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State-Space Models and Methods for MIMO Communication

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Electrical Engineering (Communication Theory and Systems)

by

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2007
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2007
To Qing and Serena
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2. C. Zhang, R. R. Bitmead, “Adaptive channel modeling for MIMO wireless


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ABSTRACT OF THE DISSERTATION

State-Space Models and Methods for MIMO Communication

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Professor Bhaskar D. Rao, Co-Chair

State-space models are proposed to represent MIMO frequency-selective wireless channels with the motivation of better model approximation and more robust channel equalization performance when the order of the channel model is lower that of the true channel. We study the MIMO channel approximation error as a function of model order and show that state-space models possess improved performance compared to more standard FIR models. We quantify the model approximation error using the upper bound of the minimum $H_\infty$ norm of the difference between the original channel and the approximated channel model. It is shown that state-space models always maintain lower $H_\infty$ approximation error than the FIR models with the same model order for either spatially uncorrelated MIMO channels or correlated ones. A recursive algorithm, based on the subspace system identification methods, to estimate the state-space channel model using training data is presented. When compared to the FIR-based Recursive Least Squares algorithm, the state-space-based channel estimator shows the ability of providing low-order models of high-quality channel approximation, while preserving comparable convergence rate. We develop a simple framework under which the equalizers for state-space channel
models can be designed using the existing methods for designing equalizers for FIR models. In particular, a MIMO MMSE-DFE equalizer is developed for state-space models. When only estimates of the channel are available to the receiver, the equalization performance is affected by the channel estimation accuracy. Because reduced-order state-space models can provide lower $H_2$ channel estimation error than reduced-length FIR models, state-space based equalizers typically exhibit significantly smaller symbol error rate than FIR-based ones. Thus, state-space channel models can be a more robust choice than FIR models in the presence of model order selection error. Numerical simulation also shows that adaptive state-space based receivers have marginally slower but comparable convergence rates compared to FIR based adaptive equalizers. Furthermore, we attempt to design a state-space channel equalizer that is especially suited for blockwise or framed data transmission such as GSM systems. A new state-space equalization scheme is developed based on the theory of fixed-interval smoothing and fixed-interval deconvolution [29]. It is combined with the recursive MOESP channel estimator to form a complete receiver processing procedure.
1

Introduction

Digital communication using multiple transmit and receive antennas has been one of the most important technical developments in modern communications. In a rich scattering environment, multi-input multi-output (MIMO) systems offer significant capacity gain at no cost of extra spectrum [14]. Furthermore, space-time channel codes [38] can be applied to build spatial redundancy in the transmitted signal such that maximum spatial diversity is achieved.

Coherent MIMO transmission schemes, including both spatial multiplexing and space-time codes, assume the availability of a channel model at the receiver. Since any channel model is essentially an approximation of the physical channel, the goal of channel modeling is to find a model that is as close to the real channel as possible and maintains manageable complexity as well.

It is the subject of this thesis to explore the application of state-space models to represent MIMO frequency-selective channels with the motivation for better model approximation and lower symbol error rate (SER) for the corresponding channel equalizers. MIMO systems have been studied in the control engineering community since the 1950s. Many theoretical tools and methods were developed to facilitate our understanding of such systems. This thesis is an attempt to borrow some of these ideas, for instance, subspace system identification methods, for the design and understanding of modern MIMO communication systems. In fact, the
interdisciplinary feature is a major theme of this work and a key contribution of the thesis. We hope that this work would spur further development connecting control theory and communication theory.

1.1 MIMO Channel Models

The design of a communication system begins with the understanding of the media through which the message will be transmitted. This understanding is usually embodied in the abstract form of a mathematical model which characterizes the major properties of the media. These properties are the ones that are deemed to be important for the design of an efficient transmitter and receiver.

Models for the single-input single-output (SISO) wireless fading channel have been developed since the early days of radio communications and have reached a certain maturity [5, 7, 18]. On the other hand, the development of MIMO wireless channel models was motivated by the findings on MIMO capacities[39] and transmission schemes [13] and is an active area of on-going research [51]. In this thesis, we shall focus on two types of discrete-time channel models: finite-length impulse response (FIR) model and state-space model.

1.1.1 FIR Models

An \( m \)-input, \( p \)-output, length-\( L \) inter-symbol interference (ISI) channel can be described by a symbol-rate sampled discrete-time FIR complex baseband model as follows

\[
y_k = \sum_{l=0}^{L-1} h_l u_{k-l} + n_k,
\]

where

- \( u_k \) - complex \( m \times 1 \) input vector at time \( k \);
- \( y_k \) - complex \( p \times 1 \) output vector at time \( k \);
- \( n_k \) - \( p \times 1 \) zero-mean circularly symmetric complex Gaussian noise vector with a
covariance matrix $N_0 I$;
$\{h_i\}_{i=0}^{L-1}$ – complex matrix channel taps of dimension $p \times m$.

Much of the efforts in modeling MIMO wireless channels has been concentrated on finding the connection between the FIR channel taps $\{h_i\}_{i=0}^{L-1}$ and the physical propagation parameters such as multipath arrival time, path amplitude, angle of arrival/angle of departure, angular spread, antenna spacing, and antenna polarization, etc [11, 16, 20, 37]. These important physical parameters affect both the deterministic and statistical characteristics of the tap values.

However, the major goal of this thesis does not lie in discovering such connections. Instead, we shall retreat to a higher level and examine the generic structure of all FIR models. We shall investigate the advantages and disadvantages of FIR models against the more general state-space models under the circumstance where the order, defined as McMillan degree, of the model is lower than the unknown true order of the physical channel. The McMillan degree of a $p \times m$ discrete-time transfer function matrix $h(z)$ is defined as the minimum number of delay elements needed to implement such a filter [41]. For a SISO transfer function, its McMillan degree is equal to the filter length minus one, which is simply the number of memories in the filter or the order of the polynomial.

FIR models have been widely used for wireless channels because of their simplicity and guaranteed stability. However, when the length of the FIR model happens to be smaller than that of the true channel, the approximation error of the model may be very poor, even though the underlying true channel is FIR [53]. This is especially true in wireless communications where the channel length varies significantly according to the surrounding environment.

On the other hand, the quality of state-space models degrades in a much more graceful way with respect to model order error. In general, reduced-order state-space models are able to provide significantly smaller channel estimation error than reduced-length FIR models [53]. At one level, this is an immediate property of the fact that the class of state-space models of a given order subsumes the class of FIR models of the same order.
1.1.2 State-Space Models

A $n$th-order state-space model is given as follows:

$$
x_{k+1} = Ax_k + Bu_k
$$

$$
y_k = Cx_k + Du_k + n_k,
$$

where $x_k$ is the $n \times 1$ state vector at time $k$. Any proper rational transfer function matrix $h(z)$ can be realized by state-space systems [6]. This realization is not unique. Among all the equivalent state-space realizations of a given transfer function, the ones with the minimum number of state variables are called minimal realizations and this number of states is equal to the McMillan degree, or the order, of the system.

State-space models are a more general class of models of which FIR models are a special case. In fact, the FIR model (1.1) is realized by the state-space model, with appropriate truncations of zero submatrices, in controllable canonical form with

$$
A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
I & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 \\
0 & 0 & I & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
I \\
0 \\
\vdots \\
0
\end{bmatrix},
$$

$$
C = [h_1, \ldots, h_{L-1}], \quad D = h_0,
$$

$$
x_k = \begin{bmatrix}
u_{k-1} \\
u_{k-2} \\
\vdots \\
u_{k-L+1}
\end{bmatrix}.
$$

A finite-order ($n < \infty$) state-space system can have an equivalent impulse response with an infinite length. It is this fact that makes state-space models more robust to model order selection error than FIR models. Suppose that the true channel has an impulse response of length $L_0$

$$
\{c_0, c_1, \ldots, c_{L_0-1}\},
$$
and the order (McMillan degree) of the true channel is $n_0$. Without knowing the true length $L_0$ a priori, a system designer might choose an FIR channel model of a reduced length $L_0 - 1$,

$$\{h_0, h_1, \ldots, h_{L_0-2}\}.$$ 

Then it is unlikely for the model to catch the energy contained in the tail tap $c_{L_0-1}$. When the energy of the tail tap happens to be significant, there would be a considerable mismatch between the chosen FIR model and the true channel. On the other hand, if a state-space model with a reduced order $n_0 < n$ is applied for the channel, the corresponding impulse response could have a length $L \geq L_0$, i.e.

$$\{\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{L_0-1}, \tilde{h}_{L_0}, \ldots\},$$

so that it can capture all the taps of the true channel, as shown in Figure 1.1. It could happen that an $n_0$th-order state-space model would have non-zero tap values beyond tap index $L_0 - 1$. However, these tail taps can be very small or nearly negligible for a reasonably good state-space approximation.

## 1.2 Problem Statement and Current State of the Art

The goal of this thesis work is to explore the benefits of applying state-space models for MIMO wireless channels and to design the corresponding receivers based on state-space channel models. We would like to compare the state-space-based receivers to the conventional FIR-based receivers. The comparison results, both advantages and disadvantages, will be helpful for system designers to understand the new approaches and to apply them in appropriate circumstances.

In particular, we are interested in the following problems:

1. **Model Quality:**
   - Can we define a measure of model quality against the true channel?
2. Channel Estimation:

- How to estimate the state-space channel models when given training data?
- How is the channel estimation error of the state-space algorithms when compared to FIR ones?
- How to estimate the state-space channel recursively when the training data is non-contiguous or when the channel is time-variant?
- How is the convergence behaviour of the recursive algorithm?

3. Channel Equalization:

- How to design equalizers with state-space channel models?
How is the performance of state-space equalizers compared to FIR ones?

What is the effect of channel estimation error?

State-space channel models have been used extensively by many authors to track the temporal variation of SISO wireless fading channels for TDMA systems [55, 24, 40, 4] and CDMA systems [33]. In this case, the FIR channel taps are represented by the state variables in the state-space model. Thus the state equation establishes an auto-regressive (AR) model for the time variation of the channel. Normally the AR model is configured to match the assumed underlying power spectral density of channel, such as the well-known “Jakes Model” [18]. Often times 1st-order or 2nd-order AR models (AR-1 or AR-2) are used for simplicity. Increasing the order of the AR model will result in better approximation at the expense of higher complexity. The output equation then represents the linear convolution of the input data and the time-varying channel impulse response with noise added. Recently, similar ideas are extended to the modeling of time-varying multi-antenna fading channels [49, 21].

Based on the above setting, optimal time-varying channel estimation or tracking can be implemented with Kalman Filter using training data [24, 40, 21] since the channel taps are modeled as state variables. Furthermore, the tracking device can also work in a decision feedback mode whereby the detected symbols at the receiver are used to refine the channel estimates.

Once estimates of the channel taps are obtained, channel equalizers can be constructed using the standard design approaches for SISO channels [32, 4]. The optimum maximum likelihood sequence estimator for MIMO channels was developed in [43]. Infinite-length minimum mean-squared error decision feedback equalizer (MMSE-DFE) has been recently proposed for MIMO frequency-selective channels in [8, 50]. The design of finite-length MIMO MMSE decision feedback equalizers is developed in [2].

Note that all the previous work mentioned above assumes an underlying FIR channel model. Although state-space models are used for channel tracking, they
are constrained to certain structures in which the FIR channel taps are included explicitly. In this thesis, we propose to release this FIR constraint on the model and to apply a generic state-space model for the wireless fading channel. With this idea, we shall proceed to answer the questions brought about at the beginning of this section.

1.3 Contributions

Corresponding to the three classes of main problems proposed in Section 1.2, our contributions also lie in those three areas.

1. State-Space Channel Modeling:

- We propose to use generic state-space models for MIMO frequency-selective channels for the benefit of high modeling quality under channel order uncertainty.
- We quantify the model approximation error using the upper bound of the minimum $H_\infty$ norm of the difference between the original channel and the approximated channel model [52].
- We show that state-space models always maintain lower $H_\infty$ approximation error than the FIR models with the same model order for either spatially uncorrelated MIMO channels or correlated ones.
- Particularly, the performance advantage of state-space models becomes more evident as the spatial correlation increases due to their ability to capture the spatial structure.

2. State-Space Channel Estimation:

- We propose a training-based recursive estimation algorithm [53] for state-space channel models based on recursive subspace system identification (SSI). The algorithm evolves from the recursive Ordinary
• MOESP algorithm [46, 28] which belongs to the family of SSI methods.

• We show that the proposed state-space channel estimation algorithm provides lower-order models with smaller $H_2$ channel estimation error compared to FIR-based Recursive Least Squares (RLS) channel estimator. This fact makes state-space models a robust choice against model order uncertainties.

• We make necessary modifications on SSI methods, which traditionally assume contiguous training sequences, for utilizing non-contiguous training data that commonly exists in communication systems. We show that, when the length of the training sequence, is sufficiently large compared to the order or the McMillan degree of the model of the MIMO channel, the modified non-contiguous-data approach retains similar performance to the original contiguous-data approach.

• We apply a new RLS-like exponential forgetting scheme on the training data to improve the adaptation rate of the modified recursive MOESP algorithm.

• We show that the proposed state-space channel estimation algorithm exhibits comparable convergence rate to the FIR-based Recursive Least Squares channel estimator. This addresses the concern in using state-space models to represent the channel is that they may result in slower convergence rate of the adaptive channel estimator since the number of parameters to be estimated in a state-space model is generally greater than that of an FIR model of the same order.

3. State-Space Channel Equalization:

• We show that smaller $H_2$ channel estimation error of the reduced-order state-space models would lead to lower symbol error rate for channel equalization when compared to FIR models. Hence state-space models
are robust to model uncertainties in terms of not only model quality measures but also channel equalization performance.

- We propose a framework to represent the blockwise input-output data relationship of a state-space model in a similar way to that of an FIR model. This framework is very convenient because, by realizing the similarity in blockwise data models, the equalizers for state-space models can be designed using modifications of the existing algorithms for FIR ones such as [1, 2].

- We develop a finite-length MIMO MMSE decision feedback equalizer for state-space channel models.

- We investigate the effect of channel estimation error on the state-space channel equalization performance. We show that the effective signal-to-noise ratio (SNR) at the decision point of the MIMO MMSE-DFE equalizer is related to $H_2$ channel estimation errors under the assumption that the decisions of previously detected symbols are all correct. The analysis and simulation results demonstrate that the smaller is the $H_2$ channel estimation error, the higher is the effective SNR, and, in turn, the lower is the symbol error rate.

- We carry out a statistical numerical performance comparison of a state-space-based receiver and an FIR-based receiver corresponds to this in showing that the former provides significantly smaller symbol error rate than the latter for reduced-order channel models.

- We show that the symbol error probability convergence rate of the proposed state-space receiver is comparable to that of the FIR-based receiver.

Besides the above main contributions, we also attempt to design a state-space channel equalizer that is especially suited for blockwise or framed data transmission such as GSM systems. A new state-space equalization scheme is developed based
on the theory of fixed-interval smoothing and fixed-interval deconvolution [29]. It is combined with the recursive MOESP channel estimator to form a complete receiver processing procedure.

1.4 Outline

Most chapters of the thesis are modified versions of published journal papers, submitted journal manuscripts, conference papers, and technical notes. The modifications are mainly more detailed description of the theoretical tools, more discussion on simulation setups, and more numerical results. Thus, the chapters are mostly self-contained and can be read in any order.

This thesis is organized as follows.

Chapter 2 discusses the advantage of state-space models over FIR models in terms of robust model approximation quality under model order uncertainty. The model approximation error is quantified using the upper bound of the minimum \( H_\infty \) norm of the difference between the original channel and the approximated channel model. It is shown that state-space models always maintain lower \( H_\infty \) approximation error than the FIR models with the same model order for either spatially uncorrelated MIMO channels or correlated ones. Chapter 2 also contains the development of a state-space recursive channel estimation algorithm based on subspace system identification methods. We propose a modification of the traditional SSI algorithms to suit non-contiguous training data. A RLS-like weighting scheme is used to improve the convergence rate. Numerical simulation results using practical channel models are also presented to compare the state-space-based recursive channel estimation algorithm to the FIR-based recursive least squares (RLS) channel estimator.

