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The Most Efficient Implementation of the IQML Algorithm

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***Abstract*— The recent work by Clark and Scharf showed a new implementation of the IQML (iterative quadratic maximum likelihood) algorithm, which requires at each iteration the computational flops of the order N^2 where N is the dimension of signal vector (or length of data sequence). They also indicated that the implementation of other related algorithms such as the Steiglitz-McBride algorithm would also require the order N^2 of computations. We present here a better way of implementation which requires the computational flops of the order N. This better way of implementation is shown in detail for the IQML algorithm. Following the same idea shown in this paper, one can also straightforwardly design the order N implementation of the Steiglitz-McBride algorithm. Our implementation is also the most efficient in that no implementation can be made less than order N.**

I. INTRODUCTION

For parameter estimation of superimposed exponentials and its related applications, the iterative quadratic maximum likelihood (IQML) algorithm has been found to be very useful. Recently, the IQML algorithm shown in [1], [2] and its predated equivalents shown in [3], [4] have been reexamined in [5], [6]. In particular, the computational complexity of the IQML algorithm was discussed by Clark and Sharf [6]. They exploited an eigendecomposition property of circular matrix in the implementation of the IQML algorithm and showed that the resulting algorithm would require the computational flops of the order N^2 where N is the dimension of signal vector. They also indicated that the implementations previously proposed by others (i.e., [1] and [4]) would require the same order of computations.

In this correspondence, we present a better way of implementing the IQML algorithm and show that the flops required by the new implementation are of the order N. In this implementation, we apply the Cholesky decomposition of a banded Hermitian matrix, which is in contrast to the idea of computing the inverse of a circular matrix [1], [6]. The detailed implementation of the IQML algorithm and its flop count are shown next. We note that one can also design the order N implementation of the Steiglitz-McBride algorithm by removing all redundant computations.

II. IMPLEMENTATION OF THE IQML ALGORITHM

It is easy to verify that at each iteration the IQML algorithm computes such a vector $\mathbf{b} = [b_0, b_1, \cdots, b_p]^T$ that

$$\mathbf{b}^H \mathbf{Y}^H (\mathbf{B}\mathbf{B}^H)^{-1} \mathbf{Y}\mathbf{b} \tag{1}$$

is minimized, subject to $b_0 = 1$, where ^{T} denotes the transpose, ^{H} the conjugate transpose, $\mathbf{Y}$ is a $(N-p) \times (p+1)$ Hankel matrix, and $\mathbf{B}$ the $(N-p) \times N$ matrix defined as

$$\mathbf{B} = \begin{bmatrix} b'_p & \cdots & b'_1 & b'_0 & & & \\ & b'_p & \cdots & b'_1 & b'_0 & & \\ & & & \cdots & & & \\ & & & b'_p & \cdots & b'_1 & b'_0 \end{bmatrix} \tag{2}$$

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where $\mathbf{b} = [b_0, b_1, \cdots, b_p]^T$ is the solution vector from the previous iteration, and $b'_0 = 1$. It should be noted that (a) the notations used here mainly follow those in [6]; (b) the dimension $p+1$ of the vector $\mathbf{b}$ is often much smaller than N the dimension of signal vector (p is also referred to as the model order of signal, and N is also the length of data sequence.); and (c) in array processing where a number of data sequences are processed, an expression similar to (1) needs to be minimized at each iteration [2, eq. 25] and hence the following discussion can be similarly applied.

The inverse of the $(N-p) \times (N-p)$ matrix $\mathbf{BB}^H$ was seen as a major part of computation. To speed up the computation, Kumaresan *et al.* [1] suggested to write $\mathbf{BB}^H$ into a circular matrix plus a matrix component, and then apply the fast Fourier transform (FFT) to obtain the inverse $(\mathbf{BB}^H)^{-1}$. In [6], Clark and Scharf suggested to treat $\mathbf{B}$ as a submatrix of a circular matrix and then apply the FFT (with a processing to solve the ill-condition problem of the circular matrix) to achieve computational efficiency. As shown in [6], the resulting algorithms from the above two approaches require at each iteration the order N^2 of flops.

