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Beyond Qubits: Qudits for Quantum Computing and Many-Body Simulation in
Superconducting Circuits

by

Noah Goss

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Irfan Siddiqi, Chair

Professor K. Birgitta Whaley

Professor Harry Levine

Spring 2026

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Superconducting Circuits

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Abstract

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Doctor of Philosophy in Physics

University of California, Berkeley

Professor Irfan Siddiqi, Chair

Quantum processors based on superconducting circuitry have reached the scale of hundreds of qubits and demonstrated computational capabilities that challenge state-of-the-art classical methods for carefully designed problems. Despite these advances, fundamental questions persist over how to best improve and scale the computational power of this emerging technology. Most transmon-based quantum processors encode information in two-level systems, or qubits, leaving the rich multilevel structure of the transmon largely unexploited. In this thesis, we demonstrate how the capabilities of these devices can be meaningfully extended by harnessing this intrinsic multilevel structure to encode qudits — d -level quantum systems that access a larger and more connected Hilbert space than their qubit counterparts. We begin by demonstrating universal single-qudit operation before turning to microwave-activated entanglement to realize scalable two-qudit gates. We present a high-fidelity, fully entangling two-qudit CZ gate based on a novel tunable cross-Kerr mechanism. We further explore two-qudit gates that selectively entangle subspaces of the full Hilbert space via two-photon multipartite Raman transitions, employing these operations to generate large entangled states across a transmon array. To address the noisy nature of near-term qudit computation, we introduce and characterize error suppression techniques for qudit circuits, including dynamical decoupling and quantum error mitigation. We then synthesize these directions by exploring simulations of quantum systems that have innate multilevel interactions and are therefore well tailored to qudit operation. First we investigate digital quantum simulations of collective neutrino oscillations and the spin-1 truncated Abelian Higgs model, finding error mitigation methods critical for improving circuit performance. Finally, out of equilibrium — or Floquet — spin-1 systems are realized using an array of 15 superconducting qutrits to probe a novel $\mathbb{Z}_3$ discrete time crystalline (DTC) phase of matter with intrinsically chiral interactions. This DTC result demonstrates that native qudit hardware provides powerful access to non-equilibrium phases rooted in higher discrete symmetries, opening a route to a broader landscape of dynamical orders beyond the $\mathbb{Z}_2$ Ising paradigm.

In Memory of My Father

And if you find her poor, Ithaka won't have fooled you. Wise as you will have become, so full of experience, you'll have understood by then what these Ithakas mean.

C.P. Cavafy

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Chapter 1

Introduction

The first flickering lights of evidence for a strange atomic world lurking beyond our sight can be traced to landmark experiments of the late 19th and early 20th century. These experiments revealed a world governed by unfamiliar laws, unlike anything physicists had encountered before; in those early days from which nascent quantum theory arose, investigations of blackbody radiation, the photoelectric effect, and the observation of the discrete spectral lines of Hydrogen at once captivated and confounded. To make sense of these experiments, the new quantum theory was developed by titanic figures like Planck, Einstein, Bohr, Dirac, Heisenberg, and Schrödinger, ultimately providing a concrete description of phenomena like wave-particle duality, quantized energy levels, and the uncertainty principle. The so-called “first quantum revolution” grew from a recognition that these seemingly esoteric notions of the physical laws governing very small things could have profound technological implications for those of us who live in the macroscopic world, bearing transformational fruit such as transistors, lasers, MRI machines, and semiconductor technology. In many ways, this first quantum revolution birthed our modern information age where silicon reigns supreme and massive computational power has enabled the internet, smartphones, and increasingly powerful artificial intelligence.

A second quantum revolution is now taking shape. In the first quantum revolution most quantum-derived technologies related to material properties at a macroscopic scale and generally relied upon gross control over quantum ensembles often interacting strongly with an outside environment. In this second quantum revolution, rather than engineering materials that collectively exhibit quantum characteristics, quantum scientists now seek to isolate, prepare, and manipulate individual coherent quantum systems, exploiting the unique resources of superposition and entanglement as programmable phenomena. While potentially transformative, this is also a profoundly more demanding endeavor; quantum states are inherently fragile and easily corrupted upon interaction with their environment in a process known as decoherence. Taming this fragility has been one of the central experimental challenges of the past several decades. Yet the payoff for overcoming these challenges can be worthwhile, promising technological capabilities that without quantum resources are thought to be impossible in computing and sensing.

The idea of quantum computing, or performing computation with elementary quantum objects instead of classical bits, can be traced back to the beginning of the 1980s when Richard Feynman and Yuri Manin independently proposed the concept [70, 159]. Their central insight was that exactly simulating quantum systems on classical computers was fundamentally intractable, since the resources required to faithfully represent a quantum system grow exponentially with its size, and that the natural solution was to build a computer that was itself quantum mechanical. Feynman, famously put it more bluntly: “Nature isn’t classical, dammit, and if you want to make a simulation of nature, you’d better make it quantum mechanical.” Whereas a classical bit must be either 0 or 1, a quantum bit, or qubit, can exist in a superposition of both simultaneously, and where classical bits are independent, qubits can be entangled, meaning the state of one qubit has a special correlation with the state of another. A system of N qubits thus can potentially exist in a superposition of 2^N states simultaneously, yielding the “quantum parallelism” that is responsible for computational power of a quantum computer. Moreover, the potential utility of a quantum computer has become better understood with advances in quantum algorithms, with the most famous examples including Shor’s algorithm which provides an exponential speed up for factoring [230], Grover’s algorithm which provides a quadratic speedup for searching unstructured databases [92], and algorithms for simulating quantum systems [151].

The promise of these quantum algorithms galvanized scientists and engineers to confront a fundamental question: how does one actually build a quantum computer? Early experimental efforts were often sobering as decoherence severely limits the fidelity of quantum operations, and some of the aforementioned algorithms can require executing billions of quantum gates. For a time this seemed an insurmountable obstacle, and a pessimistic view took hold that quantum computation might be a beautiful theoretical idea but a practical impossibility. The field was rescued from this pessimism by one of its most profound theoretical developments. In 1995, Peter Shor proposed a way forward by pioneering quantum error correction (QEC), where logical qubits are encoded in the redundancy of many physical qubits, creating a pathway to logical qubit lifetimes far in excess of the lifetime of the physical qubits of which they are composed [231]. This was followed rapidly by the development of the fault tolerance threshold theorem, which established that if the error rate of a quantum computer’s physical components could be brought below a certain critical threshold, arbitrarily long and reliable quantum computations were in principle unlocked by increasing the code size [5]. Quantum error correction transformed the outlook of the field entirely, converting the question of whether a useful quantum computer could ever be built into the more tractable and optimistic engineering question of how.

As it currently stands, there are many actively pursued physical platforms for realizing quantum computation. A non-exhaustive list of these platforms includes: superconducting circuits, trapped ions, photonics, neutral atoms, spin qubits, and topological qubits. In this thesis, our attention will be fixed to superconducting circuits which are, at this point, a well established and promising player within this competitive field. However, within the context of each of these platforms there are some recurring questions being asked by the community:

- What are the best unit-cell architectural choices for encoding quantum information?
- What are the best methods for performing quantum operations (i.e. logic gates, read-out, reset)?
- What are the best methods for scaling up the available Hilbert space for quantum computation?
- How can we best realize effective QEC?
- Can near-term quantum computers provide scientific insight into classically difficult problems?

In this thesis, we will explore quantum computation with superconducting qudits, that is, superconducting quantum systems with more than two computational energy levels. In the context of these pressing questions, the advantages and challenges of these multilevel approaches are now beginning to be explored across several of the platforms mentioned above. The main takeaway I hope the reader is left with is that qudit approaches expand the toolbox for operation with existing quantum processing technology, squeezing more computational juice out of our devices in the lab. We will return in Section 1.5 to providing more justification for the value of pursuing qudit operation.

1.1 A Brief History of Superconducting Circuits

As Rolf Landauer famously remarked, information is physical — not merely an abstract entity but something that must be embodied in a physical system where the details of that embodiment matter a great deal. This statement, is perhaps, especially relevant in the context of *quantum information*, where our quantum control is ultimately analog and we must carefully balance coupling to our qubits with their coupling to the environment. In this thesis we will explore quantum computing with superconducting circuits. It is an amazing and not entirely obvious thing that superconducting circuits can be treated as a system with a few quantum mechanical degrees of freedom. We now turn to providing an abbreviated history of this field.

The landmark experimental result (for which the 2025 Nobel prize in physics was awarded) of energy level quantization and macroscopic tunneling in a Josephson junction circuit was achieved at UC Berkeley in 1985 by Professor John Clarke, Michel Devoret, and John Martinis [161, 51]. Their pioneering work affirmatively answered Anthony Leggett’s question about whether a degree of freedom comprised of macroscopic number of atoms can behave quantum-mechanically¹. Clarke, Devoret, and Martinis’ work showed that superconducting circuits could behave as “artificial atoms”, leading to early direct experimental evidence of quantum coherence in superconducting circuits including Rabi oscillations in a charge qubit

¹This story is from Steven Girvin’s talk, *A Brief History of Superconducting Quantum Computing*

in 1999 [179] and Ramsey oscillations in a quantronium qubit in 2002 [258]. In the following years, significant progress was made to establish superconducting circuits as a viable platform for quantum computing, such as the demonstration of longer coherence times in phase qubits [162], the first two-qubit gates [197, 239], and the circuit quantum electrodynamics (circuit QED) framework for strong coupling between a single photon and an artificial atom [16, 263]. For a few years, superconducting qubits stalled as the popular charge and phase qubits were severely limited by intense charge and flux noise respectively. The introduction of the transmon qubit in 2007 and its experimental realization heralded a significant advance, giving the field a simple and scalable superconducting qubit that was robust to charge noise [136, 221]. In the time since its introduction, reported transmon coherence times have improved by over 3 orders of magnitude, crossing the millisecond threshold [232, 18].

The maturation of the transmon as a reliable and scalable qubit platform set the stage for a remarkable decade of experimental progress. On the algorithmic and computational side, Google’s demonstration of quantum advantage on the Sycamore processor in 2019 [10] marked a watershed moment, showing for the first time that a superconducting quantum processor could perform a sampling task that was classically intractable at the time, even if the practical utility of that specific task remained debated. More recently, superconducting processors have demonstrated proof of principle quantum error correction in a surface code memory experiment with exponential improvements in increasing code distance up to distance 7 [6], representing a critical milestone on the road to fault-tolerant quantum computing. In parallel, the promising contents of circuit QED toolkit has expanded beyond the transmon. The fluxonium qubit [160], which includes a parallel shunt superinductance to the transmon circuit has emerged as a compelling alternative offering long coherence times and high-fidelity gates [184, 52]. The development of bosonic codes, in which quantum information is encoded in the infinite-dimensional Hilbert space of a harmonic oscillator mode rather than in a two-level system, has opened an alternative approach for hardware-efficient quantum error correction, employing cat qubits [171, 205] and GKP qubits [89, 30]. It is an exciting time to be working in the field of circuit QED and this thesis will aim to make the case for non two-level system, or qudit approaches, as yet another promising route to realizing effective quantum operation.

1.2 A Review of Some Concepts in Quantum Mechanics

Unitary Evolution

The state of a quantum system evolves in time according to the Schrödinger equation,

$$i\hbar\partial_t |\psi(t)\rangle = H |\psi(t)\rangle, \quad (1.1)$$

where the Hamiltonian H is a Hermitian operator encoding the total energy of the system. For a particle of mass m moving in a potential $V(x)$, the Hamiltonian takes the familiar form,

$$H = \frac{p^2}{2m} + V(x), \quad (1.2)$$

where x and $p = -i\hbar\nabla$ are now quantum mechanical operators (often displayed with a $\hat{}$ symbol) satisfying the canonical commutation relation $[x, p] = i\hbar$. In the position representation, the Schrödinger equation becomes

$$i\hbar\partial_t\psi(x, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x) \right] \psi(x, t). \quad (1.3)$$

An equivalent formulation, the Heisenberg picture, keeps the state fixed and instead allows operators to evolve according to

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[H, \hat{O}] + \frac{\partial\hat{O}}{\partial t}, \quad (1.4)$$

where the first term generates the unitary part of the evolution and the second accounts for any explicit time dependence of the operator. Both pictures are equivalent and related by a unitary transformation; which one is more convenient depends on the problem at hand. The formal solution to the Schrödinger equation defines the time evolution operator $U(t, t_0)$ via

$$i\hbar\frac{dU(t, t_0)}{dt} = H U(t, t_0), \quad U(t_0, t_0) = I, \quad (1.5)$$

which for a time-independent Hamiltonian yields,

$$U(t, t_0) = e^{-i(t-t_0)H/\hbar}. \quad (1.6)$$

Since H is Hermitian, U is unitary implying $UU^\dagger = U^\dagger U = I$. Geometrically, U acts as a rotation in Hilbert space, preserving the norm of any state vector. A direct consequence of the linearity of U is that quantum mechanics is itself linear:

$$U(\alpha|a\rangle + \beta|b\rangle) = \alpha U|a\rangle + \beta U|b\rangle. \quad (1.7)$$

The famous no-cloning theorem, with deep implications in quantum information, follows directly from the linearity of quantum mechanics. Suppose there existed a unitary U that clones an arbitrary quantum state, meaning

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle, \quad (1.8)$$

for any state $|\psi\rangle$. Applying this cloning hypothesis to two states $|\psi\rangle$ and $|\phi\rangle$ individually gives

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle, \quad U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle. \quad (1.9)$$

Now consider acting with U on the superposition $|\psi\rangle + |\phi\rangle$ (suppressing normalization for clarity). By linearity of U ,

$$U((|\psi\rangle + |\phi\rangle) \otimes |0\rangle) = U(|\psi\rangle \otimes |0\rangle) + U(|\phi\rangle \otimes |0\rangle) \quad (1.10)$$

$$U((|\psi\rangle + |\phi\rangle) \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle + |\phi\rangle \otimes |\phi\rangle. \quad (1.11)$$

But the cloning hypothesis applied directly to the superposition state demands

$$U((|\psi\rangle + |\phi\rangle) \otimes |0\rangle) = (|\psi\rangle + |\phi\rangle) \otimes (|\psi\rangle + |\phi\rangle) \quad (1.12)$$

$$U((|\psi\rangle + |\phi\rangle) \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes |\phi\rangle + |\phi\rangle \otimes |\psi\rangle + |\phi\rangle \otimes |\phi\rangle. \quad (1.13)$$

These two expressions are equal only if $|\psi\rangle \otimes |\phi\rangle + |\phi\rangle \otimes |\psi\rangle = 0$, which holds only when $|\psi\rangle$ and $|\phi\rangle$ are orthogonal but not for arbitrary states. We have therefore reached a contradiction: no single unitary can clone quantum states. Unlike classical information where we take it completely for granted that we can clone the state of a register of bits, in the context of qubits, such an operation is not allowed by nature. We will return shortly to the notion of entanglement implicit in these expressions.