Chapter 3 investigates the indirect MIMO channel equalization problem with state-space models, that is, given a (possibly estimated) state-space channel model, how to design the corresponding channel equalizers. After a brief review of equalizer design for FIR models, we shall present an observation that the blockwise
input-output data relationship of state-space models can be arranged in a similar form to that of FIR models. Based on this data representation, the design of a MIMO MMSE-DFE equalizer for state-space channel models can be implemented using the existing method for FIR models assuming perfect channel knowledge at the receiver. In Section 3.4, the MIMO MMSE-DFE is designed to accommodate channel estimation errors and the analysis of the SNR at the decision point is presented. Numerical comparison between a state-space-based receiver and an FIR-based receiver is provided and shows that the former provides significantly smaller symbol error rate than the latter for reduced-order channel models while exhibiting marginally slower but comparable convergence rate.

Chapter 4 describes our attempt to design a state-space channel equalizer that is especially suited for blockwise or framed data transmission such as GSM systems. The proposed equalization scheme exploits the whole received data block for decoding every bit in it and is based on fixed-interval deconvolution [29]. This equalizer is combined with the recursive MOESP channel estimator to form a complete receiver processing procedure.

Chapter 5 concludes this thesis and provides suggestions on future research directions.
2 MIMO Channel Estimation with State-Space Models

2.1 Overview

Digital communication using multiple transmit and receive antennas has been one of the most important technical developments in modern communications. In a rich scattering environment, multi-input multi-output (MIMO) systems offer significant capacity gain at no cost of extra spectrum [14]. Furthermore, space-time channel codes [38] can be applied to build spatial redundancy in the transmitted signal such that maximum spatial diversity is achieved. Coherent space-time processing schemes assume the availability of a channel model at the receiver. Therefore, this model needs to be estimated at the receiver end.

Since any channel model is essentially an approximation of the physical channel, the goal of channel modeling is to find a model that is as close to the real channel as possible and maintains manageable complexity as well. FIR models have been widely used for wireless channels for their simplicity and guaranteed stability. On the other hand, state-space models are able to provide better modeling performance since they are a more general class of models that includes FIR models as a subset. It is found that state-space models are able to provide low-order models of high-
quality channel approximation.

Furthermore, when the channel is frequency-selective and has multiple inputs and multiple outputs, an FIR model can be very non-parsimonious and contain excessive redundancy since it represents the sub-channels of a MIMO channel with separately parametrized finite-length impulse responses. In situations where there exists correlation between the sub-channels, it may be beneficial to adopt a state-space model which models the whole channel as a single entity and hence captures the structure in the MIMO channel while allowing a more parsimonious description of it.

It is the subject of this chapter to explore the application of state-space models to represent MIMO frequency-selective wireless channels in the hope for better model approximation performance and more parsimonious parametrization. We quantify the model approximation error using the upper bound of the minimum $H_\infty$ norm of the difference between the original channel and the approximated channel model. It is shown that state-space models always maintain lower $H_\infty$ approximation error than the FIR models with the same model order for either spatially uncorrelated MIMO channels or correlated ones. Particularly, the performance advantage of state-space models becomes more evident as the spatial correlation increases due to their ability to capture the spatial structure.

For quasi-static or slowly-varying channels, training-based channel estimation is very common in practice. Therefore, an algorithm for estimating MIMO state-space models using training data is needed. We propose such a state-space channel estimation algorithm based on recursive subspace system identification (SSI). The simulations of this chapter demonstrate comparable (but marginally slower) adaptation performance of state-space methods but improved approximation performance compared to FIR-based recursive least squares (RLS) methods.

Subspace system identification (SSI) algorithms are a group of methods that identify a MIMO state-space system in a straightforward way using numerically robust computation tools such as singular value decomposition (SVD) and QR factorization [47, 42]. Furthermore, SSI algorithms provide a direct way to control
the complexity of the estimated channel model. The order of the channel model can be selected by the user by choosing the number of the largest singular values of the estimated extended observability matrix to include. In addition, the modeling error is assessable as it is related to the sum of the neglected Hankel singular values of a full-order model.

One issue with SSI-based MIMO channel estimation is that the training sequences need not be contiguous in the data stream in practical mobile communication systems. Instead, they appear as the overhead or mid-amble of a frame of data. More specifically, in the Global System for Mobile communications (GSM), a 26-bit-long segment in the middle of each 156-bit frame is allocated for the insertion of the training sequence which is the only data known to both the transmitter and the receiver [35]. Since the traditional SSI methods assume the availability of a contiguous input-output data stream, they need to be modified to suit the situation of MIMO channel estimation. It will be shown that, when the length of the training sequence, \( N_t \), is sufficiently large compared to the order or the McMillan degree of the model of the MIMO channel, the modified non-contiguous-data approach retains similar performance to the original contiguous-data approach.

Furthermore, a recursive version of the SSI algorithms is needed when the channel becomes time-varying as is fundamentally the case in mobile communications. In order to update the channel estimate adaptively upon receiving new training data, a recursive SSI algorithm is proposed based on the recursive ordinary Multivariable Output-Error State Space (MOESP) model identification algorithm in [28]. The difference between the recursive MOESP algorithm in [28] and our modified recursive MOESP lies in the way they weight the input-output data with the exponential forgetting factor, which is normally included in recursive algorithms to adjust the effect of past input-output data on the current channel estimate. The proposed recursive SSI algorithm for non-contiguous data is tested on trial channels by computer simulation and exhibits faster convergence than the one in [46, 28].

This chapter is organized as follows. Section 2.2 reviews the problem of adap-
tive channel equalization and estimation and describes the motivation for using state-space models for MIMO channels. We shall investigates the channel approximation error of state-space models and compares it to that of FIR models. Special attention is given to spatially correlated channels whose spatial structure is accessible to state-space models, but generally not to FIR models, to produce low-order models of good approximation quality. Section 2.3 provides a detailed discussion of the state-space-based channel estimation algorithm used in the simulation study in Section 2.2. We present the state-space channel model and propose a modification of the traditional SSI algorithms to suit non-contiguous training data. A modified recursive MOESP is also developed for identifying time-varying channels. Numerical simulation results using practical channel models are also presented to compare the state-space-based recursive channel estimation algorithm to the FIR-based recursive least squares (RLS) channel estimator. Section 2.4 provides the conclusion to the chapter.

2.2 State-Space Models for MIMO Wireless Channels

2.2.1 Adaptive Channel Equalization

In wireless communications, the transmitted signal is reflected and attenuated by various objects before it reaches the receive antenna. The received signal is a sum of delayed and weighted versions of the original signal. When the maximum delay is significant enough compared to the symbol period of the transmission, the interaction between these waves causes the so-called inter-symbol interference (ISI). The single-input-single-output ISI channel can be well described by a symbol-sampled discrete-time FIR model as follows

\[ y_k = \sum_{i=0}^{L-1} h_i u_{k-i} + n_k, \quad (2.1) \]
where
\[ u_k, y_k \] \(- k\)th input and output symbol, respectively;
\[ h_i \] \(- length-L\) channel impulse response;
\[ n_k \] \(- additive white Gaussian noise.\)

The received symbol at time \( k, y_k \), contains not only the contribution from the transmitted symbol at time \( k, u_k \), but also that from the symbols preceding \( u_k \), i.e. \( u_{k-1}, \ldots, u_{k-L+1} \). In order to reduce the ISI, an equalizer is employed at the receiver to compensate for the channel effect.

Most equalization schemes assume that a channel model is known to the receiver. However, in most cases the channel characteristics are unknown a priori. Moreover, the channel response tends to be time-varying in wireless mobile communication systems. Therefore, the equalizer has to be designed to be capable of capturing and adjusting to the possibly time-varying channel characteristics.

In training-based equalization schemes, a pre-selected training sequence, known to both the transmitter and the receiver, is transmitted through the channel and captured by the receiver. The receiver uses the input-output data either to estimate the channel response and then compute the coefficients of the equalizer or to estimate directly the coefficients of the equalizer which could be regarded as some type of inverse of the channel. At this stage, the equalizer is working in the \textit{training mode}. Following the training period, during which the coefficients of the equalizer are computed, the equalizer estimates from the channel output the unknown information-bearing data symbols that are transmitted after the training sequence. In return, the estimated input can be used to adjust the coefficients of the equalizer to adapt it to the variation of the channel. This is called the \textit{decision-directed mode} of the adaptive equalizer. The to-be-proposed recursive MOESP state-space channel estimation algorithm can be used in both training stage and decision-directed stage to produce estimates of channel models.

In accordance with the training-based equalization algorithm, a training sequence is inserted in each frame, such as the one shown in Figure 2.1, to aid the channel estimation at the receiver. The training sequence is now to both the
receiver and the transmitter.

<table>
<thead>
<tr>
<th>Training sequence</th>
<th>Data sequence</th>
<th>Training sequence</th>
<th>Data sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Frame</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.1: Frame structure for training-based channel estimation.

In this chapter we assume that the receiver works in the indirect manner, that is, it uses the input-output data to estimate the channel response first and then computes the coefficients of the equalizer. Furthermore, the channel is assumed to be quasi-static, i.e. it is time-invariant within each frame and changes independently from frame to frame. Our main interest is the estimation algorithm the receiver uses to estimate the channel.

With the advent of multiple antenna systems, the channel becomes a MIMO system so that MIMO system identification needs to be considered when estimating the channel. One approach is to build the MIMO model as a sequence of “stacked” or “vectorized” SISO channels using the fact that an FIR MIMO channel model has nearly the same form as its SISO counterpart except that each tap of the FIR MIMO channel is a gain matrix instead of a scalar as in (2.3) [15, 2]. This approach inherits the advantage of guaranteed stability of FIR models, and the channel estimation can usually be performed via RLS-type or LMS-type algorithms. However, because the subchannels are modeled separately in an FIR MIMO model, it can be redundant and inefficient for modeling a MIMO channel, especially when there exists correlation or structure between the subchannels.

2.2.2 State-Space Channel Models

Another approach to representing ISI channels is to use a state-space model

\[
\begin{align*}
\mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k \\
\mathbf{y}_k &= C\mathbf{x}_k + D\mathbf{u}_k + \mathbf{n}_k,
\end{align*}
\] (2.2)
where
\( \mathbf{u}_k \) – \( m \times 1 \) input vector at time \( k \);
\( \mathbf{y}_k \) – \( p \times 1 \) output vector at time \( k \);
\( \mathbf{x}_k \) – \( n \times 1 \) state vector at time \( k \);
\( \mathbf{n}_k \) – \( p \times 1 \) white Gaussian noise vector at time \( k \).

For the purpose of investigating the channel approximation performance of both state-space and FIR models, a reference channel is generated as the true channel. Then we compare the upper bounds of the minimum \( H_\infty \) channel approximation error of both state-space models and FIR models. Numerical results show the superiority of state-space models in providing high-quality channel representation.

**Generation of Reference MIMO Channels**

In order to evaluate the channel approximation performance of state-space and FIR models, a true or reference physical channel ought to be given as the reference of comparison. The ideal reference channel would be obtained directly through channel measurement. However, for simulation study, we could also use synthesized symbol-rate-sampled FIR MIMO channel models as the reference.

It may seem that using FIR reference channels is not fair to state-space models since FIR models would have the advantage of possessing the same structure as the reference channel. However, since we never know the true order of the physical channel in practice, any channel model we adopt is only an approximation of the physical channel and generally has a lower order than the true one. Therefore, we can maintain the fairness of the comparison by considering only the approximation performance of state-space or FIR models with orders strictly less than the true order of the reference channel.

The reference MIMO channel is assumed to be composed of multiple finite impulse responses. Assuming all the subchannels have the same length \( L \), an FIR MIMO channel with \( m \) transmit antennas and \( p \) receive antennas can be represented by assembling all the taps of the subchannel impulse responses with
the same delay $\tau$ into a matrix $H_{\tau}$ and representing the channel as a series of matrices

$$[H_0, H_1, \ldots, H_{L-1}], \quad H_{\tau} = \begin{bmatrix}
h_{11}(\tau) & h_{12}(\tau) & \cdots & h_{1m}(\tau) \\
h_{21}(\tau) & h_{22}(\tau) & \cdots & h_{2m}(\tau) \\
\vdots & \vdots & \ddots & \vdots \\
h_{p1}(\tau) & h_{p2}(\tau) & \cdots & h_{pm}(\tau)
\end{bmatrix}. \quad (2.3)$$

The simulated channels are taken to be square symmetric (i.e. $m = p$), and the length of the impulse responses is set to 6. The channel taps are generated as zero-mean circularly-symmetric complex Gaussian (ZMCSCG) random variables with equal covariance. The widely used exponential power delay profile (PDP) [31] is applied to all the subchannels to control the distribution of the average power of each tap,

$$E[\|H_l\|_F] = e^{-\lambda l}, \quad l = 0, 1, \ldots, L - 1, \quad (2.4)$$

where $\|\cdot\|_F$ represents the Frobenius norm of a matrix and $\lambda$ is a constant which decides how fast the tail of the impulse response decades. The simulations presented in this section were also performed using the COST 207 power delay profiles [12] and yielded conceptually similar results.

The MIMO channel is assumed to be uncorrelated in delay but correlated in spatial dimension, i.e.

$$E(\text{vec}(H_k)\text{vec}(H_l)^H) = \begin{cases} 
0, & k \neq l \\
R, & k = l
\end{cases},$$

where $R$ is a non-zero covariance matrix and $\text{vec}(\cdot)$ is an operator that stacks the columns of a matrix on top of each other to form a vector. We adopt a common model for the spatial correlation structure of $H_{\tau}$ [51],

$$H_{\tau} = (R_{RX})^{1/2}H_w(R_{TX})^{T/2}, \quad (2.5)$$

where $H_w$ is a $p \times m$ matrix with IID ZMCSCG elements. $R_{TX}$ and $R_{RX}$ are, respectively, the transmit covariance matrix and receive covariance matrix. For
Table 2.1: Configuration of the uniform linear arrays exploited by the base station (BS) and the mobile unit (MU) in an uplink transmission.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>TX/MU</th>
<th>RX/BS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Spread ∆</td>
<td>30°</td>
<td>5°</td>
</tr>
<tr>
<td>Angle of Arrival φ</td>
<td>−90°  ~ 90°</td>
<td>−90°  ~ 90°</td>
</tr>
<tr>
<td>Baseline Length (L_{bs})</td>
<td>0.3 m</td>
<td>1.0 m</td>
</tr>
<tr>
<td>Antenna Spacing (D/\lambda_c)</td>
<td>(\frac{L_{bs}}{(m-1)\lambda_c})</td>
<td>(\frac{L_{bs}}{(p-1)\lambda_c})</td>
</tr>
</tbody>
</table>

spatially white MIMO channels, both \(R_{TX}\) and \(R_{RX}\) are identity matrices. For spatially correlated MIMO channels, under the assumption of uniformly distributed angle of arrival, \(R_{TX}\) and \(R_{RX}\) can be modeled with closed-form expressions as functions of angular spread \(\Delta\), angle of arrival \(\phi\) and normalized antenna spacing \(D/\lambda_c\), where \(D\) is the antenna spacing and \(\lambda_c\) the carrier wavelength. See [34] for detailed development of the closed-form expressions.

The antenna configuration for spatially correlated MIMO channels, summarized by Table 2.1, is selected to simulate the uplink transmission. We assume that both the base station and the mobile unit exploit uniform linear arrays with fixed baseline length \(L_{bs}\) of 1.0 meter and 0.3 meters, respectively. Since the matrix taps of the channel impulse response in (2.3) generally correspond to paths from various clusters, their angles of arrival are generated as a uniformly distributed random variable on \([-90°, 90°]\). However, for simplicity, all the taps are assumed to have the same transmitter and receiver angular spread.

