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ELECTROSTATIC BOUNCE MODES IN MIRROR PLASMAS

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Abstract

Bounce modes are plasma waves which result from coherent bounce motion of trapped particles. Such modes can occur in mirror devices when the relative spread in electron bounce frequencies is small. When electron bounce modes have perpendicular wavelengths comparable with typical ion gyroradii and frequencies near the ion gyrofrequency or some harmonic, they couple effectively with the ions and can be destabilized by a loss-cone ion distribution. We derive a one-dimensional integral dispersion equation for electron bounce modes in a mirror plasma and solve it numerically for the wave eigenfrequencies. Predicted threshold densities for bounce mode instabilities in the Baseball I and Baseball II devices are compared with experimental values. Good agreement is found for Baseball II, but in Baseball I the predicted thresholds are consistently below observed values.

1. INTRODUCTION

Instabilities in low-density mirror devices such as Baseball I (BBI),^{1,2} Phoenix II,^{3,4} and Ogra II⁵ have been attributed to electron plasma waves driven by free energy from the anisotropic ion distribution. Results from BBI seemed to support this model: The waves had frequencies near low harmonics of the ion gyrofrequency, and observed threshold densities showed the linear dependence on the ratio W_e/W_i of electron thermal energy to ion energy predicted by infinite medium theory.² However, the densities at which instability occurred in the Baseball II (BBII) experiment were well below theoretical predictions and were unaccountably sensitive to changes in plasma length and the buildup procedure.⁶

To explain the BBII results, we develop a theoretical model of electrostatic mirror plasma instabilities that includes the effects of electron bounce motion. This periodic motion can result in strong plasma resonances when the electron bounce frequency ω_b has a spread $\Delta\omega_b \equiv \langle |\omega_b - \langle \omega_b \rangle| \rangle$ small enough to satisfy

$$\frac{\omega_b^2}{\omega \Delta\omega_b} \gg 1. \quad (1.1)$$

Electrons then retain substantial phase coherence on successive transits and can support standing wave modes. In typical hot-ion mirror plasmas, electrons often meet this coherence criterion because they are confined

along flux lines principally by an electrostatic potential that is approximately quadratic. Bounce modes can have wavelengths along the magnetic field comparable with the plasma length, and they are distinguished by the strong plasma response at frequencies near harmonics of the mean bounce frequency. Ions can couple effectively to electron bounce modes that have frequencies near harmonics of the ion gyrofrequency and have wavelengths perpendicular to the magnetic field that are about twice typical ion gyroradii. If the ion perpendicular energy distribution is peaked, then the free energy associated with the anisotropy can be transferred to the wave, and instability occurs when this ion drive exceeds electron Landau damping. This instability mechanism was neglected by previous treatments using fluid models of electron behavior, and it can significantly reduce density thresholds for instability in some mirror plasmas.

We treat electron bounce modes by solving a one-dimensional integral equation for the wave potential along a flux line. To retain bounce resonance effects, accurate electron bounce trajectories are used to calculate the plasma response to the wave. Our approach is similar to that of Beasley et al.^{7,8} The integral equation formulation avoids the limitation to short parallel wavelength modes found in Wentzel-Kramers-Brillouin (WKB) treatments of bounce resonance phenomena,^{9,10,11} and we improve upon previous solutions of the integral dispersion equation^{4,12,13,14} by using more realistic models of the equilibrium fields and distribution functions. A derivation of the integral equation is presented in section 2 along with the methods used to solve the

equation for marginally stable waves and to calculate instability threshold densities. Section 3 summarizes the numerical results, first reviewing bounce mode dispersion characteristics and then discussing threshold densities for bounce mode instabilities. In section 4, thresholds predicted by the model for the BBI and BBII devices are compared with experimental values. Reference 15 gives a more detailed presentation of the numerical methods and results.

2. BOUNCE MODE FORMALISM

MODEL

Electrostatic bounce modes in a finite mirror plasma are obtained from an appropriate integral eigenvalue equation for the electrostatic potential of small-amplitude waves. This equation is reduced to a matrix eigenvalue equation by Fourier analysis and solved numerically.

The physical situation considered retains important features of typical hot-ion mirror plasmas. Both the magnetic field strength B and electrostatic potential ϕ are assumed to be monotonic functions of the distance s along flux lines, cutting off at the endpoints $\pm s_{\text{max}}$. Particles which reach either endpoint with a nonzero velocity along a flux line are considered to escape confinement immediately. For a particle of species α with a charge q_α and mass m_α , we assume that total particle energy $E = m_\alpha (v_\parallel^2 + v_\perp^2) + q_\alpha \phi$ and magnetic moment $\mu = m_\alpha v_\perp^2 / 2B$ are exact constants of the motion, where $v_\parallel$ and $v_\perp$ are velocity components respectively along and perpendicular to the local magnetic field. The equilibrium distribution function F_α on a flux line can then in general be written in terms of these constants. An appropriate form for electrons is

$$F_e(E, \mu) = C_e (\mu B_{\text{max}} + \psi_{\text{max}} - E) \exp(-E/W_e). \quad (2.1)$$

This describes a distribution that is approximately thermalized at energies well below the cutoff value $E = \mu B_{\max} + \psi_{\max}$ but drops linearly to zero near the loss boundaries. Here $B_{\max}$ and $\psi_{\max}$ are the values of B and electrostatic potential energy $\psi \equiv -e\phi$ at $s_{\max}$, and the normalization constant C_e is chosen so that the electron number density is given by

$$N(s) = 4\pi \frac{N_0 B(s)}{m_e^2} \int_0^\infty d\mu \int_{\mu B(s) + \psi(s)}^{\mu B_{\max} + \psi_{\max}} dE \frac{F_e(E, \mu)}{|v_{e\parallel}(s; E, \mu)|}, \quad (2.2)$$

where $v_{e\parallel} = \pm \left[2(E - \mu B - \psi)/m_e \right]^{1/2}$ is the parallel electron velocity, and the subscript 0 hereafter labels quantities at $s = 0$. In general, the ion distribution on a flux line is also a function E and μ . However, for the modes we consider the ion response depends principally on spatially local properties of the distribution, and these features are conveniently modeled by a function of the form

$$F_i(s, v_{\parallel}, v_{\perp}) = \frac{N(s)}{N_0} F_{\parallel}(v_{\parallel}) F_{\perp}(v_{\perp}^2). \quad (2.3)$$

Here $F_{\parallel}$ and $F_{\perp}$ are unit normalized according to $2 \int_0^\infty dv_{\parallel} F_{\parallel} = 1$ and $\pi \int_0^\infty dv_{\perp}^2 F_{\perp} = 1$, and since we assume a quasineutral equilibrium plasma, N is set equal to the electron number density expression (2.2). The functions chosen for $F_{\perp}$ model two simple situations: At low density, a plasma decays principally by charge exchange with the background gas.

In this case, an initially monoenergetic ion distribution such as that created by neutral-beam injection will remain peaked about some mean value $V_1 \equiv \langle v_1^2 \rangle^{1/2}$, and a delta function

$$F_1(v_1^2) = \frac{1}{\pi} \delta(v_1^2 - V_1^2) \quad (2.4)$$

is an appropriate approximation. At higher densities, electron drag and ion-ion collisions can spread the perpendicular distribution during the plasma lifetime. A general broadened distribution

$$F_1(v_1^2) = \frac{4}{\pi V_1^2} v_1^2 \exp\left(-\frac{2v_1^2}{V_1^2}\right) \quad (2.5)$$

is then used. In practice, conditions such as the finite neutral-beam cross section, multiple energy components in the beam, and spreading of the distribution by electron drag and collisions during buildup result in more complicated ion distributions, but the effect of ions on plasma stability is still expected to be between the two extreme cases modeled by (2.4) and (2.5). For the ion parallel velocity distribution, a Gaussian form

$$F_{||}(v_{||}) = \frac{1}{\pi^{1/2} V_{||}} \exp\left(-\frac{v_{||}^2}{V_{||}^2}\right) \quad (2.6)$$

is normally chosen, where $V_{||}^2 \equiv \langle v_{||}^2 \rangle$ is the mean squared parallel velocity, and a ratio

$$\frac{v_{\parallel}^2}{v_{\perp}^2} = \frac{2(B_{\max} - B_0)}{4B_{\max} + B_0}, \quad (2.7)$$

appropriate for magnetically confined ions, is used.