Quantum Two-Level Systems

The paradigmatic quantum mechanical system is a two-level system, often referred to as a spin 1/2 particle or a qubit. A generic qubit lives on the so-called “Bloch Sphere” and can be written as,

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad (1.14)$$

with normalization $|a|^2 + |b|^2 = 1$. Often, a qubit state is more conveniently parameterized in terms of polar and azimuthal angles θ and ϕ as,

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle. \quad (1.15)$$

The algebra of two-level systems is analyzed by a complete set of observables known as the Pauli matrices,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.16)$$

If we add in the identity, then the Paulis form a matrix basis that can describe any operator acting on the Hilbert space of an individual qubit system,

$$\mathcal{O} = \frac{1}{2} \sum_i \text{Tr}(\mathcal{O}\sigma_i)\sigma_i \quad (1.17)$$

where we regularly switch between notation $X \leftrightarrow \sigma_X$ and $\{\sigma_i\} = \{I, \sigma_X, \sigma_Y, \sigma_Z\}$. A generic qubit Hamiltonian can be written in the form,

$$H = E + B\mathbf{n} \cdot \vec{\sigma}, \quad (1.18)$$

with $\vec{\sigma} = (X, Y, Z)$, which yields eigenvalues $\lambda_{\pm} = E \pm B$ and gives dynamics defined by the unitary (setting $\hbar = 1$)

$$U(t) = e^{-iHt} = e^{-iEt} \left(\cos(Bt)I + \sin(Bt)\mathbf{n} \cdot \vec{\sigma} \right). \quad (1.19)$$

The Paulis are standard in description of qubit states and dynamics; later in this chapter, we will turn our attention to describing similar sets of operators for describing multilevel systems.

Density Matrices

While our previous description was of the unitary evolution of pure states, typically speaking, the ability to describe such a system requires knowledge of all relevant degrees of freedom within a closed system. In practice, experiments do not deal with pure states, but rather exist in the “open quantum system” setting. To describe our experiments, therefore, we need to use the ensemble operator known as the density matrix ρ . The density matrix is a nonnegative Hermitian operator with $\rho = \rho^\dagger \geq 0$ and normalization $\text{Tr}(\rho) = 1$. It is often expressed in terms of a convenient choice of basis states $\{|\psi_j\rangle\}$ as

$$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|. \quad (1.20)$$

The expectation values of operators are given as

$$\langle O \rangle = \text{Tr}(O\rho) \sim \sum_n \rho_{nn} \langle n| O |n \rangle. \quad (1.21)$$

We can define projectors onto pure state as $\rho_\chi = |\psi_\chi\rangle \langle \psi_\chi|$. Normally, in an experimental setting, we will not have knowledge of the degrees of freedom of the environment, and therefore will typically describe our system as

$$\rho_{sys} = \text{Tr}_{env}(|\Psi\rangle \langle \Psi|). \quad (1.22)$$

For the case of a simple qubit, the density matrix is given as

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \sum_{i,j=0}^1 \rho_{i,j} |i\rangle \langle j| = \frac{1}{2}I + \frac{1}{2} \sum_{\alpha=x,y,z} S_\alpha \sigma^\alpha, \quad (1.23)$$

where we can define a qubit state in terms of the vector $\vec{S} = (S_x, S_y, S_z)$. This geometric picture is commonly referred to as the “Bloch sphere,” where pure states having $\text{Tr}(\rho^2) = 1 \implies |\vec{S}| = 1$ have a vector projecting all the way onto the surface of the sphere, and in general mixed states have Bloch vectors of length less than 1.

Entangled States

At the heart of quantum computing is the concept of entanglement, where two quantum systems possess a special form of correlations such that they can no longer be described independently. A hallmark signature of entanglement is that tracing out one subsystem of a pure entangled state leaves the remaining subsystem in a mixed state. We illustrate this with the canonical example of a Bell state. Consider the maximally entangled two-qubit state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad (1.24)$$

whose density matrix is,

$$\rho_{AB} = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad (1.25)$$

written in the ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. This density matrix encodes all information about the joint system of qubits A and B , including their entanglement. To extract a description of qubit A alone (as seen by an observer with no access to qubit B) we perform the partial trace over B , which amounts to marginalizing over all possible states of the traced-out subsystem,

$$\rho_A = \text{Tr}_B(\rho_{AB}) = \sum_{j \in \{0,1\}} (I_A \otimes \langle j|_B) \rho_{AB} (I_A \otimes |j\rangle_B). \quad (1.26)$$

Evaluating this explicitly,

$$\begin{aligned} \rho_A &= \frac{1}{2} (\langle 0|_B (|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) |0\rangle_B \\ &\quad + \langle 1|_B (|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) |1\rangle_B) \\ &= \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2} I. \end{aligned} \quad (1.27)$$

The result is the maximally mixed state, i.e. a state with no coherences and equal probability of being found in $|0\rangle$ or $|1\rangle$. This is a direct consequence of the entanglement in $|\psi\rangle$: because the state of qubit A is entangled with qubit B , an observer who cannot access B cannot assign a definite pure state to qubit A , and must instead describe it as a classical statistical mixture. The degree of this mixedness is quantified by the entanglement entropy $S = -\text{Tr}[\rho_A \log \rho_A]$, which equals $\log 2$ for the maximally mixed state, reflecting one ebit of entanglement in $|\psi\rangle$. More generally, a bipartite pure state is entangled if and only if its reduced density matrix is mixed, making the partial trace a natural diagnostic of entanglement.

Master Equation and Lindbladians

In general, the time evolution of a density matrix in an experimental setting will need to be described by an equation that accounts for the decoherence due to coupling to unknown

degrees of freedom (i.e. the environment). This is typically described by the Lindblad master equation

$$\partial_t \rho = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}_t(\rho), \quad (1.28)$$

where $\mathcal{L}_t$ is a linear superoperator encoding the effect of the environment on the system. In general if the coupling is weak enough and the bath is large enough that it does not hold onto knowledge of the system's state, then we can make the Markovian approximation that $\mathcal{L}_t \rightarrow \mathcal{L}$. Typically, in this limit, one expresses the Lindbladian as

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_i \gamma_i (L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}) \quad (1.29)$$

where the γ_i describe the damping rates of the jump operators L_i . Often one shortens this expression by defining

$$\gamma_i D[L_i](\rho) = \gamma_i (L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}). \quad (1.30)$$

In the next chapter, we will revisit this master equation formulation to understand the decoherence dynamics of a multilevel artificial atom.

1.3 From Qubit to Qudit Quantum Logic

In the context of quantum algorithms and stabilizer codes, quantum logic is almost always formulated, at least in part, in terms of the qubit Pauli and Clifford groups, where the Pauli matrices were already introduced in our discussion of quantum two-level system dynamics. In this section, we begin by quickly reviewing the qubit Pauli and Clifford groups before discussing how they can be generalized to the setting of multilevel quantum systems, or qudits.

The Qubit Pauli and Clifford groups

We recall from our earlier description of two-level systems that their algebra is conveniently described by a complete (when combined with the identity) set of observables known as the Pauli matrices,

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.31)$$

The Pauli group is the group generated by these elements with prefactors $\pm 1, \pm i$

$$G = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \quad (1.32)$$

The qubit Clifford group $\mathcal{C}$ is the set of operators that normalize the Pauli group, i.e.

$$\forall C \in \mathcal{C} : CPC^\dagger \rightarrow P' \in G \quad (1.33)$$

The qubit Pauli group serves as a natural basis for analyzing qubit systems as the operators within the group have the following helpful properties:

- they form a projective group under matrix multiplication (i.e. the elements are defined up to an overall phase factor, which is physically irrelevant);
- they are unitary;
- they are a trace-orthogonal basis for the space of operators (i.e. $\text{Tr}(g_i^\dagger g_j) = C\delta_{ij}, \forall g_i, g_j \in G$ with C a constant); and
- they correspond to natural Hamiltonians.

As we will see the following section, there are two natural ways to port the machinery of the qubit Pauli group to a higher dimensional setting.

Qudit Groups and Bases

Unfortunately, in the context of qudit logic, no set of operators exist with all of the same properties as the qubit Pauli group. We thus need to separate some of the properties when analyzing qudits. We define two natural generalizations of the Pauli operators to higher dimensions, the *Weyl* and *Gell-Mann* operators that, taken together, satisfy all of the properties of the single-qubit Pauli group. In general, the Weyl operators allow one to more naturally generalize the machinery of qubit-based algorithms and benchmarking routines. In contrast, the Gell-Mann matrices correspond more immediately to the underlying physical operations performed on a qudit based quantum processor, such as Rabi oscillations and Z gates in a two-level subspace of the qudit. In what follows, we refer to D as the dimension of the qudit, and $d = D^n$ as the dimension of the Hilbert space for n qudits.

The Gell-Mann Basis

To construct the Gell-Mann operators, we can begin by embedding the single-qubit Pauli operators into two-dimensional subspaces of the higher-dimensional qudit space. Specifically, for a d -dimensional qudit, we can define these operators as

$$X^{jk} = |j\rangle\langle k| + |k\rangle\langle j| , \quad (1.34)$$

$$Y^{jk} = i|j\rangle\langle k| - i|k\rangle\langle j| , \quad (1.35)$$

$$Z^{jk} = |j\rangle\langle j| - |k\rangle\langle k| , \quad (1.36)$$

where $0 \leq j \leq k \leq d-1$. We note that while the Z^{jk} are Hermitian operators, they are not linearly independent and thus cannot serve as a sufficient basis for qudit tomography. We can therefore extend the set $\{X^{jk}, Y^{jk} : 0 \leq j < k < d\}$ to a trace-orthogonal basis with the inclusion of additional diagonal operators:

$$W^j = -j|j\rangle\langle j| + \sum_{0 \leq k < j} |k\rangle\langle k| \quad (1.37)$$

for $1 \leq j < d$. The combined set $\mathbb{G}_d = \{I, X^{jk}, Y^{jk} : 0 \leq j < k < d\} \cup \{W^j : 1 \leq j < d\}$ forms the *Gell-Mann basis* for qudit dimension d . Like the qubit Pauli matrices, every element of the Gell-Mann group is both traceless and Hermitian, and thus serves as a suitable choice for constructing qudit transfer matrices.

The Weyl Group

Having constructed the qudit Gell-Mann basis, we now turn our attention to generalizing the qubit Pauli operators over the entire qudit space rather than embedding them in two-level subspaces of the qudit. This is known as the *Weyl group*, from which we can generalize qubit-based quantum algorithms and codes to qudits, as well as generalize the Clifford group to higher dimensions. We begin by defining $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, the additive group of the integers modulo d . Using $\mathbb{Z}_d$, we can generalize the qubit X and Z operators to the qudit space as

$$X = \sum_{j \in \mathbb{Z}_d} |j \oplus_d 1\rangle\langle j|, \quad (1.38)$$

$$Z = \sum_{j \in \mathbb{Z}_d} \exp\left(\frac{2\pi i}{d} j\right) |j\rangle\langle j|, \quad (1.39)$$

where $\oplus_d$ denotes addition modulo d . We note here that both X and Z compose to the identity under d rounds of self-multiplication, which is the natural generalization of the qubit X and Z operators squaring to the identity. The Weyl basis follows as a trace-orthogonal basis over $\mathbb{C}^{d \times d}$ defined as $\mathbb{W}_d = \{W_{xz} = X^x Z^z : x, z \in \mathbb{Z}_d\}$. Often in the literature, the qudit X operator is referred to as the “shift” operator as it increments the qudit state modulo d , and the qudit Z operator as the “clock” operator as it applies phases corresponding to multiples of the d -th root of unity. Finally, we can define the n -qudit Weyl group by taking the n -fold tensor product of all single-qudit Weyl matrices: $\mathbb{W}_{D,n} = \mathbb{W}_d^{\otimes n}$. It is worth noting that one can define multi-qudit gates directly from the definitions of the Weyl matrices, similar to how the Hamiltonians for two-qubit gates are often defined in terms of n -qubit Paulis.

Finally, we note that the Weyl operators form a unitary-1 design, i.e., given $\mathbb{W}_d$, for any operator (or density matrix) p ,

$$\frac{1}{|\mathbb{W}_d|} \sum_{U \in \mathbb{W}_d} U p U^\dagger = \int_{U(d)} U p U^\dagger d\mu_{\text{haar}}(U). \quad (1.40)$$

Or, stated more plainly, when acting via conjugation over the whole Weyl group, one recovers an approximately depolarizing channel. This fact will have meaningful consequences in later chapters where we use this group both for tailoring (or twirling) noise, and for benchmarking qudit quantum gates.

The n -Qudit Clifford Group

One important caveat about the Weyl basis is that, owing to the fact that it is not closed under multiplication, it is not a proper group. However, every element of the closure of the Weyl basis is related to an element of the Weyl basis up to an overall phase. We can therefore get rid of this overall phase by considering solely the adjoint action of the Weyl operators, and therefore defining a proper group we refer to as the “extended Weyl group,” $\mathbb{E}W_d = U(1)\mathbb{W}_d$ and the “extended n -qudit Weyl group” as $\mathbb{E}W_{d,n} = U(1)\mathbb{W}_{d,n}$, where $U(1)$ is the 1-dimension unitary group, allowing for arbitrary phases. Finally, with all this formalism defined, we can naturally define the n -qudit Clifford group to be the set of operators that normalize the extended Weyl group, or stated explicitly, the set $\mathbb{C}_{d,n}$ where:

$$\mathbb{C}_{d,n} = \{U \in U(d^n) : U\mathbb{E}W_{d,n}U^\dagger = \mathbb{E}W_{d,n}\} . \quad (1.41)$$

1.4 A DiVincenzo Criteria For Qudits

We present a systematic definition of qudits in the context of quantum computing by extending DiVincenzo’s criteria [54], originally developed to identify viable qubit platforms for computing, to d -level systems [153].