### 2.2.3 Upper Bounds of Minimum \(H_\infty\) Approximation Error

A multi-input multi-output linear system can be represented by its transfer function matrix \(G(e^{j\omega})\). The \(H_\infty\) norm of any stable system is defined as

\[
\|G\|_\infty \triangleq \max_{\omega} \bar{\sigma}[G(e^{j\omega})],
\]

where \(\bar{\sigma}[\cdot]\) represents the largest singular value of a matrix.
Supposing that $G(e^{j\omega})$ is the reference channel and $\hat{G}(e^{j\omega})$ is an approximation of $G(e^{j\omega})$, then the $H_\infty$ approximation error is given by

$$\epsilon_\infty = \max_\omega \bar{\sigma}[\hat{G}(e^{j\omega}) - G(e^{j\omega})].$$

(2.6)

The order of a MIMO system, as indicated by the McMillan degree of its transfer function matrix, is equal to the number of Hankel singular values of the system. Assuming that the Hankel singular values of an $n$th-order system are given by

$$\sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq \sigma_n(G),$$

a common approach to produce a $p$th-order ($p < n$) approximation of the original system is to keep only the largest $p$ Hankel singular values $\{\sigma_i(G)\}_{i=1}^p$ and remove the relatively small ones. As shown by [54] Theorem 7.11, the $H_\infty$ norm of the difference between a MIMO system and its optimal low-order state-space approximation is upperbounded by the sum of the neglected Hankel singular values,

$$\epsilon_{\infty, opt} \leq \sum_{i=p+1}^{n} \sigma_i(G).$$

(2.7)

See [54] for a detailed treatment of model approximation and Hankel singular values.

However, if we exert the constraint that the low-order approximated model must have finite-length impulse responses (FIR), the upper bound given in (2.7) no longer holds because FIR models only form a subset of state-space models. A procedure called “Nehari Shuffle” is proposed in [22] to produce a low-order length-$q$ FIR approximation to a MIMO linear system, which may or may not be FIR. If we express the transfer function of the original system as

$$G(z) = \sum_{k=0}^{q-1} g_k z^{-k} + z^{-(q-1)} \sum_{l=1}^{\infty} g_{l+q-1} z^{-l},$$

\[ \hat{G}^\text{head}(z) \] \[ \hat{G}^\text{tail}(z) \]
Figure 2.2: Upper bounds of minimum $H_\infty$ channel approximation error w.r.t reference channels of size $3 \times 3$ and $10 \times 10$. 

(a) Reference channels: $3 \times 3$ with order/McMillan degree 15.

(b) Reference channels: $10 \times 10$ with order/McMillan degree 50.
then an upper bound of the approximation error of the “Nehari Shuffle” procedure is given by the sum of all the Hankel singular values of the tail system \( G_{\text{tail}}(z) \),

\[
\epsilon_{\infty,NS} \leq \sum_{i=1}^{n_t} \sigma_i(G_{\text{tail}}),
\]

(2.8)

where \( n_t \) is the order of \( G_{\text{tail}}(z) \).

Figure 2.2 shows the upper bound of the minimum \( H_\infty \) approximation error of state-space models, \( \epsilon_{\infty,\text{opt}} \), and that of FIR models, \( \epsilon_{\infty,NS} \), for both spatially white channels and spatially correlated channels. Both of the full-order state-space model and the full-length FIR model achieve zero \( H_\infty \) approximation error. However, more attention should be given to the performance of low-order, instead of full-order, state-space models and FIR models as discussed in the beginning of Section 2.2.2.

Several observations on the performance of the low-order approximated models can be made. First, the number of low-order FIR models is significantly smaller than that of the low-order state-space models. This is because the order of an generic \( p \times m \) length-\( L \) FIR model, one whose subchannels do not share any common zeros, is given by

\[
\min\{m, p\} \times (L - 1).
\]

Since the order of an FIR model can only be reduced by decreasing the length of the impulse responses, the step size of reducing the order of an FIR model is at least \( \min\{m, p\} \). On the other hand, the step size of reducing the order of an state-space model is 1. Therefore, state-space models provide us with many more choices for low-order approximation of the reference channel than FIR models.

Secondly, for a fixed order of which there exists an FIR model, the approximation error of state-space models is significantly smaller than that of FIR models. Furthermore, when we increase the dimension of the MIMO channel from \( 3 \times 3 \) to \( 10 \times 10 \), the spatial correlation in the reference channels also increases because of the fixed-length array baselines at both the mobile unit and the base station. In turn, the performance gap becomes more evident as the spatial correlation in-
creases due to state-space models’ ability to represent the MIMO channel as a whole instead of a group of separate impulse responses as in FIR models.

2.3 Subspace System Identification for MIMO Channel Estimation

2.3.1 Subspace System Identification

The MIMO FIR model can be fitted by forming a parameter vector from the columns of $H_\tau$ in (2.3) and a regressor vector from a suitable rearrangement of the input signals at the transmit antennas. Then the Recursive Least Squares algorithm can be applied to solve this vector estimation problem.

For state-space models, a special class of estimation algorithms is needed. Subspace system identification (SSI) refers to a class of relatively recent algorithms, such as MOESP [47] and N4SID [42], which apply input-output system identification methods to determine directly a state-space realization of a system. The key idea of SSI methods is to estimate the extended observability matrix through the projection of future input-output data onto past input-output data based on the relationship between Hankel matrices of the input and output given by

$$Y_{0,i,t} = \Gamma_i X_{0,t} + H_i U_{0,i,t} + N_{0,i,t}, \quad i > n,$$

(2.9)

where $U_{0,i,t}$, $Y_{0,i,t}$ and $N_{0,i,t}$ are Hankel matrices of the input, output and noise, respectively, with the form of

$$Y_{0,i,t} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{t-i+1} \\ y_1 & y_2 & \cdots & y_{t-i+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{i-1} & y_i & \cdots & y_t \end{bmatrix}.

(2.10)$$

$X_{0,t}$ is a matrix containing the state vectors

$$X_{0,t} = \begin{bmatrix} x_0 & \cdots & x_{t-i+1} \end{bmatrix}.$$
\( \Gamma_i \) and \( H_i \) are, respectively, the extended observability matrix and the matrix of Markov coefficients,

\[
\begin{align*}
\Gamma_i &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-2} \\ CA^{i-1} \end{bmatrix} \\
H_i &= \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{i-2}B & \cdots & D \end{bmatrix}.
\end{align*}
\]

(2.11)

The system matrices \( A, B, C \) and \( D \) are computed based on an estimate of the extended observability matrix, \( \hat{\Gamma}_i \), which can be found by performing an LQ decomposition on the Hankel input-output data matrices as follows [47].

\[
\begin{bmatrix}
U_{0,i,t} \\
Y_{0,i,t}
\end{bmatrix} = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} Q,
\]

(2.12)

and

\( \hat{\Gamma}_i = R_{22} \).

See Appendix A or [47, 42, 25] for in-depth treatments of subspace system identification algorithms. Note that the identified state-space realization is not in a canonical form.

Based on the channel model given in (2.2), SSI methods require the input to satisfy the following requirements for the channel to be identifiable.

1. The input \( u_k \) is uncorrelated with the additive Gaussian white noise \( n_k \).
2. The input \( u_k \) is persistently exciting of order of at least twice the maximum order of the channel.
3. The symbols in the input sequence are contiguous and, for consistency, the number of inputs goes to infinity.

The first assumption is usually satisfied for wireless communication systems. The second one requires the training sequence to maintain a certain structure.
Also, notice that the third assumption places limitation on the application of SSI methods to channel estimation in wireless communication systems where the training sequences are usually not contiguous in time. Instead, these symbols are placed in the overhead or mid-amble of a frame and are separated by data symbols, as shown in Figure 2.1. Furthermore, realistic channels are often time-varying which suggests the use of a recursive version of SSI algorithms. Therefore, SSI needs to be reformulated before it can be applied to channel estimation.

2.3.2 SSI for Non-contiguous Data

With non-contiguous training data, the state evolution of the received data must be restarted at the frame boundaries. This is at variance with the standard formulation of SSI.

Consider the evolution of two contiguous $N_t$-symbol-long blocks of received data, with the first block commencing at time 0 and the second commencing immediately thereafter at $N_t$. For simplicity, we omit the noise and write the blocked state equations as,

\[ Y_{0,i,N_t-1} = \Gamma_i X_{0,N_t-1} + H_i U_{0,i,N_t-1}, \]
\[ Y_{N_t,i,2N_t-1} = \Gamma_i X_{N_t,2N_t-1} + H_i U_{N_t,i,2N_t-1}. \]

In standard SSI approaches, these are combined to form a new matrix equation,

\[ Y_{0,i,2N_t-1} = \Gamma_i X_{0,2N_t-1} + H_i U_{0,i,2N_t-1}. \]

This absorbs the data vectors into the Hankel structure of the new $U$ and $Y$ matrices. This adds further columns to the equation to be solved for the observability matrix $\Gamma_i$.

Next consider the availability of discontinuous $N_t$-symbol-long blocks of received data with the first block commencing at time 0 and the second at some later time $t$ with $t > N_t$. Then, we still achieve the relationships,

\[ Y_{0,i,N_t-1} = \Gamma_i X_{0,N_t-1} + H_i U_{0,i,N_t-1}, \]
\[ Y_{t,i,t+N_t-1} = \Gamma_i X_{t,t+N_t-1} + H_i U_{t,i,t+N_t-1}. \]
But now the absorption of the data into individual Hankel matrices is no longer possible, because of the non-contiguity of the received data.

We may, however, write an augmented equation composed from the above set,

\[
\begin{bmatrix}
Y_{0,i,N_t-1} 
\end{bmatrix} = \Gamma_i \begin{bmatrix}
X_{0,N_t-1} & X_{t,t+N_t-1} 
\end{bmatrix} + H_i \begin{bmatrix}
U_{0,i,N_t-1} 
\end{bmatrix} 
\]

This set of equations to be solved for $\Gamma_i$ is comparable to the contiguous-data set of equations. It has the same number of rows, $i$, and has $i - 1$ fewer columns. When the length of the training sequence, $N_t$, is sufficiently large compared to the dimension of the generalized observability matrix, $\Gamma_i$ (which depends on the state dimension of the model), then the non-contiguous-data approach is similar in its estimation power to the contiguous-data approach, as shown in Figure 2.3.

Figure 2.3 shows the mean $H_2$ channel estimation error of the MOESP algorithms with non-contiguous training sequences of various length, $N_t$. The x axis represents the length of the training data block, $N_t$. For each value of $N_t$, all the 160-symbol-long training data is divided to $\left\lceil \frac{160}{N_t} \right\rceil$ non-contiguous blocks based on which the MOESP algorithm for non-contiguous data is run over 100 random $2 \times 2$ 6-tap channels. The channels are generated in the same way as described in Section 2.3.4 with a spatial correlation coefficient $\rho = 0$, i.e. spatially uncorrelated subchannels. The full order of the channels is 12 and $i$ is set to be 13 in the MOESP algorithm. It can be seen that, as $N_t$ increases beyond 30, the $H_2$ error of non-contiguous training becomes very close to the contiguous training case where $N_t = 160$.

2.3.3 Recursive SSI for Time-varying Channels

When the channel is time-varying, channel estimation needs to be carried out in a recursive manner so that any newly received data can be used to produce an up-to-date estimate of the channel. An exponential forgetting factor should also be able to be included to reduce the effect of distant past data.
Figure 2.3: $H_2$ Channel estimation error of the MOESP algorithms with non-contiguous training data, $i = 13$.

Recursive SSI algorithms have been developed to meet this need. A recursive MOESP scheme was proposed in [46, 28], whose key idea is to update the estimate of the extended observability matrix with the newly received data using LQ factorization.

To give a brief summary of the recursive ordinary MOESP algorithm in [46, 28], consider that at time $t$ we have an LQ decomposition as in (2.12)

$$
\begin{bmatrix}
U_{0,i,t} \\
Y_{0,i,t}
\end{bmatrix} =
\begin{bmatrix}
R_{11}(t) & 0 \\
R_{21}(t) & R_{22}(t)
\end{bmatrix} Q(t).
$$

It was shown in [47] that $R_{22}(t)$ is an estimate of the extended observability matrix at time $t$, $\Gamma_i(t)$, in the sense that they share the same column space.

The task of a recursive MOESP scheme would be to compute $R_{22}(t + 1)$, an estimate of $\Gamma_i(t+1)$, based on the LQ factorization at time $t$ and the newly received data at time $t + 1$. Suppose at time $t + 1$ a new set of input-output data, $u_{t+1}$ and
$y_{t+1}$, becomes available. We can form the following data vectors

$$
\phi_u(t+1) = \begin{bmatrix}
u_{t-i+2} \\
\vdots \\
u_t \\
u_{t+1}
\end{bmatrix}, \quad \phi_y(t+1) = \begin{bmatrix}
y_{t-i+2} \\
\vdots \\
y_t \\
y_{t+1}
\end{bmatrix}.
$$

Then the new input-output Hankel data matrix at time $t+1$ can be formed by appending the above vectors to the right-hand end of the Hankel data matrix at time $t$ and applying an exponential forgetting factor $\lambda \in (0, 1]$. We perform an LQ factorization

$$
\begin{bmatrix}
U_{0,i,t+1} \\
Y_{0,i,t+1}
\end{bmatrix} = \begin{bmatrix}
\lambda U_{0,i,t} & \phi_u(t+1) \\
\lambda Y_{0,i,t} & \phi_y(t+1)
\end{bmatrix}
\begin{bmatrix}
Q(t) & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
R_{11}(t+1) & 0_{mi \times pi} & 0_{mi \times 1} \\
R_{21}(t+1) & \lambda R_{22}(t) & \phi(t+1)
\end{bmatrix}
= \begin{bmatrix}
R_{11}(t+1) & 0 \\
R_{21}(t+1) & R_{22}(t+1)
\end{bmatrix} Q(t+1),
$$

where the $pi \times 1$ vector $\phi(t+1)$ is obtained through a series of Givens rotations on the large lower triangular matrix. $Q(t+1)$ is a matrix with orthogonal rows, i.e. $Q(t+1)Q(t+1)^H = I$. Thus $R_{22}(t+1)$ is obtained through updating the LQ decomposition with the new data.

Notice that, if the aforementioned recursive procedure is carried out from time 0 to time $t$, the effective Hankel output data matrix used by the MOESP algorithm
is given by
\[
Y^\lambda_{0,i,t} = \begin{bmatrix}
\lambda^{t-i+1}\phi_y(i-1), \ldots, \lambda\phi_y(t-1), \phi_y(t) \\
\lambda^{t-i+1}y_0 & \ldots & \lambda y_{t-i} & y_{t-i+1} \\
\vdots & \vdots & \vdots \\
\lambda^{t-i+1}y_1 & \ldots & \lambda y_{t-2} & y_{t-1} \\
\lambda^{t+i-1}y_{i-1} & \ldots & \lambda y_{t-1} & y_t
\end{bmatrix},
\]
(2.15)

with \(U^\lambda_{0,i,t}\) having similar form. The input-output data \(\{y_\tau\}_{\tau=0}^t\) and \(\{u_\tau\}_{\tau=0}^t\) are not weighted according to the time instant they are received or transmitted, but according to which column they are positioned in the weighted Hankel matrices \(Y^\lambda_{0,i,t}\) and \(U^\lambda_{0,i,t}\). Within each column of \(Y^\lambda_{0,i,t}\) or \(U^\lambda_{0,i,t}\), the newest data enjoys the same weight as the past data. Therefore, the channel estimation algorithms using data Hankel matrices with this weighting structure may not be able to respond promptly to the channel variation represented in the newest input-output data. That is, this weighting approach may result in a relative slow convergence rate of the estimation algorithm.

