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On the Characterization of Convex Domains With Non-Compact Automorphism Group

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In Loving Memory of

Duncan K. Law

ABSTRACT OF THE DISSERTATION

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by

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Dr. Bun Wong, Chairperson

In the field of several complex variables, the Greene-Krantz Conjecture has yet to be proven.

Conjecture 0.0.1 (Greene-Krantz Conjecture)**.** *Let D be a smoothly bounded domain in $\mathbb{C}^n$. Suppose there exists $\{g_j\} \subset Aut(D)$ such that $\{g_j(x)\}$ accumulates at a boundary point $p \in \partial D$ for some $x \in D$. Then ∂D is of finite type at p.*

The purpose of this dissertation is to prove the following result, yielding further evidence to the probable veracity of this conjecture.

Theorem 0.0.2. *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary. Suppose that there is a sequence $\{g_j\} \subset Aut(D)$ such that $\{g_j(z)\}$ accumulates at a boundary point some point $z \in D$. Then if $p \in \partial D$ is such an orbit accumulation point, ∂D contains no non-trivial analytic variety passing through p.*

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Chapter 1

Introduction

In any category, it is natural to ask which objects are equivalent. Over the past four decades, there has been an increase in interest in this question with regards to bounded domains $D \subset \mathbb{C}^n$. In one dimension, the task of classification is already complete: due to the Riemann Mapping Theorem, any simply connected bounded domain in the complex plane is biholomorphic to the unit disc. It may seem plausible, then, that an analogous result holds for bounded domains in $\mathbb{C}^n$. Unfortunately, this is not the case. Thus, in order to begin classifying all bounded domains in $\mathbb{C}^n$, one must first restrict to a smaller collection of domains, satisfying some additional property. This prompted a rise in the study bounded domains in $\mathbb{C}^n$ from the perspective of the group of automorphisms. Specifically, some began restricting their study to bounded domains in $\mathbb{C}^n$ with non-compact automorphism group. In 1989, Robert Greene and Steven Krantz formulated what has become known as the Greene-Krantz conjecture, in order to aid in the classification of such domains. Many have attempted to prove it over the subsequent years, but it remains unproven today. A precise statement of the conjecture is as follows:

Conjecture 1.0.3 (Greene-Krantz Conjecture:)**.** *Let D be a smoothly bounded domain in $\mathbb{C}^n$. Suppose there exists $\{g_j\} \subset Aut(D)$ such that $\{g_j(x)\}$ accumulates at a boundary point $p \in \partial D$ for some $x \in D$. Then ∂D is of finite type at p.*

In 2014, Lee, Thomas, and Wong proved the following result in support of the Greene-Krantz conjecture:

Theorem 1.0.4. *Let D be a smoothly bounded convex domain in $\mathbb{C}^n$. Suppose that there is a sequence $\{g_j\} \subset Aut(D)$ such that $\{g_j(z)\}$ accumulates non-tangentially at some boundary point for all $z \in D$. Then there does not exist a non-trivial analytic disc on ∂D passing through any orbit accumulation point on the boundary.*

Notice that the above result has a weaker conclusion than that of the Greene-Krantz conjecture, since finite type implies the absence of a non-trivial analytic disc, but the reverse implication does not hold in general.

In this paper, the condition of non-tangential convergence to the boundary will be removed in order to get one step closer to proving the Greene-Krantz Conjecture. The removal of the non-tangential condition is possible due to work by Kang-Tae Kim, in which it is shown that under similar hypotheses $Aut(D)$ contains a non-compact 1-parameter subgroup. That is, in this paper we give a proof of the following result:

Theorem 1.0.5. *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary. Suppose that there is a sequence $\{g_j\} \subset Aut(D)$ such that $\{g_j(z)\}$ accumulates at a boundary point* *for some point $z \in D$. Then if $p \in \partial D$ is such an orbit accumulation point, ∂D contains no non-trivial analytic variety passing through p.*

The remainder of this dissertation will proceed as follows:

In Chapter 2, a general study of bounded domains in $\mathbb{C}^n$ with non-compact automorphism group will be presented, as well as a discussion of the concept of finite type. Emphasis will be placed on key examples and results that assist in the ultimate goal of classification. The purpose of this chapter is to provide the reader with the basic definitions that will be used throughout this dissertation, as well as a general framework from which one can begin to understand the importance of the main result.

In Chapter 3, invariant metrics and measures, two very important tools that will be utilized throughout the proof of the main theorem, will be introduced. A short discussion of their important properties will follow.

The important previous results called upon in the proof of the main theorem will be presented in Chapter 4.

The main result will be proven in Chapter 5. Visualization of complex domains in $\mathbb{C}^n$ for $n \geq 2$ can be difficult, therefore, in order to increase transparency, two separate proofs will be given, one in $\mathbb{C}^2$ and a generalized version in $\mathbb{C}^n$. The basic idea behind both proofs is the Poincaré Theorem, which states that the ball and the polydisc are not biholomorphic. In particular, assuming the domain D is not variety-free at a boundary accumulation point p, the boundary ∂D will be geometrically flat along this variety. Then, near a strongly pseudoconvex boundary point, the domain D looks like a ball, whereas, near a flat boundary point, the domain D looks like a polydisc. The non-compactness of the automorphism group allows one to mediate between these two types of boundary points, giving rise to a contradiction. The likeness to a ball and a polydisc near a strongly pseudoconvex and flat boundary point respectively, is codified precisely by the quotient of the Carathéodory and Kobayashi measures.

Chapter 2

Background

2.1 Domains with non-compact automorphism group

Throughout this dissertation, a bounded domain in $\mathbb{C}^n$ will be denoted by D, and the automorphism group of D will be denoted by $Aut(D)$. Note that an element g of $Aut(D)$ is a biholomorphic map from D onto itself. $Aut(D)$ is not only a group, the operation being function composition, it is also a topological group, the topology being given by the compact-open topology. Notice that D being a subset of $\mathbb{C}^n$ implies that D is equipped with a metric, and thus the compact-open topology of $Aut(D)$ coincides with the topology of uniform convergence on compact sets. Furthermore, H. Cartan showed that $Aut(D)$ is a Lie Group. Here, the focus will be on domains whose automorphism groups are non-compact. Before a precise definition is presented, a few other definitions are needed.

Definition 2.1.1. *Let G be a topological group and X a topological (Hausdorff) space. G acts on X if there exists a continuous map $\phi : G \times X \to X$, $\phi(g,x) = gx$, such that $\phi(e,x) = ex = x$ for all $x \in X$ and $\phi(gg', x) = \phi(g, \phi(g', x))$ for all $g, g' \in G$ and $x \in X$.*

Definition 2.1.2. *Let G and X be as in the previous definition. The orbit of $x \in X$ under the action of G is the set $\{\phi(g,x) : g \in G\} \subset X$.*

Definition 2.1.3. *A map $f : D \to \tilde{D}$, where $D \subset \mathbb{C}^n$ and $\tilde{D} \subset \mathbb{C}^m$, is called proper if for any compact set $\tilde{K} \subset \tilde{D}$, the set $f^{-1}(\tilde{K})$ is compact in D.*

Notice that this is equivalent to the following: for any sequence $\{z_j\} \subset D$ which has no limit point in D, the sequence $\{f(z_j)\}$ has no limit point in $\tilde{D}$.

Definition 2.1.4. *If G and X are as defined in definition 2.1.1 and are locally compact, then the action of G on X is proper if the map $G \times X \to X \times X$, defined by $(g,x) \mapsto (\phi(g,x), x)$, is proper.*

It is known that the action of $Aut(D)$ on D is proper, and Montel's Theorem gives that for every sequence of holomorphic functions $\{f_j : D \to D'\}$, with $D, D' \subset \mathbb{C}^n$ and D' a bounded domain, there exists a subsequence $\{f_{j_n}\}$ which converges to a holomorphic function $f : D \to \mathbb{C}^n$. Thus if $\{g_j\} \subset Aut(D)$, there are two cases: either the limiting holomorphic function g is in $Aut(D)$, or it is not. In the case at hand, $g_j(z) \longrightarrow p \in \partial D$ as $j \longrightarrow \infty$ for some $z \in D$, which shows that if g denotes $\lim_{j\to\infty} g_j$, $g \notin Aut(D)$. The fact that the action is proper implies g maps all of D into its boundary, that is, $g(D) \subset \partial D$. Therefore, the orbit of any point $z \in D$ is non-compact. The point p is called a *boundary accumulation point* for the action of $Aut(D)$ on D. More precisely, $p \in \partial D$ is a boundary accumulation point for the action of $Aut(D)$ on D if there exists a point $z \in D$ and a sequence $\{g_j\} \subset Aut(D)$ such that $g_j(z) \longrightarrow p$ as $j \longrightarrow \infty$.

Conversely, assume $D \subset \mathbb{C}^n$ is bounded, with $p \in \partial D$ a boundary orbit accumulation point. Then it can be shown that $Aut(D)$ is non-compact:

Claim 2.1.5. *If ∂D contains a boundary orbit accumulation point p, then $Aut(D)$ is non-compact.*

Proof. Assume, by way of contradiction, that $Aut(D)$ is compact. Then, for any sequence $\{\phi_j\} \subset Aut(D)$, there exists a subsequence $\{\phi_{j_\nu}\} \subset \{\phi_j\}$ such that $\phi_{j_\nu} \longrightarrow \phi \in Aut(D)$. Consider the sequence $\{g_j\} \subset Aut(D)$. By assumption, there exists $\{g_{j_\nu}\} \subset \{g_j\}$ such that $g_{j_\nu} \longrightarrow g \in Aut(D)$ as $\nu \longrightarrow \infty$. In particular, $g(q) = p \in D$ for some $q \in D$, which implies that $p \in D \cap \partial D$, contradicting the fact that D is open. Therefore, $Aut(D)$ is compact. $\square$

Thus, there is no loss in assuming that $Aut(D)$ being non-compact implies at least one orbit of the action of $Aut(D)$ on D is non-compact.

What follows are a few examples of bounded domains with non-compact automorphism groups.

Example 2.1.6. *The unit disc in* $\mathbb{C}$*.*

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc, where $|\cdot|$ denotes the Euclidean norm of z in $\mathbb{C}$. Then,

$$Aut(\Delta) = \left\{ e^{i\theta} \frac{z-a}{1-\overline{a}z} : a \in \Delta,\ \theta \in [0, 2\pi] \right\}.$$

To determine why $Aut(\Delta)$ is non-compact, the following proposition is needed:

Proposition 2.1.7. *Let D be a bounded domain in $\mathbb{C}^n$ with a transitive automorphism group, i.e. let D be homogeneous. Then $Aut(D)$ is non-compact.*

Proof. Let D be a bounded domain in $\mathbb{C}^n$ with a transitive automorphism group. That is, given any two points $a, b \in D$, there exists $\phi \in Aut(D)$ such that $\phi(a) = b$. Let $z \in D$. By they transitivity of $Aut(D)$, the orbit of z is

$$\{w \in D : w = \phi(z), \text{ for some } \phi \in Aut(D)\} = D.$$

Since D is open, it is not compact, hence the orbit of z is non-compact, and hence $Aut(D)$ is non-compact. □

This proposition can now be used to show that $Aut(\Delta)$ is non-compact, because it is known that $Aut(\Delta)$ is transitive. Specifically, given any two points $a, b \in \Delta$, let

$$\phi_a(z) := \frac{z-a}{1-\overline{a}z} \quad \text{and} \quad \phi_{-b} := \frac{z+b}{1+\overline{b}z}\ .$$

Then both ϕ_a and ϕ_{-b} are in $Aut(\Delta)$. Furthermore, $\phi_{-b} \circ \phi_a(a) = \phi_{-b}(0) = b$, which implies that $Aut(\Delta)$ is transitive. Thus, but the previous proposition, $Aut(\Delta)$ is non-compact.

Example 2.1.8. *The unit ball in* $\mathbb{C}^n$.

Let $B_n := \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \|z\| := \Sigma |z_k|^2 < 1\}$ denote the unit ball. To write down the elements of $Aut(B_n)$ explicitly, recall that

$$U(n) := \{A \in M_n(\mathbb{C}) : A\overline{A}^t = \overline{A}^t A = I\}$$

is the Lie group, under matrix multiplication, of unitary matrices. Importantly, the elements of $U(n)$ preserve the Euclidean norm, that is, they correspond to complex rotations. Furthermore, consider the collection of maps $\{\phi_a\}$, where

$$\phi_a(z_1, \ldots, z_n) := \left(\frac{z_1 - a}{1 - \overline{a} z_1}, \frac{\sqrt{1 - |a|^2} z_2}{1 - \overline{a} z_1}, \ldots, \frac{\sqrt{1 - |a|^2} z_n}{1 - \overline{a} z_1} \right), \text{ for } |a| < 1.$$

Note that $\phi_a(a, 0, \ldots, 0) = (0, \ldots, 0)$ and that ϕ_a is an automorphism of the ball. Therefore, $Aut(B_n)$ is the group generated by $U(n)$ and $\{\phi_a\}$. That is, every automorphism of the ball is a composition of elements from $U(n)$ or $\{\phi_a\}$. Again, it can be shown that $Aut(B_n)$ is non-compact because:

Claim 2.1.9. *$Aut(B_n)$ is transitive.*

Proof. Choose $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in B_n$. Then there exists $\Phi_a \in U(n)$ such that $\Phi_a(a) = (a_1, 0, \ldots, 0)$, i.e. Φ_a rotates the a onto the z_1-axis. Choose $\Phi_{-b} \in U(n)$ such that $\Phi_{-b}(b_1, 0, \ldots, 0) = b$, i.e. Φ_{-b} is the inverse of Φ_b. let ϕ_{a_1}, ϕ_{-b_1} be the automorphisms of the ball as described above. Note that $\phi_{-b_1} = (\phi_{b_1})^{-1}$. Then $(\Phi_{-b} \circ \phi_{-b} \circ \phi_a \circ \Phi_a)(a) = \Phi_{-b}(\phi_{-b}(\phi_a(\Phi_a(a)))) = \Phi_{-b}(\phi_{-b}(\phi_a(a_1, 0, \ldots, 0))) = \Phi_{-b}(\phi_{-b}(0)) = \Phi_{-b}(b_1, 0, \ldots, 0) = b$. More succinctly, $(\Phi_{-b} \circ \phi_{-b} \circ \phi_a \circ \Phi_a)(a) = b$, and hence $Aut(B_n)$ is transitive. □

Therefore, by Proposition 2.1.7, $Aut(B_n)$ is non-compact.

