University of California, Berkeley
Department of Agricultural &
Resource Economics

CUDARE Working Papers

Year 2011, September	*Paper 1117R*
(Year 2011, June	*Paper 1117)*

Discounting and Confidence

Christian P. Traeger

Discounting and Confidence

Christian P. Traeger

UC Berkeley, 207 Giannini Hall #3310, CA 94720-3310, USA
traeger@berkeley.edu

CUDARE Working Paper No. 1117, September 2011

Abstract: The paper analyzes the discount rate under uncertainty. The analysis complements the probabilistic characterization of uncertainty by a measure of confidence. Special cases of the model comprise discounting under smooth ambiguity aversion as well as discounting under a disentanglement of risk aversion from aversion to intertemporal substitution. The paper characterizes the general class of preferences for which uncertainty implies a reduction of the discount rate. It also characterizes how the more comprehensive description of uncertainty changes the discount rate with respect to the standard model. The paper relates different results in the literature by switching between different risk measures. It presents a parametric extension of the Ramsey discounting formula that takes into account confidence into future growth estimates and a measure of aversion to the lack of confidence. If confidence decreases in the futurity of the growth forecast, the discount rates have a falling term structure even in the case of an iid growth process.

JEL Codes: D61, Q54, D81, D90

Keywords: uncertainty, discounting, climate change, ambiguity, confidence, subjective beliefs, prudence, pessimism, expected utility, intertemporal substitutability, intertemporal risk aversion

1 Introduction

How does uncertainty about future economic development affect the optimality of investments? At the agent level, the consumption discount rate determines intertemporal trade-offs and precautionary saving. At the aggregate level, the social discount rate evaluates long-term projects including climate change related mitigation and adaptation projects, investments in basic research, national defense, infrastructure projects, and projects involving irreversible changes like biodiversity loss. An important determinant of these rates and their term structure is the uncertainty over economic growth and the set of future consumption possibilities. The paper presents a new model for the (social or consumption) discount rate under uncertainty. The model unifies and generalizes a variety of previously derived discounting formulas. It captures uncertainty in terms of risk, ambiguity, and confidence labeled beliefs. The paper comprehensively characterizes the preferences implying that the social discount rate decreases in the face of uncertainty. A parametric special case extends the stochastic Ramsey equation to account for general forms of uncertainty. I discuss the relation between confidence, aversion to the lack of confidence, and the term structure of discount rates.

From Leland (1968) we learned that a decision maker should increase his savings for the future under uncertainty, if his (absolute) Arrow-Pratt risk aversion decreases in wealth. In Leland's analysis, Arrow-Pratt risk aversion simultaneously characterizes aversion to intertemporal substitution. Gollier (2002) showed that Leland's reasoning continues to hold when disentangling risk aversion from the propensity to smooth consumption over time (see as well Kimball & Weil 2009). These papers, together with a large body of related analysis, assume that uncertainty is described by a uniquely given probability distribution (risk). Recently, Gierlinger & Gollier (2008) challenge the findings by analyzing more general forms of uncertainty. The authors employ Klibanoff, Marinacci & Mukerji's (2005) model of smooth ambiguity aversion and show that a decreasing coefficient of absolute ambiguity aversion is not sufficient to ensure a reduction of the discount rate in the face of uncertainty. Here, the discount rate includes a 'pessimism term' that can only be signed by restricting the form of uncertainty (lottery domain). The current paper reconstitutes Leland's original findings for settings of general uncertainty, including the case of smooth ambiguity aversion. An appropriate change in the risk measure eliminates the 'pessimism effect'. The discount rate increases or decreases under uncertainty whenever (generalized) Arrow Pratt measures of risk aversion increase or decrease in consumption. The paper also provides a general characterization how the discount rate in the comprehensive uncertainty model differs from the discount rate obtained in the intertemporally additive expected utility standard model. The generalized uncertainty framework builds on Traeger (2010) and incorporates as special cases the model based on Kreps & Porteus (1978), Epstein & Zin (1989), and Weil (1990) disentangling risk aversion from intertemporal substitutability and the model of smooth ambiguity aversion by Klibanoff et al. (2005) and Klibanoff, Marinacci & Mukerji (2009). The general framework employs multi-layer probabilistic beliefs that are indexed by a confidence measure.

Ever since Keynes (1921) and Ellsberg (1961), economists and decision theorists have expressed their concern that a standard probability distribution cannot capture uncertainty comprehensively. Arrow & Hurwicz (1972) developed an axiomatic framework that evaluates sets of possible outcomes in the complete absence of probabilities. In the framework of this paper, a related decision criteria can arise in the limit of a complete lack of confidence. However, even if decision makers form probabilistic beliefs, they might not be fully confident that these beliefs are correct, unless they face objective lotteries. Such objective lotteries are mostly encountered in casinos or experiments. In contrast, the probabilistic descriptions of a real life future result from a wide spectrum of sources and methods and ranges from careful derivations based long time series and frequent observations to mere guesstimates. A frequent illustration of the differences in uncertainty not captured in a probability distribution is based on the analysis of two mutually exclusive events. In absence of any information on the likelihood of these events, the principle of insufficient reason suggests assigning equal probabilities. Similarly, if the events correspond to heads and tails in the toss of a fair coin, a decision maker would generally assign equal probabilities. Yet, in the first situation the decision maker's guess is based on complete ignorance, while in the second situation he faces well known objective probabilities. Both uncertainties differ in a dimension not captured by the probability distributions themselves. It is a measure of confidence (or of subjectivity) that distinguishes the two situations. Experiments based on the famous Ellsberg (1961) paradox have shown that these differences matter for behavior. Decision theorists have explored this concern in depth over the last two decades. The current paper builds on a recent extension of the widespread smooth ambiguity model of Klibanoff, Marinacci, & Mukerji (2005, 2009) by Traeger (2010). The latter analysis extends the classical von Neumann & Morgenstern (1944) axioms, underlying the expected utility model, to a setting where lotteries are distinguished by their degree of confidence. Limiting cases of the model include the Arrow & Hurwicz (1972) criterion for decision making under ignorance and Gilboa & Schmeidler's (1989) maximin expected utility. In particular, the setting permits to model a decision maker who employs any combination of these decision criteria conditional on the confidence in his description of the uncertainty he faces. The current paper applies this framework to derive a discounting formula that takes into account not just probabilities, but also confidence.

A related distinction of different types of uncertainty has reached the applied policy arena in a field where recent research has proven the primordial importance of selecting the right discount rate. The guidance notes of the Intergovernmental Panel on Climate Change (AR4) ask the lead authors to distinguish between three different types of uncertainty: "unpredictability", "structural uncertainty", and "value uncertainty". The subsequent economic assessment and cost benefit analysis was not, yet, able to incorporate these distinctions. In recent work, Gierlinger & Gollier (2008) and Traeger (2008) apply the smooth ambiguity framework by Klibanoff et al (2005, 2009) to models of social discounting. These models, however, can only capture two types of uncertainty: objective versus subjective uncertainty. In real world applications, purely objective probabilities are rare. Measures of subjective uncertainty stretch from careful econometric analysis with a limited number of observations to pure guesstimates. The current paper suggests a framework in which decision makers, or scientists, base their uncertainty evaluation not only on probabilistic descriptions of future uncertainty, but also on the way that these probabilities were informed and, thus, their confidence in these probabilistic beliefs. The analysis translates both of these informations into the social (or consumption) discount rate. The adopted preference framework builds on a behaviorally as well as normatively attractive set of axioms that preserves time consistency and a minimally modified version of von Neumann-Morgenstern's independence axiom. Moreover, the framework keeps separate what is distinct: the desire to smooth consumption over time and the (various degrees of) uncertainty aversion. A growing body of work shows that such a distinction is crucial to explaining behavior under risk, like the equity premium and the risk free rate puzzles (Vissing-Jørgensen & Attanasio 2003, Bansal & Yaron 2004, Bansal, Kiku & Yaron 2010).

2 Background and Representation

Current consumption is certain and denoted x_0. I employ the unidimensional notation $u'(x_0)$ for derivatives. However, outcomes can also be multidimensional unless stated otherwise.[1] The decision maker considers investing a (marginal) unit of a good into a productive project that pays one unit plus the (average) yearly interest r in period T. The minimal interest required to make the agent invest into the project is the (risk free) social discount rate or consumption discount rate. Under certainty it is characterized by the pure rate of time preference δ and the ratio of marginal utilities in the future and in the present:

$$r = \delta - \frac{1}{T} \ln \left[\frac{u'(x_T)}{u'(x_0)} \right] \ , \tag{1}$$

where x_T denotes future consumption. In the one dimensional setting, the curvature of u characterizes the desire to smooth consumption over time.

This paper derives the modifications of equation (1) necessary to account for uncertainty over the future. In particular, the formulas will account for the in-

[1] In general, outcomes can be elements of a compact metric continuously differentiable manifold. For the general case $u'(x_0)$ is short for $\partial_0^\alpha u(x_0)$, denoting the directional derivative along consumption change α at consumption point x_0. Here, α is a curve determining the consumption change in the present period. Similarly $u'(x_1)$ is then short for $\partial_1^\beta u(x_1)$ denoting changes along curve β in period 1, and $u'(x_T)$ is then short for $\partial_T^\gamma u(x_T)$.

formedness of probabilities describing future uncertainty. Probabilities can be objective like for the toss of a coin or the spin of a roulette wheel. But probabilities can also derive from a small number of observations (or simulations), or they can base on the principle of insufficient reason. The latter principle states that the agent should assign equal probability mass to all events, if he has no information about their likelihood.The preference representation employed here allows the agent to distinguish different types of probabilities by means of a degree of confidence (or subjectivity). Traeger (2010) derives such a preference representation by enriching the well known von Neumann & Morgenstern (1944) framework for decision making under uncertainty with a dimension of confidence. Special cases of the model disentangle Arrow Pratt risk aversion from intertemporal substitution like in Epstein & Zin (1989), Weil (1990), and Kreps & Porteus (1978) [KP model], or represent smooth ambiguity aversion as in Klibanoff et al. (2005) and Klibanoff et al. (2009) [KMM model]. The representation builds on a set of functions $\{f^s\}_{s\in S}$ that characterize a measure of intertemporal risk aversion. Each index $s \in S$ corresponds to a particular degree of confidence (or subjectivity), complementing the probabilistic measure of uncertainty. The set S is an arbitrary finite set of confidence descriptions. I assume that utility u and the risk aversion functions f^s are increasing and concave. In the case of two periods and a single lottery over future consumption, which is characterized by the probability measure p of degree of subjectivity s, the welfare evaluation writes as

$$u(x_0) + e^{-\delta}\, {f^s}^{-1} \left[\mathrm{E}_p f^s \circ u(x_1) \right] ,$$

where the superindex "$^{-1}$" denotes the inverse and "$\circ$" denotes function composition. The operator $\mathcal{M}^f_p \equiv f^{-1} \mathrm{E}_p f$ takes a generalized mean of whatever follows to its right. A concave function f implies that the result of the generalized mean $\mathcal{M}^f_p$ returns a smaller value than the expected value operator itself (Hardy, Littlewood & Polya 1964). A concave function f^s characterizes intertemporal risk aversion with respect to uncertainty with degree of confidence s. Intertemporal risk aversion can be interpreted as a measure of risk aversion with respect to utility gains and losses. Equation (3) will give a choice theoretic interpretation.