We use the linearized Vlasov-Maxwell equations to derive a one-dimensional integral equation for the wave potential ϕ along a field line. For species α , the perturbation f_{α} of the equilibrium distribution function F_{α} due to ϕ is given to lowest order in perturbed quantities by the linearized Vlasov equation

$$\frac{df_{\alpha}}{dt}(\underline{x}, \underline{v}, t) = \frac{q_{\alpha}}{m_{\alpha}} \nabla \phi(\underline{x}, \underline{v}, t) \cdot \frac{\partial F_{\alpha}}{\partial \underline{v}}(\underline{x}, \underline{v}). \quad (2.8)$$

When F_{α} is written in terms of E and μ , (2.8) can be formally integrated to give

$$\begin{aligned} f_{\alpha}(\underline{x}, \underline{v}, t) = & q_{\alpha} \frac{\partial F_{\alpha}(E, \mu)}{\partial E} \int_{-\infty}^t dt' \left[\frac{d\phi(\underline{x}', t)}{dt} \Big|_{\underline{x}'} - \frac{\partial \phi(\underline{x}', t')}{\partial t} \right] \\ & + q_{\alpha} \frac{\partial F_{\alpha}(E, \mu)}{\partial \mu} \int_{-\infty}^t dt' \frac{\underline{v}' \cdot \nabla \phi(\underline{x}', t')}{B(\underline{x}')}, \end{aligned} \quad (2.9)$$

where $\underline{x}'_{\alpha}$ and $\underline{v}'_{\alpha}$ are particle position and velocity at time $t' < t$ on an unperturbed trajectory intersecting phase point $(\underline{x}, \underline{v})$ at time t , and the usual assumption that f_{α} vanishes at $t = -\infty$ has been made. To simplify the time integrals in (2.9), we neglect the wave structure perpendicular to flux lines and represent the perturbed quantities by

$$\phi(\underline{x}, t) = \phi(s) \exp(i \underline{k}_{\perp} \cdot \underline{x} - i \omega t) \quad (2.10)$$

and

$$f_{\alpha}(\underline{x}, \underline{v}, t) = f_{\alpha}(s; E, \mu, \theta) \exp(i \underline{k}_{\perp} \cdot \underline{x} - i \omega t), \quad (2.11)$$

where $\underline{k}_{\perp}$ is the wavevector component perpendicular to the magnetic field and $\theta \equiv \cos^{-1}(\underline{k}_{\perp} \cdot \underline{v} / |\underline{k}_{\perp}| |\underline{v}|)$ is the gyration phase angle. To account approximately for the change in flux surface cross section with s , we take $k_{\perp}^2/B$ to be constant along a field line.¹⁶ Since the local gyrofrequency $\Omega_{\alpha} = q_{\alpha} B / m_{\alpha} c$ and gyroradius $a_{\alpha} = v_{\perp} / \Omega_{\alpha}$ are both effectively constant over a gyroperiod, we may take $\underline{k}_{\perp} \cdot \underline{x} = k_{\perp} a_{\alpha} \sin \theta$ in the ϕ and f_{α} phase factors. Noting that $k_{\perp} a_{\alpha} = \frac{c}{q_{\alpha}} (2m_{\alpha} \mu)^{1/2} (k_{\perp}^2/B)^{1/2}$ is an approximate constant of the motion, we then write the $\exp(i \underline{k}'_{\perp} \cdot \underline{x}'_{\alpha})$ factor in the time integrals of (2.9) in the form

$$\exp(i \underline{k}_{\perp} \cdot \underline{x}) \exp(i \underline{k}'_{\perp} \cdot \underline{x}'_{\alpha} - i \underline{k}_{\perp} \cdot \underline{x}) = \exp(i \underline{k}_{\perp} \cdot \underline{x}) \exp[i k_{\perp} a_{\alpha} (\sin \theta' - \sin \theta)]$$

and expand it in a double Bessel series using the relation¹⁷

$$\exp(i x \sin \theta) = \sum_{j=-\infty}^{\infty} J_j(x) \exp(i j \theta), \quad (2.12)$$

where J_j is a j th order Bessel function of the first kind.

This expansion allows the f_{α} expression (2.9) to be rewritten as

$$f_{\alpha}^{\pm}(s; E, \mu, \theta) = q_{\alpha} \left\{ \phi(s) \frac{\partial F_{\alpha}(E, \mu)}{\partial E} + i \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[\omega \frac{\partial F_{\alpha}(E, \mu)}{\partial E} + \frac{j \Omega_{\alpha}}{B} \frac{\partial F_{\alpha}(E, \mu)}{\partial \mu} \right] \right. \\ \left. J_j(k_1 a_{\alpha}) J_m(k_1 a_{\alpha}) \int_{-\infty}^t dt' \phi(s_{\alpha}^{\pm'}) \exp[i(j-m)\theta] \exp[-ij\Delta^{\pm}\theta_{\alpha}(t', t) - i\omega(t'-t)] \right\}, \quad (2.13)$$

where the $\pm$ superscripts designate quantities on positive and negative going trajectories at s , and

$$\Delta^{\pm}\theta_{\alpha}(t', t) \equiv \theta - \theta(s_{\alpha}^{\pm'}) = \int_{t'}^t dt'' \Omega_{\alpha}(s_{\alpha}^{\pm'')}. \quad (2.14)$$

When (2.13) is integrated to give the number density perturbation

n_{α} , the integral is nonzero only for $j = m$. Explicitly, n_{α} is given by

$$n_{\alpha}(s) = \frac{2\pi q_{\alpha} N_0 B(s)}{m_{\alpha}^2} \int_0^{\infty} d\mu \int_{\mu B(s) + \psi(s)}^{\mu B_{\max} + \psi_{\max}} dE \frac{1}{|v_{\alpha}(s; E, \mu)|} \left\{ 2 \frac{\partial F_{\alpha}(E, \mu)}{\partial E} \phi(s) \right. \\ \left. + i \sum_{j=-\infty}^{\infty} \left[\omega \frac{\partial F_{\alpha}(E, \mu)}{\partial E} + \frac{j \Omega_{\alpha}}{B} \frac{\partial F_{\alpha}(E, \mu)}{\partial \mu} \right] J_j^2(k_1 a_{\alpha}) \right. \\ \left. \sum_{\pm} \int_{-\infty}^t dt' \phi(s_{\alpha}^{\pm'}) \exp[-ij\Delta^{\pm}\theta_{\alpha}(t', t) - i\omega(t'-t)] \right\}. \quad (2.15)$$

Since $k_1^2 \gg \frac{1}{\phi} \frac{\partial^2 \phi}{\partial s^2}$ for the long parallel wavelength modes considered here, we discard the $\frac{\partial^2 \phi}{\partial s^2}$ term in the linearized Poisson equation and obtain an integral equation for ϕ along a flux line

$$-k_1^2(s) \phi(s) = 4\pi \sum_{\alpha} q_{\alpha} n_{\alpha}(s), \quad (2.16)$$

with n_{α} given by (2.15).

The integral equation (2.16) is reduced to matrix form by introducing a series representation of ϕ on the interval $|s| \leq s_{\max}$. For convenience, we assume that the equilibrium fields are symmetric about $s = 0$. The wave potential is then an even or odd function of s ,¹⁵ and appropriate basis functions for representing ϕ are sinusoids having the same parity as the wave and satisfying the boundary condition $\phi(s_{\max}) = 0$ implicit in (2.16). We denote these functions by

$$\begin{aligned} \text{sc}(k_m s) &= \sin(k_m s) && \text{for even } \phi \\ &= \cos(k_m s) && \text{for odd } \phi, \end{aligned} \quad (2.17)$$

where k_m is defined

$$k_m \equiv \left(m + \frac{\sigma-1}{2} \right) \frac{\pi}{s_{\max}},$$

and σ is zero for even eigenfunctions and unity for odd. The Fourier series representation of ϕ is then

$$\phi(s) = \sum_{m=1}^{\infty} \hat{\phi}_m \text{sc}(k_m s), \quad (2.18)$$

where $\hat{\phi}_m$ is the Fourier transform of ϕ over the plasma length

$$\hat{\phi}_m \equiv \frac{2}{s_{\max}} \int_0^{s_{\max}} ds \phi(s) \text{sc}(k_m s). \quad (2.19)$$

We replace ϕ in (2.15) and (2.16) by this series and multiply the

integral equation by B_0/B to cancel the s dependence of k_1^2 . Fourier transforming both sides in the manner of (2.19) then gives a matrix eigenvalue equation for the Fourier coefficients of ϕ

$$\sum_{m=1}^{\infty} (k_{10}^2 \lambda_{De0}^2 \delta_{mn} - R_{mn}^e - R_{mn}^i) \hat{\phi}_m = 0, \quad (2.20)$$

where the response matrix R_{mn}^α for either species is defined

$$R_{mn}^\alpha = 8\pi q_\alpha \frac{\lambda_{De0}^2}{s_{\max}} \int_0^{s_{\max}} ds \hat{n}_m^\alpha(s) sc(k_n s), \quad (2.21)$$

with

$$\begin{aligned} \hat{n}_m^\alpha(s) = & 2\pi \frac{q_\alpha N_0 B_0}{m_\alpha^2} \int_0^\infty d\mu \int_{\mu B(s)+\psi(s)}^{\mu B_{\max}+\psi_{\max}} dE \frac{1}{|v_{\alpha u}(s;E,\mu)|} \left\{ 2 \frac{\partial F_\alpha(E,\mu)}{\partial E} sc(k_m s) \right. \\ & + 1 \sum_{j=-\infty}^{\infty} \left[\omega \frac{\partial F_\alpha(E,\mu)}{\partial E} + \frac{j\Omega_\alpha}{B} \frac{\partial F_\alpha(E,\mu)}{\partial \mu} \right] J_j^2(k_1 a_\alpha) \\ & \left. \sum_{\pm} \int_{-\infty}^t dt' sc(k_m s_\alpha^\pm) \exp[-ij\Delta^\pm \theta_\alpha(t',t) - i\omega(t'-t)] \right\}. \end{aligned} \quad (2.22)$$

Here the square of the electron Debye length $\lambda_{De0}^2 \equiv W_e / (4\pi e^2 N_0)$ has been introduced as a scale factor to make the response matrices dimensionless.

3. DERIVATION OF THE ELECTRON RESPONSE MATRIX

Since mirror plasma bounce modes depend on electrons retaining phase coherence longer than a half bounce period, accurate trajectories must be used in the electron orbit integral in (2.22). Orbits of collisionless electrons are in general periodic with a velocity-dependent period ω_b given by the integral

$$\frac{\pi}{2\omega_b(E, \mu)} = \int_0^{s_{\max}} \frac{ds}{|v_{e\parallel}(s; E, \mu)|} \quad (2.23)$$

The perturbed potential along an electron trajectory is likewise periodic, and it is conveniently represented by a Fourier series in harmonics of ω_b . This representation gives a time dependence that can be exactly integrated and leads to an integral expression for R_{mn}^e that is suitable for numerical evaluation.

