Simulation of equity return properties using GBM and modified URN models

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Abstract

We have been presented the properties of asset return by simulation within the empirical data. However, is it possible to illustrate properties by statistical analysis? Most currently existing models fail to reproduce all these statistical features. In this paper, we will elaborate the properties by applying different statistical models: Geometric Brownian Motion and Ehrenfest URN. We will focus on the following properties: distributional properties, tail properties and extreme fluctuations, path-wise regularity, linear and nonlinear dependence of returns in time and across stocks. In this project, I will use S&P 500 index return as the data and apply it with the models to compare the results with empirical data.

1. Geometric Brownian Model (GBM)

Scottish biologist Robert Brown discovered Brownian motion while studying pollen particles floating in water through a microscope. Brownian motion has contributed a lot to thermodynamic's foundations and is also one of the key components of statistical physics. It is also widely used in finance when modeling random behavior that evolves over time.

Before applying the Geometric Brownian Model to the data, I first plotted the graph of empirical data of S&P 500 index return, as shown in Figure 1.1. To easily compare the data in different periods, I also divided the time periods into several parts in a time interval of 10 years. The results are presented in two ways: a line chart and a histogram, corresponding to Figure 1.2 and Figure 1.3.

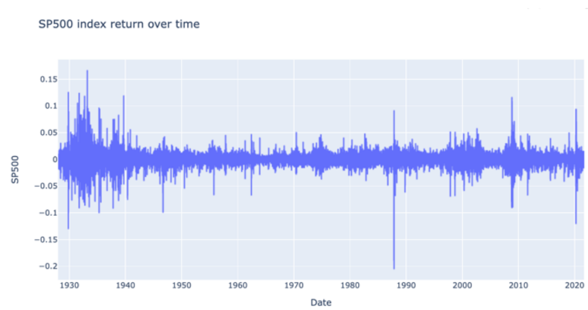
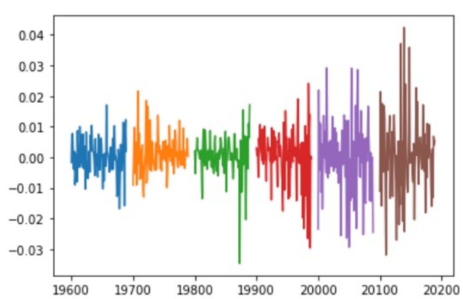


Figure 1.1: SP500 index return



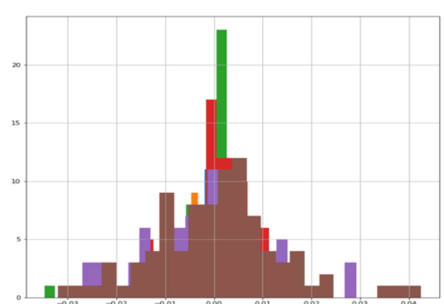


Figure 1.2

Figure 1.3

Definition 1.1

Geometric Brownian Motion formula:

$$St = S0e((\mu - \sigma^{2}/2)t + \sigma Wt)$$

Here, “St” represents the stock price at time t and in the case of S&P500, “μ” (return of stock) is roughly 9% annually and “σ” (volatility) is roughly 16% annually. We assume that μ is the return of stock and σ is the volatility. To test whether μ is roughly about 9%, we could apply the formula of it:

$$\mu = \frac{1}{n}\sum_{k=1}^{n} \left(\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}\right)$$

As well as :

$$\mu_{annually} = \left(1 + \mu_{daily}\right)^{252} - 1$$

To test whether σ is roughly about 16%, we could apply the following formula:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n} \left(\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} - \mu\right)^{2}}$$

As well as:

$$\sigma_{annually} = \sqrt{252}\sigma_{daily}$$

I used python to test it and here are the results Figure 1.4 and Figure 1.5:

```
sp500_list = []
for i in range(len(df['SP500'])):
    sp500_list.append(df['SP500'][i])
#sp500_list
miu = sum(sp500_list)/len(sp500_list)
miu_annually = ((1+miu)**252)-1
print("The annually return of SP500 since 1928 is: ", miu_annually)
```

```
The annually return of SP500 since 1928 is:  0.08058781300911733
```

```
import math
value = 0
for i in range(len(sp500_list)):
    value += (sp500_list[i]-miu)**2
sigma_daily = math.sqrt(value/(len(sp500_list)-1))
sigma_annually = sigma_daily*math.sqrt(252)
print("The annually volatility of SP500 since 1928 is: ", sigma_annually)
```

```
The annually volatility of SP500 since 1928 is:  0.19001760315371574
```

Figure 1.4: Annual return test *Figure 1.5:Annual volatility rest*

We can clearly see that the python test results are very close to the realistic value. It is reliable because we only select certain periods of time to prove and thus the bias of value is acceptable. In the following codes and explanation, I will keep the value of μ to 0.09 and σ to 0.16.