Example 2.1.10. *The unit polydisc in $\mathbb{C}^n$.*

let $\Delta_n \coloneqq \{z = (z_1, \ldots, z_n) : |z_k| < 1 \text{ for all } 1 \leq k \leq n\}$ denote the unit polydisc in $\mathbb{C}^n$. Notice that $\Delta_n = \Delta \times \cdots \times \Delta$, n times. Then

$$Aut(\Delta_n) = \left\{ \phi(z) = \phi(z_1, \ldots, z_n) \coloneqq \left(e^{i\theta_1} \frac{z_{\sigma(1)} - a_1}{1 - \overline{a}_1 z_{\sigma(1)}}, \ldots, e^{i\theta_n} \frac{z_{\sigma(n)} - a_n}{1 - \overline{a}_n z_{\sigma(n)}} \right) \right\},$$

where $a \in \Delta_n$, $0 \leq \theta_k \leq 2\pi$, and $\sigma \in S_n$, where S_n is the symmetric group on n letters. Then as usual, the fact that $Aut(\Delta_n)$ is non-compact is a result of the following claim.

Claim 2.1.11. *$Aut(\Delta_n)$ is transitive.*

Proof. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \Delta_n$. Consider the following automorphisms of Δ_n:

$$\phi_a(z) = \left(\frac{z_1 - a_1}{1 - \overline{a}_1 z_1}, \ldots, \frac{z_n - a_n}{1 - \overline{a}_n z_n} \right)$$

and

$$\phi_{-b}(z) = \left(\frac{z_1 + b_1}{1 + \overline{b}_1 z_1}, \ldots, \frac{z_n + b_n}{1 + \overline{b}_n z_n} \right).$$

Then, $(\phi_{-b} \circ \phi_a)(a) = \phi_{-b}(0) = b$, which implies that that $Aut(\Delta_n)$ is transitive. □

Thus it follows, that $Aut(\Delta_n)$ is non-compact.

Example 2.1.12. *The ellipsoid, or "egg" domain in* $\mathbb{C}^2$.

Let $E_m := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ be the egg domain in $\mathbb{C}^2$, where $m \in \mathbb{Z}^+$. Then,

$$Aut(E_m) = \left\{ (z_1, z_2) \mapsto \left(\frac{z_1 - a}{1 - \bar{a} z_1}, \left(\frac{\sqrt{1 - |a|^2}}{1 - \bar{a} z_1} \right)^{1/m} z_2 \right) : |a| < 1 \right\}.$$

In this case, $Aut(E_m)$ is non-compact because E_m has a boundary orbit accumulation point.

Claim 2.1.13. *The point* $(1, 0)$ *is a boundary orbit accumulation point for the action of* $Aut(E_m)$ *on* E_m.

Proof. Choose a_j, $0 \leq a_j \leq 1$, such that as $j \longrightarrow \infty$, $a_j \longrightarrow 1$. let $z = (z_1, z_2) \in E_m$ and $\phi_{a_j} \in Aut(E_m)$. Then (1,0) is a boundary orbit accumulation point of the action of $Aut(E_m)$ on E_m, since $\phi_{a_j}(z) \longrightarrow (1, 0)$ as $j \longrightarrow \infty$. This implies that $Aut(E_m)$ is non-compact by Claim 2.1.5. □

After examining this collection of examples, it is natural to ask if any of these domains are biholomorphic. More generally, can the original desire for a higher-dimensional Riemann Mapping theorem be found for the set of bounded domains with non-compact automorphism group? As the following theorem of Poincaré demonstrates, without the imposition of additional conditions upon the domains under consideration, no such result holds.

Theorem 2.1.14 (Poincaré's Theorem)**.** *The ball* B_n *is not biholomorphic to the polydisc* Δ_n *for* $n \geq 2$.

Because this result of Poincaré is so central to the purpose of this paper, a detailed proof is provided below preceded by all necessary definitions and preliminary arguments.

Definition 2.1.15. *For a bounded domain D, the automorphism group $Aut(D)$ becomes a topological group by defining a distance between two automorphisms as follows:*

$$d(\phi_1, \phi_2) = \sup_{z \in D} |\phi_1(z) - \phi_2(z)|.$$

Then let $Aut^{Id}(D)$ denote the subgroup of all automorphisms in the connected component of the identity. Further, given $a \in D$, let $Aut_a(D)$ denote the subgroup of automorphisms which leave a invariant.

Poincaré noticed that when observing biholomorphisms of D, the automorphisms groups contain essential information:

Lemma 2.1.16. *If D_1 is biholomorphic to D_2, then the respective automorphism groups $Aut(D_1)$ and $Aut(D_2)$ are isomorphic groups. Furthermore, if there are $a_1 \in D_1$ and $a_2 \in D_2$ for which there exists a biholomorphic map $f : D_1 \to D_2$ with $f(a_1) = a_2$, then $Aut_{a_1}(D_1)$ and $Aut_{a_2}(D_2)$ are isomorphic groups. In addition, $Aut^{Id}(D_1)$ and $Aut^{Id}(D_2)$ are isomorphic groups, as are $Aut^{Id}_{a_1}(D_1)$ and $Aut^{Id}_{a_2}(D_2)$.*

Proof. Let $\phi : D_1 \to D_2$ be a biholomorphic map from D_1 to D_2. Then

$$\phi \mapsto f \circ \phi \circ f^{-1}$$

is a group homomorphism from $Aut(D_1)$ to $Aut(D_2)$. Because the map is invertible, it is a group isomorphism. □

Proposition 2.1.17. *$Aut_0^{Id}(B_n)$ is non-abelian.*

Proof. Consider the non-abelian special unitary group $SU(n)$ of all $n \times n$ matrices A such that $AA^* = Id_n$ and $det(A) = 1$. $SU(n)$ is a subgroup of $Aut_0^{Id}(B_n)$, because $A \in SU(n)$ defines a biholomorphism $z \mapsto Az$ on B_n which leaves 0 invariant. $\square$

Proposition 2.1.18. *For every $a \in \Delta_n$, the group $Aut_a^{Id}(\Delta_n)$ is abelian.*

The proof of Proposition 2.1.18 follows directly from the following results due to Henri Cartan:

Proposition 2.1.19 (Cartan Uniqueness Theorem)**.** *Let D be a bounded domain in $\mathbb{C}^n$ and let $a \in D$. If $f \in Aut_a(D)$ satisfies $f'(a) = 1$, then $f(z) = z$ for all $z \in D$.*

Proof. It can be assumed that $a = 0$ after a change of coordinates (replacing D with $D - a$) if necessary. Then since D is bounded, $\overline{D} \subset \Delta_n(0, R)$ for some $R > 0$. Recall that every $f \in Aut_0(D)$ has a Taylor expansion centered at the origin, $f(z) = \sum_n a_n z^n$. Cauchy's estimate gives that $|a_n| \leq Mr^{-n}$, where r is such that $\Delta_n(0, r) \subset D$ and $M = \sup_{z \in \overline{D}} |f(z)|$. Then by assumption, f has a Taylor expansion

$$f(z) = z + f_N(z) + \cdots$$

where f_k are n-tuples of homogeneous polynomials of degree k, and where N is choses to be the smallest possible. Then the kth iterate $f^k = f \circ \cdots \circ f$ of f has Taylor expansion

$$f^k(z) = z + k\dot{f}_N(z)$$

which violates the about Cauchy estimate for large k unless $f_N = 0$. But if $f(z) = z$ in $\Delta_n(0, r)$, then $f(z) = z$ in D by the principle of analytic continuation. □

Definition 2.1.20. *A bounded domain $D \subset \mathbb{C}^n$ is called a circular domain if $z \in D$ implies that $k_\theta(z) = e^{i\theta}$ for all $z \in D$ and all $\theta \in \mathbb{R}$.*

Corollary 2.1.21 (Cartan)**.** *Let D be a bounded circular domain in $\mathbb{C}^n$ and assume that $0 \in D$ and $f \in Aut_0(D)$. Then f is linear.*

Proof. Assuming D is a circular domain and $0 \in D$, one has that $k_\theta \in Aut_0(D)$. Define

$$g = k_{-\theta} \circ f^{-1} \circ k_\theta \circ f.$$

Then $g'(0) = k'_{-\theta}(0) \circ (f^{-1})'(0) \circ k'_\theta \circ f'(0) = Id$, so that by the previous proposition $g(z) = z$. This implies that

$$k_\theta \circ f = f \circ k_\theta.$$

If $f = (f_1, f_2, \ldots, f_n)$, then $f_j(e^{i\theta} z) = e^{i\theta} f_j(z)$. Let $f_j(z) = \sum_k a_k z^k$. Then

$$e^{i\theta} a_k = e^{i|k|\theta} a_k,$$

implying that $a_k = 0$ for all $|k| \geq 1$. □

Corollary 2.1.22. *Every $f = (f_1, f_2, \ldots, f_n) \in Aut(\Delta_n)$ has the form*

$$f_j(z) = e^{i\theta_j} \frac{z_{p(j)} - a_j}{1 - \bar{a}_j z_{p(j)}},$$

where $\theta_j \in \mathbb{R}$, $a \in \Delta_n$, and p is a permutation of the multi-index $j = (j_1, j_2, \ldots, j_n)$.

Proof. The map $f_j(z)$ is clearly an automorphism. Denote f_j by σ_a if $\theta_j = 0$ and $p = Id$. Then given $f \in Aut(\Delta_n)$, the automorphism $\sigma_a \circ f$ leaves 0 invariant. One can therefore assume that $f \in Aut_0(\Delta_n)$. Because $f(\Delta_n) \subset \Delta_n$, we have $\sum_{k=1}^{n} |A_{kj}| \leq 1$. However, by choosing sequences $z^{(n)} = (0, \ldots, 0, 1 - \frac{1}{n}, 0, \ldots, 0)$ converging to the distinguished boundary $\mathbb{T}^n$ of Δ_n, one sees that the sequence

$$f(z^{(n)}) = (1 - \frac{1}{n})(A_{1j}, \ldots, A_{2j})$$

converges to the distinguished boundary of Δ_n. Therefore

$$|A_{q(j)j}| :- \max_{k=1,\ldots,n} |A_{kj}| = 1.$$

Then since $\sum_{k=1}^{n} |A_{kj}| \leq 1$, one has that A_{jk} is a permutation matrix which has non-vanishing entries of norm 1 only at entries $A_{q(j)j}$. If p is the inverse permutation of q, then $f_k(z) = A_{k,p(k)} z_{p(k)}$ with $|A_{k,p(k)}| = 1$. □

Finally one can prove the famous result of Poincaré, theorem 2.1.14:

Proof. Assume by way of contradiction, that f is a biholomorphic map between$B_n(0,1)$ and $\Delta_n(0,1)$. From Lemma 2.1.16 and the transitivity of $Aut(\Delta_n(0,1))$, one can conclude that $Aut_0^{Id}(B_n(0,1))$ and $Aut_{f(0)}^{Id}(\Delta_n(0,1))$ are isomorphic as groups. But by Proposition 2.1.17, $Aut_0^{Id}(B_n(0,1))$ is non-abelian, while Proposition 2.1.18 shows that $Aut_{f(0)}^{Id}(\Delta_n(0,1))$ is abelian. This is a contradiction. □

With this result in hand, a logical next question to ask might be whether or not a classification result holds for a subset of the bounded domains in $\mathbb{C}^n$ with non-compact automorphism group. In other words, if some additional constraint is placed upon the domains under consideration, can any classification be obtained? The answer is yes. In the following section a discussion of the notion of *pseudoconvexity*, together with a known classification result, will be presented.

2.2 Convexity, pseudoconvexity, and a characterization theorem of the ball

Before introducing the Ball Characterization Theorem, the definitions of convexity and pseudoconvexity will be given, along with several important properties of pseudoconvex domains. For further in depth study, please refer to books by Steven G. Krantz [6] and R.C. Gunning [3].

Definition 2.2.1. *A bounded domain $D \subset \mathbb{C}^n$ is (geometrically) convex if D contains the entire line segment joining any pair of its points.*

All convex domains are domains of holomorphy, unfortunately, convexity is not preserved under holomorphic mappings. For example, the unit disc $\Delta \subset \mathbb{C}$ is convex, but its image under the mapping $z \mapsto (4+z)^4$ is not. Thus some less rigid geometric condition is required to characterize bounded domains, preferably a biholomorphically invariant version of convexity.

