Title
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When Was Coase Right?

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1 Introduction

Ronald Coase [7] emphasized that the amount of damage that one party causes to another typically depends on the actions of both parties. Coase argued that, regardless of the way that the law assigns liability, if the perpetrator and recipient are able to bargain freely, they are likely to reach an efficient outcome.

Coase's paper consists of a series of examples and insightful discussions. He made no claims of a formal theorem based on explicit assumptions. The term "Coase Theorem" seems to originate with George Stigler, who explained Coase's ideas in his textbook *The Theory of Price* [13], pp 110-114. Stigler asserted that the Coase theorem establishes two results:

Claim 1. Under perfect competition, private and social costs will be equal.

Claim 2. The composition of output will not be affected by the manner in which the law assigns liability for damage.

Claim 1 of Stigler's version of the Coase theorem can be interpreted as a statement that private bargaining "in the absence of transaction costs" will lead to a Pareto optimal level of externality-producing actions.[1]

Stigler's Claim 2 would follow from Claim 1 if it were true that every Pareto optimal allocation has the same level of externality-producing activities, regardless of the way that private goods are distributed.

In a paper called *What is the Coase Theorem?*, Leo Hurwicz [9] explored the validity of Stigler's Claim 2. Hurwicz refers to this claim as the *Coase independence phenomenon.* He noted that the assumption of quasi-linear utility functions (linear in private goods) is *sufficient* for Coase independence. Hurwicz

*This paper is dedicated to the memory of Richard Cornes and Leo Hurwicz, with whom it was my privilege to share thoughts and puzzlements about this topic. I am grateful to Di Wang of UCSB for steering me to the Chipman-Tiang paper, for useful discussions and for help with diagrams.

[1]Paul Samuelson [12], Dan Usher [14] and others argue that although Condition 1 can be made a tautology with an appropriate definition of "transaction costs", such an interpretation is not helpful for understanding the informational obstacles that often prevent bargaining to an efficient outcome.

maintained that the quasi-linearity assumption is necessary as well as sufficient for Coase independence. This would imply that Coase independence applies only to a very limited class of environments, which rule out income effects on willingness to pay for externality reduction.

John Chipman and Guoqian Tian [5] revisited Hurwicz's contribution. They examined a simple model with two agents–a polluter and a pollutee–and and two commodities–a consumer good and pollution. Chipman and Tian proposed a less stringent characterization of Coase independence than that offered by Hurwicz.[2] They showed that, with their interpretation, Coase independence is consistent with a broader class of preferences that allow the possibility of income effects. These preferences can be represented by utility functions that are "uniformly affine in private goods", taking the form

$$u(x_i, y) = A(y)x_i + B_i(y). \tag{1}$$

This family of preferences was shown in earlier work by Ted Bergstrom and Richard Cornes [4] to be sufficient for the Pareto efficient quantity of public goods to be independent of the distribution of private goods. Bergstrom and Cornes also showed essentially that a community will have Coase independence *regardless of the technology* if and only if preferences of all community members can be represented by utility of the form given in Equation 1. Bergstrom and Cornes, did not explicitly consider harmful externalities, but interpreted their result as applying to public goods that harm nobody.

This paper revisits the contributions of Hurwicz, Chipman and Tian, and Bergstrom and Cornes. We extend the Bergstrom-Cornes results to include economies in which there can be harmful externalities and we extend the Chipman-Tian results to a broader class of economies.

Preferences are representable in the form of Equation 1 if and only if utility possibility sets, contingent on levels of y are parallel straight lines. In this paper, we show that the notion of y-contingent utility possibility sets is a powerful tool for understanding the relation between wealth distribution and efficient public choice. We clarify the sense in which parallel contingent utility possibility sets are necessary for Coase indendence.

We present an example of an economy with specified technology that has Coase independence but does not have parallel contingent utility possibility sets. It turns out that such examples always depend on a specified technology. We show that Coase independence is guaranteed *regardless of the details of technology* if and only if utility functions take the uniformly affine form seen in Equation 1.

We conclude with two applications in which Coase independence plays an interesting role. One of these, which was identified by Gary Becker [2], explores the implications of Coase independence for the effects of divorce law. The other application builds on Coase's discussion of the externalities from a neighborhood

[2] As Chipman and Tian point out, Hurwicz implicitly assumed that at least one consumer has quasi-linear preferences. If this is the case, then there is Coase independence only if all consumers have quasi-linear preferences.

fish-and-chips shop. In the fish-and-chips example, we show that even with identical preference and strong Coase independence, the rich and poor may have opposing preferences about whether the optimal level of an externalitys higher or lower than they would would prefer.

2 Technology and Utility Possibility Frontiers

2.1 Feasible Allocations

Consider a community with n consumers. There are m public variables and one private good. We refer to public *variables* rather than public *goods*, since we allow the possibility that some of these variables represent externalities that are disliked by some or all consumers. A vector y that specifies the level of each public variable is a *public choice*. Assume that there is an initial aggregate endowment of W units of private goods and a cost function $c(\cdot)$ such that given the public choice y, the amount of private goods available to be distributed among the n consumers is $W - c(y)$. Stating this more formally:

Assumption 1 (Endowment and Technology)**.** *There is an initial endowment $W > 0$ of private goods and a compact set $Y \subset \Re^m_+$ of possible public choices. There is a continuous cost function $c(\cdot)$ defined on Y such that $x = W - c(y)$ is the amount of private goods available when the public choice is y.*

Definition 1 (Feasible allocations)**.** *The set of feasible allocations is*

$$\mathcal{F} = \left\{ (x_1, \ldots, x_n, y) | (x_1, \ldots, x_n) \geq 0, y \in Y, \text{ and} \sum_i x_i = W - c(y) \right\}.$$

Definition 2 (Interior allocations)**.** *A feasible allocation $(x_1, \ldots, x_n, y)$ is said to be* interior *if $x_i > 0$ for all consumers i.*

2.2 Contingent Utility Possibility Sets

An adaptation of the *utility possibility set*, which was introduced by Paul Samuelson [10], is useful for exploring necessary and sufficient conditions for Coase independence. We define the y-contingent utility possibility set (y-CUPS) for any vector $y \in Y$ as the set of utility distributions that are possible if the public choice is y and the corresponding amount $W - c(y)$ of private goods is divided among consumers.