**Recursive MOESP with an Improved Weighting Scheme**

In order to improve the convergence rate, we propose that the exponential forgetting factor should be applied according to the time at which the data symbols are transmitted or received. This weighting scheme originates from the weighting approach used by the RLS algorithm. The new effective Hankel output matrix \(\tilde{Y}^\lambda_{0,i,t}\) is given by
\[
\tilde{Y}^\lambda_{0,i,t} = \begin{bmatrix}
\lambda^i y_0 & \ldots & \lambda^i y_{t-i} & \lambda^{i-1}y_{t-i+1} \\
\vdots & \vdots & \vdots \\
\lambda^i y_1 & \ldots & \lambda^{i-1}y_{t-2} & \lambda y_{t-1} \\
\lambda^{i+i-1}y_{i-1} & \ldots & \lambda y_{t-1} & y_t
\end{bmatrix},
\]
(2.16)

with the new effective Hankel input data matrix \(\tilde{U}^\lambda_{0,i,t}\) having the same form.
Given the new Hankel data matrices, it is necessary to develop the channel identification procedure accordingly. Note that $\tilde{Y}_{0,i,t}^\lambda$ and $\tilde{U}_{0,i,t}^\lambda$ can be written as

\begin{align}
\tilde{Y}_{0,i,t}^\lambda &= \Lambda_p Y_{0,i,t} \Lambda, \quad \text{(2.17)} \\
\tilde{U}_{0,i,t}^\lambda &= \Lambda_m U_{0,i,t} \Lambda, \quad \text{(2.18)}
\end{align}

where $p$ and $m$ are, respectively, the number of outputs and that of inputs. $\Lambda_p$, $\Lambda_m$ and $\Lambda$ are given by

\begin{align*}
\Lambda_p &= \text{diag}\{\lambda^{-1} I_p, \ldots, \lambda^{I_p}, I_p\}, \\
\Lambda_m &= \text{diag}\{\lambda^{-1} I_m, \ldots, \lambda^{I_m}, I_m\}, \\
\Lambda &= \text{diag}\{\lambda^{-1}, \ldots, \lambda, 1\}.
\end{align*}

With (2.17) and (2.18), the original Hankel matrix relationship given in (2.9) can be rearranged to

\begin{align}
\Lambda_p Y_{0,i,t} \Lambda &= \underbrace{\Lambda_p \Gamma_i X_{0,t} \Lambda}_{\triangleq \Gamma_i^\lambda} + \underbrace{\Lambda_p H_i \Lambda_m^{-1} \Lambda_m U_{0,i,t} \Lambda}_{\triangleq H_i^\lambda} + \Lambda_p N_{0,i,t} \Lambda, \\
\tilde{Y}_{0,i,t}^\lambda &= \Gamma_i^\lambda X_{0,t}^\lambda + H_i^\lambda \tilde{U}_{0,i,t}^\lambda + \tilde{N}_{0,i,t}^\lambda. \quad \text{(2.19)}
\end{align}

Now (2.19) can be used as the basis of the recursive MOESP algorithm in place of (2.9). As a comparison, the conventional recursive MOESP is based on the relationship

\begin{align}
Y_{0,i,t} \Lambda &= \Gamma_i X_{0,t} \Lambda + H_i \Lambda_m^{-1} \Lambda_m U_{0,i,t} \Lambda + N_{0,i,t} \Lambda. \quad \text{(2.20)}
\end{align}

Since (2.19) produces estimates of $\Gamma_i^\lambda$ and $H_i^\lambda$ as opposed to $\Gamma_i$ and $H_i$ of (2.20), it is necessary to modify the procedure for computing $A$, $B$, $C$ and $D$ in terms of the new structure of $\Gamma_i^\lambda$ and $H_i^\lambda$.

**Estimating C and A**

Form the following data vectors that contain the newest input-output data at time
\[ t + 1 \]
\[ \tilde{\phi}_u(t + 1) = \begin{bmatrix} \lambda^{i-1} u_{t-i+2} \\ \vdots \\ \lambda u_t \\ u_{t+1} \end{bmatrix}, \quad \tilde{\phi}_y(t + 1) = \begin{bmatrix} \lambda^{i-1} y_{t-i+2} \\ \vdots \\ \lambda y_t \\ y_{t+1} \end{bmatrix}. \]

Then, since \( \tilde{Y}_{0,i,t}^\lambda \) and \( \tilde{U}_{0,i,t}^\lambda \) satisfy the relation
\[
\begin{bmatrix} \tilde{U}_{0,i,t+1}^\lambda \\ \tilde{Y}_{0,i,t+1}^\lambda \end{bmatrix} = \begin{bmatrix} \lambda \tilde{U}_{0,i,t}^\lambda & \tilde{\phi}_u(t + 1) \\ \lambda \tilde{Y}_{0,i,t}^\lambda & \tilde{\phi}_y(t + 1) \end{bmatrix},
\]
the same procedure as described in (2.14) can be applied to update the estimate of the extended observability matrix at time \( t + 1 \), \( \hat{\Gamma}_i^\lambda(t + 1) \), based on which \( \hat{C}(t + 1) \) and \( \hat{A}(t + 1) \) can be computed using the following relation
\[
C = \lambda^{-i+1} \Gamma_i^\lambda(1 : p, :), \quad (2.21)
\]
\[
\Gamma_i^\lambda(1) A = \lambda \Gamma_i^\lambda(2), \quad (2.22)
\]
where the matrix index notation of MATLAB is used and
\[
\Gamma_i^\lambda(1) = \Gamma_i^\lambda(1 : (i - 1)p, :) = \begin{bmatrix} \lambda^{i-1} C \\ \lambda^{i-2} CA \\ \vdots \\ \lambda CA^{i-2} \end{bmatrix},
\]
\[
\Gamma_i^\lambda(2) = \Gamma_i^\lambda(p + 1 : ip, :) = \begin{bmatrix} \lambda^{i-2} CA \\ \lambda^{i-3} CA^2 \\ \vdots \\ CA^{i-1} \end{bmatrix}.
\]

**Estimating B and D**

In order to estimate \( B \) and \( D \), we define the orthogonal projection onto the left nullspace of \( \Gamma_i^\lambda \) as
\[
P = I - \Gamma_i^\lambda(\Gamma_i^{\lambda H}\Gamma_i^\lambda)^{-1}\Gamma_i^{\lambda H},
\]
where \((\cdot)^{H}\) represents the complex conjugate transpose of a matrix. By using \(P\) to remove the term \(\Gamma_{i}^{\lambda}X_{0,t}^{\lambda}\) in (2.19), we obtain

\[
P\tilde{Y}_{0,i,t}^{\lambda} = PH_{i}^{\lambda} + P\tilde{N}_{0,i,t}^{\lambda} \tilde{U}_{0,i,t}^{\lambda} \triangleq V
\]

\[
K = PH_{i}^{\lambda} + V. \tag{2.23}
\]

By decomposing \(K\), \(P\) and \(V\) into

\[
K = \begin{bmatrix} K_{1} & K_{2} & \ldots & K_{i} \end{bmatrix}
\]

\[
P = \begin{bmatrix} P_{1} & P_{2} & \ldots & P_{i} \end{bmatrix}
\]

\[
V = \begin{bmatrix} V_{1} & V_{2} & \ldots & V_{i} \end{bmatrix},
\]

where \(\{K_{i}\}_{i=1}^{i} \), \(\{P_{i}\}_{i=1}^{i} \) and \(\{V_{i}\}_{i=1}^{i} \) are \(ip \times p\) matrices, we can re-organize (2.23) to be

\[
\begin{bmatrix} K_{1} \\ K_{2} \\ \vdots \\ K_{i} \end{bmatrix} = \begin{bmatrix} P_{1} & P_{2} & P_{3} & \ldots & P_{i} \\ P_{2} & P_{3} & \ldots & P_{i} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{i} & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda \Gamma_{i}^{(1)} \end{bmatrix} \begin{bmatrix} D \\ B \end{bmatrix} + \begin{bmatrix} V_{1} \\ V_{2} \\ \vdots \\ V_{i} \end{bmatrix}. \tag{2.24}
\]

Therefore, the Least Squares solution for \(B\) and \(D\) can be obtained from (2.24). This procedure can be simplified by applying LQ decomposition as proposed in [47].

In summary, (2.21), (2.22) and (2.24) are very similar to their counterparts in the original MOESP algorithm [47] except they are modified to accommodate the new RLS-like exponential weighting scheme in (2.16).

Figure 2.4 shows the learning curves of the recursive MOESP algorithms with different weighting scheme applied to a 2 \(\times\) 2 6-tap MIMO channel as described in Section 2.2.2 with \(\rho = 0.8\). The RLS-like weighting approach has a clear advantage over the conventional one since the former exhibits a faster convergence rate and maintains the same steady-state performance as the latter. It is also worthwhile to point out that the recursive MOESP with RLS-like weighting has nearly the same computation complexity as the conventional one.
2.3.4 Performance of the Recursive State-Space MIMO Channel Estimation Algorithm

In this section, we present the performance of the state-space-based recursive MOESP channel estimation algorithm compared to the FIR-based recursive least squares (RLS) channel estimator. Besides the comparison in terms of steady-state channel estimation error, we are also interested in the comparison of convergence rate because one tends to expect state-space methods would exhibit a much slower convergence than FIR methods due to the larger number of parameters in state-space models. In the simulation, it is found that the convergence rate of the state-space method and that of the FIR method are comparable.

Steady-State Performance

A computer experiment was carried out in order to compare the modeling performance of state-space models and that of FIR models. For a given reference channel, non-recursive batch versions of both the state-space-based ordinary
Figure 2.5: $H_2$ channel estimation error of state-space-based batch MOESP algorithm and FIR-based least squares algorithm.
MOESP algorithm [45] and FIR-based least squares algorithm are applied to produce estimated channel models with various orders. The estimation performance is measured in terms of the $H_2$ norm of the difference between the estimated channel models and the reference channel. Notice we only compare the batch estimation methods in this section.

Figure 2.5 shows the channel estimation performance of both the state-space-based MOESP algorithm and the FIR-based least squares algorithm. The $H_2$ channel estimation error is presented as a function of model order with the full order being 15. The order of a model is defined as the McMillan degree of its transfer function matrix.

Several observations on the modeling performance can be made. Firstly, state-space models can provide better model approximation performance that FIR models for both spatially white MIMO channels and spatially correlated ones except for the full-order models. The full-order FIR models have the advantage that they have exactly the same structure as the reference channel. In practice, however, without knowing the true order of the physical channel, any channel model we adopt is only an approximation of the physical channel and generally has a lower order than the true one. Therefore, it is more important to examine the performance of reduced-order models. Then the advantage of state-space models over FIR models is evident. It comes from the fact that the state-space model is a more general class of models which includes FIR models as a subset.

Secondly, the number of low-order FIR models is significantly smaller than that of the low-order state-space models. This is because the order of an generic $p \times m$ length-$L$ FIR model, one whose subchannels do not share any common zeros, is given by

$$\min\{m, p\} \times (L - 1).$$

Since the order of an FIR model can only be reduced by decreasing the length of the impulse responses, the step size of reducing the order of an FIR model is at least $\min\{m, p\}$. On the other hand, the step size of reducing the order of an
state-space model is 1. Therefore, state-space models provide us with many more choices for low-order approximation of the reference channel than FIR models.

It can also been seen that FIR models are not able to take advantage of the spatial correlation structure in MIMO channels since the two FIR curves for both spatially white channels and for correlated channels nearly overlap. In contrast, state-space models use the spatial correlation to improve the model approximation performance, although the improvement does not appear to be significant.

Furthermore, the channel power delay profile has a large impact on how the channel approximation performance degrades when the model order decreases away from the full order. The reduced-order models for channels with large tail taps (Figure 2.5 (a)), suffer bigger loss in channel approximation performance compared to those for channels with small tail taps (Figure 2.5 (b)). However, for either type of power delay profile, state-space models always exhibit more graceful performance degradation than FIR models.

**Transient Performance**

The spatially correlated MIMO channels are assumed to be quasi-static and are generated in the same way as discussed in Section 2.2.2. The training sequences are random binary sequences that are uncorrelated in both spatial dimension and time dimension. Contiguous training data is used to obtain the learning curves of the channel estimation algorithms. For a fixed 6-tap MIMO FIR channel with temporally and spatially white input symbol sequence, both the recursive MOESP and RLS channel estimators were run until they reached a stationary value. Then, at time $t = 0$, the channel was changed independently and the identification methods allowed to reconverge. This is depicted in the upper plots of Figures 2.6(a) and 2.6(b) for $2 \times 2$ channels with SNR=20dB and illustrates the transient performance or learning rate of the identifiers. Here SNR is defined to be ratio of the average power of transmitted symbols at each antenna over the average noise power at each antenna.
Re-converging performance of MOESP and RLS

(a) $2 \times 2$ channels with random power delay profile ($\lambda = 0$).

(b) $2 \times 2$ channels with exponential decaying power delay profile ($\lambda = 0.5$).

Figure 2.6: $H_2$ estimation error of recursive MOESP and RLS for $2 \times 2$ spatially correlated channels.
For the upper figures, the horizontal axis shows the number of training symbols sent after time $t = 0$. The vertical axis shows the $H_2$ error between the true channel and the model. There are separate curves for each different model order, with the dashed curves corresponding to the FIR models and the solid or dotted curves corresponding to state-space models. The items of interest are the steady-state error values, which indicate the modeling accuracy, and the learning or adaptation rate. The ordering of the curves shows that lower-order models achieve poorer steady-state accuracy and faster learning. The curves themselves are the result of averaging over 20 channel (initial and final) realizations and, for each channel realization, 100 training symbol sequence realizations.

The lower figures are derived from the upper figures and plot one point for each model order and identification method. The vertical axis is the steady-state modeling error and the horizontal axis shows the input symbol number from which the channel model error remains below 110% of the steady-state value. This latter figure gives a measure of convergence rate where smaller abscissa values imply faster adaptation.

In terms of steady-state $H_2$ channel estimation error, state-space models provide low-order models of high-quality approximation, whereas the reduced-length FIR models fail to provide comparable estimation error performance. This performance difference between reduced-order state-space models and reduced-length FIR models was also shown in Section 2.2.2 where batch MOESP and batch least squares channel estimators were considered. Therefore, when the true order or length of the physical channel is unknown, the state-space model is a more robust channel modeling approach than the FIR model because the former is less sensitive to model order selection error than the latter.

The convergence rate of the recursive MOESP algorithm is in general slower than that of the RLS algorithm, but the difference stays in a comparable range. The faster convergence of RLS is to be expected because the (spatially and temporally) white training sequence is optimal for FIR model structure [25]. However, for the ultimate application of these models for the equalization of the MIMO
channel, where the models themselves might be used to capture FIR or state-space approximations to the channel inverse, it is not immediately apparent which method would be faster. One suspects that, since MOESP includes an RLS section to produce the $B$ and $D$ matrices of the state-variable realization, that RLS would always be faster than SSI of the same order.

The demonstration that subspace system identification methods can deliver a high-performance (low-error) MIMO channel model with adaptation rate comparable to that of FIR-based RLS methods, validates their further consideration for the adaptive MIMO channel equalization problem. Figure 2.6 illustrates these features. In the next section we shall move on to present in more detail the necessary modifications to a standard SSI algorithm to achieve this performance in this signal environment.

### 2.4 Conclusions

A direct modeling approach for MIMO wireless communication channels using state-space models is proposed to achieve possible parsimonious parametrization. A recursive subspace system identification (SSI) algorithm for non-contiguous training data is derived for channel estimation in multiple-antenna communication systems. The numerical results show that the state-space-based channel estimation algorithm is capable of providing low-order models of high-quality channel approximation. When the true order of the physical MIMO channel is unknown, the performance of the state-space-based channel estimator is less sensitive, or more robust, to the error in model order selection than that of the the FIR-based RLS algorithm, while preserving comparable convergence rate.

The results included in this thesis have been published in the following articles.