1. **Scalable, well-characterized d -level systems:** The d basis states must be well-defined, individually addressable, and sufficiently isolated. The information processing device should be scalable, accommodating a large number of such systems to perform tasks of practical utility.
2. **Initialisation:** Qudits are first prepared in a reference state that serves as a natural starting point of a computation.
3. **Coherence:** Once initialised, the decoherence time for qudits across all d levels should significantly exceed typical gate time to allow for performing a computational task involving execution of multiple gates.
4. **Universality:** A universal qudit quantum computer should be able to efficiently approximate any unitary operator acting on the full Hilbert space to arbitrary precision. Operationally, this universality is typically decomposed into two ingredients: arbitrary single-qudit gates (4a) and at least one class of entangling two-qudit gates (4b) (see Box II).
5. **Measurement:** A qudit processor requires a measurement apparatus that can distinguish all d basis states.

As we will show in this thesis, superconducting qudits meet all the criteria of this extended DiVincenzo criteria.

1.5 More Levels More Fun: Why Qudits?

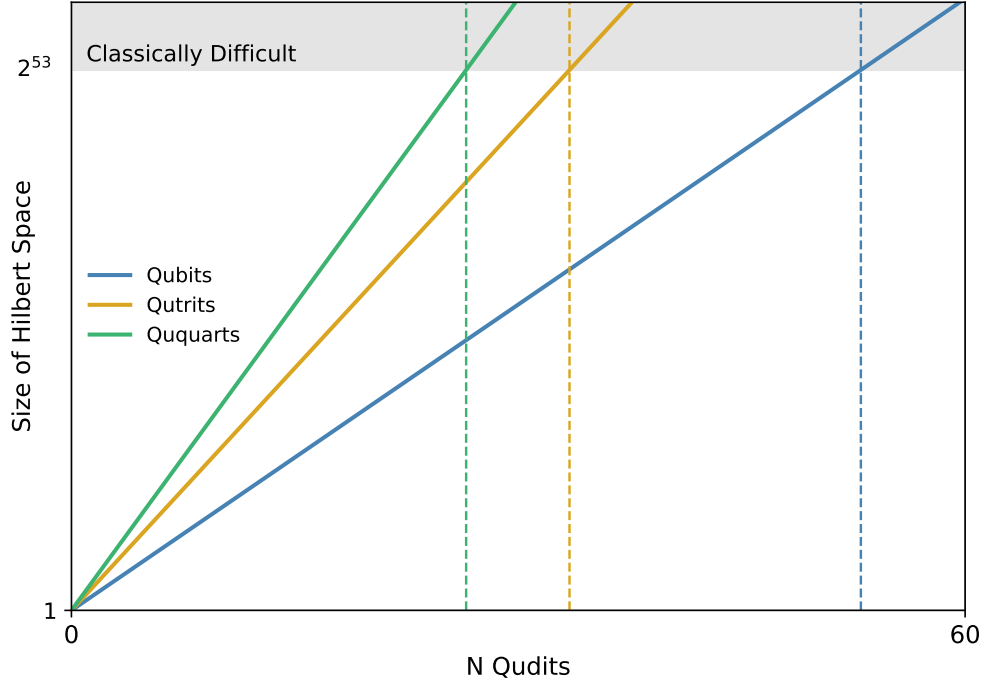


Figure 1.1: **Qudit Hilbert Space scaling.** A general N qudit state $|\Psi\rangle$ lives in a Hilbert space of size $|\Psi\rangle = d^N$. Here we plot the scaling of this Hilbert space for a system of N qudits where the shaded “classically difficult” region (based off the 2019 demonstration of quantum advantage with 53 qubits in Ref. [10]) is included as an approximate heuristic.

As illustrated in Fig. 1.1, a general N qudit state $|\Psi\rangle$ lives in a Hilbert space of size $|\Psi\rangle = d^N$, meaning that for a fixed number N of physical quantum units, higher dimensional qudit encodings provide exponentially more computational resources. A system of N ququarts, for instance, spans a Hilbert space equivalent to that of $2N$ qubits, suggesting that qudits may offer a more resource-efficient path to the large Hilbert spaces demanded by fault tolerant quantum computation. Early demonstrations of quantum operations that strained the computational capability of classical supercomputers to recreate were performed with 53 qubits [10], therefore in Fig. 1.1, we denote this region as “classically difficult,” although we note here that this is only an approximate heuristic, and as classical and quantum methods continue to evolve, we expect this regime of difficulty to remain in flux. Beyond this raw scaling advantage, qudits offer a number of specific advantages in the key areas of computing, error correction, and quantum simulation.

In the context of computing, we note that access to qudit operations has the potential to benefit the execution of algorithms written in terms of qubit gates. In particular, many

algorithms, such as Grover’s or Shor’s, that employ subroutines based on classical Boolean logic or arithmetic employ N -body Toffoli gates. Since quantum hardware often does not have access to native N -body entangling operations, it is standard for N -body qubit gates to be decomposed into the native two-qubit gates present on the hardware, often incurring a large overhead in gate operations. Notably, accessing qudit-based entangling operations and employing auxiliary levels within the qudit state space as built-in ancillae can significantly reduce the number of two-body entangling operations required to generate an N -body qubit gate, with asymptotic advantages demonstrated in the large N limit [208, 142, 82, 186, 187]. On the measurement side, the use of auxiliary levels can also aid in characterizing and suppressing qubit readout errors [57, 41, 42], and implementing generalized qubit measurements described by informationally complete positive operator-valued measures [71, 240].

In the realm of quantum simulation, qudits offer novel approaches that more naturally mirror the structure of many of the relevant physical systems one might want to simulate. For example, in this thesis we will explore quantum simulations that naturally map to qudit hardware such as higher spin systems, collective neutrino oscillations, and lattice gauge theories. Access to native qudit hardware enables explorations of these physical systems with significantly reduced quantum gate counts [94, 167].

Finally, truly scalable and arbitrarily reliable qudit computation is expected to rely on logical encoding through quantum error correction [86]. By embedding logical qudits into redundantly encoded states of many physical qudits, stabilizer-based codes provide a pathway toward fault-tolerant computation, often realizing memories with higher code thresholds than their analogous qubit counterparts [266, 8, 127, 237]. In the context of fault-tolerant computing, universal qudit operation can be achieved with established techniques such as lattice surgery [49] and through highly efficient magic state distillation routines [225, 204, 32, 31, 226]. A physical qudit Hilbert space can also be used to encode a logical qubit or qudit, which can itself be employed within a larger error-correcting code. As an example, tailoring amplitude-damping events to monitored non-computational levels allows photon loss to be converted to erasure errors on logical qubits, which can be corrected more efficiently than Pauli errors [273, 139]. Such qudit-based erasure qubits have been experimentally demonstrated in trapped ions [227, 207], both transmon [206, 148] and fluxonium [149] superconducting circuits, and neutral atoms [155, 220]. Whereas qubit error correction has seen increasingly mature experimental demonstrations, the development of comparable frameworks for qudits remains at an early stage. Nonetheless, a beyond-break-even quantum memory in $d = 3, 4$ has been demonstrated using a qudit Gottesman-Kitaev-Preskill (GKP) encoding in the bosonic modes of a superconducting cavity [22].

Within the superconducting circuit platform, qudits arise naturally; the weakly anharmonic energy level structure of the transmon means that higher levels are always present and in principle accessible, yet they have historically been regarded as “non-computational” states to be avoided rather than a resource to be exploited. This thesis argues for a different perspective, and investigates superconducting qudit operation and computation, addressing in this context the same central questions that animate the broader field: how to encode, manipulate, scale, and protect quantum information as efficiently as possible. Nonetheless

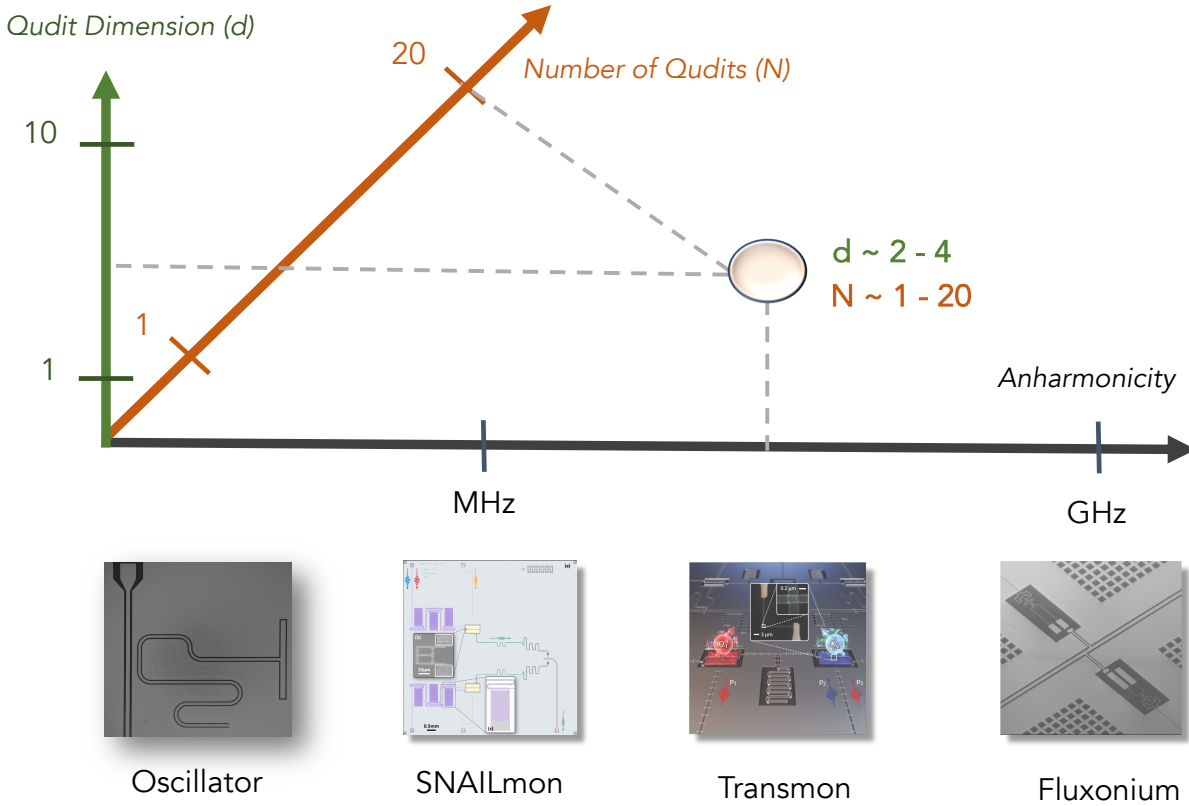


Figure 1.2: **Thesis Roadmap.** In this thesis, we focus on operating the transmon as a multilevel quantum system. The intermediate level of anharmonicity for typical transmon circuits positions it as a natural qudit, with easily addressable and coherent higher energy levels, and the ability to use the Fock states as computational state. In this thesis, we focus on encoding qubits, qutrits, and ququarts in transmon circuits, and ultimately scale up our experiments to the scale of 20 qudits.

incorporation of higher levels for qudit operation does not come as a “free lunch,” and extensive discussion of the tradeoffs and challenges of operating these multilevel systems is discussed throughout the text.

1.6 Thesis Structure and Results Summary

This thesis presents an exploration of quantum computing with superconducting transmon qudits, proceeding from single-qudit control and readout through entangling operations and error suppression, and culminating in quantum simulations of physical systems that are naturally suited to multilevel quantum hardware. As shown in Fig. 1.2, we employ the

intermediately anharmonic circuit of the transmon as our physical qudit, focusing on scalable techniques that scale the local qudit dimension from $d = 2$ to $d = 3$ and $d = 4$, which enable us to move from single and two-qudit demonstrations to eventually performing qudit operation on a 20 transmon device. We note that while we focus in this text on scalable techniques for qudits of relatively low local dimension d , there has been recent complementary work in single qudit experiments to scale up the local dimension in superconducting devices to $d = 12$ [38, 264, 215]. We briefly summarize the structure and key results of each chapter below.

In chapter 2 we provide a background treatment of superconducting circuits, circuit QED, and the transmon and discuss some specific aspects of these topics to multilevel operation. In chapter 3 we explore single qudit operation including gates, readout, and specifically touch on the calibration and benchmarking of these operations. In chapters 4 and 5 we explore entangling operations between qudits. Specifically, in chapter 4 we present a first implementation of a high-fidelity qutrit CZ gate based on a microwave driven tunable cross-Kerr interaction that natively entangles to the full two-qutrit Hilbert space. In chapter 5 we investigate gates that entangle a subspace of the two-qudit space via a two photon multipartite Raman process. After exploring single and two-qudit operation, in chapter 6 we investigate methods to suppress errors in qudit circuits, including dynamical decoupling and quantum error mitigation. We then synthesize these directions in chapters 7 and 8 by exploring simulations of quantum systems that have innate multilevel interactions and are therefore well tailored to qudit operation. First in chapter 7 we investigate digital quantum simulations of collective neutrino oscillations and the spin-1 truncated Abelian Higgs model, finding error mitigation methods critical for improving circuit performance. Finally, in chapter 8 out of equilibrium — or Floquet — spin-1 systems are realized using an array of 15 superconducting qutrits to probe a novel $\mathbb{Z}_3$ discrete time crystalline (DTC) phase of matter with intrinsically chiral interactions. Limitations of current approaches and future directions are discussed throughout the text.