In general, multiple layers of uncertainty that are characterized by different

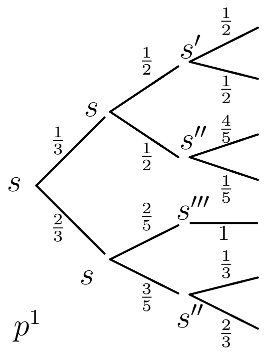

Figure 1 depicts an example of multi-layer uncertainty in tree form. The indices s, s', s'' and s''' label the different degrees of confidence of the uncertainty nodes. Collapsing the first two layers into a single layer would not change the evaluation as both layers share the same degree of subjectivity. Moreover, the degree of subjectivity s''' of the degenerate node is irrelevant for the evaluation (a degenerate node expresses certainty).

degrees of confidence come together in determining future outcomes. For example, economic growth in some developing country is governed by a probability distribution given political stability of the regime. Political stability itself is governed by a probability distribution of low confidence. Adding another layer, the likelihood of political stability might itself depend on regional climate. Future changes of the regional climate are described by probabilistic estimates that derive from cross model comparisons of various deterministic climate models. The reader is invited to rank or put his or her label on the confidence corresponding to this type of probability. An example of a multi-layer uncertainty description is depicted by the tree in Figure 1. Each uncertainty node is characterized by a subjectivity or, equivalently, confidence label. The representation derived by Traeger (2010) implies that a lottery of subjectivity s within this uncertainty tree has to be evaluated by the generalized mean that is characterized by the intertemporal risk aversion function f^s. A tree as in Figure 1 features lotteries over lotteries. I start by labeling the root lottery to the left as lottery p^1. Lottery p^1 is a lottery over different lotteries p^2 in the next uncertainty layer. The lotteries p^2 are lotteries over lotteries p^3 in the last layer, which are lotteries over future outcomes. The uncertain scenario depicted in Figure 1 involves four different lotteries p^3 with three different degrees of subjectivity.[2] In general, let there be $N \in \mathbb{N}$ layers of uncertainty. Moreover, let $\hat{s}(p)$ denote the degree of subjectivity

[2]The hierarchical structure of lotteries over lotteries can be visualized by indexing a lottery in layer i with $\theta_1, \ldots, \theta_{i-1}$, where θ_j characterizes the risk states going along with lottery p^j. If the lottery p^i has a continuous distribution, the uncertainty layer $i+1$ features an uncountable set of lotteries p^{i+1}. A formal characterization of the general lottery space is given in Traeger (2010) and employs Borel measures over disjoint unions of Borel algebras corresponding to the different degrees of subjectivity.

of a given lottery. Then, the general preference representation can be written as

$$u(x_0) + e^{-\delta}\, \mathcal{M}_{p^1}^{f^{\hat{s}(p^1)}} \mathcal{M}_{p^2}^{f^{\hat{s}(p^2)}} \cdots \mathcal{M}_{p^N}^{f^{\hat{s}(p^N)}} u(x_1) = u(x_0) + e^{-\delta} \left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{f^{\hat{s}(p^i)}} \right] u(x_1). \quad (2)$$

Each generalized mean deducts a risk premium that depends on the amount of uncertainty (described by p^i) and on the degree of confidence $\hat{s}(p^i)$. I denote certainty equivalent utility in the i-th layer by $m^i(p^i) \equiv \mathcal{M}_{p^i}^{f^{\hat{s}(p^i)}} \cdots \mathcal{M}_{p^N}^{f^{\hat{s}(p^N)}} u(x_1)$ and $m^{N+1}(x_1) \equiv u(x_1)$ or, dropping the argument, simply by m^i.

The following characterization of intertemporal risk aversion adapted from Traeger (2010) provides a choice theoretic intuition for the concept of intertemporal risk aversion.[3] Let a decision maker be indifferent between the two combinations of first and second period outcomes $(\overline{x_1}, \underline{x_2})$ and $(\underline{x_1}, \overline{x_2})$, where $u(\overline{x_1}) > u(\underline{x_1})$ and $u(\overline{x_2}) > u(\underline{x_2})$. An intertemporal risk averse decision maker prefers the certain consumption path $(\overline{x_1}, \underline{x_2})$ (or equivalently $(\underline{x_1}, \overline{x_2})$) over a lottery that yields with equal probabilities either the path $(\underline{x_1}, \underline{x_2})$ or the path $(\overline{x_1}, \overline{x_2})$. Formally this condition can be written as

$$(\overline{x_1}, \underline{x_2}) \sim (\underline{x_1}, \overline{x_2}) \quad \Rightarrow \quad (\overline{x_1}, \underline{x_2}) \;\succeq_t\; (\underline{x_1}, \underline{x_2}) \oplus_s^{\frac{1}{2}} (\overline{x_1}, \overline{x_2}) \,, \quad (3)$$

where $\oplus_s^{\frac{1}{2}}$ denotes a probability one half mixture with degree of subjectivity s. If this mixture represents an objective lottery, like in the case of a coin toss, equation (3) captures intertemporal risk aversion with respect to objective lotteries. Note that an agent described by the intertemporally additive standard model is always indifferent between the certain path and the lottery in equation (3).

In an intertemporal setting, uncertainty affects welfare in two distinct ways. First, a stochastic variable generates fluctuations over time. A decision maker with a preference for smooth consumption paths dislikes these fluctuations. This effect of risk is captured by the utility function and is part of standard model. Second, a decision maker can be intrinsically risk averse (averse to risk per se). This effect is captured by intertemporal risk aversion. A different way to measure risk aversion in the general setting is as follows. Let the curvature of u continue to

[3]The interpretation employs two uncertain periods while equation (2) only introduced uncertainty over a single period. See section 4 for the formal extension of the model to multiple uncertain periods.

measure aversion to intertemporal substitution. Eliminate the direct measure of intrinsic risk aversion by means of intertemporal risk aversion. Instead, measure total risk aversion by the functions

$$g^s = f^s \circ u \tag{4}$$

for all $s \in S$. Both, the concavity of f^s and the concavity of u translate into the curvature of the functions g^s. Hence, the functions g^s jointly capture both sources of risk aversion. As discussed in Traeger (2007) and Traeger (2010) these functions g^s characterize Arrow Pratt risk aversion with respect to lotteries of degree of confidence s. Note that this interpretation in terms of Arrow Pratt risk aversion only holds in the one-commodity setting. Equation (4) implies a further characterization of intertemporal risk aversion. The statement that $f^s = g^s \circ u^{-1}$ is concave implies that Arrow Pratt risk aversion dominates aversion to intertemporal substitution. Hence, an intertemporal risk averse agent prefers to substitute into the certain future rather than into an uncertain risk state. The assumptions that u and f^s are increasing and concave imply the same characteristics also for g^s.

3 The Present versus the Future

The model of this section analyzes the case where future payoffs are collected in a single uncertain future period. This is the setting of Leland (1968), Gollier (2002), Gierlinger & Gollier (2008), Traeger (2008), and Kimball & Weil (2009). It also is the setting of most analytic discussions of social discounting in the economics of climate change, a research area that currently is home to the hottest debate over the "right" discount rate, given that the discount rate is the single most important determinant of optimal greenhouse gas mitigation levels (Nordhaus 2007). I introduce multiple layers of uncertainty characterized by differing degrees of confidence. The model relates to the representations of the social discount rate under KP preferences analyzed in Gollier (2002) and under smooth ambiguity KMM preferences examined in Gierlinger & Gollier (2008). I compare, unify and generalize the results of these papers. I derive a sufficient condition under which the social discount rate decreases with uncertainty, and a sufficient condition under which for the social discount rate is smaller than in the standard model. A special case of these conditions characterizes when smooth ambiguity aversion decreases the social discount rate. The conditions depend on preferences only and hold for all uncertain scenarios (lotteries).

3.1 Multi-commodity setting: The social discount rate in terms of intertemporal risk aversion

The following proposition extends the expression for the social discount rate in equation (1) incorporating uncertainty and uncertainty attitude.

Proposition 1: The social discount rate under preferences of the form given in equation (2) is

$$r = \delta - \ln \left[\left\{ \prod_{i=1}^{N} \underbrace{\frac{\mathrm{E}_{p^i} f^{\hat{s}(p^i)\prime}(m^{i+1})}{f^{\hat{s}(p^i)\prime}(m^i)}}_{\substack{\text{prudence term} \\ \text{confidence level } \hat{s}(p^i)}} \ \mathrm{E}_{p^i} \ \underbrace{\frac{f^{\hat{s}(p^i)\prime}(m^{i+1})}{\mathrm{E}_{p^i} f^{\hat{s}(p^i)\prime}(m^{i+1})}}_{\substack{\text{pessimism term} \\ \text{confidence level } \hat{s}(p^i)}} \right\} \frac{u'(x_1)}{u'(x_0)} \right] . \quad (5)$$

The expected value operator E_{p^i} acts on everything carrying an index $i + 1$ for the next uncertainty layer: lotteries p^{i+1} and certainty equivalents m^{i+1}. In particular, the expected value operator printed in large also acts on the $i + 1$ entries of the subsequent product term (with E_{p^N} acting on $u'(x_1)$). I label the fractions with the expected value in the numerator prudence terms. The name is based on Proposition 2 below. The fractions with the expected value in the denominator are weights. They increase the weight given to events with high marginals (generally low outcomes), and reduces the weight of events with low marginals (generally high outcomes). Therefore, these weights gain the name pessimism term as they bias probabilities to give more weight to bad outcomes. Both names were assigned by Gierlinger & Gollier (2008) in a special case described below.

Proposition 2: A prudence term of confidence level s reduces the social discount rate for all uncertain scenarios, if and only if, the function f^s exhibits decreasing absolute risk aversion $\text{AIRA}_s = -\frac{f^{s\prime\prime}}{f^{s\prime}}$, which is equivalent to $-\frac{f^{s\prime\prime\prime}}{f^{s\prime\prime}} > -\frac{f^{s\prime\prime}}{f^{s\prime}}$.

Only relying on smoothness of the function f^s, the condition is $f^{s\prime} \circ f^{s^{-1}}$ convex.

I call the term $-\frac{f^{s\prime\prime\prime}}{f^{s\prime\prime}}$ *absolute intertemporal prudence* with respect to confidence level s. The name prudence relates to Kimball's (1990) work on third order derivatives of utility functions in the context of precautionary savings. In this terminology the condition in Proposition 2 states that the prudence term reduces the social discount rate if prudence dominates risk aversion (for a given confidence level). The condition is equivalent to a falling degree of absolute intertemporal risk aversion AIRA_S. Thus, the prudence term conforms with Leland (1968) finding in the standard model. The following intuition explains why the third order derivative or the change of risk aversion is crucial. Assume that a decision maker is less risk averse at higher welfare levels. Then, saving for the future not only increases expected future consumption, but also reduces the risk premium accounting for future uncertainty. Thus, the decision maker has an additional incentive to save for the future under uncertainty.

Sufficient conditions for which the pessimism term decreases the social discount rate are more intricate. In a simplified version of the model, Gierlinger & Gollier (2008) analyze the term. They use the KMM framework of smooth ambiguity aversion by Klibanoff et al. (2005), which accounts for two types of lotteries: objective ($s = obj$) and subjective ($s = subj$). The decision maker exhibits intertemporal risk aversion only with respect to subjective lotteries characterized by f^{subj}, but not with respect to objective lotteries (f^{obj} is the identity/absent from the model). Moreover, the smooth ambiguity model assumes that the decision maker faces a subjective lottery over an objective lottery.

Corollary 1: [KMM model of smooth ambiguity aversion]

In a setting with subjective over objective lotteries and intertemporal risk neutrality with respect to objective risk, the social discount rate collapses to the form

$$r = \delta - \ln \left[\underbrace{\frac{\mathrm{E}_{p^{subj}} {f^{subj}}'(m^{obj})}{{f^{subj}}'(m^{subj})}}_{\text{amb. prudence term}} \mathrm{E}_{p^{subj}} \underbrace{\frac{{f^{subj}}'(m^{obj})}{\mathrm{E}_{p^{subj}} {f^{subj}}'(m^{obj})}}_{\text{pessimism term}} \mathrm{E}_{p^{obj}} \frac{u'(x_1)}{u'(x_0)} \right] .$$

Here $\mathrm{E}_{p^{obj}}$ takes the expectation with respect to an objective lottery over outcomes x_1, while $\mathrm{E}_{p^{subj}}$ takes expectations over the objective lotteries p^{obj} (and the objective certainty equivalent $m^{obj} = m^{obj}(p^{obj})$). In this framework, f^{subj} corresponds to Klibanoff et al.'s (2005) measure of smooth ambiguity aversion. In this context, Gierlinger & Gollier (2008) already derived that decreasing absolute ambiguity aversion (AIRA_{subj}) implies that the prudence term reduces the social discount rate. They also discuss in detail sufficient conditions for the pessimism term to decrease the social discount rate. In general, these conditions are no longer mere preference restrictions but also involve restrictions regarding the underlying lotteries. Section 3.2 discusses a different formulation of the social discount rate that avoids these complications.

