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Zeros of Dirichlet L-functions over Function Fields and Connections to Random Matrix Theory

DISSERTATION

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DOCTOR OF PHILOSOPHY

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by

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DEDICATION

To all of my teachers, mentors, family and friends.

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VITA

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ABSTRACT OF THE DISSERTATION

Zeros of Dirichlet L-functions over Function Fields and Connections to Random Matrix Theory

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We study the one-level density of zeros for several families of Dirichlet L-functions over function fields and prove results which support the connection between zeros of families of L-functions and statistics of eigenvalues of random matrices.

In Chapter 1, we introduce definitions of various objects of relevance, such as Dirichlet characters and Dirichlet L-functions over number fields, and present analogous ones over function fields $\mathbb{F}_q(t)$. We discuss the construction of order ℓ Dirichlet characters over $\mathbb{F}_q[t]$ specifically in Section 1.3.1, for both the Kummer setting ($q \equiv 1 \pmod{\ell}$) and the non-Kummer setting ($q \not\equiv 1 \pmod{\ell}$). Section 1.4 dedicates to results that build connections between statistics of the Riemann zeta function and families of L-functions and random matrix theory; we also define and discuss the one-level density of zeros here in detail. Section 1.5 outlines the rest of the thesis, including statements of main theorems and a remark on the average order of non-vanishing at low-lying heights.

In Chapter 2, we study the one-level density of zeros for cubic and quartic Dirichlet L-functions over function fields in the Kummer setting. We prove the general explicit formula for order ℓ Dirichlet L-functions in Lemma 2.1 and evaluate the main terms and error terms for each order. As a consequence of Theorems 1.1 and 1.2, we prove that the cubic and

quartic families have unitary symmetry, supporting the philosophy of Katz and Sarnak.

In Chapter 3, we study the one-level density of zeros for cubic, quartic and sextic Dirichlet L-functions over function fields in the non-Kummer setting. We discuss a crucial construction of non-Kummer characters in Section 3.2, motivated by the works of Baier and Young, and David, Florea and Lalin. Similar to the Kummer setting, we evaluate the main terms and error term of the one-level density and prove that the families of cubic, quartic and sextic Dirichlet L-functions have unitary symmetry.

Appendix A somewhat extends the construction of non-Kummer characters in Section 3.2 to include characters of order equal to a Mersenne prime.

Chapter 1

Introduction

1.1 Dirichlet Characters

The following definitions and facts about Dirichlet characters can be found in Chapter 1 of [15] and Section 4.2 of [37].

In 1837, Dirichlet introduced arithmetic functions called Dirichlet characters to study primes in arithmetic progression.

Definition 1.1. *A function $\chi : \mathbb{Z} \to \mathbb{C}^{\times}$ is a* ***Dirichlet character*** *modulo $q \in \mathbb{N}$ if for all integers n, m:*

- $\chi(nm) = \chi(n)\chi(m)$,
- $\chi(n) \neq 0 \iff (n, q) = 1$,
- $\chi(n + q) = \chi(n)$.

The character $\chi_0(n) = 1$ for all $(n, q) = 1$ is called the ***principal character*** *modulo*

*q. Furthermore, when χ has a period exactly q (on nonzero outputs), then χ is called a **primitive character**, and the modulus q is called its **conductor**.*

When a character $\chi \pmod q$ is imprimitive, there exists a proper nontrivial factor $q_1 \mid q$ and a primitive character $\chi_1 \pmod{q_1}$ such that

$$\chi(n) = \begin{cases} \chi_1(n) & \text{if } (n,q) = 1, \\ 0 & \text{if } (n,q) > 1. \end{cases}$$

In this case, we say that χ_1 induces χ and the conductor of χ is q_1.

One can also think of these Dirichlet characters as an extension of some group homomorphisms

$$f : (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C}^{\times}.$$

Namely, we can define

$$\chi(n) = \begin{cases} f([n]_q) & \text{if } \ (n,q) = 1, \\ 0 & \text{if } \ (n,q) \neq 1. \end{cases}$$

Using this interpretation and facts from group theory, we can deduce that the number of characters modulo q is $\left|(\mathbb{Z}/q\mathbb{Z})^{\times}\right| = \phi(q)$ and

$$\sum_{n=1}^{q} \chi(n) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $(n,q)=1$,

$$\sum_{\chi}\chi(n)=\begin{cases}\phi(q) & \text{if } n\equiv 1 \pmod{q},\\ 0 & \text{otherwise,}\end{cases}$$

where the sum is over all Dirichlet characters modulo q. These two summation formulas are referred to as orthogonality relations.

For example, there are $\phi(12)=4$ Dirichlet characters modulo 12. They are the following.

	χ_0	χ_a	χ_b	χ_c
1	1	1	1	1
5	1	1	-1	-1
7	1	-1	1	-1
11	1	-1	-1	1

Table 1.1: All characters modulo 12.

Observe that characters χ_0, χ_a and χ_b are imprimitive. By examining their periods, we see that χ_a is induced by a character modulo 4 and χ_b is induced by a character modulo 6 .

In general, Dirichlet characters can be defined on any ring of integers $\mathcal{O}_K$ for a global field K. Those over $\mathbb{F}_q[t]$ will be discussed in Section 1.3.1. Some features of Dirichlet L-functions differ depending on the value of χ on $(\mathcal{O}_K)^\times$, so we have the following definition.

Definition 1.2. *Let χ be a Dirichlet character on $\mathcal{O}_K$. Then*

$$\chi \text{ is } \begin{cases}\text{even} & \text{if } \chi\left((\mathcal{O}_K)^\times\right)=1,\\ \text{odd} & \text{otherwise.}\end{cases}$$

Observe that $\mathbb{Z}^\times=\{1,-1\}$, thus χ is even if $\chi(-1)=1$. If $\chi(-1)=-1$, then χ is odd.

1.2 Dirichlet L-functions

We refer the reader to [15] for some historical background and the proof of Dirichlet's theorem.

Long before the time of Dirichlet, it was conjectured that, for $(a, q) = 1$, there are infinity many primes in the sequence

$$a, a + q, a + 2q, \ldots.$$

Dirichlet proved this statement for prime q in 1837, and for general q in 1839. He was motivated by Euler's proof for the infinitude of primes, which showed

$$\sum_{p \text{ prime}} \frac{1}{p^s} \to \infty \quad \text{as } s \to 1^+.$$

In a key step, Dirichlet showed

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n} \neq 0$$

when $\chi \neq \chi_0$, the principal character. We now name the following functions after him.

Definition 1.3. *Let χ be a Dirichlet character modulo q and $s = \sigma + it$ be a complex variable. For $\sigma > 1$, the **Dirichlet L-series** is defined as*

$$L(s, \chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

*After analytic continuation to the whole complex plane, we also let $L(s, \chi)$ denote the **Dirichlet L-function** associated to χ.*

We note that when $\chi \neq \chi_0$, $L(s, \chi)$ is an entire function; when χ_0 is the principal character modulo q, $L(s, \chi_0)$ has a pole at $s = 1$ with residue $\phi(q)/q$.

Let $\chi \pmod q$ be a Dirichlet character and $\Re(s) > 1$. Since χ is completely multiplicative, we can express the L-function as the product,

$$L(s,\chi) = \prod_{\substack{p \text{ prime} \\ p \nmid q}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{1.1}$$

Riemann zeta function

The Riemann zeta function can be interpreted as a Dirichlet L-function with $\chi_0 \pmod 1$. We have for $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = L(s,1),$$

and

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1.2}$$

Equations (1.1) and (1.2) are equivalent to the fact that every natural number has a unique prime decomposition.

Furthermore, for the principal character $\chi_0 \pmod q$

$$L(s,\chi_0) = \zeta(s) \prod_{\substack{p \text{ prime} \\ p \mid q}} \left(1 - \frac{1}{p^s}\right).$$

Functional Equations

In 1860, Riemann proved the functional equation for the zeta function and obtained that $\zeta(s)$ is analytic on the whole complex plane except at $s = 1$. For $\Re(s) > 0$, let

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

denote the gamma function, and let

$$\xi\left(s\right):=\frac{1}{2}s\left(s-1\right)\zeta\left(s\right)\Gamma\left(s/2\right)\pi^{-s/2}. \tag{1.3}$$

($\xi(s)$ is sometimes referred to as the completed zeta function.) Riemann showed that $\xi\left(s\right)$ is entire, and $\xi\left(s\right)=\xi\left(1-s\right)$ for all s. Since $\zeta\left(s\right)\neq 0$ for $\Re(s)\geq 1$, and $\Gamma\left(s/2\right)$ has simple poles at $0,-2,-4,-6\ldots$, $\zeta\left(s\right)$ has simple zeros at $-2,-4,-6,\ldots$. We call those the **trivial zeros**. All non-trivial zeros of zeta are in the critical strip $0<\Re(s)<1$. In fact, the famous Riemann Hypothesis states that all non-trivial zeros of the Riemann zeta function lie on the line $\Re(s)=1/2$.

Similar to the zeta function, we have functional equations for Dirichlet L-functions. These functional equations differ for even and odd characters as defined in (1.2), so we use the following notation. Let

$$\kappa=\begin{cases}0 & \text{if } \chi\left(-1\right)=1,\\ 1 & \text{if } \chi\left(-1\right)=-1.\end{cases}$$

Now, for χ a primitive character modulo $q>1$, the function

$$\xi\left(s,\chi\right)=L\left(s,\chi\right)\Gamma\left(\frac{s+\kappa}{2}\right)\left(q/\pi\right)^{(s+\kappa)/2}$$

is entire, and $\xi\left(s,\chi\right)=\xi\left(1-s,\overline{\chi}\right)$ for all s. For more details on functional equations for Dirichlet L-functions, please see Section 10 of [37].

1.3 Background in Function Fields

For more background on function fields, we refer the reader to [43].

Many features of $\mathbb{F}_q[t]$ are similar to those of $\mathbb{Z}$. For example, both rings are principal ideal domains; they both have infinitely many primes, and finitely many units, $\mathbb{F}_q^\times$ and {-1,1} respectively. Yet, many problems which are intractable over $\mathbb{Z}$ can be solved over $\mathbb{F}_q[t]$. For instance, the Riemann Hypothesis, still open over $\mathbb{Z}$, was proven for curves over finite fields by Weil in 1948 [46]. The Generalized Riemann Hypothesis (GRH), which states that all zeros of Dirichlet L-functions $L(s,\chi)$ lie on the line $\Re(s) = 1/2$, is true over function fields, but still open over number fields.

The table below shows the dictionary from $\mathbb{Z}$ to $\mathbb{F}_q[t]$.

Over Number Fields	Over Function Fields
$\mathbb{Z}$	$\mathbb{F}_q[t]$
$\mathbb{Q}$	$\mathbb{F}_q(t)$
n positive integer	F monic polynomial
p prime number	P irreducible (prime)
$\lvert n\rvert = \lvert\mathbb{Z}/n\mathbb{Z}\rvert$	$\lvert F\rvert_q := \lvert\mathbb{F}_q[t]/F\rvert = q^{\deg(F)}$

Table 1.2: Number fields to function fields analogies.

List of notations

We use the following notations in this document.

- Let $\mathcal{M}_q$ denote the set of monic polynomials in $\mathbb{F}_q[t]$ and $\mathcal{M}_{q,d}$ be those monic polynomials of degree d. (Note that $|\mathcal{M}_{q,d}| = q^d$.)

- Let $\mathcal{P}_q$ be the monic irreducible polynomials in $\mathbb{F}_q[t]$ and $\mathcal{P}_{q,d}$ be those monic irreducible polynomials of degree d.

- Let $\mathcal{H}_q$ be the monic squarefree polynomials in $\mathbb{F}_q[t]$ and $\mathcal{H}_{q,d}$ be those monic squarefree polynomials of degree d. (Note that for $d \geq 2$, we have $|\mathcal{H}_{q,d}| = q^d\left(1 - \frac{1}{q}\right)$.)

- Let $d(f)$ denote the degree of the polynomial f. If convenient, we also use $\deg(f)$.

- Let $e(x) := e^{2\pi i x}$.

- Let $|f|_{q^n} := q^{nd(f)}$ define the norm of f in $\mathbb{F}_{q^n}[t]$, and we write $|f| := q^{d(f)}$ if $f \in \mathbb{F}_q[t]$.

Some familiar arithmetic functions are defined analogously over $\mathbb{F}_q[t]$. For example, for $c \in \mathbb{F}_q^\times$, the Mobius function is

$$\mu(F) = \begin{cases} (-1)^k & \text{if } F = cP_1P_2\cdots P_k, \\ 0 & \text{if } F \text{ is not squarefree.} \end{cases}$$

The von Mangoldt function Λ is defined analogously as

$$\Lambda(F) = \begin{cases} \deg(P) & \text{if } F = cP^\alpha \text{ for some } c \in \mathbb{F}_q^\times \text{ and } \alpha \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the von Mangoldt function, the Prime Polynomial Theorem can be written as

$$\sum_{f \in \mathcal{M}_n} \Lambda(f) = q^n.$$

Lastly, we write the function field analogue of the familiar Perron's formula.

Lemma 1.1 (Perron's Formula). *If the generating series $\mathcal{S}(u) = \sum_{f \in \mathcal{M}_q} a(f) u^{d(f)}$ is absolutely convergent in $|u| \leq r < 1$, then*

$$\sum_{f \in \mathcal{M}_{q,d}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{S}(u)}{u^d} \frac{du}{u},$$

where, as usual, $\oint$ denote the integral over the circle oriented counterclockwise.

1.3.1 Order ℓ Dirichlet Characters

The works described in the later sections involve families of Dirichlet L-functions of a specific order. For example, we will compute the average behavior of low-lying zeros of cubic Dirichlet L-functions $L(s,\chi)$ over function fields, such that $\chi^3 = 1$. Hence, we discuss order ℓ Dirichlet characters over function fields more specifically.

First, analogous to Definition 1.1, we define Dirichlet characters $\chi : \mathbb{F}_q[t] \to \mathbb{C}^\times$ as follows.

Definition 1.4. *A function $\chi : \mathbb{F}_q[t] \to \mathbb{C}^\times$ is a* ***Dirichlet character*** *modulo a polynomial m of positive degree if for all $a, b \in \mathbb{F}_q[t]$:*

- $\chi(ab) = \chi(a)\chi(b)$,
- $\chi(a) \neq 0 \iff (a,m) = 1$,
- $\chi(a+m) = \chi(a)$.

The character $\chi_0(a) = 1$ for all $(a,m) = 1$ is called the principal character modulo m. Furthermore, when χ has a period exactly m (on non-zero outputs), then χ is called a ***primitive character****, and the modulus m is called the* ***conductor****. We use the notation $\chi_m(\cdot)$ to denote the Dirichlet character with conductor m.*

Now, order ℓ Dirichlet characters $\chi_m(a)$ over $\mathbb{F}_q[t]$ can be defined based on the ℓ^{th} residue symbol $\left(\frac{a}{m}\right)_\ell$, which are defined analogously as residue symbols over number fields. These ℓ^{th} residue symbols are defined when $q \equiv 1 \pmod{\ell}$, i.e., when $\mathbb{F}_q^\times[t]$ contains an ℓ^{th} root of unity. (Since units are of degree 0, we will simply write $\mathbb{F}_q^\times$ instead.) We can extend the definition of the ℓ^{th} residue symbol to $\mathbb{F}_q[t]$ when $q \not\equiv 1 \pmod{\ell}$ and thus Dirichlet characters in these settings differ. We call the case when $q \equiv 1 \pmod{\ell}$ the **Kummer setting** and when $q \not\equiv 1 \pmod{\ell}$ the **non-Kummer** setting.

Primitive characters in the Kummer setting

First, we define the ℓ^{th} residue symbol $\left(\frac{f}{P}\right)_\ell$ on primes P over $\mathbb{F}_q[t]$ and the associated primitive character with conductor P. Then, we extend multiplicatively to general ℓ^{th} power-free conductors H. The following definition can be seen in Chapter 3 of [43].

Definition 1.5. *Let $f, P \in \mathbb{F}_q[t]$ and P a prime polynomial that does not divide f. The ℓ^{th} Jacobi symbol $\left(\frac{f}{P}\right)_\ell$ is the unique element of $\mathbb{F}_q^\times$, such that*

$$\left(\frac{f}{P}\right)_\ell \equiv f^{\frac{|P|-1}{\ell}} \pmod{P}.$$

We can extend this definition multiplicatively to any residue symbol $\left(\frac{f}{H}\right)_\ell$, where H is a monic polynomial in $\mathbb{F}_q[t]$. We also note that $\left(\frac{f}{P}\right)_\ell$ is an ℓ^{th} root of unity in $\mathbb{F}_q^\times$.

Given the definition above, we define primitive characters with a prime conductor P in the Kummer setting.

Definition 1.6. *Let Ω_ℓ be a fixed isomorphism from the ℓ^{th} roots of unity $\mu_\ell \subseteq \mathbb{C}^\times$ to the ℓ^{th} roots of unity in $\mathbb{F}_q^\times$. We define $\chi_P(f) = 0$ if $P \mid f$, and otherwise*

$$\chi_P(f) = \Omega_\ell^{-1}\left(\left(\frac{f}{P}\right)_\ell\right).$$

Similarly to above, extending multiplicatively to a monic polynomial $H = P_1^{e_1} \cdots P_s^{e_s}$ with distinct primes factors P_i, we have..

$$\chi_H = \chi_{P_1}^{e1} \cdots \chi_{P_s}^{e_s}, \tag{1.4}$$

where $\chi_H^\ell = 1$ with conductor $H' = P_1 \cdots P_s$. Furthermore, χ_H is a primitive character if and only if $1 \leq e_i \leq \ell - 1$ for all i. For instance, when $\ell = 3$, χ_H is primitive if and only if

$e_i \in \{1, 2\}$ as in [17].

Observe that grouping the primes factors by their exponents, we can write

$$H = F_1 F_2^2 \cdots F_{\ell-1}^{\ell-1},$$

where the F_i's are monic squarefree polynomials and pairwise coprime. Thus, given these F_i's, we can consider the conductor

$$H' = F_1 F_2 \cdots F_{\ell-1}, \tag{1.5}$$

which corresponds to the original primitive character

$$\chi_H = \chi_{F_1} \chi_{F_2}^2 \cdots \chi_{F_{\ell-1}}^{\ell-1}. \tag{1.6}$$

Defining the family $\mathcal{F}_\ell^{\mathrm{K}}(g)$

In Chapter 2, we study the family of primitive cubic Dirichlet characters of genus g in the Kummer setting. We denote this family by $\mathcal{F}_3^{\mathrm{K}}(g)$. Here, we give the general definition of $\mathcal{F}_\ell^{\mathrm{K}}(g)$ for a prime $\ell \geq 3$. We also work with $\mathcal{F}_4^{\mathrm{K}}(g)$ (defined separately below) in Chapter 2.

Let χ_ℓ be a fixed order ℓ character such that on the group of units $\mathbb{F}_q^\times$,

$$\chi_\ell(\alpha) = \Omega_\ell^{-1}\left(\alpha^{\frac{q-1}{\ell}}\right). \tag{1.7}$$

A character on $\mathbb{F}_q[t]$ is even if it is the principal character on $\mathbb{F}_q^\times$, and odd otherwise. Thus in the Kummer case, any order ℓ character on $\mathbb{F}_q[t]$ falls into ℓ classes depending on its restriction to $\mathbb{F}_q^\times$: it is either the principal character χ_0 or it is χ_ℓ^j for some integer $1 \leq j < \ell$. The character is even in the first case, and odd otherwise.

For some $\alpha \in \mathbb{F}_q^\times$, we have by definition

$$\chi_{F_1 F_2{}^2 \cdots F_{\ell-1}^{\ell-1}}(\alpha) = \Omega_\ell^{-1}\left(\alpha^{\frac{q-1}{\ell}(d_1+2d_2+\cdots+(\ell-1)d_{\ell-1})}\right),$$

where $d_i := \deg(F_i)$. Thus this character is even if and only if $d_1+2d_2+\cdots+(\ell-1)d_{\ell-1} \equiv 0 \pmod{\ell}$. Furthermore, using the Riemann-Hurwitz formula, such as the version given in (1.9), the degree of H' is given by the following.

$$d(H') = \begin{cases} \frac{2g+2\ell-2}{\ell-1} & \text{if } d_1+2d_2+\cdots+(\ell-1)d_{\ell-1} \equiv 0 \pmod{\ell}, \\ \frac{2g+2\ell-2}{\ell-1} - 1 & \text{if } d_1+2d_2+\cdots+(\ell-1)d_{\ell-1} \not\equiv 0 \pmod{\ell}. \end{cases}$$

For convenience, we restrict to the case of odd primitive characters whose restriction to $\mathbb{F}_q^\times$ is χ_ℓ as seen in (1.7). This means $d_1+2d_2+\cdots+(\ell-1)d_{\ell-1} \equiv 1 \pmod{\ell}$ and $\deg(H') = \dfrac{2g+2\ell-2}{\ell-1} - 1$. Hence, using the notation in (1.5) we have

$$\mathcal{F}_\ell^{\mathrm{K}}(g) := \{H' : \ H' = F_1 F_2 \cdots F_{\ell-1},\ F_i \text{ squarefree and pairwise coprime}, \\ \deg(H') = \frac{2g}{\ell-1} + 1,\ d_1+2d_2+\cdots+(\ell-1)d_{\ell-1} \equiv 1 \pmod{\ell}\}. \tag{1.8}$$

Defining $\mathcal{F}_4^{\mathrm{K}}(g)$

For the family of quartic Dirichlet L-functions in the Kummer setting, we consider curves of the affine model

$$Y^4 = F_1(t) F_3^3(t),$$

where $F_1(t), F_3(t)$ are squarefree and $(F_1(t), F_3(t)) = 1$. This correspond to characters of order four, such that $\chi^4 = 1$ and χ^2 remains primitive. As in the cubic case, we consider characters whose restriction to the units $\mathbb{F}_q^\times$ is χ_4 as given in (1.7).