For electrons, the $\hat{n}_m^e$ expression (2.22) is simplified by observing that typical electron gyroradii are small compared with perpendicular wavelengths for modes of interest. The Bessel function argument $k_{\perp} a_e$ in (2.22) is then small enough that only the $j = 0$ term contributes significantly to the sum. Using the small argument approximation $J_0(k_{\perp} a_e) \approx 1$ reduces (2.22) to

$$\begin{aligned} \hat{n}_m^e(s) = & - \frac{2\pi e N_0 B_0}{m_e^2} \int_0^{\infty} d\mu \int_{\mu B(s) + \psi(s)}^{\mu B_{\max} + \psi_{\max}} dE \frac{1}{|v_{e\parallel}(s; E, \mu)|} \frac{\partial F_e(E, \mu)}{\partial E} \left[2 \operatorname{sc}(k_m s) \right. \\ & \left. + i\omega \int_{-\infty}^t dt' \left(\operatorname{sc}[k_m s_e^+(t'; E, \mu, t)] + \operatorname{sc}[k_m s_e^-(t'; E, \mu, t)] \right) \exp[-i\omega(t' - t)] \right] , \end{aligned} \quad (2.24)$$

where both positive and negative going trajectories coincide with s when $t' = t$.

In terms of a trajectory $s_e(t; E, \mu)$ that satisfies the conditions $s_e(0; E, \mu) = 0$ and $v_{e||}(0; E, \mu) > 0$, the trajectories in (2.24) may be written

$$\begin{aligned} s_e^+(t'; E, \mu, t) &= s_e(t' - t_0; E, \mu) \\ s_e^-(t'; E, \mu, t) &= s_e(2t - t' - t_0; E, \mu) , \end{aligned} \quad (2.25)$$

where t_0 is the last time at which $s_e^+(t'; E, \mu, t)$ crosses $s = 0$. Explicitly,

$$t_0(s; E, \mu) = t - \int_0^s \frac{ds'}{|v_{e||}(s'; E, \mu)|} . \quad (2.26)$$

Since $s_e(t; E, \mu)$ is an odd function of time, $sc(k_m s)$ along that path has the same symmetry in time as in position and so may be represented by an appropriate sine or cosine series of the form

$$sc[k_m s_e(t; E, \mu)] = \sum_{\ell=0}^{\infty} T_{\ell m}(E, \mu) sc[(2\ell + \sigma)\omega_b(E, \mu)t] , \quad (2.27)$$

where

$$T_{\ell m}(E, \mu) \equiv \frac{4\omega_b(E, \mu)}{\pi} \int_0^{\pi/2\omega_b(E, \mu)} dt' sc[(2\ell + \sigma)\omega_b(E, \mu)t'] sc[k_m s_e(t'; E, \mu)] . \quad (2.28)$$

When $sc(k_m s_e^+)$ in (2.24) are replaced by appropriate Fourier series using (2.25) and (2.27), the time integral can be evaluated analytically, and the $\hat{n}_m^e$ expression becomes

$$\hat{n}_m^e(s) = - \frac{4\pi e N_0 B_0}{m_e^2} \int_0^\infty d\mu \int_{\mu B_0}^{\mu B_{\max} + \psi_{\max}} dE \frac{1}{|v_{e\parallel}(s; E, \mu)|} \frac{\partial F_e(E, \mu)}{\partial E} \quad (2.29)$$

$$\left[sc(k_m s) - \sum_{\ell=0}^{\infty} \frac{\omega^2 T_{\ell m}(E, \mu) sc[(2\ell + \sigma)\omega_b(E, \mu)(t - t_0)]}{\omega^2 - (2\ell + \sigma)^2 \omega_b^2(E, \mu)} \right].$$

Substituting (2.29) into (2.21) and changing the order of integration leads finally to an expression for the electron response matrix:

$$R_{mn}^e = \frac{2\pi^2 W_e^2 B_0}{m_e^2 s_{\max}} \int_0^\infty d\mu \int_{\mu B_0}^{\mu B_{\max} + \psi_{\max}} dE \frac{1}{\omega_b(E, \mu)} \frac{\partial F_e(E, \mu)}{\partial E} \quad (2.30)$$

$$\left[S_{mn}(E, \mu) - \sum_{\ell=0}^{\infty} \frac{\omega^2}{\omega^2 - (2\ell + \sigma)^2 \omega_b^2(E, \mu)} T_{\ell m}(E, \mu) T_{\ell n}(E, \mu) \right].$$

Here the definition

$$S_{mn}(E, \mu) \equiv \frac{4\omega_b(E, \mu)}{\pi} \int_0^{\pi/2\omega_b(E, \mu)} dt' sc[k_m s_e(t'; E, \mu)] sc[k_n s_e(t'; E, \mu)] \quad (2.31)$$

has been introduced, and (2.26) and (2.28) have been used to reduce the second bracketed term.

The response matrix (2.30) is put into a form better suited for numerical evaluation by a change of variables. When restoring forces on a particle vary linearly with position as in quadratically varying ψ and B fields, then particles execute simple harmonic motion

$$s(t; E, \mu) = s_t(E, \mu) \sin[\omega_b(E, \mu)t], \quad (2.32)$$

with turning point s_t defined by $E - \mu B(s_t) - \psi(s_t) \equiv 0$. The orbit integrals S_{mn} and T_{lm} in this case are expressible in terms of Bessel functions of the first kind depending only on s_t :

$$S_{mn}(E, \mu) = J_0[(k_m - k_n)s_t(E, \mu)] + (-1)^\sigma J_0[(k_m + k_n)s_t(E, \mu)] \quad (2.33)$$

$$T_{lm}(E, \mu) = 2 J_{2l+\sigma}[k_m s_t(E, \mu)] . \quad (2.34)$$

Except in strongly anharmonic fields (2.33) and (2.34) remain good approximations of the orbit integrals, and when s_t is held constant, S_{mn} and T_{lm} are in general weakly dependent functions of μ . Because of this slow variation, it is convenient to rewrite the R_{mn}^e expression (2.30) using the dimensionless turning point $\chi = s_t/s_{\max}$ instead of E :

$$R_{mn}^e = \frac{2\pi^2 W_e^2 B_0}{m_e^2 s_{\max}} \int_0^1 d\chi \int_0^\infty d\mu \frac{1}{\omega_b(\chi, \mu)} \frac{\partial F_e(\chi, \mu)}{\partial \chi} \left[S_{mn}(\chi, \mu) - \sum_{\ell=0}^{\infty} \frac{\omega^2}{\omega^2 - (2\ell+\sigma)^2 \omega_b^2(\chi, \mu)} T_{1m}(\chi, \mu) T_{1n}(\chi, \mu) \right], \quad (2.35)$$

where all functions of χ are obtained from E-dependent forms by the substitution $E = \mu B(\chi s_{\max}) + \psi(\chi s_{\max})$.

Since the strongest μ dependence of the integrand in (2.35) arises from F_e and from the resonant factor multiplying T_{1m} , a useful approximate expression for R_{mn}^e is obtained by using the Bessel function forms (2.33) and (2.34) for S_{mn} and T_{1m} and factoring them outside the μ integration

$$R_{mn}^e \approx \frac{2\pi^2 W_e^2 B_0}{m_e^2 s_{\max}} \int_0^1 d\chi \left[\left(J_0[(\bar{k}_m - \bar{k}_n)\chi] + (-1)^\sigma J_0[(\bar{k}_m + \bar{k}_n)\chi] \right) G_1(\chi) - 4 \sum_{\ell=0}^{\infty} J_{2\ell+\sigma}(\bar{k}_m \chi) J_{2\ell+\sigma}(\bar{k}_n \chi) G_2(\chi) \right], \quad (2.36)$$

where the μ integrals G_1 and G_2 are defined by

$$G_1(\chi) = \int_0^\infty d\mu \frac{1}{\omega_b(\chi, \mu)} \frac{\partial F_e(\chi, \mu)}{\partial \chi} \quad (2.37)$$

and

$$G_2(\chi) = \int_0^\infty d\mu \frac{1}{\omega_b(\chi, \mu)} \frac{\omega^2}{\omega^2 - (2\ell+\sigma)^2 \omega_b^2(\chi, \mu)} \frac{\partial F_e(\chi, \mu)}{\partial \chi}, \quad (2.38)$$

and $\bar{k}_m \equiv k_{m \max} s = \left(m + \frac{\sigma-1}{2}\right)\pi$ is the dimensionless parallel wave number for Fourier modes. The R_{mn}^e expression (2.36) is exact for quadratically varying equilibrium fields and closely approximates (2.35) except for strongly anharmonic B or ψ .