Definition 2.2.2. *Let $D \subset \mathbb{C}^n$ be a bounded domain with a C^2 boundary (i.e. the defining funcrtion ρ for the boundary is C^2). Then ∂D is pseudoconvex at p if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0 \quad \text{for all } w \in T_p^{1,0}(\partial D),$$

where

$$T_p^{1,0}(\partial D) := \left\{ w \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0 \right\}.$$

$T_p^{1,0}$ is called the *complex tangent space* to the boundary, ∂D, at the point p. If the inequality above is strict, then p is called a *strongly pseudoconvex* point. That is, a point $p \in \partial D$ is a point of pseudoconvexity (resp. strong pseudoconvexity) if the complex Hessian (also known as the Levi form) is positive semi-definite (resp. positive definite) at p on the complex tangent space. If every point $p \in \partial D$ is a point of pseudoconvexity (resp. strong pseudoconvexity), then the domain D is said to be *pseudoconvex* (resp. *strongly pseudoconvex*). For the sake of simplicity, let $T_p(\partial D) := T_p^{1,0}(\partial D)$ for the remainder of this dissertation.

What follows are a few important properties of pseudoconvex domains.

(1) Pseudoconvexity is independent of the choice of defining function.

Proof. Let ρ and $\tilde{\rho}$ be two defining functions of ∂D in a neighborhood U of p, for $p \in \partial D$. Then there exists a C^1 function h defined in U such that $\tilde{\rho} = h\rho$, where $h(z) > 0$ for all $z \in U$. Hence,

$$\begin{aligned}
\frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(p) &= \frac{\partial^2 (\rho h)}{\partial z_j \partial \bar{z}_k}(p) = \frac{\partial}{\partial z_j}\left(\frac{\partial(\rho h)}{\partial \bar{z}_k}(p)\right) \\
&= \frac{\partial}{\partial z_j}\left(\frac{\partial \rho}{\partial \bar{z}_k}(p) \cdot h(p) + \rho(p) \cdot \frac{\partial h}{\partial \bar{z}_k}\right) \\
&= \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) \cdot h(p) + \frac{\partial \rho}{\partial \bar{z}_k}(p) \cdot \frac{\partial h}{\partial z_j}(p) + \frac{\partial \rho}{\partial z_j}(p) \cdot \frac{\partial h}{\partial \bar{z}_k}(p) + \rho(p) \cdot \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}(p) \\
&= \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) \cdot h(p) + \frac{\partial \rho}{\partial \bar{z}_k}(p) \cdot \frac{\partial h}{\partial z_j}(p) + \frac{\partial \rho}{\partial z_j}(p) \cdot \frac{\partial h}{\partial \bar{z}_k}(p),
\end{aligned}$$

where the last equality follows from the fact that $\rho(p) = 0$. Therefore,

$$
\begin{aligned}
\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k &= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \\
&\quad + \sum_{j,k=1}^{n} \left(\frac{\partial \rho}{\partial \bar{z}_k}(p) \cdot \frac{\partial h}{\partial z_j}(p) + \frac{\partial \rho}{\partial z_j}(p) \cdot \frac{\partial h}{\partial \bar{z}_k}(p) \right) w_j \bar{w}_k \\
&= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \\
&\quad + 2\mathrm{Re} \sum_{j,k=1}^{n} \left(\frac{\partial \rho}{\partial z_j}(p) \cdot \frac{\partial h}{\partial \bar{z}_k}(p) w_j \bar{z}_k \right) \\
&= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \quad \text{if } \; w \in T_p(\partial D).
\end{aligned}
$$

Therefore, since $h(p) > 0$, $p \in \partial D$ is a pseudoconvex point with respect to ρ if and only if it is a pseudoconvex point with respect to $\tilde{\rho}$. Therefore, the pseudoconvexity of a boundary point is irrespective of the choice of defining function. □

(2) Pseudoconvexity is preserved under biholomorphic mappings.

Proof. Let $\Phi : D \to \mathbb{C}^n$ be biholomorphic onto its image, and let D' denote the image $\Phi(D)$. Further, assume that Φ is biholomorphic in a neighborhood of $p \in \partial D$. Then $\Phi(z) = \Phi(z_1, \ldots, z_n) = (\Phi_1(z_1), \ldots, \Phi_n(z_n)) = (z'_1, \ldots, z'_n)$. Let $\rho : U \to \mathbb{R}$ be a local defining function for ∂D, for U an open set. Then $\tilde{\rho} \coloneqq \rho \circ \Phi^{-1}$ is a local defining function for $\partial D'$. Choose $p \in \partial D$ and $w \in T_p(\partial D)$. Then $\Phi(p) \in \partial D'$ and $w' \in T_{\Phi(p)}(\partial D')$, where

$$
w' = \begin{pmatrix} w'_1 \\ \vdots \\ w'_n \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial z_1}(p) & \ldots & \frac{\partial \Phi_1}{\partial z_n}(p) \\ \vdots & & \vdots \\ \frac{\partial \Phi_n}{\partial z_1}(p) & \ldots & \frac{\partial \Phi_n}{\partial z_n}(p) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \sum \frac{\partial \Phi_1}{\partial z_j}(p) w_j \\ \vdots \\ \sum \frac{\partial \Phi_n}{\partial z_j}(p) w_j \end{pmatrix}.
$$

Now, since $\tilde{\rho} \coloneqq \rho \circ \Phi^{-1}$, one can see that $\rho = \tilde{\rho} \circ \Phi$, implying that

$$\begin{aligned}\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) = \frac{\partial^2 (\tilde{\rho} \circ \Phi)}{\partial z_j \partial \bar{z}_k}(p) &= \frac{\partial}{\partial z_j}\left(\frac{\partial(\tilde{\rho} \circ \Phi)}{\partial \bar{z}_k}(p)\right) \\ &= \sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_l}(\Phi(p)) \cdot \frac{\partial \Phi_m}{\partial z_j}(p) \cdot \frac{\partial \bar{\Phi}_l}{\partial \bar{z}_k}(p)\end{aligned}$$

by the chain rule. Hence,

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k &= \sum_{j,k=1}^{n}\left(\sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_l}(\Phi(p)) \cdot \frac{\partial \Phi_m}{\partial z_j}(p) \cdot \frac{\partial \bar{\Phi}_l}{\partial \bar{z}_k}(p)\right) w_j \bar{w}_k \\ &= \sum_{l,m=1}^{n}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_l}(\Phi(p)) \cdot \frac{\partial \Phi_m}{\partial z_j}(p) w_j \cdot \frac{\partial \bar{\Phi}_l}{\partial \bar{z}_k}(p) \bar{w}_k\right) \\ &= \sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_l}(\Phi(p)) w'_m \bar{w}'_l \ ,\end{aligned}$$

which implies that the Levi form is preserved under biholomorphic mappings. In other words, pseudoconvexity is preserved under biholomorphism. □

(3) If $p \in \partial D$ is a strongly pseudoconvex point, then there exists a neighborhood U containing p such that for all $q \in \partial D \cap U$, q is strongly pseudoconvex.

To prove this result, a technical lemma (that can be found in Chapter 3 of [6]) is needed.

Lemma 2.2.3. *If D is strongly pseudoconvex, then D has a defining function $\tilde{\rho}$ such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq C|w|^2$$

for all $p \in \partial D$ and $w \in \mathbb{C}^n$, where $C > 0$.

Using this lemma, we can now prove property **(3)**.

Proof. By Lemma 2.2.3, there exists a defining function $\tilde{\rho}$ for D such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq C|w|^2$$

for all $w \in \mathbb{C}^n$. In particular,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k > 0$$

for all $w \neq 0$, $w \in \mathbb{C}^n$. Since $\tilde{\rho}$ is C^2, the function

$$q \xrightarrow{\Phi} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k$$

is continuous in a neighborhood U of p, which implies that for all $q \in U \cap \partial D$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k > 0$$

for all $w \neq 0$, $w \in \mathbb{C}^n$ by the continuity of Φ. This implies that $q \in U \cap \partial D$ is strongly pseudoconvex, completing the proof. $\square$

Notice that the analogous result for pseudoconvex boundary points is false (see the examples below for details).

(4) Every domain in $\mathbb{C}$ with a C^2 boundary is vacuously pseudoconvex.

Proof. Let D be a domain in $\mathbb{C}$ with C^2 boundary. That is, the defining function ρ is C^2. Then for all $p \in \partial D$,

$$\nabla \rho(p) = \frac{\mathrm{d}\rho}{\mathrm{d}z}(p) \neq 0,$$

which implies that $w \in T_p(\partial D)$ if and only if $w = 0$. This implies that $T_p(\partial D) = \{0\}$, giving that D is pseudoconvex, since the condition for pseudoconvexity in one dimension,

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}z\mathrm{d}\bar{z}}(p)w\bar{w} \geq 0,$$

is always satisfied. □

To better illustrate this important notion of pseudoconvexity, a few examples are presented below. For clarity and in order to keep calculations simple, only domains in $\mathbb{C}^2$ are considered.

Example 2.2.4. *The unit ball B_2.*

Recall that $B_2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) := |z_1|^2 + |z_2|^2 - 1 < 0\}$. The complex Hessian for ρ is the matrix

$$\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies that

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k = |w_1|^2 + |w_2|^2 > 0$$

for all $w \in \mathbb{C}^2$, $w \neq 0$. Since this is true for every $p \in \partial D$, the unit ball B_2 is pseudoconvex.

Example 2.2.5. *The ellipsoid, or "egg" domain E_m.*

Recall that $E_m \coloneqq \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) \coloneqq |z_1|^2 + |z_2|^{2m} - 1 < 0\}$. The complex Hessian for ρ is the matrix

$$\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right) = \begin{pmatrix} 1 & 0 \\ & \\ 0 & m^2 z_2^{m-1} \bar{z}_2^{m-1} \end{pmatrix},$$

which implies that for any $p \in \partial D$,

$$\begin{aligned} \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k &= \begin{pmatrix} \bar{w}_1 & \bar{w}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ & \\ 0 & m^2 z_2^{m-1} \bar{z}_2^{m-1} \end{pmatrix} \begin{pmatrix} w_1 \\ \\ w_2 \end{pmatrix} \\ &= |w_1|^2 + m^2 |p_2|^{2m-2} |w_2|^2, \end{aligned}$$

which is greater than 0 only if $|p_2|^{2m-2} \neq 0$. That is, any point $p = (p_1, p_2) \in \partial D$ is a point of strong pseudoconvexity if $p_2 \neq 0$. Therefore, points of pseudoconvexity are of the form $(e^{i\theta}, 0)$, for the complex Hessian is positive semi-definite at these points (take $w = (w_1, w_2) \in \mathbb{C}^2$, $w_1 = 0$).

One can now return to the question of classification, for adding the restriction of strong pseudoconvexity to the set of bounded domains yields the following important result proved originally by Bun Wong [12]. J.P. Rosay later described a local version of this result (see [9]) using the same method introduced in [12].

Theorem 2.2.6. *Let D be a strongly pseudoconvex bounded domain with smooth boundary in $\mathbb{C}^n$ with non-compact automorphism group. Then D is biholomorphic to the unit ball B_n.*

Consider also the following local version of the above theorem due to Bun Wong:

Theorem 2.2.7. *Let $D \subset \mathbb{C}^n$ be any bounded domain with a strongly pseudoconvex boundary point $p \in \partial D$. Suppose further that there exist $K \Subset D$, $\{z^j\} \in K$, and $\{g_j\} \in Aut(D)$ such that $\{g_j(z^j)\} \longrightarrow p$. Then D is biholomorphic to the unit ball $B_n \subset \mathbb{C}^n$.*

Given these results, the problem of classifying smoothly bounded strongly pseudoconvex domains with non-compact automorphism group is complete. What happens, then, if the hypothesis of strong pseudoconvexity is relaxed to pseudoconvexity? Can one obtain any classification in this case? Once again, the answer is yes only after the addition of further conditions upon the domain. This classification is illustrated by the following result (see [1]):

Theorem 2.2.8. *Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ of finite type with non-compact automorphism group such that the Levi form of ∂D has no more than one zero eigenvalue at any point. Then D is biholomorphic to the ellipsoid $E_m \subset \mathbb{C}^{n+1}$, where*

$$E_m = \{(w, z_1, z_2, \dots, z_n) \in \mathbb{C}^{n+1} : |w|^2 + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 < 1\}$$

for some integer $m \geq 1$.

This result classifies all smoothly bounded pseudoconvex domains of finite type. This raises the question: What is finite type? A discussion of the concept of finite type in two dimensions follows in the proceeding section.

2.3 Concept of finite type

The concept of finite type encapsulates the geometry of a boundary point. Think, for example, about the ball B in $\mathbb{C}^n$, and consider a point $q \in \partial B$. Since the boundary of the ball has positive curvature, and a complex line is flat, it is straight forward to see that any complex line tangent to ∂B at q passes through only one boundary point, namely q. More precisely, no complex line (equivalently, no affine analytic disc) can have geometric order of contact with ∂B at q greater than two because the differential geometric structures disagree at the level of second derivatives. When looking for a biholomorphically invariant version of this idea, we turn to the notion of a strongly pseudoconvex boundary point. That is, no analytic disc can osculate to better than first order tangency to a strongly pseudoconvex boundary point.