The Riemann-Hurwitz formula [43, 35] states that for characters of genus g and order ℓ

$$2g + 2\ell - 2 = \sum_{i=1}^{\ell-1} (\ell - (\ell, i))\, d_i + (\ell - (\ell, d)), \tag{1.9}$$

where $d_i = \deg(F_i)$ and $d = \sum_{i=1}^{\ell-1} i d_i$. For $\ell = 4$ and for the affine model above, we have

$$2g + 6 = 3d_1 + 3d_3 + 3,$$

which implies that the degree of the conductor is

$$\frac{2g+6}{3} - 1 = d_1 + d_3. \tag{1.10}$$

Note that (1.10) gives an analogous formula to the degree of the conductor when ℓ is prime; if $d_2 \neq 0$, then the relation between the genus and the conductor degree becomes more complicated compared to the prime case. Lastly, since $d_1 + d_3 = \dfrac{2g}{3} + 1$, there are $\dfrac{2g}{3}$ non-trivial zeros for each L-function.

Let $\mathcal{F}_4^{\mathrm{K}}(g)$ denote the conductors of primitive quartic characters of genus g. Following the discussion above and the notation in (1.5) by denoting the conductor as H', we have

$$\begin{aligned} \mathcal{F}_4^{\mathrm{K}}(g) := \{H' :\ & H' = F_1F_3,\ F_i \text{ squarefree and pairwise coprime}, \\ & \deg(H') = \frac{2g}{3} + 1,\ d_1 + 3d_3 \equiv 1 \pmod 4\}. \end{aligned} \tag{1.11}$$

Before defining characters in the non-Kummer setting, we give the condition for perfect reciprocity. We will consider the case of perfect reciprocity for convenience in the later sections. The following is the reciprocity law derived from Theorem 3.5 in [43].

Lemma 1.2 (Order ℓ Reciprocity)**.** *Let $a, b \in \mathcal{M}_q$ be relatively prime polynomials, and let*

χ_a and χ_b be the order ℓ^{th} characters defined above. If $q \equiv 1 \pmod{2\ell}$, then

$$\chi_a(b) = \chi_b(a).$$

Primitive characters in the non-Kummer setting

Recall that when $q \not\equiv 1 \pmod{\ell}$, we have characters in the non-Kummer setting.

First, we consider the case when the character χ has a prime conductor P. Let n_q be the multiplicative order of $q \pmod{\ell}$, such that

$$q^{n_q} \equiv 1 \pmod{\ell}. \tag{1.12}$$

If χ_P is an order ℓ character over $\mathbb{F}_q[t]$ in the non-Kummer setting, then n_q must divide the degree of P, since $\ell \mid (q^{n_q} - 1)$ and

$$\chi_P^{\ell} = \chi_P^{q^{n_q}-1} = \chi_P^{|P|-1} = 1.$$

Recall that $|P| = q^{d(P)}$ denotes the size of the prime P. Thus, we define the ℓ^{th} Jacobi symbol and the associated order ℓ primitive character with a prime conductor P the same way as in the Kummer setting when the degree of P is divisible by n_q.

Definition 1.7. *Let n_q be defined as in (1.12) and P a prime of degree divisible by n_q. Then*

1. *for any $f \in \mathbb{F}_q[t]$ such that $P \nmid f$, the ℓ^{th} Jacobi symbol $\left(\frac{f}{P}\right)_\ell$ is the unique element of $\mathbb{F}_{q^{n_q}}^\times$ such that*
$$f^{\frac{|P|-1}{\ell}} \pmod{P} \equiv \left(\frac{f}{P}\right)_\ell.$$

2. *Let Ω_ℓ be a fixed isomorphism from the ℓ^{th} roots of unity $\mu_\ell \subseteq \mathbb{C}^\times$ to the ℓ^{th} roots of*

unity in $\mathbb{F}_{q^{n_q}}^{\times}$. *We define* $\chi_P(f) = 0$ *if* $P \mid f$, *and otherwise*

$$\chi_P(f) = \Omega_\ell^{-1}\left(\left(\frac{f}{P}\right)_\ell\right).$$

For works in the non-Kummer setting, it is crucial to have another (perhaps more natural) description for these characters. We give more details in Section 3.2.

1.3.2 The Riemann zeta function and Dirichlet L-functions over function fields

We refer to [43] for background on the zeta function over function fields.

For $\Re(s) > 1$, the zeta function over $\mathbb{F}_q[t]$ is defined to be

$$\zeta_q(s) := \sum_{f \in \mathcal{M}_q} \frac{1}{|f|_q^s} = \prod_{P \in \mathcal{P}_q} \left(1 - \frac{1}{|P|_q^s}\right)^{-1},$$

where $\mathcal{M}_q$ (given in the list of notations in Section 1.3) is the set of monic polynomials over $\mathbb{F}_q[t]$, analogous to the positive integers on $\mathbb{Z}$. Since $|\mathcal{M}_{q,d}| = q^d$, we have

$$\zeta_q(s) = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}.$$

It is convenient to make the change of variable $u = q^{-s}$, therefore we use the notation

$$\mathcal{Z}_q(u) := \zeta_q(s) = \frac{1}{1 - qu}.$$

For an order ℓ character χ with conductor h, the Dirichlet L-function attached to χ is defined by

$$L_q(s,\chi) := \sum_{f\in\mathcal{M}_q} \frac{\chi(f)}{|f|_q^s}.$$

With the same change of variable $u = q^{-s}$, we have

$$\mathcal{L}_q(u,\chi) := L_q(s,\chi) = \sum_{f\in\mathcal{M}_q} \chi(f)u^{d(f)} = \prod_{\substack{P\in\mathcal{P}_q\\ P\nmid h}} \left(1-\chi(P)u^{d(P)}\right)^{-1}.$$

Now let C be a curve of genus g with conductor h over $\mathbb{F}_q(t)$ and let the function field of C be a cyclic degree ℓ extension of the base field. From the Weil conjectures, the zeta function of the curve C can be written as

$$\mathcal{Z}_C(u) = \frac{\mathcal{P}_C(u)}{(1-u)(1-qu)},$$

where $\mathcal{P}_C(u)$ is a polynomial of degree $2g$. Furthermore, we have that from [16], when χ is an odd character

$$\mathcal{P}_C(u) = \prod_{i=1}^{\ell-1} \mathcal{L}_q(u,\chi^i),$$

and when χ is even

$$\mathcal{P}_C(u) = (1-u)^{-\ell+1} \prod_{i=1}^{\ell-1} \mathcal{L}_q(u,\chi^i).$$

The Riemann Hypothesis for curves over function fields [46] states that all non-trivial zeros of $\mathcal{L}_q(u,\chi)$ are on the circle $|u| = q^{-\frac{1}{2}}$. Recall that $e(x) = e^{2\pi i x}$. Hence we can express the L-function in terms of its zeros

$$\mathcal{L}_q(u,\chi) = (1-u)^b \prod_{j=1}^{d(h)-1-b} \left(1-u\sqrt{q}e(\theta_{j,h})\right), \tag{1.13}$$

where $b = 1$ if χ is even, and $b = 0$ otherwise.

Similar to the number field setting,"completed" L-functions with primitive characters χ have functional equations over function fields. Let

$$\Lambda_q(u,\chi) := (1-bu)^{-1}\mathcal{L}_q(u,\chi)$$

denote the "completed" L-function where b is defined as above. Then

$$\Lambda_q(u,\chi) = \omega(\chi)\left(\sqrt{q}u\right)^{d(h)-1-b}\Lambda_q\left(\frac{1}{qu},\overline{\chi}\right), \tag{1.14}$$

with $|\omega(\chi)| = 1$ [45]. This implies that angles of zeros of $\mathcal{L}_q(u,\chi)$ corresponds to the negative of angles of zeros of $\mathcal{L}_q(u,\overline{\chi})$.

Given a Dirichlet character of genus g, we have the following relation. To simplify notations, let

$$D_\ell(g) = \frac{2g+2\ell-2}{\ell-1}. \tag{1.15}$$

Using the Riemann-Hurwitz formula [43], the degree of the conductor is

$$d(h) = \begin{cases} D_\ell(g) & \text{if } \chi \text{ is even,} \\ D_\ell(g)-1 & \text{if } \chi \text{ is odd.} \end{cases} \tag{1.16}$$

Thus, there are $D_\ell(g)-2$ non-trivial zeros for each L-function.

When evaluating the error term in sections such as 2.4.2, we use following results from [17] to bound the size of L-functions. Note that these results hold for characters of any order.

Lemma 1.3. *Let χ be a primitive order ℓ character of conductor h defined over $\mathbb{F}_q[t]$. Then for $\Re(s) \geq 1/2$ and for all $\epsilon > 0$,*

$$|L_q(s,\chi)| \ll q^{\epsilon d(h)}.$$

Lemma 1.4. *Let χ be a primitive order ℓ character of conductor h defined over $\mathbb{F}_q[t]$. Then for $\Re(s) \geq 1$ and for all $\epsilon > 0$,*

$$|L_q(s, \chi)| \gg q^{-\epsilon d(h)}.$$

Note that Lemma 1.3 is analogous to the Lindelöf Hypothesis, since $q^{\epsilon d(h)} = |h|^{\epsilon}$.

1.4 Spectral interpretation of the non-trivial zeros of the zeta and L-functions

In the early 1900s, Hilbert and Pólya independently stated that the non-trivial zeros of the Riemann zeta function correspond to eigenvalues of a self-adjoint operator. Although there seems to be no first-hand account due to Hilbert, Pólya discussed some historical details in his correspondence with Odlyzko [40]. While in Göttingen studying analytic number theory with Landau, Pólya was asked by Landau one day if there is a physical reason for the truth of the Riemann Hypothesis (RH). He answered that if the non-trivial zeros of the $\xi(s)$ function (1.3) are associated to a physical problem, then RH would be equivalent to the fact that eigenvalues of the physical problem to are real. This remark was never published, but it became known and remembered.

There was little evidence at the time suggesting the connection between zeros of the zeta function and the spectrum of matrices, but many results since then have supported such an idea. Notably, Montgomery's pair correlation conjecture [36], Katz and Sarnak's philosophy [31, 32] and conjectures on moments of the zeta function and L-functions using random matrix theory by Keating and Snaith [34, 33]. We discuss these works in this section.

1.4.1 Montgomery's pair correlations of zeros

Let $Z(T) = \{s = \sigma + i\gamma : \zeta(s) = 0, 0 \leq \sigma < 1, 0 < \gamma < T\}$ denote the set of zeros in the critical strip up to height T and $\hat{\gamma} = \frac{\gamma \log T}{2\pi}$ be the normalized imagarinary part of s. To study the zeros of the Riemann zeta function, Montgomery [36] computed the pair correlation of the normalized zeros, and found that, assuming the Riemann Hypothesis, for a test function f with the support of its Fourier transform $\hat{f}$ in $(-1, 1)$,

$$\lim_{T\to\infty} \frac{1}{|Z(T)|} \sum_{\substack{s,s'\in Z(T) \\ \alpha \leq \hat{\gamma}-\hat{\gamma}' \leq \beta}} f\left(\hat{\gamma} - \hat{\gamma}'\right) = \int_\alpha^\beta f(x) \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right) dx. \tag{1.17}$$

In an encounter with the renowned physicist Freeman Dyson in 1972, Montgomery discovered that the function $1-\left(\frac{\sin(\pi x)}{\pi x}\right)^2$ in the integrand above (what we refer to as the distribution function today) also appears in statistics of eigenvalues of large random complex Hermitian or unitary matrices [36]. Since then, this connection has been supported by extensive numerical and theoretical tests, such as those done by Odlyzko in [38]. It deepened to give random matrix models for moments of the zeta function and the low-lying zeros of families of L-functions [31, 32, 34, 33].

One-level density of zeros by Özlük and Snyder

Before discussing the works of Katz and Sarnak in 1999, it may be worth highlighting the results obtained in 1993 by Özlük and Snyder, the former of which was a student of Montgomery. The authors studied the average behavior of low-lying zeros of quadratic Dirichlet L-functions in a statistic called the one-level density. We discuss their results below and give more details in Section 1.4.4.

Assuming the Generalized Riemann Hypothesis, Özlük and Snyder computed the one-level density of quadratic Dirichlet L-functions and found that this family has a symplectic symmetry. To determine the "symmetry type" of the family for quadratic L-functions, one does it in the same way as in (1.17) for zeros of zeta. (The zeros of zeta up to height T actually behave similarly to zeros of families of L-functions.)

The authors' results also shed light on the important question of non-vanishing of L-functions. Studying the non-vanishing of the zeta function was key in proving the Prime Number Theorem in the late 1800s, and studying the non-vanishing of L-functions, besides being part of the proof of Dirichlet's theorem, gives arithmetic information about objects such as the class number and the rank of an elliptic curves. Using their computation, Özlük and Snyder showed that more than 93.75% of L-functions in the family do not vanish at the central point $s = 1/2$. This corresponds to their result holding for test functions ϕ with Fourier transform $\hat{\phi}$ supported in $(-2, 2)$. In fact, Chowla's conjecture [9] states that $L(1/2, \chi) \neq 0$ for any Dirichlet character. Obtaining the one-level density result for $\hat{\phi}$ supported in $(-A, A)$ for any A is equivalent to 100% non-vanishing as predicted by Chowla's conjecture.

1.4.2 Katz and Sarnak's philosophy

The works by Katz and Sarnak [31, 32] further strengthened the connection between random matrix theory and statistics of zeros of L-functions. In their works in 1999, the two authors computed the one-level density of zeros for Dirichlet L-functions for curves over finite fields as the genus of the curve g and the size of the finite field q both tend to infinity. They showed that for some families of L-functions, their statistics of zeros follow distribution laws of eigenvalues of classical groups. This lead them to predict that, in general, statistics of families of L-functions have a spectral interpretation, i.e., there is a symmetry type associated to each family given by the classical groups, such as the group of symplectic, unitary and orthogonal matrices. This is referred to as Katz and Sarnak's philosophy.

For example, in the case of quadratic characters, let $\mathcal{F}_X$ denote the family of L-functions with conductors $c_f \leq X$ (as f varies over $\mathcal{F}_X$) and, following the notation in [31], let $\Delta(f, \phi)$ be the sum over imaginary part of zeros γ_f

$$\Delta(f, \phi) = \sum_{\gamma_f} \phi\left(\frac{\gamma_f \log c_f}{2\pi}\right),$$

where $\phi \in \mathcal{S}(\mathbb{R})$ is a test function in the Schwartz space. The one-level density of zeros in this family is

$$W(X, \mathcal{F}, \phi) = \frac{1}{\#\mathcal{F}_X} \sum_{c_f \leq X} \Delta(f, \phi),$$

which tends to the integral

$$W(X, \mathcal{F}, \phi) \to \int_{-\infty}^{\infty} \phi(x)\, w(Sp)(x)\, dx$$

as $X \to \infty$ and for any test function whose Fourier transform is supported in $(-2, 2)$. The function $w(Sp)(x) = 1 - \dfrac{\sin 2\pi x}{2\pi x}$ is called the one-level scaling density for symplectic matrices.

In general, Katz and Sarnak found that

$$w(G)(x) = \begin{cases} 1 & \text{if } G = U \text{ or } SU, \\ 1 - \frac{\sin 2\pi x}{2\pi x} & \text{if } G = Sp, \\ 1 + \frac{1}{2}\delta_0(x) & \text{if } G = O, \\ 1 + \frac{\sin 2\pi x}{2\pi x} & \text{if } G = SO\,(\text{even}), \\ \delta_0(x) + 1 - \frac{\sin 2\pi x}{2\pi x} & \text{if } G = SO\,(\text{odd}), \end{cases}$$

with the Fourier transform given in [16],

$$\hat{w}\,(G)\,(x) = \begin{cases} \delta_0(x) & \text{if } G = U \text{ or } SU, \\ \delta_0(x) - \frac{1}{2}\eta(x) & \text{if } G = Sp, \\ \frac{1}{2} + \delta_0(x) & \text{if } G = O, \\ \delta_0(x) + \frac{1}{2}\eta(x) & \text{if } G = SO\,(\text{even})\,, \\ 1 + \delta_0(x) - \frac{1}{2}\eta(x) & \text{if } G = SO\,(\text{odd})\,, \end{cases} \tag{1.18}$$

where

$$\eta(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ \frac{1}{2} & \text{if } |x| = 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Confirming Katz and Sarnak's philosophy by computing the symmetry type of several families of Dirichlet L-functions over function fields is a goal of works described in this document.

1.4.3 Moments conjectures of the Riemann zeta function and L-functions

Motivated by the connection of non-trivial zeros of the Riemann zeta function to eigenvalues of random unitary matrices, such as Montgomery's pair correlation result discussed in 1.4.1, Keating and Snaith computed moments of the characteristic polynomial of unitary matrices, averaged over the group $U\,(N)$ with respect to the Haar measure (over the circular unitary ensemble in random matrix theory). They found that for $Z\left(U,\theta\right) = \det\left(1 - Ue^{-i\theta}\right)$,

$$\langle\left|Z\left(U,\theta\right)\right|^{s}\rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma\left(j\right)\Gamma\left(j+s\right)}{\left(\Gamma\left(j+s/2\right)\right)^{2}} \tag{1.19}$$

for any $\theta \in \mathbb{R}$ and $\Re(s) > -1$.

This led them to conjecture asymptotics for moments of the zeta function, a difficult problem that traces its origin from Hardy and Littlewood in the early 20$^{\text{th}}$ century. Studying moments of the zeta function on the critical line $s = 1/2$ leads to results about the size of $\zeta(1/2 + it)$ and progresses towards the Lindelöf Hypothesis, which states that

$$|\zeta(1/2 + it)| \ll t^{\epsilon}, \text{ for any } \epsilon > 0 \text{ as } t \to \infty.$$

The $2k^{\text{th}}$ moment of the zeta function on the critical line is defined to be the integral

$$I_k\left(T\right) = \int_{T}^{2T} \left|\zeta(1/2 + it)\right|^{2k} dt.$$

Hardy and Littlewood computed the 2$^{\text{nd}}$ moment in 1916 [25] and Littlewood's student, Ingham, computed the 4^{th} moment in 1927 [29]. For higher moments, only bounds for have been computed, such as those assuming the Riemann Hypothesis in [42, 28] and unconditionally in [41, 5, 26, 27]. Using Equation (1.19), Keating and Snaith conjectured that

$$\lim_{T\to\infty} \int_{T}^{2T} \left|\zeta(1/2 + it)\right|^{2k} dt = a_k g_k T \left(\log T\right)^{k^2},$$

where

$$g_k = \frac{G^2\left(1+k\right)}{G\left(1+2k\right)},$$

for the Barnes G-function

$$G\left(1+z\right)=\left(2\pi\right)^{z/2}e^{-\left[(1+\gamma)z^2+z\right]/2}\prod_{n=1}^{\infty}\left[\left(1+z/n\right)^n e^{-z+z^2/(2n)}\right], \tag{1.20}$$

The a_k term is a known arithmetic factor

$$a_k=\prod_{p \text{ prime}}\left\{\left(1-1/p\right)^{k^2}\left(\sum_{m=0}^{\infty}\left(\frac{\Gamma\left(k+m\right)}{m!\Gamma\left(k\right)}\right)^2 p^{-m}\right)\right\}.$$

Note that we have $g_0=1$ and for integer $k\geq 1$,

$$g_k=\prod_{j=1}^{k-1}\frac{j!}{(j+k)!}.$$

In particular, $g_1=1$ and $g_2=1/12$, which match those computed by Hardy and Littlewood and Ingham respectively; $g_3=42/9!$ and $g_4=24024/16!$, which match those conjectured by Conrey and Ghosh [13], and Conrey and Gonek [14] respectively using number theoretic arguments.

As discussed in Section 1.4.2, Katz and Sarnak predicted that statistics of zeros of families of L-functions correspond to the distribution of eigenvalues of random matrices in the classical groups $U(N)$, $O(N)$ or $USp(2N)$ [31, 32]. Motivated by these works, Keating and Snaith [33] investigated moments of L-functions, applying the method they developed for the unitary family in [34] to other families suggested to exhibit non-unitary symmetry types. For example, the family of quadratic Dirichlet L-functions is suggested to have symplectic symmetry, since statistics of zeros for the family, such as the one-level density of zeros, obey the distribution law of the group of symplectic matrices.

For instance, in the symplectic family $USp(2N)$, the eigenvalues of $U\in USp(2N)$ lie on the unit circle and come in complex conjugate pairs $e^{i\theta_1},e^{-i\theta_1},e^{i\theta_2},e^{-i\theta_2},\ldots,e^{i\theta_N},e^{-i\theta_N}$. Thus the characteristic polynomial related to such a matrix U can be written as

$$Z_U(\theta) = \prod_{i=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right)\left(1 - e^{i(-\theta_n - \theta)}\right).$$

Keating and Snaith computed k^{th} powers of $Z_U(\theta)$, averaged over $USp(2N)$ with respect to the Haar measure and obtained that

$$\left\langle Z_U\left(0\right)^k \right\rangle_{USp(2N)} = N^{k/2+k^2/2} 2^{s^2/2} \times \frac{G\left(1+s\right)\sqrt{\Gamma\left(1+s\right)}}{\sqrt{G\left(1+2s\right)\Gamma\left(1+2s\right)}} \tag{1.21}$$

as $N \to \infty$, where G is the Barnes G-function given in (1.20). We note that for integer k, Equation (1.21) simplifies to

$$\left\langle Z_U\left(0\right)^k \right\rangle_{USp(2N)} = N^{k/2+k^2/2} \left(\prod_{j=1}^{k} (2j-1)!!\right)^{-1}.$$

The authors then conjectured that in the case of quadratic Dirichlet L-functions,

$$\frac{1}{D^*} \sum_{|d| \le D} L\left(1/2, \chi_d\right)^k \sim a_k g_k \left(\log D\right)^{k(k+1)/2}$$

where D^* is the number of quadratic characters in the sum and

$$g_k = 2^{-\frac{k(k+1)}{2}} \left(\prod_{j=1}^{k} (2j-1)!!\right)^{-1},$$

with the arithmetic factor given explicitly in [22],

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{\frac{k(k+1)}{2}} \left(1 + \frac{1}{p}\right)^{-1} \left(\frac{1}{2}\left(\left(1 - \frac{1}{\sqrt{p}}\right)^{-k} + \left(1 + \frac{1}{\sqrt{p}}^{-k}\right)\right) + \frac{1}{p}\right).$$

By Equation (1.4.3), $g_1 = 1$, $g_2 = \frac{1}{3}$, $g_3 = \frac{1}{45}$ and $g_4 = \frac{1}{4725}$, which agree precisely with those Conrey and Farmer reported in [11], supporting this conjecture.