3 MIMO Channel Equalization With State-Space Models

3.1 Overview

In digital communication systems, channel equalization techniques have been developed to mitigate the negative effects of inter-symbol interference (ISI) that is usually caused by frequency-selective channels [32]. Generally, equalizers are designed assuming the knowledge of the channel model except for some adaptive systems that directly compute equalizer coefficients from the training data. Recently some attention has been paid to the robustness of equalization with respect to channel model uncertainties [9, 17, 10].

An $H_\infty$ criterion for linear equalization has been proposed in [9], in lieu of the commonly used zero-forcing and minimum mean square error (MMSE) criteria, to improve equalizers’ robustness, with the belief that $H_\infty$ estimators are more immune to model uncertainties and lack of statistical information on the interference. The same idea is extended to multi-input multi-output (MIMO) channels for both linear equalization [17] and decision feedback equalization [10]. These $H_\infty$-optimal equalizers are generically infinite impulse response (IIR) even though they are based on finite impulse response (FIR) channel models.
In this work we take a different path to improve equalizer robustness by modeling MIMO frequency-selective channels with state-space models, which are typically IIR, instead of FIR models. FIR channel models have been widely used for their simplicity and guaranteed stability. However, when the length of the FIR model happens to be smaller than that of the true channel, the approximation error of the model may be very poor, even though the underlying true channel is FIR [53]. This is especially true in wireless communications where the channel length varies significantly according to the surrounding environment. On the other hand, the quality of state-space models degrades in a much more graceful way with respect to model order error. In general, reduced-order state-space models are able to provide significantly smaller channel estimation error than reduced-length FIR models [53]. At one level, this is an immediate property of the fact that the class of state-space models of a given order subsumes the class of FIR models of the same order.

In this chapter, we shall show that smaller $H_2$\textsuperscript{1} channel estimation error of the reduced-order state-space models would lead to lower symbol error rate for channel equalization. Hence state-space models shall be robust to model uncertainties in terms of not only model quality measures but also channel equalization performance. We should demonstrate this feature.

In order to design equalizers for state-space channel models in a fashion directly comparable to their FIR counterparts, we propose to represent the blockwise input-output data relationship of a state-space model in a similar way to that of an FIR model. This representation is very convenient because, by realizing the similarity

\[ \|G\|_\infty \triangleq \max_\omega \sigma[G(e^{j\omega})], \]

where $\sigma[\cdot]$ represents the largest singular value of a matrix. Its $H_2$ norm is defined as

\[ \|G\|_2 \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(G^H(e^{j\omega})G(e^{j\omega})) \, d\omega. \]

\textsuperscript{1}The $H_\infty$ norm of any stable system with transfer function $G(e^{j\omega})$ is defined as [54]
in blockwise data models, the equalizers for state-space models can be designed using modifications of the existing algorithms for FIR ones such as [1, 2]. As an example, a finite-length MIMO MMSE decision feedback equalizer (DFE) is developed for state-space models in this chapter. The tap value of the equalizer is computed indirectly via a channel estimate.

Assuming that the decisions of previously detected symbols are all correct, we show that the effective signal-to-noise ratio (SNR) at the decision point of the MIMO MMSE-DFE equalizer is related to $H_2$ channel estimation errors. The analysis and simulation results demonstrate that the smaller is the $H_2$ channel estimation error, the higher is the effective SNR. Moreover, the numerical performance comparison of a state-space-based receiver and an FIR-based receiver corresponds to this in showing that the former provides significantly smaller symbol error rate than the latter for reduced-order channel models.

This chapter is organized as follows. Section 3.2 presents FIR and state-space channel models. We shall see that the blockwise input-output data relationship of state-space models can be arranged in a similar form to that of FIR models. Based on this data representation, Section 3.3 discusses the design of a MIMO MMSE-DFE equalizer for state-space channel models using the existing method for FIR models assuming perfect channel knowledge at the receiver. In Section 3.4, the MIMO MMSE-DFE is designed to accommodate channel estimation errors and the analysis of the SNR at the decision point is presented. Numerical comparison between a state-space-based receiver and an FIR-based receiver is provided in Section 3.5. The paper concludes in Section 3.6.
3.2 Blockwise Data Models for Channel Equalization

3.2.1 FIR Models

An \(m\)-input, \(p\)-output, length-\(L\) inter-symbol interference (ISI) channel can be described by a symbol-rate sampled discrete-time FIR complex baseband model as follows

\[
y_k = \sum_{l=0}^{L-1} h_l u_{k-l} + n_k,
\]

where \(u_k\) is the complex \(m \times 1\) input vector at time \(k\), and \(y_k\) the complex \(p \times 1\) output vector at time \(k\). \(n_k\) is the \(p \times 1\) zero-mean circularly symmetric complex Gaussian noise vector with a covariance matrix \(N_0 I\). \(\{h_l\}_{l=0}^{L-1}\) are the complex matrix channel taps of dimension \(p \times m\).

Given a length-\(N_f\) block of the input-output data \(\{y_{k+\Delta}, \ldots, y_{k+\Delta-N_f+1}\}\) and \(\{u_{k+\Delta}, \ldots, u_{k+\Delta-N_f-L+2}\}\), the FIR channel model (3.1) can be rewritten as follows:

\[
\begin{bmatrix}
y_{k+\Delta} \\
\vdots \\
y_{k} \\
\vdots \\
y_{k+\Delta-N_f+1}
\end{bmatrix}_{k} \triangleq Y_k = \begin{bmatrix}
h_0 \cdots h_{L-1} & 0 & \cdots & 0 \\
0 & h_0 & \cdots & h_{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & h_0 \cdots h_{L-1}
\end{bmatrix}_{L \times L} \begin{bmatrix}
u_{k+\Delta} \\
\vdots \\
u_{k} \\
\vdots \\
u_{k+\Delta-N_f-L+2}
\end{bmatrix}_{k}_{L \times m} \triangleq H_{FIR} U_{FIR,k} + \begin{bmatrix}
n_{k+\Delta} \\
\vdots \\
n_k \\
\vdots \\
n_{k+\Delta-N_f+1}
\end{bmatrix}_{k} \triangleq N_k,
\]

or more compactly

\[
Y_k = H_{FIR} U_{FIR,k} + N_k.
\]
The above blockwise data model (3.2) is commonly used for the design of finite-length equalizers [2], with $\Delta$ representing the equalizer’s decision delay and $N_f$ the length of the equalizer (feedforward) filter.

### 3.2.2 State-Space Models

Another approach to representing ISI channels is to use a state-space model [52],

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k \\
    y_k &= Cx_k + Du_k + n_k, 
\end{align*}
\]

(3.4)

where $x_k$ is the $n \times 1$ state vector at time $k$.

Correspondingly, the blockwise data model for a state-space system is given by

\[
\begin{bmatrix}
    y_{k+\Delta} \\
    \vdots \\
    y_k \\
    \vdots \\
    y_{k+\Delta-N_f+1}
\end{bmatrix}
\]

\[\triangleq \Gamma \begin{bmatrix}
    x_{k+\Delta-N_f+1} \\
    \vdots \\
    CA
\end{bmatrix}
\]

\[\triangleq H \begin{bmatrix}
    u_{k+\Delta} \\
    \vdots \\
    u_k \\
    \vdots \\
    u_{k+\Delta-N_f+1}
\end{bmatrix}
\]

\[\triangleq H \begin{bmatrix}
    n_{k+\Delta} \\
    \vdots \\
    n_k \\
    \vdots \\
    n_{k+\Delta-N_f+1}
\end{bmatrix},
\]

(3.5)
or

\[ Y_k = \Gamma x_{k+\Delta-N_f+1} + H U_k + N_k, \quad (3.6) \]

where \( \Gamma \) is the so-called extended observability matrix and \( H \) the matrix of Markov parameters.

Using the well-known relationship between a transfer function and the corresponding state-space realizations, the blockwise data model (3.5) can be written in terms of the impulse response as follows:

\[
\begin{bmatrix}
  y_{k+\Delta} \\
  \vdots \\
  y_k \\
  \vdots \\
  y_{k+\Delta-N_f+1}
\end{bmatrix}
=\begin{bmatrix}
  h_N \ h_{N_f+1} \ \cdots \\
  \vdots \\
  h_2 \ h_3 \ \cdots \\
  h_1 \ h_2 \ \cdots \\
  \vdots \\
\end{bmatrix}
\begin{bmatrix}
  u_{k+\Delta-N_f} \\
  \vdots \\
  u_{k+\Delta-N_f-1} \\
  \vdots \\
\end{bmatrix}
=\Gamma x_{k+\Delta-N_f+1}, \text{ distant past}
\]

\[
+\begin{bmatrix}
  h_0 \ \cdots \ h_{L-1} \ 0 \\
  0 \ h_0 \ \cdots \ 0 \\
  \vdots \\
  0 \ \cdots \ 0 \ h_0 \\
\end{bmatrix}
\begin{bmatrix}
  u_{k+\Delta} \\
  \vdots \\
  u_{k+\Delta-N_f+1} \\
\end{bmatrix}
=HU_k, \text{ recent past}
\]

\[
+\begin{bmatrix}
  n_{k+\Delta} \\
  \vdots \\
  n_{k+\Delta-N_f+1}
\end{bmatrix}.
\quad (3.7)
\]

It is clear that the FIR blockwise model (3.2) is a special case of (3.5) or (3.7) when the impulse response \( \{h_i\} \) has a finite length \( L \), and the zero columns (those beyond block column \( L-1 \)) of the leftmost matrix are omitted together with the corresponding \( (L-1) \)-onward block rows of the infinite input matrix. In fact, the FIR model (3.2) is realized by the state-space model (3.5), with appropriate
truncations of zero submatrices, in controllable canonical form with

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
I & 0 & \cdots & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & I & 0
\end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_k = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-L+1} \end{bmatrix},
\]

\[
C = [h_1, \ldots, h_{L-1}], \quad D = h_0.
\]

Equation (3.7) is essentially the blockwise input-output relationship for an IIR channel model with the right-hand side divided into three parts: the contribution from the distant past input, the contribution from the recent past input, and the noise. FIR channel models are necessarily a subset of IIR channel models. Moreover, a length-\(L\) FIR model will have a McMillan degree at most \((L - 1) \times \min\{m, p\}\), and so can be completely represented by a state-space model of state dimension equal to this maximal McMillan degree. There are two immediate consequences of this observation; the McMillan degree overbound for length-\(L\) FIR models must be a multiple of \(\min\{m, p\}\), and the model approximation properties of the class of state-space models is more powerful than that of FIR models of the same McMillan degree. In the specific case of \(H_2\) model approximation properties, IIR state-space models have greater approximation power than their FIR McMillan-degree-equivalent counterparts.

Reduced-order model approximation is a broad subject embodying a range of approaches for differing approximation criteria with known full-order models, see [30]. In our treatment, we shall be dealing with models fitted to observed input-output data using techniques from System Identification. Our thesis (validated by simulation experiments) is that the identified state-space models truly outperform identified FIR models when the identification is based on least squares criteria.

The modeling quality comparison between state-space models and FIR ones for MIMO frequency-selective channels is quantitively investigated in [52] and [53]. In particular, a recursive MIMO channel estimation algorithm based on subspace system identification is developed for state-space models [53] in MIMO telecommu-
communications and compared to the Recursive Least Squares (RLS) channel estimator based on FIR channel models. The fact that low-order state-space models provide better channel approximation than low-order FIR models is demonstrated by Figure 3.1 from [53]. The $H_2$ MIMO channel estimation error of both the state-space-based MOESP algorithm and the FIR-based least squares algorithm is presented as a function of model order with the full order being 15.

![Figure 3.1: $H_2$ channel estimation error for $3 \times 3$ channels with uniform power delay profiles](image)

Based on the fact that the approximation quality of state-space models is more robust to model order error, we would expect to see the same robustness in equalization performance. This motivates us to develop equalizers for state-space channel models and to examine their performance in contrast to the equalizers based on FIR models.
3.3 Equalization with State-Space Channel Models

In this section, after a brief review of equalizer design for FIR channel models, we shall discuss how to extend existing FIR design approach to construct equalizers for state-space channel models. MIMO MMSE Decision Feedback Equalizers (DFE) will be used as an example to illustrate the design procedure.

3.3.1 Review: Equalization with FIR Models

For FIR channel models, the block diagram of a MIMO MMSE-DFE receiver is shown in Figure 3.2. It consists of a feedforward filter

$$W^H = [W_0^H, W_1^H, \ldots, W_{N_f-1}^H]$$

with $N_f$ matrix taps $W_i^H$ of size $p \times m$, and a feedback filter

$$B^H = [B_0^H, B_1^H, \ldots, B_{N_b-1}^H]$$

with $N_b$ matrix taps $B_i^H$ of size $m \times m$, where $(\cdot)^H$ represents the complex conjugate transpose (Hermitian transpose) of a matrix. Estimates of FIR channel impulse response matrices, $\{h_l\}_{l=0}^{L-1}$, can be obtained by algorithms such as least mean squares (LMS) and recursive least squares (RLS) through the use of training data. In this section, it is assumed that a model of the channel is already available to the receiver. The effects of channel estimation error will be studied in Section 3.4.

By defining

$$g_l \triangleq \begin{bmatrix} h_{l+\Delta} \\ \vdots \\ h_{l+\Delta-N_f+1} \end{bmatrix},$$

the blockwise input-output relationship of FIR channel models (3.2) can be written
in a convolutional form as follows:

\[
Y_k = H_{FIR}U_{FIR,k} + N_k
\]

\[
= \sum_{l=-\Delta}^{\Delta} \sum_{l'<0} g_l u_{k-l'} + N_k
\]

\[
= g_0 u_k + \sum_{-\Delta \leq l < 0} g_l u_{k-l}
\]

\[
+ \sum_{0 < l < \Delta + N_f + L - 2} g_l u_{k-l} + N_k. \quad (3.9)
\]

The received signal vector \(Y_k\) contains the contribution from: the desired symbol \(u_k\), the interference from future symbols, the interference from past symbols, and noise.

The output of the feedforward filter is given by

\[
W^H \mathbf{y}_k = W^H g_0 u_k + \sum_{-\Delta \leq l < 0} W^H g_l u_{k-l}
\]

\[
+ \sum_{0 < l < \Delta + N_f + L - 2} W^H g_l u_{k-l} + W^H N_k. \quad (3.10)
\]

Assuming that the previously detected symbols are all correct and the length of the feedback filter is long enough \((N_b \geq \Delta + N_f + L - 1)\), the interference from past symbols can be completely cancelled by constructing the feedback filter as

\[
B_l^H = \begin{cases} 
0, & l = 0 \\
-W^H g_l, & l > 0.
\end{cases} \quad (3.10)
\]

Then the signal at the input of the decision device (decision point) is given by

\[
v_k = W^H g_0 u_k + \sum_{-\Delta \leq l < 0} W^H g_l u_{k-l} + W^H N_k. \quad (3.11)
\]

The feedforward filter \(W\) is chosen to satisfy the following minimum mean-square error criterion

\[
W_{MMSE} = \arg \min_W tr(E[||u_k - v_k||^2]), \quad (3.12)
\]

where \(tr(\cdot)\) denotes the trace of a matrix.
A different but equivalent approach to derive the MMSE-DFE is simply to use the blockwise data model (3.3) and define

\[ \tilde{W} \triangleq \begin{bmatrix} W \\ B \end{bmatrix}, \quad Z_k \triangleq \begin{bmatrix} Y_k \\ D_k \end{bmatrix}, \quad D_k \triangleq \begin{bmatrix} u_k \\ \vdots \\ u_{k-N_b+1} \end{bmatrix}, \]

where \( D_k \) represents the previously detected symbols which are the same as the transmitted symbols assuming correct detection. Note that the current symbol vector \( u_k \) is also included in \( D_k \). In the SISO case, \( u_k \) is not available to the feedback filter and thus the corresponding filter tap \( B_0 \) is equal to 0. However, in a MIMO system where detection schemes such as successive interference cancellation are applicable, the decisions on other data streams can be used to facilitate the detection of one data stream. Hence \( B_0 \) can be non-zero and take such forms as a lower triangular matrix [2].