Definition 3 (y-Contingent Utility Possibility Set)**.** *The y-contingent utility possibility set for an economy with initial wealth W and cost function $c(\cdot)$ x is*

$$UP(W, y) = \left\{ (u_1(x_1, y), \ldots, u_n(x_n, y)) \,|\, (x_1, \ldots, x_n,) \geq 0, \sum_i x_i = W - c(y) \right\}.$$

The full utility possibility set is then the union of the y-conditional utility possibility sets over all $y \in Y$.

Definition 4 (Utility possibility set)**.** *The utility possibility set for an economy with initial wealth W and the set Y of possible public choices is*

$$UP^*(W,Y) = \cup_{y \in Y} UP(W,y).$$

3 Coase independence

Coase independence, as discussed by Stigler and by Hurwicz can take two possible forms, which we call *weak Coase independence* and *strong Coase independence.*

Let us define a public choice y to be *always Pareto efficient* for an economy if every feasible allocation with this public choice is Pareto optimal; no matter how the private goods are divided among consumers. More formally:

Definition 5 (Always Pareto efficient)**.** *For an economy with a set $\mathcal{F}$ of feasible allocations and a set Y of possible public choices, a public choice $y^* \in Y$ is* always Pareto efficient *if every allocation $\{(x_1, \ldots, x_n, y^*) \geq 0 | \sum x_i = W - c(y^*)\}$ is Pareto optimal.*

We say that an economy displays weak Coase independence if there is some public choice y^* that is always Pareto efficient.

Definition 6 (Weak Coase independence)**.** *An economy with a set $\mathcal{F}$ of feasible allocations and a set Y of possible public choices, satisfies* weak Coase independence *if there is some feasible public choice $y^* \in Y$ that is always Pareto efficient.*

An economy is said to display strong Coase independence if one and only one public choice y^* is possible at an interior Pareto optimum.

Definition 7. *An economy with a set $\mathcal{F}$ of feasible allocations and a set Y of possible public choices satisfies* strong Coase independence *if there is some public choice $y^* \in Y$ such that an interior allocation is Pareto optimal if and only if the public choice is y^*.*

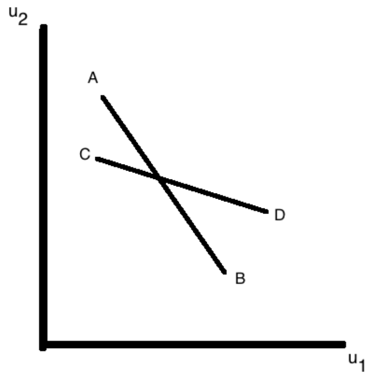
Example 1 shows contingent utility possibility frontiers for an economy that has neither strong nor weak Coase independence.

Example 1

There are two consumers and the set Y has two elements, y and y'. In Figure 1, the line segment AB shows the y-contingent utility possibility set and CD shows the y'-contingent utility possibility set.

Since the two lines cross, some of the allocations possible with public choice y are Pareto dominated by allocations possible only with public choice y', and some of the allocations possible with y' are Pareto dominated by allocations

Figure 1: Crossing contingent utility possibility sets



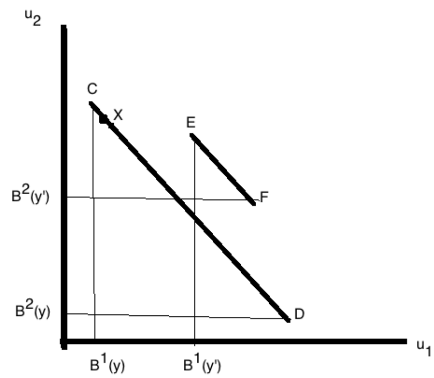
only possible with public choice y. Thus, neither y nor y' is always Pareto optimal. It follows that this economy does not satisfy either weak or strong Coase independence.

Example 2 shows an economy that has weak, but not strong, Coase independence.

Example 2

The set $Y = \{y, y'\}$ consists of two public choice vectors y and y'. In Figure 2, the y-contingent utility possibility set is the line EF and the y'-contingent utility possibility set is the line CD. The endpoints of CD and EF correspond to allocations in which all of the private goods go to one or the other of the two consumers.[3]

Figure 2: Two parallel y-CUPS.



In Example 2, the public choice y' is always Pareto optimal, since all of the points on the line EF are Pareto optimal. Therefore the economy satisfies weak Coase independence. However, not every interior Pareto optimum lies on EF. The utility distribution x shown on line CD is interior since both consumers have positive private consumption. It is also Pareto optimal and strictly preferred by Consumer 2 to any point on the line segment EF. Thus the economy in this example does not have strong Coase independence.

[3] The utility possibility curves were drawn for consumers with utility functions $u_1(x_1, y) = A(y)x_1 + B^1(y)$ and $u_2(x_2, y) = A(y)x_2 + B^2(y)$, where there are two possible social choices y and y', such that $c(y) < c(y')$. The line segment CD of y-contingent utility distributions is longer than the segment EF of y' consistent utility distributions because $c(y) < c(y')$, which means that greater inequality of private income is possible with y than with y'.

4 Parallel y-CUPS and uniformly affine preferences

4.1 Parallel y-contingent utility possibility sets

Example 1 shows that if contingent utility possibility sets cross, then an economy has neither weak nor strong Coase independence. If, as in Example 2, contingent utility possibility frontiers are parallel, there will be no such crossings. In this case, there must be at least weak Coase independence.

The y-contingent utility possibility sets are parallel over the set $\mathcal{F}$ of feasible allocations if there exist utility functions $u_i(x_i, y)$ for each consumer i such that for all $(x_1, \ldots, x_n, y) \in \mathcal{F}$, the y-contingent utility possibility sets lie in parallel hyperplanes, with the sum of utilities constant for each y. More formally:

Definition 8 (Parallel contingent utility possibility sets)**.** *Contingent utility possibility sets are said to be* parallel *on a set $\mathcal{F}$ of feasible allocations if there is a real-valued $F(\cdot)$ with domain Y and for every consumer i there exists a utility function $u_i(x_i, y)$ that represents preferences of i such that the y-contingent utility possibility set $UP(W, y)$ is contained in the hyperplane*

$$\{(u_1, \ldots, u_n) | \sum_i u_i = F(y)\}.$$

A familiar example of utility functions that imply parallel contingent utility possibility sets is the quasilinear family.