Chapter 2

Superconducting Circuits

2.1 Suggested Reading

In Chapter 1, we provided a brief historical overview of the field of superconducting circuits. Now, in this chapter we will provide an abridged technical treatment of superconducting circuits, starting from circuit quantization of the LC oscillator and nonlinear transmon circuit, before moving on to describe some specific aspects of operation of the transmon as a multi-level artificial atom. This chapter will only scratch the surface of what has become a very rich landscape of circuit quantization and quantum mechanical circuit analysis. We therefore suggest the reader interested in a more complete technical treatment of these matters check out the following resources:

- **Circuit Quantum Electrodynamics** *Blais, Grimsmo, Girvin, and Devoret* [17]: This is a modern review of the field of circuit QED providing an accessible and thorough treatment of a large set of key concepts, where the authors themselves are responsible for many of the foundational contributions to the field.
- **Lecture Notes on Quantum Electrical Circuits** *Ciani, DiVincenzo, and Terhal* [46]: This textbook treats a range of introductory to advanced topics including circuit quantization, black-box quantization, non-reciprocal circuits, and noise. Additionally, there are completed exercises within the text that are particularly instructive to readers.
- **Quantum Information and Quantum Optics with Superconducting Circuits** *Ripoll* [212]: This book provides a comprehensive introduction to the theory of superconducting circuits and their applications in quantum information processing and quantum optics. It covers topics such as circuit quantization, open quantum systems, qubit control, and quantum error correction, with a focus on bridging the gap between theoretical foundations and experimental implementations in superconducting quantum hardware, with example problems provided in each chapter.

2.2 Quantizing Circuits

To describe the quantum behavior of superconducting circuits, we employ the framework of circuit quantization. This process involves identifying the classical degrees of freedom of a circuit and promoting them to quantum operators that obey specific commutation relations. This approach allows us to treat collective electronic excitations as quasiparticles, such as photons in a resonator or excitations in a qubit.

The Classical LC Oscillator

Consider a simple linear oscillator consisting of an inductor L and a capacitor C in parallel. The state of the system is characterized by the node flux $\Phi(t)$, defined as the time integral of the voltage $V(t)$ across the capacitor:

$$\Phi(t) = \int_{-\infty}^t V(t') dt'. \quad (2.1)$$

The capacitor stores energy in the form of an electric field, and the inductor stores energy in the form of a magnetic field. The total energy of the system, which corresponds to the classical Hamiltonian $\mathcal{H}$, is the sum of the capacitive and inductive energies:

$$\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \quad (2.2)$$

where Q is the charge on the capacitor. In this formulation, Φ and Q are conjugate variables. This can be seen by looking at the Lagrangian of the system, $\mathcal{L} = T - V$, where the "kinetic" energy is provided by the capacitor and the "potential" energy by the inductor:

$$\mathcal{L} = \frac{C\dot{\Phi}^2}{2} - \frac{\Phi^2}{2L}. \quad (2.3)$$

From the Lagrangian, we can derive the conjugate momentum to the flux Φ :

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}, \quad (2.4)$$

and from this point we can recover the classical Hamiltonian in Eq. 2.2 using the Legendre transformation $\mathcal{H} = \dot{\Phi}Q - \mathcal{L}$.

Quantization of the Circuit

To quantize the LC circuit, we promote the classical variables Φ and Q to operators $\hat{\Phi}$ and $\hat{Q}$ acting on a Hilbert space. We impose the canonical commutation relation:

$$[\hat{\Phi}, \hat{Q}] = i\hbar. \quad (2.5)$$

The Hamiltonian operator for the quantum LC oscillator becomes:

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}. \quad (2.6)$$

This expression is mathematically identical to the Hamiltonian of a mechanical quantum harmonic oscillator (QHO), where $\hat{Q}$ plays the role of momentum p and $\hat{\Phi}$ plays the role of position x . To solve for the energy eigenvalues, we introduce the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators:

$$\hat{\Phi} = \sqrt{\frac{\hbar Z}{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{Q} = \frac{1}{i\sqrt{\hbar Z/2}}(\hat{a} - \hat{a}^\dagger) \quad (2.7)$$

where $Z = \sqrt{L/C}$ is the characteristic impedance of the circuit. Substituting these into the Hamiltonian, we obtain the familiar form:

$$\hat{H} = \hbar\omega_{LC} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (2.8)$$

where $\omega_{LC} = 1/\sqrt{LC}$ is the resonance frequency of the circuit. The LC oscillator is a linear system with equally spaced energy levels, $E_n = \hbar\omega_{LC}(n + 1/2)$. It is natural to ask, can this system be used as a qubit? Because the energy gap $\Delta E = \hbar\omega_{LC}$ is the same for all n , a linear LC oscillator cannot function as a qubit (at least not without adding in ancillary nonlinear modes such as a coupled qubit), as any drive at ω_{LC} would excite the system to all higher levels (i.e. driving displacements on coherent states).

Another natural question to ask would be: when would this circuit behave quantum mechanically? Generally speaking, for an LC circuit, to behave in a quantum mechanical manner, we must realize two things: first, the circuit must be low enough loss such that the spectral linewidth of the energy states is less than their separation (this motivates us to create circuits out of superconductors), and second the ambient temperature of our system must satisfy $k_B T \ll \hbar\omega$ (this motivates us to put our circuits into very cold cryogenic systems).

To create a circuit that behaves as a qubit (and later a qudit), however, one must go a step further and introduce a *nonlinearity* into the system to create anharmonic energy spacing between adjacent levels. In superconducting circuits, this is typically achieved by replacing the linear inductor L with a non-linear inductor, such as a Josephson junction (JJ). The JJ provides a cosine-shaped potential, which makes the energy levels non-equidistant, allowing us to isolate a collection of energy levels, typically the lowest ones in the potential well, as a computational subspace.

Nonlinear Quantum Circuits

To transform the linear LC oscillator into a qubit, we must introduce an anharmonic element that breaks the degeneracy of the energy level spacing. In superconducting circuits, this is achieved using a JJ.

The Josephson Junction

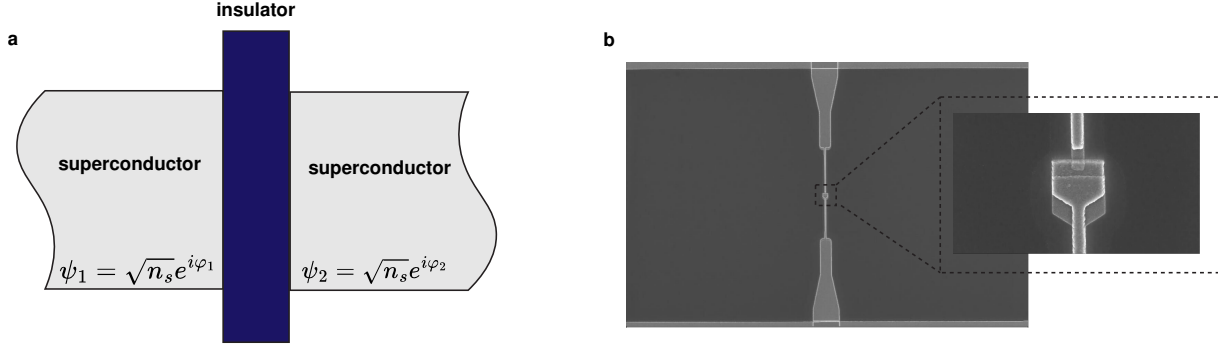


Figure 2.1: **The Josephson Junction** **a**, A schematic of a JJ. Two superconducting islands with distinct superconducting wave functions (and phases φ_i) are separated by a weak insulating barrier. **b** A scanning electron microscopy image of a typical Al/AlOx/Al JJ in the “Dolan bridge” style (image courtesy of Kan-Heng Lee).

A Josephson junction consists of two superconducting electrodes separated by a thin insulating barrier (typically Aluminum Oxide). In Fig. 2.1, we present a schematic and a scanning electron microscopy image of a typical JJ used in superconducting qubit experiments. The physics of the junction is described by phase difference $\varphi = \varphi_1 - \varphi_2$ across the barrier, where φ is the phase of the superconducting wavefunction (or order parameter) in the Ginzburg-Landau theory. The dynamics are governed by the two Josephson relations:

$$I = I_c \sin \varphi, \quad V = \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \quad (2.9)$$

where I_c is the critical current of the junction and $\Phi_0 = h/2e$ is the magnetic flux quantum. The energy stored in a Josephson junction is given by:

$$U(\varphi) = -E_J \cos \varphi, \quad (2.10)$$

where $E_J = \frac{I_c \Phi_0}{2\pi}$ is the Josephson energy. By expanding the cosine term for small φ , we see that $U(\varphi) \approx -E_J + \frac{1}{2} E_J \varphi^2$. This allows us to treat the JJ as a non-linear inductor with a phase-dependent inductance:

$$L_J(\varphi) = \frac{\Phi_0}{2\pi I_c \cos \varphi}. \quad (2.11)$$

Because the inductance changes with the current flowing through it, the resulting potential is not parabolic, leading to the anharmonicity required to isolate the $|0\rangle$ and $|1\rangle$ states, and therefore yielding the critical nonlinearity to make qubits out of superconducting circuits.

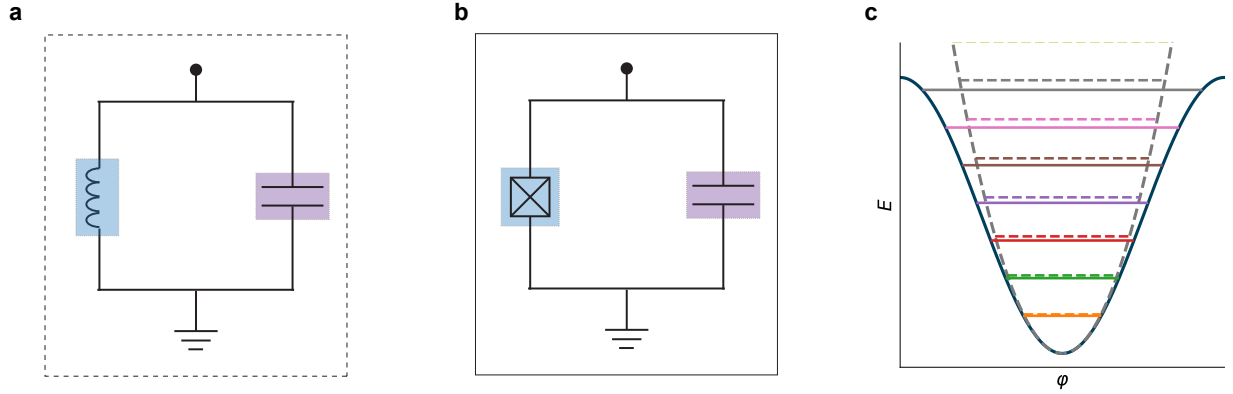


Figure 2.2: **Comparing the Transmon to a Linear Oscillator** **a**, The lumped element circuit model of an LC oscillator. The capacitive “kinetic” energy component is shown in purple and the inductive “potential” energy component is shown in blue. **b**, The lumped element model of the transmon qubit where the linear inductor is replaced with a JJ. **c**, Comparing the transmon and LC resonator potentials and associated energy levels where $\omega_{LC} = \omega_q$ for a typical transmon with $E_J/E_C = 100$.

The Transmon Qubit

The simplest and most common form of a superconducting qubit is realized by replacing the linear inductor in an LC circuit with the nonlinear JJ element (see Fig. 2.2). More specifically, the transmon (transmission-line shunted plasma oscillation) qubit is designed to operate in a regime where it is insensitive to charge noise [136]. This is achieved by shunting the Josephson junction with a large external capacitor. The Hamiltonian of the Transmon is:

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos \hat{\varphi}, \quad (2.12)$$

where $E_C = e^2/2C_\Sigma$ is the charging energy (with C_Σ being the total capacitance) and $\hat{n} = \hat{Q}/2e$ is the number of Cooper pairs transferred across the junction. The term n_g represents the effective offset charge induced by the environment. Often we will drop the hats on operators, but we leave them in this equation to distinguish the quantum operators from n_g which is a c-number.

In early “charge qubits,” $E_C \gtrsim E_J$, making the energy levels highly sensitive to fluctuations in n_g (charge noise). The Transmon operates in the regime $E_J/E_C \gg 1$. In this limit, the energy levels become nearly independent of n_g , exponentially suppressing charge noise. While this reduction in noise comes at the cost of slightly reduced anharmonicity, the transmon remains extremely popular due to its long coherence times, ease of fabrication, and proven track record for performing high-fidelity gates and scaling up.

Duffing Oscillator Approximation

Often throughout this thesis, we will want to treat the transmon analytically within a Hamiltonian or Master Equation. To do so it is quite useful to describe the system in terms of second quantization. To achieve this, we expand the cos term in the Hamiltonian and approximate the transmon as a Duffing (or Kerr) Oscillator.

$$\cos \varphi \approx 1 - \varphi^2/2 + \varphi^4/24. \quad (2.13)$$

Upon doing so, our Hamiltonian now becomes,

$$H = 4E_C n^2 + \frac{E_J}{2} \varphi^2 - \frac{E_J}{24} \varphi^4. \quad (2.14)$$

First we can identify the harmonic part of the potential will give us QHO oscillator solutions with $\omega_p = \sqrt{8E_J E_C}$. To solve for the φ^4 contribution, we need to write n, φ as ladder operators (expressing them in terms of their zero point fluctuations)

$$\varphi = \varphi_{zpf}(a^\dagger + a) \quad (2.15)$$

$$n = i n_{zpf}(a^\dagger - a) \quad (2.16)$$

We know that $[\varphi, n] = i \rightarrow \varphi_{zpf} n_{zpf} = 1/2$, therefore

$$4E_C n_{zpf}^2 = E_J \varphi_{zpf}^2 \quad (2.17)$$

$$\varphi_{zpf} = \left(\frac{2E_C}{E_J}\right)^{1/4}, \quad n_{zpf} = \frac{1}{2} \left(\frac{E_J}{2E_C}\right)^{1/4}. \quad (2.18)$$

We can now solve for the φ^4 contribution

$$\frac{E_J}{24} \varphi^4 = \frac{E_J}{24} \varphi_{zpf}^4 (a^\dagger + a)^4 = \frac{E_C}{12} (a^\dagger + a)^4. \quad (2.19)$$

We will use the RWA to only hold onto number conserving contributions to this term, which will yield $(a^\dagger + a)^4 \approx 6a^\dagger a^\dagger a a + 12a^\dagger a$.

$$= \frac{E_C}{2} a^\dagger a^\dagger a a + E_C a^\dagger a. \quad (2.20)$$

Therefore our total Hamiltonian becomes:

$$H = (\sqrt{8E_J E_C} - E_C) a^\dagger a - \frac{E_C}{2} a^\dagger a^\dagger a a. \quad (2.21)$$

We can define the qubit frequency, as $\omega_q = \sqrt{8E_J E_C} - E_C$ and the anharmonicity $\eta = -E_C$, which allows us to arrive at the common form for the approximate transmon Hamiltonian written in terms of bosonic operators,

$$H = \omega_q a^\dagger a + \frac{\eta}{2} a^\dagger a^\dagger a a. \quad (2.22)$$

Flux Tunability via SQUIDS

In the basic Transmon Hamiltonian described previously, the qubit frequency ω_q is determined by the parameters E_J, E_C . Since E_J depends on the critical current of the Josephson junction, which is fixed during the fabrication process, the qubit frequency is static. However, many quantum computing architectures benefit from the ability to tune the qubit frequency to bring it into resonance with other qubits for two-qubit gates or to shift it away from noise sources. This is achieved by replacing the single Josephson junction with a DC-SQUID (Superconducting Quantum Interference Device).