Moments conjecture and results over function field

In 2014, Andrade and Keating [2] gave conjectures for moments of quadratic Dirichlet L-functions over function fields, adapting the approach in [12] over number fields. The method used in [12] by Conrey, Farmer, Keating, Rubinstein and Snaith is often referred to as the "recipe". For the ideas used in the "recipe" over function fields, Florea gave a nice presentation in Section 1.9 of [22].

Let q be a fixed odd prime, $\mathcal{H}_{2g+1}$ the set of monic squarefree polynomials of degree $2g+1$, and $X(s) = q^{-1/2+s}$. Andrade and Keating conjectured that

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(1/2, \chi_D\right)^k = \sum_{D\in\mathcal{H}_{2g+1}} Q_k\left(\log_q |D|\right)\left(1+o(1)\right),$$

where Q_k is the polynomial of degree $k(k+1)/2$ given by the k-fold residue

$$Q_k(x) = \frac{(-1)^{k(k+1)/2}2^k}{k!}\frac{1}{2\pi i}\oint\cdots\oint\frac{G\left(z_1,\ldots,z_k\right)\Delta\left(z_1^2,\ldots,z_k^2\right)^2}{\prod_{j=1}^k z_j^{2k-1}}q^{\frac{x}{2}\prod_{j=1}^k z_j}dz_1\ldots dz_k,$$

with $\Delta\left(z_1,\ldots,z_k\right)$ being the Vandermonde determinant given by

$$\Delta\left(z_1,\ldots,z_k\right) = \prod_{1\le i\le j\le k}\left(z_j - z_i\right),$$

and the function

$$G\left(z_1,\ldots,z_k\right) = A\left(\frac{1}{2};z_1,\ldots,z_k\right)\prod_{j=1}^k X\left(\frac{1}{2}+z_j\right)^{-1/2}\prod_{1\le i\le j\le k}\zeta_q\left(1+z_i+z_j\right),$$

in which $A\left(\frac{1}{2};z_1,\ldots,z_k\right)$ is the Euler product, absolutely convergent when $|\Re(z_j)| < \frac{1}{2}$, defined by

$$A\left(\frac{1}{2};z_1,\dots,z_k\right)=\prod_{\substack{P \text{ moinc}\\ \text{irreducible}}}\prod_{1\le i\le j\le k}\left(1-\frac{1}{|P|^{1+z_i+z_j}}\right)$$
$$\times\left(\frac{1}{2}\left(\prod_{j=1}^{k}\left(1-\frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}+\prod_{j=1}^{k}\left(1+\frac{1}{|P|^{\frac{1}{2}+z_j}}\right)\right)^{-1}+\frac{1}{|P|}\right)\left(1+\frac{1}{|P|}\right)^{-1}.$$

For the first moment, the conjecture matches the result computed by the same authors in 2012 [1]. Florea computed the asymptotics for the second and third moments in [21] and the fourth moment in [20], all matching the prediction of the conjecture.

Compare to the wealth of literature on quadratic Dirichlet L-functions, there are few works on higher order families of Dirichlet L-functions. Over number fields, Baier and Young [3] computed the smoothed mean value of cubic Dirichlet L-functions and obtained that more than $Q^{\frac{6}{7}-\epsilon}$ cubic characters with conductors at most Q satisfy $L(1/2,\chi)\neq 0$. More recently, David, Florea and Lalin [17, 18] computed the mean value of the cubic family over function fields and found results corresponding to those with a unitary symmetry. The works of these authors motivated the projects described in this thesis, where we study the same cubic family and similar families of quartic and sextic L-functions.

1.4.4 One-level density of zeros

The one-level density of zeros studies the average behavior of low-lying zeros for families of L-functions, $L(s,\chi)$ in the complex plane. More specifically, the one-level density formula in the Kummer case can be seen in (2.3), and non-Kummer case in (1.27). As in the case of moments, this statistic on families of L-functions corresponds to the distribution of eigenvalues of random matrices in the classical group.

In the works of Katz and Sarnak, the authors computed the one-level density for families of L-functions for curves over finite fields by taking both the genus and the size of the field to infinity. This allowed them to use deep equidistribution theorems by Deligne. Near the end of 2010s, Rudnick considered the one-level density in the hyperelliptic ensemble

$$Y^2 = F(t), \quad \text{for } F(t) \in \mathcal{H}_{2g+1},$$

without taking q to infinity. One cannot use the equidistribution theorems by Deligne in this case and the computation depended more on the arithmetic of the family.

Rudnick [44] proved that for $\mathcal{H}_{2g+1}$ the set of monic, squarefree polynomials of degree $2g+1$ over $\mathbb{F}_q[t]$, $\phi(\theta) = \sum_{|n|\leq N} \hat{\phi}(n)\, e(n\theta)$ a real, even trigonometric polynomial, where $e(x) = e^{2\pi i x}$ and $\Phi(2g\theta) = \phi(\theta)$, the one-level density of quadratic Dirichlet L-functions is

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \sum_{j=1}^{2g} \Phi(2g\theta_{j,D}) = \hat{\Phi}(0) - \frac{1}{g} \sum_{n \leq g} \hat{\Phi}(n/g) + \frac{dev(\Phi)}{g} + o(1/g).$$

Here we have

$$dev(\Phi) = \hat{\Phi}(0) \sum_{P \in \mathcal{P}} \frac{d(P)}{|P|^2 - 1} - \frac{\hat{\Phi}(1)}{q-1},$$

where the sum is over all monic irreducible polynomials P, and $d(P)$ denotes its degree. Bui and Florea computed the one-level density in the same family, obtained the result above along with extra lower order terms not predicted by the powerful Ratios Conjecture by further restricting the support of $\hat{\phi}$. Additionally, using the optimization in [30], Bui and Florea showed that more than 94% of the L-functions in the family do not vanish at the central point.

Katz and Sarnak predicted that the one-level density of zeros for higher order Dirichlet L-functions should have the unitary symmetry. As discussed in 1.3.1, we need to consider two different settings (Kummer and non-Kummer) for higher order Dirichlet characters. In the cubic Kummer setting over number fields under GRH, Cho and Park [8] studied the one-level density of cubic L-functions and obtained results matching those predicted by the Ratios Conjecture. David and Güloğlu computed the one-level density of a thin family of cubic Dirichlet L-functions under GRH and obtained a positive proportion of non-vanishing at $s = 1/2$ [19]. Gao and Zhao [23, 24] studied the one-level density of thin families of quartic and sextic L-functions over number fields under GRH, and obtained at least 5% and 2/45 of non-vanishing respectively.

Below is a possibly non-exhaustive summary of one-level density results for Dirichlet L-functions over number fields. All results assumed the truth of the Generalized Riemann Hypothesis.

Over number fields	Kummer	non-Kummer	Notable contributions
Cubic	Cho and Park [8]		matched the prediction of the Ratios Conjecture [10]
	David and Güloğlu [19]		showed $\geq 2/13$ non-vanishing for the thin family
Quartic	Gao and Zhao [23] for thin family		at least 5% non-vanishing for the thin family
Sextic	Gao and Zhao [24] for thin family		at least 2/45 non-vanishing for the thin family

Table 1.3: One-level density of zeros results over number fields.

Although there are no one-level density results in the non-Kummer setting over number fields in the literature (nor function fields besides the work in this thesis), we note that the work of Baier and Young [3] on the mean value of cubic Dirichlet L-functions studies the cubic non-Kummer family. The cubic non-Kummer family over function fields was studied by David, Florea and Lalin [17, 18].

1.5 Outline

We summarize Chapters 2 and 3 in this section. We also note an application of one-level density results, such as Theorems 1.3 and 1.5, to the question of non-vanishing at low-lying heights. This outline is written to be read as independently from the rest of this thesis as possible.

1.5.1 One-level density of cubic and quartic families in the Kummer setting

In Chapter 2, we compute the one-level density of zeros of cubic and quartic Dirichlet L-functions in the Kummer setting over function fields. In Section 2.1, we briefly discuss relevant works and reference results we obtained. We also prove the explicit formula for any order ℓ Dirichlet L-functions over function fields, which rewrites the sum over zeros to a sum over prime power.

For the cubic case, we consider the family of L-functions $\mathcal{L}_q(u, \chi_H)$ given by primitive cubic characters χ_H of genus g, where H denotes its conductor. Let $\mathcal{F}_3^{\mathrm{K}}(g)$ be the set of conductors of this family as seen in (1.8). Recall that for convenience, χ_H is odd and each $\mathcal{L}_q(u, \chi_H)$ has g non-trivial zeros. Furthermore, the Riemann Hypothesis over function fields implies that, under the change of variable $u = q^{-s}$, all non-trivial zeros lie on the circle $|u| = q^{-1/2}$, parametrized by their angles $\{\theta_{j,H}\}_{j=1}^{g}$.

Given a test function $\phi(\theta)$ in the Schwartz space $\mathcal{S}(\mathbb{R})$, the one-level density of zeros of cubic Dirichlet L-functions in the Kummer setting is

$$\mathcal{D}_3^{\mathrm{K}}(\phi, g) = \frac{1}{|\mathcal{F}_3^{\mathrm{K}}(g)|} \sum_{H \in \mathcal{F}_3^{\mathrm{K}}(g)} \sum_{j=1}^{g} \phi(\theta_{H,j}). \tag{1.22}$$

I proved the following.

Theorem 1.1 (Cubic Kummer setting). *Let $\phi(\theta) = \sum_{|n|\leq N}\hat{\phi}(n)e(n\theta)$ be a real, even trigonometric polynomial and $\Phi(g\theta) = \phi(\theta)$. The one-level density of zeros of cubic Dirichlet L-functions is*

$$\begin{aligned}\mathcal{D}_3^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0) &- \frac{2}{g}\sum_{1\leq n\leq N/3}\hat{\Phi}\left(\frac{3n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)} \\ &+4\mathfrak{Re}\left(\frac{h_1}{g}\sum_{1\leq n\leq N/3}\hat{\Phi}\left(\frac{3n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)\left(1-c_Q\right)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)}\right) \\ &+\frac{2h_2}{g}\sum_{1\leq n\leq N/3}\hat{\Phi}\left(\frac{3n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{2d(Q)^2|Q|^{-1}}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)^2} + O\left(q^{N/2}q^{-g/2}q^{\epsilon N}\right),\end{aligned} \tag{1.23}$$

where c_Q, h_1 and h_2 are explicitly defined in (2.10), (2.11) *and* (2.12) *respectively.*

We note that $|c_Q| < 2$, and h_1, h_2 are of order $1/g$ and the sums over n and Q are convergent in each lower order terms. Therefore, when $N < g$, all terms except for $\hat{\Phi}(0)$ vanishes as the genus g tends to infinity, as predicted by random matrix theory. The lower order terms are not predicted by random matrix theory.

For the quartic Dirichlet L-functions, we consider curves of the affine model

$$Y^4 = F_1(t)F_3^3(t),$$

where $F_1(t), F_3(t)$ are squarefree polynomials and $(F_1(t), F_3(t)) = 1$. This correspond to characters such that $\chi^4 = 1$ and χ^2 remains primitive. The Riemann-Hurwitz formula gives the relation between the genus and the degree of the conductors as seen in (1.10), which stays analogous to the prime case in this model. This implies that there are $2g/3$ non-trivial zeros for each L-function. With the definition of $\mathcal{F}_4^{\mathrm{K}}(g)$ correspondingly modified in (1.11), the one-level density of zeros in the Kummer setting for quartic Dirichlet L-functions is

$$\mathcal{D}_4^{\mathrm{K}}(\phi, g) = \frac{1}{|\mathcal{F}_4^{\mathrm{K}}(g)|} \sum_{H \in \mathcal{F}_4^{\mathrm{K}}(g)} \sum_{j=1}^{2g/3} \phi(\theta_{H,j}). \tag{1.24}$$

I proved the following statement.

Theorem 1.2. *Let $\phi(\theta) = \sum_{|n|\le N} \hat{\phi}(n)e(n\theta)$ be any real, even trigonometric polynomial and $\Phi\left(\frac{2g\theta}{3}\right) = \phi(\theta)$. The one-level density of zeros of quartic Dirichlet L-functions is*

$$\begin{aligned}\mathcal{D}_4(\phi, g) =& \hat{\Phi}(0) - \frac{3}{g} \sum_{1\le n\le N/4} \hat{\Phi}\left(\frac{6n}{g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{d(Q)}{|Q|^{2r}(1+2|Q|^{-1})} \\ &- \frac{3s_2}{g} \sum_{1\le n\le N/4} \hat{\Phi}\left(\frac{6n}{g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{2d(Q)^2|Q|^{-1}}{|Q|^{2r}(1+2|Q|^{-1})^2} + O\left(q^{N/2}q^{-G/2}q^{\epsilon N}\right),\end{aligned} \tag{1.25}$$

where $G = \frac{2g}{3}+1$ is the degree of the conductor, and s_2 is explicitly defined in (2.18).

Note that the constant s_2 is of order $1/g$. Therefore, similar to Theorem 1.1, when $N < \frac{2g}{3}+1$ every term except for $\hat{\Phi}(0)$ vanishes as the genus g tends to infinity.

Computing the limit as the genus $g \to \infty$, I proved that both families correspond to the unitary symmetry, confirming the suggestion of Katz and Sarnak's philosophy [31, 32]. We note that the condition $N < D_\ell(g) - 2$ below is equivalent to $\hat{\Phi}$ being supported in (-1,1), analogous to the Fourier transform of the test function having limited support over number fields.

Theorem 1.3 (Symmetry type of the Kummer families)**.** *Let $\phi(\theta) = \sum_{|n|\le N} \hat{\phi}(n)e(n\theta)$ be any real, even trigonometric polynomial and $\Phi((D_\ell(g)-2)\theta) = \phi(\theta)$. For $N < D_\ell(g) - 2 =$*

$\dfrac{2g}{\ell-1}$, *we have in the Kummer setting for* $\ell = 3, 4$,

$$\lim_{g\to\infty} \mathcal{D}_\ell^{\mathrm{K}}(\phi, g) = \int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{\mathcal{W}}_{U(D_\ell(g)-2)}(y)dy + o(1). \tag{1.26}$$

Here $\hat{\mathcal{W}}_{U(D_\ell(g)-2)}(y) = \delta_0(y)$ *denotes the one-level scaling density of the group of unitary matrices seen in* (1.18).

1.5.2 One-level density of cubic, quartic and sextic families in the non-Kummer setting

In Chapter 3, we study the families of cubic, quartic and sextic Dirichlet L-functions in the non-Kummer setting. As discussed in Section 1.4.4, Baier and Young [3] studied the cubic non-Kummer Dirichlet characters in their work on the mean value of cubic Dirichlet L-functions over number fields. They classified the characters as the restriction of those in the Kummer setting in $\mathbb{Z}[\omega]$ for $\omega = e^{2\pi i/3}$, and suggested that the cases of quartic and sextic characters in the non-Kummer setting should be similar. Bary-Soroker and Meisner [4] found analogous constructions over function fields and David, Florea and Lalin [17, 18] studied cubic non-Kummer characters over function fields analogously to that of Baier and Young over number fields.

We describe this interpretation of non-Kummer characters for the cubic family, and extend it to families of quartic and sextic Dirichlet L-functions. The set of conductors for the corresponding family is denoted $\mathcal{F}_\ell^{\mathrm{nK}}(g)$ and explicitly defined in (3.3).

The one-level density of order ℓ Dirichlet L-functions in the non-Kummer setting for $\ell \in$

$\{3,4,6\}$ is

$$\mathcal{D}_{\ell}^{\mathrm{nK}}(\phi, g)=\frac{1}{\left|\mathcal{F}_{\ell}^{\mathrm{nK}}(g)\right|} \sum_{H \in \mathcal{F}_{\ell}^{\mathrm{nK}}(g)} \sum_{j=1}^{\frac{2g}{\ell-1}} \phi\left(\theta_{H, j}\right). \tag{1.27}$$

I proved the following.

Theorem 1.4. *Let $\phi(\theta)=\sum_{|n| \leq N} \hat{\phi}(n) e(n \theta)$ be any real, even trigonometric polynomial and $\Phi\left(\frac{2g\theta}{\ell-1}\right)=\phi(\theta)$. The one-level density of zeros of order ℓ Dirichlet L-functions in the non-Kummer setting for $\ell \in\{3,4,6\}$ is*

$$\begin{aligned}\mathcal{D}_{\ell}^{\mathrm{nK}}(\phi, g)=&\ \hat{\Phi}(0)-\frac{\ell-1}{g} \sum_{1 \leq n \leq N} \hat{\Phi}\left(\frac{(\ell-1) n}{2g}\right) q^{-n/2} \\ &-\frac{\ell-1}{g} \sum_{1 \leq n \leq N/\ell} \hat{\Phi}\left(\frac{\ell(\ell-1) n}{2g}\right) \sum_{\substack{Q \in \mathcal{P}_{q, n/r} \\ r \geq 1}} \frac{d(Q)}{|Q|_q^{\ell r/2}\left(1+|Q|_q^{-2/m_Q}\right)^{m_Q}}+O\left(q^{N/2} q^{-D_{\ell}(g)/2} q^{\epsilon(N+g)}\right),\end{aligned} \tag{1.28}$$

where $m_Q=\operatorname{gcd}(d(Q), 2)$.

Using Theorem 1.4, I proved that the families of cubic, quartic and sextic Dirichlet L-functions in the non-Kummer setting have the unitary symmetry.

Theorem 1.5 (Symmetry type of the non-Kummer families)**.** *Let $\phi(\theta)=\sum_{|n| \leq N} \hat{\phi}(n) e(n \theta)$ be any real, even trigonometric polynomial and $\Phi\left(\left(D_{\ell}(g)-2\right) \theta\right)=\phi(\theta)$. For $N<D_{\ell}(g)-2=\frac{2g}{\ell-1}$,*

$$\lim_{g \rightarrow \infty} \Sigma_{\ell}^{\mathrm{nK}}(\Phi, g)=\int_{-\infty}^{\infty} \hat{\Phi}(y) \hat{\mathcal{W}}_{U\left(D_{\ell}(g)-2\right)}(y) d y+o(1), \tag{1.29}$$

for $\ell \in\{3,4,6\}$ in the non-Kummer setting. Here $\hat{\mathcal{W}}_{U\left(D_{\ell}(g)-2\right)}(y)=\delta_0(y)$ denotes the one-level scaling density of the group of unitary matrices as seen in (1.18)*.*

1.5.3 Remark on the application to non-vanishing at low-lying heights at the central point

Modifying results from a recent paper by Carneiro, Chirre, and Milinovich [7], and Theorems 1.3 and 1.5 above on symmetry types, one can prove results on the average order of non-vanishing at $s = 1/2 + it$ for small $t > 0$ for families considered in this thesis.

Following the notation in [7], the reproducing kernel for unitary symmetry, where the Fourier transform of the test function is in $(-\Delta, \Delta)$, is given by

$$K_{U,\pi\Delta}(w, z) = \frac{\sin \pi\Delta (z - \overline{w})}{\pi (z - \overline{w})}. \tag{1.30}$$

Given that the families of cubic and quartic Dirichlet L-functions in the Kummer setting and cubic, quartic, and sextic Dirichlet L-functions in the non-Kummer setting are of unitary symmetry, and the support of $\hat{\Phi}$ is in $(-1, 1)$, Theorem 2 of [7] states that for these families the average order of vanishing at $s = 1/2 + it$ for $t > 0$ is at most

$$\frac{1}{K_{U,\pi}(t,t) + |K_{U,\pi}(t, -t)|} = \frac{1}{1 + \left|\frac{\sin(2\pi t)}{2\pi t}\right|}. \tag{1.31}$$

Thus the average order of non-vanishing at $s = 1/2 + it$ for $t > 0$ is at least

$$n(t) = 1 - \frac{1}{1 + \left|\frac{\sin(2\pi t)}{2\pi t}\right|} < \frac{1}{2}, \tag{1.32}$$

where for $t \in (0, 0.5)$, $n(t)$ is decreasing and $\lim_{t \to 0} n(t) = \frac{1}{2}$.

Chapter 2

One-level density of zeros of Dirichlet L-functions in the Kummer setting

2.1 Introduction

As mentioned in Section 1.3.1, for families of order $\ell \geq 3$ Dirichlet characters over function fields, either $\mathbb{F}_q^\times$ contains an ℓ^{th} root of unity, i.e., $q \equiv 1 \pmod{\ell}$, or otherwise when $q \not\equiv 1 \pmod{\ell}$. The former scenario is called the Kummer setting, which is the focus of this chapter.