Remark 1. *If preferences of all consumers can be represented by utility functions of the quasi-linear form, $u_i(x_i, y) = x_i + f_i(y)$, then y-contingent utility possiblity frontiers must be parallel.*

Proof. If utility is quasi-linear, then

$$\sum_i u_i(x_i, y) = \sum_i x_i + \sum_i f_i(y).$$

An allocation with public choice y is feasible if and only if $\sum_i x_i = W - c(y)$ and $x_i \geq 0$ for all i. It follows that the y-contingent utility possibility set is contained in the hyperplane

$$\{(u_1, \ldots, u_n) | \sum u_i = F(y)\},$$

where $F(y) = W - c(y) + \sum_i f_i(y)$.

□

4.2 Uniformly affine preferences

The assumption of quasi-linearity is highly restrictive, since it rules out the possibility that one's willingness to pay for a public variable might depend on one's wealth. There is, however, a larger class of preferences that imply parallel contingent utility possibility sets and that allow for the possibility of wealth effects.

We show that an economy has parallel y-contingent utility possibility sets if and only if they can be represented by utility functions of the form

$$u_i(x_i, y) = A(y)x_i + B^i(y),$$

where $A(y)$ is common to all i and the functions $B_i(y)$ may differ between individuals.[4]

Definition 9 (Uniformly affine preferences)**.** *Preferences over the feasible set $\mathcal{F} = (X_1, \ldots, X_n, Y)$ are uniformly affine in private goods if the preferences of each consumer i over the domain (X_i, Y) can be represented by a utility function*

$$U_i(x_i, y) = A(y)x_i + B^i(y),$$

where $A(y)$ and $B^i(y)$ are continuously differentiable functions and where $A(y) > 0$ for all $y \in Y$.

If preferences are uniformly affine in private goods, then Consumer i's marginal rate of substitution between public variable j and the private good is

$$m^i_j(x_i, y) = \frac{A_j(y)}{A(y)}x_i + \frac{B^i_j(y)}{A(y)}. \tag{2}$$

where $A_j(y)$ and $B^i_j(y)$ are partial derivatives of A and B^i with respect to y_j.

The uniformly affine family includes quasilinear preferences, but also allows for a rich variety of income effects. From Equation 2, we see that when $A(y)$ is not constant, consumers' marginal rates of substitution between public variable j and the private good may increase or decrease with private consumption, depending on whether the partial derivative, $A_j(y)$, is positive or negative. Since the sign of $B^i_j(y)$ may differ between consumers, it may be that some consumers prefer more and some consumers prefer less of a public variable j. If $A_j(y)$ and $B^i_j(y)$ are of opposite signs, then consumer i may switch between favoring and opposing increased y as her wealth changes.

It is easy to show that if preferences are uniformly affine in private goods, then the corresponding y-contingent utility possibility sets are parallel.

Proposition 1. *If preferences of all consumers can be represented by continuous utility functions of the form*

$$u_i(x_i, y) = A(y)x_i + B_i(y).$$

for all $y \in Y$, then the y-contingent utility possibility sets are parallel and take the form

$$UP(W, y) = \left\{(u_1, \ldots, u_n) \geq \left(B^1(y), \ldots, B^n(y)\right) \mid \sum u_i = F(y)\right\}$$

where $F(y) = A(y)\left(W - c(y)\right) + \sum_i B^i(y)$.

[4]This class of utilities was introduced by Bergstrom and Cornes [4]. Chiappori *et al* [6] refers to these as "generalized quasi-linear utilities."

Proof. If preferences of consumer i are represented by $u_i(x_i, y) = A(y)x_i + B_i(y)$, then it must be that

$$\sum_i u_i(x_i, y) = A(y) \sum_i x_i + \sum_i B^i(y).$$

An allocation $(x_1, \ldots, x_n, y)$ is feasible if and only if $\sum x_i = W - c(y)$ and $x_i \geq 0$ for all i. Therefore the y-contingent utility possibility set is the set

$$\{(u_1, \ldots, u_n) \geq \left((B^1(y), \ldots, B^n(y)\right) \mid \sum u_i = F(y)\}$$

where $F(y) = A(y)\,(W - c(y)) + \sum_i B^i(y)$. Hence the y-contingent utility possibility sets are parallel on Y. $\square$

The condition that preferences are uniformly affine in private goods is *necessary* as well as *sufficient* for parallel contingent utility possibility sets. To show this result, we employ a standard result from the theory of functional equations. An equation that satisfies the equation $f(x + y) = g(x) + h(y)$ for all real-valued x and y is known as a Pexider functional equation. (See Aczel [1], page 142). Aczel shows that if f, g, and h are continuous and satisfy the Pexider functional equation, then there must be real numbers a, b, and c, such that $f(x) = ax + b + c$, $g(x) = ax + b$ and $h(x) = ax + c$.

This result generalizes in a straightforward way to the case of sums of n terms. Although the usual statement of the result deals with functions whose domain is the entire real line, the proof that Aczel uses for the Pexider result shows that that this result holds if the domain is the non-negative reals.[5] We have the following lemma.

Lemma 1 (Pexider functional equations)**.** *Let f_i, $i = 1, \ldots, n$, be continuous functions with domain $\Re_+$. If there is a function f such that $\sum_i f_i(x_i) = f(\sum x_i)$ for all $x_i \geq 0$, then there must exist constants a and $b_1, \ldots, b_n$ such that $f_i(x) = ax + b_i$ for $i = 1, \ldots, n$ and $f(x) = ax + \sum_i b_i$.*

A proof (which is quite elementary) can be found in Aczel[1] or in Diewert[8].

Proposition 2. *If contingent utility possibility sets are parallel on $\mathcal{F}$, then there exist functions $A(y)$ and $B^i(y)$ such that preferences of each consumer i are represented by a utility function of the form*

$$A(y)x_i + B^i(y).$$

Proof. If y-contingent utility possibility sets are parallel, preferences of each i can be represented by a utility function $u_i(x_i, y)$ such that for all allocations in $UP(W, y)$, $\sum_i u = F(y)$. The y-contingent utility possibility set consists of all vectors, $(u_1(x_1, y), \ldots, u_n(x_n, y))$ such that $\sum_i x_i = W - c(y)$ and $x_i \geq 0$ for all i. It follows if $\sum x_i = \sum x_i'$ and $x_i \geq 0$ for all i, then $\sum u_i(x_i, y) = \sum u_i(x_i', y)$. From Lemma 1, it then follows that for each $y \in Y$, the function $u_i(x, y)$ must be of the form $u_i(x_i, y) = A(y)x_i + B^i(y)$.

