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Zero Sets of Abelian Lie Algebras of Vector Fields

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Abstract. Assume M is a 3-dimensional real manifold without boundary, $\mathcal{A}$ is an abelian Lie algebra of analytic vector fields on M, and $X \in \mathcal{A}$.

Theorem If K is a locally maximal compact set of zeroes of $X \in \mathcal{A}$ and the Poincaré-Hopf index of X at K is nonzero, there is a point in K at which all the elements of $\mathcal{A}$ vanish.
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1. Introduction

Throughout this paper M denotes a real analytic, metrizable manifold that is connected and has finite dimension n, fixed at $n = 3$ in the main results.
The space of (continuous) vector fields on M endowed with the compact open topology is $\mathcal{V}(M)$, and $\mathcal{V}^r M$ is the subspace of C^r vector fields. Here r denotes a positive integer, ∞, or ω (meaning analytic); this convention is abbreviated by $1 \leq r \leq \omega$.

The *zero set* of $X \in \mathcal{V}(M)$ is $\mathsf{Z}(X) := \{p \in M \colon X_p = 0\}$. If $\mathsf{Z}(X) = \varnothing$ (the empty set), X is *nonsingular*. The zero set of a subset $\mathcal{S} \subset \mathcal{V}(M)$ is $\mathsf{Z}(\mathcal{S}) := \bigcap_{X \in S} \mathsf{Z}(S)$.

A compact set $K \subset \mathsf{Z}(X)$ is a *block of zeros* for X— called an X*-block* for short— if it lies in a precompact open set $U \subset M$ whose closure $\overline{U}$ contains no other zeros of X; such an open set is *isolating* for X, and for (X, K).

When U is isolating for X there is a unique maximal open neighborhood $\mathcal{N}_U \subset \mathcal{V}(M)$ of X with the following property (Hirsch [10]):

> *If $Y \in \mathcal{N}_U$ has only finitely many zeros in U, the Poincaré-Hopf index of $Y|U$ depends only on X and K.*

This index is an integer denoted by $\mathsf{i}_K(X)$, and also by $\mathsf{i}(X, U)$, with the latter notation implying that U is isolating for X.The index can be equivalently defined as the intersection number of $X(U)$ with the zero section of the tangent bundle

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of U ([2]); and as the the fixed-point index of the time-t map of the local flow of $X|U$ for sufficiently small $t > 0$. ([5, 9, 12].)

The celebrated POINCARÉ-HOPF Theorem [14, 23] connects the index to the Euler characteristic $\chi(M)$. A modern formulation (see MILNOR [19]) runs as follows:

Theorem 1.1 (POINCARÉ-HOPF). *Assume M is a compact n-manifold, $X \in \mathcal{V}(M)$, and $\mathsf{Z}(X) \cap \partial M = \varnothing$. If X is tangent to ∂M at all boundary points, or points outward at all boundary points then $i(X, M) = \chi(M)$. If X points inward at all boundary points, $i(X, M) = (-1)^{n-1}\chi(M)$.*

For calculations of the index in more general settings see GOTTLIEB [6], JUBIN [15], MORSE [21], PUGH [24].

The X-block K is *essential* if $\mathsf{i}_K(X) \neq 0$. When this holds every $Y \in \mathsf{N}_U(X)$ has an essential block of zeros in U (Theorem 2.3). If M is a closed manifold (compact, no boundary) and $\chi(M) \neq 0$, the Poincaré-Hopf Theorem implies $\mathsf{Z}(X)$ is an essential X-block.

C. Bonatti's proved a remarkable extension of the Poincaré-Hopf Theorem to certain pairs of commuting analytic vector fields on manifolds that need not be compact:

Theorem 1.2 (BONATTI [2]). *Assume $\mathsf{dim}\, M \leq 4$ and $\partial M = \varnothing$. If $X, Y \in \mathcal{V}^\omega(M)$ and $[X, Y] = 0$, then $\mathsf{Z}(Y)$ meets every essential X-block.*[1]

Related results are in the articles [3, 10, 11, 13, 16, 17, 22, 27].