The DC-SQUID

A DC-SQUID consists of two Josephson junctions connected in parallel within a superconducting loop. Let the two junctions have Josephson energies E_{J1} and E_{J2} , and let ϕ_1 and ϕ_2 be the phase differences across each junction. The total current flowing through the SQUID is the sum of the currents through the two branches:

$$I = I_{c1} \sin \phi_1 + I_{c2} \sin \phi_2. \quad (2.23)$$

The fluxoid quantization condition for the superconducting loop requires that the phase differences are constrained by the external magnetic flux Φ_{ext} threading the loop:

$$\phi_1 - \phi_2 = 2\pi \frac{\Phi_{ext}}{\Phi_0} \pmod{2\pi}, \quad (2.24)$$

where $\Phi_0 = h/2e$ is the magnetic flux quantum. By substituting this constraint into the total energy of the SQUID, the device can be modeled as a single “effective” junction with a tunable Josephson energy $E_{J,\text{eff}}$.

The Effect of Asymmetry

In an ideal symmetric SQUID, where $E_{J1} = E_{J2} = E_J$, the effective Josephson energy is given by:

$$E_{J,\text{eff}}(\Phi_{ext}) = 2E_J \left| \cos \left(\pi \frac{\Phi_{ext}}{\Phi_0} \right) \right|. \quad (2.25)$$

In this symmetric case, $E_{J,\text{eff}}$ can be tuned all the way to zero at $\Phi_{ext} = \Phi_0/2$. However, it is often beneficial to design asymmetric junctions to have less sharp flux tuning curves, and therefore less sensitivity to flux noise. We define an asymmetry parameter d as:

$$d = \frac{E_{J1} - E_{J2}}{E_{J1} + E_{J2}}. \quad (2.26)$$

For an asymmetric SQUID, the effective Josephson energy becomes:

$$E_{J,\text{eff}}(\Phi_{ext}) = (E_{J1} + E_{J2}) \sqrt{1 + d^2 \tan^2 \left(\pi \frac{\Phi_{ext}}{\Phi_0} \right)} \cos \left(\pi \frac{\Phi_{ext}}{\Phi_0} \right). \quad (2.27)$$

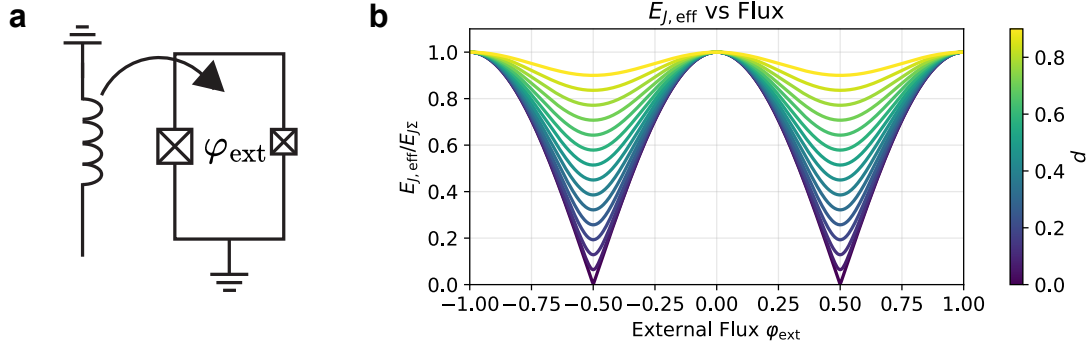


Figure 2.3: **a**, Lumped element circuit diagram of a SQUID with flux threaded through the SQUID loop. **b**, the $E_{J,\text{eff}}$ of the SQUID as a function of external flux φ_{ext} for different junction asymmetries d .

As a result of this asymmetry, $E_{J,\text{eff}}$ no longer reaches zero, but given a choice of architecture/gate mechanism, there is considerable freedom afforded to a choice of circuit parameters. Typically speaking, these DC-SQUIDs can be integrated into qubit circuits (both for use as data qubits) and as tunable coupler elements and have become a mainstay in modern quantum processing unit architectures.

Incorporation into Qubits

By replacing the single junction in a Transmon with a SQUID, the Hamiltonian becomes:

$$\hat{H} = 4E_C n^2 - E_{J,\text{eff}}(\Phi_{\text{ext}}) \cos \varphi. \quad (2.28)$$

Since the qubit frequency $\omega_q \approx \sqrt{8E_C E_{J,\text{eff}}}$, the frequency can now be modulated by varying the external flux Φ_{ext} via a nearby flux bias line.

This capability is central to the implementation of high-fidelity gates. For example, in a frequency-tunable Transmon, two qubits can be idling at different frequencies to avoid crosstalk and then brought into resonance (or a specific detuning) to execute an i SWAP or CZ gate. However, this tunability introduces a trade-off: the qubit becomes sensitive to flux noise, leading to pure dephasing (Γ_ϕ). To mitigate this, qubits are often operated at the “flux sweet spot” ($\Phi_{\text{ext}} = 0$), where $\partial\omega_q/\partial\Phi_{\text{ext}} = 0$, making the qubit first-order insensitive to flux fluctuations. Another knob we can use to engineer less flux sensitivity is to create a large SQUID junction asymmetry d , as can be seen in Fig. 2.3, which fundamentally trades off less frequency tunable for more robustness against flux noise due to smaller $\partial\omega_q/\partial\Phi_{\text{ext}}$ along the qubit’s flux tuning curve.

2.3 Transmon Qudits

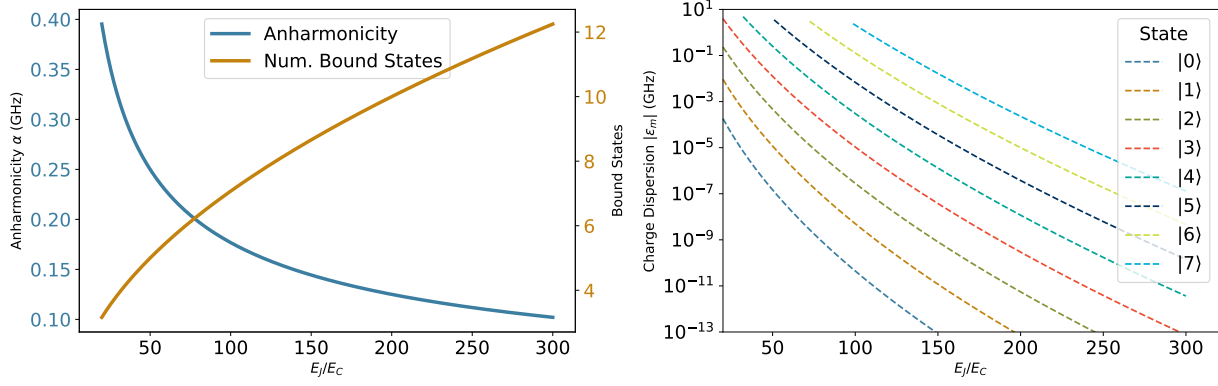


Figure 2.4: **The Transmon as a Multilevel Artificial Atom** **a**, The number of bound states and anharmonicity of a transmon with $\omega_q = 2\pi \times 5$ GHz as a function of E_J/E_C . **b**, the charge dispersion of the m -th level of that same transmon, now only plotting ϵ_m once that level becomes a bound state of the cosine potential.

We have now described the transmon as a simple nonlinear quantum circuit in which one can spectrally isolate the $|0\rangle$ and $|1\rangle$ states to create a qubit. As can be seen from our Duffing oscillator approximation of the transmon in Eq. 2.22, the anharmonic nature of the transmon means that we approximately have $E_{j+1} - E_j = \omega_q - j\eta$, meaning we can spectrally isolate and therefore coherently control more than just the $|0\rangle$ and $|1\rangle$ state. In practice, there are a few major constraints to keep in mind when designing transmon circuit parameters for multilevel, or qudit, based operation. In general, the figure of merit for these designs is the ratio of E_J/E_C , so for the purposes of our conversation, we will imagine that the qubit frequency remains fixed, and we vary this ratio.

The first of these considerations is the actual number of bound states available within the transmon well. We can approximate the number of transmon states by noting that the height of our JJ cosine well is given by $2E_J$ and approximating the energy of the n th transmon energy level as $n\omega_q = n\sqrt{8E_JE_C}$, from which one obtains the expression for the approximate number of bound states [264, 228],

$$N_{\text{levels}} \simeq \frac{2E_J}{\sqrt{8E_JE_C}} = \sqrt{\frac{2E_J}{E_C}}. \quad (2.29)$$

We face an immediate tradeoff, where if we desire to keep the qubit frequency fixed, but add more bound states, then our anharmonicity will scale as

$$\eta = \frac{\omega_q}{\sqrt{8\frac{E_J}{E_C}}}, \quad (2.30)$$

which means as we add more bound states to the well for a given ω_q , we will tradeoff less spacing between our energy levels, and therefore slower gate times.

However, having a bound state alone does not mean it will be a level with sufficient quantum coherence to be used for computation. In the previous section, we discussed how adding a shunting capacitance into the transmon circuit can make the qubits approximately insensitive to charge noise and allow us to trace out the role of n_g in the system Hamiltonian. Here, we make this precise by noting that the charge dispersion of the m -th transmon energy level is given as [136],

$$\epsilon_m \simeq (-1)^m E_C \frac{2^{4m+5}}{m!} \sqrt{\frac{2}{\pi}} \left(\frac{E_J}{2E_C} \right)^{m/2+3/4} e^{-\sqrt{8E_J/E_C}}. \quad (2.31)$$

Immediately it is worth noting that the m -th transmon energy level obtains exponentially more charge dispersion as one climbs the ladder of transmon states, but simultaneously the charge dispersion for this level is exponentially suppressed in the ratio of E_J/E_C . In Fig. 2.4b, we plot the charge dispersion of the m -th level of the transmon as a function of the ratio E_J/E_C for a fixed qubit frequency of $\omega_q = 2\pi \times 5$ GHz, only plotting the values once the transmon well becomes deep enough such that $|m\rangle$ becomes a bound state.

Finally, we note for readers that a careful modeling of highly excited states of the transmon requires careful attention to the higher Josephson harmonics beyond the standard current phase relations that may be present in a transmon's tunnel junction [271]. This in principle requires adjusting the transmon's Hamiltonian to be

$$H = 4E_C(n - n_g)^2 + \sum_{m \geq 1} E_{jm} \cos(m\varphi). \quad (2.32)$$

These higher harmonics can be critical in modeling of gates and measurement induced state transitions.

A Noise Model of a Superconducting Qudit

For a general qudit, we can consider a master equation in the presence of decoherence that considers ladder like decay events (only from $|j+1\rangle \rightarrow |j\rangle$ and ignoring heating) and dephasing between arbitrary levels:

$$\dot{\rho} = -i[H, \rho] + \sum_{j \in \mathbb{Z}_{d-1}} \Gamma_1^{j,j+1} \mathcal{D}[|j\rangle \langle j+1|] \rho + 2 \sum_{i < j} \Gamma_2^{i,j} \mathcal{D}[|i\rangle \langle i| - |j\rangle \langle j|] \rho, \quad (2.33)$$

where Γ_1 represents the $1/T_1$ decay rate and Γ_2 is related to Γ_ϕ the dephasing time but affected by both the intrinsic dephasing between the relevant levels and T_1 decay. Following, Ref. [176], if we restrict ourselves to just the qutrit Hilbert space and consider a system in

free evolution, we can solve the time dynamics due to our decay and find $\rho(t)$ given by,

$$\rho(t) = \begin{pmatrix} 1 - \rho_{11}e^{-\Gamma_1^{01}t} - \frac{\rho_{22}}{\Gamma_1^{01} - \Gamma_1^{12}}(\Gamma_1^{01}e^{-\Gamma_1^{12}t} - \Gamma_1^{12}e^{-\Gamma_1^{01}t}) & \rho_{01}e^{-\Gamma_2^{01}t} & \rho_{02}e^{-\Gamma_2^{02}t} \\ \rho_{10}e^{-\Gamma_2^{01}t} & \rho_{11}e^{-\Gamma_1^{01}t} + \frac{\rho_{22}\Gamma_1^{12}}{\Gamma_1^{01} - \Gamma_1^{12}}(e^{-\Gamma_1^{12}t} - e^{-\Gamma_1^{01}t}) & \rho_{12}e^{-\Gamma_2^{12}t} \\ \rho_{20}e^{-\Gamma_2^{02}t} & \rho_{21}e^{-\Gamma_2^{12}t} & \rho_{22}e^{-\Gamma_1^{12}t} \end{pmatrix}. \quad (2.34)$$

In general to characterize these decay mechanisms, and ultimately build error models for our quantum gates, we need to measure the decay rates in our density matrix. This is typically done through coherence measurements.