$\square$

[5] Diewert [8] also notices this in his notes on functional equations.

4.3 Coase independence and uniformly affine preferences

For the economies studied here, if preferences are uniformly affine in private goods, then there must be weak Coase independence. Stated formally:

Proposition 3. *For an economy satisfying Assumption 1, where consumer preferences are uniformly affine in private goods, there is weak Coase independence, and there is an always Pareto efficient public choice $y^* \in Y$. The public choice y^* maximizes $F(y) = A(y)\,(W - c(y)) + \sum_i B^i(y)$ on the set Y.*

Proof of Proposition 3. Assumption 1 requires that the set Y is compact. Continuity of the functions $A(y)$, $c(y)$, and $B^i(y)$ implies that $F(\cdot)$ is continuous. Therefore there exists $y^* \in Y$ such that $F(y^*) \geq F(y)$ for all $y \in Y$.

We next show that if $\sum_i x_i = W - c(y^*)$ and $x_i \geq 0$ for all i, then the allocation $(x_1, \ldots, x_n, y^*)$ is Pareto optimal. Suppose that the allocation $(x'_1, \ldots, x'_n, y)$ is Pareto superior to $(x_1, \ldots, x_n, y^*)$. Then it must be that $A(y)x'_i + B^i(y) \geq A(y^*)x_i + B^i(y^*)$ for all i, with strict inequality for some i. This implies that

$$A(y)\sum_i x'_i + \sum B^i(y) > A(y^*)\sum x_i + \sum B^i(y^*) \tag{3}$$

But $\sum_i x_i = W - c(y^*)$ and if $(x'_1, \ldots, x'_n, y)$ is feasible, it must also be that $\sum_i x'_i = W - c(y)$. Therefore if $(x'_1, \ldots, x'_n, y)$ is feasible, it must be that

$$F(y) = A(y)\,(W - c(y)) + \sum B^i(y') > A(y^*)\,(W - c(y^*)) + \sum B^i(y^) = F(y^*). \tag{4}$$

But this is impossible, since y^* maximizes $F(\cdot)$ on Y. It follows that the public choice y^* is always Pareto optimal.

□

Proposition 3 does not depend on any assumptions about convexity of preference or of feasible sets. There is weak Coase independence whenever preferences are uniformly affine.

The economy shown in Example 3 was constructed with uniformly affine preferences and thus has weak Coase independence, but lacks strong Coase independence. There will, however be strong Coase independence if preferences are convex as well as uniformly affine in private goods and if the cost function is a convex function.

Assumption 2 (Convexity assumption)**.** *Preferences of all consumers i are strictly convex. The set Y is a convex set and the function $c(y)$ is a convex function.*

Proposition 4. *Given the feasibility assumption 1 and the convexity assumption 2, if preferences are uniformly linear in private goods, then there is strong Coase independence.*

Proof of Proposition 4. According to Proposition 2, the assumption that preferences are uniformly linear in private goods implies that there exists a public choice $y^* \in Y$ that is always Pareto efficient and such that

$$F(y^*) = A(y^*)\,(W - c(y^*)) + \sum_i B^i(y^*) \geq F(y) \tag{5}$$

for all $y \in Y$. If preferences are strictly convex, then the function $F(\cdot)$ is strictly quasi-concave and hence y^* is the unique maximizer of F on the convex set Y.

We next show that any feasible interior allocation $(x_1, \ldots, x_n, y)$ where $y \neq y^*$, is Pareto dominated by a feasible allocation with public choice y^*. If $(x_1, \ldots, x_n, y)$ is feasible, it must be that $\sum x_i \leq W - c(y)$. Since y^* is the unique maximizer of $F(y)$ on Y, it follows that $F(y^*) - F(y) = \delta > 0$. For each i, let

$$x_i^* = \frac{A(y)x_i + B^i(y) + \frac{\delta}{n} - B^i(y^*)}{A(y^*)}. \tag{6}$$

Then it must be that for all i,

$$A(y^*)x_i^* + B^i(y^*) - \big(A(y)x_i + B^i(y)\big) = \frac{\delta}{n} > 0. \tag{7}$$

For $\lambda \in (0,1)$, define $x_i(\lambda) = x_i + \lambda(x_i^* - x_i)$, and $y(\lambda) = y + \lambda(y^* - y)$. Since the allocation $(x_1, \ldots, x_n, y)$ is feasible, it must be that $\sum_i x_i = W - c(y)$. Since $\sum_i x_i^* = W - c(y^*)$, and since the function $c(\cdot)$ is assumed to be convex, it follows that $\sum_i x_i(\lambda) \leq W - c\,(y(\lambda))$. Since $x_i > 0$ for all i, it follows that for λ positive but sufficiently small, $x_i(\lambda) > 0$ for all i. Therefore for sufficiently small positive values of λ, the allocation $(x_1(\lambda), \ldots, x_n(\lambda), y(\lambda))$ is feasible.

Since preferences are assumed to be strictly quasiconcave, it must be that for all i, $u_i\,(x_i(\lambda), y(\lambda)) > u_i\,(x_i, y)$. Therefore the allocation $(x_1, \ldots, x_n, y)$ cannot be Pareto optimal. It follows that the only Pareto optimal allocations have public choice y^*.

□

Example 3 shows a utility possibility set with a continuum of possible values of public choice y.

Example 3

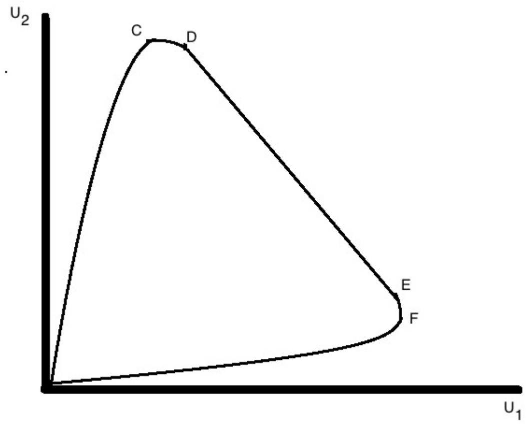
A single public good is produced at cost $c(y) = y$ and the set of possible quantities is $Y = [0, 9]$. There are two consumers with convex preferences and utility functions: $u_1(x_1, y) = x_1 y + y$ and $u_2(x_2, y) = x_2 y + y$. Total initial holdings of private goods are $W = 9$.