DERIVATION OF THE ION RESPONSE MATRIX

The assumption of a hot-ion plasma allows the ion response matrix to be considerably simplified. Since $\psi_{\max}$ in such plasmas is normally a few times the electron thermal energy W_e and small compared with W_i , ions are principally mirror-confined along flux lines. The μ dependence of the ion bounce frequency then causes the phase shift over a bounce period τ_b

$$|\omega\tau_b - j \int_0^{\tau_b} dt \Omega_i| \quad (2.39)$$

to have a thermal spread much greater than unity. This spread eliminates bounce resonance effects and makes the time integral in (2.22) sharply peaked for j and s satisfying the stationary phase condition $\omega \approx j\Omega_i$. Keeping only the principal resonant contribution to $\hat{n}_m^i$ in (2.22), we find

$$\hat{n}_m^i(s) \approx 2\pi \frac{q_i j \Omega_i N(s)}{m_i} \int_0^\infty dv_\perp^2 \frac{\partial F_\perp(v_\perp^2)}{\partial v_\perp^2} J_j^2\left(\frac{k_\perp v_\perp}{\Omega_i}\right) \int_0^{v_{\perp\max}(s, v_\perp)} dv_\parallel F_\parallel(v_\parallel) I_{jm}(s, v_\parallel, v_\perp), \quad (2.40)$$

where

$$I_{jm}(s, v_\parallel, v_\perp) \equiv i \sum_{\pm} \int_{-\infty}^t dt' \operatorname{sc}[k_m s_1^\pm(t'; v_\parallel, v_\perp, t)] \exp[ij\Delta\theta_1^\pm(t', t) - i\omega(t' - t)] \quad (2.41)$$

and

$$v_{\parallel \max}(s, v_{\perp}) \equiv v_{\perp} \left(\frac{B_{\max}}{B(s)} - 1 \right)^{1/2}.$$

The phase factor in I_{jm} is rapidly oscillating for $|t - t'| \geq (3/j\Delta\Omega v_{\parallel})^{1/3}$, where $\Delta\Omega \equiv 0.5(\partial^2 \Omega_1 / \partial s^2)_{s=0}$, and since this time is small compared with τ_b for most ions, only orbit points near s contribute significantly to the integral. A straight orbit approximation¹⁸ treating $v_{\parallel}$ as constant along trajectories is therefore justified and removes the $v_{\perp}$ dependence of I_{jm} . Also, $v_{\parallel \max}$ may be taken to be infinity with little error because the Gaussian $v_{\parallel}$ distribution (2.6) decreases rapidly for $v_{\parallel}^2 > v_{\parallel}^2 \sim v_{\perp}^2 (B_{\max} - B_0)/B_0$. These approximations separate the $v_{\parallel}$ and $v_{\perp}$ integrals in $\hat{n}_m^i$ and allow the ion response matrix (2.21) to be written approximately in the form

$$R_{mn}^i \approx \frac{2j\Lambda_1 \Omega_{i0}}{s_{\max}} \frac{W_e}{W_i} \int_0^{s_{\max}} ds \frac{N(s)}{N_0} \text{sc}(k_n s) \int_0^{\infty} dv_{\parallel} F_{\parallel}(v_{\parallel}) I_{jm}(s, v_{\parallel}), \quad (2.42)$$

where

$$I_{jm}(s, v_{\parallel}) = \frac{1}{2} \sum_{\pm} \int_{-\infty}^0 d\tau \text{sc}[k_m(s \pm v_{\parallel} \tau)] \exp\left[ij \int_0^{\tau} d\tau' \Omega_i(s \pm v_{\parallel} \tau') - i\omega \tau\right] \quad (2.43)$$

and

$$\Lambda_j \equiv \pi v_1^2 \int_0^\infty dv_1^2 \frac{\partial F_1(v_1^2)}{\partial v_1^2} J_j^2\left(\frac{k_1 v_1}{\Omega_1}\right) . \quad (2.44)$$

When $(d\Omega_1/ds)^3$ is large compared with $v_{||} \left(0.5 \frac{d^2\Omega_1}{ds^2}\right)^2$, Ω_1 in (2.43) may be approximated by its value at s . In this case, the I_{jm} expression is evaluated analytically to give

$$I_{jm}(s, v_{||}) \approx -sc(k_m s) \sum_{\pm} \frac{1}{\omega - j\Omega_1(s) \mp k_m v_{||}} . \quad (2.45)$$

For the Gaussian $v_{||}$ distribution (2.6), the ion response matrix can then be put into the symmetric form

$$R_{mn}^i \approx \frac{j\Lambda_j \Omega_{i0}}{s_{\max}} \frac{W_e}{W_i} \int_0^{s_{\max}} ds \frac{N(s)}{N_0} sc(k_m s) sc(k_n s) \left\{ \frac{1}{k_m v_{||}} Z\left[\frac{\omega - j\Omega_1(s)}{k_m v_{||}}\right] + \frac{1}{k_n v_{||}} Z\left[\frac{\omega - j\Omega_1(s)}{k_n v_{||}}\right] \right\} , \quad (2.46)$$

where

$$Z(z) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} dy \frac{\exp(-y^2)}{y - z} \quad (2.47)$$

is the plasma dispersion function for a complex argument z .¹⁹

The local R_{mn}^1 approximation (2.46) is invalid near the midplane because $d\Omega_1/ds$ vanishes. To estimate I_{jm} in this region, Ω_1 is expanded in a Taylor series about $s = 0$, and the quadratic term is retained. The argument of the exponential in (2.43) is then

$$ij \int_1^0 d\tau' \Omega_1(s \pm v_{\parallel} \tau') - i\omega\tau \approx -\tau [\omega - j\Omega_{10} - j\Delta\Omega(s^2 \pm v_{\parallel} s\tau + \frac{v_{\parallel}^2}{3} \tau^2)]. \quad (2.48)$$

Using the relation $\tau \approx -|s' - s|/v_{\parallel}$, the time integration in (2.43) is rewritten as a spatial integral, and the R_{mn}^1 expression (2.42) becomes

$$R_{mn}^1 \approx i\pi^{-1/2} \frac{j\Lambda_1 \Omega_{10}}{v_{\parallel} s_{\max}} \frac{W_e}{W_i} \int_0^{\infty} \frac{dv_{\parallel}}{v_{\parallel}} \exp\left(\frac{-v_{\parallel}^2}{V_{\parallel}^2}\right) \int_{-s_{\max}}^{s_{\max}} ds \frac{N(s)}{N_0} \text{sc}(k_n s) \int_{-s_{\max}}^{s_{\max}} ds' \text{sc}(k_m s') \\ \times \exp\left[i \frac{|s - s'|}{v_{\parallel}} \left\{ \omega - j\Omega_{10} - j\Delta\Omega \left[\left(\frac{s' + s}{2}\right)^2 + \frac{(s' - s)^2}{12} \right] \right\}\right]. \quad (2.49)$$

An expression for R_{mn}^1 that is suitable when $\omega \approx j\Omega_{10}$ is obtained by taking $N/N_0 = 1$. This approximation is appropriate because it uses the correct number density where the ions couple most strongly with the wave. The resulting expression is reduced to a single complex integral similar to that of Beasley et al.⁸ by first introducing the variables $x = (s' + s)/2$ and $y = s' - s$. When the product $\text{sc}(k_m s')\text{sc}(k_n s)$ is written in terms of exponential functions involving x and y , the integrals in (2.49) reduce to a sum of terms having the form

$$I(K_1, K_2, j) = \frac{1}{\pi v_{||}} \int_0^\infty \frac{dv_{||}}{v_{||}} \exp\left(-\frac{v_{||}^2}{2}\right) \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \exp\left[iK_1 x + iK_2 y + \frac{i|y|}{v_{||}} (C_1 - C_2 x^2 - C_3 y^2)\right], \quad (2.50)$$

where the coefficients in the exponential argument are $C_1 = \omega - j\Omega_1$, $C_2 = j\Delta\Omega$, and $C_3 = j\Delta\Omega/12$, and K_1 and K_2 are combinations of k_m and k_n . The y integration can be carried out analytically using the relation

$$\int_{-\infty}^\infty dx \exp(-\alpha x - \beta x^2) = \left(\frac{\pi}{\beta}\right)^{1/2} \exp\left(\frac{\alpha^2}{4\beta}\right), \quad (2.51)$$

and a second change of variables $z = y/v_{||}$ then gives a $v_{||}$ integral with the form of (2.51). The resulting R_{mn}^1 expression is

$$R_{mn}^1 \approx \frac{i\pi^{1/2}}{2} \frac{j\Lambda_{10}\Omega}{s_{\max}} \frac{W_e}{W_1} [I(k_m - k_n, k_m + k_n, j) + (-1)^\sigma I(k_m + k_n, k_m - k_n, j)], \quad (2.52)$$

with

$$I(K_1, K_2, j) = \int_0^\infty dz \frac{\exp\left[i(\omega - j\Omega_{10})z - \frac{K_1^2}{4j\Delta\Omega z} - \frac{K_2^2 v_{||}^2 z^2}{16(1 + \frac{j\Delta\Omega}{12} v_{||}^2 z^3)}\right]}{(j\Delta\Omega z)^{1/2} (1 + \frac{j\Delta\Omega}{12} v_{||}^2 z^3)^{1/2}}. \quad (2.53)$$