The one-level density of zeros for quadratic Dirichlet L-functions over number fields was computed by Özlük and Snyder in 1993 under the Generalized Riemann Hypothesis [39]. Many conditional one-level density results over number fields, some with the thin subfamily restriction, were computed in the works such as [8, 19, 23, 24] as mentioned in Section 1.4.4. Over function fields, Rudnick [44] and Bui and Florea [6] computed the one-level density for quadratic Dirichlet L-functions. The works of these authors and the study of cubic characters in [17, 18] of David, Florea and Lalin motivated the work in this chapter.

Let $\mathcal{F}_3^{\mathrm{K}}(g)$ be as defined in (1.8) for $\ell = 3$, the set of conductors of primitive cubic Dirichlet characters with genus g in the Kummer setting. The one-level density of zeros of cubic Dirichlet L-functions in the Kummer setting is defined in (1.22) and I proved the statement as seen in Theorem 1.1.

We highlight that for $N < g$ and $g \to \infty$, the main term matches that predicted by random matrix theory. The lower order terms are not predicted by random matrix theory.

For the quartic Dirichlet L-functions, we consider curves of the affine model

$$Y^4 = F_1(t)F_3^3(t),$$

where $F_1(t), F_3(t)$ are square-free and $(F_1(t), F_3(t)) = 1$. This corresponds to conductors $\mathcal{F}_4^{\mathrm{K}}(g)$ as defined in (1.11). The one-level density of zeros in the Kummer setting for quartic Dirichlet L-functions is defined in (1.24) and I proved Theorem 1.2.

As a consequence of Theorems 1.1 and 1.2, I prove the result in Theorem 1.3, which states that the one-level density for both families corresponds to the unitary symmetry type as q fixed and $g \to \infty$. This confirmed suggestions of Katz and Sarnak's philosophy [31, 32]. We note that the condition $N < D_\ell(g) - 2$ is equivalent to $\hat{\Phi}$ being supported in (-1,1), analogous to the Fourier transform of the test function having limited support over number fields.

2.2 The Explicit Formula

A key step in the computation of the one-level density is to rewrite the sum over the zeros of $\mathcal{L}_q(u, \chi)$ to a sum over prime powers. We call this type of equation the explicit formula. We prove the following statement for order ℓ Dirichlet L-functions for any integer $\ell \geq 3$ for both even and odd characters. (We will use the case of odd characters in Chapter 2 and even characters in Chapter 3.)

We use notations given in the beginning of Section 1.3 and recall that $\Lambda(f)$ is the von Mangoldt function given in (1.3).

Lemma 2.1. *Let $\chi_F, \overline{\chi_F}$ be Dirichlet characters of conductor $F \in \mathbb{F}_q[t]$ and $\{\theta_{j,F}\}$ be the angles of non-trivial zeros of the L-function $\mathcal{L}_q(u, \chi_F)$ as seen in* (1.13)*. For any $n \in \mathbb{N}$,*

$$-\sum_{j=1}^{D} e(n\theta_{j,F}) = \frac{b}{q^{n/2}} + \sum_{f \in \mathcal{M}_{q,n}} \frac{\Lambda(f)\chi_F(f)}{|f|^{1/2}}. \tag{2.1}$$

When $n < 0$,

$$-\sum_{j=1}^{D} e(n\theta_{j,F}) = \frac{b}{q^{|n|/2}} + \sum_{f \in \mathcal{M}_{q,|n|}} \frac{\Lambda(f)\overline{\chi_F(f)}}{|f|^{1/2}}. \tag{2.2}$$

Here $D = \deg(F) - 1 - b$ denotes the number of non-trivial zeros, in which $b = 1$ if χ_F is even and $b = 0$ if χ_F is odd.

Proof. We compute the explicit formula for $n \geq 0$ first.

In terms of its zeros

$$\mathcal{L}_q(u, \chi_F) = (1-u)^b \prod_{j=1}^{D} (1 - u\sqrt{q}e(\theta_{j,F})),$$

for some integer $b \in \{0, 1\}$, where $b = 1$ if and only if χ_F is an even character. Here $D = d(F) - 1 - b$ denotes the number of non-trivial zeros, where F is the conductor of χ_F. Alternatively, we can express the L-functions as the product over primes

$$\mathcal{L}_q(u, \chi_F) = \prod_{P \in \mathcal{P}_q} \left(1 - \chi_F(P)u^{d(P)}\right)^{-1}.$$

Using log differentiation on the two expressions of $\mathcal{L}_q(u, \chi_F)$ and setting them equal, we obtain

$$\frac{-b}{1-u} + \sum_{j=1}^{D} \frac{-\sqrt{q}e(\theta_{j,F})}{1-u\sqrt{q}e(\theta_{j,F})} = \sum_{P \in \mathcal{P}_q} \frac{d(P)\chi_F(P)u^{d(P)-1}}{1-\chi_F(P)u^{d(P)}}.$$

Now, expanding the denominators using geometric series, we have

$$\sum_{n=0}^{\infty} \left[-b + \sum_{j=1}^{D} -\sqrt{q}e(\theta_{j,F}) \left(\sqrt{q}e(\theta_{j,F}) \right)^n \right] u^n = \sum_{n=0}^{\infty} \sum_{P \in \mathcal{P}_q} d(P)\chi_F^{n+1}(P)u^{(n+1)d(P)-1}.$$

Rewriting using the von Mangoldt function, the sum over primes P on the right can be expressed as $\sum_{\substack{f=P^{n+1} \\ P \in \mathcal{P}_q}} \Lambda(f)\chi_F(f)u^{d(f)-1}$, hence we write

$$\sum_{n=0}^{\infty} \left[-b + \sum_{j=1}^{D} -\sqrt{q}e(\theta_{j,F}) \left(\sqrt{q}e(\theta_{j,F}) \right)^n \right] u^n = \sum_{n=0}^{\infty} \sum_{f \in \mathcal{M}_{q,n+1}} \Lambda(f)\chi_F(f)u^n.$$

We match corresponding terms with the same power on u from both sides and obtain

$$-\sum_{j=1}^{D} e(n\theta_{j,F}) = \frac{b}{q^{n/2}} + \sum_{f \in \mathcal{M}_{q,n}} \frac{\Lambda(f)\chi_F(f)}{|f|^{1/2}}$$

as given in the statement of the lemma.

Now for $n < 0$, using the functional equation in (1.14), we see that if $\{q^{-1/2}e(-\theta_{j,F})\}$ are zeros of $\mathcal{L}_q(u, \chi_F)$, then $\{q^{-1/2}e(\theta_{j,F})\}$ are zeros of $\mathcal{L}_q(u, \overline{\chi_F})$. Thus doing the computations above for $\overline{\chi_F}$, we derive that for $n < 0$

$$-\sum_{j=1}^{D} e(n\theta_{j,F}) = \frac{b}{q^{|n|/2}} + \sum_{f \in \mathcal{M}_{q,|n|}} \frac{\Lambda(f)\overline{\chi_F(f)}}{|f|^{1/2}}.$$

□

2.3 Preliminary Computations

In this section, we derive main terms and the error term of the one-level density computation for a general order ℓ in the Kummer setting. For $\ell = 3, 4$, we prove a lemma that helps to evaluate the character sum over the family $\mathcal{F}_\ell^{\mathrm{K}}(g)$.

Recall that the one-level density of zeros for order ℓ Dirichlet L-functions is

$$\mathcal{D}_\ell^{\mathrm{K}}(\phi, g) = \frac{1}{|\mathcal{F}_\ell^{\mathrm{K}}(g)|} \sum_{H \in \mathcal{F}_\ell^{\mathrm{K}}(g)} \sum_{j=1}^{D_\ell(g)-2} \phi(\theta_{H,j}), \tag{2.3}$$

where $\mathcal{F}_\ell^{\mathrm{K}}(g)$ denotes the set of conductors defined in (1.8) for prime ℓ and (1.11) for $\ell = 4$. $D_\ell(g) - 2 = \dfrac{2g}{\ell - 1}$ is the number of non-trivial zeros of the L-functions given in (1.15) and $\phi(\theta) = \sum_{|n| \leq N} \hat{\phi}(n)\, e(n\theta)$ is any real, even trigonometric polynomial.

Let $\Phi((D_\ell(g) - 2)\theta) = \phi(\theta)$. Since

$$\Phi((D_\ell(g) - 2)\theta_{j,F}) = \frac{1}{D_\ell(g) - 2} \sum_{|n| \leq N} \hat{\Phi}\left(\frac{n}{D_\ell(g) - 2}\right) e(n\theta_{j,F}),$$

the sum over zeros in (2.3) can be written as

$$\sum_{j=1}^{D_\ell(g)-2} \Phi((D_\ell(g) - 2)\theta_{j,F}) = \hat{\Phi}(0) + \frac{1}{D_\ell(g) - 2} \sum_{0 < |n| \leq N} \hat{\Phi}\left(\frac{n}{D_\ell(g) - 2}\right) \sum_{j=1}^{D_\ell(g)-2} e(n\theta_{j,F}).$$

Applying the explicit formula on the inner most sum by using (2.1) and (2.2) for $b = 0$ and $D = D_\ell(g) - 2$, we have

$$\sum_{j=1}^{D_\ell(g)-2} \Phi((D_\ell(g) - 2)\theta_{j,F}) \\ = \hat{\Phi}(0) - \frac{1}{D_\ell(g) - 2} \sum_{1 \leq n \leq N} \hat{\Phi}\left(\frac{n}{D_\ell(g) - 2}\right) \sum_{f \in \mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \left[\chi_H(f) + \overline{\chi_H(f)}\right].$$

Thus, we write the one-level density in (2.3) as

$$\mathcal{D}_\ell^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0) - \mathcal{A}_\ell^{\mathrm{K}}(\phi, g), \tag{2.4}$$

where we let

$$\mathcal{A}_\ell^{\mathrm{K}}(\phi, g) = \frac{1}{(D_\ell(g)-2)|\mathcal{F}_\ell^{\mathrm{K}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right) \sum_{f\in\mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{H\in\mathcal{F}_\ell^{\mathrm{K}}(g)} \left[\chi_H(f) + \overline{\chi_H(f)}\right]. \tag{2.5}$$

We decompose $\mathcal{A}_\ell^{\mathrm{K}}(\phi, g)$ as the sum

$$\mathcal{A}_\ell^{\mathrm{K}}(\phi, g) = M_\ell^{\mathrm{K}}(\phi, g) + E_\ell^{\mathrm{K}}(\phi, g),$$

where its main term $M_\ell^{\mathrm{K}}(\phi, g)$ comes from f being an ℓ^{th} power

$$\begin{aligned} &M_\ell^{\mathrm{K}}(\phi, g) \\ &= \frac{1}{(D_\ell(g)-2)|\mathcal{F}_\ell^{\mathrm{K}}(g)|} \sum_{1\le n\le N/\ell} \hat{\Phi}\left(\frac{\ell n}{D_\ell(g)-2}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r} \\ r\ge 1}} \frac{d(Q)}{|Q|^{\ell r/2}} \sum_{H\in\mathcal{F}_\ell^{\mathrm{K}}(g)} \left[\chi_H\left(Q^{\ell r}\right) + \overline{\chi_H\left(Q^{\ell r}\right)}\right], \end{aligned} \tag{2.6}$$

and the non-ℓ^{th} power contribution is

$$\begin{aligned} &E_\ell^{\mathrm{K}}(\phi, g) \\ &= \frac{2}{(D_\ell(g)-2)|\mathcal{F}_\ell^{\mathrm{K}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right) \sum_{\substack{f\in\mathcal{M}_{q,n} \\ f \text{ non-}\ell^{\text{th}} \text{ power}}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{H\in\mathcal{F}_\ell^{\mathrm{K}}(g)} \left[\chi_H(f) + \overline{\chi_H(f)}\right]. \end{aligned} \tag{2.7}$$

These equations hold for $\ell = 3, 4$ and q an odd prime power coprime to ℓ.

Character sum lemmas

Recall that for $\ell = 3$, the set of conductors of primitive cubic characters $\mathcal{F}_3^{\mathrm{K}}(g)$ as given by (1.8) is

$$\mathcal{F}_3^{\mathrm{K}}(g) := \{H : \ H = F_1F_2 \in \mathcal{H}_{q,g+1}, \ (F_1,F_2)=1, \ \deg(F_1) + 2\deg(F_2) \equiv 1 \pmod 3\}.$$

Thus we have the following lemma.

Lemma 2.2. *Let q be a prime power coprime to 6, $f \in \mathbb{F}_q[t]$ be a monic polynomial, and χ_3 be as defined in* (1.7)*. Then*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^\times}=\chi_3}} \chi(f) = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_3\equiv 1 \pmod 3}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ (F_1,f)=1}} \chi_{F_1}(f) \sum_{\substack{F_2\in\mathcal{H}_{q,d_2} \\ (F_2,F_1f)=1}} \chi_{F_2}^2(f).$$

For the quartic case, recall that we defined $\mathcal{F}_4^{\mathrm{K}}(g)$ in (1.11).

Lemma 2.3. *Let q be an odd prime power, $f \in \mathbb{F}_q[t]$ be a monic polynomial, and χ_4 be as defined in* (1.7)*. Then*

$$\sum_{\substack{\chi \text{ primitive quartic} \\ \chi^2 \text{primitive} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^\times}=\chi_4}} \chi(f) = \sum_{\substack{d_1+d_3=\frac{2g}{3}+1 \\ d_1+3d_3\equiv 1 \pmod 4}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ (F_1,f)=1}} \chi_{F_1}(f) \sum_{\substack{F_3\in\mathcal{H}_{q,d_3} \\ (F_3,F_1f)=1}} \chi_{F_3}^3(f).$$

We use these lemmas in the computations of (2.6) and (2.7) for $\ell = 3, 4$ in sections below.

2.4 Cubic Dirichlet L-functions

2.4.1 The Main Term

First we compute the following sum over the family of cubic Dirichlet L-functions with primitive cubic characters of genus g and derive the size of the family $\left|\mathcal{F}_3^{\mathrm{K}}(g)\right|$.

Lemma 2.4. *For f a monic polynomial, let*

$$\mathcal{T}_1 := \sum_{\substack{F\in\mathcal{F}_3^{\mathrm{K}}(g)\\(F,f)=1}} 1 = \sum_{\substack{d_1+d_2=g+1\\d_1+2d_2\equiv 1 \pmod 3}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\(F_1,f)=1}} \sum_{\substack{F_2\in\mathcal{H}_{q,d_2}\\(F_2,F_1f)=1}} 1.$$

Then

$$\mathcal{T}_1 = \frac{q^{g+1}}{3}\left[(g+2)\,J\left(\frac{1}{q},\frac{1}{q}\right) - \frac{\frac{d}{du_1}J(u_1,u_1)\mid_{1/q}}{q} - 2\mathfrak{Re}\left(\frac{J\left(\frac{1}{q},\frac{\zeta_3}{q}\right)\zeta_3^{1+2a}}{1-\zeta_3}\right)\right] + O\left(q^{g/3+\epsilon g}\right),$$

where

$$J(x,y) = \prod_{P\in\mathcal{P}_q}\left[\left(1+x^{d(P)}+y^{d(P)}\right)\left(1-x^{d(P)}\right)\left(1-y^{d(P)}\right)\right]\prod_{\substack{P\in\mathcal{P}_q\\P\mid f}}\left(1+x^{d(P)}+y^{d(P)}\right)^{-1},$$

and ζ_3 denotes the 3^{th} root of unity, $\zeta_3 = e^{2\pi i/3}$.

Proof. We consider the generating series for the sums over F_1 and F_2 in $\mathcal{T}_1$, which is

$$\mathcal{S}(u_1,u_2) = \sum_{\substack{F_1\in\mathcal{H}_q\\(F_1,f)=1}} u_1^{d(F_1)} \sum_{\substack{F_2\in\mathcal{H}_q\\(F_2,F_1f)=1}} u_2^{d(F_2)}.$$

Now, $\mathcal{S}(u_1,u_2)$ can be written as the product over primes as

$$\mathcal{S}(u_1,u_2)=\frac{\prod_{P\in\mathcal{P}_q}\left(1+\frac{u_1^{d(P)}}{1+u_2^{d(P)}}\right)\prod_{P\in\mathcal{P}_q}\left(1+u_2^{d(P)}\right)}{\prod_{\substack{P\in\mathcal{P}_q\\ P|f}}\left(1+\frac{u_1^{d(P)}}{1+u_2^{d(P)}}\right)\prod_{\substack{P\in\mathcal{P}_q\\ P|f}}\left(1+u_2^{d(P)}\right)}=\frac{\prod_{P\in\mathcal{P}_q}\left(1+u_1^{d(P)}+u_2^{d(P)}\right)}{\prod_{\substack{P\in\mathcal{P}_q\\ P|f}}\left(1+u_1^{d(P)}+u_2^{d(P)}\right)}.$$

Hence

$$\mathcal{S}(u_1,u_2)=\mathcal{Z}_q(u_1)\mathcal{Z}_q(u_2)J(u_1,u_2),$$

where

$$J(u_1,u_2)=\prod_{P\in\mathcal{P}_q}\left(1+u_1^{d(P)}+u_2^{d(P)}\right)\left(1-u_1^{d(P)}\right)\left(1-u_2^{d(P)}\right)\prod_{\substack{P\in\mathcal{P}_q\\ P|f}}\left(1+u_1^{d(P)}+u_2^{d(P)}\right)^{-1}$$

$$=\prod_{P\in\mathcal{P}}\left(1-u_1^{2d(P)}-u_2^{2d(P)}-(u_1u_2)^{d(P)}+(u_1^2u_2)^{d(P)}+(u_1u_2^2)^{d(P)}\right)\prod_{\substack{P\in\mathcal{P}_q\\ P|f}}\left(1+u_1^{d(P)}+u_2^{d(P)}\right)^{-1}.$$

Note that $J(u_1,u_2)$ has analytic continuation when $|u_1|<q^{-1/3}$ and $|u_2|<q^{-1/3}$.

Using Perron's formula twice, we have

$$\mathcal{T}_1=\sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \pmod 3}}\frac{1}{(2\pi i)^2}\oint\oint\frac{J(u_1,u_2)}{(1-qu_1)(1-qu_2)u_1^{d_1}u_2^{d_2}}\frac{du_2}{u_2}\frac{du_1}{u_1}.$$

Since g is fixed, we let $g\equiv b \pmod 3$ for some $b\in\{0,1,2\}$ and take the difference of equations

$$d_1+d_2=g+1,\ \text{and}\ \ d_1+2d_2\equiv 1 \pmod 3$$

to obtain that

$$d_2\equiv b \pmod 3.$$

Thus for some integer k and a fixed $a\in\{0,1,2\}$,

$$d_1=3k+a.$$

By analyzing the cases, the sum over d_1 is a sum over integers $0 \leq k \leq [g/3]$, where $[x]$ denotes the closest integer to x.

Computing the sum inside the integrals first,

$$\begin{aligned}\mathcal{T}_1 &= \sum_{k=0}^{[g/3]} \frac{1}{(2\pi i)^2} \oint \oint \frac{J(u_1,u_2)}{(1-qu_1)(1-qu_2)u_1^{3k+a}u_2^{g+1-3k-a}} \frac{du_2}{u_2}\frac{du_1}{u_1} \\ &= \frac{1}{(2\pi i)^2} \oint_{|u_1|=q^{-3}} \oint_{|u_2|=q^{-2}} \frac{J(u_1,u_2)}{(1-qu_1)(1-qu_2)(u_2^3-u_1^3)} \left[\frac{u_2^{2+a-b}}{u_1^{g+a-b}} - \frac{u_1^{3-a}}{u_2^{g+1-a}}\right] \frac{du_2}{u_2}\frac{du_1}{u_1}.\end{aligned}$$

We write the integral above as the difference of two integrals. Note that the second one vanishes since the integrand over u_1 has no poles inside the circle $|u_1| = q^{-3}$.

Hence

$$\mathcal{T}_1 = \frac{1}{(2\pi i)^2} \oint_{|u_1|=q^{-3}} \oint_{|u_2|=q^{-2}} \frac{u_2^{2+a-b} J(u_1,u_2)}{u_1^{g+a-b}(1-qu_1)(1-qu_2)(u_2^3-u_1^3)} \frac{du_2}{u_2}\frac{du_1}{u_1},$$

where the integrand has poles $u_1, \zeta_3 u_1, \zeta_3^2 u_1$ integrating over u_2. Here ζ_3 denotes the third root of unity $e^{2\pi i/3}$.

Computing the residue at the poles above, we have

$$\mathcal{T}_1 = \frac{1}{2\pi i} \oint_{|u_1|=q^{-3}} \frac{1}{3u_1^{g+1}(1-qu_1)} \left[\frac{J(u_1,u_1)}{1-qu_1} + \frac{J(u_1,\zeta_3 u_1)\zeta_3^{2+a-b}}{1-q\zeta_3 u_1} + \frac{J(u_1,\zeta_3^2 u_1)\zeta_3^{2(2+a-b)}}{1-q\zeta_3^2 u_1}\right] \frac{du_1}{u_1}. \tag{2.8}$$

Now to integrate over u_1, we write (2.8) as the sum of three integrals

$$\mathcal{T}_1 = T_1 + T_2 + T_3. \tag{2.9}$$

For each T_i, we shift the contour to $|u_1| = q^{-1/3+\epsilon}$, compute the residue at the corresponding poles at either $1/q$, $1/(\zeta_3 q)$, or $1/(\zeta_3^2 q)$ and bound the integral on the circle $|u_1| = q^{-1/3+\epsilon}$.