Now we are ready to simulate stock prices using Geometric Brownian Motion. We apply the following Brownian motion formula:

$$St = S0e((\mu-\sigma^2/2)t+\sigma Wt)$$

We also apply SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

I selected the initial price (S0) as 121.71, starting from 1981-11-20. To test its reliability, I first simulated 40days Stock Price with 100 simulations and simulated 4000 days with one simulation, as shown below denoted respectively as Figures 1.6 and 1.7.

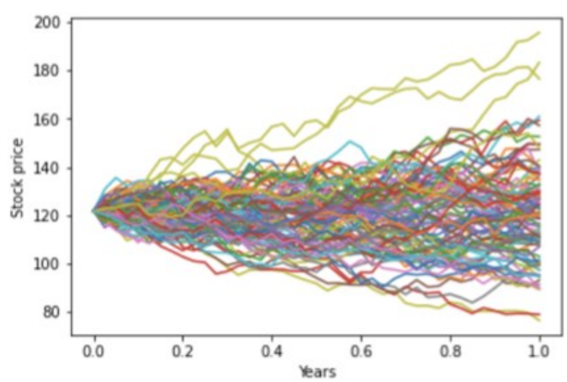


Figure 1.6: 40 days with 100 simulations

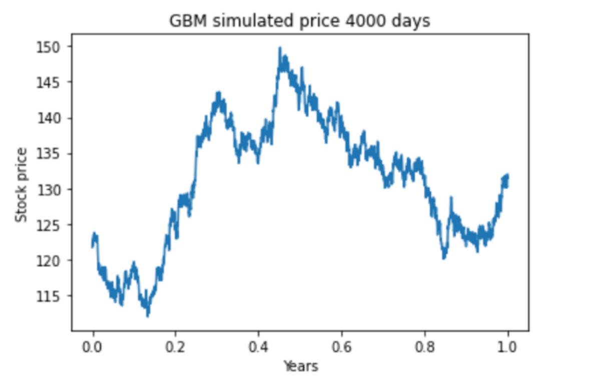


Figure 1.7: 4000 days with 1 simulation

Next, in order to compare the simulations of GBM and realistic price index daily return, I calculated the GBM price-index daily return and realistic daily price return of 4000 days starting from 1981- 11-20. The results are as follows:

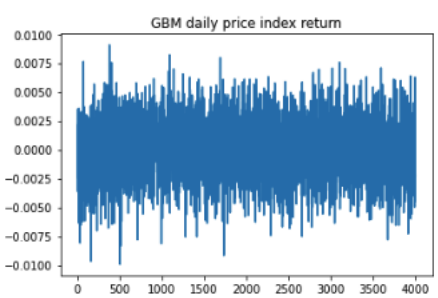


Figure 1.8 Simulated index return

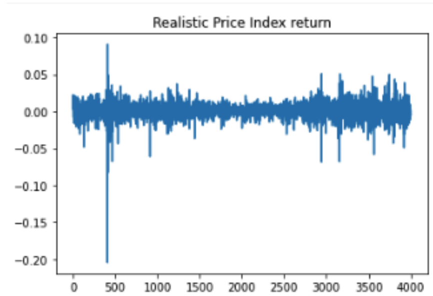


Figure 1.9: Realistic index return

We cannot directly observe a difference between the line charts. In order to visualize the difference in the same plot, I applied the KDE plot and histogram, as presented in Figure 1.10.

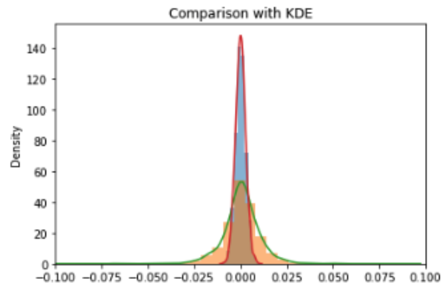


Figure 1.10

It is very clear that the GBM return is not very accurate. The range is smaller compared to the realistic return, which shows that there are many other factors influencing the stock price.

The value of parameters will also affect the accuracy of simulation. For example, with the increase of volatility from 0.10 to 0.25, the GBM will be a better fit, as shown in Figures 11 and 12.

