilibrium at network level departure time choice. This dissertation contribute to the literature of stable day-to-day departure time dynamics and congestion pricing.

Future research directions include formulating the control problem as a state estimation problem (Jin et al., 2020) to dynamically estimate deferral and advance coefficients according to travelers' responses to congestion pricing. These coefficients are essential for the day-to-day departure time dynamics. Researchers can also include the suppression and generation of travel demand in Vickrey (1991, 2020) to consider elastic demand. Researchers can also study the impacts of ex ante and ex post pricing — charging travelers before they depart from their origins (ex ante) and charging them after they arrive at their destinations (ex post) (Vickrey, 1994, 2019). In addition, researchers can apply Vickrey (2020)'s cost increment-based departure time dynamics to both route choice and corridor level departure time choice with point queue model to examine its impacts. Future research can introduce heterogeneity to Vickrey's cost increment-based dynamics to capture how different travelers respond to cordon based congestion pricing. Finally, future research can also incorporate peer-to-peer trading in departure time to improve the efficiency and equity of the system and benefit both parties who participate in such trading (Vickrey, 1994, 2019; Lloret Batlle, 2017; Nam, 2019; Masoud et al., 2017).

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