A single expression for R_{mn}^i suitable when $j\Omega_{i0} \leq \omega < j\Omega_i(s_{\max})$ is obtained by combining the local approximation (2.46) and the quasi-local form (2.52). Because of the $N/N_0 \rightarrow 1$ substitution used in deriving (2.52), the quasi-local form is valid only when $\omega \approx j\Omega_{i0}$, whereas the local form is appropriate when the wave is resonant away from the midplane. A suitable composite form may therefore be written symbolically as

$$R_{mn}^i \approx R_{mn}^L(N) - R_{mn}^L(1) + R_{mn}^{QL}, \quad (2.54)$$

where $R_{mn}^L(N)$ is the local approximation given by (2.46) using the correct number density, $R_{mn}^L(1)$ is the same integral with the change $N/N_0 \rightarrow 1$, and R_{mn}^{QL} is the quasilocal form (2.52). Explicitly, the hybrid form for R_{mn}^i is

$$R_{mn}^i \approx \frac{j\Lambda_j \Omega_{i0}}{s_{\max}} \frac{W_e}{W_i} \left\{ \frac{i\pi^{1/2}}{2} [I(k_m - k_n, k_m + k_n, j) + (-1)^\sigma I(k_m + k_n, k_m - k_n, j)] \right. \\ \left. + [II(k_m, k_n, j) + II(k_n, k_m, j)] \right\}, \quad (2.55)$$

where

$$II(k_m, k_n, j) \equiv \frac{1}{k_m V_{ii}} \int_0^\infty ds \left[\frac{N(s)}{N_0} - 1 \right] \text{sc}(k_m s) \text{sc}(k_n s) Z \left[\frac{\omega - j\Omega_i(s)}{k_m V_{ii}} \right]. \quad (2.56)$$

For the perpendicular ion distributions considered, Λ_j in (2.55) can be evaluated analytically in terms of Bessel functions and is found to depend only on $\zeta \equiv (k_{\perp 0} V_{i0} / \Omega_{i0})$. The delta function (2.4) gives

$$\Lambda_j(\zeta) = \frac{\zeta}{2} J_j(\zeta) [J_{j+1}(\zeta) - J_{j-1}(\zeta)] , \quad (2.57)$$

and a spread distribution of the form of (2.5) leads to

$$\Lambda_j(\zeta) = \frac{\zeta^2}{4} \exp\left(-\frac{\zeta^2}{4}\right) \left[2I_j\left(\frac{\zeta^2}{4}\right) - I_{j+1}\left(\frac{\zeta^2}{4}\right) - I_{j-1}\left(\frac{\zeta^2}{4}\right) \right] , \quad (2.58)$$

where I_j denotes a modified Bessel function of the first kind.

These functions are shown in Fig. 1 for $j = 1$.

CALCULATION OF marginally STABLE EIGENMODES

When electron damping due to bounce-frequency spread is sufficiently weak, the real part of the electron response largely determines the character of eigenmodes. This condition occurs in low-density hot-ion mirror plasmas where electrons are confined principally by a nearly quadratic ambipolar potential. In this case, a perturbation procedure treating $\text{Im}(R_{mn}^e)$ and R_{mn}^i as higher order quantities is used to calculate ϕ and the instability threshold densities. We obtain an equation for the Fourier coefficients $\hat{\phi}_m^0$ of the zeroth order eigenfunctions ϕ^0 by discarding higher order terms of (2.20):

$$\sum_{m=1}^{\infty} [k_{10}^2 \lambda_{De0}^2 \delta_{mn} - \text{Re}(R_{mn}^e)] \hat{\phi}_m^0 = 0 \quad (2.59)$$

This perturbation treatment is justified whenever

$$\frac{\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \text{Im}(R_{mn}^e) \hat{\phi}_m^0 \hat{\phi}_n^0 \right|}{\lambda \sum_{m=1}^{\infty} (\hat{\phi}_m^0)^2} \ll 1. \quad (2.60)$$

Even though the eigenfrequencies ω for real $\lambda \equiv (k_{10} \lambda_{De0})^2$ are in general complex, we wish to find marginally stable eigenmodes with $\gamma \equiv \text{Im}(\omega) = 0$. It is therefore convenient to specify real values of ω and calculate the corresponding λ as eigenvalues of (2.59). All of these eigenvalues are

real because $\text{Re}(R_{mn}^e)$ for real ω is a real symmetric matrix, although only positive values of λ are physically meaningful. Since the orbit integrals S_{mn} and T_{lm} in the R_{mn}^e expression (2.35) become increasingly oscillatory for higher mode numbers m and n , the elements of R_{mn}^e with the lowest values of $m+n$ are dominant, and the matrix may be truncated by discarding elements with indices above some appropriate maximum value M . When the resulting finite matrix eigenvalue equation is solved numerically, we obtain a set of at most M real eigenvalues λ along with the first M Fourier coefficients $\hat{\phi}_m^0$ for each corresponding eigenfunction.

For any zeroth order eigenmode $\hat{\phi}^0$, the plasma density at the onset of instability is determined from the imaginary part of (2.20). We first multiply (2.20) by $\hat{\phi}_n^*$ and sum over n to form a scalar equation. A Taylor expansion about the zeroth order eigenfrequency then gives to first order in the frequency shift

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \text{Im}(R_{mn}^e) + \text{Im}(R_{mn}^i) + \gamma \text{Re} \left[\frac{\partial}{\partial \omega} (R_{mn}^e + R_{mn}^i) \right] \right\} \hat{\phi}_m^0 \hat{\phi}_n^0 \approx 0.$$

Since $\gamma = 0$ at marginal stability, we obtain the condition

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\text{Im}(R_{mn}^e) + \text{Im}(R_{mn}^i)] \hat{\phi}_m^0 \hat{\phi}_n^0 = 0. \quad (2.61)$$

Since Λ_j appears in the R_{mn}^i expression (2.55) as an undetermined factor,

(2.61) is an algebraic equation for the values of Λ_j at marginal stability. If the electron and ion contributions are written respectively as

$$\Gamma^e \equiv \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \text{Im}(R_{mn}^e) \hat{\phi}_m^0 \hat{\phi}_n^0 \quad (2.62)$$

and

$$\Lambda_j \Gamma^i \equiv \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \text{Im}(R_{mn}^i) \hat{\phi}_m^0 \hat{\phi}_n^0, \quad (2.63)$$

this threshold Λ_j value for the particular mode treated is

$$\Lambda_j = \left| \frac{\Gamma^e}{\Gamma^i} \right|. \quad (2.64)$$

With either of the analytic expressions (2.57) or (2.58) for Λ_j , the corresponding Λ_j argument ζ_{th} is found by quickly converging secant method.

Introducing the density parameter $\epsilon \equiv (\omega_{pi}/\Omega_{i0})^2$, where

$\omega_{pi} \equiv (4\pi e^2 N_0/m_i)^{1/2}$ is the ion plasma frequency at $s = 0$, we

calculate the value of ϵ at marginal stability by combining ζ_{th} with the eigenvalue λ for the mode:

$$\epsilon_{th} = \frac{\zeta_{th}^2}{2\lambda} \frac{W_e}{W_i}. \quad (2.65)$$

When more than one solution of (2.61) exists, ζ_{th} is taken to be the smallest value because this choice gives the lowest threshold density ϵ_{th} . This constraint and the further requirement that Λ_j must be positive for instability to occur restrict ζ_{th} to values between the first zero and first maximum of Λ_j . For $j = 1$, this range is 1.85 to 2.89 for a peaked ion v_1 distribution and 2.49 to 4.27 for ions with a spread in v_1 . If the Λ_j value given by (2.64) is greater than the respective maxima $\Lambda_{max} = 0.385$ and $\Lambda_{max} = 0.133$ for the peaked and spread distributions, then the ion drive is insufficient to cause instability at any density, and the particular mode is said to be unconditionally stable. Since waves that couple to the first gyrofrequency harmonic have the lowest instability threshold densities, these $j = 1$ modes are the ones examined most thoroughly, and in each case the ion centerplane gyrofrequency Ω_{i0} is chosen to maximize Γ^i .

When the perturbation validity criterion (2.60) is not satisfied, the full complex matrix equation (2.20) must be solved. Since the eigenvalues are in general complex, an iterative procedure is used to find the value of Λ_j for which k_1 for a particular eigenmode is real: Λ_j is increased stepwise from zero, and the eigenvalues are recalculated with the new R_{mn}^i until $\text{Im}(k_1)$ for the mode changes sign. A secant method is then used to find the desired Λ_j value. The mode is unconditionally stable if the real k_1 condition cannot be satisfied for

$0 < \Lambda_j < \Lambda_{\text{max}}$. When a marginally stable eigenmode is found, the corresponding ϵ_{th} value is calculated from Λ_j just as in the perturbation treatment.

3. PROPERTIES OF BOUNCE MODES

EIGENVALUES

Important features of bounce-mode behavior are seen in the response of electrons confined only by a quadratic potential

$$\psi(s) = \psi_{\max} \frac{s^2}{s_{\max}^2} \quad (3.1)$$

This idealized plasma shows especially strong bounce resonance effects because all electrons have the bounce frequency

$$\omega_b = \left(\frac{2}{m_e} \frac{\psi_{\max}}{s_{\max}^2} \right)^{1/2} \quad (3.2)$$

Even though this case omits the response of ions to the wave and the mirror force on electrons, it illustrates the underlying bounce-mode mechanism.