In Example 3, the y-contingent utility possibility set is a line segment such that

$$u_1 + u_2 = (9 - y)y + y = 10y - y^2$$

$u_1 \geq y$ and $u_2 \geq y$, where the endpoints correspond to the utility distributions attained with y when all of the private goods are given to one of the consumers.

Figure 3: A utility possibility set with a continuum of possible y's



The sum of utilities is maximized at $\bar{y} = 5$. The $\bar{y}$-contingent utility possiblity set is the line segment shown as CD in Figure 3, extending between the points $D = (5, 20)$ and $E = (20, 5)$. This line segment is tangent to the curves OCD and OFE. The full utility possibility set is the area in Figure 3, which includes all points lying between the curves OCD and OFE and below the line CD. Points in the interior of the utility possibility set are reached with values of $y < \bar{y}$ and $x_i > 0$ for $i = 1$ and 2.

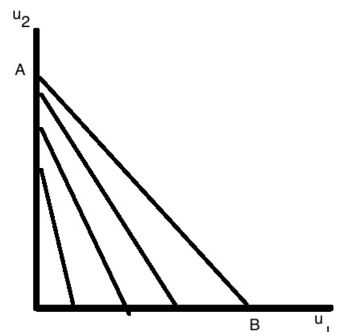
The utility possibility frontier, containing all of the Pareto efficient points, consists of all of the points on the line CD as well as the points on the two curved line segments, CD and EF. The Pareto optimal points on these curved line segments are achieved with income distributions in which one of the two consumers receives no private goods. In this example, the highest possible utility for Consumer 2 occurs at the point C. Other points on the curve CD represent outcomes in which Consumer 2 continues to receive all of the private goods and where increased amounts of the public goods paid for by Consumer 2 are beneficial to both consumers.

5 Coase independence without uniformly affine preferences

We have seen that Coase independence requires that contingent utility possibility sets do not cross. If utility possibility sets are parallel straight lines, then of course they do not cross and, as we have shown, there must be Coase independence. But since contingent possibility sets are bounded, they might not cross even if they are parallel. This is illustrated in Figure 4.

Example 4 presents an economy that does not have parallel contingent utility possibility sets, but does have strong Coase independence.

Figure 4: Coase independence without parallel contingent utility possibility sets



Example 4

There are two consumers and one public good, which is produced at zero cost and can be made available in any quantity y between 0 and 2. The set of feasible allocations is

$$\mathcal{F} = \{(x_1, x_2, y) | x_1 + x_2 = W \text{ and } y \in [0, 2]\}.$$

Consumer 1's utility function is

$$u_1(x_1, y) = x_1 + y - 1 \tag{8}$$

and person 2's utility function is

$$u_2(x_2, y) = x_2 \left(1 - \frac{1}{2}(y-1)^2\right) - y + 1. \tag{9}$$

Adding Equations 8 and 9 yields

$$\begin{aligned} u_1(x_1, y) + u_2(x_2, y) &= x_1 + x_2 - \frac{1}{2} x_2 (1-y)^2 \\ &= W - \frac{1}{2} x_2 (1-y)^2. \end{aligned} \tag{10}$$

From Equation 10, it follows that the y-contingent utility possibility frontier for $y = 1$ is the line segment

$$\{(u_1, u_2) \geq 0 | u_1 + u_2 = W\}$$

Since for all feasible allocations (x_1, x_2, y), it must be that $x_1 + x_2 = W$, it follows from Equation 10 that for all feasible allocations such that $y \neq 1$, $u_1(x_1, y) + u_2(x_2, y) < W$. From this it follows that the public choice $y = 1$ is always Pareto optimal, and that in every Pareto optimum, the public choice must be

$y = 1$.[6] Therefore this economy has strong Coase independence. Although there is strong Coase indepence, the $y-$contingent utility possiblity sets for $y \neq 1$, are not parallel. To see this, note that a y-contingent utility possibility set is described by a line segment with equation

$$\left\{(u_1, u_2)|u_1 + \left(\frac{1}{1 - \frac{1}{2}(y-1)^2}\right) u_2 = W - \frac{(y-1)^3}{1 - \frac{1}{2}(y-1)^2}\right\} \tag{11}$$

Since the slopes of these line segments are not constant as y changes, it must be that y-contingent utility possibility sets are not parallel, and therefore it must also be that they cannot be represented by utility functions that are uniform affine in private goods.

6 General Coase independence

Example 4 assumed that the public choice y is produced at zero cost. But if the cost of producing y units is cy where $c > 0$, then this example will not have Coase independence.[7] This observation illustrates a more general result. In order for there to be Coase independence regardless of the details of technology, then if private goods are "important enough," contingent utility possibility sets must be parallel. This in turn implies that preferences must be uniformly affine in private goods. Thus preferences that are uniformly affine in private goods are a necessary condition for "General Coase independence" as defined here.

Definition 10 (General Coase Independence)**.** *A set of n consumers, with preferences $\succeq_1, \ldots, \succeq_n$ satisfies general Coase independence if there is weak Coase independence for every economy with these consumers and where the set of possible public choices*

$$\mathcal{F} = \{(x_1, \ldots, x_n, y) \geq 0 | \sum_i x_i = W - c(y) \text{ and } y \in Y\}$$

where $c(\cdot)$ is any continuous function.

To establish this result, we need to develop the notions of "money metric" utility and of "compensable changes" in public choice.

6.1 Money metric utility and compensable changes

To define a money metric[8] utility function for an economy with public choices, we choose a reference public choice vector $\bar{y}$.[9] The money metric utility $\bar{u}(x_i, y)$

[6] Consider any allocation $(x_1, x_2, 1)$ such that $x_1 + x_2 = W$. If allocation $(x - 1', x_2', y)$ is Pareto supperior to $(x_1, x_2, , 1)$, then it must be that $u_1(x_1', y) + u_2(x_2', y) > u_1(x_1, 1) + u_2(x_2, 1) = W$. But we have shown that $u_1(x_1', y) + u_2(x_2', y) \leq W$ for all feasible allocations.