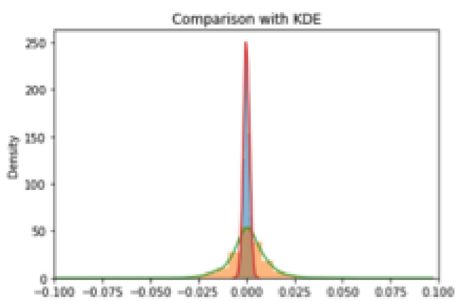


Figure 1.11: sigma = 0.1

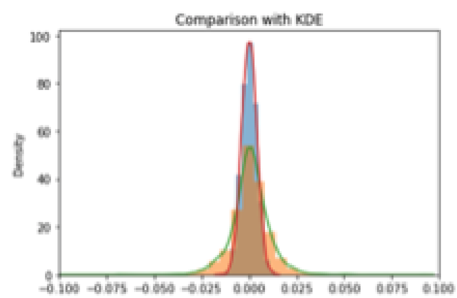


Figure 2: sigma = 0.25

1. Modified Ehrenfest Urn Model

Before moving on to the Modified URN model, I would like to introduce and implement the original URN model. The Ehrenfest Urn model is named after Paul Ehrenfest and is a discrete models that explains the exchange of gas molecules between two containers. We can formulte the molecules and containers as simple ball and urn models: the balls correspond to the molecules and the urns to the two containers.

Suppose there are two urns A and B. Urn A contains N marbles and Urn B contains none. The marbles are labeled 1,2,...N. In each step of the algorithm, a number between 1 and N is chosen randomly, with all values having equal probability. The marble corresponding to that value is moved to the opposite urn. The first step of the algorithm will always involve moving a marble from A to B. The state of the system at time n is the number of balls in urn 1, which we will denote by X_n. The first step analysis is as followed:

$$W_{n-1,n} = \frac{n}{N}, \qquad n = 1, \ldots, N$$

$$W_{n+1,n} = \frac{N-n}{N} \qquad n = 0,1, \ldots, N-1$$

I implemented the model in python and plotted the graph, and I simulated 1000 steps with 1000 balls.

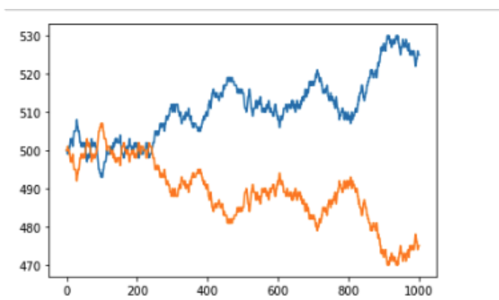


Figure 2.1: Urn model with 1000 steps and 1000 balls

We assume the random drawn ball has a replacement (creation) probability that depends on the number of balls present in the urns. We add alpha and beta to the scenario.

The first step transition probabilities are as follows:

$$W_{n-1,n} = \frac{n}{N}\frac{\beta + N - n}{\alpha + \beta + N - 1}, \qquad n = 1, \ldots, N$$

$$W_{n,n} = \frac{n}{N}\frac{\alpha + N - 1}{\alpha + \beta + N - 1} + \frac{N-n}{N}\frac{\beta + N - n - 1}{\alpha + \beta + N - 1} \qquad n = 1, \ldots, N$$

$$W_{n+1,n} = \frac{N-n}{N}\frac{\alpha+N-n}{\alpha+\beta+N-1}, \qquad\qquad n = 1, \dots, N$$

There are three cases of alpha and beta as discussed in different situations:

Case 1: 0 < α, β < 1: Long time spent around the barriers, with rapid fluctuation across the central region;

Case 2: α, β >> 1: Short time spent around the barriers, with long fluctuation across the central region. If α, β → ∞ the behavior is Bernoullian.

Case 3: α, β < 0, |α| ≥ n, |β| ≥ N − n: Very short time spent around the barriers, with long fluctuation across the central region;

I plotted each case with corresponding alpha and beta values, Figures 2.2, 2.3, and 2.4 correspond respectively to cases 1, 2 and 3.