The optimization criterion for the composite filter \( \tilde{W} \) is

\[ \tilde{W}_{MMSE} = \arg \min_{\tilde{W}} \text{tr}(E[||u_k - \tilde{W}Z_k||^2]). \]  \hspace{1cm} (3.13)

Assuming that the input sequence \( \{u_k\} \) is a spatially and temporally i.i.d. random process with zero mean and that the total transmit power \( P_s \) is equally allocated over the transmit antennas, the solution to the above vector linear MMSE problem is given by [19]

\[ \tilde{W}_{MMSE} = R_{Zu_k}^{-1} R_{Zu_k}, \]  \hspace{1cm} (3.14)
with

\begin{align*}
R_{ZZ} &= E[Z_k Z_k^H] \\
&= \begin{bmatrix} E[Y_k Y_k^H] & E[Y_k D_k^H] \\
E[D_k Y_k^H] & E[D_k D_k^H] \end{bmatrix} \\
&= \begin{bmatrix} \frac{P}{m} H_{FIR}^H H_{FIR} + N_0 I & H_{FIR} E[U_{FIR,k} D_k^H] \\
E[D_k U_{FIR,k}^H] H_{FIR}^H & \frac{P}{m} I \end{bmatrix}, \\
R_{Zu_k} &= E[Z_k u_k^H] \\
&= \begin{bmatrix} H_{FIR} E[U_{FIR,k} u_k^H] \\
0 \end{bmatrix},
\end{align*}

where \(N_0\) is the single-sided power spectral density of the noise.

The estimated input symbol at the output of the MIMO MMSE-DFE is simply

\[ v_k = \tilde{W}_{MMSE}^H Z_k. \]

![Diagram](image)

**Figure 3.2:** Decision feedback equalizer for FIR channel models.

### 3.3.2 Equalization with State-Space Models

In Section 3.2 we established that the blockwise I-O relationship for a state-space channel model (3.5) is equivalent to an IIR model (3.7). Therefore, it can
be written in a convolution form in a manner similar to FIR models (3.9) except \( g_l \) might have an infinite length.

\[
Y_k = \Gamma x_{k+\Delta-N_f+1} + HU_k + N_k
\]

\[
= \sum_{l=-\Delta}^{\infty} g_l u_{k-l} + N_k
\] (3.15)

\[
= g_0 u_k + \sum_{-\Delta \leq l < 0} g_l u_{k-l} + \sum_{0 \leq l \leq -\Delta+N_f-1} g_l u_{k-l} + \sum_{l>\Delta+N_f-1} g_l u_{k-l} + N_k.
\] (3.16)

The recursive state-space channel estimation algorithm developed in [53] is capable of providing not only estimates of matrices \( A, B, C, D \) but also of the initial state \( x_{k+\Delta-N_f+1} \). Thus by defining

\[
Y'_k \triangleq Y_k - \Gamma x_{k+\Delta+N_f+1}
\]

\[
= HU_k + N_k,
\] (3.17)

(3.16) becomes

\[
Y'_k = g_0 u_k + \sum_{-\Delta \leq l < 0} g_l u_{k-l} + \sum_{0 \leq l \leq -\Delta+N_f-1} g_l u_{k-l} + N_k.
\] (3.18)

which has the same structure as the FIR case (3.9). The fact that state-space channel estimation algorithms [53] are able to provide a corresponding initial state
estimate $x_{k+\Delta-N_f+1}$ converts the blockwise input-output data model for IIR channel models into a form equivalent to that of FIR models. Therefore the state-space channel equalizer can be designed using the same methods developed for FIR models. The key idea is the ability of the state-space channel estimator to provide the knowledge of $\Gamma x_{k+\Delta-N_f+1}$.

Figure 3.3 shows the block diagram of MIMO MMSE-DFE for state-space channel models. Assuming that the previously detected symbols are all correct and the length of the feedback filter satisfies $N_b \geq -\Delta + N_f$, the interference from past symbols can be completely cancelled by constructing the feedback filter as

$$B_l^H = \begin{cases} 0, & l = 0 \\ -W_l^Hg_l, & 0 < l \leq -\Delta + N_f - 1. \end{cases} \quad (3.19)$$

The signal at the input of the decision device (decision point) is given by

$$v_k = W^Hg_0u_k + \sum_{-\Delta \leq l < 0} W_l^Hg_{l-1}u_k - l + W^HN_k. \quad (3.20)$$

The feedforward filter $W$ is chosen to satisfy the following minimum mean-square error criterion

$$W_{MMSE} = \arg \min_W tr(E[||u_k - v_k||^2]). \quad (3.21)$$

Equivalently, if we define

$$\tilde{W} \triangleq \begin{bmatrix} W \\ B \end{bmatrix}, \quad Z_k \triangleq \begin{bmatrix} Y_k' \\ D_k \end{bmatrix}, \quad D_k \triangleq \begin{bmatrix} u_k \\ \vdots \\ u_{k-N_b+1} \end{bmatrix},$$

the composite filter $\tilde{W}$ optimized using the criterion (3.21) becomes,

$$\tilde{W}_{MMSE} = \arg \min_W tr\{E[||u_k - \tilde{W}Z_k||^2]\}. \quad (3.22)$$

Assuming that the input sequence $\{u_k\}$ is a spatially and temporally i.i.d. random process with zero mean and that the total transmit power $P_s$ is equally allocated over the transmit antennas, the MMSE solution is given by

$$\tilde{W}_{MMSE} = R_{ZZ}^{-1}R_{Zu_k}. \quad (3.23)$$
with

\[
\begin{align*}
R_{ZZ} &= E[Z_k Z_k^H] \\
&= \begin{bmatrix} E[Y_k^H Y_k^H] & E[Y_k^H D_k^H] \\
E[D_k Y_k^H] & E[D_k D_k^H] \end{bmatrix} \\
&= \begin{bmatrix} \frac{P}{m} HH^H + N_0 I & HE[U_k D_k^H] \\
E[D_k U_k^H]H^H & \frac{P}{n} I \end{bmatrix}, \\
R_{Zu_k} &= E[Z_k u_k^H] \\
&= \begin{bmatrix} HE[U_k u_k^H] \\
0 \end{bmatrix}.
\end{align*}
\]

The estimated input symbol at the output of the MIMO MMSE-DFE is simply

\[
v_k = \tilde{W}_{MMSE}^H Z_k.
\]

### 3.4 Effects of Channel Estimation Error

In the previous section, it is implicitly assumed that perfect knowledge of the channel is available to the receiver in constructing the corresponding equalizer. However, in practical systems the channel model has to be estimated. Therefore the effect of channel estimation error needs to be considered when designing equalizers. This topic has been pursued for single-input single-output channels in the existing literature such as [36]. We shall extend a similar idea to the MIMO channel case.

Because both state-space models and FIR models are either equivalent to or special cases of IIR models, we shall continue our discussion under the general framework of IIR channel models (3.15). Based on the IIR channel model, the output of the feedforward filter is

\[
W^H Y_k = W^H g_0 u_k + \sum_{-\Delta \leq l < 0} W^H g_l u_{k-l}
+ \sum_{l > 0} W^H g_l u_{k-l} + W^H N_k.
\]
If perfect channel state information is known to the receiver, the inter-symbol interference from past symbols can be completely removed by taking

\[ B_l^H = \begin{cases} 0, & l = 0 \\ -W^H g_l, & l > 0. \end{cases} \] (3.24)

However, when only estimates of the channel \{\hat{g}_l\} are available, the feedback filter will be chosen as

\[ B_l^H = \begin{cases} 0, & l = 0 \\ -W^H \hat{g}_l, & l > 0. \end{cases} \] (3.25)

Then the signal at the decision point becomes

\[ v_k = W^H \left( \hat{g}_0 u_k + \sum_{\Delta \leq l < 0} g_l u_{k-l} + \sum_{l \geq 0} \tilde{g}_l u_{k-l} + N_k \right), \] (3.26)

where \( \tilde{g}_l \triangleq g_l - \hat{g}_l \) represents channel estimation error.

By grouping the inter-symbol interference terms and the noise term as a generalized noise

\[ \tilde{N}_k = \sum_{\Delta \leq l < 0} g_l u_{k-l} + \sum_{l \geq 0} \tilde{g}_l u_{k-l} + N_k \] (3.27)

with a covariance matrix

\[ R_{\tilde{N}\tilde{N}} = \frac{P_s}{m} \sum_{\Delta \leq l < 0} g_l g_l^H + \frac{P_s}{m} \sum_{l \geq 0} \tilde{g}_l \tilde{g}_l^H + N_0 I, \] (3.28)

the optimal feedforward filter is given by

\[ W_{MMSE} = R_{\tilde{N}\tilde{N}}^{-1} \hat{g}_0 (\frac{m}{P_s} I + \hat{g}_0^H R_{\tilde{N}\tilde{N}}^{-1} \hat{g}_0)^{-1}, \] (3.29)

and the corresponding covariance matrix of the error vector, \( e_k \triangleq u_k - v_k \), is

\[ R_{ee,\text{min}} = \left( \frac{m}{P_s} I + \hat{g}_0^H R_{\tilde{N}\tilde{N}}^{-1} \hat{g}_0 \right)^{-1}. \] (3.30)

The effects of channel estimation error can be seen using the SNR at the decision point as a performance measure. This SNR performance measure is commonly used for MMSE channel equalizers because it is easy to compute and better SNR
Figure 3.4: ASNR at the decision point vs model order for a fixed transmit SNR of 25dB. $L = 6; N_f = 9, N_b = 6$ and $\Delta = 6$ for MMSE-DFE; $N_f = 15, N_b = 0$ and $\Delta = 6$ for MMSE-LE.

generally leads to lower error probability. For MIMO receivers, the arithmetic SNR (ASNR) at the input of the decision device is defined as [2]

$$\text{ASNR} \triangleq \frac{\text{tr}(R_{uu})}{\text{tr}(R_{ee,\min})}.$$  \hspace{1cm} (3.31)

From (3.26) and (3.27), it is intuitively straightforward that larger channel estimation errors $\{\tilde{g}_l\}_{l \geq 0}$ will increase the power of the generalized noise and thus reduce ASNR. Further, this formulation indicates that it is the $H_2$ modeling error that is the appropriate quantification criterion. We have observed in Figure 3.1 that reduced-order state-space models provide lower $H_2$ channel estimation error than FIR models. Therefore it is not surprising that the former gives rise to better decision point ASNR. Figure 3.4 shows the ASNR comparison between state-space-based receivers and FIR-based receivers obtained through numerical simulation. The details of the simulation setup will be provided in Section 3.5.
3.5 Numerical Results

3.5.1 Simulation Setup

In the numerical simulation, the transmitted data frames are assumed to possess the structure shown in Figure 3.5. Each frame consists of a sequence of training symbols followed by a sequence of the information-bearing data symbols. Since we assume a static channel, placing the training symbols at the beginning of a frame does not affect the channel estimation performance.

![Frame structure of the transmitted signal.](image)

Upon receiving each frame, the receiver uses the training data to estimate the channel and compute equalizer coefficients with the estimated channel. The symbol error rate (SER) performance of both the receiver using FIR channel models (Figure 3.2) and that using state-space channel models (Figure 3.3) is simulated and compared. The recursive least squares (RLS) algorithm is applied to estimate the FIR channel models, and an appropriate variant of the recursive subspace system identification estimator [53] is used to estimate the state-space channel models.

We examine both the steady-state performance and the transient performance of the two different receivers. When the recursive channel estimator is given a long enough training sequence to learn the channel and to converge, the corresponding equalizer’s symbol error rate represents the steady-state performance of the receiver. On the other hand, with a limited amount of training data, the receiver has to equalize the information-bearing data before the recursive channel estimator reaches its steady state. The symbol error rate in this scenario represents the receiver’s transient performance.
(a) SER vs model order for a fixed SNR=20dB. \( N_f = 9, N_b = 6 \) and \( \Delta = 8 \) for MMSE-DFE; \( N_f = 15, N_b = 0 \) and \( \Delta = 8 \) for MMSE-LE.

(b) SER vs SNR for MMSE-DFE with \( N_f = 9, N_b = 6, \) and \( \Delta = 8 \). The solid curves represent state-space models with order 4, 5, 6, 7, 8, 9, 10. The dotted curves represent FIR models with length 3, 4, 5, 6 or with equivalent order 4, 6, 8, 10.

Figure 3.6: Steady-state symbol error rate of 2 × 2 channels with \( L = 6 \).
The reference frequency-selective fading channels are generated using the so-called “correlation channel model”, which describes the second-order statistics of the MIMO fading channel by a transmit antenna covariance matrix and a separate receive antenna covariance matrix. Specifically, the matrix channel taps, \( \{ \mathbf{h}_l \}_{l=0}^{L-1} \), can be modeled as

\[
\mathbf{h}_l = (\mathbf{R}_{RX,l})^{1/2}\mathbf{h}_w(\mathbf{R}_{TX,l})^{T/2},
\]

where \( \mathbf{h}_w \) is a \( p \times m \) matrix with i.i.d. zero-mean circularly-symmetric complex gaussian elements. \( \mathbf{R}_{TX,l} \) and \( \mathbf{R}_{RX,l} \) are, respectively, the transmit covariance matrix and receive covariance matrix for the \( l \)-th channel tap. Under the assumption of uniformly distributed angle of arrival, \( \mathbf{R}_{TX,l} \) and \( \mathbf{R}_{RX,l} \) can be modeled with closed-form expressions as functions of angular spread \( \Delta_s \), angle of arrival \( \phi \) and normalized antenna spacing \( D/\lambda_c \), where \( D \) is the antenna spacing and \( \lambda_c \) the carrier wavelength. See [34] for detailed development of the closed-form expressions.

In the computer simulation, the length of all the subchannels \( L \) is 6 and the transmitted data are modulated with QPSK.

For MMSE-DFE equalizers, a decision delay \( \Delta \geq L - 1 \) should be chosen in order to remove the precursor ISI completely [27]. For a given equalization lag \( \Delta \), the optimal feedforward filter length is \( N_f = \Delta + 1 \). The optimal feedback filter length is \( N_b = L \). When \( \Delta, N_f, \) and \( N_b \) are selected according to the above principles, the mean square error (MSE) at the decision point is generally a non-decreasing function of \( \Delta \). In our numerical experiments, we find that \( \Delta = L+2 = 8 \) appears to be a good choice in terms of symbol error rate.

### 3.5.2 Steady-State Equalization Performance

Figure 3.6 shows the steady-state symbol error rate with different channel model orders. When the channel model has the same order as the reference channel, it is referred to as being full order which is 10 in our case. Otherwise, if the order of a model is smaller than the order of true channel, it is called a reduced-order model.
Several observations on the performance of the reduced-order models can be made. First, the number of reduced-order FIR models is significantly smaller than that of reduced-order state-space models. This is because the order of an generic \( p \times m \) length-\( L \) FIR model is given by

\[
\min\{m, p\} \times (L - 1).
\]

Since the order of an FIR model can only be reduced by decreasing the length of the impulse responses, the step size of reducing the order of an FIR model is at least \( \min\{m, p\} \). On the other hand, the step size for reducing the order of an state-space model is 1.

Secondly, for a fixed reduced order where there exists an FIR model, the symbol error rate of state-space-based receivers is significantly smaller than that of FIR-based receivers. In reality the order/length of the physical channel is usually unknown, thus it is possible that the order of the chosen channel model is smaller than that of some given physical channel realizations. In this case, the performance of state-space-based receivers degrades in a much more graceful way that that of the FIR-based ones. In other words, state-space-based receivers are robust to the error in model order selection.