[7] Calculations show that it must be that in all Pareto optimal allocations $y < 1$, and in Pareto optimal allocations the more private goods awarded to Person 2, the smaller y must be.

[8] The term "money metric" utility function seems to have been coined by Samuelson [11] who applied this idea to definitions of complementary private goods.

[9] For example, the reference vector $\bar{y}$ could represent a *status quo* public choice.

is defined to be the amount of private good that consumer i would need along with $\bar{y}$ in order to be indifferent between (x_i, y) and $(\bar{u}_i(x_i, y), \bar{y})$.

Without further assumptions, the function $\bar{u}(\cdot, \cdot)$ would not be well-defined. It might be for instance, that the outcome (x_i, y) is worse for i than any outcome with public choice $\bar{y}$. Alternatively, it might be that (x_i, y) is preferred by i to any outcome with public choice $\bar{y}$ no matter how much private expenditures would accompany $\bar{y}$.

To ensure that the function $\bar{u}$ is well-defined, we restrict its domain to outcomes that are no worse for any i than the outcome with zero consumption of private goods and public choice $\bar{y}$.

Definition 11 (No worse than the $\bar{y}$-extremes)**.** *The set of feasible allocations that are no worse than the $\bar{y}$-extremes is*

$$\bar{\mathcal{F}}^{+} = \{(x_1, \ldots, y) \in \mathcal{F} | u_i(x_i, y) \geq u_i(0, \bar{y}) \text{ for all } i\}.$$

We also assume that private goods are important enough to i so that for any public choice y, there is some amount of private goods that would be sufficient to compensate i for having public choice $\bar{y}$ rather than y.

Definition 12 (Privately compensable advantage)**.** *The advantage of public choice y over $\bar{y}$ is privately compensable for i if for every $x_i > 0$, there exists $z > 0$ such that $(z, \bar{y}) \succeq_i (x_i, y)$.*

Lemma 2 gives us conditions under which the money metric utility function $\bar{u}(cdot, \cdot)$ is well defined.

Lemma 2. *Let consumer i have preferences that are monotone increasing in private goods and assume that for all $y \in Y$, the advantage of y over $\bar{y}$ is privately compensable for i. Then for all (x_i, y) such that $(x_i, y) \succeq (0, \bar{y})$, preferences of consumer i can be represented by a money metric utility function $\bar{u}_i$ with base $\bar{y}$ defined by the condition*

$$(\bar{u}_i(x_i, y), \bar{y}) \sim_i (x_i, y)$$

where $\sim_i$ denotes indifference.

Proof. The lemma assumes that $(x_i, y) \succeq_i (0, \bar{y})$ and that for some $z > 0$, $(z, \bar{y}) \succeq_i (x_i, y)$. The assumptions that preferences are continuous and monotone increasing in private goods imply that there is exactly one real number $\bar{u}_i(x_i, y)$ such that $(\bar{u}_i(x_i, y), \bar{y}) \sim_i (x_i, y)$. Since preferences are monotone increasing in private goods, it follows that $\bar{u}_i(x_i, y)$ represents preferences of i over the set of all outcomes (x_i, y) such that $(x_i, y) \succeq (0, \bar{y})$.

□

6.2 A necessary condition for general Coase independence

We can use the money metric utility function to show that if there is general Coase independence, y-contingent utility possibility sets must be parallel. Since

y-CUPS are parallel if and only if preferences are uniformly affine in private goods, it follows that, subject to mild technical conditons, a necessary condition for general Coase independence is that preferences are uniformly affine in private goods.

Proposition 5. *If for all consumers, preferences are continuous, monotonic in private goods, and if public choices are compensable, then there is general Coase independence only if and only if y-contingent utility possibility sets are parallel on the set $\bar{\mathcal{F}}^+$ of allocations that are no worse than the $\bar{y}$-extremes.*

Proof. For each i, let $\bar{u}_i(x_i, y)$ be the money metric utility function defined with base $\bar{y}$. It is immediate from the definition of $u_i(\cdot, \cdot)$ that $\bar{u}_i(x_i, \bar{y}) = x_i$ for all $x_i \geq 0$. It then follows that for any W and any function $c(y)$, the $\bar{y}$-contingent utility possibility set is the set $\{(u_1, \ldots, u_n | \sum u_i = W - c(\bar{y})\}$.

Suppose that y-contingent utility possibility sets are not parallel on $\bar{\mathcal{F}}^+$. Then for some $\hat{y}$ there exist two allocations $(x_1, \ldots, x_n, \hat{y})$ and $(x'_1, \ldots, x'_n, \hat{y})$ in $\bar{\mathcal{F}}^+$ such that $\sum x'_i = \sum x_i$ and $\sum_i \bar{u}_i(x'_i, \hat{y}) > \sum_i \bar{u}_i(x_i, \hat{y})$.

Consider the economy with only two possible public choices $\hat{y}$ and $\bar{y}$. General Coase independence requires that for any cost function, one of the two public choices $\hat{y}$ or $\bar{y}$ is always Pareto optimal. Let us choose the function $c(\cdot)$ so that

$$\sum_i \bar{u}_i(x'_i, \hat{y}) > W - c(\bar{y}) > \sum_i \bar{u}_i(x_i, \hat{y}) \tag{12}$$

Now the $\bar{y}$-contingent utility possibility set is $(u_1, \ldots, u_n) | \sum u_i = W - c(\bar{y})$. Since according to Expression 12, $\sum_i \bar{u}_i(x'_i, \hat{y}) > W - c(\bar{y})$, it must be that the public choice $\bar{y}$ is not always Pareto optimal. But since we also have $W - c(\bar{y}) > \sum_i \bar{u}_i(x_i, \hat{y})$, it follows that the public choice $\hat{y}$ is also not always Pareto optimal. Therefore this economy has no always Pareto optimal allocation.

It follows that if y-contingent utility possibility frontiers are not parallel for all $y \in Y$, then there is not general Coase independence. Hence if there is general Coase independence, y-contingent utility possibility frontiers must be parallel on the set of feasible allocations that are the $\bar{y}$-extremes.

□

We have shown that y-contingent utility possibility frontiers are parallel if and only if preferences can be represented by utility functions that are uniformly linear in private goods. Thus we have the following corollary.