We obtain quadratic-well eigenmodes by solving (2.59) with R_{mn}^e given exactly by (2.36). For each real frequency ω , a finite set of real eigenvalues $\lambda = k_{10}^2 \lambda_{De0}^2$ is found for odd and even ϕ , and consequently, a plasma can support electron bounce modes with frequency ω only at discrete values of k_{10}^2 . Figure 2 shows the three largest eigenvalues for a quadratic potential with $\psi_{\max} = 4.5 W_e$ over a range of ω/ω_{b0} . Here ω_{b0} is the zero-energy bounce frequency, defined in general by

$$\omega_{b0} \equiv \lim_{\substack{\chi \rightarrow 0 \\ \mu \rightarrow 0}} \omega_b(\chi, \mu), \quad (3.3)$$

and given for a quadratic potential well by (3.2). The strongly resonant behavior near alternate bounce-frequency harmonics is characteristic of bounce modes and arises because electrons at a particular position always encounter the same wave phase on each transit whenever $\omega = (2\ell + \sigma)\omega_{b0}$. The even and odd modes with the largest respective eigenvalues, referred to hereafter as the principal modes, have obviously different frequency dependences. Since the sum in the electron response matrix expression (2.36) includes a nonresonant $\ell = 0$ term when ϕ is even, the principal eigenvalue remains relatively large at frequencies away from resonance, and the mode persists even at frequencies below the $\omega = 2\omega_{b0}$ resonance. For odd eigenfunctions, only resonant terms appear in the R_{mn}^e summation, so the plasma response drops off rapidly away from the resonances at $\omega = (2\ell + 1)\omega_{b0}$. In each instance, eigenfunctions for the principal even and odd modes have the longest parallel wavelengths and are well represented by the first few terms of appropriate Fourier series.

For electron modes in a finite quadratic potential well, the dimensionless parameter $\xi \equiv m_e \omega_{b0}^2 s_{\max}^2 / 2W_e$ is equal to $\psi_{\max} / W_e$ and fully characterizes the system. This quantity determines the energy at which the electron distribution is cut off, and as $\xi \rightarrow \infty$ the R_{mn}^e

expression approaches the form for Maxwellian electrons in an unbounded potential well. The value $\xi = 4.5$ used for the eigenvalues in Fig. 2 is a typical value found in Fokker-Planck studies of mirror devices.

Figure 3 shows how the eigenvalues change with ξ . The decreasing sensitivity of eigenvalues to changes in maximum potential for $\xi > 2$ is expected because 78% of the electrons in a Maxwellian distribution have energies below $2W_e$ and are unaffected by variations of $\psi_{\max}$. The increase in minimum eigenvalues found for the principal even modes as ξ is reduced arises mainly from the lowered bounce frequency and consequently greater nonresonant interaction of particles with the wave.

The principal qualitative features of quadratic well eigenvalues are seen in a simple analytic solution of (2.59). When $\xi > 2$, the response matrix R_{mn}^e for Maxwellian electrons in a finite quadratic potential is approximately

$$R_{mn}^e \approx -\frac{1}{2} \left(\frac{\pi}{\xi} \right)^{1/2} \left\{ \exp \left[-\frac{(\bar{k}_m - \bar{k}_n)^2}{4\xi} \right] + (-1)^\sigma \exp \left[-\frac{(\bar{k}_m + \bar{k}_n)^2}{4\xi} \right] \right. \\ \left. - 4 \exp \left(-\frac{\bar{k}_m^2 + \bar{k}_n^2}{4\xi} \right) \sum_{\ell=0}^{\infty} \frac{\omega^2}{\omega^2 - (2\ell + \sigma)^2 \omega_{b0}^2} I_{2\ell + \sigma} \left(\frac{\bar{k}_m \bar{k}_n}{2\xi} \right) \right\}. \quad (3.4)$$

This expression shows that R_{mn}^e is dominated by diagonal terms, and an estimate of the principal even eigenvalue is obtained by setting $\sigma = 0$

and taking only the $m = n = 1$ matrix element:

$$\lambda \approx \frac{1}{2} \left(\frac{\pi}{\xi} \right)^{1/2} \left\{ 4 \exp \left(-\frac{\pi^2}{8\xi} \right) \left[I_0 \left(\frac{\pi^2}{8\xi} \right) + \frac{\omega^2}{\omega^2 - (2\ell_{\text{res}})^2 \omega_{b0}^2} I_{2\ell_{\text{res}}} \left(\frac{\pi^2}{8\xi} \right) - \exp \left(-\frac{\pi^2}{4\xi} \right) - 1 \right] \right\} \quad (3.5)$$

Here ℓ_{res} is the largest integer satisfying the condition

$\delta_{\text{res}} \equiv [\omega - (2\ell_{\text{res}} + \sigma)\omega_{b0}] / \omega_{b0} > 0$ and identifies the most

strongly resonant contribution to the sum in (3.4). It is clear from

(3.5) that the $\ell = 0$ term of the sum leads to a positive nonresonant

contribution to the principal even eigenvalue that should persist for

$\omega < 2\omega_{b0}$ and scale roughly like $\xi^{-1/2}$. The principal odd eigenvalue is

similarly estimated by

$$\lambda \approx \frac{1}{2} \left(\frac{\pi}{\xi} \right)^{1/2} \left[\frac{4\omega^2}{\omega^2 - (2\ell_{\text{res}} + 1)^2 \omega_{b0}^2} \exp \left(-\frac{\pi^2}{2\xi} \right) I_{2\ell_{\text{res}} + 1} \left(\frac{\pi^2}{2\xi} \right) + \exp \left(-\frac{\pi^2}{\xi} \right) - 1 \right] \quad (3.6)$$

Since the nonresonant contribution to this expression is negative, the

eigenvalue should decrease toward zero as δ_{res} increases and should

become negative and therefore spurious for $\omega < \omega_{b0}$. In both

approximate eigenvalue expressions, the $I_{2\ell_{\text{res}} + \sigma}$ factor in the resonant

term becomes small for $\ell_{\text{res}} > 1$, so that resonances at higher ω are much

less pronounced. The analytic approximations given by (3.5) and (3.6)

are plotted on Fig. 2 as dotted lines. These estimates of λ differ from

the numerically calculated values mainly because the eigenvalue equation (2.59) is not in general dominated by the $m = n = 1$ element. Analysis of (3.4) shows that the resonant term of R_{mn}^e is greatest for

$$m = n \sim \frac{(2\xi)^{1/2}}{\pi} (2\ell_{\text{res}} + \sigma) + \frac{1-\sigma}{2}, \quad (3.7)$$

so that the approximations omit significant higher mode contributions when $\ell_{\text{res}} > 1$ or $\xi > 2$.

For the collisionless plasma considered, both the mirror force $-\mu \nabla_{\parallel} B$ and any anharmonic spatial dependence of ψ spread electron bounce frequencies. If the potential is a quartic function of s , $\psi(s) = A_1 s^2 + A_2 s^4$ and B is quadratic, $B(s) = B_0 + B_1 s^2$, then for $A_1 \gg A_2 s_{\text{max}}^2$ the bounce frequency is approximately

$$\omega_b(\chi, \mu) \approx \left[\frac{2}{m_e} \left(A_1 + \mu B_1 + \frac{3}{2} A_2 \chi^2 s_{\text{max}}^2 \right) \right]^{1/2}. \quad (3.8)$$

This expression shows how the two spreading mechanisms differ: Whereas μB_1 is positive for monotonic mirror fields and always increases ω_b , the χ term can have either sign. The frequency is increased by positive A_2 since the potential well is sharpened, while a negative A_2 reduces ω_b by flattening the central field. Also, the magnitude of the μ term in

(3.8) is unbounded, but the shift in ω_b^2 from the χ term is no greater than $3A_2 s_{\max}^2 / m_e$ for cutoff electron distributions. The zero-energy bounce frequency is the same here as for the pure electrostatic field (3.1) because ω_{b0} is determined only by the quadratic dependence of ψ at $s = 0$. The spread in ω_b has two principal effects on eigenmodes. Since electrons resonate at different frequencies, the strong resonances at harmonics of ω_{b0} found for quadratic ψ are shifted in frequency and broadened. In addition, the electron damping introduced by phase mixing tends to stabilize waves at frequencies where damping is strongest. Since these phenomena are independent, it is convenient to separate them by first considering the effect of bounce frequency spread on undamped electron modes.

The principal eigenvalues of (2.59), plotted in Fig. 4 for several values of $B_{\max}/B_0$, illustrate the effect of a quadratic mirror field. Since $\omega_b > \omega_{b0}$ for all electrons, the maxima of λ occur at frequencies shifted approximately $\delta_{\text{res}} = (2\ell_{\text{res}} + \sigma)^2 \langle \mu \rangle B_1 / m_e \omega_{b0}$ from the $B_{\max}/B_0 = 1$ resonances, and the maxima become less pronounced as ℓ_{res} increases because the $(2\ell_{\text{res}} + \sigma)^2$ harmonic factor spreads resonances over a wider frequency range. Since $B_1 = (B_{\max} - B_0) / s_{\max}^2$ for a quadratic B field, increasing $B_{\max}/B_0$ results in greater frequency spread and weakening of the electron response. Scaling of these mirror force effects may be estimated from an approximate $\text{Re}(R_{mn}^e)$ expression obtained from (2.36)

$$R_{mn}^e \approx -\frac{1}{2} \left(\frac{\pi}{\xi} \right)^{1/2} \left\{ \exp \left[-\frac{(\bar{k}_m - \bar{k}_n)^2}{4\xi} \right] + (-1)^\sigma \exp \left[-\frac{(\bar{k}_m + \bar{k}_n)^2}{4\xi} \right] - 4 \exp \left(-\frac{\bar{k}_m^2 + \bar{k}_n^2}{4\xi} \right) \right. \\ \left. \left[(1-\sigma) I_0 \left(\frac{\bar{k}_m \bar{k}_n}{2\xi} \right) + \sum_{\ell=1-\sigma}^{\infty} \frac{B_0 \xi}{B_{\max} - B_0} \frac{\omega^2}{\omega_{b0}^2} \frac{\exp(-\nu_\ell) E_1(\nu_\ell) I_{2\ell+\sigma} \left(\frac{\bar{k}_m \bar{k}_n}{2\xi} \right)}{(2\ell+\sigma)^2} \right] \right\}, \quad (3.9)$$

where

$$\nu_\ell \equiv \frac{B_0 \xi}{B_{\max} - B_0} \frac{\omega^2 - (2\ell+\sigma)^2 \omega_{b0}^2}{(2\ell+\sigma)^2 \omega_{b0}^2}, \quad (3.10)$$

and E_1 is the real exponential integral

$$E_1(x) = - \int_{-x}^{\infty} \frac{\exp(-y)}{y} dy. \quad (3.11)$$

The main difference between (3.9) and the $B_{\max}/B_0 = 1$ result is that the resonant factor $\omega^2/[\omega^2 - (2\ell+\sigma)^2 \omega_{b0}^2]$ in (3.4) is replaced by a factor proportional to $\exp(-\nu_\ell) E_1(\nu_\ell)$. Since this factor is peaked at $\nu_\ell = 1.4$, the maximum eigenvalues should occur for frequencies

$$\omega = (2\ell+\sigma) \left(1.4 \frac{B_{\max} - B_0}{B_0 \xi} + 1 \right)^{1/2} \omega_{b0}. \quad (3.12)$$