T_1 has a double pole at $1/q$, and we obtain

$$\begin{aligned} T_1 =& \frac{1}{2\pi i}\oint_{|u_1|=q^{-3}} \frac{J(u_1,u_1)}{3u_1^{g+1}(1-qu_1)^2}\frac{du_1}{u_1} \\ =& \frac{q^{g+1}}{3}\left[(g+2)J\left(\frac{1}{q},\frac{1}{q}\right) - \frac{\frac{d}{du_1}J(u_1,u_1)\mid_{1/q}}{q}\right] + O\left(q^{g/3+\epsilon g}\right). \end{aligned}$$

T_2 has two simple poles at $u_1 = 1/q$ and $u_1 = 1/(\zeta_3 q)$. Hence we have

$$\begin{aligned} T_2 =& \frac{1}{2\pi i}\oint_{|u_1|=q^{-3}} \frac{\zeta_3^{2+a-b}J(u_1,\zeta_3 u_1)}{3u_1^{g+1}(1-qu_1)(1-\zeta_3 qu_1)}\frac{du_1}{u_1} \\ =& \frac{q^{g+1}}{3}\left[\frac{J\left(\frac{1}{q},\frac{\zeta_3}{q}\right)\zeta_3^{1+2a}}{1-\zeta_3} + \frac{J\left(\frac{\zeta_3^2}{q},\frac{1}{q}\right)\zeta_3^{a}}{1-\zeta_3^2}\right] + O\left(q^{g/3+\epsilon g}\right). \end{aligned}$$

T_3 has two simple poles at $u_1 = 1/q$ and $u_1 = 1/(\zeta_3^2 q)$. We obtain

$$\begin{aligned} T_3 =& \frac{1}{2\pi i}\oint_{|u_1|=q^{-3}} \frac{\zeta_3^{2(2+a-b)}J\left(u_1,\zeta_3^2 u_1\right)}{3u_1^{g+1}(1-qu_1)(1-\zeta_3^2 qu_1)}\frac{du_1}{u_1} \\ =& \frac{q^{g+1}}{3}\left[\frac{J\left(\frac{1}{q},\frac{\zeta_3^2}{q}\right)\zeta_3^{2+a}}{1-\zeta_3^2} + \frac{J\left(\frac{\zeta_3}{q},\frac{1}{q}\right)\zeta_3^{2a}}{1-\zeta_3}\right] + O\left(q^{g/3+\epsilon g}\right) \\ =& \zeta_3^2 T_2. \end{aligned}$$

where the last equality holds since $J(u_1,u_2) = J(u_2,u_1)$. Hence, we can reduce (2.9) to

$$\mathcal{T}_1 = T_1 + T_2 + \zeta_3^2 T_2 = T_1 - \zeta_3 T_2.$$

Plugging in the value of T_1, we have

$$\begin{aligned} \mathcal{T}_1 = \frac{q^{g+1}}{3}&\left[(g+2)J\left(\frac{1}{q},\frac{1}{q}\right) - \frac{\frac{d}{du_1}J(u_1,u_1)\mid_{1/q}}{q} - \frac{J\left(\frac{1}{q},\frac{\zeta_3}{q}\right)\zeta_3^{1+g}}{1-\zeta_3} - \frac{J\left(\frac{\zeta_3^2}{q},\frac{1}{q}\right)\zeta_3^{2+2g}}{1-\zeta_3^2}\right] \\ &+O\left(q^{g/3+\epsilon g}\right), \end{aligned}$$

where we use the fact $2g+1 \equiv a \pmod 3$. □

The size of the family is the following, obtained by removing the coprimality condition of f in $\mathcal{T}_1$ as seen in Lemma 2.4.

Corollary 2.1. *Let* $J_0(u_1,u_2)=\prod_{P\in\mathcal{P}}\left(1+u_1^{d(P)}+u_2^{d(P)}\right)\left(1-u_1^{d(P)}\right)\left(1-u_2^{d(P)}\right)$, *then*

$$\left|\mathcal{F}_3^{\mathrm{K}}(g)\right|=\frac{q^{g+1}}{3}\left[(g+2)\,J_0\left(\frac{1}{q},\frac{1}{q}\right)-\frac{\frac{d}{du_1}J_0(u_1,u_1)\mid_{1/q}}{q}-\frac{J_0\left(\frac{1}{q},\frac{\zeta_3}{q}\right)\zeta_3^{1+g}}{1-\zeta_3}-\frac{J_0\left(\frac{\zeta_3^2}{q},\frac{1}{q}\right)\zeta_3^{2+2g}}{1-\zeta_3^2}\right]$$
$$+O\left(q^{g/3+\epsilon g}\right).$$

Now we compute the main term $M_3^{\mathrm{K}}(\phi,g)$ of the one-level density of zeros of cubic Dirichlet L-functions in the Kummer setting. We use notations in Section 1.3 as needed.

Lemma 2.5. *Recall the main term of the one-level density of zeros in the Kummer setting is given in* (2.6). *Let* $\ell=3$ *for cubic Dirichlet L-functions with characters of genus* g, *we have*

$$\begin{aligned}M_3^{\mathrm{K}}(\phi,g)=&\frac{2}{g}\sum_{1\le n\le N/3}\hat{\Phi}\left(\frac{3n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}}\frac{d(Q)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)}\\&+4\mathfrak{Re}\left(\frac{h_1}{g}\sum_{1\le n\le N/3}\hat{\Phi}\left(\frac{3n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}}\frac{d(Q)\left(1-c_Q\right)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)}\right)\\&+\frac{2h_2}{g}\sum_{1\le n\le N/3}\hat{\Phi}\left(\frac{3n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}}\frac{2d(Q)^2|Q|^{-1}}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)^2}+O\left(q^{-2g/3}q^{\epsilon g}\right),\end{aligned}$$

where c_Q,h_1,h_2 *are explicitly defined in* (2.10), (2.11) *and* (2.12) *respectively.*

Proof. Recall that for $\ell=3$, the degree of the conductor is $D_3(g)-2=g$ given by (1.16), and the main term of the one-level density of zeros described in (2.6) comes from when f is a cube. For some monic irreducible polynomial Q and some positive integer r, let

$$f = Q^{3r}.$$

Thus $\chi_F(f) = \overline{\chi_F(f)} = 1$, and we can write

$$M_3^{\mathrm{K}}(\phi, g) = \frac{2}{g|\mathcal{F}_3^{\mathrm{K}}(g)|} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{d(Q)\mathcal{T}_1}{|Q|^{3r/2}}.$$

Let

$$D_f(u_1, u_2) = \prod_{\substack{P \in \mathcal{P}_q \\ P | f}} \left(1 + u_1^{d(P)} + u_2^{d(P)}\right)^{-1}.$$

Since $f = Q^{3r}$, we have

$$D_f(u_1, u_1) = D_Q(u_1, u_1) = \left(1 + u_1^{d(Q)} + u_2^{d(Q)}\right)^{-1},$$

and

$$J(u_1, u_2) = J_0(u_1, u_2) D_Q(u_1, u_2).$$

Now, using Lemma 2.4 for $\mathcal{T}_1$, we can write the sum over Q as two sums

$$H_1 + H_2 + O\left(q^{-n} q^{g/3 + \epsilon g}\right),$$

grouped by the same order of derivation on $D_Q(u_1, u_2)$.

We have

$$H_1 = \frac{q^{g+1}}{3} \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \geq 1}} \frac{d(Q)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)} \times$$
$$\left[(g+2) J_0\left(\frac{1}{q}, \frac{1}{q}\right) - \frac{\frac{d}{du_1} J_0(u_1, u_1) \mid_{1/q}}{q} - \frac{c_Q J_0\left(\frac{1}{q}, \frac{\zeta_3}{q}\right) \zeta_3^{1+g}}{1-\zeta_3} - \frac{\overline{c_Q} J_0\left(\frac{\zeta_3^2}{q}, \frac{1}{q}\right) \zeta_3^{2+2g}}{1-\zeta_3^2} \right],$$

where we denote

$$c_Q = \frac{D_Q\left(\frac{1}{q}, \frac{\zeta_3}{q}\right)}{D_Q\left(\frac{1}{q}, \frac{1}{q}\right)}. \tag{2.10}$$

Taking the derivative of $D_Q(u_1, u_1)$, we obtain

$$H_2 = -\frac{q^{g+1} J_0\left(\frac{1}{q}, \frac{1}{q}\right)}{3} \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \geq 1}} \frac{2 d(Q)^2 |Q|^{-1}}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)^2}.$$

We now consider each $\dfrac{H_i}{|\mathcal{F}_3^{\mathrm{K}}(g)|}$.

$$\frac{H_1}{|\mathcal{F}_3^{\mathrm{K}}(g)|} = \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \geq 1}} \frac{d(Q)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)} + 2\mathfrak{Re}\left(h_1 \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \geq 1}} \frac{d(Q)\left(1-c_Q\right)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)} \right),$$

where h_1 denotes the constant

$$h_1 = \frac{q^{g+1} J_0\left(\frac{1}{q}, \frac{\zeta_3}{q}\right) \zeta_3^{1+g}}{3\,|\mathcal{F}_3^{\mathrm{K}}(g)|\left(1-\zeta_3\right)}. \tag{2.11}$$

We remark that h_1 is of order $1/g$, and when $d(Q) \equiv 0 \pmod 3$, $c_Q = 1$ and latter two terms in the sum vanish.

Let

$$h_2 = -\frac{q^{g+1} J_0\left(\frac{1}{q}, \frac{1}{q}\right)}{3\left|\mathcal{F}_3^{\mathrm{K}}(g)\right|}. \tag{2.12}$$

Thus we have

$$\begin{aligned} M_3^{\mathrm{K}}(\phi, g) =& \frac{2}{g} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{d(Q)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)} \\ &+ 4\mathfrak{Re}\left(\frac{h_1}{g} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{d(Q)\left(1-c_Q\right)}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)}\right) \\ &+ \frac{2h_2}{g} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{2d(Q)^2|Q|^{-1}}{|Q|^{3r/2}\left(1+2|Q|^{-1}\right)^2} + O\left(q^{-2g/3} q^{\epsilon g}\right), \end{aligned}$$

where h_2 is also of order $1/g$. □

2.4.2 The Error Term

Recall that if f is not a cube, we have the error term $E_3^{\mathrm{K}}(\phi, g)$ of the one-level density of zeros expressed in (2.7). We prove the following upper bound.

Lemma 2.6. *Let $f \in \mathbb{F}_q[t]$ be a monic polynomial and χ_3 as defined in* (1.7)*. Let*

$$\mathcal{T}_2 = \sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi) = g \\ \chi|_{\mathbb{F}_q^\times} = \chi_3}} \chi(f).$$

Then

$$\mathcal{T}_2 = \sum_{\substack{d_1 + d_2 = g+1 \\ d_1 + 2d_2 \equiv 1 \pmod 3}} \sum_{F_1 \in \mathcal{H}_{q,d_1}} \chi_f(F_1) \sum_{\substack{F_2 \in \mathcal{H}_{q,d_2} \\ (F_2, F_1) = 1}} \chi_f(F_2)^2 \ll g q^{g/2} q^{\epsilon d(f)}.$$

The error term for the one-level density of zeros of Dirichlet L-functions in the Kummer *setting is given in* (2.7)*. For $\ell = 3$, we have*

$$E_3^{\mathrm{K}}(\phi, g) \ll q^{N/2} q^{-g/2} q^{\epsilon N}.$$

Proof. First we give an upper bound on $\mathcal{T}_2$.

By Lemma 2.2, we have

$$\mathcal{T}_2 = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2 \equiv 1 \pmod 3}} \sum_{F_1 \in \mathcal{H}_{q,d_1}} \chi_f(F_1) \sum_{\substack{F_2 \in \mathcal{H}_{q,d_2} \\ (F_2,F_1)=1}} \chi_f(F_2)^2.$$

Then we consider the generating series for the sums over F_1 and F_2,

$$\mathcal{S}(u_1, u_2) = \sum_{F_1 \in \mathcal{H}_q} \chi_f(F_1) u_1^{d(F_1)} \sum_{\substack{F_2 \in \mathcal{H}_q \\ (F_2,F_1)=1}} \chi_f(F_2)^2 u_2^{d(F_2)}.$$

The sum over F_2 can be written as the product

$$\frac{\prod_{P \in \mathcal{P}_q} \left(1 + \chi_f(P)^2 u_2^{d(P)}\right)}{\prod_{P \mid F_1} \left(1 + \chi_f(P)^2 u_2^{d(P)}\right)},$$

thus factoring out the L-functions

$$\mathcal{S}(u_1, u_2) = \frac{\mathcal{L}\left(u_2, \chi_f^2\right)}{\mathcal{L}\left(u_2^2, \chi_f\right)} \sum_{F_1 \in \mathcal{H}_q} \frac{\chi_f(F_1) u_1^{d(F_1)}}{\prod_{P \mid F_1} \left(1 + \chi_f(P)^2 u_2^{d(P)}\right)}.$$

Similarly, we write the sum over F_1 as the product

$$\prod_{P \in \mathcal{P}_q} \left(1 + \frac{\chi_f(P) u_1^{d(P)}}{1 + \chi_f(P)^2 u_2^{d(P)}}\right) = \frac{\mathcal{L}\left(u_1, \chi_f\right)}{\mathcal{L}\left(u_1^2, \chi_f^2\right)} \prod_{P \in \mathcal{P}_q} \frac{1 + \chi_f(P) u_1^{d(P)} + \chi_f(P)^2 u_2^{d(P)}}{\left(1 + \chi_f(P) u_1^{d(P)}\right)\left(1 + \chi_f(P)^2 u_2^{d(P)}\right)}.$$

Combining the two, we obtain

$$\mathcal{S}(u_1,u_2)=\frac{\mathcal{L}\left(u_1,\chi_f\right)\mathcal{L}\left(u_2,\chi_f^2\right)}{\mathcal{L}\left(u_1^2,\chi_f^2\right)\mathcal{L}\left(u_2^2,\chi_f\right)}\prod_{P\in\mathcal{P}_q}\frac{1+\chi_f(P)u_1^{d(P)}+\chi_f(P)^2u_2^{d(P)}}{\left(1+\chi_f(P)u_1^{d(P)}\right)\left(1+\chi_f(P)^2u_2^{d(P)}\right)}.$$

Note here that the product over P is absolutely convergent for $|u_1|,|u_2|<q^{-1/2}$.

Using Perron's formula,

$$\mathcal{T}_2=\sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \pmod 3}}\frac{1}{(2\pi i)^2}\oint_{|u_1|=q^{-1/2}}\oint_{|u_2|=q^{-1/2}}\frac{\mathcal{S}(u_1,u_2)}{u_1^{d_1}u_2^{d_2}}\frac{du_1}{u_1}\frac{du_2}{u_2}.$$

Then by Lindelöf Hypothesis type bounds (Lemma 1.3 and Lemma 1.4), we obtain the following bounds on the integrals. For $i\in\{1,2\}$

$$\frac{1}{2\pi i}\oint_{|u_i|=q^{-1/2}}\frac{\mathcal{L}\left(u_i,\chi_f^i\right)}{\mathcal{L}\left(u_i^2,\chi_f^{2i}\right)u_i^{d_i}}\frac{d}{du_i}\ll q^{d_i/2}q^{\epsilon d(f)}.$$

Hence

$$\mathcal{T}_2\ll\sum_{d_1=1}^{g+1}q^{g/2}q^{\epsilon d(f)}\ll q^{g/2}q^{\epsilon d(f)}.$$

Now for $\overline{\mathcal{T}_2}$, since Lemma 1.3 and Lemma 1.4 hold for $\mathcal{L}\left(u_i,\overline{\chi_f}^i\right)$ and $\mathcal{L}\left(u_i^2,\overline{\chi_f}^{2i}\right)$ for $i\in\{1,2\}$, we have

$$\overline{\mathcal{T}_2}\ll q^{g/2}q^{\epsilon d(f)}.$$

The error term in the Kummer setting given in (2.7) for $\ell=3$ can thus be written as

$$E_3^{\mathrm{K}}(\phi,g)=\frac{1}{g\left|\mathcal{F}_3^{\mathrm{K}}(g)\right|}\sum_{1\le n\le N}\hat{\Phi}\left(\frac{n}{g}\right)\sum_{\substack{f\in\mathcal{M}_{q,n}\\ f \text{ noncube}}}\frac{\Lambda(f)}{|f|^{1/2}}\left(\mathcal{T}_2+\overline{\mathcal{T}_2}\right),$$

where trivially bound the double sum over f and n, then divide by $\left|\mathcal{F}_3^{\mathrm{K}}(g)\right|$ to obtain

$$E_3^{\mathrm{K}}(\phi, g) \ll q^{N/2} q^{-g/2} q^{\epsilon N}.$$

□

2.4.3 Proofs for $\ell = 3$ Kummer setting results

Proof of Theorem 1.1. Recall that we use notations in Section 1.3 and $\mathcal{F}_3^{\mathrm{K}}(g)$, the family of cubic Dirichlet L-functions in the Kummer setting is defined in (1.8). The one-level density of zeros for cubic Dirichlet L-functions in the Kummer setting is

$$\mathcal{D}_3^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0) - \mathcal{A}_3^{\mathrm{K}}(\phi, g), \tag{2.13}$$

where

$$\mathcal{A}_3^{\mathrm{K}}(\phi, g) = \frac{1}{g|\mathcal{F}_3^{\mathrm{K}}(g)|} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{n}{g}\right) \sum_{f \in \mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{H \in \mathcal{F}_3^{\mathrm{K}}(g)} \left[\chi_H(f) + \overline{\chi_H(f)}\right].$$

Note that $\mathcal{D}_3^{\mathrm{K}}(\phi, g)$ is defined in (1.22) and $\mathcal{A}_3^{\mathrm{K}}(\phi, g)$ is the $\ell = 3$ case in (2.5).

Using Lemma 2.5 and Lemma 2.6, we have the following result given in (1.23).

$$\begin{aligned}
\mathcal{D}_3^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0) &- \frac{2}{g} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{d(Q)}{|Q|^{3r/2}\left(1 + 2|Q|^{-1}\right)} \\
&+ 4\mathfrak{Re}\left(\frac{h_1}{g} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{d(Q)\left(1 - c_Q\right)}{|Q|^{3r/2}\left(1 + 2|Q|^{-1}\right)}\right) \\
&+ \frac{2h_2}{g} \sum_{1 \le n \le N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{2d(Q)^2|Q|^{-1}}{|Q|^{3r/2}\left(1 + 2|Q|^{-1}\right)^2} + O\left(q^{N/2} q^{-g/2} q^{\epsilon N}\right).
\end{aligned}$$

Here c_Q, h_1 and h_2 are explicitly defined in the main term lemma, Lemma 2.5, by (2.10), (2.11) and (2.12) respectively. □

Computing the limit as $g \to \infty$, we confirm the symmetry type of the family.

Proof of Theorem 1.3 for $\ell = 3$. Let $N < g$. Then

$$\lim_{g \to \infty} \mathcal{D}_3^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0),$$

since the double sums over n and Q above are of constant size.

Furthermore, we compute the two integrals below and confirm that

$$\int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{\mathcal{W}}_{U(g)}(y)dy = \hat{\Phi}(0) = \int_{-\infty}^{\infty} \hat{\Phi}(y)\delta_0(y)dy,$$

where $\mathcal{W}_{U(g)}(y) = \delta_0(y)$ denotes the one-level scaling density of the group of $g \times g$ unitary matrices. □

This proves that the symmetry type of the family is unitary and it supports Katz and Sarnak's philosophy.

2.5 Quartic Dirichlet L-functions

2.5.1 The Main Term

First we compute the following sum over the family of quartic Dirichlet L-functions and derive the size of the family $\left|\mathcal{F}_4^{\mathrm{K}}(g)\right|$.

Lemma 2.7. *For f a monic polynomial, let*

$$\mathcal{K}_1 = \sum_{\substack{d_1+d_3=\frac{2g}{3}+1 \\ d_1+3d_3\equiv 1 \pmod 4}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ (F_1,f)=1}} \sum_{\substack{F_3\in\mathcal{H}_{q,d_3} \\ (F_3,F_1f)=1}} 1.$$

Then

$$\mathcal{K}_1 = \frac{q^{G+2}}{2}\left[(G+1)\,J\left(\frac{1}{q},\frac{1}{q}\right) - \frac{\frac{d}{du_1}J(u_1,u_1)\mid_{1/q}}{q}\right] + O\left(q^{G/3+\epsilon g}\right),$$

where $G=\dfrac{2g}{3}+1$, and

$$J(x,y) = \prod_{P\in\mathcal{P}_q}\left(1+x^{d(P)}+y^{d(P)}\right)\left(1-x^{d(P)}\right)\left(1-y^{d(P)}\right)\prod_{\substack{P\in\mathcal{P}_q \\ P\mid f}}\left(1+x^{d(P)}+y^{d(P)}\right)^{-1}.$$

Proof. We first rewrite the outermost sum of $\mathcal{K}_1$.