Corollary 1. *If preferences of all consumers are continuous and y-contingent utility possibility sets are parallel for all $y \in Y$ and $W > c(y)$, then it must be that preferences of all consumers i can be represented by utility functions of the form*

$$A(y)x_i + B^i(y).$$

7 Applications

7.1 Becker's Divorce Theory

Interesting applications of Coase independence also appear in discussions of the economics of the family. Gary Becker [2] (page 331) suggested that the "Coase theorem" implies that changes in divorce law, such as requiring divorce by mutual consent rather than allowing unilateral withdrawal, would not affect divorce rates, though they might affect the division of family resources within marriages. Becker reasoned that if a married couple will always reach efficient bargains with each other about the terms of marriage, then the Coase theorem implies that they will divorce if and only if they can both be better off divorced than they would be under any arrangement of benefits within marriage.

Chiappori *et al* [6] argue that Becker's assumption of Coasian independence may not be appropriate in the case of divorce, where income distribution effects could be large, and consequently efficient outcomes that favor one partner might leave them married, while efficient outcomes that favor the other partner would have them divorced.

A connubial example

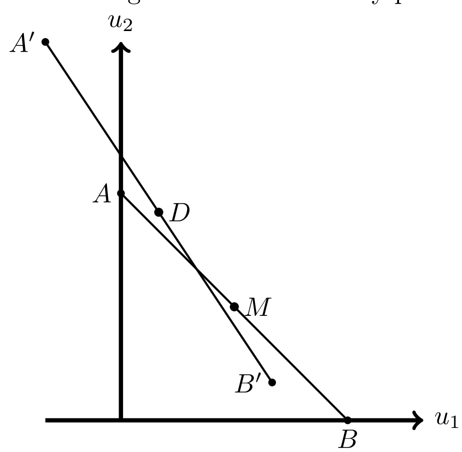
Persons 1 and 2 are currently married. They have a fixed wealth W which could be divided between them in any way. Each has a property right to $W/2$ units of private good, whether they divorce or remain married. There are two possible public choices, y_M in which they remain married, and y_D, in which they divorce. For any fixed level of private consumption, Person 1 would prefer to remain married and Person 2 would prefer to divorce. The utility functions of persons 1 and 2 are given by $u_i(x_i, y_M) = x_i$, for $i = 1, 2$, while $u_1(x_1, y_D) = x_i - 2$ and $u_2(x_2, y_D) = 1.5x_2 + 1$.

The contingent utility possibility sets are shown in Figure 5, where the lines AB and $A'B'$ show, respectively, the y_M-contingent and y_D-contingent utility possibility frontiers. Since each has property rights to half of their joint wealth W, their utility distribution is shown by the point M if they remain married and by the point D if they divorce.

We see from Figure 5 that the outcome M is Pareto optimal. Therefore, if divorce can occur only by mutual consent, the couple will stay married. There is no distribution of wealth between them that would leave both persons better off after divorce. If, on the other hand, it is possible for either to unilaterally withdraw from the marriage, Person 2 would choose to divorce, since she has a higher utility at the point D than at M.

In this example, preferences are not uniformly affine in private goods and the contingent utility possibility sets cross. According to Proposition 4, if preferences were convex and uniformly affine in private goods, then there would be strong Coase independence and Becker's conclusion that divorce rates do not depend on the details of divorce law would be correct.

Figure 5: Marriage and divorce: utility possibilities



7.2 Coase's Fish and Chips

We have shown that there can be Coase independence even if rich people are willing to pay more than poor people to reduce pollution. In fact, it is even possible that despite identical preferences, the poor may regard a public variable as a good, while the rich regard it as a bad. Coase himself offered a nice example in which this is the case.

Coase [7] (p 21) described a court case in which a fried fish shop in a "predominantly working class district was set up near houses 'of a much better character'." Occupants of these houses sought to close the shop on grounds of the "odour and fog or mist" emitted. The judge ruled that the shop must be moved, but could be allowed to locate near houses of less high character, whose occupants would be likely to find that the convenience of proximity would more than compensate for any adverse aromatic effects.

Coase's fried fish story is clearly not consistent with quasi-linear utility, since aversion to the smell of fish and chips is assumed to increase with income. Nevertheless, we can construct an economy that is qualitatively similar to Coase's fish and chips case and also exhibits strong Coase independence.

A culinary example

A community has n residents and a fish and chips shop. The public choice is the number of hours y per day that the shop is open. Where x_i is private consumption by i, residents have identical utility functions of the form

$$u_i(x_i, y) = A(y)x_i + B(y). \tag{13}$$

We assume that u_i is strictly quasi-concave and $A(y) > 0$. We also assume that $A'(y) < 0$, $B'(y) > 0$, $A''(y) < 0$ and $B''(y) < 0$. Let $W = \sum x_i$ be total income and let $\bar{W} = W/n$ be average per capita income of members the community.

Since preferences are convex and uniformly affine in private goods, Proposition 4 implies that there is a unique Pareto optimal number of hours y^* for the fish shop to be open. The Pareto number optimal of hours is y^* where y^* maximizes

$$\sum_{i=1}^{n} u_i(x_i, y) = A(y)W + nB(y) \tag{14}$$

Expression 14 is maximized when

$$A'(y^*)\frac{W}{n} + B'(y^*) = 0. \tag{15}$$

Calculations show that the greater is total community income, the smaller will be the optimal number of hours, y^*.[10]

For each resident i, we have

$$\frac{\partial u_i(x_i, y^*)}{\partial y} = A'(y^*)x_i + B'(y^*). \tag{17}$$

Since, by assumption, $A'(y^*) < 0$, it follows from Equations 15 and 17 that residents with above-average incomes will favor shorter hours and those with below-average incomes will favor longer hours for the fish shop.

8 Conclusion

One of the claims of the "Coase theorem" is that when bargaining leads to a Pareto optimal allocation, the observed public choice of externality-producing activities is independent of the assignment of property rights. This assertion can take two forms, *weak Coase independence* and *strong Coase independence.* There is weak Coase independence if there is some public choice y^* such that any feasible distribution of private goods along with public choice y^* is Pareto optimal. There is strong Coase independence if every interior Pareto optimal allocation requires the same public choice y^*.

A sufficient condition for weak Coase independence is that the utility possibility sets contingent on public choice y remain parallel as one varies y. If preferences are convex and the feasible set of allocations is convex, then there will also be strong Coase independence.