Gierlinger & Gollier (2008) classify the pessimism effect as newly arising in the ambiguity setting. However, the next lemma shows that the pessimism term can already arise in a pure risk setting. Assume there is a single lottery and no subjective risk or ambiguity. Then, a unique function f characterizes intertemporal risk aversion. The setting is a special case of Kreps & Porteus (1978) and a generalization of Epstein & Zin's (1989) and Weil's (1990) model.

Corollary 2: [KP model]

In a setting with a single lottery (no distinction of confidence, no ambiguity), the social discount rate collapses to the form

$$r = \delta - \ln \left[\underbrace{\frac{\mathrm{E} f'(u(x_1))}{f'(m)}}_{\text{prudence term}} \mathrm{E} \underbrace{\frac{f'(u(x_1))}{\mathrm{E} f'(u(x_1))}}_{\text{pessimism term}} \frac{u'(x_1)}{u'(x_0)} \right] .$$

Thus, already in a standard setting without ambiguity, an appropriate representation decomposes the social discount rate into a prudence and a pessimism term.

The smooth ambiguity model and the KP model yield similar forms for the social discount rate. Comparing the two, the smooth ambiguity model replaces outcomes in the KP setting by conditional expectations (conditional with respect to subjective uncertainty). This similarity arises because the KP preference structure allows for intertemporal risk aversion with respect to objective risk, while the KMM model allows for intertemporal risk aversion with respect to subjective risk (and assumes neutrality with respect to objective risk). Combining the two models yields prudence and pessimism effects in both uncertainty layers.

Corollary 3: [KMM merged with KP model]

In a general setting with subjective over objective lotteries the social discount rate collapses to the form

$$r = \delta - \ln \Bigg[\underbrace{\frac{\mathrm{E}_{p^{subj}} f^{subj\,\prime}(m^{obj})}{f^{subj\,\prime}(m^{subj})}}_{\text{subj. prudence term}} \mathrm{E}_{p^{subj}} \underbrace{\frac{f^{subj\,\prime}(m^{obj})}{\mathrm{E}_{p^{subj}} f^{subj\,\prime}(m^{obj})}}_{\text{subj. pessimism term}}$$

$$\underbrace{\frac{\mathrm{E}_{p^{obj}} f^{obj\,\prime}(u(x_1))}{f^{obj\,\prime}(m^{obj})}}_{\text{obj. prudence term}} \mathrm{E}_{p^{obj}} \underbrace{\frac{f^{obj\,\prime}(u(x_1))}{\mathrm{E}_{p^{obj}} f^{obj\,\prime}(u(x_1))}}_{\text{obj. pessimism term}} \frac{u'(x_1)}{u'(x_0)} \Bigg] .$$

Proposition 2 then states that the subjective prudence term reduces the social discount rate if and only if subjective prudence dominates subjective risk aversion and that the objective prudence term reduces the discount rate if and only if objective prudence dominates objective risk aversion. In the language of Leland (1968): If both risk measures AIRA_{subj} and AIRA_{obj} are decreasing in their arguments, then (at least) the prudence terms reduce the social discount rate. Observe that, in the representation of this section, the arguments of the risk aversion functions f^s are not physical wealth, but utility that measures outcome appreciation with respect to a cardinality that derives from intertemporal tradeoffs.

3.2 One-commodity setting: The social discount rate in terms of Arrow Pratt risk aversion

Gollier (2002) has shown that, in the one-commodity setting, the effect of uncertainty on the social discount rate in the KP framework can be unambiguously determined. His decomposition does not involve the pessimism term, whose sign generally depends on the underlying lottery. This section develops a similar representation of the social discount rate for the general model and pins down the overall effects of uncertainty on the social discount rate. The key is to use a representation that employs measures of Arrow Pratt risk aversion rather than measures of intertemporal risk aversion or smooth ambiguity aversion. This approach is only possible in a one-commodity setting and I assume that outcomes are drawn from a closed subset of $\mathbb{R}$ for the remainder of this section. As discussed in section 2, the functions

$$g^s = f^s \circ u$$

characterize Arrow Pratt risk aversion with respect to lotteries of degree of confidence s. The subsequent discounting formula builds on a preference representation that eliminates the f^s functions and introduces the g^s measures of risk aversion instead. In this representation, certainty equivalents are measured in real terms rather than in certainty equivalent utility. I denote the i-th layer certainty equivalent by $n^i \equiv n^i(p^i) \equiv \mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}} \cdots \mathcal{M}_{p^N}^{g^{\hat{s}(p^N)}} x_1$ and $n^{N+1} \equiv x_1$ (or equivalently $n^i = u^{-1}(m^i)$).

Proposition 3: In a one-commodity setting, the social discount rate of Proposition 1 is also characterized by

$$r = \delta - \ln \Bigg[\underbrace{\frac{u'(n^1)}{u'(x_0)}}_{\substack{\text{Arrow Pratt risk aversion} \\ \text{(combined of all levels)}}} \prod_{i=1}^{N} \mathrm{E}_{p^i} \underbrace{\frac{g^{\hat{s}(p^i)\prime}(n^{i+1})}{g^{\hat{s}(p^i)\prime}(n^i)}}_{\substack{\text{Arrow Pratt prudence} \\ \text{confidence level } \hat{s}(p^i)}} \Bigg] . \tag{6}$$

Again, the expected value operator E_{p^i} acts on lotteries p^{i+1} and certainty equivalents n^{i+1} to the right, including those in the subsequent product term. The important difference between equation (5) and (6) is that, in equation (6), the marginal utility ratio shows up on the left of the expected value operators, evaluating only the certainty equivalent n^1. Then, the essence of Proposition 2 can be applied recursively to all Arrow Pratt prudence terms yielding the following result.

Proposition 4: The social discount rate under uncertainty is lower than

I) under certainty if the coefficient of absolute Arrow Pratt risk aversion $\text{ARA}_s = -\frac{g^{s\prime\prime}}{g^{s\prime}}$ is decreasing for all confidence levels, i.e.

$$-\frac{g^{s\prime\prime\prime}}{g^{s\prime\prime}} > -\frac{g^{s\prime\prime}}{g^{s\prime}} \text{ for all } s \in S \ .$$

Only relying on smoothness of the functions $\{g^s\}_{s\in S}$, the condition is $g^{s\prime} \circ g^{s^{-1}}$ convex for all $s \in S$.

II) in the standard model (where $g^s = u \ \forall s \in S$) if, *in addition* to the conditions stated in part I, absolute Arrow Pratt risk aversion dominates utility prudence, i.e.

$$-\frac{g^{s\prime\prime}}{g^{s\prime}} > -\frac{u^{\prime\prime\prime}}{u^{\prime\prime}} \text{ for all } s \in S \ .$$

Only relying on smoothness of the functions $\{g^s\}_{s\in S}$ and u, the condition is $u' \circ g^{s^{-1}}$ concave for all $s \in S$.

In the special case of KP preferences ($\#S = 1$), part I of the proposition was derived by Gollier (2002). If the decision maker is less Arrow Pratt risk averse the higher his wealth, then uncertainty over his future income will induce higher savings (thereby effectively reducing risk aversion). The corresponding $\#S = 1$ special case of part II extends Gollier's finding by comparing the social discount rate under KP preferences to the discount rate in the standard model. In the standard model, the concavity of marginal utility is the only ingredient that reduces the social discount rate under risk. In contrast, under KP preferences, the disentangled Arrow Pratt risk aversion takes this role. If this disentangled risk aversion dominates utility prudence, then KP preferences reduce the social discount rate more. In particular, a decision maker who does not exhibit utility prudence, but exhibits Arrow Pratt risk aversion, will always choose a lower discount rate under Kreps Porteus preferences.

The special case of the smooth ambiguity model corresponds to $S = \{obj, subj\}$ and $g^{obj} = u$. The condition of smooth ambiguity aversion, i.e. that f^{subj} is concave, translates into $g^{subj} \circ u^{-1}$ concave, which means that the decision maker is more averse to subjective risk than to objective risk or intertemporal substitution. Ambiguity itself relates to second order subjective uncertainty (over objective first order lotteries).

Corollary 4: [KMM model]

I) The introduction of uncertainty in terms of ambiguity and/or objective risk decreases the social discount rate if $\text{ARA}^{subj} = -\frac{g^{subj''}}{g^{subj'}}$ and $\eta = -\frac{u''}{u'}$ are both decreasing or, equivalently, if

$$-\frac{g^{subj'''}}{g^{subj''}} > -\frac{g^{subj''}}{g^{subj'}} \text{ and } -\frac{u'''}{u''} > -\frac{u''}{u'} \; .$$

Only relying on smoothness of u and g^{subj}, the conditions are $u' \circ u^{-1}$ and $g^{subj'} \circ {g^{subj}}^{-1}$ concave. Translated into the f-representation of section 3.1 these conditions become
$u' \circ u^{-1}$ and $f^{subj'} \circ {f^{subj}}^{-1} \cdot \; u' \circ u^{-1} \circ {f^{subj}}^{-1}$ concave.

II) In an uncertain world, the introduction of ambiguity aversion reduces the social discount rate if, in addition to the conditions stated in part I, subjective Arrow Pratt risk aversion ARA^{subj} dominates utility prudence:

$$-\frac{g^{subj''}}{g^{subj'}} > -\frac{u'''}{u''} \; .$$

Only relying on smoothness of u and g^{subj}, this additional requirement is $u' \circ {g^{subj}}^{-1}$ concave.

In the statement relating to the f-representation in part I of the corollary, the expression is a (pointwise) multiplication ('$\cdot$') of two composed functions. The corresponding condition can be translated into a third order condition, however, the resulting expression extends over several lines and is of little insight. Gierlinger & Gollier (2008) have analyzed the questions answered in Corollary 4 in the f-representation. For general functional forms, they only found joint conditions on preferences and lotteries in order to identify when ambiguity aversion reduces the discount rate. In contrast, the above corollary holds for all well defined ambiguous lotteries. Note that the corollary also holds for the setting where the decision maker faces objective over subjective lotteries, which is not part of the KMM setting. Finally, observe that it would be misleading to interpret the condition on utility prudence in part I of Corollary 4 as a decreasing absolute aversion to intertemporal substitution. The comparison with the general result in Proposition 4 shows that aversion to intertemporal substitution only plays a role as entangled aversion to objective risk.

4 The multiperiod case

This section extends the discounting formula to settings with an arbitrary time horizon. In the discounted expected utility standard model, the discount rate only depends on consumption and uncertainty in the investment and the payoff period. This simplification no longer holds in the current setting, or in the special cases of KP or smooth ambiguity preferences. Here, the timing of uncertainty resolution between the investment and the payoff period influences the discount rate, and so does the uncertainty governing the post-payoff future. In general, Arrow Pratt prudence no longer characterizes fully the overall effect of uncertainty on the discount rate. The second part of this section derives a parametric discounting formula under the assumptions of normal growth rates and homothetic preferences, extending the familiar Ramsey rule stated in equation (1). Finally, I introduce a notion of aversion to the lack of confidence and discuss some implications for the term structure of the discount rate.