Because the concept of finite type is more complicated in dimension higher than two, the following discussion will be restricted to $\mathbb{C}^2$ in order to sustain comprehensibility, and will follow the discussion outlined in [6].

Definition 2.3.1. *Let $D := \{z : \rho(z) < 0\}$ be a smoothly bounded domain in $\mathbb{C}^2$, and let $q \in \partial D$. Then the analytic disc $\phi : \Delta \to \mathbb{C}^2$ is called a non-singular disc tangent to ∂D at q if $\phi(0) = q$, $\phi'(0) \neq 0$, and $(\rho \circ \phi)'(0) = 0$.*

Definition 2.3.2. *Let $D := \{z : \rho(z) < 0\}$ be a smoothly bounded domain with $q \in \partial D$. Then ∂D is of finite (geometric) type m at q if the following condition holds: There exists a non-singular disc ϕ tangent to ∂D at q such that*

$$\frac{|\rho \circ \phi(\zeta)|}{|\zeta|^m} \leq C$$

as $|\zeta| \to 0$, *but there does not exists a non-singular disc* ψ *tangent to* ∂D *at* q *such that*

$$\frac{|\rho \circ \psi(\zeta)|}{|\zeta|^{m+1}} \leq C$$

as $|\zeta| \to 0$, *where* C *is some constant. Here we call* q *a point of finite type.*

In short, the type of a boundary point measures the maximum order of contact of an analytic disc with said point. Before going through a few explicit examples to illustrate the notion of type, we present a few important properties:

Property (1) The definition of type is independent of the choice of defining function.

Proof. Let $D \subset \mathbb{C}^2$ be smooth, with defining function ρ. Let $p \in \partial D$ and $\tilde{\rho}$ be a second defining function for D. Then there exists a function h, non-vanishing in a neighborhood of ∂D, such that $\tilde{\rho} = h\rho$, and hence $\rho = \frac{1}{h}\tilde{\rho}$. Thus, for any non-singular analytic disc ϕ tangent to ∂D at p,

$$|\rho(\phi(\zeta))| = \left| \left(\frac{\tilde{\rho}}{h}\right)(\phi(\zeta)) \right| = \left| \frac{\tilde{\rho}(\phi(\zeta))}{h(\phi(\zeta))} \right|.$$

Let $p \in \partial D$ be a point of finite type m with respect to ρ. That is, suppose there exists a non-singular disc ϕ tangent to ∂D at p such that for small $|\zeta|$,

$$|\rho \circ \phi(\zeta))| \leq C|\zeta|^m.$$

Then for small $|\zeta|$ one sees that

$$\left| \frac{\tilde{\rho}(\phi(\zeta))}{h(\phi(\zeta))} \right| \leq C|\zeta|^m,$$

i.e.

$$|\tilde{\rho}(\phi(\zeta))| \leq C|h(\phi(\zeta))||\zeta|^m \leq CM|\zeta|^m$$

for small $|\zeta|$, where

$$M := \sup_{\text{small } |\zeta|} |h(\phi(\zeta))|.$$

Thus for small $|\zeta|$,

$$|\tilde{\rho}(\phi(\zeta))| \leq C_1|\zeta|^m.$$

Now suppose there exists a non-singular analytic disc ψ tangent to ∂D at p such that $|\tilde{\rho}(\psi(\zeta))| \leq C|\zeta|^{m+1}$ for small $|\zeta|$. Then,

$$|\rho(\psi(\zeta))| \leq \frac{|\tilde{\rho}(\psi(\zeta))|}{|h(\psi(\zeta))|} \leq \frac{C|\zeta|^{m+1}}{|h(\psi(\zeta))|} \leq \frac{C}{M}|\zeta|^{m+1},$$

where

$$M := \inf_{\text{small } |\zeta|} |h(\psi(\zeta))|.$$

Therefore,

$$|\rho(\psi(\zeta))| \leq C_1|\zeta|^{m+1}$$

for small $|\zeta|$. This contradicts the fact that p is a point of finite type m with respect to ρ. Therefore, p is a point of finite type m with respect to $\tilde{\rho}$, which completes the proof. $\square$

Property (2) The condition of finite type is preserved under biholomorphism.

The remainder of this section will be dedicated to examples of specific domains, in order to help provide intuition as to the concept of finite type.

Example 2.3.3. *The unit ball* $B_2 = \{z \in \mathbb{C}^2 : \rho(z) = |z_1|^2 + |z_2|^2 < 1\} = \{z \in \mathbb{C}^2 : \rho(z) = |z_1|^2 + |z_2|^2 - 1 < 0\}$.

Consider the boundary point $p = (1, 0)$. Is p a point of finite type? Observe that

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ \\ \\ \bar{z}_2 \end{pmatrix} \implies \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ \\ 0 \end{pmatrix},$$

which imples that any curve tangent to ∂D_2 at p must be of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2)),$$

after a re-parameterization (look at the Taylor expansion).

Consider the disc $\phi(\zeta) = (1, \zeta)$. It has order of contact 2 with the boundary of B_2 at p because

$$\rho(\phi(\zeta)) = \rho(1, \zeta) = |\zeta|^2.$$

So what is the maximum order of contact when ϕ is of the form $\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2))$? Observe the following computation:

$$\begin{aligned}\rho(\phi(\zeta)) &= |1 + O(\zeta^2)|^2 + |\zeta + O(\zeta^2)|^2 - 1 \\ &= |1 + O(\zeta^2)|^2 + |\zeta|^2 \cdot |1 + O(\zeta)|^2 - 1 \\ &\leq C|\zeta|^2\end{aligned}$$

for small $|\zeta|$, since

$$|1 + O(\zeta^2)|^2 \longrightarrow 1 \ \text{ as } \ |\zeta| \longrightarrow 0$$

and

$$|1 + O(\zeta)|^2 \longrightarrow 1 \ \text{ as } \ |\zeta| \longrightarrow 0.$$

Therefore, $p = (1, 0) \in \partial B_2$ is a point of finite type 2. In general, it can be shown that a strongly pseudoconvex boundary point is always of type 2.

Example 2.3.4. *The ellipsoid* $E_m = \{z \in \mathbb{C}^2 : \rho(z) = |z_1|^2 + |z_2|^{2m} - 1 < 0\}$.

Consider the boundary point $p = (1, 0)$. In order to calculate the type at p, notice that

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ \\ mz_2^{m-1}\bar{z}_2^{m-1} \end{pmatrix} \implies \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},$$

which implies that , after a re-parameterization, a non-singular analytic disc ϕ that intersects ∂E_m at p is of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2)).$$

What is the maximum order of contact of such a curve with the boundary? First, consider the simple case wherein $\phi(\zeta) = (1, \zeta)$. This curve has order of contact $2m$ at the boundary point p, because

$$\rho(\phi(\zeta)) = |\zeta|^{2m}.$$

One must now ask, can the order of contact improve? For an arbitrary curve ϕ as described above,

$$\begin{aligned}\rho(\phi(\zeta)) &= |1 + O(\zeta^2)|^2 + |\zeta + O(\zeta^2)|^{2m} - 1 \\ &= |1 + O(\zeta^2)|^2 + |\zeta|^{2m} \cdot |1 + O(\zeta)|^{2m} - 1 \\ &\leq C|\zeta|^{2m}\end{aligned}$$

for small $|\zeta|$, since

$$|1 + O(\zeta^2)|^2 \longrightarrow 1 \quad \text{as} \quad |\zeta| \longrightarrow 0$$

and

$$|1 + O(\zeta)|^{2m} \longrightarrow 1 \quad \text{as} \quad |\zeta| \longrightarrow 0.$$

Therefore, the maximum order of contact of any non-singular analytic disc tangent to ∂E_m at $p = (1, 0)$ is $2m$, which implies that p is a point of finite type $2m$. Turn now to a few examples of domains with infinite type boundary points:

Example 2.3.5. *The domain* $E_\infty = \{z \in \mathbb{C}^2 : \rho(z) = |z_1|^2 + 2e^{-1/|z_2|^2} - 1 < 0\}$.

Consider the point $p = (1, 0) \in \partial E_\infty$. Then

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ \\ \\ \frac{2e^{-1/|z_2|^2}}{z_2^2\bar{z}_2} \end{pmatrix} \implies \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ \\ 0 \end{pmatrix}.$$

And regarding the curve $\phi(\zeta) = (1, \zeta)$ which is tangent to ∂E_∞ at p, one sees that

$$\rho(\phi(\zeta)) = 2e^{-1/|\zeta|^2},$$

which implies that

$$\frac{|\rho(\phi(\zeta))|}{|\zeta|^m} = \frac{2e^{-1/|\zeta|^2}}{|\zeta|^m} \longrightarrow 0 \quad \text{as} \quad \zeta \longrightarrow 0$$

by l'Hôpital's rule, since

$$\frac{\mathrm{d}^k}{\mathrm{d}\zeta^k}(2e^{-1/|\zeta|^2})\Big|_{\zeta=0} = 0 \quad \forall k \in \mathbb{Z}^+.$$

Since this is true for any $m \in \mathbb{Z}^+$,

$$|\rho(\phi(\zeta))| \leq C|\zeta|^m$$

as $|\zeta| \longrightarrow 0$ for all $m \in \mathbb{Z}^+$, implying that $p = (1, 0)$ is a point of *infinite type*.

Example 2.3.6. *The unit polydisc* $\Delta_2 = \{z \in \mathbb{C}^2 : |z_j| < 1, j = 1, 2\}$.

Let $p = (1, 0) \in \partial\Delta_2$. In a neighborhood U of p, let $\rho(z) = |z_1| - 1$ be a local defining function for the boundary defined inside $U \cap \partial\Delta_2$. Then

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ \\ \\ 0 \end{pmatrix} \implies \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ \\ 0 \end{pmatrix}$$

and hence the non-singular analytic disc $\phi(\zeta) = (1, \zeta)$ is tangent to $\partial\Delta_2$ at p. Thus

$$\rho(\phi(\zeta)) = \rho(1, \zeta) = |1| - 1 = 0 \quad \forall\zeta \in \Delta_2,$$

which implies that

$$|\rho(\phi(\zeta))| \leq C|\zeta|^m$$

as $|\zeta| \longrightarrow 0$ for all $m \in \mathbb{Z}^+$. Therefore, $p = (1, 0) \in \partial\Delta_2$ is a point of *infinite type.*

As these examples illustrate, the greater the type at a boundary point, the flatter the boundary is in a neighborhood of that point. In particular, if there is an analytic variety in the boundary of some domain $D \subset \mathbb{C}^n$ passing through a boundary point p, then p is a point of infinite type.

Chapter 3

Invariant Metrics and Measures

3.1 Invariant metrics

Let $H(A, B)$ be the set of holomorphic mappings from A to B and let Δ be the unit disc in $\mathbb{C}$. The Kobayashi and Carathéodory metrics are defined as follows:

Definition 3.1.1. *The Kobayashi and Carathéodory metrics on $D \subset \mathbb{C}^n$ at $x \in D$ in the direction $\xi \in \mathbb{C}^n$, denoted $F_K^D(x,\xi)$ and $F_C^D(x,\xi)$, respectively, are defined by*

$$F_K^D(x,\xi) = \inf\left\{\frac{1}{\alpha} : \exists\, \phi \in H(\Delta, D) \text{ such that } \phi(0) = x,\ \phi'(0) = \alpha\xi\right\}$$

$$F_C^D(x,\xi) = \sup\left\{\left|\sum_{j=1}^{n} \frac{\partial f(x)}{\partial z_j}\xi_j\right| : \exists\, f \in H(D, \Delta) \text{ such that } f(x) = 0\right\}.$$

If $z, w \in D$, then the Kobayashi and Carathéodory pseudo-distances on D between z and w, denoted $d_K^D(z,w)$ and $d_C^D(z,w)$ respectively, are given by

$$d_K^D(z,w) = \inf_{\gamma} \int_0^1 F_K^D(\gamma(t), \gamma'(t))\mathrm{d}t, \text{ and}$$

$$d_C^D(z,w) = \sup_f \rho(f(z), f(w))$$

where $\gamma : [0,1] \to D$ is a piecewise C^1 curve connecting z and w, and where $\rho(p,q)$ is the Poincaré distance on Δ between $p, q \in \Delta$. The supremum in the Carathéodory pseudo-distance is taken over all holomorphic mappings $f : D \to \Delta$.

The Kobayashi and Carathéodory metrics satisfy the following important non-increasing property under holomorphism.