The degree of the conductor is the integer $\frac{2g}{3}+1$ given in (1.16), so $g\equiv 0 \pmod 3$. We let g be even for convenience, which implies that d_3 must also be even. The case when g is odd is the symmetric case when d_1 must be even. To simplify some notations, we let

$$v=\frac{g}{3} \;\text{ and }\; G=\frac{2g}{3}+1=2v+1. \tag{2.14}$$

Using $d_1 + d_3 = G$ and $d_1 + 3d_3 \equiv 1 \pmod 4$, we found that

$$2d_3 \equiv 0 \pmod 4 \implies d_3 \equiv 0 \pmod 2.$$

Since G is odd, for some integer k_1,

$$2k_1 + 1 = d_1 \leq G. \tag{2.15}$$

Lastly, $d_3 = G - d_1$, which simplifies to

$$d_3 = 2\left(v - k_1\right). \tag{2.16}$$

We also note that the congruence

$$d_1 + 3d_3 \equiv 1 \pmod 4$$

is satisfied for all d_1, d_3 in agreement with equations (2.15) and (2.16). Thus, we are summing over all non-negative integers $k_1 \leq v$. Hence we rewrite the sum as

$$\mathcal{K}_1 = \sum_{\substack{d_1 = k_1 \leq v \\ d_3 = 2(v - k_1)}} \sum_{\substack{F_1 \in \mathcal{H}_{q,d_1} \\ (F_1, f) = 1}} \sum_{\substack{F_3 \in \mathcal{H}_{q,d_3} \\ (F_3, F_1 f) = 1}} 1.$$

Then, similar to the cubic case, we consider the generating series for the sum over F_1 and F_3 of $\mathcal{K}_1$

$$\mathcal{S}(u_1, u_3) = \sum_{\substack{F_1 \in \mathcal{H}_q \\ (F_1, f) = 1}} u_1^{d(F_1)} \sum_{\substack{F_3 \in \mathcal{H}_q \\ (F_3, F_1 f) = 1}} u_3^{d(F_3)}.$$

We can write it as the product,

$$\mathcal{S}(u_1,u_3)=\frac{\prod_{P\in\mathcal{P}_q}\left(1+\frac{u_1^{d(P)}}{1+u_3^{d(P)}}\right)\prod_{P\in\mathcal{P}_q}\left(1+u_3^{d(P)}\right)}{\prod_{\substack{P\in\mathcal{P}_q\\P|f}}\left(1+\frac{u_1^{d(P)}}{1+u_3^{d(P)}}\right)\prod_{\substack{P\in\mathcal{P}_q\\P|f}}\left(1+u_3^{d(P)}\right)}=\frac{\prod_{P\in\mathcal{P}_q}\left(1+u_1^{d(P)}+u_3^{d(P)}\right)}{\prod_{\substack{P\in\mathcal{P}_q\\P|f}}\left(1+u_1^{d(P)}+u_3^{d(P)}\right)}.$$

Hence

$$\mathcal{S}(u_1,u_3)=\mathcal{Z}_q(u_1)\mathcal{Z}_q(u_3)J(u_1,u_3),$$

where

$$J(u_1,u_3)=\prod_{P\in\mathcal{P}_q}\left(1+u_1^{d(P)}+u_3^{d(P)}\right)\left(1-u_1^{d(P)}\right)\left(1-u_3^{d(P)}\right)\prod_{\substack{P\in\mathcal{P}_q\\P|f}}\left(1+u_1^{d(P)}+u_3^{d(P)}\right)^{-1}$$

$$=\prod_{P\in\mathcal{P}}\left(1-u_1^{2d(P)}-u_3^{2d(P)}-(u_1u_3)^{d(P)}+(u_1^2u_3)^{d(P)}+(u_1u_3^2)^{d(P)}\right)\prod_{\substack{P\in\mathcal{P}_q\\P|f}}\left(1+u_1^{d(P)}+u_3^{d(P)}\right)^{-1}.$$

Note that $J(u_1,u_3)$ has analytic continuation when $|u_1|<q^{-1/3}$ and $|u_3|<q^{-1/3}$.

Using Perron's formula twice,

$$\mathcal{K}_1=\sum_{k_1=0}^{v}\frac{1}{(2\pi i)^2}\oint\oint\frac{J(u_1,u_3)}{(1-qu_1)(1-qu_3)u_1^{2k_1+1}u_3^{2v-2k_1}}\frac{du_3}{u_3}\frac{du_1}{u_1}.$$

Computing the sum over k_1 first, we have

$$\begin{aligned}\sum_{k_1=0}^{v}\frac{1}{u_1^{2k_1+1}u_3^{2v-2k_1}}&=\frac{u_1}{u_3^{2v}\left(u_3^2-u_1^2\right)}\left[\left(\frac{u_3^2}{u_1^2}\right)^{v+1}-1\right]\\&=\frac{1}{\left(u_3^2-u_1^2\right)}\left[\frac{u_3^2}{u_1^{2v+1}}-\frac{u_1}{u_3^{2v}}\right].\end{aligned}$$

Thus

$$\mathcal{K}_1=\frac{1}{(2\pi i)^2}\oint_{|u_1|=q^{-3}}\oint_{|u_2|=q^{-2}}\frac{J(u_1,u_3)}{(1-qu_1)(1-qu_3)(u_3^2-u_1^2)}\left[\frac{u_3}{u_1^{2v+2}}-\frac{1}{u_3^{2v+1}}\right]du_3du_1.$$

We write the integral above as the difference of two integrals correspondingly and note that the second one

$$\frac{1}{(2\pi i)^2}\oint_{|u_1|=q^{-3}}\oint_{|u_2|=q^{-2}}\frac{-J(u_1,u_3)}{u_3^{2v+1}(1-qu_1)(1-qu_3)(u_3^2-u_1^2)}du_3du_1=0$$

since the integrand over u_1 has no poles inside the circle $|u_1|=q^{-3}$.

Hence

$$\mathcal{K}_1=\frac{1}{(2\pi i)^2}\oint_{|u_1|=q^{-3}}\oint_{|u_3|=q^{-2}}\frac{u_3J(u_1,u_3)}{u_1^{2v+2}(1-qu_1)(1-qu_3)(u_3^2-u_1^2)}du_3du_1,$$

where the poles of the integrand integrating over u_3 are $u_1,-u_1$.

Computing the residue at the poles above, we have

$$\mathcal{K}_1=\frac{1}{2\pi i}\oint_{|u_1|=q^{-3}}\frac{1}{2u_1^{2v+2}(1-qu_1)}\left[\frac{J(u_1,u_1)}{1-qu_1}+\frac{J(u_1,-u_1)}{1+qu_1}\right]du_1. \tag{2.17}$$

Now to integrate over u_1, we write (2.17) as the sum of two integrals

$$\mathcal{K}_1=K_1+K_{-1}$$

where for $\beta\in\{1,-1\}$,

$$K_\beta=\frac{1}{2\pi i}\oint_{|u_1|=q^{-3}}\frac{J(u_1,\beta u_1)}{2u_1^{2v+2}(1-qu_1)(1-q\beta u_1)}du_1.$$

For each K_β, we shift the contour to $|u_1|=q^{-1/3+\epsilon}$ and encounter the pole $1/q$ and $1/(q\beta)$. We compute residues at the corresponding poles and bound the integral on circle $|u_1|=q^{-1/3+\epsilon}$.

K_1 has a double pole at $1/q$, and we obtain

$$K_1=\frac{q^{G+2}}{2}\left[(G+1)\,J\left(\frac{1}{q},\frac{1}{q}\right)-\frac{\frac{d}{du_1}J(u_1,u_1)\mid_{1/q}}{q}\right]+O\left(q^{G/3+\epsilon g}\right).$$

K_{-1} has two simple poles at $u_1 = 1/q$ and $u_1 = -1/q$.

Computing the residues, we have

$$K_{-1} = \frac{q^G}{4}\left[J\left(\frac{1}{q}, \frac{-1}{q}\right) - J\left(\frac{-1}{q}, \frac{1}{q}\right)\right] + O\left(q^{G/3+\epsilon g}\right),$$

where since $J\left(\frac{1}{q}, \frac{-1}{q}\right) = J\left(\frac{-1}{q}, \frac{1}{q}\right)$, $K_{-1} = O\left(q^{G/3+\epsilon g}\right)$.

Thus

$$\mathcal{K}_1 = \frac{q^{G+2}}{2}\left[(G+1)\, J\left(\frac{1}{q}, \frac{1}{q}\right) - \frac{\frac{d}{du_1} J(u_1, u_1) \mid_{1/q}}{q}\right] + O\left(q^{G/3+\epsilon g}\right).$$

This gives the result as desired. $\square$

From Lemma 2.7, the size of the family is the following.

Corollary 2.2. *Let* $J_0(u) = \prod_{P \in \mathcal{P}} \left(1 + 2u^{d(P)}\right)\left(1 - u^{d(P)}\right)^2$, *then*

$$\left|\mathcal{F}_4^{\mathrm{K}}(g)\right| = \frac{q^{G+2}}{2}\left[(G+1)\, J_0\left(\frac{1}{q}\right) - \frac{\frac{d}{du} J_0(u) \mid_{1/q}}{q}\right] + O\left(q^{G/3+\epsilon g}\right),$$

where $G = \frac{2g}{3} + 1$ *as in* (2.14).

Now we compute the main term of the one-level density of quartic Dirichlet L-functions over function fields. We use some notations as needed given in Section 1.3. Furthermore, by Equation (1.16), we have $D_4(g) - 2 = \frac{2g}{3}$.

Lemma 2.8. *Recall the main term of the one-level density of zeros in the Kummer setting is given in* (2.6). *Let* $\ell = 4$ *for quartic Dirichlet L-functions with characters of genus* g, *we* *have*

$$M_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right) =$$
$$\frac{3}{g}\sum_{1\leq n\leq N/4}\hat{\Phi}\left(\frac{6n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)}{|Q|^{2r}\left(1+2|Q|^{-1}\right)}+\frac{3s_2}{g}\sum_{1\leq n\leq N/4}\hat{\Phi}\left(\frac{6n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{2d(Q)^2|Q|^{-1}}{|Q|^{2r}\left(1+2|Q|^{-1}\right)^2}$$
$$+\,O\left(q^{-2G/3}q^{\epsilon g}\right),$$

where s_2 is an explicit constant defined in (2.18) *and* $G=\frac{2g}{3}+1$.

Proof. Recall that for $\ell=4$, the main term of the one-level density (2.6) comes from when f is a 4^{th} power. Since $\chi_F(f)=\overline{\chi_F(f)}=1$, we have

$$M_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right)=\frac{3}{g|\mathcal{F}_4^{\mathrm{K}}(g)|}\sum_{1\leq n\leq N/4}\hat{\Phi}\left(\frac{6n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)\mathcal{K}_1}{|Q|^{2r}}.$$

Let

$$D_f(u)=\prod_{\substack{P\in\mathcal{P}_q\\ P|f}}\left(1+2u^{d(P)}\right)^{-1},$$

then

$$J(u,u)=J_0(u)D_f(u).$$

Furthermore, if $f=Q^{4r}$ for some monic irreducible polynomial Q and some integer r, then

$$D_f(u)=D_Q(u)=\left(1+2u^{d(Q)}\right)^{-1}.$$

Now using Lemma 2.7 for $\mathcal{K}_1$, we can write the sum over Q as two sums

$$H_1+H_2+O\left(q^{-n}q^{G/3+\epsilon g}\right)$$

grouped by the same order of derivation on $D_Q\left(u\right)$. Here $G=\dfrac{2g}{3}+1$ as given in (2.14).

We have

$$H_1 = \frac{q^{G+2}}{2}\left[(G+1)J_0\left(\frac{1}{q}\right) - \frac{\frac{d}{du}J_0\,(u)\;|_{1/q}}{q}\right]\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)}{|Q|^{2r}\left(1+2|Q|^{-1}\right)},$$

and

$$H_2 = \frac{q^{G+2}J_0(\frac{1}{q})}{2}\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{2d(Q)^2|Q|^{-1}}{|Q|^{2r}\left(1+2|Q|^{-1}\right)^2}.$$

Let s_i denote the constant obtained by the coefficient of H_i divided by $\left|\mathcal{F}_4^{\mathrm{K}}(g)\right|$.

We note that $s_1 = 1$, and we have

$$s_2 = \frac{q^{G+2}J_0(\frac{1}{q})}{2\left|\mathcal{F}_4^{\mathrm{K}}(g)\right|}. \tag{2.18}$$

Therefore

$$M_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right) =$$

$$\frac{3}{g}\sum_{1\leq n\leq N/4}\hat{\Phi}\left(\frac{6n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)}{|Q|^{2r}\left(1+2|Q|^{-1}\right)} + \frac{3s_2}{g}\sum_{1\leq n\leq N/4}\hat{\Phi}\left(\frac{6n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{2d(Q)^2|Q|^{-1}}{|Q|^{2r}\left(1+2|Q|^{-1}\right)^2}$$

$$+\,O\left(q^{-2G/3}q^{\epsilon g}\right),$$

where s_2 is of order $1/g$. □

2.5.2 The Error Term

Recall that if f is a non-4^{th} power, the error term contribution to the one-level density of quartic Dirichlet L-functions over function fields is

$$E_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right) = \frac{3}{2g\,|\mathcal{F}_4^{\mathrm{K}}(g)|} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{3n}{2g}\right) \sum_{\substack{f \in \mathcal{M}_{q,n} \\ f \text{ non-}4^{\text{th}} \text{ powers}}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{F \in \mathcal{F}_4^{\mathrm{K}}(g)} \left(\chi_F(f) + \overline{\chi_F(f)}\right).$$

We prove the following upper bound.

Lemma 2.9. *Let $f \in \mathbb{F}_q[t]$ be a monic polynomial and χ_4 as defined in* (1.7). *Let*

$$\mathcal{K}_2 = \sum_{\substack{\chi \text{ primitive quartic} \\ \chi^2 \text{ primitive} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_4}} \chi(f).$$

Then

$$\mathcal{K}_2 = \sum_{\substack{d_1+d_3=G \\ d_1+3d_3 \equiv 1 \pmod 4}} \sum_{F_1 \in \mathcal{H}_{q,d_1}} \chi_f(F_1) \sum_{\substack{F_3 \in \mathcal{H}_{q,d_3} \\ (F_3,F_1)=1}} \chi_f(F_3)^3 \ll G q^{G/2} q^{\epsilon d(f)},$$

for $G = \dfrac{2g}{3} + 1$.

The error term for the one-level density of zeros of Dirichlet L-functions in the Kummer setting is given in (2.7). *For $\ell = 4$, we have*

$$E_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right) \ll q^{N/2} q^{-G/2} q^{\epsilon N}.$$

Proof. We consider the generating series of $\mathcal{K}_2$

$$\mathcal{S}(u_1, u_3) = \sum_{F_1 \in \mathcal{H}_q} \chi_f(F_1) u_1^{d(F_1)} \sum_{\substack{F_3 \in \mathcal{H}_q \\ (F_3,F_1)=1}} \chi_f(F_3)^3 u_3^{d(F_3)}.$$

First, the sum over F_3 can be written as the product

$$\frac{\prod_{P\in\mathcal{P}_q}\left(1+\chi_f(P)^3u_3^{d(P)}\right)}{\prod_{P|F_1}\left(1+\chi_f(P)^3u_3^{d(P)}\right)},$$

thus

$$\mathcal{S}(u_1,u_3)=\prod_{P\in\mathcal{P}_q}\left(1+\chi_f(P)^3u_3^{d(P)}\right)\sum_{F_1\in\mathcal{H}_q}\frac{\chi_f(F_1)u_1^{d(F_1)}}{\prod_{P|F_1}\left(1+\chi_f(P)^3u_3^{d(P)}\right)}.$$

Writing the sum over F_1 as the product as well, we combine the two and obtain

$$\mathcal{S}(u_1,u_3)=\prod_{P\in\mathcal{P}_q}\left(1+\chi_f(P)u_1^{d(P)}+\chi_f(P)^3u_3^{d(P)}\right).$$

Thus

$$\mathcal{S}(u_1,u_3)=\frac{\mathcal{L}\left(u_1,\chi_f\right)\mathcal{L}\left(u_3,\chi_f^3\right)}{\mathcal{L}\left(u_1^2,\chi_f^2\right)\mathcal{L}\left(u_3^2,\chi_f^2\right)}\prod_{P\in\mathcal{P}_q}\frac{1+\chi_f(P)u_1^{d(P)}+\chi_f(P)^3u_3^{d(P)}}{\left(1+\chi_f(P)u_1^{d(P)}\right)\left(1+\chi_f(P)^3u_3^{d(P)}\right)},$$

where the product over P is absolutely convergent for $|u_1|,|u_3|<q^{-1/2}$.

Using Perron's formula,

$$\mathcal{K}_2=\sum_{\substack{d_1+d_3=G\\ d_1+3d_3\equiv 1 \pmod 4}}\frac{1}{(2\pi i)^2}\oint_{|u_1|=q^{-1/2}}\oint_{|u_3|=q^{-1/2}}\frac{\mathcal{S}(u_1,u_3)}{u_1^{d_1}u_3^{d_3}}\frac{du_1}{u_1}\frac{du_3}{u_3}.$$

Then, we use the Lindelöf Hypothesis type results (Lemma 1.3 and Lemma 1.4) to obtain a bound for each of the following integrals.

For $\beta\in\{1,3\}$, we have

$$\frac{1}{2\pi i}\oint_{|u_\beta|=q^{-1/2}}\frac{\mathcal{L}\left(u_\beta,\chi_f^\beta\right)}{\mathcal{L}\left(u_\beta^2,\chi_f^{2\beta}\right)u_\beta^{d_\beta}}\frac{du_\beta}{u_\beta}\ll q^{d_\beta/2}q^{\epsilon d(f)},$$

thus trivially bounding the outer sum, we have

$$\mathcal{K}_2 \ll G q^{G/2} q^{\epsilon d(f)}.$$

Since Lemma 1.3 and Lemma 1.4 hold for $\mathcal{L}\left(u_\beta, \overline{\chi_f}^\beta\right)$ and $\mathcal{L}\left(u_\beta^2, \overline{\chi_f}^{2\beta}\right)$ for $\beta \in \{1,3\}$, we have

$$\overline{\mathcal{K}_2} \ll G q^{G/2} q^{\epsilon d(f)}.$$

The error term in the Kummer setting is defined as in (2.7) for $\ell = 4$.

$$E_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right) = \frac{3}{2g\left|\mathcal{F}_4^{\mathrm{K}}(g)\right|} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{3n}{2g}\right) \sum_{\substack{f \in \mathcal{M}_{q,n} \\ f \text{ non-}4^{th} \text{ powers}}} \frac{\Lambda(f)}{|f|^{1/2}} \left(\mathcal{K}_2 + \overline{\mathcal{K}_2}\right).$$

We trivially bound the double sum over f and n, then divide by $\left|\mathcal{F}_4^{\mathrm{K}}(g)\right|$ to obtain

$$E_4^{\mathrm{K}}(\phi, g) \ll q^{N/2} q^{-G/2} q^{\epsilon N}.$$

□

2.5.3 Proofs of the Kummer setting results for $\ell = 4$

Proof of Theorem 1.2. We compute the one-level density of zeros of quartic Dirichlet L-functions in the Kummer setting. We use some notations in Section 1.3 and recall that $\mathcal{F}_4^{\mathrm{K}}(g)$ denotes the family of quartic Dirichlet L-functions in the Kummer setting as in (1.11). We have

$$\mathcal{D}_4^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0) - \mathcal{A}_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right), \tag{2.19}$$

where

$$\mathcal{A}_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right) = \frac{3}{2g|\mathcal{F}_4^{\mathrm{K}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{3n}{2g}\right) \sum_{f\in\mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{F\in\mathcal{F}_4^{\mathrm{K}}(g)} \left[\chi_F(f) + \overline{\chi_F(f)}\right].$$

Note that $\mathcal{D}_4^{\mathrm{K}}(\phi, g)$ is defined in (1.24) and $\mathcal{A}_4^{\mathrm{K}}\left(\phi, \frac{2g}{3}\right)$ is given by setting $\ell = 4$ in (2.5).

Using Lemma 2.8 and Lemma 2.9, we obtain the following result in (1.25).

$$\begin{aligned}
&\mathcal{D}_4^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0) \\
&-\frac{3}{g} \sum_{1\le n\le N/4} \hat{\Phi}\left(\frac{6n}{g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{d(Q)}{|Q|^{2r}\left(1+2|Q|^{-1}\right)} - \frac{3s_2}{g} \sum_{1\le n\le N/4} \hat{\Phi}\left(\frac{6n}{g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{2d(Q)^2|Q|^{-1}}{|Q|^{2r}\left(1+2|Q|^{-1}\right)^2} \\
&+O\left(q^{N/2}q^{-G/2}q^{\epsilon N}\right),
\end{aligned}$$

where s_2 is an explicit constant defined in (2.18) and $G = \dfrac{2g}{3} + 1$. □

Using the theorem, we confirm the symmetry type of the family.

Proof of Theorem 1.3 for $\ell = 4$. Let $N < 2g/3$. Then

$$\lim_{g\to\infty} \mathcal{D}_4^{\mathrm{K}}(\phi, g) = \hat{\Phi}(0),$$

since the double sums over n and Q are $o(1)$ as $g \to \infty$.

Furthermore, by computing the two integrals, we confirm that

$$\hat{\Phi}(0) = \int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{\mathcal{W}}_{U(2g/3)}(y)dy = \int_{-\infty}^{\infty} \hat{\Phi}(y)\delta_0(y)dy,$$

where $\mathcal{W}_{U(2g/3)}(y) = \delta_0(y)$ denotes the one-level scaling density of the group of unitary matrices. □

Thus the symmetry type of the family is unitary. This supports Katz and Sarnak's philosophy.

Chapter 3

One-level density of zeros of cubic, quartic and sextic Dirichlet L-functions in the non-Kummer setting

3.1 Introduction

In this chapter, we study the low-lying zeros of order ℓ Dirichlet L-functions over function fields for $\ell \in \{3, 4, 6\}$ and when $\mathbb{F}_q^\times$ does not contain an ℓ^{th} root of unity. For a general order $\ell \geq 3$, we say this is the non-Kummer setting. In terms of congruences modulo ℓ, this is equivalent to $q \not\equiv 1 \pmod{\ell}$. We describe the cubic non-Kummer characters over function fields as appeared in [17, 18], previously studied by David, Florea and Lalin, and extend that interpretation to quartic and sextic families in Section 3.2.

Let $\mathcal{F}_\ell^{\text{nK}}(g)$ denote the set of conductors for the family of order ℓ characters in the non-

Kummer setting as defined in (3.3). The one-level density of zeros of order ℓ Dirichlet L-functions in the non-Kummer setting for $\ell \in \{3, 4, 6\}$ is defined in (1.27) and I proved the statement in Theorem 1.4.

Using Theorem 1.4, I proved that the families of cubic, quartic and sextic Dirichlet L-functions in the non-Kummer setting have the unitary symmetry in Theorem 1.5. We note that the condition $N < D_\ell(g) - 2$ is equivalent to $\hat{\Phi}$ being supported in (-1,1), analogous to the Fourier transform of the test function having limited support over number fields.