4.1 The general case

Let the decision maker evaluate a project with payoffs in period T. In general, the time horizon influences the discount rate, even if it surpasses the time of the project payoffs. I assume a planning horizon $\bar{T} \geq T$.[4] The resulting social or consumption discount rate corresponds to the equilibrium interest rate on a zero coupon bond with maturity in period T. Uncertainty in period t is captured by N_t layers of uncertainty. Lottery p_t^1 is a lottery over lotteries p_t^2 in the next lower uncertainty layer, continuing down to lotteries $p_t^{N_t-1}$ over $p_t^{N_t}$. The final layer of uncertainty in period t, $p_t^{N_t}$, characterizes uncertainty over outcomes x_t and over the remaining future p_{t+1}^1. A degree of confidence $\hat{s}(p_t^i) \in S$ characterizes each lottery p_t^i. This construction generalizes Kreps & Porteus's (1978) concept of temporal lotteries. For a detailed description see Traeger (2010). Preferences are extended recursively to the multiperiod case. They are stationary giving rise to the existence of a constant pure rate of time preference δ. If the decision maker adopts a finite planning horizon $\bar{T}$, then $W_{\bar{T}} = u(x_{\bar{T}})$ captures welfare in the last period after all uncertainty has resolved. If the decision maker's planning horizon coincides with the time of payoff T, then $W_T = u(x_T)$. Welfare in earlier periods is obtained by recursively calculating[5]

$$W_{t-1}(x_{t-1}, p_t^1) = u(x_{t-1}) + e^{-\delta} \left[\prod_{i=1}^{N_t} \mathcal{M}_{p_t^i}^{f^{\hat{s}(p_t^i)}} \right] W_t(x_t, p_{t+1}^1) \tag{7}$$

for $t \in 1, \ldots, T$.

I denote certainty equivalent welfare in uncertainty layer i of period t for some given lottery p_t^i by $m_t^i(p_t^i) \equiv \left[\prod_{j=i}^{N_t} \mathcal{M}_{p_t^j}^{f^{\hat{s}(p_t^j)}} \right] W_t(x_t, p_{t+1}^1)$ or, dropping the argument, simply by m_t^i. Moreover, I use $m_t^{N_t+1} \equiv W_t(x_t, p_{t+1}^1)$. Recall that the social discount rate in the multiperiod setting was defined as the yearly average.

[4]In the case of an infinite time horizon, I assume that future consumption grows sufficiently slow that the welfare functional converges. For an infinite time horizon, welfare is generally obtained as a fix point of equation (7) under some stationarity assumption, or by explicitly spelling out the equations of motion and the formulating the Bellman equation. Assuming a positive time preference and a stationary consumption process beyond some point in time $\hat{T} > \bar{T}$ would permit the decision maker to calculate the value $W_{\hat{T}}$ in the infinite time horizon setting and then to simply work through the relevant years of the project recursively.

[5]The recursion calculates welfare at the end of a given period when uncertainty only remains about future outcomes. To obtain welfare at the onset of period $t-1$ simply apply $\prod_{i=1}^{N_{t-1}} \mathcal{M}_{p_{t-1}^i}^{f_{t-1}^{\hat{s}(p_{t-1}^i)}}$ to equation (7).

Proposition 5: The social discount rate in the multiperiod setting for payoffs in period T is

$$r = \delta - \frac{1}{T} \ln \left[\left\{ \prod_{t=1}^{T} \prod_{i=1}^{N} \underbrace{\frac{\mathrm{E}_{p_t^i} f^{\hat{s}(p_t^i)\prime}(m_t^{i+1})}{f^{\hat{s}(p_t^i)\prime}(m_t^i)}}_{\substack{\text{prudence term} \\ \text{confidence level } \hat{s}(p^i)}} \ \mathrm{E}_{p^i} \underbrace{\frac{f^{\hat{s}(p_t^i)\prime}(m_t^{i+1})}{\mathrm{E}_{p^i} f^{\hat{s}(p_t^i)\prime}(m_t^{i+1})}}_{\substack{\text{pessimism term} \\ \text{confidence level } \hat{s}(p^i)}} \right\} \frac{u'(x_T)}{u'(x_0)} \right].$$

For $i < N_T$ the expected value operator $\mathrm{E}_{p_t^i}$ acts on lotteries p_t^{i+1} and certainty equivalents m_t^{i+1}. The expected value operator $\mathrm{E}_{p_t^{N_t}}$ acts on $m_t^{N_t+1} = x_t$ and, for $t < T$, on the lottery p_{t+1}^1 characterizing uncertainty in the next period. The form for the discount rate in Proposition 5 does not depend on whether the decision maker applies a finite planning horizon coinciding with the payoff time T, some larger finite horizon, or an infinite planning horizon. However, the evaluation of the certainty equivalent utility levels do depend on the time horizon. In the case of an infinite time horizon the $m_t^i(p_t^i)$ depend on an infinite consumption process. Proposition 2 also applies in the general setting.

Proposition 2': A prudence term of confidence level s reduces the social discount rate, if and only if, the function f^s exhibits decreasing absolute risk aversion $\mathrm{AIRA}_s = -\frac{f^{s\prime\prime}}{f^{s\prime}}$: $\quad -\frac{f^{s\prime\prime\prime}}{f^{s\prime\prime}} > -\frac{f^{s\prime\prime}}{f^{s\prime}}$. Only relying on smoothness of the functions $\{f^s\}_{s\in S}$, the condition is $f^{s\prime} \circ f^{s^{-1}}$ convex for all $s \in S$.

Once more, the assumption of a one-commodity setting permits a translation of the risk aversion measures into Arrow Pratt terms by using $g^s = f^s \circ u$ for $s \in S$. The corresponding representation of the social discount rate employs the definitions of the certainty equivalents in real terms $n_t^i(p_t^i) = u^{-1}\left(m_t^i(p_t^i)\right)$ including $n_0^{N_0+1} \equiv x_0$.

Proposition 6: In a one-commodity setting, the social discount rate expressed by means of Arrow Pratt risk aversion for

I) a decision maker adopting the time horizon $\bar{T} = T$ coinciding with the time of the payoff is

$$r = \delta - \frac{1}{T} \ln \left[\prod_{t=1}^{T} \frac{u'(n_t^1)}{u'(n_{t-1}^{N_{t-1}+1})} \left[\prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)\prime}(n_t^{i+1})}{g^{\hat{s}(p_t^i)\prime}(n_t^i)} \right] \right]. \tag{8}$$

II) a decision maker adopting a time horizon $\bar{T} > T$ (possibly infinite) is

$$r = \delta - \frac{1}{T} \ln \left[\left\{ \prod_{t=1}^{T} \frac{u'(n_t^1)}{u'(n_{t-1}^{N_{t-1}+1})} \left[\prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)\prime}(n_t^{i+1})}{g^{\hat{s}(p_t^i)\prime}(n_t^i)} \right] \right\} \frac{u'(x_T)}{u'(n_T^{N_T+1})} \right].$$

Note that, once more, also the evaluation of the certainty equivalents n_t^i depend on the time horizon. In equation (8) the use of Arrow Pratt risk aversion shifts once more marginal utility to the left of the period's expectation operator. In a two period setting, this shift enabled the powerful statements of Proposition 4. However, in the multiperiod setting these marginal utilities also depend on uncertainty and consumption levels in other periods. Because of this interdependence of welfare the proof underlying Proposition 4 no longer applies. Moreover, if the planning horizon exceeds that of the project, the interdependence with future welfare introduces an additional marginal utility term to the right of the last expectation operator. Assuming that uncertainty only resolves in period $T = \bar{T}$ recovers a separable welfare function and Proposition 4 applies again.

Corollary 5: [Uncertainty resolves only in period $T = \bar{T}$]

If the planning horizon of the agent coincides with the time of the payoff $\bar{T} = T$ and there is no uncertainty resolving in earlier periods, then Proposition 4 also holds for the multiperiod setting.

4.2 Isoelastic preferences and normal uncertainty

Isoelastic preferences are arguably the most prominent specification in economics. They imply decreasing absolute coefficients of aversion and underly the usual parametric formulation of the Ramsey discounting formula. I extend the parametric Ramsey rule to the setting of this paper. Let $X \subset \mathbb{R}$ describe an aggregate consumption commodity. I assume that consumption growth is uncertain and described by a normal distribution. Growth from one period to the next is captured by a single uncertainty layer. The growth rate $g_t = \ln \frac{x_{t+1}}{x_t} \sim N(\mu_t, \sigma_t; s_t)$ is distributed normally, where $s_t \in S$ labels confidence. The isoelastic preferences can be represented by the functions $u(x_t) = \frac{x_t^\rho}{\rho}$ and $f^s(z) = (\rho z)^{\frac{\alpha_s}{\rho}}$, implying $g^s = x_t^{\alpha_s}$. In a setting without confidence, these preferences correspond to those used by Epstein & Zin (1989) and Weil (1990) to disentangle risk preferences from intertemporal substitutability. I employ the inverse of the intertemporal elasticity of substitution $\eta = 1 - \rho = -\frac{u''}{u'}$ for measuring the decision maker's propensity to smooth consumption over time. Epstein & Zin (1989) measure Arrow Pratt risk aversion as a function of α. Employing instead a measure of intertemporal risk aversion allows me to flesh out the contribution differing from the standard model (where the intertemporal risk aversion term vanishes). Moreover, the carefully chosen risk measure will simplify the formula for the discount rate significantly. A straight forward numerical measure of intertemporal risk aversion would be a coefficient of relative intertemporal risk aversion calculated as

$$\text{RIRA}_s = -\frac{f^{s\prime\prime}(z)}{f^{s\prime}(z)}\,|z| = \begin{cases} 1 - \frac{\alpha_s}{\rho} \text{ for } \rho > 0 \\ \frac{\alpha_s}{\rho} - 1 \text{ for } \rho < 0 \end{cases} \tag{9}$$

for all $s \in S$ and $\rho \neq 0 \Leftrightarrow \eta \neq 1$. The absolute in the definition arises because the utility function u can be negative. Such a measure is suggested in the case of smooth ambiguity aversion by Klibanoff et al. (2009). However, the Epstein-Zin specification implies that the measure RIRA_s goes to infinity for $\rho \to 0$, when utility switches from the positive to the negative domain. I therefore introduce a renormalized measure

$$\zeta_s = \begin{cases} \text{RIRA}_s\,|1-\eta^2| & \text{for } \eta \neq 1 \\ -2\alpha_{s_t} & \text{for } \eta = 1 \end{cases}$$

which is continuous at $\eta = 1$ and positive, if and only if, the decision maker is intertemporal risk averse.

Proposition 7: The social discount rate for payoffs in period T is

$$r = \delta + \frac{1}{T}\left[\sum_{t=1}^{T} \eta\mu_t - \eta^2\frac{\sigma_t^2}{2} - \zeta_{s_t}\frac{\sigma_t^2}{2}\right]. \tag{10}$$

The first term in the sum captures how expected growth reduces future marginal utility deriving from a unit of consumption. The second term corresponds to the risk term of the standard model. It is caused by aversion to intertemporal fluctuations generated by the stochastic process. The last term captures intertemporal risk aversion. Common values for η in the standard model are between 1 and 2. However, the disentangled approach generally estimates values of η smaller than unity: Aversion to intertemporal substitution turns out smaller when it no longer has to simultaneously play the role of risk aversion. Vissing-Jørgensen & Attanasio (2003), Bansal & Yaron (2004), and Bansal et al. (2010) identify $\eta = \frac{2}{3}$ as a reasonable estimate. I adopt Kocherlakota's (1996) estimates of $\mu = 1.8\%$ and $\sigma = 3.6\%$, based on an annual time series of 90 years for the US, to compare the different contributions. Then, the growth term ranges $1.2\% - 3.6\%$ for $\eta \in [\frac{2}{3}, 2]$ and the second term ranges $0.03\% - 0.3\%$, which is close to negligible. Thus, intertemporal growth trends are highly significant for the social discount rate and for the saving decision, while wiggles are not. Relative risk aversion, measured in the Arrow Pratt sense, generally is assumed to range $5 - 10$ giving rise to $\zeta \in [7^2/_9, 15^5/_9]$ for $\eta = \frac{2}{3}$, $\zeta \in [8, 18]$ for $\eta = 1$, and $\zeta \in [9, 24]$ for $\eta = 2$. The last contribution in equation (10) then ranges $0.5\% - 1.6\%$. Thus, intertemporal risk aversion reduces the social discount rate significantly, while risk has a negligible effect in the standard model.