Lemma 3.1.2. *Let $D \subset \mathbb{C}^n$ and $\hat{D} \subset \mathbb{C}^m$, and suppose there exists a holomorphism between them $\Psi : D \to \hat{D}$. Then for and $p \in D$ and $\xi \in \mathbb{C}^n$,*

$$F_K^D(p,\xi) \geq F_K^{\hat{D}}(\Psi(p), \Psi_*(p)\xi)$$

and

$$F_C^D(p,\xi) \geq F_C^{\hat{D}}(\Psi(p), \Psi_*(p)\xi)$$

Proof. Beginning with the Kobayashi case, let $\phi \in \mathrm{Hol}(\Delta, D)$ such that $\phi(0) = p$ and $\phi'(0) = \alpha\xi$. Then consider $\Psi \circ \phi \in \mathrm{Hol}(\Delta, \hat{D})$. $(\Psi \circ \phi)(0) = \Psi(p)$ and $(\Psi \circ \phi)'(0) =$

$\Psi_*(\phi(0))\phi'(0) = \Psi_*(p)\alpha\xi = \alpha\Psi_*(p)\xi$. Thus

$$F_K^{\hat{D}}(\Psi(p), \Psi_*(p)\xi) \leq \frac{1}{\alpha}.$$

Now taking the infimum over all possible ϕ yields

$$F_K^D(p, \xi) \geq F_K^{\hat{D}}(\Psi(p), \Psi_*(p)\xi).$$

For the Carathéodory case, let $\phi \in \mathrm{Hol}(\hat{D}, \Delta)$ such that $\phi(\Psi(p)) = 0$. Then consider $\phi \circ \Psi \in \mathrm{Hol}(D, \Delta)$. $(\phi \circ \Psi)(p) = \phi(\Psi(p)) = 0$, hence $F_C^D(p, \xi) \geq |(\phi \circ \Psi)_*(p)\xi| = |\phi_*(\Psi(p))(\Psi_*(p)\xi)|$. So taking the supremum over all possible ϕ yields

$$F_C^D(p, \xi) \geq F_C^{\hat{D}}(\Psi(p), \Psi_*(p)\xi).$$

$\square$

Both metrics and distances above satisfy the non-increasing property under holomorphism. That is, given $D \subset \mathbb{C}^n$ and $\hat{D} \subset \mathbb{C}^m$, a holomorphism between them $\Psi : D \to \hat{D}$, and $p, q \in D$, $\xi \in \mathbb{C}^n$, the following inequalities hold:

$$F^D(p, \xi) \geq F^{\hat{D}}(\Psi(p), \Psi_*(p)\xi)$$

$$d^D(p, q) \geq d^{\hat{D}}(\Psi(p), \Psi(q)).$$

Thus both the Kobayashi and Carathéodory metrics and distances are invariant under biholomorphism.

3.2 Invariant measures

The definitions of the Kobayashi and Carathéodory metrics can now be extended to define the respective measures. Let B_k represent the complex k-dimensional unit ball and let Δ_k denote the complex k-dimensional unit polydisc.

Definition 3.2.1. *Let $D \subset \mathbb{C}^n$ be a domain, $p \in D$, and $\xi_1, \xi_2, \ldots \xi_n \in T_p^{\mathbb{C}} D$ be linearly independent vectors on the complex tangent space to D at p for $1 \leq m \leq n$. One can find an (m, m) volume for M on D such that $M(\xi_1, \ldots, \xi_m, \bar{\xi}_1, \ldots, \bar{\xi}_m) = 1$. Let $U = B_{m-j} \times \Delta_j$ for $0 \leq j \leq m$, and let $\mu_m = \prod_{j=1}^{m} \left(\frac{i}{2} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j\right)$. We define the Kobayashi and Carathéodory m-measures with respect to U as follows:*

$$K_U^D(p; \xi_1, \ldots, \xi_m) = \inf \Bigg\{ \frac{1}{\alpha} : \exists \Phi \in H(U, D) \text{ such that } \Phi(0) = p \text{ and } \Phi^*(0) M = \alpha \mu_m, \text{ for some } \alpha > 0 \Bigg\}, \textit{ and}$$

$$C_U^D(p; \xi_1, \ldots, \xi_m) = \sup \Bigg\{ \beta : \exists \Phi \in H(D, U) \text{ such that } \Phi(p) = 0 \text{ and } \Phi^*(p) \mu_m = \beta M, \text{ for some } \beta > 0 \Bigg\}.$$

Both the Kobayashi and the Carathéodory measures satisfy the non-increasing property under holomorphic mappins. That is,

Proposition 3.2.2. *Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^{n'}$ be domains, and let $U = B_{m-j} \times \Delta_j$, for $0 \leq j \leq m$ and $m \leq \min\{n, n'\}$. Let $p \in D_1$, and for $j = 1, \ldots, m$, let $\xi_j \in T_p^{\mathbb{C}} D_1$ be* *linearly independent. If $\phi \in H(D_1, D_2)$ is such that $\phi_(p)\xi_j$'s are linearly independent, then*

$$K_U^{D_1}(p;\xi_1,\ldots,\xi_m) \geq K_U^{D_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m), \text{ and}$$

$$C_U^{D_1}(p;\xi_1,\ldots,\xi_m) \geq C_U^{D_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m).$$

Proof. Let M be an (m,m) volume form on D_1 such that $M(\xi_1,\ldots,\xi_m,\bar{\xi}_1,\ldots,\bar{\xi}_m) = 1$. And let $\Phi : U \to D_1$ be a holomorphic mapping such that $\Phi(0) = p$ and $\Phi^*(0)M = \alpha\mu_m$. Consider $h = \phi \circ \Phi : U \to D_2$. Let M' be an (m,m) volume form on D_2 such that $\phi^*(p)M' = M$. Then $h(0) = \phi(p)$ and

$$h*(0)M' = \Phi^*(0)(\phi^*(p)M') = \Phi^*(0)(M) = \alpha\mu_m.$$

Hence $1/\alpha \geq K_U^{D_2}(\phi(p), M)$ and $\inf 1/\alpha \geq K_U^{D_2}(\phi(p), M)$. The second inequality follows similarly. $\square$

The following remarks are important well known results that will prove useful in the proof of the main theorem.

Remark 3.2.3. *Let $D \subset \mathbb{C}^n$, $p \in D$, and $\xi_1,\ldots\xi_m \in T_p^{\mathbb{C}}D$ where $1 \leq m \leq n$, be linearly independent vectors. If $U = B_{m-j} \times \Delta_j$, for $0 \leq j \leq m$, then*

$$\frac{C_U^D(p;\xi_1,\ldots,\xi_m)}{K_U^D(p;\xi_1,\ldots,\xi_m)} \leq 1.$$

Remark 3.2.4. *Let $D \subset \mathbb{C}^n$, $p \in D$, and $\xi_1, \ldots \xi_m \in T_p^{\mathbb{C}} D$ where $1 \leq m \leq n$, be linearly independent vectors. Let $U = B_{m-j} \times \Delta_j$. We have*

$$\frac{C_U^D(p; \xi_1, \ldots, \xi_m)}{K_U^D(p; \xi_1, \ldots, \xi_m)} = 1$$

if and only if D is biholomorphic to U.

Remark 3.2.5. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded convex domain and let $p \in \partial D$ be a strongly pseudoconvex boundary point. Let V be a neighborhood of p. Then we have*

$$\frac{K_U^D(z; \xi_1, \ldots, \xi_m)}{K_U^{D \cap V}(z; \xi_1, \ldots, \xi_m)} \longrightarrow 1, \quad \frac{C_U^D(z; \xi_1, \ldots, \xi_m)}{C_U^{D \cap V}(z; \xi_1, \ldots, \xi_m)} \longrightarrow 1, \text{ as } z \longrightarrow p.$$

Remark 3.2.6. *Let D be a smoothly bounded convex domain. The domain D near a strongly pseudoconvex boundary point can be approximated by ellipsoids which are biholomorphic to balls. Since B_m and $B_{m-j} \times \Delta_j$, for $j \geq 1$, are not biholomorphic and since the Kobayashi and Carathéodory measures are localizable near a strongly pseudoconvex boundary point by Remark 3.2.5, we have*

$$\begin{aligned} &\frac{C_U^D(z; \xi_1, \ldots, \xi_m)}{K_U^D(z; \xi_1, \ldots, \xi_m)} < L < 1, \quad U = B_{m-j} \times \Delta_j, \ j \geq 1 \\ &\frac{C_U^D(z; \xi_1, \ldots, \xi_m)}{K_U^D(z; \xi_1, \ldots, \xi_m)} \longrightarrow 1, \quad U = B_m. \end{aligned}$$

Chapter 4

Automorphism Groups and Analytic Varieties

4.1 Automorphism groups of $D \subset \mathbb{C}^n$

In order to remove the condition of non-tangential convergence, we rely on the following 2004 result of Kang-Tae Kim (see [4]), which is a modification of work by Sidney Frankel (see [2]). For completeness, the idea of the proof is reproduced below.

Theorem 4.1.1. *If a bounded convex domain $D \subset \mathbb{C}^n$ possesses a non-compact automorphism orbit accumulating at a boundary point with sphere contact inside, then the automorphism group contains a non-compact 1-parameter subgroup.*

The condition of sphere contact inside simply means that one can draw a ball of radius $\epsilon > 0$ tangent to the boundary accumulation point, which lies completely inside D. Notice that this condition is satisfied given that the boundary is C^1 smooth. The following summary of the proof of Theorem 4.1.1 begins with the construction of a version of Pinchuk's scaling process (see [8]).

Centering and stretching sequence: Let D be our convex domain in $\mathbb{C}^n$, and assume we have a sequence $Q = \{p_j \in \Omega : j = 1, 2, 3, \dots\}$ that converges to a point $q \in \partial D$. Then we can define a sequence of *centering maps*, $\{\psi_j\}$, as follows. For a fixed j , choose a boundary point $q_j \in \partial D$ such that

$$\|p_j - q_j\| = \min_{x \in \partial D} \|p_j - x\|.$$

Note that such q_j is not necessarily uniquely determined, but the choice of q_j determines a unique supporting real hyperplane to D at q_j. Next choose a unitary transformation $T_j : \mathbb{C}^n \to \mathbb{C}^n$ such that the complex affine transformation $\psi_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\psi_j(z) = T_j(z - q_j)$, satisfies the relation

$$\psi_j(D) \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : \mathrm{Im}(z_n) > 0\},$$

for all $z \in \mathbb{C}^n$. Here the supporting hyperplane to $\psi_j(D)$ is defined by the equation $\mathrm{Im}(z_n) = 0$.

After "centering" the sequence Q, we introduce a change of coordinates and scaling factors. Fixing j and $p_j \in Q$, consider the complex orthogonal complement $V_{n-1}^{(j)} \subset \mathbb{C}^n$ of the line joining the origin and the point $\psi_j(p_j)$, and the "projected slice"

$$D_{n-1}^{(j)} = \{z \in V_{n-1} : z + \psi_j(p_j) \in \psi_j(D)\}.$$

When $V_{n-1}^{(j)}$ is equipped with the Hermitian inner product inherited from $\mathbb{C}^n$, $D_{n-1}^{(j)}$ is a domain in $V_{n-1}^{(j)}$ containing the origin. Choose a point from $\partial D_{n-1}^{(j)}$ that is closest to the origin, denote it by $x_{n-1}^{(j)}$. Notice that such a point is not necessarily unique. Next, let $V_{n-2}^{(j)}$ denote the complex orthogonal complement in $V_{n-1}^{(j)}$ of the vector $x_{n-1}^{(j)}$, and let

$$D_{n-2}^{(j)} = D_{n-1}^{(j)} \cap V_{n-1}^{(j)}.$$

Again, we pick a (not necessarily unique) point $x_{n-2}^{(j)} \in \partial D_{n-2}^{(j)}$ that is closest to the origin. Continue this process until it is no longer possible to proceed. Then we obtain mutually orthogonal vectors $x_1^{(j)}, \ldots x_{n-1}^{(j)}$. Adjoin to this sequence the vector $x_n^{(j)} = \psi_j(p_j)$. Then the set of vectors $\{e_l^{(j)}\}$ given by

$$e_l^{(j)} = \frac{x_l^{(j)}}{\|x_l^{(j)}\|} \ , \quad (l = 1, 2, \ldots, n)$$

form an orthonormal basis for $\mathbb{C}^n$. Now consider, for each j, the complex linear mapping $\Lambda_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Lambda_j(e_l^{(j)}) = \lambda_l^{(j)} e_l^{(j)} \ , \quad (l = 1, 2, \ldots, n)$$

where $\lambda_l^{(j)}$ is defined as $\lambda_l^{(j)} = (\|x_l^{(j)}\|)^{-1}$ for $l = 1, 2, \dots, n$. Finally, we compose the complex affine linear maps, resulting in a *Pinchuck stretching sequence* $\{\sigma_j\}$ defined as follows:

$$\sigma_j = \Lambda_j \circ \psi_j : \mathbb{C}^n \to \mathbb{C}^n \ , \quad (j = 1, 2, \dots).$$

Each σ_j is a "stretching" followed by the "centering" of the sequence Q.

Scaling the automorphism orbits: Choose Q to be the particular sequence in Theorem 1.0.4 That is, let $Q = \{p_j\} = \{g_j(x)\}$, where $x \in D$ and $g_j \in Aut(D)$. Then, considering the *scaling sequence*

$$\omega_j(z) = \sigma_j \circ g_j(z)$$

which is the "normalization" of Pinchuck's stretching sequence by the non-compact automorphisn sequence g_j, we can capitalize on the following convergence theorems.