In addition, full-order FIR models have lower symbol error rate than full-order state-space models because the former has exactly the same structure as the true channels. However the performance difference is nearly negligible. With model orders which are greater than the true channel model, one would expect the FIR models to outperform their IIR counterparts because their parsimony is better. However, the performance difference is very small as shown in Figure 3.6(a).

### 3.5.3 Transient Equalization Performance

Figures 3.7(a) and 3.7(b) illustrate the transient performance of MMSE linear equalizers and MMSE-DFE equalizers, respectively, for \( 2 \times 2 \) channels with \( \text{SNR}=20\text{dB} \). The horizontal axis shows the number of training symbols sent after
(a) MMSE linear equalizers with $N_f = 15$, $N_b = 0$, and $\Delta = 8$.

(b) MMSE-DFE equalizers with $N_f = 9$, $N_b = 6$, and $\Delta = 8$.

Figure 3.7: Transient symbol error rate for a fixed SNR=20dB of $2 \times 2$ channels with $L = 6$. The solid curves represent state-space models with order 4, 5, 6, 7, 8, 9, 10. The dotted curves represent FIR models with length 3, 4, 5, 6 or with equivalent order 4, 6, 8, 10. Note that the steady-state values here match the corresponding values indicated in Figure 3.6.
time $t = 0$. The vertical axis shows the symbol error rate of the equalizers based on FIR or state-space channel models with various orders. There are separate curves for each different model order, with the dotted curves corresponding to the FIR models and the solid curves corresponding to state-space models.

The convergence rate of the state-space-based equalizers is in general slower than that of the FIR-based ones, but the difference stays in a comparable range. It is seen that some of the state-space SER learning curves show sudden fluctuation before settling down to the convergence level. However the estimation error learning curves of the recursive state-space channel estimator, shown in [53], do not have such behavior. Therefore, for state-space models, the effect of channel estimation error on the equalization performance might not be proportional. In other words, before the recursive channel estimator converges, small channel estimation error does not imply smaller symbol error rate of the equalizer for state-space channel models.

### 3.6 Conclusions

State space models are proposed to represent MIMO frequency-selective wireless channels with the motivation of better model approximation performance and more robust channel equalization performance with respect to the mismatch between the order of the channel model and that of the true channel. We proposed a simple framework under which the equalizer design for state-space channel models can proceed using the existing methods for designing equalizers for FIR models. In particular, a MIMO MMSE-DFE equalizer is developed for state-space models assuming either perfect channel knowledge at the receiver or estimated channel knowledge. The performance comparison of a state-space-based receiver and an FIR-based receiver shows that the former provides significantly smaller symbol error rate than the latter for reduced-order channel models while exhibiting marginally slower but comparable convergence rate. This implies that state-space channel models can be a more robust choice than FIR ones in the presence of
model order selection error.

The results included in this thesis have been published in the following articles.


Channel Equalization via State-Space Deconvolution

Blockwise transmission format is widely used in practical telecommunication systems. For instance, in GSM systems each user is allocated a time slot which contains the information bits, the training bits and other overhead bits used for the purposes of synchronization and guard period. The training bits are used by a recursive channel estimation algorithm, for instance the recursive MOESP developed in Chapter 2, to update the current knowledge of the channel. Then the channel equalizer is calculated using the newly obtained channel estimate and then is used to recover the information bits.

<table>
<thead>
<tr>
<th>Bits</th>
<th>T</th>
<th>Information</th>
<th>F</th>
<th>Training</th>
<th>F</th>
<th>Information</th>
<th>T</th>
<th>GP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>57</td>
<td>1</td>
<td>26</td>
<td>1</td>
<td>57</td>
<td>3</td>
<td>8.5</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.1: GSM frame structure. “T” represents the tail bits, “F” the flag bits, and “GP” the guard period.

The justification for the design of such a blockwise transmission scheme is that the delay requirement of the communication application, such as voice communication, can be met with a sufficiently small block duration. The receiver can wait until receiving the whole block of bits before decoding them without causing any
noticeable delay by the user. Therefore, technically the receiver could use the whole received block for the decoding of each single bit in it. For the MIMO MMSE-DFE receiver, this implies that the length of the feedforward filter needs to be long enough to span the block length. However, as in GSM systems, a feedforward filter with 57 taps is certainly unrealistic in terms of complexity issues.

In this chapter, we shall touch upon a state-space channel equalization scheme based on fixed-interval deconvolution [29] that exploits the whole received data block for decoding every bit in it. To illustrate the idea, consider a received data frame with the training sequence as its mid-amble. Then, assuming the channel matrices $A$, $B$, $C$, $D$, and the statistics of the input and noise are known or estimated by the state-space channel estimation algorithm, the state vectors within this frame can be estimated using the fixed-interval smoothing algorithm from Kalman filtering theory [3]. In fact, the fixed-interval smoothing algorithm can be expanded to estimate the channel input vectors within the frame. The resulting algorithm is called the fixed-interval deconvolution algorithm [29].

A brief review of Kalman filtering and an introduction to fixed-interval smoothing and fixed-interval deconvolution is given in this chapter. A detailed application of the deconvolution algorithm is discussed in Section 4.4 where the recursive MOESP channel estimation algorithm and the fixed-interval deconvolution channel equalization algorithm will be combined to form a complete receiver processing procedure.

4.1 Kalman Filter

Before we discuss fixed-interval smoothing, it is necessary to list briefly some important results of Kalman filter. The state-space channel model is given as usual

\begin{align*}
\mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k \\
y_k &= C\mathbf{x}_k + \mathbf{n}_k
\end{align*}  \quad (4.1, 4.2)
where $u_k$ and $n_k$ are assumed to be mutually uncorrelated zero-mean Gaussian white sequences. That is,

$$
E\{u_i u_j^H\} = Q_i \delta_{ij},
$$

$$
E\{n_i n_j^H\} = R_i \delta_{ij},
$$

$$
E\{u_i n_j^H\} = 0,
$$

where $\delta_{ij}$ is the Kronecker delta function. The initial condition of the state vectors $x_0$ is assumed to be Gaussian with mean $\bar{x}_0$ and covariance $\Sigma_0$.

Let $\hat{x}_{k+1|k}$ denote the estimate of $x_{k+1}$ given measurements of the output up to and including time $k$, i.e. $y_0, y_1, \ldots, y_k$. Then the Kalman predictor is given by

$$
\hat{x}_{k+1|k} = (A - K_k C)\hat{x}_{k|k-1} + K_k y_k,
$$

$$
\Sigma_{k+1|k} = A \Sigma_{k|k-1} A^H - A \Sigma_{k|k-1} C^H (C \Sigma_{k|k-1} C^H + R_k)^{-1} C \Sigma_{k|k-1} A^H + Q_k,
$$

where $K_k$ is the Kalman predictor gain given by

$$
K_k = A \Sigma_{k|k-1} C^H (C \Sigma_{k|k-1} C^H + R_k)^{-1}.
$$

### 4.2 Fixed-Interval Smoothing

The smoothing problem deals with state estimates $\hat{x}_{k|j}$ for $k < j$. There are three different types of smoother.

- **fixed-interval** smoother:
  
  $\hat{x}_{k|N}$, $k = 0, 1, \ldots, N - 1$, where $N$ is the fixed length of the interval.

- **fixed-point** smoother:
  
  $\hat{x}_{k|j}$, $j = k + 1, k + 2, \ldots$, where $k$ is a fixed positive integer.

- **fixed-lag** smoother:
  
  $\hat{x}_{k|k+L}$, $k = 0, 1, \ldots$, where $L$ is a fixed positive integer.
In communication systems, data is usually transmitted and received in the form of frames. Therefore, the fixed-interval smoother becomes a natural choice for estimating the states within a given frame of data. The length of the smoothing interval is equal to the length of the frame.

From the fundamental theorem of estimation theory, we know that the minimum-variance smoother is a conditional mean estimator.

\[
\hat{x}_{k|N} = E\{x_k|y_1, y_2, \ldots, y_N\}, k < N
\]

There are many different approaches to the derivation of recursive smoothers. The one we adopt for channel equalization is given by the following theorem.

**Theorem 1. Fixed-Interval Smoothing:**

(a) The optimal fixed-interval smoothed estimate \( \hat{x}_{k|N} \) is

\[
\hat{x}_{k|N} = \hat{x}_{k|k-1} + \Sigma_{k|k-1}r_{k|N}
\]

where \( k = N - 1, N - 2, \ldots, 1 \). The \( n \times 1 \) vector \( r_{k|N} \) is called Residual State Vector. It satisfies the following backward-recursive equation.

\[
r_{k|N} = \Phi_k^H r_{k+1|N} + C_k^H (C\Sigma_{k|k-1}C_k^H + R_k)^{-1}(y_k - C\hat{x}_{k|k-1})
\]

where \( k = N, N - 1, \ldots, 1 \) and \( r_{N+1|N} = 0 \). \( \Phi_k \) is defined as

\[
\Phi_k = A[I - K_kC]
\]

(b) The smoothing error covariance matrix \( \Sigma_{k|N} \) is

\[
\Sigma_{k|N} = \Sigma_{k|k-1} - \Sigma_{k|k-1}S_{k|N}\Sigma_{k|k-1}
\]

where \( k = N - 1, N - 2, \ldots, 1 \). The \( n \times n \) matrix \( S_{k|N} \) is the covariance matrix of \( r_{k|N} \) and satisfies the backward-recursive equation

\[
S_{k|N} = \Phi_k^H S_{k+1|N} \Phi_k + C_k^H (C\Sigma_{k|k-1}C_k^H + R_k)^{-1}C_k
\]

where \( k = N, N - 1, \ldots, 1 \) and \( S_{N+1|N} = 0 \).
Proof: For the proof, see [29] page 64.

From Equation (4.4) we can see that the fixed-interval smoother includes a one-step-ahead predictor \( \hat{x}_{k|k-1} \) and the residual state vector \( r_{k|N} \) that contains the information from the rest of the received data. So the fixed-interval smoothing is a two-pass procedure.

- **The forward process:**
  A Kalman predictor is run from the beginning of the interval \( (k = 1) \) to the end \( (k = N) \). The state prediction \( \hat{x}_{k|k-1} \) and its covariance \( \Sigma_{k|k-1} \) is stored for later use.

- **The backward process:**
  The residual state vector \( r_{k|N} \) is computed by Equation (4.5) from \( k = N \) to \( k = 1 \) using \( \hat{x}_{k|k-1} \) and \( \Sigma_{k|k-1} \) obtained from the forward process.

At any time instant \( k \) in the backward process, the fixed-interval smoother \( \hat{x}_{k|N} \) can be computed using Equation (4.4) after obtaining the corresponding residual state vector. However, as we will see later, the fixed-interval deconvolution algorithm does not need to compute \( \hat{x}_{k|N} \) explicitly. All it needs to estimate the input \( u_k \), i.e. equalize the channel, is the residual state vector \( r_{k|N} \) which has to be computed using the state prediction \( \hat{x}_{k|k-1} \) and its covariance \( \Sigma_{k|k-1} \) from the forward process.

### 4.3 Fixed-Interval Deconvolution

Assuming a frame of data \( y_1, y_2, \ldots, y_N \) has been received, we want to obtain the minimum-variance linear estimates of \( u_k, \hat{u}_{k|N} \ (k = 1, 2, \ldots, N) \), from the frame of received data. This process is called **fixed-interval deconvolution**. Notice that \( \hat{u}_{k|N} \) is in fact a fixed-interval smoothed estimate of \( u_k \). If we operate on both sides of Equation (4.1) with \( E\{ \cdot | y_1, y_2, \ldots, y_N \} \), the equation becomes

\[
B\hat{u}_{k|N} = \hat{x}_{k+1|N} - A\hat{x}_{k|N} \quad (4.8)
\]
Therefore \( \hat{u}_{k|N} \) can be directly computed from two fixed-interval smoothed estimates of the states. However, it is shown in [29] that it is not necessary to compute the states estimates explicitly to obtain the input estimate. The new fixed-interval deconvolution algorithm is given in the following theorem.

**Theorem 2.** Fixed-interval deconvolution:

(a) A two-pass fixed-interval smoother for \( u_k \) is

\[
\hat{u}_{k|N} = Q_k B^H r_{k+1|N}
\]

where \( k = N - 1, N - 2, \ldots, 1 \) and \( r_{k+1|N} \) is the residual state vector given in Equation (4.5).

(b) The smoothing error covariance \( \Psi_{k|N} \) is

\[
\Psi_{k|N} = Q_k - Q_k B^H S_{k+1|N} B Q_k
\]

where \( S_{k+1|N} \) is given in Equation (4.7)

**Proof:** For the proof, see [29] page 69.

Notice from Equation (4.9) that only the residual state vector is needed to compute the smoothed estimate of the input. Therefore, the fixed-interval deconvolution algorithm will be to run the same forward and backward process as in fixed-interval smoothing and compute the input estimates during the backward process using Equation (4.9).

### 4.4 Recursive State-Space Receiver for Blockwise Transmission

We now have described the basic state-space-based algorithms for both the MIMO channel estimation and equalization. We assume a simplified GSM-type frame structure shown in Figure 4.2; the channel is quasi-static, i.e. it only changes from frame to frame; the channel variation between two adjacent frames for a single user is large enough to be taken as statistically independent. The quasi-static
Figure 4.2: Frame structure of transmitted/received data.

assumption on the channel is reasonable for TDMA systems in that the two adjacent data frames of a single user are in fact separated in time by other users’ data because of the time division multiple access scheme. Accordingly, the processing of the received data should be carried out block by block since the channel information obtained on the previous block is not helpful to the understanding of the following one.

In the following procedure, we shall combine the fix-interval deconvolution equalizer with the recursive SSI channel estimator to form a complete state-space-based receiver.

**SSI Channel Estimation:**

1. Choose $n$ and $i$.
   - $n$ is the order of the estimated channel (Equation (4.2));
   - $i$ is the order of the extended observability algorithm, $i > n$ (Equation (2.11));

2. After receiving Frame #1, apply the SSI algorithm developed in Chapter 2 to obtain the estimates $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\hat{x}_{t_1}$, and $\hat{x}_{t_2}$.

**Data Decoding from $t_0$ to $t_1$ with the Backward Model:**

3. Using the method in [44] to construct a corresponding backward state-space model, $\{\hat{A}^B, \hat{B}^B, \hat{C}^B, \hat{D}^B\}$, from $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$.

4. With the backward state-space model $\{\hat{A}^B, \hat{B}^B, \hat{C}^B, \hat{D}^B\}$, run the Kalman predictor (the forward process in fixed-interval deconvolution) from $t_1$ to $t_0$ with the initial state given by $\hat{x}_{t_1}$ as obtained by the SSI channel estimator.
in Step 2. Note that since the backward model is used here, the forward process starts from \( t_1 \) to \( t_0 \), not from \( t_0 \) to \( t_1 \). Store the state estimate \( \hat{x}_{k-1|k} \) and its covariance matrix \( \Sigma_{k-1|k} \) for \( \forall k \in [t_0, t_1] \).

5. Run the backward process for the backward state-space model from \( t_0 \) to \( t_1 \) to compute \( r_{k|N} \) and \( \hat{u}_{k|N} \) for \( \forall k \in [t_0, t_1] \) from the stored state estimates and covariance matrices at last step.

![Diagram](image)

Figure 4.3: Channel equalization of a single frame using fixed-interval deconvolution with the backward and the forward state-space models.

**Data Decoding from \( t_2 \) to \( t_3 \) with the Forward Model:**

6. With the forward state-space model \( \{\hat{A}, \hat{B}, \hat{C}, \hat{D}\} \), run the Kalman predictor (the forward process in fixed-interval deconvolution) from \( t_2 \) to \( t_3 \) with the initial state given by \( \hat{x}_{t_2} \) as obtained by the SSI channel estimator in Step 2. Store the state estimate \( \hat{x}_{k|k-1} \) and its covariance matrix \( \Sigma_{k|k-1} \) for \( \forall k \in [t_2, t_3] \).