4.3 Aversion to the lack of confidence and the term structure of discount rates

The numerical reasoning in the previous section assumed constant growth rates and confidence. My interest is in analyzing the situation where confidence in the normal distributions governing growth decreases the further an agent (or society as a whole) looks into the future. In order to capture reactions to decreasing confidence, I introduce the notion of aversion to subjectivity. For this purpose, I assume that the set of subjectivity descriptions is completely ordered by a binary relation $\rhd \subset S^2$, such that $s \rhd s'$ denotes that lotteries labeled s are considered more subjective (or less confident) then lotteries labeled s'. I keep μ_t and σ_t fix at not necessarily constant levels. Following Traeger (2010), I define a decision maker as [*strictly*] *averse to* the *subjectivity* of belief or the *lack of confidence* in beliefs if for all $s, s' \in S$

$$\begin{aligned} s \rhd s' \quad \Leftrightarrow \quad & (x_1, ..., x_T) \oplus_{s'}^{\frac{1}{2}} (x'_1, ..., x'_T) \quad \succeq_T \ [\succ_T] \ (x_1, ..., x_T) \oplus_{s}^{\frac{1}{2}} (x'_1, ..., x'_T) \\ & \forall\, x_1, ..., x_T, x'_1, ..., x'_T \in X \text{ [with non-indifferent paths]} \\ \Leftrightarrow \quad & f^s \circ (f^{s'})^{-1} \text{ [strictly] concave }. \end{aligned}$$

The equivalence of the two lines on the right hand side is shown in the cited paper. The definition of smooth ambiguity aversion is the special case of aversion to subjectivity where $s = subj$ and $s' = obj$ and, for the definition of Klibanoff et al. (2009), f^{obj} is the identity. I aim at a representation of discount rates where subjectivity and confidence are measured on a scale from zero to one, rather than in abstract terms. For this purpose, I introduce the space of all order preserving maps from the abstract space of confidence descriptions onto the unit interval

$$\begin{aligned} \mathcal{S}^* = \{S^* : S \to [0,1] \mid \ & S^*(s) > S^*(s') \Leftrightarrow s \rhd s' \text{ and} \\ & \exists\, \underline{s}, \overline{s} \in S \text{ s.th. } S^*(\underline{s}) = 0, S^*(\overline{s}) = 1\}\,, \end{aligned}$$

which map the label indicating objectivity or most confidence to zero and the label indicating highest subjectiveness or least confidence to unity. The following proposition expresses the social discount rate in terms of subjectivity and aversion to subjectivity (lack of confidence).

Proposition 8: Let a decision maker exhibit isoelastic preferences, intertemporal risk aversion to objective lotteries, and aversion to the lack of confidence. Let the growth rate be normally distributed as laid out above. Then, there exist parameters $\eta \in \mathbb{R}$, $\lambda \in [0, 1)$, and $\zeta \geq 0$ and a map $S^* \in \mathcal{S}^*$ such that the discount rate for a payoff in period T is

$$r = \delta + \frac{1}{T} \left[\sum_{t=1}^{T} \eta \mu_t - \eta^2 \frac{\sigma_t^2}{2} - \frac{\zeta}{1 - \lambda s_t^*} \frac{\sigma_t^2}{2} \right] , \tag{11}$$

where $s_t^* = S^*(s_t) \in [0, 1]$.

For a decision maker who is *strictly* intertemporal risk averse with respect to objective lotteries and satisfies *strict* aversion to the lack of confidence in beliefs it is $\zeta, \lambda > 0$.

The parameter λ captures aversion to subjectivity or lack in confidence. A decision maker who is indifferent to subjectivity obeys the KP model an exhibits $\lambda = 0$. His consumption discount rate features a time-constant contribution from intertemporal risk aversion, measured by ζ, weighting the volatility. For decision makers with aversion to subjectivity, the parameter ζ only reflects intertemporal risk aversion to objective lotteries. As subjectivity increases, the denominator $1-\lambda s^*$ decreases the discount rate by increasing risk aversion. I am interested in the case where confidence in the normal description of growth uncertainty decreases the further the agent looks into the future: $s^*_{t+1} > s^*_t \, \forall \, t \in \{1, ..., T\}$. An example is a decision maker who takes into consideration that technological uncertainty, climate change, or social tensions make future growth less and less predictable. Such an agent invests more into projects with certain future payoffs because he accounts for his increasing lack of confidence. Note that the combination of a complete lack of confidence $s_t = 1$ and an extreme aversion to the lack of confidence $\lambda \to 1$ results in an Arrow & Hurwicz (1972) type criterion for decision making under ignorance. Such a decision maker would only pay attention to the worst possible outcome.[6]

Proposition 8 brings the discount rate into a suitable form to discuss the term structure of the discount rate in relation to confidence in beliefs. I employ again Kocherlakota's (1996) growth estimates of $\mu = 1.8\%$ and $\sigma = 3.6\%$ and choose $\eta = \frac{2}{3}$ based on Vissing-Jørgensen & Attanasio (2003), Bansal & Yaron (2004), and Bansal et al. (2010). The graphs in Figure 2 depict the average yearly discount rate for a payoff in period T assuming a pure rate of time preference of

[6]The limit $\lambda \to 1$ corresponds to $\gamma_{s^*} \to \infty$ for $s^* \to 1$, giving rise to full weight on the minimal element carrying positive probability mass. The normal distribution has full support on $\mathbb{R}$ so that the decision maker puts all weight on however small possibility of dying of hunger (or worse). If we offer such an agent a zero coupon bond enabling a sure transfer into the future and allowing him not to worry about starvation in that period he would pay an infinite amount for the first marginal transfer. This is reflected by the discount rate going to infinity if both, λ and s, approach unity. Obviously, the underlying growth model in combination with the offer of a certain transfer would be too simple a model in order to support the decisions of such an agent. A "dismal theorem" interpretation of equation (11) would be a misinterpretation (or extrapolation beyond applicability) of the model.

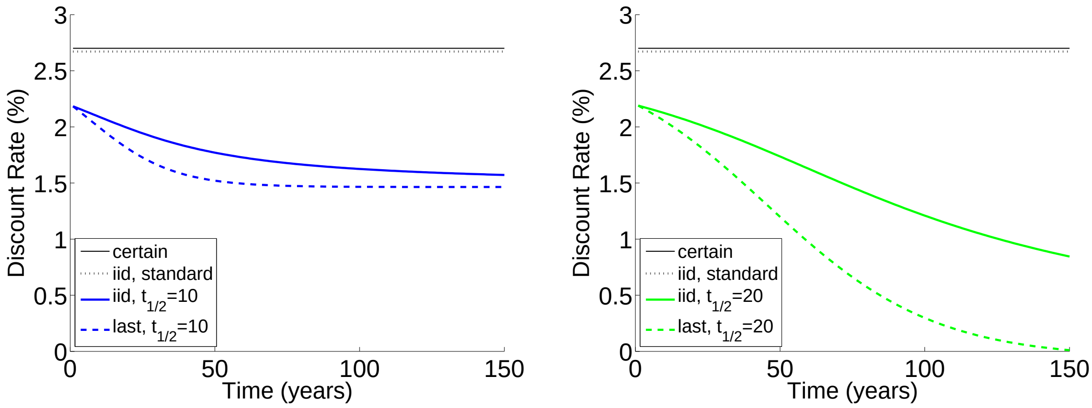

Figure 2 depicts the term structure for decreasing confidence. The thick, solid lines (colored) represent an idd growth process with exponentially decreasing confidence. For the dashed lines, all uncertainty resolves in the payoff period (abscissa). The model parameters are chosen such that implied Arrow Pratt risk aversion is RRA = 5 in the present and RRA = 10 in year 30. The blue (dark) graphs on the left assume a confidence half life of $t_{\frac{1}{2}} = 10$, implying an aversion to the lack of confidence of $\lambda = 0.61$. The green (light) graphs on the right assume a confidence half life of $t_{\frac{1}{2}} = 20$, implying an aversion to the lack of confidence of $\lambda = 0.83$. For comparison, the thin, solid line (black) at the top represents certainty and the thin, dotted line an iid growth process in the intertemporally additive expected utility standard model.

$\delta = 1.5\%$. The solid black line at the top of the graphs in Figure 2 depicts the discount rate under certainty. The dashed black line right underneath depicts the discount rate under iid growth uncertainty in the standard model. The formulas correspond to the first summand and to the first two summands in equation (11), respectively. The figure shows that uncertainty has almost no effect in the standard model. For higher values of η, the absolute value and the difference between the two lines would be larger. However, for any value of η the term structure will be flat and the decision maker discounts the future at a constant rate.

In contrast, a decrease of confidence in futurity implies a falling term structure, even under iid uncertainty. The term structure is determined by the intrinsic aversion to objective risk, the loss in confidence over time, and the aversion to the lack of confidence. A quantitative simulation of the term structure has to fix these values. I use an Arrow Pratt risk aversion coefficient of RRA = 5 for aversion to objective risk. This value lies at the lower end of what is measured for relative risk aversion in disentangled approaches. The assumption is equivalent to $\zeta = 7.2$. Moreover, I assume that confidence, measured as $1 - s_t^*$, declines exponentially to zero at rate γ. The exponential distribution is reasonable because it lacks memory. The assumption translates into the functional form $s_t^* = 1 - \exp(-\gamma t)$. I fix the values for the decay rate of confidence and the aversion to the lack of confidence as follows. I assume that the Arrow Pratt measure of relative risk aversion increases to RRA = 10 over the course of the next 30 years. This risk aversion coefficient is on the high side of estimates, but by no means an upper bound. I fix the remaining degree of freedom by an assumption on the half life of confidence ($t_{\frac{1}{2}} = \frac{\ln 2}{\gamma}$). The representation in Proposition 8 has cardinalized the degree of confidence $1 - s_t^*$ on an interval from unity to zero. The blue (thick) lines on the left of Figure 2 assume that the decision maker's confidence over the distribution governing growth from on period to the next decreases to half its value during the course of 10 years. The solid line represents the iid growth model of equation (11). The dashed lines represent a model where all uncertainty resolves in the payoff period.[7] Both discount rates start at approximately 2.2%. The KP model with RRA = 5 would yield a constant term structure at this level, which is a significant reduction with repect to the standard model. As the agent becomes less confident over the future, the discount rate falls. By construction, Arrow Pratt risk aversion with respect to the growth realization in year 30 is RRA = 10. At this point, confidence has already fallen to an eigth and the curves slowly starts to flatten. The dashed line at year 30 depicts a discount rate of 1.7% corresponding to the (constant) rate that would obtain in the KP model with RRA = 10. The solid line lags behind because it calculates the discount rate for the case where part of the uncertainty resolves in the closer future, over which the decision maker is more confident. In the long run, the solid line converges to the dashed line.

The graph on the right of Figure 2 analyzes the term structure for a longer half life of confidence, $t_{\frac{1}{2}} = 20$ years. Aversion to the lack of confidence is once more determined by the requirement that after 30 years the implied Arrow Pratt measure should be RRA = 30. Because confidence falls slower in the right graph, this calibration implies a higher degree of aversion to the lack of

[7] The variance is adjusted to imply the same uncertainty over consumption in period T as in the case of the iid growth scenario. The scenario with last period uncertainty resolution is captured by equation (11) with $\sigma_t = 0\%$ for $t \in \{0, ..., T-1\}$ and $\sigma_T = 3.6\% * \sqrt{T}$.

confidence ($\lambda = 0.83$ as opposed to $\lambda = 0.61$ for the graph on the left). In year 30, the dashed line once more depicts the discount rate of 1.7% corresponding to the KP model with RRA = 10. However, confidence falls slower than in the simulation on the left. Thus, confidence keeps falling more significantly beyond year 30 and the dashed line only starts to flatten at the end of the century. The solid line, representing the iid model, once more lags behind because uncertainty partially resolves in the closer future where the decision maker is more confident about his probabilistic beliefs.[8]

5 Conclusions

The future is uncertain. Probabilistic beliefs over the future range from well founded estimates to guesstimates. The present framework captures differences in way probabilities have been informed by a confidence index. Different confidence classes of probabilistic beliefs give rise to different degrees of risk aversion. The widespread smooth ambiguity model is a special case with two classes of probabilities. The paper analyzes how such general forms of uncertainty affect the consumption discount rate of individual agents and the social discount rate for aggregate cost benefit analysis.