Lemma 4.1.2. *The scaling sequence $\omega_j : D \to \mathbb{C}^n$ $(j = 1, 2, \dots)$ introduced above has the following convergence property: every subsequence of $\{\omega_j\}$ admits a subsequence that converges uniformly on compact sets to a biholomorphic embedding, say $\hat{\omega}$ of D into $\mathbb{C}^n$.*

Furthermore, notice that

$$\hat{\omega}_j(x) = (1, 0, \dots, 0) \text{ for } j = 1, 2, \dots$$

Let $B(0; R)$ represent the open ball in $\mathbb{C}^n$ with radius R centered at the origin, and let

$$\tau : \mathbb{C}^n \to \mathbb{C}^n \text{ be the map given by } \tau(z_1, \dots, z_n) = (z_1 - 1, 0, \dots, 0)$$

Then the following result, usually called *the local Hausdorff set-convergence* reveals how $\omega_j(D)$ tends to $\hat{\omega}(D)$ as j tends to ∞. For a proof, refer to Kim-Krantz [5].

Lemma 4.1.3. *Let $R > 0$ be arbitrarily given. Then, for every $\epsilon > 0$ there exists $N > 0$ such that for every $j > N$ we have*

$$(1-\epsilon)[(\tau \circ \omega_j(D)) \cap B(0;R)] \subset (\tau \circ \hat{\omega}(D)) \cap B(0;R) \subset (1+\epsilon)[(\tau \circ \omega_j(D)) \cap B(0;R)].$$

Boundary of scaled limit domains: In this section, we show that the local Hausdorff set limit of the sequence $\sigma_j(D)$ contains a real line in its boundary. First, recall that the Banach selection theorem implies the existence of a subsequence of $\sigma_j(D)$ that converges to a convex, not necessarily bounded domain in $\mathbb{C}^n$, in the sense of local Hausdorff set convergence.

Let us momentarily forget the automorphism sequences here, and simply consider a point sequence $Q = \{p_j : j = 1, 2, \dots\} \subset D$ that converges to a boundary point $q \in \partial D$. Then, as we did previously during the construction of the Pinchuck stretching sequences, choose a boundary point $q_j \in \partial D$ that closest to p_j for each $j = 1, 2, \dots$. Now consider the sets

$$\Sigma_j = \{z \in D : z - q_j = \lambda(p_j - q_j) \text{ for some } \lambda \in \mathbb{C}\}$$

which we call the *j-th principal slice* of D. Then focusing on the sequence $\sigma_j(\Sigma_j)$, let us restrict to the closed ball $\bar{B}(0;R)$, for arbitrary $R > 0$, and consider the usual Hausdorff limit of the sequence $\sigma_j(\Sigma_j) \cap \bar{B}(0;R)$. If q is a smooth boundary point in the sense that there is sphere contact from inside D, then this Hausdorff limit coincides with the set

$$\{(z_1, 0, \ldots, 0) \in \mathbb{C}^n : \mathrm{Re}(z_1) \geq 0\} \cap \bar{B}(0; R).$$

And since $R > 0$ is arbitrary, we can conclude: *if $q \in \partial D$ is a smooth boundary point in the sense that it admits sphere contact from inside D, then the local Hausdorff limit domain, say $\hat{D}$, of the sequence $\sigma_j(D)$ has a real one-dimensional straight line in its boundary.*

Proof of Theorem 4.1: Let $g_j \in Aut(D)$ and $x \in D$ be such that $g_j(x)$ converges to $q \in \partial D$, where q admits sphere contact from inside D. Then using the arguments of the previous section, consider the sequence $Q = \{g_j(x)\}$ for $j = 1, 2, \ldots$. Since each g_j is an automorphism of D, we have $g_j(D) = D$ for all j. Thus

$$\omega_j(D) = \sigma_j \circ g_j(D) = \sigma_j(D)$$

where ω_j is the scaling sequence introduced above. Therefore, the scaled limit domain $\hat{\omega}(D) = \hat{D}$ as described above. In particular, $\hat{\omega}(D)$ has a real one-dimensional straight line, say l, in its boundary.

Recall that the convex hull of a straight line and a point away from the line is a parallel strip. Due to the convexity of D and this fact, it becomes clear that every point of the domain $\hat{D}$ admits a line contained in D through that point, which is in fact a parallel translation of l. Let $v \in \mathbb{C}^n$ be a direction vector of l. Then the map

$$f_t(z) = z + tv$$

defines an automorphism of $\hat{\omega}(D) = \hat{D}$ for every $t \in \mathbb{R}$. Finally, since D is biholomorphic to $\hat{\omega}(D)$, this shows that $Aut(D)$ admits a non-compact 1-parameter subgroup.

$\square$

4.2 Characterization of the bidisc by its automorphism group

The well-known 1995 result of Bun Wong [14] uses the automorphism group of domains under specific restrictions to show biholomorphism to the bidisc, Δ_2. We restate the theorem here:

Theorem 4.2.1. *Let D be a bounded domain in $\mathbb{C}^2$. Suppose there exists a noncompact sequence $g_j \subset Aut(D)$ such that,*

1. *$W = \{(\lim_{j\to\infty} g_j)(D)\}$ is a complex variety of positive dimension contained in ∂D*
2. *W is contained in an open subset $U \subset \partial D$ such that the boundary of U is C^1, and there is an open set $N \subset \mathbb{C}^2$ such that $N \cap \partial D = U$ and $N \cap D$ is convex.*
3. *There exists a point $x \in D$ such that $\{g_j(x)\}$ converges to $p \in W \subset \partial D$ non-tangentially.*

Then D is biholomorphic to Δ_2.

The full proof can be found in [14], and the proof of the main theorem (Theorem 1.0.5) in section 5.1 will follow the same basic outline (see parts (A), (B), and (C)). A few relevant lemmas are listed below, as they will be invoked in the proof of the main theorem.

Lemma 4.2.2. (see Theorem E in [12]) *Let D be a bounded domain in $\mathbb{C}^n$. Suppose there exists a point $x \in D$ such that $|M_D^E(x)| = |M_D^C(x)|$, where M^E and M^C are defined with respect to the unit polydisc $\Delta_n \subset \mathbb{C}^n$, then D is biholomorphic to the polydisc, Δ_n. If M_D^E and M_D^C are defined with respect to the unit ball $B_n \subset \mathbb{C}^n$ and the condition of the lemma is satisfied with respect to these measures, then D is biholomorphic to the unit ball.*

Here $M_D^E(x)$ is the differential Eisenman-Kobayashi measure on D, (an (n,n)-form on D) and recall that it is given by

$$M_D^E(z) = |M_D^E(z)| \prod_{j=1}^{n} \left(\frac{i}{2} dz_j \wedge d\bar{z}_j \right)$$

where $|M_D^E|$ is a local function on D defined by

$$|M_D^E| = \inf\{|\det f'(0)|^{-2} : \ f : \Delta_n \to D, \ \text{a holomorphism with } f(0) = z\}$$

and $M_D^C(x)$ is the differential Carathéodory measure on D, (an (n,n)-form on D) given by

$$M_D^C(z) = |M_D^C(z)| \prod_{j=1}^{n} \left(\frac{i}{2} dz_j \wedge d\bar{z}_j \right)$$

where $|M_D^C|$ is a local function on D defined by

$$|M_D^C| = \sup\{|\det f'(z)|^2 : \ f : D \to \Delta_n, \ \text{a holomorphism with } f(z) = 0\}.$$

From these definitions, we obtain the following facts:

Lemma 4.2.3. *The following are true:*

1. $|M_D^E| \geq |M_D^C|$

2. *Given a holomorphism between two complex manifolds $f : D_1 \to D_2$, we have*

$$f^*(M_{D_2}^E) \leq M_{D_1}^E, \textit{ and}$$

$$f^*(M^C_{D_2}) \le M^C_{D_1}$$

Consequently, both of these measures are preserved under biholomorphism.

3. *Let* $\tilde{D}$ *be the universal cover of* D *and let* $\pi : \tilde{D} \to D$ *be the covering projection. Then* $M^E_{\tilde{D}} = \pi^*(M^E_D)$. *Consequently, for all* $z \in D_2 \subset D_1$, *we have* $|M^C_{D_1}(z)| \le |M^C_{D_2}(z)|$ *and* $|M^E_{D_1}(z)| \le |M^E_{D_2}(z)|$.

For proofs and thorough explanations of the above lemmas and definitions, please refer to [6] pages 429-462.

Lemma 4.2.4. *Let* $V \Subset \mathbb{C}^2$ *be a bounded convex domain. Let* z_0 *be a regular point for some complex variety* W *contained in the boundary* ∂V. *Then* W *is completely contained in a complex line* H, *which is the complex linear subspace of a supporting real hyperplane* ∂V *at* z_0.

Proof. Let H be a complex line tangential to W at z_0. Since V is convex, one can choose a hyperplane L of real dimension three such that L is supporting ∂V at z_0. Moreover, L contains H as a complex linear subspace. It is now possible to choose a complex coordinate system (z_1, z_2) with origin at z_0 such that

1. $L = \{(z_1, z_2) : \mathrm{Re}(z_2) = x_2 = 0\}$, where $z_2 = x_2 + iy_2$, and

2. V is completely contained in the half space defined by $\{(z_1, z_2) : \mathrm{Re}(z_2) = x_2 < 0\}$.

Then considering the holomorphic function z_2 restricted to W, namely z_2/W, one can see that $x_2/W = \mathrm{re}(z_2/W)$. That is, x_2/W attains a maximum at an interior point $z_0 \in W$.

This implies that $x+2/W \equiv 0$ on W by the maximum modulus principle. However, seeing as x_2/W is the real past of the holomorphic function z_2/W on W, the imaginary part $y_2/W \equiv 0$ as well. Thus $W \subset H \subset L$. □

From these two lemmas it becomes clear that $g(D) = \lim_{j\to\infty} g_j(D)$ can be considered as an open subset in a complex line H. Now let Ω be the upper half plane $\{z : \mathrm{Im}(z) > 0\}$, and supposing p to be the origin of the z-plane, let Γ_p^θ denote the cone in Ω with vertex at p, whose angle between the imaginary axis and its edges is θ, where $0 \leq \theta < \frac{\pi}{2}$. Then in the following lemma, assume that $z = (0, y)$ is a point in Γ_p^θ on the y-axis.

Lemma 4.2.5. *The distance, with respect to d_Ω^K, from a point z to the boundary of the cone Γ_p^θ, is given by $d_\Omega^K(z, \partial\Gamma_p^\theta) = 2\ln(\tan\theta + \sec\theta)$.*

For proof of this lemma, which is elementary, please refer to Theorem 5.1 in [13].

4.3 Analytic variety in the boundary is a ball

The purpose of this section is to show that an analytic variety of dimension m in the boundary of D must me biholomorphic to a complex m-ball. The idea is summarized in the following propositions, the first of which illustrates that the boundary of D is geometrically flat along any analytic disc. The last, and most important, proposition is an improvement upon Proposition 5 found in [7].

It may first be helpful to examine what is meant by *geometrically flat*. Let ρ be a defining function for D. Then we can define the gradient normal of ρ by

$$n = \frac{\nabla\rho}{|\nabla\rho|}$$

The boundary, ∂D, is called geometrically flat along an analytic variety if the gradient normals to the boundary are all parallel along the analytic variety.

Proposition 4.3.1. *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain. If $\phi : \Delta \to \partial D$ is a holomorphic mapping, then ∂D is geometrically flat along $\phi(\Delta)$.*

Proof. Let $D = \{\rho < 0\}$ and $p = \phi(0) \in \partial D$. Let $H = \{\mathrm{Re}(h) = 0\}$ be the real tangent plane to ∂D at p, where h is a linear holomorphic function. Since D is convex, one sees that $\bar{D} \subset \{\mathrm{Re}(h) \leq 0\}$. Consider $f(\zeta) = h \circ \phi(\zeta)$. Then f is a holomorphic function on Δ and satisfies $\mathrm{Re}(f(\zeta)) \leq 0$ for all $\zeta \in \Delta$, and $\mathrm{Re}(f(0)) = 0$. By the maximum principle for harmonic functions, $\mathrm{Re}(f(\zeta)) = 0$ for all $\zeta \in \Delta$. Therefore $f \equiv 0$ on Δ and hence $h \equiv 0$ on $\phi(\Delta)$. $\square$

Definition 4.3.2. *Let $H \subset \mathbb{C}^n$, and $q \in H$. Define the maximal chain of analytic discs on H through q, denoted Δ_q^H, as follows:*

$$\Delta_q^H = \{x \in H : \text{there exists a finite chain of analytic discs joining } z \text{ and } q\},$$

that is, there exist holomorphic maps $\phi_1, \phi_2, \ldots, \phi_k : \Delta \to \mathbb{C}^n$ such that $\phi_j(\Delta) \subset H$ for $1 \leq j \leq k$, and there exist $z_i \in H$ and $a_i, b_i \in \Delta$ for $a \leq i \leq k$ such that $\phi_j(a_j) = z_{j-1}$ and $\phi_j(b_j) = z_j$, where $z_0 = q$ and $z_k = z$. We say that Δ_q^H is trivial if $\Delta_q^H = \{q\}$.

Proposition 4.3.1 above implies that if $D \Subset \mathbb{C}^n$ is a smoothly bounded convex domain, then ∂D is geometrically flat along $\Delta_p^{\partial D}$ for all $p \in \partial D$. Furthermore, as described in Theorem 1 of [7], if $D \Subset \mathbb{C}^n$ is a smoothly bounded convex domain, then $\Delta_p^{\partial D}$ is linearly convex for all $p \in \partial D$. That is, if $z, w \in \Delta_p^{\partial D}$, then $t \cdot z + (1-t)w \in \Delta_p^{\partial D}$ for all $t \in [0,1]$.