The weakening of principal mode resonances seen in Fig. 4 for higher ℓ_{res} results from the $I_{2\ell+\sigma}$ factor in the resonant term of (3.9), and

this effect together with the greater spread of resonances at higher ℓ_{res} causes bounce-resonance effects to become inconsequential for $(2\ell_{\text{res}} + \sigma)(B_{\text{max}} - B_0)/(2\xi B_0) \gg 1$. Since nonresonant terms in R_{mn}^e are unaffected by ω_b , λ for the principal even mode shows that same limiting value and ξ dependence for $\delta_{\text{res}} \approx 2$ as the $B_{\text{max}}/B_0 = 1$ case.

If a nonquadratic potential

$$\psi(s) = \psi_{\text{max}} \left[(1-\alpha) \frac{s^2}{s_{\text{max}}^2} + \alpha \frac{s^4}{s_{\text{max}}^4} \right] \quad (3.13)$$

is combined with the quadratic B field, the anharmonic potential term has two main effects on bounce modes. The $(1-\alpha)$ factor, introduced in the quadratic term of ψ to keep ψ_{max} constant, changes ξ by a $(1-\alpha)^{-1}$ factor by altering ω_{b0} . Since the R_{mn}^e expression (3.9) remains an acceptable approximation when $\alpha/(1-\alpha)$ is small, this ξ change affects the relative magnitude of the resonant term and, through ν_ℓ , the frequencies at which the principal eigenvalues are greatest. In addition, the quartic term in (3.13) causes a shift of $3\alpha\psi_{\text{max}} \langle \chi^2 \rangle / (m_e s_{\text{max}}^4)$ in the mean squared bounce frequency, which further shifts the λ maxima. These changes are seen in the variation of principal mode eigenvalues with α , shown in Fig. 5. It is evident that a positive α broadens the resonance, while negative α tends to cancel the mirror force spread in ω_b and to sharpen resonances.

When terms for electron damping and the ion response are included in the eigenvalue equation for ϕ , they affect bounce modes principally

by stabilizing the waves at frequencies near the $\omega = (2l+\sigma)\omega_{b0}$ resonances, where damping is strongest. Even though eigenvalues are in general altered by the added terms, $\text{Im}(R_{mn}^e)$ is normally small enough for parameters of interest that the validity condition (2.60) for a perturbation treatment of ϕ is satisfied whenever marginally stable modes exist. Consequently, $\text{Re}(R_{mn}^e)$ is still the dominant matrix term in these cases, and the eigenvalues calculated from the undamped electron equation (2.59) are close approximations to correct values.

INSTABILITY THRESHOLD DENSITIES

The stability of bounce modes is governed by the relative magnitudes of the electron damping and ion drive terms of R_{mn}^e and R_{mn}^i . For a quadratic B field and the quartic potential (3.13) with $|\alpha| \ll 1$, a single-mode approximation of the electron contribution (2.62) is

$$\Gamma^e \approx -2^{1/2} \pi \frac{\omega_{b0}}{2\Delta\omega_b} \frac{1}{\xi_{l_{res}}^{1/2}} \frac{\exp\left[-\frac{\omega_{b0} \delta_{res}}{(2l_{res} + \sigma)\Delta\omega_b}\right]}{2l_{res} + \sigma}, \quad (3.14)$$

where

$$\xi_{res} \equiv \left[\frac{\omega^2}{(2l_{res} + \sigma)^2 \omega_{b0}^2} - \frac{3\alpha}{m_e \omega_{b0}^2 S_{max}^2} \frac{B_0}{B_{max} - B_0} \right] \xi, \quad (3.15)$$

and the relative spread in bounce frequency

$$\frac{\Delta\omega_b}{\omega_{b0}} \approx \frac{B_{max} - B_0}{2B_0 \xi}, \quad (3.16)$$

has been assumed small. An upper band on Γ^i is obtained by taking $\omega = j\Omega_{i0}$ in (2.53) and suppressing the spatial dependence of ϕ :

$$\Gamma^i \approx 2 \frac{W_e}{W_i} \left(\frac{B_0}{B_{max} - B_0} \right)^{2/3} \left(\frac{\omega_{s_{max}}}{V_{||}} \right)^{1/3}. \quad (3.17)$$

Substituting these approximations into (2.64) gives an estimate of Λ_j at marginal stability:

$$\Lambda_j = \left| \frac{\Gamma^e}{\Gamma^i} \right| \approx 5 \frac{W_i}{W_e} \left(\frac{\xi}{\xi_{res}} \right)^{1/2} \left(\frac{\omega_{b0}}{2\Delta\omega_b} \frac{v_{||}}{\omega s_{max}} \right)^{1/3} \frac{\exp \left[- \frac{\omega_{b0} \delta_{res}}{(2\ell_{res} + \sigma)\Delta\omega_b} \right]}{2\ell_{res} + \sigma} \quad (3.18)$$

The frequency dependence of the instability threshold density is controlled principally by the exponential factor in (3.18). When the factor is of order unity, the low ℓ_{res} bounce modes in a plasma with $W_i \geq W_e$ are usually stable because $\Lambda_j > \Lambda_{max} \lesssim 0.385$. However, the factor may become exponentially small as δ_{res} increases. In this case, ζ_{th} remains approximately equal to the first zero of Λ_j independent of ω , and the frequency dependence of threshold density parameter ϵ_{th} then comes mainly from the λ^{-1} factor in (2.65). The transition between these two frequency regimes occurs for $\delta_{res} \approx (2\ell_{res} + \sigma)\Delta\omega_b/\omega_{b0}$.

Calculated threshold densities for the three longest-wavelength modes, plotted in fig. 6, show the expected frequency dependences. Values are calculated from the full complex eigenvalue equation (2.20) using protons with a sharply peaked $v_{||}$ distribution as the ion species. Both B and ψ are taken to be quadratic functions of s , with $B_{max}/B = 1.5$ and $\psi_{max}/W_e = 4.5$. The energy ratio $W_e/W_i = 4.5 \times 10^{-3}$ is low enough all modes are unconditionally stable just above resonances, and at frequencies where electron damping is exponentially small, thresholds for all except the principal even mode increase rapidly with frequency due to decreasing λ . For the principal even mode, ϵ_{th} varies gradually with ω in the

$\Lambda_j/\Lambda_{\max} \ll 1$ region because λ approaches a constant nonzero value. The nonresonant even mode below $\omega = 2\omega_{b0}$ is unstable independent of density since it is undamped when the fields vary quadratically. Threshold densities for ions with a spread in v_1 differ from the peaked distribution results of Fig. 6 in two respects: Since the first zero of Λ_j for the broad v_1 distribution is higher than the peaked distribution value by a factor 1.3, the ϵ_{th} curves are greater by a factor of 1.7 at frequencies where $\Lambda_j/\Lambda_{\max}$ is small. Also, the smaller $\Lambda_{\max}$ value for a spread distribution results in a larger frequency interval above resonances in which modes are stable. The principal mode threshold densities show parametric dependences that are consistent with (3.18): Changing W_e/W_i mainly shifts the minimum threshold due to the linear W_e/W_i dependence of the ϵ_{th} expression (2.65). In contrast, any increase in $\Delta\omega_b/\omega_{b0}$ tends to stabilize the modes near resonances by reducing the exponential decrease of Λ_j with δ_{res} in (3.18), but causes relatively little change in the minimum ϵ_{th} . This stabilization results physically from more effective electron damping, and the principle effect on the threshold curves is a widening of the frequency intervals above resonances where the damping stabilizes bounce modes. According to (3.16), $\Delta\omega_b/\omega_{b0}$ increases with $B_{\max}/B_0$, whereas higher $\psi_{\max}$ or α reduces the quantity by raising ω_{b0} . The pattern of strong bounce resonances seen in Fig. 6 is found to vanish when ℓ_{res} is large enough that particle resonances from different bounce harmonics begin to overlap. In typical cases, the resonance structure is substantially lost for $\omega/\omega_{b0} \gtrsim 9$.