A prudence term reduces the discount rate under uncertainty, whenever absolute measures of uncertainty aversion are falling. If uncertainty aversion is measured in terms of smooth ambiguity aversion or, more generally, intertemporal risk aversion, the prudence effect is complemented by a pessimism effect. The pessimism term has an ambiguous effect on discount rates. Intertemporal risk aversion and smooth ambiguity aversion capture an intrinsic aversion to risk, as opposed to an aversion caused by the desire for consumption smoothing and

[8]In year 150 the dashed line reaches zero and eventually even becomes slightly negative. Note that a negative consumption discount rate in this model does not imply that the welfare function diverges. In contrast, the comprehensive model yields lower welfare levels under uncertainty than does the standard model with the high discount rate. The interpretation is that, if society has the means to transfer a certain unit of consumption into a highly uncertain future described by low confidence probabilities, then it should do so, even when it has to accept a small depreciation of the transfer. However, a larger rate of pure time preference will imply a strictly positive discount rate with the same assumptions on confidence and aversion. Moreover, see footnote 6 on the limitations of the normal growth model in this context.

fluctuations over time. In a one commodity setting, generalized Arrow Pratt measures of risk aversion jointly measure intrinsic risk aversion and aversion to intertemporal fluctuations. Using these measures to represent preferences eliminates the pessimism term. Then, the prudence effect uniquely characterizes the discount rates. Thus, the discount rates fall under uncertainty, whenever these absolute measures of Arrow Pratt risk aversion are falling in consumption for all confidence levels. This finding confirms Leland's (1968) intuition on precautionary savings, which was recently challenged in the context of ambiguity aversion. The intuition is that, if risk aversion falls in consumption level, an increase in future baseline consumption mitigates the effect of uncertainty aversion and, thus, increases the incentive to save. Similarly, it makes cost benefit analysis pay more attention to (certain) future payoffs. This intuition holds as long as the aversion measure captures all of the uncertainty aversion present in the model. The effect of uncertainty on discount rates is larger in the general model than in the (all-entangling) standard model, if the absolute Arrow Pratt measures of (disentangled) risk aversion dominate absolute utility prudence.

In an application, the paper extended the Ramsey formula for isoelastic preferences and normal growth to the setting of general uncertainty. A new contribution arises that is proportional to volatility and a measure of intrinsic risk aversion: smooth ambiguity aversion or, more generally, intertemporal risk aversion. The new term reduces the discount rate significantly in the presence of uncertainty. In order to analyze the term structure of discount rates, risk aversion has to be related across different confidence levels. An assumption of aversion to the lack of confidence plays this role. If confidence decreases over time, the term structure is falling even in the case of iid growth. The further the agent, or society, look into the future, the lower the confidence and the higher is the incentive to increase baseline consumption. For cost benefit analysis, the finding implies the use of hyperbolic discount rates. They reduce the extreme devaluation of long-run consequences of current actions implied by exponential discounting, while maintaining standard discount rates for the short term. Many of the examples cited in the introduction exhibit long-run benefits of current actions - or long-run costs of current inaction. Here, a reduced social discount rate implies that more projects should be carried out, e.g. mitigation of greenhouse gas emissions. A different perspective on the finding is as follows. The standard model contains the implicit assumption that long-run uncertainties are of the same type as flipping a coin. This implicit assumption can result in a bias against precautionary action ensuring future consumption levels.

Appendix

A Proofs

Proof of Proposition 1: First observe that $\partial_{x_0} W_0 = \partial_{x_0} u(x_0) = u'(x_0)$ and

$$\partial_{x_1} \mathcal{M}_p^{f^{\hat{s}(p)}} u(x_1) = \frac{\mathrm{E}_p \, f^{\hat{s}(p)\prime} \left[u(x_1)\right] u'(x_1)}{f^{\hat{s}(p)\prime}\big[\mathcal{M}_p^{f^{\hat{s}(p)}} u(x_1)\big]} \; .$$

Then

$$\begin{aligned}
\partial_{x_1} \left[\prod_{i=j}^{N} \mathcal{M}_{p^i}^{f^{\hat{s}(p^i)}}\right] u(x_1) = \; & \mathrm{E}_{p^j} \, \frac{f^{\hat{s}(p^j)\prime} \left[m^{j+1}\right]}{f^{\hat{s}(p^j)\prime}\big[\mathcal{M}_{p^j}^{f^{\hat{s}(p^j)}} m^{j+1}\big]} \, \partial_{x_1} \left[\prod_{i=j+1}^{N} \mathcal{M}_{p^i}^{f^{\hat{s}(p^i)}}\right] u(x_1) \\
= & \left[\prod_{i=j}^{N} \mathrm{E}_{p^i} \, \frac{f^{\hat{s}(p^i)\prime} \left[m^{i+1}\right]}{f^{\hat{s}(p^i)\prime}\big[m^i\big]}\right] u'(x_1)
\end{aligned}$$

and the discount rate becomes

$$r = -\ln\left[\frac{\partial_{x_1} W_0}{\partial_{x_0} W_0}\right] = \delta - \ln\left[\left\{\prod_{i=1}^{N} \mathrm{E}_{p^i} \, \frac{f^{\hat{s}(p^i)\prime}(m^{i+1})}{f^{\hat{s}(p^i)\prime}(m^i)}\right\} \frac{u'(x_1)}{u'(x_0)}\right] \; .$$

The second version stated in the proposition is obtained by expanding numerator and denominator with $\mathrm{E}_{p^i} \, f^{\hat{s}(p^i)\prime}(m^{i+1})$. □

Proof of Proposition 2: For two continuously differentiable functions h and f with $h > 0$ and f strictly monotone on a non-degenerate domain holds

$$\frac{\mathrm{E} h\,(z)}{h\,[f^{-1} \mathrm{E}\, f(z)]} > 1 \quad \Leftrightarrow \quad \mathrm{E}\, h \circ f^{-1}(y) > h f^{-1}\, \mathrm{E}\, y \quad \Leftrightarrow \quad h \circ f^{-1} \text{ convex} \; .$$

For $h<0$ it follows analogously $h\circ f^{-1}$ concave. Moreover,

$$h\circ f^{-1} \text{ convex} \quad \Leftrightarrow \quad h''>\frac{f''}{f'}h' \quad \Leftrightarrow \quad \begin{cases} -\frac{f''}{f'}>-\frac{h''}{h'} \;\wedge\; h'>0 \text{ or} \\ -\frac{f''}{f'}<-\frac{h''}{h'} \;\wedge\; h'<0 \end{cases} .$$

Then with $h=f'$, noting that $f'>0$ and $f''<0$ by assumption, I find the condition

$$\frac{\mathrm{E}_{p^i}\, f^{\hat{s}(p^i)\prime}(m^{i+1})}{f^{\hat{s}(p^i)\prime}(m^i)}>1 \quad \Leftrightarrow \quad \frac{\mathrm{E}_{p^i}\, f^{\hat{s}(p^i)\prime}(m^{i+1})}{f^{\hat{s}(p^i)\prime}\left({f^{\hat{s}(p^i)}}^{-1}\mathrm{E}_{p^i}\, f^{\hat{s}(p^i)}(m^{i+1})\right)}>1$$

$$\Leftrightarrow f^{\hat{s}(p^i)\prime}\circ {f^{\hat{s}(p^i)}}^{-1} \text{ convex} \quad \Leftrightarrow \quad -\frac{f^{\hat{s}(p^i)\prime\prime}}{f^{\hat{s}(p^i)\prime}}<-\frac{f^{\hat{s}(p^i)\prime\prime\prime}}{f^{\hat{s}(p^i)\prime\prime}} ,$$

which is equivalent to $\mathrm{AIRA}^{\hat{s}(p^i)}=-\frac{f^{\hat{s}(p^i)\prime\prime}}{f^{\hat{s}(p^i)\prime}}$ decreasing. □

Corollaries 1 - 3 are immediate consequences of Proposition 1.

Proof of Proposition 3: Replacing the functions f^s by $g^s=f^s\circ u$ for all $s\in S$ and observing that $f^s\circ f^{\bar{s}-1}=g^s\circ u^{-1}\circ u\circ g^{\bar{s}-1}$ I obtain the welfare representation

$$u(x_0)+e^{-\delta}\,u^{-1}\left[\prod_{i=1}^{N_t}\mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right]x_1 .$$

The required welfare changes then become $\partial_{x_0}u(x_0)=u'(x_0)$ and

$$e^{-\delta}\partial_{x_1}\,u\left[\prod_{i=1}^{N_t}\mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right]x_1=e^{-\delta}\,u'(n^1)\prod_{i=j}^{N}\mathrm{E}_{p^i}\,\frac{g^{\hat{s}(p^i)\prime}(n^{i+1})}{g^{\hat{s}(p^i)\prime}(n^i)} ,$$

which results in the discount rate stated in equation (6). □

Proof of Proposition 4:

Part I: Let r^{cert} denote the discount rate when receiving the expected future payoff $\bar{x}$ with certainty. Comparison with the discount rate r in the uncertain scenario yields

$$r < r^{cert} \quad \Leftrightarrow \quad \frac{u'(n^1)}{u'(x_0)} \prod_{i=1}^{N} \mathrm{E}_{p^i} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)} > \frac{u'(\bar{x})}{u'(x_0)} \ .$$

Assume $g^{s\prime} \circ g^{s^{-1}}$ convex for all $s \in S$. Define

$$\tilde{A}_j(p^j) \equiv \prod_{i=j}^{N} \mathrm{E}_{p^i} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)}$$

for all $j = 1, \ldots, N'$. Recursive application of Proposition 2 for $j = N, N-1, \ldots, 1$ ensures that all $\tilde{A}_j(p^j)$ are larger than unity:

$$\tilde{A}_j(p^j) = \prod_{i=j}^{N} \mathrm{E}_{p^i} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)} = \mathrm{E}_{p^j} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)} \tilde{A}_{j+1} > \mathrm{E}_{p^j} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)} > 1 \ .$$

Thus a sufficient condition for $r < r^{cert}$ is

$$u'(n^1) > u'(\bar{x}) \quad \Leftrightarrow \quad u'\left(\left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right] x_1\right) > u'\left(\left[\prod_{i=1}^{N_t} \mathrm{E}_{p^i}\right] x_1\right) \ ,$$

which is satisfied by the assumptions $g^{s\prime} > 0$ and $u'', g^{s\prime\prime} < 0$ for all $s \in S$, because an increasing concave function g^s makes the generalized mean smaller than the expected value.