Notice that if D is a smoothly bounded convex domain in $\mathbb{C}^n$, and if there is a sequence $\{g_j\} \subset Aut(D)$ such that $g_j(z) \longrightarrow p \in \partial D$ for some $z \in D$ where $\Delta_p^{\partial D}$ is non-trivial, then there exists a non-constant holomorphic surjective mapping $g : D \to \Delta_p^{\partial D}$ such that $g_j \longrightarrow g$ after taking a subsequence if necessary.

One can prove that $\Delta_p^{\partial D}$ is a convex open set in the m-dimensional complex subspace of $T_p(\partial D)$, where $m = \dim_{\mathbb{C}} \Delta_p^{\partial D}$. All of the necessary arguments can be found in [7] except for the following modification of Proposition 4 (from [7]) in which the condition of non-tangential approach is actually not needed. To be complete and precise, a proof of the following amelioration of Proposition 4 is provided.

Proposition 4.3.3. *let $D \Subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose $\{\phi_j\} \subset Aut(D)$ accumulates at a boundary point $p \in \partial D$ for some point $q \in D$, that is, $\{\phi_j(q)\} \longrightarrow p \in \Delta_p^{\partial D} \subset \partial D$. Then $\phi = \lim \phi_j$, passing through a subsequence if necessary, is a surjective holomorphic map from D onto $\Delta_p^{\partial D}$.*

Proof. If $\Delta_p^{\partial D}$ consists of only one point, then there is nothing to prove. Suppose then, that $z \in \Delta_p^{\partial D}$, $z \neq p$. One can assume that both z and p are interior points of $\Delta_p^{\partial D}$, otherwise one can use the maximum principle argument in Corollary 4 of [7] to eliminate the possibility that p or z may be boundary points. Let S be a relatively compact holomorphic disc in $\Delta_p^{\partial D}$ containing p and z. Denote by $\{p_j = \phi_j(q)\}$ the sequence in D converging to p. For sufficiently large j, one can translate S into D such that its image S_j contains p_j and $dist(z_j, z) \longrightarrow 0$ as $j \longrightarrow \infty$, where the distance is the Euclidean distance and z_j is the image of z in S_j via this translation. It then follows from the distance decreasing property of the Kobayashi metric that $d_D^K(p_j, z_j) \leq c = d_S^K(p, z)$ where c is a constant independent of j. Since the Kobayashi metric is invariant under biholomorphism, $\phi_j^{-1}(z_j) \in \{x \in D : d_D^K(q, x) \leq c\}$, which is a compact set in D due to the fact that d_D^K is a complete metric when D is a bounded convex open set. Therefore $\{\phi_j^{-1}(z_j)\}$ converges, passing through a subsequence if necessary, to a point $\tilde{z} \in D$. It is elementary to see that $\phi(\tilde{z}) = z$. This completes the proof. □

Given this surjective holomorphic mapping, there exists a sequence of points $\{q_j\} \subset D$ such that $\{q_j\} \longrightarrow q \in \partial D$ and that $\{g(q_j)\} \subset \Delta_p^{\partial D}$ converge to a point in $\Delta_p^{\partial D}$ for some strongly pseudoconvex boundary point $q \in \partial D$.

The preceding discussion implies the following result, which is an improvement upon Proposition 5 in [7].

Proposition 4.3.4. *Let $D \Subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose $\Delta_p^{\partial D}$ is not trivial for some $p \in \partial D$. If there exists $\{g_j\} \subset Aut(D)$ such that $g_j(z) \to \Delta_p^{\partial D}$ for all $z \in D$, then $\Delta_p^{\partial D}$ is biholomorphic to a complex m-ball, where m is the complex dimension of $\Delta_p^{\partial D}$.*

Proof. Let $q \in D$ be arbitrarily close to a strongly pseudoconvex boundary point, and let $\lim_{j\to\infty} g_j(q) = g(q) = p \in \Delta_p^{\partial D}$. For simplicity, let $q_j = g_j(q)$ and let $V = \Delta_p^{\partial D}$.

Let $\xi_1, \ldots, \xi_m$ be m linearly independent complex tangent vectors to V at p. It can be assumed that V lies in the $z_2 \ldots z_{m+1}$ plane, where $\mathrm{Re}(z_1)$ is the outward normal direction. Let $p \in V' \Subset V$ be a relatively compact open subset, then for large enough j, V' can be moved inside D using the parallel translation that sends p to q_j. Let V'_j be the image of V' under this translation.

Then, defining the Carathéodory and Kobayashi measures with respect to the complex unit m-ball (i.e., let U of definition 3.2.1 be B_m), one sees that

$$\frac{C_U^D(q;(g_j^{-1})_*(q_j)\xi_l)}{K_U^D(q;(g^{-1})_*(p)\xi_l)} \leq \frac{C_U^{V'_j}(q_j;\xi_l)}{K_U^D(q;(g^{-1})_*(p)\xi_l)} \leq \frac{C_U^{V'}(p;\xi_l)}{K_U^V(p;\xi_l)},$$

where ξ_l represents the set of m-vectors $\xi_1, \ldots, \xi_m$. Note that $(g^{-1})_*\xi_l$ should be interpreted as the pre-image vector of ξ_l, which is well-defined since the rank of g is m along $\Delta_p^{\partial D}$.

Then as $j \to \infty$, let $V' \to V$. Then, because q was chosen to be close to a strongly pseudoconvex boundary point, the left hand side of the above inequality approaches 1.

Meanwhile, the right hand side is always less than or equal to 1. Therefore,

$$\frac{C_U^V(p;\xi_l)}{K_U^V(p;\xi_l)} = 1$$

and hence V is biholomorphic to a complex m-dimensional ball. $\square$

Chapter 5

Proof of Main Theorem

5.1 In $\mathbb{C}^2$

In proving Theorem 1.0.5, we use the arguments discussed previously, and begin with a proof of the simplified result in $\mathbb{C}^2$:

Theorem 5.1.1. *Let D be a bounded convex domain in $\mathbb{C}^2$ with C^2 boundary. Suppose that there is a sequence $\{g_j\} \subset Aut(D)$ such that $\{g_j(x)\}$ accumulates at a boundary point for some point $x \in D$. Then if $p \in \partial D$ is such an orbit accumulation point, ∂D contains no non-trivial analytic variety passing through p.*

Proof. Let D be a bounded convex domain in $\mathbb{C}^2$ with C^2 boundary, and let $\{g_j\} \subset Aut(D)$ be a sequence of automorphisms such that $\{g_j(x)\}$ converges to a point $p \in \partial D$ for a point $x \in D$. Suppose, by way of contradiction, that there exists a non-trivial analytic variety W in ∂D at p.

Notice that the sequence $\{g_j\} \subset Aut(D)$ is noncompact, and hence $g(D) = \lim_{j\to\infty} g_j(D) \subset \partial D$. Then since D is bounded and convex, conditions (1) and (2) of Theorem 4.2.1 are satisfied. That is, $g(D) = W$ is contained in an open subset $U \subset \partial D$ such that U has C^1 boundary and there is an open set $N \subset \mathbb{C}^2$ such that $N \cap D$ is convex and $N \cap \partial D = U$.

(A) Open product set around the flat boundary

Consider the product domain M given by $M = \Omega \times g(D)$, where Ω denotes the upper half plane $\{z : \mathrm{Im}(z) > 0\}$ and $g(D)$ is a connected bounded open set in H as described in 4.2.4. And consider the subdomain of M, $V_\theta = \Gamma_p^\theta \times g(D)$, where Γ_p^θ is a cone inside Ω with vertex at the boundary accumulation point p. By 4.2.4, $g(D)$ is flat, and hence one can view M as a subdomain in $\mathbb{C}^2$ such that:

- M lies in the same half space as D (the one created by dividing space along the supporting hyperplane L described in 4.2.4).
- $\partial M \cap \partial D$ contains $g(D)$.
- The upper half plane Ω is perpendicular to $g(D)$.

Lemma 5.1.2. *For D, $\{g_i\}$, and V_θ as above, if K is a compact subset inside D, then there is an angle $0 < \theta_0 < \frac{\pi}{2}$ such that $g_j(K) \subset V_\theta$, for any $\theta \geq \theta_0$, if j is sufficiently large.*

Proof. Since g_j converges to g on compact sets, we see that $g_j(K) \subset M$ for sufficiently large j. Thus $\pi_2(g_j(K))$ is a compact subset of $g(D)$ for sufficiently large j. Here $\pi_2 : M \to g(D)$ is the projection of $M = \Omega \times g(D)$ onto its second component. Furthermore, we observe the following:

1. In the above theorem, we assume there is a point $x \in D$ such that $\{g_j(x)\}$ converges to $p \in \partial D$, but we do not assume non-tangential convergence, that is, convergence within a cone. If convergence is indeed non-tangential, then since $g_j(x)$ is contained in M for large j, we see that $\{\pi_1(g_j(x))\}$ converges to p on the upper half plane Ω within a cone Γ_p^{ψ}, for an angle $0 < \psi < \frac{\pi}{2}$, where $\pi_1 : M \to \Omega$ is the projection of $M = \Omega \times g(D)$ onto its first component.

   If, however, convergences is tangential, one can make adjustments to the sequence $\{g_j(x)\} \subset D$ as in the following discussion:

   Notice that the hypotheses of Lemma 4.1.1 are satisfied by the hypotheses of this theorem (C^1 boundary is sufficient for sphere contact inside), and hence $Aut(D)$ contains a 1-parameter subgroup. As detailed in section 4.1, automorphisms of this 1-parameter subgroup are defined by $f_t(z) = z + tv$ for each $t \in \mathbb{R}$. That is, f_t moves points in D along a path of real one dimension, whose direction under the Pinchuk scaling sequence is parallel to the line contained in the boundary. For each fixed j, let $T_j = \{f_t(g_j(x)) : t \in \mathbb{R}\}$ denote the path of $g_j(x)$ under the map f_t for all t. Then, since M has real dimension three and is perpendicular to $g(D)$, one sees that $M \cap T_j \neq \emptyset$. let $f_j(g_j(x))$ denote one such point in the intersection. Then, since the composition of automorphisms f_t and g_j is again an automorphism, we can replace our original sequence $\{g_j(x)\}$ with the translated sequence $\{f_j(g_j(x))\}$ which is now contained in M for large j. Then as above, $\{\pi_1(f_j(g_j(x)))\}$ converges to p on the upper half plane Ω within a cone Γ_p^{ψ}, for an angle $0 < \psi < \frac{\pi}{2}$.

2. Let $y \in K$ be a point such that $d = d_D^K(x,y) = \sup\{d_D^K(x,w) : w \in K\}$. Consider an increasing sequence of relatively compact open subsets $\{D_i\}$, where $D_i \Subset D_{i+1}$ and $\bigcup_{i=1}^{\infty} D_i = D$. We can assume D_1 contains both x and K. For fixed i, $g_j(D_i) \subset M$ for sufficiently large j, thus the composition $\pi_1 \circ g_j : D_i \to \Omega$ is a well-defined holomorphism. The distance decreasing property gives that $d_i = d_{D_i}^K(x,y) \geq d_\Omega^K(\pi_1(g_j(x)), \pi_1(g_j(y)))$ for large j, then letting $i \to \infty$, we see (by a normal family argument) that $\lim_{i\to\infty} d_i = d$. Combining the above arguments we conclude that $d_\Omega^K(\pi_1(g_j(x)), \pi_1(g_j(y))) \leq d + \epsilon$ for sufficiently large j, where ϵ is a positive constant.

Combining Lemma 4.2.5 and the result from observation 1 that $\pi_1(g_j(p))$ lies inside a cone Γ_p^ψ, we see that $\{\pi_1(g_j(y))\}$ also lies inside a cone Γ_p^θ for sufficiently large j and sufficiently large θ, $0 < \psi \leq \theta < \frac{\pi}{2}$. Therefore $\{\pi_1(g_j(K))\}$ must lie inside Γ_p^θ for large j, completing the proof that $g_j(K) \subset V_\theta$ for sufficiently large j. □

(B) Existence of an open set around the flat boundary whose universal covering is biholomorphic to a bidisc

Let K be a compact subset of D, and consider the subdomain of M given by $V_K = \Gamma_{r_K}^{\theta_K} \times H_K$, where H_K, θ_K, and r_K are described as follows:

H_K: As $g_j \to g$ on compacta, there is a relatively compact open subset $H_K \Subset g(D)$ such that $\pi_2(g_j(K)) \subset H_K$ for large j.

r_K and θ_K: Since a neighborhood of the boundary of D containing $g(D)$ is assumed to be C^1 and convex, one can choose $0 < \theta_K < \frac{\pi}{2}$ big enough and $r_K > 0$ small enough such that the cone $\Gamma_{r_K}^{\theta_K} \subset \Omega$, with vertex at p, angle θ_K, and radius r_K, satisfies the properties in the following lemma (Lemma 5.1.3). To be precise,

$$\Gamma_{r_K}^{\theta_K} = \left\{ z = x + iy \in \Omega : \tan^{-1}\left(\left|\frac{x}{y}\right|\right) < \theta_K,\ 0 < \theta_K < \frac{\pi}{2},\ \sqrt{x^2+y^2} < r_K \right\}.$$

Lemma 5.1.3. *Assuming all assumptions of the main theorem (Theorem 1.0.5), suppose K is a compact subset in D. Then:*

1. *$g_j(K) \subset V_K$ for sufficiently large j.*
2. *V_K is holomorphically covered by a bidisc.*
3. *$V_K \subset D$.*

Proof. (1) and (3) follow directly from the above discussion. (2) follows from the fact that both $\Gamma_{r_K}^{\theta_K}$ and H_K are covered holomorphically by the unit disc. □

(C) Domain biholomorphic to the bidisc

Recall that by lemma 4.2.2, if $|M_D^E(x)| = |M_D^C(x)|$, for one point $x \in D$, then D is biholomorphic to the bidisc, Δ_2.