Our main result is an extension of Bonatti's Theorem:

Theorem 1.3. *Let M be a connected 3-manifold and $\mathcal{A} \subset \mathcal{V}^\omega(M)$ an abelian Lie algebra of analytic vector fields on M. Assume $X \in \mathcal{A}$ is nontrivial and $\mathsf{Z}(X) \cap \partial M = \varnothing$. If K is an essential X-block, then $\mathsf{Z}(\mathcal{A}) \cap K \neq \varnothing$.*

The proof, in Section 3, relies heavily on Bonatti's Theorem. An analog for surfaces is proved in HIRSCH [10, Thm. 1.3].

1.1. Application to attractors. The *interior* $\mathsf{Int}(L)$ of a subset $L \subset M$ is the union of all open subsets of M contained in L.

Fix a metric on M. If $Q \subset M$ is closed, the minimum distance from $z \in M$ to points of Q is denoted by $\mathsf{dist}(z, Q)$.

Let $X \in \mathcal{V}^1(M)$ have local flow Φ. An *attractor* for X (see [1, 4, 7, 25]) is a nonempty compact set $P \subset M$ that is invariant under Φ and has a compact neighborhood $N \subset M$ such that

$$\Phi_t(N) \subset (N)$$

and

[1] "The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of X, as an analytic Y-invariant set." —P. MOLINO [20]

$$\lim_{t\to\infty} \mathsf{dist}(\Phi_t(x,P),P) = 0 \text{ uniformly in } x \in N. \tag{1}$$

Such an N can be chosen so that

$$t > s \geq 0 \implies \Phi_t(N) \subset \mathsf{Int}(\Phi_s(N)). \tag{2}$$

Henceforth (2) is assumed. By F. W. WILSON [28, Thm. 2.2], we choose N so that:

$$\textit{N is a compact } C^1 \textit{ submanifold and X is inwardly transverse to } \partial N.^2 \tag{3}$$

Theorem 1.4. *Let M, $\mathcal{A}$ and X be as in* Theorem 1.3. *If $P \subset M$ is a compact attractor for X and $\chi(P) \neq 0$, then $\mathsf{Z}(\mathcal{A}) \cap P \neq \varnothing$.*

Proof. P is a proper subset of M, as otherwise M is a closed 3-manifold having nonzero Euler characteristic, contradicting the classical result that odd-dimensional closed manifolds have zero Euler characteristic (e.g., HIRSCH [8, Thm. 5.2.5]); Spanier[26, Thm. 6.2.18]. Fix N as above and note that $\chi(N) \neq 0$.

By (3) and the Poincaré-Hopf Theorem 1.1 there is an essential X-block $K \subset N \setminus \partial N$, and $K \subset P$ by (1). Standard homology theory and (2) imply that the inclusion map $P \hookrightarrow N$ induces an isomorphisms on singular homology, hence $\chi(N) = \chi(P) \neq 0$.

The conclusion follows from Theorem 1.3 applied to the data $M', \mathcal{A}', X'$:

$$M' := N, \quad \mathcal{A}' := \{Y|N\colon Y \in \mathcal{A}\}, \quad X' := X|N. \qquad \blacksquare$$

Example 1.5. Denote the inner product of $x, y \in \mathbb{R}^3$ by $\langle x, y\rangle$ and the norm of x by $\|x\|$. Let $B_r \subset \mathbb{R}^3$ denote the open ball about the origin of radius $r > 0$.

- *Assume $\mathcal{A}$ is an abelian Lie algebra of analytic vector fields on an open set $M \subset \mathbb{R}^3$ that contains $\overline{B}_r$. Let $X \in \mathcal{A}$ and $r > 0$ be such that*

  $$\|x\| = r \implies \langle X_p, p\rangle < 0.$$

  Then $\mathsf{Z}(\mathcal{A}) \cap B_r \neq \varnothing$.

Proof. This is a consequence of Theorem 1.4: $\overline{B}_r$ contains an attractor for X because X inwardly transverse to $\partial\overline{B}_r$ and $\chi(\overline{B}_r) = 1$.