The calculated instability threshold densities for the principal even modes are insensitive to details of the physical model. According to (2.65), any change in ϵ_{th} when W_e/W_i is fixed requires variation in either λ or ζ_{th} . The eigenvalue studies in the preceding section indicate that λ for the principal even mode is approximately constant over the frequency range where electron damping is weak enough to allow instability, and the minimum value is principally affected by ψ_{max}/W_e rather than by the plasma distribution functions or spatial dependences of the fields. Similarly, ζ_{th} is effectively constant over most of the frequency range of possible instability so long as $(2\ell_{res} + \sigma)\Delta\omega_b/\omega_{b0} \lesssim 1$, since Λ_j/Λ_{max} is exponentially small. Changing $F_{\parallel}$ does alter this ζ_{th} value by shifting the first zero of Λ_j , but for plausible distributions such as multiply peaked functions or superpositions of peaked and spread functions, the first zero is bracketed by values for the forms of (2.4) and (2.5). Using different $F_{\parallel}$ or $V_{\parallel}/V_i$ expressions normally alters Γ^i in (2.64) by a factor of the order of unity. Since this change principally affects the size of the frequency range in which modes are stable, the choice of these functions has negligible effect on the minimum ϵ_{th} in typical devices. Only when the principal even mode is actually suppressed, either by stronger damping or by weakened ion drive, is a substantial change in the minimum ϵ_{th} expected.

4. APPLICATION TO MIRROR EXPERIMENTS

Qualitatively, the bounce mode instability mechanism is consistent with many features of the disruptions observed in BBI and BBII and in earlier mirror devices. The modes were found experimentally to have wavelengths along field lines comparable with the plasma length,²⁰ and the polarization of radio frequency emissions indicated plasma currents along the magnetic field.¹ Both the rapid wave growth¹ and the fact that instabilities occurred near the ion gyrofrequency or low harmonics^{1,21} suggest that the waves were driven by resonant coupling to the peaked ion perpendicular energy distribution. Also, the strong dependence of instability threshold densities on the plasma length found in BBII⁶ is expected of modes involving electron bounce motion. Such spatial effects as the radial and azimuthal localization of BBII disruptions²¹ and the ion betatron modes evident in DCX²² are not reproduced by the theoretical treatment, nor are nonlinear phenomena like the bursts of ejected electrons from BBI and BBII.^{1,21} These effects are clearly outside the scope of the linearized, one-dimensional model used here.

To test quantitatively whether electron bounce modes were an important instability mechanism in BBI and BBII, we compare the experimental threshold densities with theoretical values for bounce modes. No experimental data exist to check other predictions of the theory, such as bounce mode dispersion characteristics and spatial wave forms. Our general approach is to choose parameters in the model to simulate BBI and BBII plasmas, and then to compare predicted threshold values and parametric dependences with the findings of a large number of experimental

runs. Since most idealizations in the model are appropriate for low-density mirror devices, calculated threshold densities should be close to experimental values if destabilized bounce modes were the dominant cause of instability in the two machines.

A typical sample of theoretical and experimental threshold densities for BBI and BBII is shown in Fig. 7. All ϵ_{th} values correspond to densities at the mirror center at the onset of instability. Experimental thresholds, plotted in Fig. 7 as circles for BBI and as squares for BBII, were determined from separate measurements of N_0 , B_0 , and W_i with W_e/W_i estimated from ψ_{max} . Although no error limits are shown, the experimental threshold densities are certain only within $\pm 50\%$, and W_e/W_i for each data point has about a $\pm 25\%$ uncertainty. The corresponding theoretical thresholds are minima found by varying N_0 and B_0 in (2.20) over the range of midplane values and choosing ω in each case to give the strongest wave-ion coupling. Minimum ϵ_{th} values are calculated for both a sharply peaked and spread ion v_i distribution. Since the actual distribution is expected to fall between these extreme cases, we plot the range of ϵ between the ϵ_{th} for the peaked and spread distributions as a vertical bar in Fig. 7 for each W_e/W_i value. Except for a few low values of W_e/W_i , all the theoretical ϵ_{th} correspond to the principal even mode. Even though the calculated thresholds here are obtained using the cutoff Maxwellian electron distribution (2.1) and a quadratic

potential with $\psi_{\max}/W_e = 4.5$, the results are insensitive to these choices. Another series of BBII tests was carried out to determine the effect of reducing the plasma length. A metal limiter was placed on the mirror axis, and the threshold density was found to vary sharply with the limiter position. Experimental instability thresholds for these tests are plotted in Fig. 8 along with lines bracketing the range of threshold values expected from theory.

The BBII threshold densities show close agreement with theoretical predictions. In Fig. 7, about 70% of the experimental values fall between the thresholds calculated for the spread and peaked perpendicular ion distributions, and none of the other data points is farther away from the theoretical range than experimental uncertainties could explain. Many of the BBII thresholds found in the axial limiter tests likewise fall within the range of predicted values, as Fig. 8 shows. The experimental densities are closer to thresholds expected for a peaked distribution for smaller W_e/W_i , while for larger energy ratios the experimental values are near to predictions for ions with a spread in v_i . This behavior is consistent with the expectation that at higher threshold densities electron drag spread the initially peaked ion distribution. Since the electron drag time defined by Spitzer²³ is around 30 s for the BBII cases with the lowest ϵ_{th} and varies inversely with N_0 , the effect should become important above $\epsilon_{th} \approx 0.1$. In the collisional regime, a further adjustment of the theoretical ϵ_{th} is needed to account for the decrease of W_i of about 15% due to electron drag for tests of the duration reported.

The corresponding 15% increase in the calculated thresholds improves the agreement with experimental thresholds.

The BBI threshold densities presented in Fig. 7 are systematically higher than the theoretical values. Even though experimental values show the approximately linear dependence on W_e/W_i expected from (2.65), typical threshold densities are greater than the theoretical predictions for ions with a spread in v_i by a factor of 2 to 3, and since plasma densities remain relatively low, no correction of W_i for electron drag is justified. Also, the scatter of experimental thresholds from a linear W_e/W_i dependence is markedly greater than that of calculated values. These discrepancies between theory and experiment are greater than experimental uncertainties and cannot be eliminated by reasonable changes in the plasma potential profile or the ion velocity distribution functions. The most likely explanation of the high observed BBI thresholds is that principal even eigenmode was stabilized in that experiment. Stabilization of the mode would allow density to increase during buildup to the thresholds for odd modes or shorter-wavelength even modes. Since these densities are higher by factors of 2 to 5 than ϵ_{th} for the principal even mode, they are near observed values, and because small changes in parameters can destabilize the plasma, more scatter of threshold values would be expected. The principal even mode is stable if $|\Gamma^e/\Gamma^i|$ in (3.17) remains larger than Λ_{max} over the frequency range of interest. This condition may have been satisfied in BBI for two main reasons: Since the plasma buildup procedure in BBI gave greater ion parallel velocity

spread than in BBII, the potential profile may have been flatter. The greater $\Delta\omega_b$ due to this anharmonic potential would increase Γ^e away from resonances. Also, the larger ion gyroradii in BBI would make the ion drive for bounce mode instabilities more sensitive to radial localization of the waves. If these effects made $|\Gamma^e/\Gamma^i|$ in BBI higher than calculated by a factor of the order of 10, the principal even mode would have been stable according to theory. To verify this conjecture would require both considerably more detailed plasma model than the present one and information about the plasma that is not available from experimental data.

5. SUMMARY

The present numerical model of electrostatic modes improves on infinite-medium analyses principally in using complete electron histories in calculating the perturbed number density. Retaining the periodicity of orbits allows mathematically for the regeneration of density perturbations that causes bounce modes. In addition, solving the matrix eigenvalue equation for the plasma normal modes avoids the short wavelength restriction of WKB formalism. Our model is more complete than previous work on electrostatic mirror plasma modes since terms for the ion response and electron damping are retained in the eigenmode calculations and a wide range of equilibrium fields and plasma distribution functions may be used. The principal simplifications are the neglect of perpendicular wave structure and use of idealized models for the unperturbed fields and distributions.

We find that electron bounce modes can be an important instability mechanism in low-density mirror plasmas when the conditions for coupling to the ions are met. The instability threshold densities calculated for bounce modes in BBII are significantly lower than the predictions of infinite-medium theory and show close agreement with experimental values. For BBI parameters, where the calculation is more sensitive to changes in electron damping and ion response, idealizations in the model result in predicted bounce mode thresholds consistently below those observed.

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Figure Captions

- Fig. 1. Λ_1 vs ζ for peaked and spread v_1 distribution functions.
- Fig. 2. Eigenvalues for electron modes when $B_{\max}/B_0 = 1.0$ and $\xi = 4.5$. Dotted lines are analytic approximations: (a) even modes, (b) odd modes.
- Fig. 3. Variation of principal electron mode eigenvalues with ξ when $B_{\max}/B_0 = 1.0$: (a) even modes, (b) odd modes.
- Fig. 4. Variation of principal undamped electron mode eigenvalues with $B_{\max}/B_0$ when $\xi = 4.5$: (a) even modes, (b) odd modes.
- Fig. 5. Variation of principal undamped electron mode eigenvalues with α in a quartic potential when $B_{\max}/B_0 = 1.5$ and $\psi_{\max}/W_e$: (a) even modes, (b) odd modes.
- Fig. 6. Threshold densities for the longest-wavelength modes when $W_e/W_1 = 4.5 \times 10^{-3}$, $B_{\max}/B_0 = 1.5$, and $\xi = 4.5$: (a) even modes, (b) odd modes.
- Fig. 7. Theoretical and experimental threshold densities for BBI and BBII when $\xi = 4.5$.
- Fig. 8. Theoretical and experimental threshold densities for BBII when axial length is varied and $\xi = 4.5$.

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Fig. 1

XBL 789-11075

Fig. 2

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Fig. 3

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Fig. 4

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Fig. 5

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Fig. 6

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Fig. 7

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Fig. 8

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