Contingent utility possibility sets are parallel if and only if preferences of all individuals can be represented in the functional form

$$U_i(x_i, y) = A(y)x_i + B_i(y), \tag{18}$$

[10] Totally differentiating Equation 15 with respect to W, we find that

$$\frac{dy^*}{dW} = \frac{-A'(y^*)}{A''(y^*)W + nB''(y^*)} \tag{16}$$

Since by assumption, $A'(y^*) < 0$, $B''(y^*) < 0$ and $A''(y^*) < 0$, it follows that y^* is a decreasing function of W.

which we call "uniformly affine in private goods." This family of preferences includes quasi-linear preferences, but also allows a rich variety of preferences in which there are income effects.

Examples can be found of economies with strong Coase independence, although preferences can not be represented by utility functions of the form in Equation 18. However if one wishes to impose conditions on preferences alone that will guarantee Coase independence regardless of technology, then, subject to minor restrictions, it must be the case that preferences can be represented by utilities functions of this form.

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Appendix: A Proof using differentiability

Some may find it instructive to see an alternative proof of the necessity of parallel indifference curve, given the assumptions preferences are convex and utility functions are differentiable. Similar proofs can be found in [3] and in [5].

A sketch of the proof is as follows: If preferences are convex and if there is general Coase independence, then every possible allocation is Pareto optimal for some cost function. All interior Pareto optima all satisfy the Samuelson condition that the sum of marginal rates of substitution equal marginal cost. If there is weak Coase Independence, this implies that if private goods are redistributed, the sum of marginal rates of substitution must remain constant. It then follows from Lemma 1 that marginal rates of substitution for all i take the form $m_i(x_i, y) = \alpha(y)x_i + \beta_i(y)$. Applying standard tricks of partial differentiation theory we can show that this will be the case if and only if preferences can be represented by a utility function of the form $A(y) + B^i(y)$ and hence y-contingent utility possibility sets are parallel.

Here is a more formal treatment: Let there be n consumers, one private good and one public variable. Consumers have convex preferences representable by differentiable utility functions, defined over all $(x_i, y) \geq 0$. There is a differentiable cost function $c(y)$ for the public variable. The set of feasible allocations consists of all allocations $(x_1, \ldots, x_n, y) \geq 0$ such that $\sum_i x_i + c(y) = W$ for some constant W.

Proposition 6. *If an economy as described in this section satisfies General Coase Independence, then it must be that preferences of each consumer i can be represented by a utility function of the functional form*

$$u_i(x_i, y) = A(y)x_i + B^i(y).$$

Proof. Since preferences are convex and differentiable, for any interior allocation $(x_1, \ldots, x_n, y)$, there is some cost function cy, such that this allocation is Pareto optimal. To see why this is true, we observe that for an economy with convex preferences and linear cost function, the Samuelson condition setting sum of marginal rates of substitution equal to marginal cost is both necessary and sufficient for an allocation to be Pareto optimal. Therefore the allocation $(x_1, \ldots, x_n, y)$ is Pareto optimal for an economy in which

$$c = \sum_{i=1}^{n} m_i(x_i, y) \tag{19}$$

where $m_i(x_i, y)$ is i's marginal rate of substitution between y and x_i. □

General Coase independence requires that if $(x_1 \ldots x_n, y)$ is a Pareto efficient allocation, then so is every allocation $(x'_1, \ldots, x'_n, y) > 0$ such that $\sum_{i=1}^n x'_i = \sum_{i=1}^n x_i$. But this implies that if $\sum_i x'_i = \sum x_i$, then

$$\sum m_i(x'_i, y) = \sum m_i(x_i, y) = c. \tag{20}$$

It follows that

$$\sum_{i=1}^{n} m_i(x_i, y) = m\left(\sum_{i=1}^{n} x_i, y\right) \tag{21}$$

for some function $m(x, y)$.

Equation 21 is therefore a *Pexider* functional equation, and it follows from Lemma 1 that there exist real numbers $\alpha(y)$ and $(\beta_1(y), \ldots, \beta_n(y))$ such that

$$m_i(x_i, y) = \alpha(y)x_i + \beta_i(y). \tag{22}$$

for $i = 1, \ldots, n$.

We now show that when marginal rates of substitution are of the functional form are linear in x_i as in Equation 22, then preferences can be represented by utility functions that are also linear in x_i

For any $(x_i, y) > 0$, the indifference curve that passes through (x_i, y) consists of the set of all (x', y') such that $u_i(x'_i, y') = u_i(x_i, y)$. We can define a function $x'_i(y')$ implicitly so that

$$u_i(x'_i(y), y') = u_i(x_i, y) \tag{23}$$

Then it must be that

$$\frac{\partial x'_i(y')}{\partial y'} = -m_i(x_i(y'), y') \tag{24}$$

and hence from Equation 22 it follows that

$$\frac{\partial x_i(y')}{\partial y'} = -\alpha(y')x'_i(y') - \beta^i(y') \tag{25}$$

Equation 25 is a first-order linear differential equation. To solve this equation, it is useful to multiply both sides of the equation by an "integrating factor"

$$e^{\int_0^{y'} \alpha(s)ds}.$$

Equation 25 is seen to be equivalent to

$$e^{\int_0^{y'} \alpha(s)ds}\left(\frac{\partial x'_i(y')}{\partial y'} + \alpha(y')x'_i(y')\right) = -e^{\int_0^{y'} \alpha(s)ds}\beta_i(y')) \tag{26}$$

Integrating both sides of Equation 26, we have

$$x'_i(y')e^{\int_0^{y'} x(s)ds} = -\int_0^{y'}\left(e^{\int_0^{y'} \alpha(s)ds}\beta_i(y')\right) + c \tag{27}$$

where c is a constant of integration. We can define

$$A(y') = e^{\int_0^{y'} x(s)ds}$$

and

$$B^i(y') = \int_0^{y'}\left(e^{\int_0^{y'} \alpha(s)ds}\beta_i(y')\right)$$

Substituting into Equation 27 and rearranging terms, we then have the result that

$$A(y')x_i' + B^i(y') = c, \tag{28}$$

for all (x_i', y') such that $u(x_i', y') = u(x_i, y)$. It follows that there exist functions $A(\cdot)$ and $B_i(\cdot)$ such that preferences of all consumers can be represented by a utility function of the form $A(y)x_i + B^i(y)$.