For a qubit, the standard approach to measuring T_1 and T_2 is well established: a T_1 measurement consists of preparing the qubit in $|1\rangle$, waiting a variable delay time t , and measuring the excited state population, which decays as $e^{-\Gamma_1 t}$. A T_2 (Ramsey) measurement consists of preparing a superposition state via a $\pi/2$ pulse (with some detuning to induce oscillations), allowing free evolution for a time t , applying a second $\pi/2$ pulse, and measuring. The resulting oscillation envelope decays as $e^{-\Gamma_2 t}$, where $\Gamma_2 = \Gamma_1/2 + \Gamma_\phi$. A $T_{2,\text{echo}}$ (Hahn echo) measurement inserts a refocusing π pulse at $t/2$ to remove the effects of low-frequency noise, isolating the intrinsic dephasing rate. Generally speaking, coherence measurements that incorporate more echo like pulses (such as the CPMG sequence) provide a better estimate of gate fidelities due to system coherence, but further discussions of dynamical decoupling will be given later in this thesis.

For a qudit, these coherence measurements generalize naturally by restricting to a two-level subspace $\{|i\rangle, |j\rangle\}$ of the full d -level system. To measure T_1^{ij} for adjacent levels $|j\rangle \rightarrow |i\rangle$ (where $j = i + 1$), one prepares the system in $|j\rangle$ using a sequence of π pulses (e.g., for a qutrit, preparing $|2\rangle$ requires a π_{01} pulse followed by a π_{12} pulse) and then monitors the population decay of the $|j\rangle$ state as a function of time. From the solution for $\rho(t)$ above, we see that the diagonal elements decay exponentially with rates set by the Γ_1^{ij} , though cascaded decay.

Similarly, to measure T_2^{ij} between levels $|i\rangle$ and $|j\rangle$, one performs a Ramsey experiment within the $\{|i\rangle, |j\rangle\}$ subspace. This requires first preparing the system in $|i\rangle$ (if $i \neq 0$) and then applying a $\pi/2$ pulse resonant with the $|i\rangle \leftrightarrow |j\rangle$ transition (which if not natively available can also be compiled) to create a superposition $(|i\rangle + |j\rangle)/\sqrt{2}$. After a variable delay t , a second $\pi/2$ pulse is applied and the resulting population is measured. Alternatively, one can modify this experiment to measure the echo coherence time by adding in a π pulse in the relevant qudit subspace at $t/2$. The coherence $\rho_{ij}(t)$ decays as $e^{-\Gamma_\phi^{ij}t}$, from which we can extract the total dephasing rate $\Gamma_2^{ij} = \Gamma_\phi^{ij} + \Gamma_1^{ij}/2$ for adjacent transitions. The full set of decay and dephasing rates extracted from these measurements then parameterizes the master equation above and provides the basis for constructing realistic noise models for qudit-based quantum gates.

Limits due to Charge Noise

It is natural for one to want to estimate what ratio of E_J/E_C would be required in this setting for generating an m -th transmon level with reasonable coherence, not limited by charge noise. Systematic experimental studies have recently probed the coherence of up to 10 levels in the transmon [264] for large E_J/E_C . It is in general difficult to estimate the effects of charge dispersion analytically. As an absolute worst case estimate, we could imagine that the charge noise is bad enough that it completely broadens the qubit linewidth to be equal to ϵ_m , in which case we would have $T_2^{i,m} \sim 1/|\epsilon_m|$ (for $i < m$). In general, we can build up intuition by considering how a qubit system dephases, starting from considering a qubit Hamiltonian subject to some longitudinal noise shifting around the qubit frequency,

$$H_q = \frac{1}{2}(\omega_q + \delta\omega_q(t))\sigma_z. \quad (2.35)$$

In such a situation, a superposition state of the qubit will accumulate a phase,

$$\varphi(t) = \langle\omega_q\rangle t + \frac{\partial\omega_q}{\partial\lambda} \int_0^t \delta\lambda(t') dt' \quad (2.36)$$

where $\frac{\partial\omega_q}{\partial\lambda}$ is the qubit's longitudinal sensitivity to the coupled noise source. If one assumes the noise is Gaussian distributed then we can take an ensemble average over all Gaussian distributed stochastic process $\delta\lambda(t)$ to find,

$$\langle e^{i\delta\varphi(t)} \rangle = e^{-1/2\langle\delta\varphi^2(t)\rangle}. \quad (2.37)$$

The decay of the off-diagonal matrix element of the density operator can then be calculated,

$$\rho_{01}(t) = e^{i\omega_q t} \exp \left[-\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_\lambda(\omega) \frac{\sin^2(\omega t/2)}{(\omega/2)^2} \right], \quad (2.38)$$

where $S_\lambda(\omega) = \int_{-\infty}^{\infty} dt' \langle \lambda(t') \lambda(0) \rangle e^{-i\omega t'}$ is the power spectral density of the noise.

At lower frequencies, charge noise is typically modeled as $1/f$ noise, and therefore the spectral density takes the form

$$S_Q(\omega) = A_Q^2 \left(\frac{2\pi \times 1 \text{ Hz}}{\omega} \right)^{\gamma_Q}, \quad (2.39)$$

with quasi-universal values $A_Q^2 = (10^{-3}e)^2/\text{Hz}$ at 1 Hz, and $\gamma_Q \approx 1$ [138].

In Ref. [136], the authors calculate two potential limits to T_2 due to charge noise in the transmon circuit. For small fluctuations in the noise setting $\lambda \rightarrow n_g$, they obtain the expression,

$$T_2^{01} \sim \frac{\hbar}{A_Q \pi |\epsilon_1|}, \quad (2.40)$$

from which we can straightforwardly assume that in this small n_g oscillation limit, the dephasing of the $\{|i\rangle, |m\rangle\}$ ($i < m$) will always be limited by the m -th level as it is exponentially more sensitive to n_g , giving,

$$T_2^{im} \sim \frac{\hbar}{A_Q \pi |\epsilon_m|}. \quad (2.41)$$

Additionally, the authors calculate a worst case dephasing rate by assuming the offset charge randomly switches between $[0,1]$, from which we can extract the worst case bound,

$$T_2^{im} \sim \frac{4\hbar}{e^2 \pi |\epsilon_m|}. \quad (2.42)$$

These values can be used as a guide for choosing transmon circuit parameters to avoid being limited by charge noise if one aims to encode m levels. As a figure of merit estimation, in the worst case limit here, a charge dispersion of 300 KHz implies a T_2 limit of about $1 \mu s$. In practice, this number may provide an overly pessimistic view of the coherence of the relevant transition, as dynamical decoupling approaches can often significantly counter low frequency noise induced dephasing.

2.4 A Superconducting QPU

We have now described how the transmon circuit can be engineered to create a multilevel artificial atom, possibly hosting a number of quantum coherent bound states out of which we can perform qudit operation. Now we turn our attention to understanding how these transmon circuits can be coupled together to perform entangling gates and to additional linear modes for readout. While the operation of single and two qudit gates is the focus of later chapters of this thesis, here we describe the general toolbox of circuit QED [16], with terminology and methods analogous to cavity QED (where optical cavities are couple to real atoms for investigations of quantum light-matter interactions).

Transmon-Resonator Coupling

A standard way in cavity QED for one to read out the state of an atom, is to place it in a cavity. In the world of superconducting circuits, we take a similar approach by coupling our transmon artificial atom to an LC resonator. This is realized by engineering a coupling capacitance between the transmon and the resonator, which can be thought of as an electric dipole coupling, yielding an interaction term in the Hamiltonian,

$$H_{\text{int}} = -g(a - a^\dagger)(b - b^\dagger) \quad (2.43)$$

where a and b are respectively the bosonic annihilation operators of the transmon and cavity respectively. The coupling strength between the two elements is given by [17],

$$g = \omega_r \frac{C_g}{C} \left(\frac{E_J}{2E_C} \right)^{1/4} \frac{\pi Z_0}{R_K}, \quad (2.44)$$

where ω_r is the resonator angular frequency, C_g is the coupling capacitance, C is the transmon capacitance, Z_0 is the resonator impedance, and $R_K = h/e^2$ is the resistance quantum. Throughout this manuscript, we will simplify Eq. 2.43 as,

$$H_{\text{int}} = -\hbar g(ab^\dagger + a^\dagger b), \quad (2.45)$$

where we have made the rotating wave approximation, as terms that do not preserve excitations (like $a^\dagger a^\dagger$) will be perturbatively suppressed by factors of $g/|\omega_q + \omega_r|$ (in comparison to excitation number preserving terms that have prefactors $g/|\omega_r - \omega_q|$) and will therefore have negligible effects on the dynamics of our system. Putting this all together, we end up with the system Hamiltonian,

$$H = \omega_q a^\dagger a + \frac{\eta}{2} a^\dagger a^\dagger a a + \omega_r b^\dagger b + g(a^\dagger b + ab^\dagger). \quad (2.46)$$

In the next chapter, we will describe how we can employ this system of an artificial atom coupled to an LC resonator to readout and distinguish multiple qudit states.

Coupling Qudits

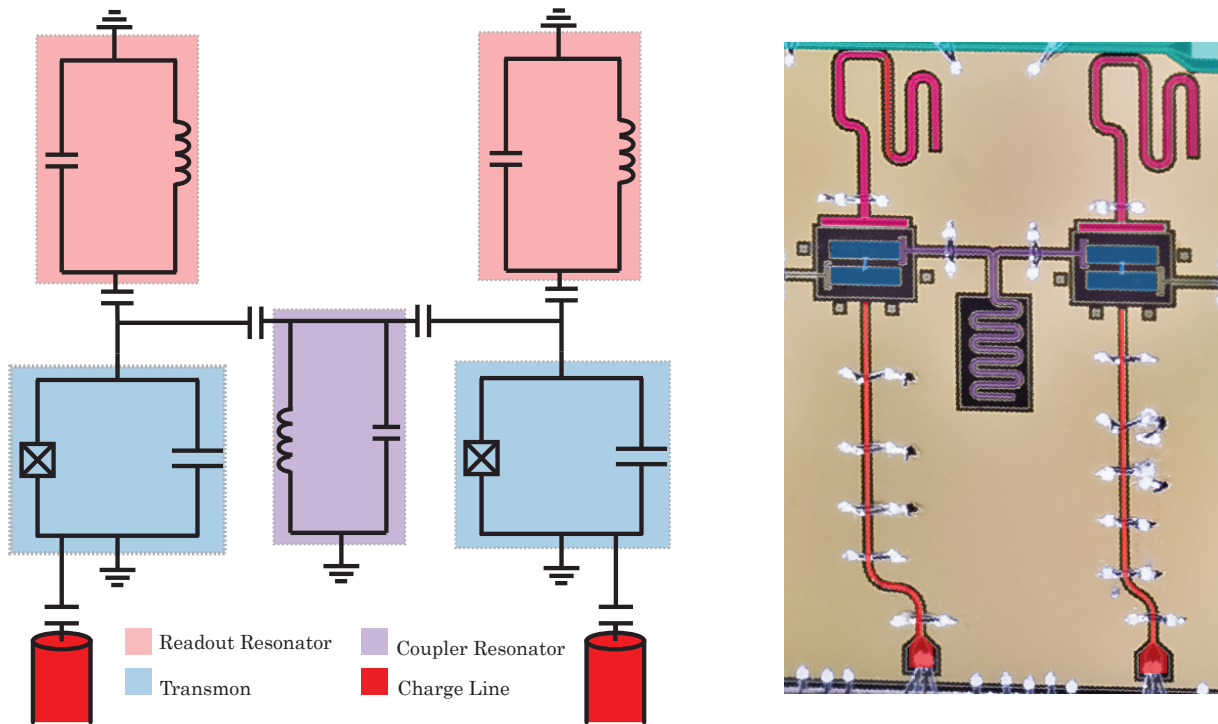


Figure 2.5: Unit Cell of 8 Qubit Ring QPU.

As can be seen in the figure depicting the unit cell of the transmon processor design utilized for most work in this thesis (Fig. 2.5), our transmon circuits are coupled together through a coupling resonator. Such a coupling scheme has a few benefits,

- The lack of flux tunable elements tends to produce higher coherence times;
- Placing a coupling resonator element allows one to space out the qubits more which can help with microwave crosstalk;
- It reduces the number of coaxial lines required in the dilution refrigerator for controlling a processor, only requiring a single charge line (excluding the multiplexed readout line) per transmon.

We can derive the effective Hamiltonian of this coupled two transmon resonator system, using a Schrieffer-Wolf transformation to eliminate the high frequency (and therefore unpopulated) coupling resonator mode. Our system Hamiltonian is,

$$H = \omega_r b^\dagger b + \sum_{i=1,2} \omega_{q,i} a_i^\dagger a_i + \frac{\eta_i}{2} a_i^\dagger a_i^\dagger a_i a_i + \sum_{i=1,2} g_{ic} (a_i^\dagger b + a_i b^\dagger). \quad (2.47)$$

To simplify the calculations, we just consider a system of two qubits coupled to a resonator, although we note that for higher level couplings, to get convergence with experimental results, one needs to either include the resonator in the system or use the SW transformation to find effective couplings between the higher levels directly. For our calculation, let's consider two qubits coupled to a shared linear bus. Truncating each transmon to its two lowest levels, the Hamiltonian is $H = H_0 + V$, where,

$$H_0 = \omega_r b^\dagger b + \sum_{i=1,2} \frac{\omega_{q,i}}{2} \sigma_z, \quad (2.48)$$

$$V = \sum_{i=1,2} g_{ic} (\sigma_i^+ b + \sigma_i^- b^\dagger), \quad (2.49)$$

in the dispersive regime $|\Delta_i| \equiv |\omega_{q,i} - \omega_r| \gg g_{ic}$. We apply a unitary transformation $\tilde{H} = e^S H e^{-S}$, expanding via BCH to second order,

$$\tilde{H} = H_0 + V + [S, H_0] + \frac{1}{2}[S, V] + \dots \quad (2.50)$$

We choose S to cancel V at first order via $[S, H_0] = -V$, which is satisfied by

$$S = \sum_{i=1,2} \frac{g_{ic}}{\Delta_i} (\sigma_i^+ b - \sigma_i^- b^\dagger). \quad (2.51)$$

The effective Hamiltonian is then $\tilde{H} = H_0 + \frac{1}{2}[S, V]$. Evaluating the commutator and retaining only the qubit-qubit exchange terms (as the non excitation preserving terms can

be ignored via the RWA), we find,

$$\frac{1}{2}[S, V] = \frac{g_{1c}g_{2c}}{2} \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right) (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+), \quad (2.52)$$

where the two terms arise from the two orderings of S and V in the commutator, each contributing a factor of $1/\Delta_i$. The effective qubit-qubit coupling is therefore

$$g_{\text{eff}} = \frac{g_{1c}g_{2c}}{2} \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right). \quad (2.53)$$

Often throughout this text, we will employ the approximation for the system Hamiltonian of the transmon-resonator-transmon unit cell in our processors (Fig. 2.5),

$$H = \sum_{i=1,2} \omega_{q,i} a_i^\dagger a_i + \frac{\eta_i}{2} a_i^\dagger a_i^\dagger a_i a_i + g(a_1^\dagger a_2 + a_1 a_2^\dagger). \quad (2.54)$$

where we often write $g_{\text{eff}} \rightarrow g$

Qudit Quantum Crosstalk

An immediate consequence of the direct coupling between our qudits is that this coupling is “always-on” and will lead to a hybridization that entangles our qudits even when they are idling. This can serve as a major source of errors when one attempts to perform isolated single qudit operations, an issue with which we will get more intimately familiar throughout this thesis.