3.2 Cubic, Quartic and Sextic Dirichlet Characters in the Non-Kummer Setting

Recall that we first discussed order ℓ non-Kummer characters in Section 1.3.1. In this section, we talk about works on cubic non-Kummer characters [3, 17, 18] and use results in [4] to generalize the cubic case to the quartic and sextic case.

Baier and Young [3] observed that over number fields, if $p \equiv 1 \pmod 3$ is the prime conductor of a cubic non-Kummer Dirichlet character, then $p = \pi\overline{\pi}$ over $\mathbb{Z}[\omega]$ for $\omega = e^{2\pi i/3}$ and $\pi, \overline{\pi}$ are primes with $N(\pi) = p$. Thus χ_p corresponds to either χ_π or $\overline{\chi_\pi}$.

Over function fields, David, Florea and Lalin [17, 18] found an analogous result for the cubic non-Kummer characters. Since in the non-Kummer setting $q^2 \equiv 1 \pmod 3$, characters with a prime conductor $P \in \mathbb{F}_q[t]$ have degree divisible by 2. Using results from the work of Bary-Soroker and Meisner [4], $P = \pi\overline{\pi}$ over $\mathbb{F}_{q^2}[t]$, and χ_P corresponds to either χ_π or $\overline{\chi}_\pi$ restricted to $\mathbb{F}_q[t]$.

Extending this construction to higher orders, since the Lemma 2.9 in [4] about the splitting of P works for ℓ not necessarily a prime, we have the following lemma.

Lemma 3.1. *Let $q \not\equiv 1 \pmod{\ell}$ and n_q the multiplicative order of $q \pmod{\ell}$, i.e., n_q is the smallest integer such that $q^{n_q} \equiv 1 \pmod{\ell}$. Let $P \in \mathcal{P}_q$ be a prime factor of the conductor of an order ℓ character. Then*

$$P = \pi_1 \cdots \pi_{n_q} \in \mathbb{F}_{q^{n_q}}[t], \tag{3.1}$$

with $\{\pi_i\}_1^{n_q}$ being Galois conjugates.

Note that for q an odd prime power, if $q \not\equiv 1 \pmod{\ell}$, then $q^2 \equiv 1 \pmod{\ell}$ for $\ell = 4, 6$. Thus conductors of quartic and sextic characters have prime factors of even degree and those primes split into two primes in $\mathbb{F}_{q^2}[t]$. Therefore, as suggested by Baier and Young, the non-Kummer characters for the quartic and sextic case are indeed similar to the cubic one.

Lemma 3.1 implies that

$$|P| = q^{d(P)} = q^{n_q d(\pi_i)} = |\pi_i|_{q^{n_q}}$$

for any $i \in \{1, 2, \ldots, n_q\}$. We give the definition of χ_π below and note its relation to χ_P for general multiplicative order n_q. One can also use the definition given in Section 1.3.1 for the Kummer case by setting the size of the base field equal to q^{n_q}.

Definition 3.1. *Let $P \in \mathcal{P}_q$ be the conductor of a primitive order ℓ character in the non-Kummer setting, with the splitting $P = \pi_1 \pi_2 \cdots \pi_{n_q}$ over $\mathbb{F}_{q^{n_q}}[t]$. Let $\pi \in \mathcal{P}_{q^{n_q}}$ be one of its prime factors.*

1. *For any $f \in \mathbb{F}_q[t]$ such that $P \nmid f$, the ℓ^{th} Jacobi symbol $\left(\frac{f}{\pi}\right)_\ell$ is the unique element of $\mathbb{F}_{q^{n_q}}^\times$ such that*
$$f^{\frac{|\pi|_{q^{n_q}}-1}{\ell}} \pmod{\pi} \equiv \left(\frac{f}{\pi}\right)_\ell.$$
2. *Let Ω_ℓ be a fixed isomorphism from the ℓ^{th} roots of unity $\mu_\ell \subseteq \mathbb{C}^\times$ to the ℓ^{th} roots of*

unity in $\mathbb{F}_{q^{n_q}}^\times$. *We define* $\chi_\pi(f) = 0$ *if* $\pi \mid f$, *and otherwise*

$$\chi_\pi(f) = \Omega_\ell^{-1}\left(\left(\frac{f}{\pi}\right)_\ell\right).$$

Note that by definition, one of the prime factors π dividing P satisfies

$$\left(\frac{f}{\pi}\right)_\ell = \left(\frac{f}{P}\right)_\ell,$$

and

$$\chi_\pi(f) = \chi_P(f). \tag{3.2}$$

Thus, we have a (non-canonical) definition of the Jacobi symbol and associated character with a prime conductor in the non-Kummer setting.

Focusing on $\ell = 3, 4, 6$, we extend the definition above multiplicatively to general conductors H of genus g, where

$$H = F_1F_2, \quad H \in \mathcal{H}_{q,D_\ell(g)}, \text{ and } (F_1, F_2) = 1,$$

similar to those given in (1.5). Note that for quartic characters such that $\chi^4 = 1$ and χ^2 remains primitive, the conductors should be $H = F_1F_3$ following the notations used in Section 1.3.1; similarly $H = F_1F_5$ for sextic characters, $\chi^6 = 1$ and χ^2, χ^3 remains primitive. This does not change the correspondence below.

Using (3.2), we can thus consider characters χ_F with certain conductors $F \in \mathbb{F}_{q^2}[t]$ restricting their input to $\mathbb{F}_q[t]$. We observe that the corresponding conductors $F \in \mathbb{F}_{q^2}[t]$ must also be squarefree polynomials, and if $P = \pi\overline{\pi} \mid H$, since $(F_1, F_2) = 1$, π and $\overline{\pi}$ do not both appear in the factors of F. Since each prime $P \mid H$ is represented by a prime (either π or $\overline{\pi}$) of degree $\frac{\deg(P)}{2}$, we have $\deg(F) = \frac{D_\ell(g)}{2}$. Thus we have the one-to-one correspondence between the set of conductors H and $\mathcal{F}_\ell^{\mathrm{nK}}(g)$, given by

$$\mathcal{F}_\ell^{\mathrm{nK}}(g) = \{F : \; F \in \mathcal{H}_{q^2, D_\ell(g)/2}, \; P \mid F \implies P \notin \mathbb{F}_q[t]\}. \tag{3.3}$$

We have proved the following result for $\ell = 3, 4$ and 6, where the cubic case first appeared in [17].

Lemma 3.2. *Let $f \in \mathcal{M}_q$. For $\ell = 3$,*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} \chi(f) = \sum_{\substack{F \in \mathcal{H}_{q^2, D_3(g)/2} \\ P|F \implies P \notin \mathbb{F}_q[t]}} \chi_F(f).$$

For $\ell = 4$, we have

$$\sum_{\substack{\chi \text{ primitive order } 4 \\ \text{genus}(\chi)=g \\ \chi^2 \text{primitive}}} \chi(f) = \sum_{\substack{F \in \mathcal{H}_{q^2, D_4(g)/2} \\ P|F \implies P \notin \mathbb{F}_q[t]}} \chi_F(f).$$

For $\ell = 6$

$$\sum_{\substack{\chi \text{ primitive order } 6 \\ \text{genus}(\chi)=g \\ \chi^2, \chi^3 \text{ primitive}}} \chi(f) = \sum_{\substack{F \in \mathcal{H}_{q^2, D_6(g)/2} \\ P|F \implies P \notin \mathbb{F}_q[t]}} \chi_F(f).$$

3.3 Preliminary Computations

Recall that the one-level density of zeros for order ℓ Dirichlet L-functions in the non-Kummer setting is

$$\mathcal{D}_\ell^{\mathrm{nK}}(\phi, g) = \frac{1}{|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{H \in \mathcal{F}_\ell^{\mathrm{nK}}(g)} \sum_{j=1}^{\frac{2g}{\ell-1}} \phi(\theta_{H,j}). \tag{3.4}$$

where $\mathcal{F}_\ell^{\mathrm{nK}}(g)$ denotes the set of conductors defined in (3.3), $D_\ell(g) - 2 = \dfrac{2g}{\ell - 1}$ given in (1.15) is the number of non-trivial zeros of the L-functions and $\phi(\theta) = \sum_{|n| \leq N} \hat{\phi}(n) e(n\theta)$

is any real, even trigonometric polynomial.

Let $\Phi\left((D_\ell(g)-2)\theta\right)=\phi(\theta)$. Then

$$\Phi\left((D_\ell(g)-2)\theta_{j,F}\right)=\frac{1}{D_\ell(g)-2}\sum_{|n|\le N}\hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right)e(n\theta_{j,F}),$$

and the sum over zeros in (3.4) can be written as

$$\sum_{j=1}^{D_\ell(g)-2}\Phi((D_\ell(g)-2)\theta_{j,F})=\hat{\Phi}(0)+\frac{1}{D_\ell(g)-2}\sum_{0<|n|\le N}\hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right)\sum_{j=1}^{D_\ell(g)-2}e(n\theta_{j,F}).$$

Using the explicit formula (Lemma 2.1) for $b=1$ and $D=D_\ell(g)-2$, we rewrite the inner most sum above and obtain

$$\begin{aligned}\sum_{j=1}^{D_\ell(g)-2}\Phi((D_\ell(g)-2)\theta_{j,F})=&\hat{\Phi}(0)-\frac{2}{D_\ell(g)-2}\sum_{1\le n\le N}\hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right)q^{-n/2}\\&-\frac{1}{D_\ell(g)-2}\sum_{1\le n\le N}\hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right)\sum_{f\in\mathcal{M}_{q,n}}\frac{\Lambda(f)}{|f|^{1/2}}\left[\chi_H(f)+\overline{\chi_H(f)}\right].\end{aligned}$$

Thus we write the one-level density in (3.4) as

$$\mathcal{D}_\ell^{\mathrm{nK}}(\phi,g)=\hat{\Phi}(0)-\frac{2}{D_\ell(g)-2}\sum_{1\le n\le N}\hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right)q^{-n/2}-\mathcal{A}_\ell^{\mathrm{nK}}(\phi,g),\qquad(3.5)$$

where

$$\begin{aligned}&\mathcal{A}_\ell^{\mathrm{nK}}(\phi,g)\\&=\frac{1}{(D_\ell(g)-2)\left|\mathcal{F}_\ell^{\mathrm{nK}}(g)\right|}\sum_{1\le n\le N}\hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right)\sum_{f\in\mathcal{M}_{q,n}}\frac{\Lambda(f)}{|f|^{1/2}}\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\P|F\Rightarrow P\notin\mathbb{F}_q[t]}}\left[\chi_F(f)+\overline{\chi_F(f)}\right].\end{aligned}\qquad(3.6)$$

We decompose $\mathcal{A}_\ell^{\mathrm{nK}}(\phi, g)$ as the sum

$$\mathcal{A}_\ell^{\mathrm{nK}}(\phi, g) = M_\ell^{\mathrm{nK}}(\phi, g) + E_\ell^{\mathrm{nK}}(\phi, g),$$

where the main term comes from when f is an ℓ^{th} power

$$\begin{aligned} M_\ell^{\mathrm{nK}}(\phi, g) =& \frac{1}{(D_\ell(g)-2)\left|\mathcal{F}_\ell^{\mathrm{nK}}(g)\right|} \sum_{1 \le n \le N/\ell} \hat{\Phi}\left(\frac{\ell n}{D_\ell(g)-2}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \ge 1}} \frac{d(Q)}{|Q|^{\ell r/2}} \\ & \times \sum_{\substack{F \in \mathcal{H}_{q^2, D_\ell(g)/2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[t]}} \left[\chi_F\left(Q^{\ell r}\right) + \overline{\chi_F\left(Q^{\ell r}\right)}\right], \end{aligned} \tag{3.7}$$

and the non-ℓ^{th} power contribution is

$$\begin{aligned} E_\ell^{\mathrm{nK}}(\phi, g) =& \frac{1}{(D_\ell(g)-2)\left|\mathcal{F}_\ell^{\mathrm{nK}}(g)\right|} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right) \sum_{\substack{f \in \mathcal{M}_{q,n} \\ f \text{ non-}\ell^{\mathrm{th}} \text{ power}}} \frac{\Lambda(f)}{|f|^{1/2}} \\ & \times \sum_{\substack{F \in \mathcal{H}_{q^2, D_\ell(g)/2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[t]}} \left[\chi_F(f) + \overline{\chi_F(f)}\right]. \end{aligned} \tag{3.8}$$

Lastly, we use the following fact to select conductors of the three non-Kummer families studied in Section 3.4.

Lemma 3.3. *Let F be a polynomial in $\mathbb{F}_{q^2}[t]$. Then*

$$\sum_{\substack{D|F \\ D \in \mathbb{F}_q[t]}} \mu(D) = \begin{cases} 1 & \text{if } F \text{ has no prime divisors in } \mathbb{F}_q[t] \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We observe that the sum above can be written as the product

$$\sum_{\substack{D|F \\ D \in \mathbb{F}_q[t]}} \mu(D) = \prod_{\substack{P|F \\ P \in \mathcal{P}_q}} (1 + \mu(P)).$$

If F is divisible by a prime $P \in \mathbb{F}_q[t]$, then $1+\mu(P)=0$, making the product on the right vanish. $\square$

3.4 The non-Kummer setting

3.4.1 The Main Term

First we compute the main term in (3.7) of the one-level density of zeros of Dirichlet L-functions for $\ell = 3, 4$ and 6 in the non-Kummer setting.

Lemma 3.4. *Let $D_\ell(g)$ be the degree of conductors of primitive order ℓ characters over $\mathbb{F}_q[t]$ as seen in (1.16) for $\ell = 3, 4$ or 6. For a fixed $f \in \mathcal{M}_{q,n}$, we have*

$$\sum_{\substack{F \in \mathcal{H}_{q^2, D_\ell(g)/2} \\ P | F \Rightarrow P \notin \mathbb{F}_q[t] \\ (F,f)=1}} 1 = q^{D_\ell(g)} \left(q^{-2} - q^{-4}\right) \frac{E_0\left(1/q^2\right)}{E_f\left(1/q^2\right)} \prod_{\substack{\pi \in \mathcal{P}_{q^2} \\ \pi | f}} \left(1 + |\pi|_{q^2}^{-1}\right)^{-1} + O\left(q^{D_\ell(g)/2+\epsilon g}\right),$$

where

$$E_h(u) = \prod_{\substack{P \in \mathcal{P}_q \\ P|h \\ d(P) \equiv 0 \pmod 2}} \left(1 - \frac{u^{d(P)}}{(1+u^{d(P)/2})^2}\right) \prod_{\substack{P \in \mathcal{P}_q \\ P|h \\ d(P) \equiv 1 \pmod 2}} \left(1 - \frac{u^{d(P)}}{1+u^{d(P)}}\right), \tag{3.9}$$

and $E_0(u)$ is the product over all primes in $\mathcal{P}_q$.

Proof. Since

$$\sum_{\substack{D \in \mathbb{F}_q[t] \\ D | F}} \mu(D) = \begin{cases} 1 & \text{if } F \text{ has no prime divisors in } \mathbb{F}_q[t], \\ 0 & \text{otherwise.} \end{cases}$$

the generating series for the sum over F can be written as

$$\mathcal{S}(u) = \sum_{\substack{F\in\mathcal{H}_{q^2} \\ P|F \Rightarrow P\notin\mathbb{F}_q[t] \\ (F,f)=1}} u^{d(F)} = \sum_{\substack{F\in\mathcal{H}_{q^2} \\ (F,f)=1}} u^{d(F)} \sum_{\substack{D\in\mathbb{F}_q[t] \\ D|F}} \mu(D) = \sum_{\substack{D\in\mathbb{F}_q[t] \\ (D,f)=1}} \mu(D)u^{d(D)} \sum_{\substack{F\in\mathcal{H}_{q^2} \\ (F,Df)=1}} u^{d(F)}. \quad (3.10)$$

Writing the inner sum over F as the product over primes,

$$\sum_{\substack{F\in\mathcal{H}_{q^2} \\ (F,Df)=1}} u^{d(F)} = \prod_{\substack{\pi\in\mathcal{P}_{q^2} \\ \pi\nmid Df}} \left(1+u^{d(\pi)}\right) = \frac{Z_{q^2}(u)}{Z_{q^2}(u^2)\prod_{\substack{\pi\in\mathcal{P}_{q^2} \\ \pi|Df}}\left(1+u^{d(\pi)}\right)}.$$

Thus (3.10) above is

$$\mathcal{S}(u) = \frac{1-q^2u^2}{(1-q^2u)\prod_{\substack{\pi\in\mathcal{P}_{q^2} \\ \pi|f}}\left(1+u^{d(\pi)}\right)} \sum_{\substack{D\in\mathbb{F}_q[t] \\ (D,f)=1}} \frac{\mu(D)u^{d(D)}}{\prod_{\substack{\pi\in\mathcal{P}_{q^2} \\ \pi|D}}\left(1+u^{d(\pi)}\right)}.$$

Similarly, the sum over D can be written as the product

$$\begin{aligned}\sum_{\substack{D\in\mathbb{F}_q[t] \\ (D,f)=1}} \frac{\mu(D)u^{d(D)}}{\prod_{\substack{\pi\in\mathcal{P}_{q^2} \\ \pi|D}}\left(1+u^{d(\pi)}\right)} &= \prod_{\substack{P\in\mathcal{P}_q \\ P\nmid f}} \left(1-\frac{u^{d(P)}}{\prod_{\substack{\pi\in\mathcal{P}_{q^2} \\ \pi|P}}\left(1+u^{d(\pi)}\right)}\right) \\ &= \prod_{\substack{P\in\mathcal{P}_q \\ P\nmid f \\ d(P)\equiv 0 \pmod 2}} \left(1-\frac{u^{d(P)}}{(1+u^{d(P)/2})^2}\right) \prod_{\substack{P\in\mathcal{P}_q \\ P\nmid f \\ d(P)\equiv 1 \pmod 2}} \left(1-\frac{u^{d(P)}}{1+u^{d(P)}}\right),\end{aligned} \quad (3.11)$$

where the last equality follows from lemma 2.9 in [4]. We denote this product by $\dfrac{E_0(u)}{E_f(u)}$ where $E_h(u)$ is as defined in (3.9).

Note that $\dfrac{E_0(u)}{E_f(u)}$ is absolutely convergent for $|u| < 1/q$. We can see this by expanding the denominator of the fraction in each product of (3.11), where for $j \in \{0,1\}$, each term can be written as

$$\prod_{\substack{P\in\mathcal{P}_q,\ P\nmid f\\ d(P)\equiv j \pmod 2}} \left(1-u^{d(P)} \ + B\left(u\right)\right)$$

where $B(u)$ contains u^α for $\alpha > d(P)$.

Hence

$$\mathcal{S}(u) = \frac{1-q^2u^2}{(1-q^2u)\prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi|f}}\left(1+u^{d(\pi)}\right)} \times \frac{E_0(u)}{E_f(u)},$$

which is absolutely convergent for $|u|<1/q^2$.

Using Perron's formula (Lemma 1.1), we first integrate along a circle of radius $|u| = 1/q^{2+\epsilon}$ and then shift the contour to $|u| = 1/q^{1+\epsilon}$, where we encounter a simple pole at $u = 1/q^2$. Let $\mathcal{S}(u)(1-q^2u) = F(u)$. For a small circle around the origin, for example, $|u| = 1/q^{100}$, we have

$$\begin{aligned}\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]\\ (F,f)=1}} 1 =& \frac{1}{2\pi i}\oint_{|u|=1/q^{100}} \frac{F(u)}{u^{D_\ell(g)/2}(1-q^2u)}\frac{du}{u}\\ =& -\operatorname{Res}\left(u=1/q^2\right) + \frac{1}{2\pi i}\oint_{|u|=1/q^{1+\epsilon}} \frac{F(u)}{u^{D_\ell(g)/2}(1-q^2u)}\frac{du}{u}.\end{aligned}$$

We bound the integral and compute the residue to obtain

$$\begin{aligned}\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]\\ (F,f)=1}} 1 =& -\lim_{u\to 1/q^2} \frac{F(u)(u-1/q^2)}{u^{D_\ell(g)/2}(1-q^2u)} + O\left(q^{D_\ell(g)/2+\epsilon D_\ell(g)/2}\right)\\ =& \frac{F(1/q^2)q^{D_\ell(g)}}{q^2} + O\left(q^{D_\ell(g)/2+\epsilon g}\right)\\ =& q^{D_\ell(g)}\left(q^{-2}-q^{-4}\right)\frac{E_0(1/q^2)}{E_f(1/q^2)}\prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi|f}}\left(1+|\pi|_{q^2}^{-1}\right)^{-1} + O\left(q^{D_\ell(g)/2+\epsilon g}\right).\end{aligned}$$

This gives the result stated in Lemma 3.4. □

Now we compute the size of the family $|\mathcal{F}_\ell^{\mathrm{nK}}(g)|$ in the non-Kummer setting. The proof is similar to the previous lemma.