7. Run the backward process for the forward state-space model from \( t_3 \) to \( t_2 \) to compute \( r_{k|N} \) and \( \hat{u}_{k|N} \) for \( \forall k \in [t_2, t_3] \) from the stored state estimates and covariance matrices at last step.

8. Repeat Step 1 to Step 7 for other data frames.
Conclusions and Future Directions

5.1 Conclusions

Our main effort has been applying state-space models for MIMO wireless channels and designing the transceiver algorithms accordingly. The work can be roughly divided into three parts although they are closely related to each other: state-space channel modeling, state-space channel estimation, and state-space channel equalization.

We propose applying state-space models, instead of the more commonly used FIR models, for MIMO frequency-selective channels for the benefit of high modeling quality under channel order uncertainty. The modeling quality can be quantified by an upper bound of the minimum $H_\infty$ norm of the difference between the original channel and the approximated channel model. It is shown that state-space models always maintain lower $H_\infty$ approximation error than the FIR models with the same model order for either spatially uncorrelated MIMO channels or correlated ones.

The estimation of a state-space MIMO channel model can be accomplished by the proposed recursive subspace system identification (SSI) algorithm for non-
contiguous training data with accelerated rate of convergence. Numerical results show that the state-space-based channel estimation algorithm is capable of providing low-order models of high-quality channel approximation. When the true order of the physical MIMO channel is unknown, the performance of the state-space-based channel estimator is less sensitive, or more robust, to the error in model order selection than that of the the FIR-based RLS algorithm, while preserving slower but comparable convergence rate.

The robustness of state-space models against the mismatch between the order of the channel model and that of the true channel can be found not only in channel estimation error but also in channel equalization performance. In order to design equalizers for state-space channel models, a simple input-output data relationship framework ensures that the equalizer design for state-space channel models can proceed using the existing methods for designing equalizers for FIR models. In particular, a MIMO MMSE-DFE equalizer is developed for state-space models assuming either perfect channel knowledge at the receiver or estimated channel knowledge. The performance comparison of a state-space-based receiver and an FIR-based receiver shows that the former provides significantly smaller symbol error rate than the latter for reduced-order channel models while exhibiting marginally slower but comparable convergence rate. This implies that state-space channel models can be a more robust choice than FIR ones in the presence of model order selection error.

Besides the above main aspects of our work, a preliminary attempt is made to design a state-space channel equalizer that is especially suited for blockwise or framed data transmission such as GSM systems. A new state-space equalization scheme is developed based on the theory of fixed-interval smoothing and fixed-interval deconvolution. It is combined with the recursive MOESP channel estimator to form a complete receiver processing procedure.
5.2 Future Directions

In this section we shall make a few suggestions of the pathways along which further development of our work can be carried out.

- **Direct State-Space Channel Equalization:**
  Channel equalization can be performed in two different manners: indirect or direct. In the indirect equalization schemes, the coefficients of the channel model are estimated using training data and then are used to calculate the tap values of the equalizer filter. This is the approach we have taken in this thesis. The concatenation of channel estimation and equalization provides robust performance but could suffer relatively high complexity.

In the direct equalization approach, the receiver uses the training data to compute the equalizer directly without explicitly estimating the channel model. Many practical communication systems use this approach because of its simplicity and low computational cost. The design of a direct equalizer with state-space channel models would be an interesting problem for future exploration.

We imagine that the direct equalizer could take the form of a linear dynamic system characterized a state-space expression. Thus both the wireless channel and the equalizer are represented as state-space systems. Subspace system identification (SSI) methods can be applied to estimate the direct state-space equalizer taking the channel output as its input and the known transmitted training data as its output.

One foreseeable problem of this scheme is the coloring in the input random process to the SSI algorithms introduced by the non-ideal frequency response of the channel. SSI algorithms generally require white input to produce consistent estimates, thus the colored input may lead to a degraded quality of the equalizer.
• **Effects of Mobility:**
The development of channel estimation and equalization schemes in the thesis assumes a static channel model. However, realistic wireless channels are always time-variant in nature. It is important to study how their performance will be effected by the time variation of the channel. The design of competent channel tracking algorithms becomes necessary when the time variation is fast enough.

• **Smith-McMillan Form of MIMO Transfer Function Matrices:**
Since a frequency-selective MIMO FIR channel can be represented a polynomial transfer function matrix, the Smith-McMillan form of polynomial matrices may be another theoretical tool in system theory that can be used to further our understanding in the structure of MIMO communication channels.

Given a polynomial matrix $H(z)$, it can be decomposed into the Smith form [41]

$$H(z) = U(z)\Sigma(z)V(z),$$

where both $U(z)$ and $V(z)$ are unimodular square polynomial matrices, that is, their inverses exist and are also polynomial matrices. $\Sigma(z)$ is a diagonal polynomial matrix with the same dimension as $P(z)$. The same decomposition holds if $H(z)$ is a rational transfer matrix representing a causal system except $\Sigma(z)$ would be a diagonal rational matrix as well. This is called the Smith-McMillan form.

The SVD decomposition of the scalar channel matrix of MIMO flat fading channels is used intensively in calculating capacity and designing transceivers for such channels [39]. Similar to the SVD decomposition, the Smith-McMillan form converts the MIMO frequency-selective channel $P(z)$ into a series of SISO channels represented by the non-zero diagonal entries in $\Sigma(z)$. Therefore, it may lead to a nice way of calculating the capacity of MIMO frequency-selective channels. Furthermore, the fact that $U(z)$ and $V(z)$ are unimodular
square matrices implies that FIR filters are sufficient for the design of the precoders and the equalizers.
Appendix A

Basics of Subspace System Identification

Subspace system identification (SSI) methods refer to a group of algorithms developed for estimating multivariable state-space system realizations, which includes “Numerical algorithms for Subspace State Space System IDentification” (N4SID) [42], “Multivariable Output-Error State sPace” (MOESP) model identification [45], and “Canonical Variate Analysis” (CVA) [23], etc. One advantage of SSI methods is that users have simple and few design variables. Another advantage is that they have robust numerical properties and relatively low computational complexity because SSI methods are implemented using numerically robust computational tools such as Singular Value Decomposition (SVD) and QR factorization. This appendix is intended to outline the basic principles and procedures underlying the SSI algorithms. For more detailed treatment, please refer to [48, 25] and the references therein.

A $n$th-order $m$-input $p$-output state-space system is given as follows:

$$
\begin{align*}
\mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k \\
\mathbf{y}_k &= C\mathbf{x}_k + D\mathbf{u}_k + \mathbf{n}_k,
\end{align*}
$$

(A.1)

where $\mathbf{u}_k$ is the $m \times 1$ input vector at time $k$, $\mathbf{y}_k$ the $p \times 1$ output vector, $\mathbf{n}_k$ the
additive Gaussian noise, and \( x_k \) the \( n \times 1 \) state vector. The identification goal is to estimate a realization of matrices \( A, B, C, D \) and, possibly, the initial state vector \( x_0 \) given a block of input-output data \( \{u_k\} \) and \( \{y_k\} \).

The key equation on which most SSI methods are based is the following block-wise input-output data model.

\[
Y_{1,i,M} = \Gamma X_{1,M} + HU_{1,i,M} + N_{1,i,M}
\]

where

\[
Y_{1,i,M} = [Y_1, \ldots, Y_M] = \begin{bmatrix} y_1 & y_2 & \cdots & y_M \\ y_2 & y_3 & \cdots & y_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_i & y_{i+1} & \cdots & y_{i+M-1} \end{bmatrix},
\]

\[
U_{1,i,M} = [U_1, \ldots, U_M] = \begin{bmatrix} u_1 & u_2 & \cdots & u_M \\ u_2 & u_3 & \cdots & u_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_i & u_{i+1} & \cdots & u_{i+M-1} \end{bmatrix},
\]

\[
X_{1,M} = \begin{bmatrix} x_1 & \cdots & x_M \end{bmatrix},
\]

\[
\Gamma = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix},
\]

\[
H = \begin{bmatrix} 0 & \cdots & 0 & D \\ 0 & \cdots & 0 & D & CB \\ \vdots & \ddots & \vdots & \vdots \\ 0 & D & CB & \cdots & CA^{i-3}B \\ D & CB & CAB & \cdots & CA^{i-2}B \end{bmatrix}.
\] (A.2)

\( \Gamma \) is the extended observability matrix. \( H \) is the matrix of Markov parameters. \( i \) is called the “Horizon” of the SSI algorithm and is chosen by users as one of the
design variables. See [26] for a detailed discussion on the choices of the design variables and their effects on the performance of SSI algorithms.

Generally, SSI methods consist of two steps to estimate a state-space system. The first step is to estimate the subspace of the extended observability matrix $\Gamma$. Various ways of estimating this subspace give rise to different flavors of the algorithm, such as N4SID, MOESP, and CVA. In the second step, the structure of the extended observability matrix is used to find a system model estimate, that is, a realization of $A, B, C, D$ and $x_0$.

**Estimating $\Gamma$**

Starting with forming the data according to the key equation

$$Y_{1,i,M} = \Gamma X_{1,M} + HU_{1,i,M} + N_{1,i,M}, \quad (A.3)$$

$\Gamma$ can be estimated with the following steps.

- Remove the $U_{1,i,M}$ term.
  
  By finding the projection matrix to the orthogonal complement space of the row space of $U_{1,i,M}$
  
  $$P^\perp = I - U_{1,i,M}^+ U_{1,i,M}, \quad (A.4)$$

  the contribution from the input in (A.3) can be removed by applying the projection to both sides of the equation as below,

  $$Y_{1,i,M} P^\perp = \Gamma X_{1,M} P^\perp + HU_{1,i,M} P^\perp + N_{1,i,M} P^\perp,$$

  $$= \Gamma X_{1,M} P^\perp + N_{1,i,M} P^\perp. \quad (A.5)$$

- Remove the noise term.

  We can find $pi \times 1$ instrumental variables $\psi_k$ to “average out” the noise term
in (A.5).

\[ \Psi = [\psi_1, \ldots, \psi_M] \]

\[
\frac{1}{M} Y_{1,i,M} P^\perp \Psi^H = \Gamma \underbrace{\frac{1}{M} X_{1,M} P^\perp \Psi^H}_{\text{full rank}} + \underbrace{\frac{1}{M} N_{1,i,M} P^\perp \Psi^H}_{0 \text{ as } M \to \infty} \tag{A.6}
\]

A good choice of \( \psi_k \) is the past input-output data

\[
\psi_k = [y^T_{k-1}, \ldots, y^T_{k-s_1}, u^T_{k-1}, \ldots, u^T_{k-s_2}]^T
\]

When the amount of the training data is large enough, \( i.e., \) as \( M \to \infty \), the range of the left hand side of (A.6) is equal to the range of \( \text{Gamma} \).

The above steps can be implemented by LQ factorization. If we ignore the noise for simplicity, then

\[
\begin{bmatrix} U_{1,i,M} \\ Y_{1,i,M} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},
\]

and

\[ R(\Gamma) = R(L_{22}). \]

When noise is present, \( L_{22} \) is corrupted by the noise terms in \( Y_{1,i,M} \). In this case, singular value decomposition (SVD) can be used to extract the signal subspace of \( L_{22} \) as an estimate of \( R(\Gamma) \).

\[
L_{22} = \begin{bmatrix} U_1 & U_1^\perp \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_1^H \\ (V_1^\perp)^H \end{bmatrix}
\]

where \( U_1 \) is a \( pi \times n \) matrix, and

\[ U_1 = \Gamma T \]

for some nonsingular matrix \( T \). Note that if there is no noise, matrix \( \Sigma_0 \) is equal to zero.
Estimating a State-Space Model

Finding $A$ and $C$

Once an estimate of $\Gamma$ is available, $C$ can be obtained immediately by realizing that $C$ is a submatrix of $\Gamma$.

$$\Gamma = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}$$

$$C = \Gamma(1 : i, :)$$

In order to find $A$, we shall use the so-called shift-invariance property of $\Gamma$, that is,

$$\Gamma^{(1)} A = \Gamma^{(2)},$$

where

$$\Gamma^{(1)} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-2} \end{bmatrix}, \quad \Gamma^{(2)} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{i-1} \end{bmatrix}.$$ 

Then $A$ can be obtained by least squares,

$$A = \Gamma^{(2)} \Gamma^{(1)} +.$$ 

Finding $B$, $D$, and $x_0$

Assuming $A$ and $C$ are known and the initial state $x_0$ equals zero, then

$$y_k = C(qI - A)^{-1} Bu_k + D u_k + n_k$$

is linear in $B$ and $D$, where $q$ is the delay operator [25]. In order to solve for $B$ and $D$, we can organize the above expression in a standard linear regression form

$$y_k = W \begin{bmatrix} vec(B) \\ vec(D) \end{bmatrix} + n_k,$$
where $W$ is the $p \times (mn + mp)$ regressor. $\text{vec}(\cdot)$ is the operation that forms a vector from a matrix by stacking its columns on top of each other. The elements of $B$ and $D$ can be solved by least squares.

When the initial state $x_0$ is non-zero, we have

$$y_k = C(qI - A)^{-1}x_0\delta_k + C(qI - A)^{-1}Bu_k + Du_k + n_k,$$

where $\delta_k$ is the Kronecker Delta function in $k$. Similarly, this expression can be written in a linear regression form

$$y_k = \tilde{W} \begin{bmatrix} x_0 \\ \text{vec}(B) \\ \text{vec}(D) \end{bmatrix} + n_k,$$

then $B$, $D$ and $x_0$ can be solved by least squares.

**MATLAB Toolbox**

Subspace-based system identification algorithms are implemented by the command `n4sid` in MATLAB’s System Identification Toolbox. The following is the help description provided by MATLAB.

**n4sid** Estimates a state-space model using a sub-space method.

MODEL = N4SID(DATA) or MODEL = N4SID(DATA,ORDER)

MODEL: Returned as the estimated state-space model in the IDSS format.
DATA: The output input data as an IDDATA object. See HELP IDDATA.
ORDER: The order of the model (Dimension of state vector). If entered as a vector (e.g. 3:10) information about all these orders will be given in a plot. (Note that input delays (NK, see below) larger than 1 will be appended as extra states, giving a resulting model of higher order.) If ORDER is entered as 'best', the default order among 1:10 is chosen. This is the default choice.
ORDER can also be an IDSS model object, in which case all model structure and algorithm properties are taken from this object.
By MODEL = N4SID(DATA,ORDER,Property_1,Value_1, ...., Property_n,Value_n) all properties associated with the model structure and the algorithm can be affected. See IDPROPS IDSS and IDPROPS ALGORITHM for a list of Property/Value pairs.

Useful model structure properties are

'Focus' : ['Prediction'|'Simulation'|'Filter'|'Stability'].
  'Simu' and 'Stab' guarantee a stable model.

'nk': row vector of delays from the different inputs.
The initial state is always estimated, but delivered in MODEL only if 'InitialState' = 'Estimate'.

If 'DisturbanceModel' = 'None', the K-matrix is returned as 0, and a stable model is guaranteed. Default is 'DisturbanceModel' = 'Estimate'.

Computing the covariance information takes most of the time. Setting 'CovarianceMatrix' = 'None' suppresses these calculations.

The algorithm is affected by the properties

'N4Weight': ['Auto'|'MOESP'|'CVA'] Determines the weightings before the SVD. 'Auto' makes an automatic choice.

'N4Horizon': Determines the prediction horizons used by the algorithm.
  N4Horizon = [r, sy, su], where
  r: the maximum prediction horizon
  sy: The number of past outputs used in the predictors
  su: The number of past inputs used in the predictors
  If N4Horizon has several rows, each row will be tried.
  N4Horizon = 'Auto' (default) estimates reasonable horizons.
  In case 'DisturbanceModel' = 'None', this default choice uses sy = 0.

'Trace': ['On'|'Off'] 'On' gives info to screen about fit and choice of N4Horizon

'MaxSize': No matrix with more than maxsize elements is formed. Loops are used instead when necessary.
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