Let N be as in the aforementioned satisfied condition (2) of lemma 4.2.1. Since $N \cap D$ is a convex open set, one can consider a increasing sequence of relatively compact open subsets $\{N_k\}_{k=1}^{\infty}$ inside $N \cap D$ such that $N_k \Subset N_{k+1}$ and $\bigcup_{k=1}^{\infty} N_k = N \cap D$. Then the satisfied condition (1) of lemma 4.2.1 yields that $g(D) \subset N \cap \partial D$. Now one can choose a subsequence $\{g_k\} \subset \{g_j\}$ such that $\bigcup_{k=1}^{\infty} D_k = D$ and $D_k \Subset D_{k+1}$, where $D_k = g_k^{-1}(N_k)$. Observe that each D_k is simply connected since each D_k is biholomorphic to a convex open set.

As outlined in part (B) above, for a compact subset $K \subset D$, there is an open set V_k around the flat boundary containing the accumulation point p, whose universal covering is biholomorphic to a bidisc. Going forward, consider the compact subset $K = D_k$ and denote the corresponding V_K by V. Then for sufficiently large j, $g_j(D_k) \subset V$. And since D_k is cimply connected, there is a holomorphic lifting map $f_j : D_k \to \tilde{V}$ of $g_j : D_k \to V$ satisfying the commutative diagram:

$$\begin{array}{ccc} & & \tilde{V} \\ & f_j \nearrow & \downarrow\pi \\ D_k & \xrightarrow{g_j} & V \subset D \end{array}$$

Here $\tilde{V}$ represents the universal cover of V, and π is the covering projection. Let $x_j = g_j(x)$ and let $y_j = f_j(x)$, then $\pi(y_j) = x_j$.

Since $f_j(D_k) \subset \tilde{V}$, we can use the measure decreasing property of holomorphic maps, Lemma 4.2.3 part (2), to obtain

$$|M^C_{D_k}(x)| \geq |\det(df_j(x))| \cdot |M^C_{\tilde{V}}(y_j)|. \tag{5.1}$$

Then considering the inclusion map $V \hookrightarrow D$ as a holomorphic map and again applying Lemma 4.2.3 part (2), we get

$$|M^E_D(x_j)| \leq |M^E_V(x_j)|. \tag{5.2}$$

The differential Eisenman-Kobayashi measure is preserved under covering projections by Lemma 4.2.3 part (3), and hence

$$|M^E_{\tilde{V}}(y_j)| = |\det(d\pi(y_j))| \cdot |M^E_V(x_j)|. \tag{5.3}$$

Finally, it follows once again by the measure preserving property of biholomorphic maps, Lemma 4.2.3 part (2), that

$$|M^E_D(x)| = |\det(dg_j(x))| \cdot |M^E_D(x_j)|. \tag{5.4}$$

We now combine the above four (in)equalities (5.1, 5.2, 5.3, and 5.4) and cancel all determinants of Jacobian matrices as an application of the chain rule to the map $g_j = \pi \circ f_j$. This results in the following inequality:

$$\frac{|M^C_{D_k}(x)|}{|M^E_D(x)|} \geq \frac{|M^C_{\tilde{V}}(y_j)|}{|M^E_{\tilde{V}}(y_j)|}. \tag{5.5}$$

Then letting $k \to \infty$, it is clear that $|M^C_{D_k}(x)| \to |M^C_D(x)|$, and hence

$$1 \geq \frac{|M^C_D(x)|}{|M^E_D(x)|} \geq \frac{|M^C_{\tilde{V}}(y_j)|}{|M^E_{\tilde{V}}(y_j)|} = 1$$

where the last equality comes from the fact that $\tilde{V}$ is biholomorphic to Δ_2. Therefore, $|M^E_D(x)| = |M^C_D(x)|$ and hence D is biholomorphic to Δ_2.

(D) Domain is biholomorphic to the unit ball in $\mathbb{C}^2$

Since D is biholomorphic to the bidisc Δ_2, and because the bidisc is homogeneous, one can bring any point to a point arbitrarily close to a strongly pseudoconvex boundary point by a biholomorphism, if such a point exists.

Notice that the assumption that D is a bounded domain with C^2 boundary implies the existence of a strongly pseudoconvex point in ∂D. It follows from Theorem 2.2.7 that D is biholomorphic to the unit ball $B_2 \subset \mathbb{C}^2$.

However, by Poincaré's Theorem (refer to Theorem 2.1.14), one knows that there exists no biholomorphism between the bidisc Δ_2 and the unit ball B_2 in $\mathbb{C}^2$. Therefore, we have a contradiction, and must conclude that there exists no analytic variety W in the boundary of D passing through the point p. $\square$

5.2 In $\mathbb{C}^n$

Generalizing the above result to dimension $n \geq 3$ requires a bit more care, and the proof illustrates a generalization of Poincaré's theorem 2.1.14.

Theorem 5.2.1. *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary. Suppose that there is a sequence $\{g_j\} \subset Aut(D)$ such that $\{g_j(z)\}$ accumulates at a boundary point for some point $z \in D$. If $p \in \partial D$ is such an orbit accumulation point, then ∂D contains no non-trivial analytic variety at p.*

Proof. Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary, and let $\{g_j\} \subset Aut(D)$ be a sequence of automorphisms such that $\{g_j(z)\}$ converges to some boundary point for all $z \in D$. Let g denote $\lim_{j\to\infty} g_j$, and let p be an orbit accumulation point of g_j, and suppose, by way of contradiction, that $g(D) \subset \partial D$ contains a non-trivial analytic variety at p. Then $g(D)$ is contained in $\Delta_p^{\partial D}$, and for simplicity, let S denote $\Delta_p^{\partial D}$.

Let m denote the complex dimension of S, and let the Re z_1 direction be the outward normal direction along S, with S lying on the complex $z_2 z_3 \cdots z_{m+1}$ plane. Then notice that $\dim_{\mathbb{C}} S = m \geq 1$, and S has codimension $n - m$. For each $s \in S$, let N_s denote the set of all real lines simultaneously perpendicular to both S and $\text{Im}(z_1)$ at s. Then $\dim_{\mathbb{R}} N_s = 2n - 2m - 1$ for all $s \in S$. Further, let $\tilde{N}$ denote the collection of all such N_s. That is, let

$$\tilde{N} = \bigcup_{s \in S} N_s.$$

Then $\tilde{N}$ is a $(2n - 2m - 1)$-bundle over S and $\dim_{\mathbb{R}} \tilde{N} = 2n - 1$.

Choose a point $q \in D$ close to a strongly pseudoconvex boundary point, whose existence is guaranteed due to the fact that the boundary of D is C^2 smooth. Then $g_j(q) \to p$ for some $p \in S$.

Due to the aforementioned result of K.-T. Kim (theorem 4.1.1), there exists a 1-parameter subgroup of $Aut(D)$, say $\{T_t\}_{t\in\mathbb{R}}$. Just as illustrated in the $\mathbb{C}^2$ case above, the automorphisms of this 1-parameter subgroup can be used to translate each $g_j(q)$ into $\tilde{N}$. More specifically, for each g_j, one can choose a $T_j \in \{T_t\}_{t\in\mathbb{R}}$ such that $(T_j \circ g_j)(q) \in \tilde{N}$. For simplicity, let q_j denote $(T_j \circ g_j)(q)$. Then for j large enough, each q_j sits on N_s for some $s \in S$.

Let $\Gamma_{\epsilon,r}$ be a wedge domain in $\mathbb{C}$ with radius less than r, defined by

$$\Gamma_{\epsilon,r} = \left\{ z \in \mathbb{C} : \frac{\pi}{2} + \epsilon < \arg z < \frac{3\pi}{2} - \epsilon, |z| < r \right\}.$$

Let $S' \Subset S$ be a relatively compact open set biholomorphic to the unit ball in $\mathbb{C}^n$ such that S' contains p. Consider the product domain $A^r_\epsilon = \Gamma_{\epsilon,r} \times S' \subset D$. Let $p = 0$, $q_j = (T_j \circ g_j)(q)$, and $\tilde{q}_j$ be the projection of q_j onto the $z_1 z_2 \cdots z_{m+1}$ plane. That is, if $q_j = (a_1, \ldots, a_n)$, then $\tilde{q}_j = (a_1, a_2, \ldots, a_{m+1}, 0, \ldots, 0)$. Then $\tilde{q}_j \to p$.

Consider the holomorphic mapping $f_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by $f_j(z) = (h_1(z), \ldots, h_n(z))$ for $1 \le k \le n$, where

$$h_k(z_1, z_2, \ldots, z_n) = \begin{cases} z_k, & \text{if } k = 1, \ldots, m+1 \\ \frac{a_k \cdot \overline{a_1}}{|a_1|^2} z_1, & \text{if } k = m+2, \ldots, n. \end{cases}$$

Note that when f_j is restricted to S, f_j is the identity mapping and $f_j(\tilde{q}_j) = q_j$. Then one can find $\epsilon > 0$ and $r > 0$ such that $f_j(A_\epsilon^r) \subset D$ given j large enough.

Now define the Kobayashi and Carathéodory metrics with respect to $U = \Delta \times B_m$, let ξ_j be the unit vector in the z_j direction, and let D_k be the exhaustion of D, that is, $D_k \nearrow D$. Then

$$\begin{aligned}\frac{C_U^{D_k}(q; (\tilde{g}_j^{-1} \circ f_j)_*(\tilde{q}_j)\xi_l)}{K_U^D(q; (\tilde{g}_j^{-1} \circ f_j)_*(\tilde{q}_j)\xi_l)} &\geq \frac{C_U^{\tilde{g}_j(D_k)}(q_j; (f_j)_*(\tilde{q}_j)\xi_l)}{K_U^D(q_j; (f_j)_*(\tilde{q}_j)\xi_l)} \\ &\geq \frac{C_U^{A_{\epsilon,r}}(\tilde{q}_j; \xi_l)}{K_U^{A_{\epsilon,r}}(\tilde{q}_j; \xi_l)}\end{aligned}$$

where $\tilde{g}_j = T_j \circ g_j$ and ξ_l represents the set of $m+1$ vectors $\xi_1, \ldots, \xi_{m+1}$. Note that the first $(m+1) \times (m+1)$ complex Jacobian of f_j is the identity and hence $(f_j)_*\xi_l$ is well-defined for $l = 1, \ldots, m+1$. The second inequality is a result of the measure decreasing property while using the inclusion map of $A_{\epsilon,r}$ into D for the Kobayashi measure, and using the projection mapping of $\mathbb{C}^n$ onto the $z_1 z_2 \cdots z_{m+1}$ plane for the Carathéodory measure. For j and k large enough, one can assume that the projection of $\tilde{g}_j(D_k)$ lies inside $A_{\epsilon,r}$ for some ϵ and r. Note that the Jacobian matrix of the projection is the identity along the $z_1 \cdots z_{m+1}$ direction, and hence $\xi_1, \ldots, \xi_{m+1}$ all remain unchanged.

Observe that as $j, k \to \infty$, the left hand side of the above inequality approaches a constant strictly less than 1. However, the right hand side is always equal to one, and one can exhaust S by S' so that the above argument works for all k and large j, where S is biholomorphic to a ball by proposition 4.3.4. This yields a contradiction, as desired. $\square$

Chapter 6

Conclusions

This result supports the veracity of the Greene-Krantz conjecture and illustrates a useful generalization of Poincaré's Theorem which states that B_n is not biholomorphic to the polydisc Δ_n for $n \geq 2$, by showing that B_{m+1} is not biholomorphic to $B_m \times \Delta$.

Work in this area is far from complete. While the result of this dissertation advances one step closer to the Greene-Krantz conjecture, the hypothesis is significantly stronger, and the conclusion weaker. In order to continue advancing towards the full conjecture, the author is interested in pursuing the following problem, restricted to smoothly bounded convex domains:

Problem 6.0.2. *Let D be a smoothly bounded convex domain in $\mathbb{C}^n$. Suppose there exists $\{g_j\} \subset Aut(D)$ such that $\{g_j(x)\}$ accumulates at a boundary point $p \in \partial D$ for some $x \in D$. Then ∂D is of finite type at p.*

The author believes that the lack of analytic variety passing through p, as proved in this dissertation, can be extended to reach the desired conclusion of Greene and Krantz, that of finite type, in the case of smoothly bounded convex domains.

Furthermore, it is believed by the author, that the use of the 1-parameter sub group of $Aut(D)$, as described by K.-T. Kim in [4], is not necessary to the proof of the main result of this dissertation. While the use of the 1-parameter subgroup gives intuitive geometric insight into the crux of the proof, perhaps a different method of proof might give further useful insights. It would be of interest, therefore, to pursue an alternate proof, without the use of this 1-parameter subgroup.

And finally, it is the obvious goal of the author to prove the Greene-Krantz conjecture itself.

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