When working in the standard doubly rotating frame of a two-qudit system that removes all the local phases, i.e., performing a rotating frame transformation defined by the unitary,

$$U = \exp \left\{ -it \left[\sum_{j=1}^{d-1} E_{0j} I \otimes |j\rangle \langle j| + \sum_{l=1}^{d-1} E_{l0} |l\rangle \langle l| \otimes I \right] \right\}, \quad (2.55)$$

we are left with the d -level generalized cross-Kerr or ZZ quantum crosstalk Hamiltonian,

$$H = \sum_{i,j=1}^{d-1} \alpha_{ij} |ij\rangle \langle ij|, \quad (2.56)$$

where the cross-Kerr terms are given by $\alpha_{ij} = (E_{ij} - E_{i0}) - (E_{0j} - E_{00})$ (with $\hbar = 1$ everywhere) and with I defined as the single qudit identity operator. Notably, as we encode a local dimension d in each transmon, we will be contending with $(d-1)^2$ quantum crosstalk terms.

To solve for these dispersive energy shifts, we employ perturbation theory and expand $\alpha_{ij} = \alpha_{ij}^{(0)} + \alpha_{ij}^{(1)} + \dots$ where $\alpha_{ij}^{(n)} = E_{ij}^{(n)} + E_{00}^{(n)} - (E_{i0}^{(n)} + E_{0j}^{(n)})$ and $E_{ij}^{(n)}$ is the n th order

correction to the energy of state $|ij\rangle$. Here we do so for the setting of two coupled qutrits. The first two terms $\alpha_{ij}^{(0)}$ and $\alpha_{ij}^{(1)}$ are zero because in the unperturbed system, H_0 , the transmons are not interacting and the perturbation V only maps states $|ij\rangle$ to states $|i'j'\rangle$ that are orthogonal to $|ij\rangle$. The first contribution comes at second order where the static coupling of strength g between the transmons generates a static cross-Kerr interaction with rates (for a full derivation see Appendix B)

$$\alpha_{11}^{(2)} = \frac{4\eta g^2}{(\eta - \Delta)(\eta + \Delta)}, \quad (2.57)$$

$$\alpha_{21}^{(2)} = \frac{2\eta g^2(5\eta - 4\Delta)}{\Delta(\eta - \Delta)(2\eta - \Delta)}, \quad (2.58)$$

$$\alpha_{12}^{(2)} = -\frac{2\eta g^2(5\eta + 4\Delta)}{\Delta(\eta + \Delta)(2\eta + \Delta)}, \quad (2.59)$$

$$\alpha_{22}^{(2)} = \frac{16\eta g^2}{(\eta + \Delta)(\eta - \Delta)}, \quad (2.60)$$

where $\Delta = \omega_c - \omega_t$ and we have set $\eta_c = \eta_t = \eta$ in order to arrive at more compact expressions. As can be seen, a harsh design constraint is imposed by these quantum crosstalk terms; the effective coupling strength g limits the rate of two-qudit gates, but the quantum crosstalk terms which affect single qudit gates grow as g^2 . In chapter 4 we will introduce a driven method that can tune these cross-Kerr terms and effectively improve their nulling (and more importantly be employed to perform high-fidelity two qudit gates). In chapter 6, we will introduce dynamical decoupling approaches that can help limit their impact on qudit dynamics. Nonetheless, in the standard fixed frequency devices we use throughout this thesis, these quantum crosstalk terms pose a serious challenge, and are considerably more difficult to deal with than in qubits. Thankfully, a straightforward way out of this issue is to employ tunable coupler elements. In chapter 8, we will introduce experiments on a device involving a grid of transmons coupled together with tunable transmon couplers, and show an order of magnitude improvement in the nulling of the cross-Kerr terms, opening a pathway to simultaneous high-fidelity one and two-qudit operations.

Purcell Filter

In general, to put wires on our qudits and control them or gain information about their degrees of freedom, we must couple them to the environment. When doing so, we open up pathways for loss or decoherence. One such pathway is Purcell decay, which will take place if our qudit circuit can emit a photon through the transmission line coupled to its readout resonator. In general, the Purcell decay rate for a general qudit transition in the absence of a filter is [136, 264],

$$\Gamma_{1, \text{purcell}}^{i,j} = \kappa_r \frac{g_{qr} |\langle j | n | i \rangle|^2}{2\pi(\omega_i - \omega_j - \omega_r)^2}, \quad (2.61)$$

where κ_r is the resonator linewidth. One might desire to get around this constraint by coupling a qudit to a readout resonator with weak environmental coupling, therefore engineering a smaller κ_r , but this presents a harsh tradeoff as it will lead to slow readout times and limit readout fidelity. Thankfully, there is a straightforward way out through filter engineering. By introducing an extra mode acting as a *Purcell filter*, one can modify the impedance environment at the qubit frequency and effectively maintain a high Q qubit strongly coupled to a low- Q cavity [209, 118]. Following the derivation in Ref. [118], one can estimate the Purcell limited T_1 of the higher levels of a transmon as,

$$T_1^{j,j+1} \leq \frac{1}{\kappa_r} \left(\frac{\Delta_{jr}}{\sqrt{j}g_{qr}} \right)^2 \left(\frac{2\Delta_{jr}}{\omega_{j,j+1}/Q_F} \right), \quad (2.62)$$

where $\Delta_{qr} = \omega_{j,j+1} - \omega_r$, Q_F is the filter mode quality factor, and we have factored in the bosonic enhancement mapping $g \rightarrow \sqrt{j}g$.

A Quantum Processor

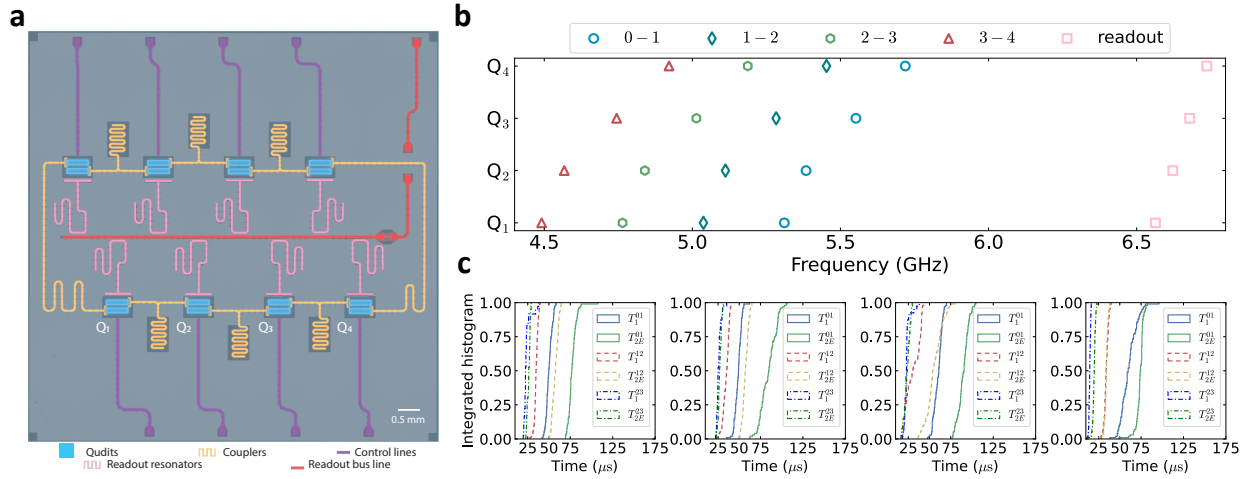


Figure 2.6: **A Qudit Quantum Processor.** (a) False-color micrograph of a representative qudit-ring device. The qudits (blue) are transmon circuits. They are pair-wise connected using resonator couplers (yellow). Each qudit is capacitively connected to a microwave control line (purple) and dispersively coupled to a readout resonator (pink), which are mutually connected to a readout feed line (red). (b) Frequency allocation of 4 qudit subset of the chip. (c) Coherence time statistics of the relevant qudit transitions, each one obtained from 100 individual measurements. Data for $Q_1 - Q_4$ are positioned in order from left to right.

A typical quantum processor (QPU) made out of superconducting circuits combines many of the elements we have discussed in our discussion of the unit cell of the 8 qubit chip in

Fig. 2.6. In principle, a QPU needs to support all the necessary operations to perform quantum computation: namely single and two qudit gates, readout, and reset. The device employed in much of the work in this thesis, features 8 transmons coupled together via resonator buses in a ring geometry. The transmon circuits are each coupled to a designated readout resonator. The readout resonators of all 8 transmons are coupled to a single multiplexed readout bus which acts as a multiplexed bandpass filter to limit Purcell decay. The qudit-qudit coupling is achieved through coupling via the resonator mediated static coupling previously described. Later in this thesis, we will work with a very different QPU architecture in which all qubits are coupled in a square grid lattice, with tunable qubits and couplers. In Chapter 8 when we encounter this device, we will comment more about its specific features in relation to multilevel operation.

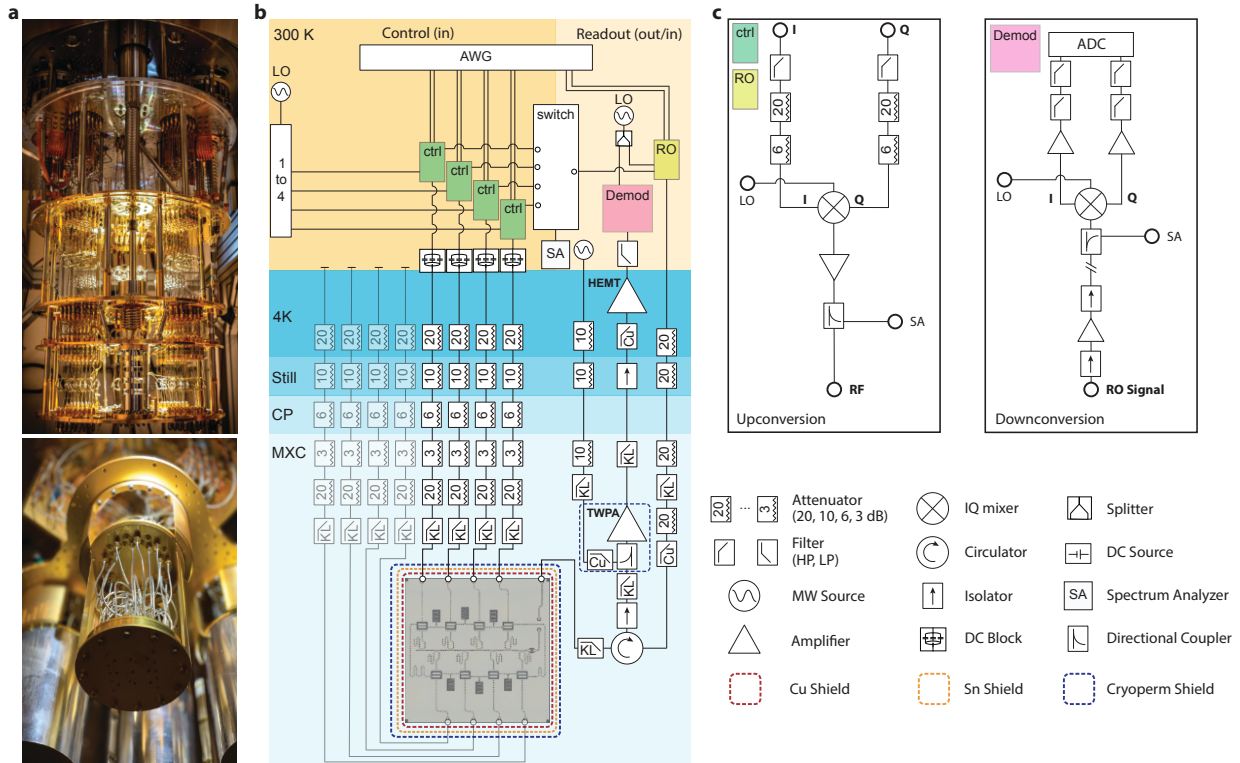


Figure 2.7: **a**, A BlueFors XLD dilution refrigerator in Berkeley lab. **b**, Cryogenic wiring diagram. **c**, Room temperature control and readout circuitry. (Figure reproduced with permission from Ref. [185])

The ground plane of the device is made of Niobium and the junctions are Al/AlOx/Al tunnel junctions as described earlier in this chapter. Niobium becomes superconducting at approximately 9.2 K and aluminum becomes superconducting at approximately 1.2 K. While the onset of superconductivity is a necessary condition for device operation, it is far from

sufficient for quantum coherent behavior. The qubit transition frequency $\omega_q/2\pi \sim 5$ GHz sets a characteristic temperature scale $T^* = \hbar\omega_q/k_B \approx 240$ mK, above which thermal fluctuations are sufficient to scramble the qubit state although even at this temperature the qubit is far too hot to operate effectively. For high-fidelity operation we require $k_B T \ll \hbar\omega_q$, so that the thermal occupation of the excited states is negligibly small. This is achieved by mounting our QPUs inside dilution refrigerators (see Fig. 2.7), which exploit the thermodynamic properties of $^3\text{He}/^4\text{He}$ mixtures to reach base temperatures of 10–20 mK, more than two orders of magnitude below the superconducting transition of Aluminum. Cooling the qubit chip to millikelvin temperatures is necessary but not sufficient: every electrical line connected to the device is a potential channel through which room-temperature thermal noise and high-energy radiation can reach the QPU. To mitigate this, the QPU is surrounded by multiple layers of shielding, including radiation shields and cryoperm magnetic shields, which attenuate stray magnetic fields that would otherwise introduce flux noise. Control and readout signals are routed to and from the device via coaxial cables, with attenuation carefully distributed across each temperature stage of the cryostat. A typical scheme places attenuators at the 4K, Still, 100 mK (cold plate), and mixing chamber stages (see Fig. 2.7b), thermalizing the noise on each input line to the local stage temperature and ensuring that by the time a signal reaches the qubit, the effective noise temperature of the line is close to that of the mixing chamber. Output lines carrying weak readout signals are equipped with cryogenic isolators to prevent amplifier back-action from reaching the qubit, and are amplified at the mixing chamber stage by parametric amplifiers that nearly saturate the quantum limit known as TWPAs [156] and by High Electron Mobility Transistor (HEMT) amplifiers anchored at the 4 K stage before being routed to room-temperature electronics.