Part II: The comparison of r w.r.t. to the discount rate r^{std} obtained under standard preferences where $g^s = u \ \forall s \in S$ implies the condition

$$r < r^{std} \quad \Leftrightarrow \quad \frac{u'(n^1)}{u'(x_0)} \prod_{i=1}^{N} \mathrm{E}_{p^i} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)} > \frac{u'(n_u^1)}{u'(x_0)} \prod_{i=1}^{N} \mathrm{E}_{p^i} \frac{u'(n_u^{i+1})}{u'(n_u^i)} \ ,$$

where $n_u^j = \left[\prod_{i=j}^{N_t} \mathcal{M}_{p^i}^{u}\right] x_1$ is the certainty equivalent calculated with u characterizing the generalized mean. The denominator in the last term of the equation does not depend on the expected value operator immediately to the left and cancels with the numerator in the preceding term, reducing the right hand side to

$\left[\prod_{i=1}^{N} \mathrm{E}_{p^i}\right] \frac{u'(x_1)}{u'(x_0)}$. Rearranging implies

$$r < r^{std} \quad \Leftrightarrow \quad \frac{u'\left(\left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right] x_1\right)}{\left[\prod_{i=1}^{N} \mathrm{E}_{p^i}\right] u'(x_1)} \prod_{i=1}^{N} \mathrm{E}_{p^i} \frac{g^{\hat{s}(p^i)'}(n^{i+1})}{g^{\hat{s}(p^i)'}(n^i)} > 1 \; .$$

As shown in part I) the product term is larger than unity. Thus, a sufficient condition for $r < r^{cert}$ is

$$u'\left(\left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right] x_1\right) > \left[\prod_{i=1}^{N} \mathrm{E}_{p^i}\right] u'(x_1) \; . \tag{12}$$

If u' is concave the condition is satisfied as u concave and g^S increasing and concave for all s imply

$$u'\left(\left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right] x_1\right) > u'\left(\left[\prod_{i=1}^{N} \mathrm{E}_{p^i}\right] x_1\right) > \left[\prod_{i=1}^{N} \mathrm{E}_{p^i}\right] u'(x_1) \; .$$

If u' is convex, using concavity of u, condition (12) translates into

$$\left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{g^{\hat{s}(p^i)}}\right] x_1 < {u'}^{-1}\left(\left[\prod_{i=1}^{N} \mathrm{E}_{p^i}\right] u'(x_1)\right) = \left[\prod_{i=1}^{N_t} \mathcal{M}_{p^i}^{u'}\right] x_1 \; .$$

This condition is satisfied if all the mean values characterized by g^s, $s \in S$ are smaller than those characterized by u', which is satisfied if $g^s \circ {u'}^{-1}$ is concave for all $s \in S$ or, equivalently, $\frac{-u'''}{u''} < -\frac{g''}{g'}$ (see proof of Proposition 2). □

Corollary 4 is an immediate consequence of Proposition 3. □

Proof of Proposition 5: The welfare change from an infinitesimal change in current consumption still is $\partial_{x_1} W_0 = \partial_{x_1} u(x_0) \equiv u'(x_0)$, while the future change becomes

$$\begin{aligned} \partial_{x_T} W_0 &= e^{-\delta} \, \partial_{x_T} \left[\prod_{i=1}^{N_t} \mathcal{M}_{p_t^i}^{f_t^{\hat{s}(p_t^i)}}\right] W_1(x_1, p_1^1) \\ &= e^{-\delta} \left[\prod_{i=1}^{N_1} \mathrm{E}_{p_t^i} \frac{f^{\hat{s}(p_1^i)'}\left[m_1^{i+1}\right]}{f^{\hat{s}(p_1^i)'}\left[m_1^i\right]}\right] \partial_{x_T} W_1(x_t, p_{t+1}^1) \end{aligned}$$

$$= e^{-\delta t} \left[\left\{ \prod_{t=1}^{T} \prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{f^{\hat{s}(p_t^i)}{}'(m_t^{i+1})}{f^{\hat{s}(p_t^i)}{}'(m_t^i)} \right\} u'(x_T) \right] .$$

Note that the only change for a longer time horizon is that the m_t^i are calculated for longer consumption stream. Then

$$r = -\frac{1}{t} \ln \left[\frac{\partial_{x_T} W_0}{\partial_{x_0} W_0} \right] = \delta - \frac{1}{t} \ln \left[\left\{ \prod_{t=1}^{T} \prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{f^{\hat{s}(p_t^i)}{}'(m_t^{i+1})}{f^{\hat{s}(p_t^i)}{}'(m_t^i)} \right\} \frac{u'(x_T)}{u'(x_0)} \right] .$$

□

Proof of Proposition 2': Same as for Proposition 2. □

Proof of Proposition 6: Replacing the functions f^s by $g^s = f^s \circ u$ I obtain the welfare representation $V_{\bar{T}}(x_{\bar{T}}) = x_{\bar{T}}$ and

$$W_{t-1}(x_{t-1}, p_t^1) = u(x_{t-1}) + e^{-\delta}\, u \left\{ \left[\prod_{i=1}^{N_t} \mathcal{M}_{p_t^i}^{g_t^{\hat{s}(p_t^i)}} \right] u^{-1} \circ W_t(x_t, p_{t+1}^1) \right\}$$

for $t \in 1, \ldots, \bar{T}$. First, observe that $\partial_{x_0} W_0 = u'(x_0)$ still holds. Moreover,

$$\begin{aligned}
\partial_{x_T} W_0 &= e^{-\delta}\, u'(n_1^1)\, \partial_{x_T} \left[\prod_{i=1}^{N_t} \mathcal{M}_{p_t^i}^{g_t^{\hat{s}(p_t^i)}} \right] u^{-1} \circ W_1(x_t, p_{t+1}^1) \\
&= e^{-\delta}\, u'(n_1^1) \left[\prod_{i=1}^{N_1} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_1^i)'}\left[n_1^{i+1}\right]}{g^{\hat{s}(p_1^i)'}\left[n_1^i\right]} \right] \partial_{x_T} u^{-1} \circ W_1(x_t, p_{t+1}^1) \\
&= e^{-\delta}\, u'(n_1^1) \left[\prod_{i=1}^{N_1} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_1^i)'}\left[n_1^{i+1}\right]}{g^{\hat{s}(p_1^i)'}\left[n_1^i\right]} \right] \frac{\partial_{x_T} W_1(x_t, p_{t+1}^1)}{u'(n_1^{N_1+1})} \\
&= e^{-\delta(T-1)} \left\{ \prod_{t=1}^{T-1} u'(n_t^1) \left[\prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)} \right] \frac{1}{u'(n_t^{N_t+1})} \right\} \partial_{x_T} W_{T-1} \\
&= e^{-\delta(T)} \left\{ \prod_{t=1}^{T} u'(n_t^1) \left[\prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)} \right] \frac{1}{u'(n_t^{N_t+1})} \right\} \partial_{x_T} W_T \\
&= e^{-\delta(T)} u'(n_1^1) \left\{ \prod_{t=1}^{T} \left[\prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)} \right] \frac{\delta_{T,t} + 0_{T,t}\; u'(n_{t+1}^1)}{u'(n_t^{N_t+1})} \right\} \partial_{x_T} W_T \\
&= e^{-\delta T} \left\{ \prod_{t=1}^{T} \left[\frac{u'(n_t^1)}{\delta_{1,t} + 0_{1,t}\; u'(n_{t-1}^{N_{t-1}+1})} \prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)} \right] \right\} \frac{\partial_{x_T} W_T}{u'(n_T^{N_T+1})} \qquad (13)
\end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise and $0_{i,j} = 0$ if $i = j$ and 1 otherwise. The last two equations will both be used to derive two alternative formulations. The further calculation differs depending on the time horizon:

I) *Time horizon* $\bar{T} = T$ and $W_T(\cdot) = W_T(x_T) = u(x_t)$. Then $\partial_{x_T} W_T = u'(x_T) = u'(n_T^{N_T+1})$ and the last derivative cancels the last u'-denominator, resulting in

$$\begin{aligned}
\partial_{x_T} W_0 &= e^{-\delta t} u'(n_t^1) \left\{ \prod_{t=1}^{T-1} \left[\prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)} \right] \frac{u'(n_{t+1}^1)}{u'(n_t^{N_t+1})} \right\} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_T^i)'}(n_T^{i+1})}{g^{\hat{s}(p_T^i)'}(n_T^i)} \\
&= e^{-\delta T} \left\{ \prod_{t=1}^{T} \left[\frac{u'(n_1^1)}{\delta_{1,t} + 0_{1,t}\; u'(n_{t-1}^{N_{t-1}+1})} \prod_{i=1}^{N_t} \mathrm{E}_{p_t^i} \frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)} \right] \right\}
\end{aligned}$$

and the discount rate becomes

$$r = -\frac{1}{t}\ln\left[-\frac{\partial_{x_T}W_0}{\partial_{x_0}W_0}\right]$$

$$= \delta - \frac{1}{t}\ln\left[\frac{u'(n_t^1)}{u'(x_0)}\left\{\prod_{t=1}^{T-1}\left[\prod_{i=1}^{N_t}\mathrm{E}_{p_t^i}\frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)}\right]\frac{u'(n_{t+1}^1)}{u'(n_t^{N_t+1})}\right\}\mathrm{E}_{p_t^i}\frac{g^{\hat{s}(p_T^i)'}(n_T^{i+1})}{g^{\hat{s}(p_T^i)'}(n_T^i)}\right]$$

$$= \delta - \frac{1}{t}\ln\left[\prod_{t=1}^{T}\left[\frac{u'(n_t^1)}{u'(n_{t-1}^{N_{t-1}+1})}\prod_{i=1}^{N_t}\mathrm{E}_{p_t^i}\frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)}\right]\right] .$$

where in the second line by definition $n_0^{N_0+1} = x_0$.

II) *Time horizon* $\bar{T} > T$. Then $\partial_{x_T}W_T(\cdot) = \partial_{x_T}W_T(x_T, p_{T+1}) = u'(x_T)$ as before, but $u'(n_T^{N_T+1}) \neq u'(x_T)$, so that the last terms no longer cancel and by equation (13) the discount rate becomes

$$r = \delta - \frac{1}{t}\ln\left[\left\{\prod_{t=1}^{T}\frac{u'(n_t^1)}{u'(n_{t-1}^{N_{t-1}+1})}\left[\prod_{i=1}^{N_t}\mathrm{E}_{p_t^i}\frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)}\right]\right\}\frac{u'(x_T)}{u'(n_T^{N_T+1})}\right]$$

$$= \delta - \frac{1}{t}\ln\left[\left\{\prod_{t=1}^{T}\left[\prod_{i=1}^{N_t}\mathrm{E}_{p_t^i}\frac{g^{\hat{s}(p_t^i)'}(n_t^{i+1})}{g^{\hat{s}(p_t^i)'}(n_t^i)}\right]\frac{u'(n_t^1)}{u'(n_t^{N_t+1})}\right\}\frac{u'(x_T)}{u'(x_0)}\right] .$$

Note that

$$n_t^i(p_t^i) \equiv \Big[\prod_{i=j}^{N_t}\mathcal{M}_{p_t^i}^{g^{\hat{s}(p_t^i)}}\Big]u^{-1}\circ W_t(x_t, p_{t+1}^1)$$

$$= u^{-1}\Big[\prod_{i=j}^{N_t}\mathcal{M}_{p_t^i}^{f^{\hat{s}(p_t^i)}}\Big]\circ W_t(x_t, p_{t+1}^1) = u^{-1}\left(m_t^i(p_t^i)\right) .$$

□

Proof of Corollary 5: In this case the general formula collapses to

$$r = \delta - \frac{1}{t}\ln\left[\left[\prod_{t=1}^{T-1}\frac{u'(n_t^1)}{u'(n_{t-1}^1)}\right]\left[\frac{u'(n_T^1)}{u'(n_{T-1}^{N_{T-1}+1})}\prod_{i=1}^{N_T}\mathrm{E}_{p_T^i}\frac{g^{\hat{s}(p_T^i)'}(n_T^{i+1})}{g^{\hat{s}(p_T^i)'}(n_T^i)}\right]\right]$$

$$= \delta - \frac{1}{t}\ln\left[\frac{u'(n_T^1)}{u'(x_0)}\left[\prod_{i=1}^{N_T}\mathrm{E}_{p_T^i}\frac{g^{\hat{s}(p_T^i)'}(n_T^{i+1})}{g^{\hat{s}(p_T^i)'}(n_T^i)}\right]\right] .$$

But then the same reasoning as in the proof of Proposition 4 can be applied. □

Proof of Proposition 7: At time $\bar{T}$ I have $m^2_{\bar{T}} = u_{\bar{T}}(x_{\bar{T}}) = \frac{x^\rho_{\bar{T}}}{\rho}$ and

$$m^1_{\bar{T}} = f^{-1}_{s_{\bar{T}}}\left[\mathrm{E}_{\bar{T}-1} f_{s_{\bar{T}}} \circ u_{\bar{T}}(x_{\bar{T}})\right] = \tfrac{1}{\rho}\left[\mathrm{E}_{\bar{T}-1}\left(\rho \frac{x^\rho_{\bar{T}}}{\rho}\right)^{\frac{\alpha_{s_{\bar{T}}}}{\rho}}\right]^{\frac{\rho}{\alpha_{s_{\bar{T}}}}}$$

where expectations are evaluated given $x_{\bar{T}-1}$ so that

$$\begin{aligned} m^1_{\bar{T}} &= \tfrac{x^\rho_{\bar{T}-1}}{\rho}\left[\mathrm{E}_{\bar{T}-1}\left(\frac{x^\rho_{\bar{T}}}{x^\rho_{\bar{T}-1}}\right)^{\frac{\alpha_{s_{\bar{T}}}}{\rho}}\right]^{\frac{\rho}{\alpha_{s_{\bar{T}}}}} = \tfrac{x^\rho_{\bar{T}-1}}{\rho}\left[\mathrm{E}_{\bar{T}-1} e^{g_{\bar{T}}\alpha_{s_{\bar{T}}}}\right]^{\frac{\rho}{\alpha_{s_{\bar{T}}}}} \\ &= \tfrac{x^\rho_{\bar{T}-1}}{\rho}\left[e^{\alpha_{s_{\bar{T}}}\mu_{\bar{T}} + \alpha^2_{s_{\bar{T}}}\frac{\sigma^2_{\bar{T}}}{2}}\right]^{\frac{\rho}{\alpha_{s_{\bar{T}}}}} = \tfrac{x^\rho_{\bar{T}-1}}{\rho} \underbrace{e^{\rho\mu_{\bar{T}} + \rho\alpha_{s_{\bar{T}}}\frac{\sigma^2_{\bar{T}}}{2}}}_{\equiv B_{\bar{T}}} . \end{aligned}$$