2. Background material

Lemma 2.1 (INVARIANCE). *If $T, S \in \mathcal{A}$ then $\mathsf{Z}(S)$ is invariant under T.*

Proof. Let $\Phi := \{\Phi_t\}_{t\in\mathbb{R}}$ and $\Psi := \{\Psi_s\}$ denote the local flows of T and S, respectively. If $t, s \in \mathbb{R}$ are sufficiently close to 0, because $[T, S] = 0$ we have

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t,$$

[2] This means X_p is not tangent to ∂N if $p \in \partial N$.

and

$$\mathsf{Z}(S) = \mathsf{Fix}(\Psi) := \bigcap_s \mathsf{Fix}(\Psi_s),$$

where Fix denotes the fixed point set. Suppose $p \in \mathsf{Z}(S)$. Then $p \in \mathsf{Fix}(\Psi)$, and

$$\Psi_s \circ \Phi_t(p) = \Phi_t \circ \Psi_s(p) = \Phi_t(p).$$

Consequently $\Phi_t(p) \in \mathsf{Fix}(\Psi_s)$ for sufficiently small $|t|, |s|$, implying the conclusion. ■

A closed set $Q \subset M$ is an *analytic subspace of* M, or *analytic in* M, provided Q has a locally finite covering by zero sets of analytic maps defined on open subsets of M. This is abbreviated to *analytic space* when the ambient manifold M is clear from the context. The connected components of analytic spaces are also analytic spaces.

Analytic spaces have very simple local topology, owing to the theorem of ŁOJASIEWICZ [18]:

Theorem 2.2 (TRIANGULATION). *If T is a locally finite collection of analytic spaces in M, there is a triangulation of M such that each element of T is covered by a subcomplex.*

The proof of Theorem 1.3 uses the following folk theorem:

Theorem 2.3 (STABILITY). *Assume $X \in \mathcal{V}(M)$ and $U \subset M$ is isolating for X.*

(a) *If $\mathsf{i}(X, U) \neq 0$ then $\mathsf{Z}(X) \cap U \neq \varnothing$.*

(b) *If $Y \in \mathcal{V}(M)$ is sufficiently close to X, then U is isolating for Y and $\mathsf{i}(Y, U) = \mathsf{i}(X, U)$.*

Proof. See HIRSCH [10, Thm. 3.9]. ■

Let $\mathsf{Z}(\mathcal{S})$ denote the set of common zeros of a subset $\mathcal{S} \subset \mathcal{V}^\omega(M)$.

Proposition 2.4. *The following conditions hold for every $\mathcal{S} \subset \mathcal{A}$:*

(a) *$\mathsf{Z}(\mathcal{S})$ is analytic in M.*

(b) *Every zero dimensional $\mathcal{A}$-invariant set lies in $\mathsf{Z}(\mathcal{A})$.*

Proof. Left to the reader. ■

3. Proof of Theorem 1.3

Recall the hypotheses of the Main Theorem:

- *M is a 3-dimensional manifold,*

- *$\mathcal{A} \subset \mathcal{V}^\omega(M)$ is an abelian Lie algebra,*
- *$X \in \mathcal{A}$ is nontrivial, $\mathsf{Z}(X) \cap \partial M = \varnothing$, and K is an essential block of zeroes for X.*

The conclusion to be proved is: $\mathsf{Z}(A) \cap K \neq \varnothing$. It suffices to show that $\mathsf{Z}(A)$ meets every neighborhood of K, because $\mathsf{Z}(\mathcal{A})$ is closed and K is compact.

Case I: $\dim \mathcal{A} = d < \infty$. The special case $d \leq 2$ is covered by Bonatti's Theorem. We proceed by induction on d:

Induction Hypothesis

- $\dim \mathcal{A} = d + 1,\ d \geq 2$.
- *The zero set of every d-dimensional subalgebra of $\mathcal{A}$ meets K.*

Arguing by contradiction, we assume *per contra*:

(PC) $\mathsf{Z}(\mathcal{A}) \cap K = \varnothing$.

An important consequence is:

(A) $\dim K \leq 2$.

For otherwise $\dim K = 3$, which entails the contradiction that X is trivial: X is analytic and vanishes on a 3-simplex in the connected 3-manifold M.

The Stability Theorem (2.3) implies X has a neighborood $\mathcal{N}_U \subset \mathcal{V}^\omega(X)$ with the following property:

(B) $Y \in \mathcal{N}_U \implies$ *U is isolating for Y and* $\mathsf{i}(Y, U) = \mathsf{i}(X, U) \neq 0$.