Corollary 3.1. *For $\ell = 3, 4$ and 6, the size of the family of Dirichlet L-functions of order ℓ is*

$$|\mathcal{F}_\ell^{\mathrm{nK}}(g)| = q^{D_\ell(g)} \left(q^{-2} - q^{-4}\right) E_0(1/q^2) + O\left(q^{D_\ell(g)/2+\epsilon g}\right)$$

Proof. The number of primitive characters of order ℓ with conductor of degree $D_\ell(g)$ is given by

$$|\mathcal{F}_\ell^{\mathrm{nK}}(g)| = \sum_{\substack{F \in \mathcal{H}_{q^2, D_\ell(g)/2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[t]}} 1.$$

Using the same method as in Lemma 3.4, its generating series can be written as

$$\mathcal{S}(u) = \sum_{\substack{F \in \mathcal{H}_{q^2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[t]}} u^{d(F)} = \sum_{F \in \mathcal{H}_{q^2}} u^{d(F)} \sum_{\substack{D \in \mathbb{F}_q[t] \\ D|F}} \mu(D) = \sum_{D \in \mathbb{F}_q[t]} \mu(D) u^{d(D)} \sum_{\substack{F \in \mathcal{H}_{q^2} \\ (F,D)=1}} u^{d(F)}.$$

The sum over F is

$$\sum_{\substack{F \in \mathcal{H}_{q^2} \\ (F,D)=1}} u^{d(F)} = \prod_{\substack{\pi \in \mathcal{P}_{q^2} \\ \pi \nmid D}} \left(1 + u^{d(\pi)}\right) = \frac{Z_{q^2}(u)}{Z_{q^2}(u^2) \prod_{\substack{\pi \in \mathcal{P}_{q^2} \\ \pi | D}} \left(1 + u^{d(\pi)}\right)},$$

so we can rewrite $\mathcal{S}(u)$ as

$$\mathcal{S}(u) = \frac{1 - q^2u^2}{1 - q^2u} \sum_{D \in \mathbb{F}_q[t]} \frac{\mu(D) u^{d(D)}}{\prod_{\substack{\pi \in \mathcal{P}_{q^2} \\ \pi | D}} \left(1 + u^{d(\pi)}\right)}.$$

Now the sum over D can be written as the product

$$\sum_{D\in\mathbb{F}_q[t]} \frac{\mu(D)u^{d(D)}}{\prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi|D}}\left(1+u^{d(\pi)}\right)} = \prod_{P\in\mathcal{P}_q}\left(1-\frac{u^{d(P)}}{\prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi|P}}\left(1+u^{d(\pi)}\right)}\right)$$
$$= \prod_{\substack{P\in\mathcal{P}_q\\ d(P)\equiv 0 \pmod 2}}\left(1-\frac{u^{d(P)}}{(1+u^{d(P)/2})^2}\right)\prod_{\substack{P\in\mathcal{P}_q\\ d(P)\equiv 1 \pmod 2}}\left(1-\frac{u^{d(P)}}{1+u^{d(P)}}\right),$$

which we denote by $E_0(u)$ as in Lemma 3.4. Note that $E_0(u)$ is absolutely convergent for $|u|<1/q$ as in the same lemma. Thus

$$\mathcal{S}(u) = \frac{(1-q^2u^2)\,E_0(u)}{1-q^2u},$$

which is absolutely convergent for $|u|<1/q^2$.

Using Perron's formula, we first integrate along a circle of radius $|u|=1/q^{2+\epsilon}$, then we shift the contour to $|u|=1/q^{1+\epsilon}$ and encounter a simple pole at $u=1/q^2$.
Let $\mathcal{S}(u)(1-q^2u)=F(u)$. We have

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} 1 = \frac{1}{2\pi i}\oint_{|u|=1/q^{100}} \frac{F(u)}{u^{D_\ell(g)/2}(1-q^2u)}\frac{du}{u}$$
$$= -\mathrm{Res}\left(u=1/q^2\right) + \frac{1}{2\pi i}\oint_{|u|=1/q^{1+\epsilon}} \frac{F(u)}{u^{D_\ell(g)/2}(1-q^2u)}\frac{du}{u}.$$

Bounding the integral and computing the residue we obtain

$$\left|\mathcal{F}_\ell^{\mathrm{nK}}(g)\right| = \sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} 1 = -\lim_{u\to 1/q^2}\frac{F(u)(u-1/q^2)}{u^{D_\ell(g)/2}(1-q^2u)} + O\left(q^{D_\ell(g)/2+\epsilon D_\ell(g)/2}\right)$$
$$= \frac{F(1/q^2)q^{D_\ell(g)}}{q^2} + O\left(q^{D_\ell(g)/2+\epsilon g}\right)$$
$$= q^{D_\ell(g)}\left(q^{-2}-q^{-4}\right)E_0\left(1/q^2\right) + O\left(q^{D_\ell(g)/2+\epsilon g}\right).$$

□

Using our previous results, we compute the main term $M_\ell^{\mathrm{nK}}(\phi, D_\ell(g)-2)$ given in (3.7). We recall some notations in Section 1.3 as needed.

Lemma 3.5. *Let $m_Q = \gcd(2, d(Q))$. The main term of the one-level density of zeros of Dirichlet L-functions with order ℓ characters for $\ell = 3, 4$ and 6 in the non-Kummer setting as given in (3.7) is*

$$M_\ell^{\mathrm{nK}}(\phi, D_\ell(g)-2) = \\ \frac{\ell-1}{g} \sum_{1\le n\le N/\ell} \hat{\Phi}\left(\frac{\ell(\ell-1)n}{2g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{d(Q)}{|Q|^{\ell r/2}\left(1+|Q|^{-2/m_Q}\right)^{m_Q}} + O\left(q^{-D_\ell(g)/2+\epsilon g}\right).$$

Proof. Recall that we use $\mathcal{F}_\ell^{\mathrm{nK}}(g)$ as in (3.3) to denote the family of order ℓ Dirichlet L-functions in the non-Kummer setting. In this case, the number of non-trivial zeros is $D_\ell(g) - 2 = \dfrac{2g}{\ell-1}$. Thus the main term, which comes from the ℓ^{th} power polynomials, is $M_\ell^{\mathrm{nK}}\left(\phi, \frac{2g}{\ell-1}\right)$ as follows.

$$M_\ell^{\mathrm{nK}}\left(\phi, \frac{2g}{\ell-1}\right) = \frac{\ell-1}{2g|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{1\le n\le N/\ell} \hat{\Phi}\left(\frac{\ell(\ell-1)n}{2g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{d(Q)}{|Q|^{\ell r/2}} \\ \times \sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} \chi_F(Q^{\ell r}) + \overline{\chi_F(Q^{\ell r})}. \tag{3.12}$$

Since $\chi_F(Q^{\ell r}) = \overline{\chi_F(Q^{\ell r})} = 1$, we have

$$M_\ell^{\mathrm{nK}}\left(\phi, \frac{2g}{\ell-1}\right) = \frac{\ell-1}{g|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{1\le n\le N/\ell} \hat{\Phi}\left(\frac{\ell(\ell-1)n}{2g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{d(Q)}{|Q|^{\ell r/2}} \sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} 1. \tag{3.13}$$

Using Lemma 3.4, the sum over Q can be written as

$$\begin{aligned} & q^{D_\ell(g)}\left(q^{-2}-q^{-4}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)E_0(1/q^2)}{|Q|^{\ell r/2}E_f(1/q^2)}\prod_{\substack{\pi\in\mathbb{F}_{q^2}[t]\\ \pi|Q}}\left(1+|\pi|_{q^2}^{-1}\right)^{-1}+O\left(q^{D_\ell(g)/2+\epsilon g}\right) \\ &= q^{D_\ell(g)}\left(q^{-2}-q^{-4}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)E_0(1/q^2)}{|Q|^{\ell r/2}E_f(1/q^2)\left(1+|Q|^{-2/m_Q}\right)^{m_Q}}+O\left(q^{D_\ell(g)/2+\epsilon g}\right). \end{aligned} \tag{3.14}$$

We divide (3.14) by $|\mathcal{F}_\ell^{\mathrm{nK}}(g)|$ and obtain

$$\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)}{|Q|^{\ell r/2}\left(1+|Q|^{-2/m_Q}\right)^{m_Q}}+O\left(\frac{Nq^{-D_\ell(g)/2+\epsilon g}}{g}\right).$$

Note here that the error term from dividing the size of the family is of the same size as the error term above. Now using the equation above and (3.13) gives the stated result. □

3.4.2 The Error Term

Recall that the error term in (3.8) comes from the non-ℓ^{th} power contributions. To compute the error term we first prove the following lemma.

Lemma 3.6. *Let f be a monic non-ℓ^{th} power polynomial in $\mathbb{F}_q[t]$. Then*

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}}\chi_F(f)\ll q^{D_\ell(g)/2}q^{\epsilon(d(f)+D_\ell(g))}.$$

Proof. We first write the sum over F as

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} \chi_F(f) = \sum_{F\in\mathcal{H}_{q^2,D_\ell(g)/2}} \chi_F(f) \sum_{\substack{D\in\mathbb{F}_q[t]\\ D|F}} \mu(D)$$
$$= \sum_{\substack{D\in\mathbb{F}_q[t]\\ d(D)\le D_\ell(g)/2\\ (D,f)=1}} \mu(D) \sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2-d(D)}\\ (F,D)=1}} \chi_F(f),$$

where the inner sum over F has the generating series

$$\sum_{F\in\mathcal{H}_{q^2}} \chi_F(f)u^{d(F)} = \prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi\nmid Df}} 1+\chi_\pi(f)u^{d(\pi)} = \frac{\mathcal{L}_{q^2}(u,\chi_f)}{\mathcal{L}_{q^2}(u^2,\chi_f^2)} \prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi|D\\ \pi\nmid f}} \frac{1-\chi_\pi(f)u^{d(\pi)}}{1-\chi_\pi^2(f)u^{2d(\pi)}}.$$

We evaluate the inner sum above using Perron's formula. Thus

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2-d(D)}\\ (F,D)=1}} \chi_F(f) = \frac{1}{2\pi i}\oint \frac{\mathcal{L}_{q^2}(u,\chi_f)}{\mathcal{L}_{q^2}(u^2,\chi_f^2)u^{D_\ell(g)/2-d(D)}} \prod_{\substack{\pi\in\mathcal{P}_{q^2}\\ \pi|D\\ \pi\nmid f}} \frac{1-\chi_\pi(f)u^{d(\pi)}}{1-\chi_\pi^2(f)u^{2d(\pi)}}\frac{du}{u},$$

where we integrate along a circle of radius $|u|=q^{-1}$ around the origin.

The Lindelöf bound in Lemma 1.3 for the L-function in the numerator gives

$$|\mathcal{L}_{q^2}(u,\chi_f)| \ll q^{2\epsilon d(f)},$$

and the lower bound in Lemma 1.4 for the L-function in the denominator gives

$$|\mathcal{L}_{q^2}(u^2,\chi_f^2)| \gg q^{-2\epsilon d(f)}.$$

Hence

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2-d(D)}\\ (F,D)=1}} \chi_F(f) \ll q^{D_\ell(g)/2-d(D)}q^{4\epsilon d(f)+2\epsilon d(D)}.$$

Now, trivially bounding the sum over D gives

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} \chi_F(f) \ll \sum_{m=0}^{D_\ell(g)/2} \sum_{\substack{D\in\mathbb{F}_q[t]\\ d(D)=m}} q^{D_\ell(g)/2-m} q^{4\epsilon d(f)+2\epsilon m} \ll q^{D_\ell(g)/2} q^{\epsilon(d(f)+D_\ell(g))}.$$

This concludes the proof of the above lemma. □

Using Lemma 3.6, we bound the contribution from $E_\ell^{\mathrm{nK}}(\phi, D_\ell(g)-2)$.

Lemma 3.7. *The contribution of non-ℓ^{th} power terms is bounded by*

$$E_\ell^{\mathrm{nK}}(\phi, D_\ell(g)-2) \ll q^{N/2} q^{-D_\ell(g)/2} q^{\epsilon(N+g)}.$$

Proof. Recall that

$$E_\ell^{\mathrm{nK}}(\phi, D_\ell(g)-2)$$
$$= \frac{1}{(D_\ell(g)-2)\,|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right) \sum_{\substack{f\in\mathcal{M}_{q,n}\\ f \text{ non-}\ell^{\mathrm{th}} \text{ power}}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} \left[\chi_F(f) + \overline{\chi_F(f)}\right],$$

as in (3.8).

Lemma 3.6 implies that,

$$\sum_{\substack{F\in\mathcal{H}_{q^2,D_\ell(g)/2}\\ P|F\Rightarrow P\notin\mathbb{F}_q[t]}} \overline{\chi_F(f)} \ll q^{D_\ell(g)/2} q^{\epsilon(d(f)+D_\ell(g))}.$$

Thus

$$E_\ell^{\mathrm{nK}}(\phi, D_\ell(g)-2) \ll \frac{\ell-1}{g|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{(\ell-1)n}{2g}\right) \sum_{\substack{f\in\mathcal{M}_n \\ f \text{ non-}\ell^{\text{th}} \text{ powers}}} \frac{\Lambda(f) q^{D_\ell(g)/2} q^{\epsilon(n+D_\ell(g))}}{|f|^{1/2}}$$

$$\ll \frac{\ell-1}{g|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{(\ell-1)n}{2g}\right) \frac{nq^n}{q^{n/2}} \left(q^{D_\ell(g)/2} q^{\epsilon(n+D_\ell(g))}\right)$$

$$\ll q^{N/2} q^{-D_\ell(g)/2} q^{\epsilon(N+g)},$$

which gives the result above. □

3.4.3 The Non-Kummer Setting Results

Proof of Theorem 1.4. Recall some notations in Section 1.3. Since $D_\ell(g)-2 = \dfrac{2g}{\ell-1}$, the one-level density in the non-Kummer setting is

$$\mathcal{D}_\ell^{\mathrm{nK}}(\phi, g) = \hat{\Phi}(0) - \mathcal{A}_\ell^{\mathrm{nK}}\left(\phi, \frac{2g}{\ell-1}\right) - \frac{\ell-1}{g} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right) q^{-n/2}, \tag{3.15}$$

where as in (3.6)

$$\mathcal{A}_\ell^{\mathrm{nK}}\left(\phi, \frac{2g}{\ell-1}\right)$$
$$= \frac{1}{(D_\ell(g)-2)\,|\mathcal{F}_\ell^{\mathrm{nK}}(g)|} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D_\ell(g)-2}\right) \sum_{f\in\mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{\substack{F\in\mathcal{H}_{q^2, D_\ell(g)/2} \\ P|F \Rightarrow P\notin\mathbb{F}_q[t]}} \left[\chi_F(f) + \overline{\chi_F(f)}\right].$$

From Lemma 3.5 and Lemma 3.7, we write Equation (3.15) above as

$$\mathcal{D}_\ell^{\mathrm{nK}}(\phi, g) = \hat{\Phi}(0) - \frac{\ell-1}{g} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{(\ell-1)n}{2g}\right) q^{-n/2}$$
$$- \frac{\ell-1}{g} \sum_{1\le n\le N/\ell} \hat{\Phi}\left(\frac{\ell(\ell-1)n}{2g}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r} \\ r\ge1}} \frac{d(Q)}{|Q|^{\ell r/2}\left(1+|Q|^{-2/m_Q}\right)^{m_Q}} + O\left(q^{N/2} q^{-D_\ell(g)/2} q^{\epsilon(N+g)}\right),$$

where $m_Q = \gcd(d(Q), 2)$. This is the result in Theorem 1.4. $\square$

Using the theorem above, we prove the symmetry type of the family below.

Proof of Theorem 1.5. Let $N < \dfrac{2g}{\ell - 1}$. Then

$$\lim_{g\to\infty} \mathcal{D}_\ell^{\mathrm{nK}}(\phi, g) = \hat{\Phi}(0),$$

since the double sums over n and Q above are $o(1)$ as $g \to \infty$.

Furthermore, compute both integeral to confirm that

$$\hat{\Phi}(0) = \int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{\mathcal{W}}_{U(D_\ell(g)-2)}(y)dy = \int_{-\infty}^{\infty} \hat{\Phi}(y)\delta_0(y)dy,$$

where $\mathcal{W}_{U(D_\ell(g)-2)}(y) = \delta_0(y)$ denotes the one-level scaling density of the group of unitary matrices. $\square$

This proves the symmetry types of the families are unitary and it supports the philosophy of Katz and Sarnak.

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Appendix A

Order ℓ Dirichlet characters in the non-Kummer setting for ℓ a Mersenne prime

The works above required the multiplicative order n_q of $q \pmod{\ell}$ to be 2. We discuss a special case when n_q is the prime exponent of a Mersenne prime. Suppose n_q is a prime number p, and $\phi(\ell) = 2^p - 2$. Since

$$\binom{p}{1} + \cdots + \binom{p}{p-1} = 2^p - 2,$$

the number of order ℓ primitive characters with conductor P equals to the number of conductors $F \neq P$, and F a product of distinct primes in the splitting of P over $\mathbb{F}_{q^p}[t]$. More precisely, let P splits as in (**??**) and let $S(P)$ denote the set

$$S(P) = \{F \in \mathbb{F}_{q^p}[t] : F \neq P \text{ and } F \text{ squarefree consisting of products of } \pi_i\}. \tag{A.1}$$

For distinct $\{e_i\}$ coprime to ℓ, we have

$$\left|\{\chi_P^{e_1}, \cdots, \chi_P^{e_{\phi(\ell)-1}}\}\right| = |S(P)|.$$

Furthermore, under certain conditions on $q \pmod{\ell}$, we have the equivalence of characters as seen in (**??**). Let $q \equiv 2 \pmod{\ell}$ for convenience. (In general, we can take $q \equiv 2^k \pmod{\ell}$ for some positive integer $k < p$. This follows from the observation that a complete set of residues modulo p is preserved under multiplication by an integer k for $1 \leq k \leq p-1$.) The q^{th}-power Frobenius map ψ defined on $\mathbb{F}_{q^p}[t]$ (as seen in Lemma 2.9 of [4]) acts on the prime factors of P. For $\pi_i = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$,

$$\psi\left(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0\right) = a_n^q t^n + a_{n-1}^q t^{n-1} + \cdots + a_0^q$$

where ψ raises the coefficients of π_i to the power q. For convenience, we can order the prime factors of P in the following way. Let $\pi_1 = \pi$ as given in Remark **??**, and

$$\psi(\pi_i) = \begin{cases} \pi_{i+1}, & 1 \leq i \leq p-1, \\ \pi_1, & i = p. \end{cases} \tag{A.2}$$

Then, the character

$$\chi_{\pi_{i+1}} = \chi_{\psi(\pi_i)} = \chi_{\pi_i}^2,$$

and we have the equivalence of characters

$$\{\chi_P^{e_1}, \cdots, \chi_P^{e_{\phi(\ell)-1}}\} = \{\chi_F|_{\mathbb{F}_q[t]} : F \in S(P)\},$$

since each integer e_i has a unique binary representation.

We have shown the following.

Lemma A.1. *Let $\ell = 2^p - 1$ for some prime number p be a Mersenne prime. Suppose χ_P is a primitive order ℓ character over $\mathbb{F}_q[t]$ with the prime conductor P and $q \equiv 2^k \pmod{\ell}$ for some integer $1 \leq k < p$. Let χ_{π_i} be the characters defined in Definition 3.1. Then*

$$\{\chi_P, \chi_P^2, \cdots, \chi_P^{2^p-2}\} = \{\chi_F : F \in S(P)\}|_{\mathbb{F}_q[t]},$$

where the characters on the right are restricted to $\mathbb{F}_q[t]$ and $S(P)$ is defined in (A.1).

The septic non-Kummer characters

As an example, we compute explicitly the equivalence of character for septic Dirichlet L-functions when $q \equiv 2 \pmod 7$. The case for $q \equiv 4 \pmod 7$ is exactly the same with some reordering.

The multiplicative order of 2 in $(\mathbb{Z}/7\mathbb{Z})^\times$ is $3 = n_q$. Therefore, all prime conductors $P \in \mathbb{F}_q[t]$ in the non-Kummer setting have degrees divisible by 3, and P splits over $\mathbb{F}_{q^3}[t]$ as

$$P = \pi_1 \pi_2 \pi_3$$

for some primes π_i in the extension following the ordering given in (A.2). Thus, restricting the characters to $\mathbb{F}_q[t]$, we have

$$\begin{aligned}
&\chi_{\pi_1} = \chi_P,\\
&\chi_{\psi(\pi_1)} = \chi_{\pi_2} = \chi_P^2, \quad \chi_{\psi(\pi_2)} = \chi_{\pi_3} = \chi_P^4,\\
&\chi_{\pi_1\pi_2} = \chi_P^3, \; \chi_{\pi_1\pi_3} = \chi_P^5, \; \chi_{\pi_2\pi_3} = \chi_P^6.
\end{aligned}$$

Note it is not crucial to take the ordering in (A.2). Without it, we have, for a fixed positive integer α less than 7 and restricting the characters to $\mathbb{F}_q[t]$,

$$\chi_{\pi_1} = \chi_P^{\alpha},$$
$$\chi_{\psi(\pi_1)} = \chi_{\pi_2} = \chi_P^{2\alpha}, \quad \chi_{\psi(\pi_2)} = \chi_{\pi_3} = \chi_P^{4\alpha},$$
$$\chi_{\pi_1\pi_2} = \chi_P^{3\alpha},\ \chi_{\pi_1\pi_3} = \chi_P^{5\alpha},\ \chi_{\pi_2\pi_3} = \chi_P^{6\alpha}.$$

Since multiplying by α is an automorphism on $(\mathbb{Z}/7\mathbb{Z})^{\times}$,

$$\{\chi_P^{\alpha}, \chi_P^{2\alpha}, \cdots, \chi_P^{6\alpha}\} = \{\chi_P, \chi_P^2, \cdots, \chi_P^6\}.$$

Thus we have shown the following.

Lemma A.2. *Let χ_P be a primitive septic character over $\mathbb{F}_q[t]$ with the prime conductor P for some $q \equiv 2, 4 \pmod 7$. Let χ_{π_i} be the characters defined in Definition 3.1. Then*

$$\{\chi_P, \chi_P^2, \cdots, \chi_P^6\} = \{\chi_{\pi_1}, \chi_{\pi_2}, \chi_{\pi_3}, \chi_{\pi_1\pi_2}, \chi_{\pi_1\pi_3}, \chi_{\pi_2\pi_3}\}|_{\mathbb{F}_q[t]},$$

where the characters on the right are restricted to $\mathbb{F}_q[t]$.