For earlier periods, I recursively find that the certainty equivalent utility levels m^2_t (uncertain only about the future) and m^1_t (uncertain about current and future consumption) are of the form:

$$\begin{aligned} m^2_t &= \tfrac{x^\rho_t}{\rho} + e^{-\delta}\, m^1_{t+1} = \tfrac{x^\rho_t}{\rho}\left\{1 + e^{-\delta}\, B_{t+1}\right\} , \\ m^1_t &= f^{-1}_{s_t}\left[\mathrm{E}_{t-1} f_{s_t}(m^2_t)\right] = \tfrac{1}{\rho}\left[\mathrm{E}_{t-1}\left(\rho\frac{x^\rho_t}{\rho}\right)^{\frac{\alpha_{s_t}}{\rho}}\right]^{\frac{\rho}{\alpha_{s_t}}} \left\{1 + e^{-\delta}\, B_{t+1}\right\} \\ &= \tfrac{x^\rho_{t-1}}{\rho} \underbrace{e^{\rho\mu_t + \rho\alpha_{s_t}\frac{\sigma^2_t}{2}}\left\{1 + e^{-\delta}\, B_{t+1}\right\}}_{\equiv B_t} = \tfrac{x^\rho_{t-1}}{x^\rho_t}\, e^{\rho\mu_t + \rho\alpha_{s_t}\frac{\sigma^2_t}{2}}\, m^2_t . \end{aligned} \tag{14}$$

Then I can evaluate the expressions

$$\frac{f^{s_t\prime}(m^2_t)}{f^{s_t\prime}(m^1_t)} = \frac{\alpha_{s_t}\left(\rho\, m^2_t\right)^{\frac{\alpha_{s_t}}{\rho}-1}}{\alpha_{s_t}\left(\rho\, m^1_t\right)^{\frac{\alpha_{s_t}}{\rho}-1}} = \left(\frac{m^2_t}{m^1_t}\right)^{\frac{\alpha_{s_t}-\rho}{\rho}} = \left(\frac{x_t}{x_{t-1}}\right)^{\alpha_{s_t}-\rho} e^{-(\alpha_{s_t}-\rho)(\mu_t + \alpha_{s_t}\frac{\sigma^2_t}{2})} \tag{15}$$

by virtue of equation (14). In the payoff period I have $u'(x_T) = x^{\rho-1}_T$ and

$$\begin{aligned} \frac{f^{s_T\prime}(m^2_T)}{f^{s_T\prime}(m^1_T)} u'(x_T) &= \left(\frac{x_T}{x_{T-1}}\right)^{\alpha_{s_T}-\rho} e^{-(\alpha_{s_T}-\rho)(\mu_T + \alpha_{s_T}\frac{\sigma^2_T}{2})}\, x^{\rho-1}_T \\ &= x^{\rho-1}_{T-1}\left(\frac{x_T}{x_{T-1}}\right)^{\alpha_{s_T}-1} e^{-(\alpha_{s_T}-\rho)(\mu_T + \alpha_{s_T}\frac{\sigma^2_T}{2})} . \end{aligned}$$

The following calculation takes expectations. The expression coincides with the one in the previous line for $t = T$ and $D_{T+1} = 1$.

$$x_{t-1}^{\rho-1}\ \mathrm{E}_{t-1}\ \left(\frac{x_t}{x_{t-1}}\right)^{\alpha_{s_t}-1} e^{-(\alpha_{s_t}-\rho)(\mu_t+\alpha_{s_t}\frac{\sigma_t^2}{2})}\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1}\ e^{(\alpha_{s_t}-1)\mu_t+(\alpha_{s_t}-1)^2\frac{\sigma_t^2}{2}} e^{-(\alpha_{s_t}-\rho)(\mu_t+\alpha_{s_t}\frac{\sigma_t^2}{2})}\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1}\ \exp\left((\alpha_{s_t}-1)\mu_t + (\alpha_{s_t}-1)^2\frac{\sigma_t^2}{2} - (\alpha_{s_t}-\rho)(\mu_t+\alpha_{s_t}\frac{\sigma_t^2}{2})\right)\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1}\ \exp\left(-\mu_t + (-2\alpha_{s_t}+1)\frac{\sigma_t^2}{2} + \rho(\mu_t+\alpha_{s_t}\frac{\sigma_t^2}{2})\right)\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1}\ \exp\left((\rho-1)\mu_t + (1-2\alpha_{s_t}+\rho\alpha_{s_t})\frac{\sigma_t^2}{2}\right)\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1}\ \exp\left(-(1-\rho)\mu_t + \left(1-\frac{\alpha_{s_t}}{\rho}\right)\left(1-(1-\rho)^2\right)\frac{\sigma_t^2}{2}\right)\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1}\ \underbrace{\exp\left(-\eta\mu_t + \eta^2\frac{\sigma_t^2}{2} + \mathrm{RIRA}_{s_t}\,|1-\eta^2|\frac{\sigma_t^2}{2}\right)}_{\equiv D_t}\ \prod_{\tau=t}^T D_{\tau+1}$$

$$= x_{t-1}^{\rho-1} \prod_{\tau=t-1}^T D_{\tau+1} \tag{16}$$

employing the definitions $\eta = 1-\rho$ and equation (9). The easiest way to verify the third last line is by expansion. Recursively employing equation (16) and making use of equation (15) yields

$$r = \delta - \frac{1}{t}\ln\left[\frac{1}{u'(x_0)}\left\{\prod_{t=1}^T \mathrm{E}_{p_t^i}\ \frac{f^{s_t\prime}(m_t^2)}{f^{s_t\prime}(m_t^1)}\right\}u'(x_T)\right] = \delta - \frac{1}{t}\ln\left[\frac{x_0^{\rho-1}}{x_0^{\rho-1}}\prod_{t=1}^T D_{t+1}\right]$$

implying equation (10) stated in the proposition for $\eta \neq 1$. For $\eta = 1$ find that the problem is smooth and $\lim_{\eta\to1}\mathrm{RIRA}_{s_t}\,|1-\eta^2| = \lim_{\rho\to0}(1-\frac{\alpha_{s_t}}{\rho})\rho(2-\rho) = -2\alpha_{s_t}$. □

Proof of Proposition 8: With defining $\zeta = \zeta_{\underline{s}} = \mathrm{RIRA}_{S^*(s^{obj})}\,|1-\eta^2|$ equation (11) is an immediate consequence of equation (11) for the set of least subjective lotteries. Fixing the parameter λ by the requirement $\frac{\zeta}{1-\lambda} = \zeta_{\overline{s}} \Rightarrow \lambda = 1 - \frac{\zeta}{\mathrm{RIRA}_{\overline{s}}}$ implies validity of the representation also for the most subjective lotteries (labeled $\overline{s}$). Moreover, the parameter $\lambda \in [0, 1)$ because the assumption of aversion to objective risk makes $\zeta > 0$ and the assumption of aversion to the subjectivity of belief implies $\zeta \leq \zeta_{\overline{s}}$ with a strict inequality in the case of strict aversion to the subjectivity of belief. By aversion to the subjectivity of belief it also follows that $\zeta_{S*^{-1}(s_t^*)}$ is monotonous in s_t^* for all $S^* \in \mathcal{S}^*$ because a more subjective lottery implies a higher degree of intertemporal risk aversion (Traeger 2010). In particular, there exists a map S^* on the set of finite elements $s \in S$ such that $\frac{\zeta}{1-\lambda s_t^*} = \zeta_{s_t}$. □

References

Arrow, K. & Hurwicz, L. (1972), An optimality criterion for decision-making under ignorance, *in* C. Carter & J. Ford, eds, 'Uncertainty and expectations in economics: Essays in honour of G.L.S. Shackl', Basil Blackwell, Oxford, pp. 1–11.

Bansal, R., Kiku, D. & Yaron, A. (2010), 'Long run risks, the macroeconomy, and asset prices', *American Economic Review: Papers & Proceedings* **100**, 542–546.

Bansal, R. & Yaron, A. (2004), 'Risks for the long run: A potential resolution of asset pricing puzzles', *The Journal of Finance* **59**(4), 1481–509.

Ellsberg, D. (1961), 'Risk, ambiguity and the savage axioms', *Quarterly Journal of Economics* **75**, 643–69.

Epstein, L. G. & Zin, S. E. (1989), 'Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework', *Econometrica* **57**(4), 937–69.

Gierlinger, J. & Gollier, C. (2008), 'Socially efficient discounting under ambiguity aversion', *Working Paper* .

Gilboa, I. & Schmeidler, D. (1989), 'Maxmin expected utility with non-unique prior', *Journal of Mathematical Economics* **18**(2), 141–53.

Gollier, C. (2002), 'Time horizon and the discount rate', *Journal of Economic Theory* **75**, 229–253.

Hardy, G., Littlewood, J. & Polya, G. (1964), *Inequalities*, 2 edn, Cambridge University Press. first puplished 1934.

Keynes, J. M. (1921), *A Treatise on Probability*, Macmillan, London.

Kimball, M. S. (1990), 'Precautionary saving in the small and in the large', *Econometrica* **58**(1), 53–73.

Kimball, M. & Weil, P. (2009), 'Precautionary saving and consumption smoothing across time and possibilities', *Journal of Money, Credit and Banking* **41**. First published as NBER Working Paper # 3976 in 1992.

Klibanoff, P., Marinacci, M. & Mukerji, S. (2005), 'A smooth model of decision making under ambiguity', *Econometrica* **73**(6), 1849–1892.

Klibanoff, P., Marinacci, M. & Mukerji, S. (2009), 'Recursive smooth ambiguity preferences', *Journal of Economic Theory* **144**, 930–976.

Kocherlakota, N. R. (1996), 'The equity premium: It's still a puzzle', *Journal of Economic Literature* **34**, 42–71.

Kreps, D. M. & Porteus, E. L. (1978), 'Temporal resolution of uncertainty and dynamic choice theory', *Econometrica* **46**(1), 185–200.

Leland, H. E. (1968), 'Saving and uncertainty: The precautionary demand for saving', *The Quarterly Journal of Economics* **82**(3), 465–473.

Nordhaus, W. D. (2007), 'A review of the Stern review on the economics of climate change', *Journal of Economic Literature* **45**(3), 686–702.

Traeger, C. (2007), 'Intertemporal risk aversion - or - wouldn't it be nice to know whether robinson is risk averse?'. CUDARE Working Paper No 1103.

Traeger, C. (2008), 'The social discount rate under intertemporal risk aversion and ambiguity', *CUDARE working paper 1092* .

Traeger, C. (2010), ‘Subjective risk, confidence, and ambiguity’, *CUDARE working paper 1103* .

Vissing-Jørgensen, A. & Attanasio, O. P. (2003), ‘Stock-market participation, intertemporal substitution, and risk-aversion’, *The American Economic Review* **93**(2), 383–391.

von Neumann, J. & Morgenstern, O. (1944), *Theory of Games and Economic Behaviour*, Princeton University Press, Princeton.

Weil, P. (1990), ‘Nonexpected utility in macroeconomics’, *The Quarterly Journal of Economics* **105**(1), 29–42.