Let $G_d(\mathcal{A})$ denote d-dimensional Grassmann manifold of d-dimensional linear subspaces $\mathcal{B}$ of $\mathcal{A}$; these are abelian subalgebras.

The nonempty set

$$G_d(\mathsf{N}_U) := \{\mathcal{B} \in G_d(\mathcal{A}) \colon\ \mathcal{B} \cap \mathsf{N}_U \neq \varnothing\}$$

is open in $G_d(\mathcal{A})$, hence it is a d-dimensional analytic manifold.

Bonatti's Theorem and (B) imply:

(C) $\mathsf{Z}(\mathcal{B}) \cap K \neq \varnothing$ for all $\mathcal{B} \in G_d(\mathsf{N}_U)$.

A key topological consequence of (C) is:

(D) *If $\mathcal{B}$ and $\mathcal{B}'$ are distinct elements of $G_d(\mathsf{N}_U)$, then $\mathsf{Z}(\mathcal{B}) \cap K$ and $\mathsf{Z}(\mathcal{B}') \cap K$ are disjoint.*

This holds because $\mathcal{B} \cup \mathcal{B}'$ spans $\mathcal{A}$, hence (PC) implies

$$\big(\mathsf{Z}(\mathcal{B}) \cap K\big) \bigcap \big(\mathsf{Z}(\mathcal{B}') \cap K\big) = \mathsf{Z}(\mathcal{A}) \cap K = \varnothing.$$

Each set $\mathsf{Z}(\mathcal{B}) \cap K$ is invariant (Lemma 2.1) and therefore has positive dimension by (PC). Moreover:

(E) *The set* $\Gamma_U := \{\mathcal{B} \in G_d(\mathsf{N}_U)\colon \mathsf{dim}\, \mathsf{Z}(\mathcal{B}) \cap K = 2\}$ *is finite.*

For otherwise (D) implies K contains an infinite sequence of pairwise disjoint compact subsets that are 2-dimensional and analytic in M. But this is impossible by (A) and the Triangulation Theorem 2.2.

(E) shows that $\Gamma_U = \varnothing$ provided U is small enough. Therefore we can assume:

(F) $\mathsf{dim}\, \mathsf{Z}(\mathcal{B}) \cap K = 1$ *for all* $\mathcal{B} \in G_d(\mathsf{N}_U)$.

The set $Q := \{(\mathcal{B}, p) \in G_d(\mathsf{N}_U) \times M\colon p \in \mathsf{Z}(\mathcal{B}) \cap K\}$ is analytic in $G_d(\mathsf{N}_U) \times M$ (Proposition 2.4). The natural projections

$$\pi_1\colon Q \to G_d(\mathsf{N}_U), \quad \pi_2\colon Q \to K$$

are analytic, π_1 is surjective, π_2 is injective by (D).

The sets $\mathsf{Z}(\mathcal{B})\cap K$ are therefore pairwise disjoint, and each is a 1-dimensional analytic subspaces of Q by (F). Therefore $\mathsf{dim}\, Q = \mathsf{dim}\, G_d(\mathcal{A}) + \mathsf{dim}\,(\mathsf{Z}(\mathcal{B}) \cap K) \leq \mathsf{dim}\, K$, whence $\mathsf{dim}\, Q = d + 1 \leq 2$. But this is impossible because $d \geq 2$ by the Induction Hypothesis. This completes the inductive proof of the Main Theorem in Case I.

Case II: $\mathsf{dim}\, \mathcal{A}$ *is infinite.* Consider the family $\mathfrak{F}$ of compact subsets of K:

$$\mathfrak{F} := \{\, \mathsf{Z}(\mathcal{A}') \cap K\colon \mathcal{A}' \subset \mathcal{A} \text{ is a finite-dimensional subalgebra}\}.$$

Evidently $\bigcap_{S\in\mathfrak{F}} S = \mathsf{Z}(\mathcal{A})\cap K$. Case I shows every finite subset of $\mathfrak{F}$ has nonempty intersection. As K is compact, all the elements of $\mathfrak{F}$ have nonmpty intersection, proving $\mathsf{Z}(\mathcal{A}) \cap K \neq \varnothing$. ■

References

[1] Akin, E., "The general topology of dynamical systems," Graduate Studies in Mathematics, 1. Amer. Math. Soc., Providence, RI ,1993.