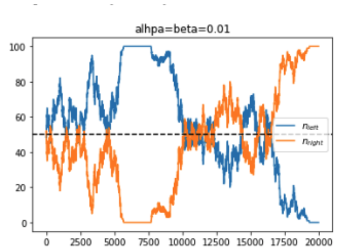


Figure 2.2: case 1

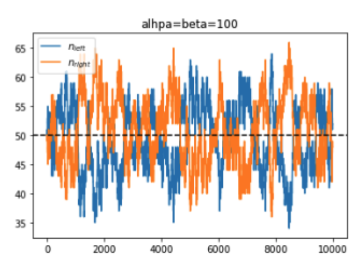


Figure 2.3: case 2

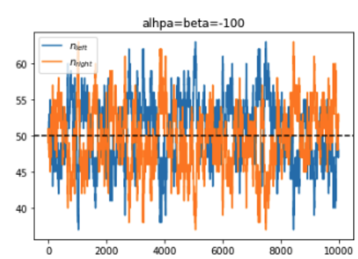


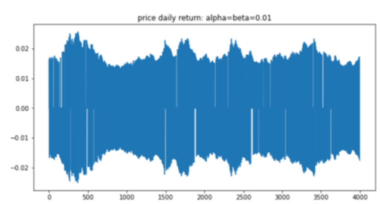
Figure 2.4: case 3

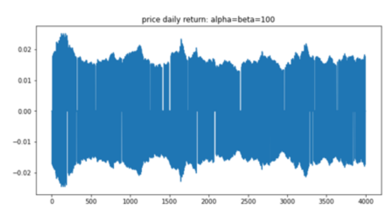
When alpha = beta = 0.01, the graph depicts a horizontal line from steps 5000 to 7500, and it is deviates from the other steps interval. When there are 99 balls in the left urn and 1 ball in the right urn, alpha and beta don't have much influence on the first formula because they are much less than N. In this case, the first formula is fully dependent on N. So, the chance of replacing one ball from left to right is very low. Therefore, there are multiple steps during which no ball is chosen and replaced from the right urn which means that the number of balls in left and right will remain the same. And that's why the line is horizontal from steps 5000 to 7500.

When it comes to case two, alpha = beta = 100, the fluctuation is rapidly floating from the central line. And that is because alpha and beta now are much greater than N which means they dominate the first step probability formula. Therefore, the probability is nearing 0.5. In this case, balls are replaced many times between two urns in a balanced motion.

To summarize, we pick the particle with probability n / N (in proportion to how many there are in the urn), but we move it with probability roughly (N-n) / N (in proportion to how many there are in the other urn). Therefore, if we are at large n, we tend to stay at large n.

After analyzing the situation of add-up values (alpha and beta), we connect the model to two traders: bull and bear. A bull market is the condition of a financial market in which prices are rising or are expected to rise. The term “bull market” is most often used to refer to the stock market but can be applied to anything that is traded, such as bonds, real estate, currencies, and commodities. Bull market is when investors get very bullish and start buying a lot of stocks driving up their prices and often causing bubbles. Because prices of securities rise and fall essentially continuously during trading, the term ”bull market” is typically reserved for extended periods in which a large portion of security prices are rising. Bull markets tend to last for months or even years. In short, a bull trader takes a lot of risk, and a bear trader is very risk averse. I plotted the daily returns in 3 cases, with 4000 days and 100 trade, as shown in Figure 2.5, 2.6 and 2.7.





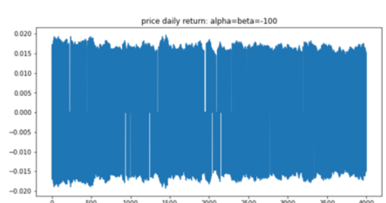
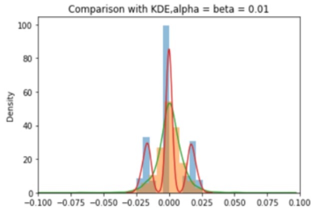


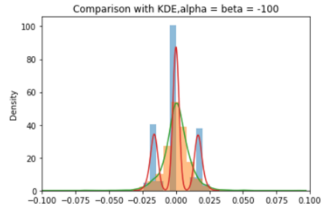
Figure 2.5: case 1

Figure 2.6: case 2

Figure 2.7: case 3

To compare with empirical data, I, again, used S&P500 datasets to plot the realistic return starting from 1981.11.20, and used the KDE plot to compare the difference. The plots are as followed:





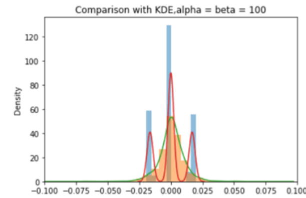


Figure 2.8: case 1 *Figure 2.9: case 2* *Figure 2.10: case 3*

To compare the difference between different alpha and beta values, I also plotted the three cases in one graph to directly see the results.