In the following, we treat $\mathbf{BB}^H$ as a banded Hermitian matrix with the (single-sided) bandwidth p. The resulting algorithm requires the order N of flops, instead of N^2! (Each flop here means roughly a complex multiplication and a complex addition [7].)

Step 1: Compute $\mathbf{BB}^H$.

This $(N-p) \times (N-p)$ product matrix can be seen as the autocorrelation matrix of the sequence $b'_0, b'_1, \cdots, b'_p$ with its (i,j)th element defined by

$$c_{i-j} = \begin{cases} \sum_{k=0}^{p-i+j} b'_k \quad b'^*_{k+i-j}, & \text{for } i \geq j \\ c_{j-i^*}, & \text{for } i < j \\ 0, & \text{for } |i-j| > p \end{cases} \quad (3)$$

This step (or equivalently computing c_τ for $\tau = 0, 1, \cdots, p$) requires

$$\frac{1}{2} \quad (p+1)(p+2) \qquad \text{flops.} \quad (4)$$

Step 2: Cholesky decomposition $\mathbf{R}^H\mathbf{R}$ of $\mathbf{BB}^H$.

Since $\mathbf{BB}^H$ is $(N-p) \times (N-p)$ positive-definite banded Hermitian matrix with the (single-sided) bandwidth p, one can follow (a complex version of) Algorithm 5.3–5 in [7] to obtain the factorization $\mathbf{R}^H\mathbf{R}$ where $\mathbf{R}$ is $(N-p) \times (N-p)$ non-singular upper triangular matrix with p upper bandwidth. This requires [7, p. 96]

$$(N-p)\left(\frac{1}{2}p^2 + \frac{3}{2}p\right) - \frac{1}{3}p^3 - \frac{3}{2}p^2 \qquad \text{flops} \quad (5)$$

and $N-p$ square roots. Note that this step can be easily accomplished by using the IMSL (version 1.0) routine LFSQHL.

Step 3: Compute $(\mathbf{R}^H)^{-1}\mathbf{Y}$.

Since $\mathbf{R}$ is nonsingular banded upper triangular, a simple modification of Algorithm 5.3–2 or 5.3–3 in [7] can yield the product $(\mathbf{R}^H)^{-1}\mathbf{Y}$ after

$$(N-p)(p+1)^2 - \frac{1}{2}p^2(p+1) \qquad \text{flops.} \quad (6)$$

Step 4: Compute $((\mathbf{R}^H)^{-1}\mathbf{Y})^H(\mathbf{R}^H)^{-1}\mathbf{Y}$.

This product is identical to $\mathbf{Y}^H(\mathbf{BB}^H)^{-1}\mathbf{Y}$. With consideration of the Hermitian structure in this matrix, we know that this step requires

$$\frac{1}{2}(N-p)(p+2)(p+1) \qquad \text{flops.} \quad (7)$$

Step 5: Compute the "smallest" eigenvector of $\mathbf{Y}^H \ (\mathbf{BB}^H)^{-1} \ \mathbf{Y}$.

Since this matrix has the dimension $(p+1) \times (p+1)$, the computational order of this step must be less than N.

Steps 1–5 complete one (any one) iteration required by the IQML algorithm. (The additional constraints as mentioned in [1] are not considered here. A nonlinear constraint on $\mathbf{b}$ could require another search procedure and hence its flop count would be difficult to obtain.) Adding all the flops shown in (4)–(7) leads to

$$\text{Flop}_{\text{IQML}} \cong (2p^2 + 5p + 1)N \quad (8)$$

which (plus $N-p$ square roots) are the total flops required by the IQML algorithm for each iteration assuming $N \gg p$. Note that the flops can be further reduced if the Cholesky decomposition is replaced by the $L-D-L^H$ decomposition [7], which avoids the $N-p$ square roots. The Toeplitz structure in $\mathbf{BB}^H$ can also be exploited to reduce computations by following Rialan-Scharf [8]. But the order of computations can never be made less than N.

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