[2] Bonatti, C., *Champs de vecteurs analytiques commutants, en dimension* 3 *ou* 4*: existence de zéros communs*, Bol. Soc. Brasil. Mat. (N. S.) **22** (1992), 215–247.

[3] Bonatti, C., and B. Santiago, *Existence of common zeros for commuting vector fields on 3-manifolds,*
`arXiv:1504.06104 (2015).`

[4] Conley, C. "Isolated Invariant Sets and the Morse Index," CBMS Regional Conference Series in Mathematics **38**, Amer. Math. Soc., Providence, R.I., 1978.

[5] Dold, A., "Lectures on Algebraic Topology," Die Grundlehren der matematischen Wissenschaften bf52, 2nd edition, Springer, New York etc., 1972.

[6] Gottlieb, D., *A de Moivre like formula for fixed point theory*, in: "Fixed Point Theory and its Applications, Berkeley, CA, 1986," Contemporary Mathematics **72** Amer. Math. Soc., Providence, RI, 1988.

[7] Hale, J., "Asymptotic behavior of dissipative systems," Mathematical Surveys and Monographs **25**. Amer. Math. Soc., Providence, RI, 1988.

[8] Hirsch, M. W., "Differential Topology," Springer-Verlag, Berlin, etc., 1976

[9] —, *Fixed points of local actions of nilpotent Lie groups on surfaces*, `arXiv:1405.2331 (2014).`

[10] —, *Zero sets of Lie algebras of analytic vector fields on real and complex 2-manifolds,*
`arXiv:1310.0081v2 (2015).`

[11] —, *Common zeroes of families of smooth vector fields on surfaces,* Geom. Dedicata 182 (2016), 43–49.

[12] —, *Fixed points of local actions of nilpotent Lie groups on surfaces*, Ergodic Theory and Dynamical Systems, to appear.

[13] Hirsch, M. W., and A. Weinstein, *Fixed points of analytic actions of supersoluble Lie groups on compact surfaces*, Ergod. Theor. Dyn. Sys. **21** (2001), 1783–1787.

[14] Hopf, H., *Vektorfelder in Mannifgfaltigkeiten*, Math. Annalen **95** (1925), 340–367

[15] Jubin, B., *A generalized Poincaré-Hopf index theorem,*
`arXiv.org/abs/0903.0697v2, (2009)`

[16] Lima, E., *Common singularities of commuting vector fields on 2-manifolds*, Comment. Math. Helv. **39** (1964), 97–110.

[17] —, *Commuting vector fields on* S^3, Ann. of Math. **81** (1965), 70–81.

[18] Łojasiewicz, S., *Triangulation of semi-analytic sets*, Annali Scuola Norm. Sup. Pisa (3) **18** (1964), 449–474.

[19] Milnor, J., "Topology from the differentiable viewpoint," University of Virginia Press, Charlottesville, Virginia 1965.

[20] Molino, P., *Review of Bonatti [2]*, Math Reviews 93h:57044, Amer. Math. Soc. (1993).

[21] Morse, M., *Singular Points of Vector Fields Under General Boundary Conditions*, Amer. J. Math. **52** (1929), 165–178.

[22] Plante, J., *Elementary zeros of Lie algebras of vector fields,* Topology **30** (1991), 215–222.

[23] Poincaré, H., *Sur les courbes définies par une équation différentielle,* J. Math. Pures Appl. **1** (1885), 167–244.

[24] Pugh, C., *A generalized Poincaré index formula,* Topology **7** (1968), 217–226.

[25] Smale, S., *Differentiable dynamical systems,* Bull. Amer. Math. Soc. **73** (1967), 747–817.

[26] Spanier, E. H., "Algebraic Topology," McGraw Hill, 1966.

[27] Turiel, F.-J., *Analytic actions on compact surfaces and fixed points*, Manuscripta Mathematica **110** (2003), 195–201.

[28] Wilson, F. W., *Smoothing derivatives of functions and applications,* Trans. Amer. Math. Soc. **139** (1969), 413–428.

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