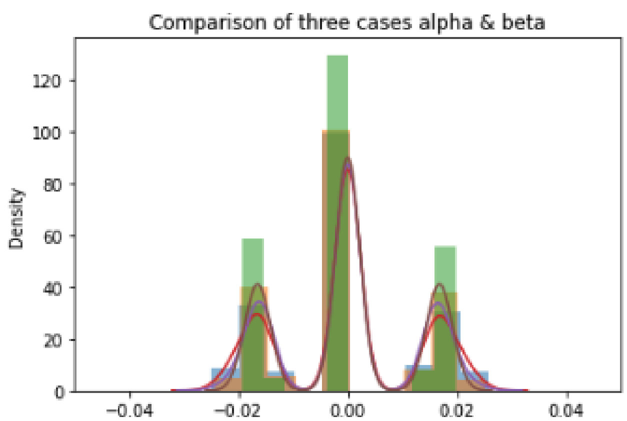


Figure 2.11

We can see from the plots that the Modified Urn Model is still not a reliable model. The range is bigger, and the fluctuation is huge. There are many other factors which will affect the performance of the model. For example, we could also have a neutral trader who does not trade in certain situations, which creates new cases and scenarios.

So far, we have been setting the price change as unit price increment 1. However, in the real stock market, bull or bear traders won't keep the increment as 1. Therefore, the prices change is N(1, sigma) when a bear becomes a bull. I revised the code and add use the method of numpy.random.normal (miu, sigma) to change the price increment. Here are the results of all three cases under the new scenario.

Figure 2.12: Normalized Case 1

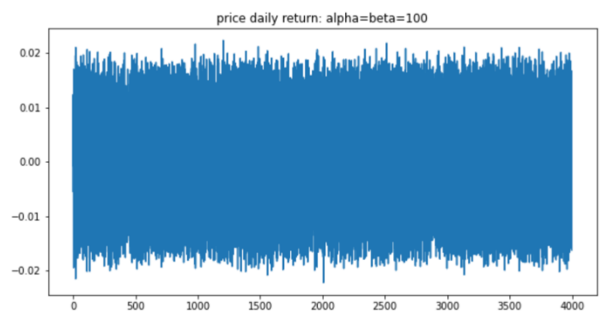


Figure 2.13: Normalized Case2

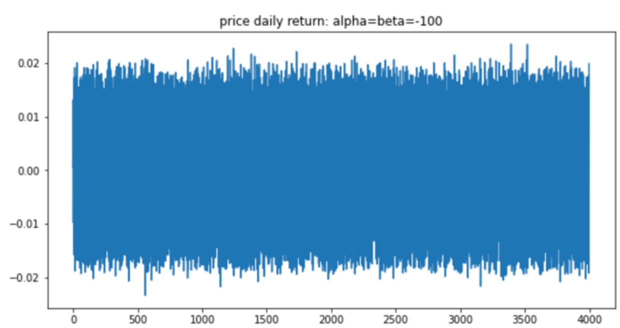
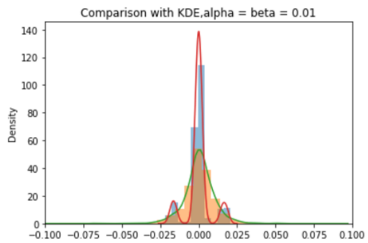
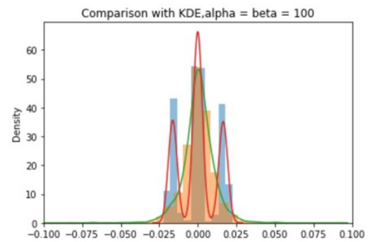


Figure 2.14: Normalized Case3

We can see from the Figures 2.12, 2.13 and 2.14 that the curves are smoother than Figures 2.8, 2.9, 2.10. Results become more realistic and more similar when it comes to comparing realistic price increments. I also use KDE plots to process the comparison, as shown below in Figures 2.15, 2.16 and 2.17.





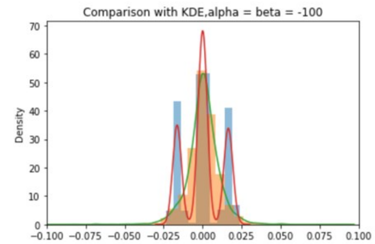


Figure 2.15 *Figure 2.16* *Figure 2.17*

It is clear that the error is not significant smaller. There is still some gap between the curves because there are so many other factors/randomness which could have affected the results, such as the value of alpha and beta, the value of sigma, the population of agents/traders and the simulations time steps. There even may exist a third trader who does not do any action under certain circumstances, which we call a neutral trader. Therefore, the simulation of a modified URN model is not very precise to the real data given so much randomness.

Conclusion

In this paper, I simulated Geometric Brownian motion and modified the Ehrenfest URN model using the S&P 500 dataset. Through the results, we can see that both models reproduce some features but do not accurately fit the data. Both models could be enhanced by including additional parameters or factors.

The next step of the project is to dive deeper into the other complicated models, such as GAMMA and Fractional Brownian Motion model. We aim to explore and understand the utilization of these models in a realistic market.

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