Chapter 3

Single Qudit Control and Readout

So far in this thesis, we have described some of the benchmark capabilities an experimentalist needs to realize to build a quantum computer that can operate with multilevel logic (the DiVincenzo criteria). In the last chapter, we outlined how we can engineer circuits with innate multilevel and spectrally resolvable structure in superconducting circuits. The next critical piece is realizing how to control and readout individual multilevel systems. In this chapter, we will outline how single shot multilevel readout and arbitrary single qudit gates are realized in transmon qudits. These operations serve as necessary ingredients for all of the explorations of entangling operations and qudit based quantum simulations we will explore later in this thesis.

3.1 Readout of a Multilevel Artificial Atom

To meet all the requirements of the Divincenzo criteria for qudit based quantum computing, it is necessary to have access to readout of the encoded states within our multilevel circuits. In this section, we will explore the standard approach to dispersive readout of transmon circuits, and how those techniques can be applied for high-fidelity and single-shot readout of multiple levels of the circuit with a single readout pulse.

The Jaynes-Cummings Model and Qubit Dispersive Readout

In chapter 2, we showed how coupling a transmon circuit to an LC resonator yielded the Hamiltonian,

$$H = \omega_q a^\dagger a + \frac{\eta}{2} a^\dagger a^\dagger a a + \omega_r b^\dagger b + g(a^\dagger b + a b^\dagger). \quad (3.1)$$

To gain intuition for this system, let's first neglect the higher levels of the transmon and treat the circuit as a two-level virtual atom,

$$H = \frac{1}{2} \omega_q \sigma_z + \omega_r b^\dagger b + g(a^\dagger b + a b^\dagger). \quad (3.2)$$

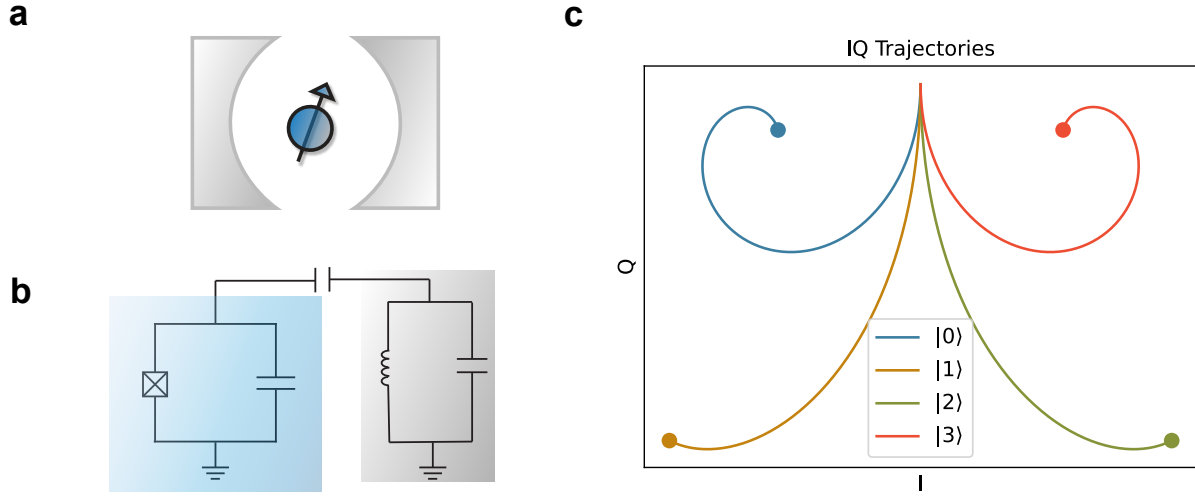


Figure 3.1: **Dispersive Readout Schematic.** **a-b**, A schematic representation of the equivalence between an atom in a cavity (realizing the famous Jaynes-Cummings Hamiltonian) and an artificial atom (transmon circuit) coupled to an LC resonator (acting as a cavity mode). **c**, Numerical simulations of the cavity evolution as a function of four different transmon states where I and Q represent the two cavity quadratures.

Eq. 3.2 is the famous Jaynes-Cummings model from quantum optics, which describes the system of a two-level atom interacting with the quantized degrees of freedom of a bosonic field or cavity and serves as the cornerstone of cavity QED (from which circuit QED gets its name). This model can be solved exactly, but for the purposes of readout in our setting, we will always desire to treat the cavity field and atom as approximately independent systems, working in the so-called “dispersive” limit where $|\Delta| = |\omega_q - \omega_r| \gg g$ and we can treat the qubit-cavity coupling as a perturbation on the uncoupled systems eigenstates. We can again perform a standard Schrieffer-Wolf transformation to remove the coupling, yielding the dispersive Hamiltonian

$$H' = \omega_r a^\dagger a + \tilde{\omega}_a/2\sigma_z + \chi a^\dagger a \sigma_z, \quad (3.3)$$

where the qubit has experienced a “Lamb shift” from the presence of the cavity yielding $\tilde{\omega} = \omega_q + \frac{g^2}{\Delta}$ and the state dependent dispersive shift is $\chi = \frac{g^2}{\Delta}$. This dispersive shift term implies that the cavity’s frequency will shift based off of the qubit’s state, which can be probed (i.e. readout) via a pulse on the cavity; it was this insight that led to the first proposals and demonstrations of the dispersive readout of an artificial atom in a superconducting circuits [16, 263]. We can specifically investigate the time dynamics of the cavity under a drive by considering the quantum Langevin equation [17],

$$\dot{a}(t) = i[\hat{H}_s, a(t)] - \frac{\kappa}{2}a(t) + \sqrt{\kappa}b_{in}(t), \quad (3.4)$$

where κ is the cavity linewidth, and $\hat{b}_{in}$ is the input field. Solving this equation, we find that the complex field amplitudes (setting $\langle \hat{a} \rangle_\sigma = \alpha_\sigma$) evolve as [17],

$$\dot{\alpha}_e(t) = -i\Omega(t) - (\Delta_r + \chi)\alpha_e(t) - \kappa\alpha_e(t)/2, \quad (3.5)$$

$$\dot{\alpha}_g(t) = -i\Omega(t) - (\Delta_r - \chi)\alpha_g(t) - \kappa\alpha_g(t)/2, \quad (3.6)$$

where $\Omega(t)$ is the cavity drive and Δ_r is the detuning of the drive from the cavity frequency.

Multilevel Dispersive Readout

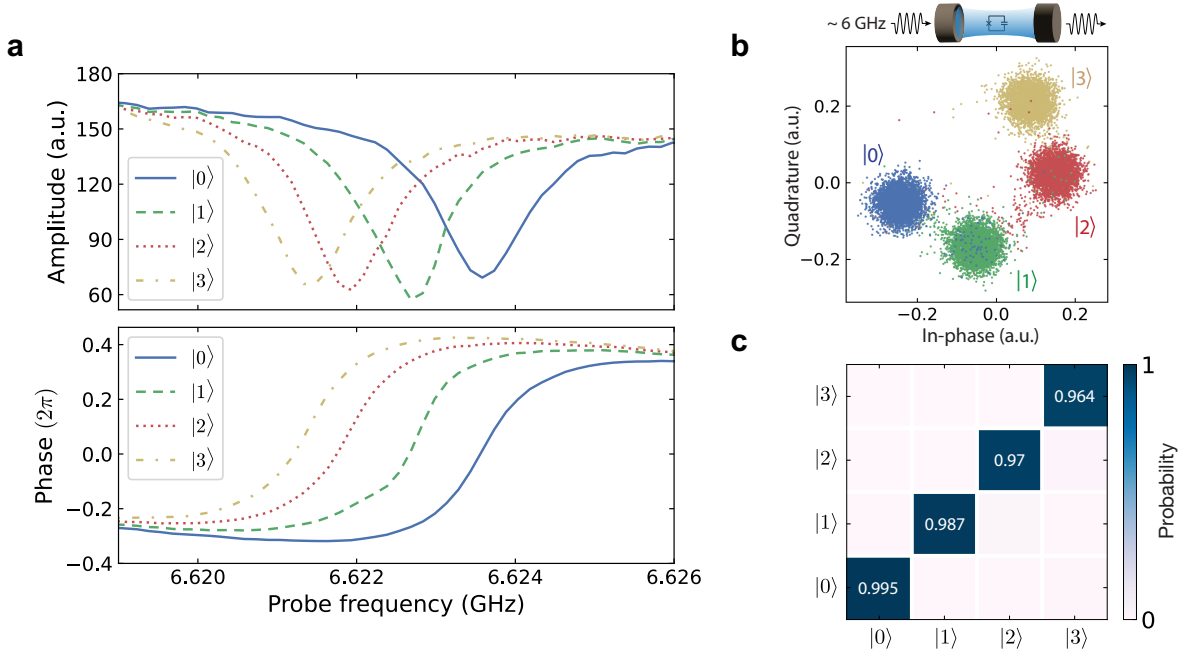


Figure 3.2: **Single Shot Readout of a Four-Level Qudit System**, **a**, The amplitude and phase response of an LC resonator dispersively coupled to a transmon qudit (note that the resonator is readout in reflection). **b**, The integrated IQ signals of the dispersive readout of the four-level system. **c**, The assignment fidelities.

Let's now consider the setting where instead of two-level atom coupled to our cavity, we instead consider our multilevel transmon circuit. This system can be well approximated by the linear dispersive Hamiltonian given in Ref. [15],

$$H_{JC}^D = \omega_r b^\dagger b + \sum_{k=0}^{d-1} \omega_k |k\rangle \langle k| + \sum_{k=1}^{d-1} \chi_{k-1} |k\rangle \langle k| - \sum_{k=0}^{d-1} (\chi_k - \chi_{k-1}) |k\rangle \langle k| b^\dagger b. \quad (3.7)$$

Here, we express the Hamiltonian for d levels of the qudit in its own eigenbasis. Following Eq. 3.7, the cavity now experiences a dispersive shift of $\chi_k - \chi_{k-1}$ when the qudit is in state $|k\rangle$, where $\chi_k = g_k^2/\Delta_{kr}$ and $g_k \approx \sqrt{k}g$. We can also find now the cavity field equations as,

$$\dot{\alpha}_k(t) = -i\Omega(t) - (\Delta_r + \chi_k)\alpha_k(t) - \kappa\alpha_k(t)/2. \quad (3.8)$$

In Fig. 3.1, we display an example evolution of the cavity coherent states for four-level readout of transmon, where for simplicity we set $\chi_{k+1} - \chi_k = \chi, \forall k$. In general, when thinking about readout parameters for single shot readout of a qudit system, one ideally wants to target a situation where $\chi \approx 2\kappa/d$, where d is the basis dimension of the qudit you are encoding [33]. The basic intuition for this is one wants the dispersive shifts to be large enough to separate the displaced states of the cavity, but not so large relative to κ that one cannot distinguish the states. However, this constraint can be relaxed by incorporating multiple readout tones [264].

Fig. 3.2 shows the experimentally measured phase and amplitude responses of a resonator mode when a dispersively coupled qudit is prepared in $|0\rangle, |1\rangle, |2\rangle$ and $|3\rangle$. For the given χ_k 's and resonator linewidths on that device (varying from 1 to 2 MHz), it is possible to choose a probe frequency for each resonator that distinguishes all four relevant qudit states with single shot separability (see Fig. 3.2b-c). While we do observe that the readout fidelity degrades for higher levels of the transmon, this can largely be attributed to T1 decay during the readout. Finally, although we do not explore it in this thesis, we note here that access to higher level readout of transmons can prove to be a useful tool to study the problem of transmon ionization or measurement induced state transition errors that plague dispersive readout [265].

3.2 Single Qudit Gates

Having now established how to readout qudit states by dispersively coupling the transmon to a readout resonator, we will turn our attention to performing arbitrary logic operations in individual transmon qudits. To start we will build up intuition from single qubit gates, and then describe how we calibrate and benchmark single qudit gates. Additionally, we will touch on limitations to these gates due to decoherence and quantum crosstalk.

Rabi Oscillations and Virtual Z Gates

Let's start by considering the Hamiltonian of a driven transmon in the Duffing oscillator approximation,

$$H = \omega_q a^\dagger a + \frac{\eta}{2} a^\dagger a^\dagger a a + \Omega(t) \cos(\omega_d t + \phi)(a + a^\dagger), \quad (3.9)$$

where $\Omega(t)$ is the microwave drive envelope at the carrier frequency ω_d .

To understand the effect of this drive, it is useful to move into a rotating frame defined by the unitary $U_0 = \exp(i\omega_d a^\dagger a t)$, which removes the fast oscillation at ω_d from the problem.

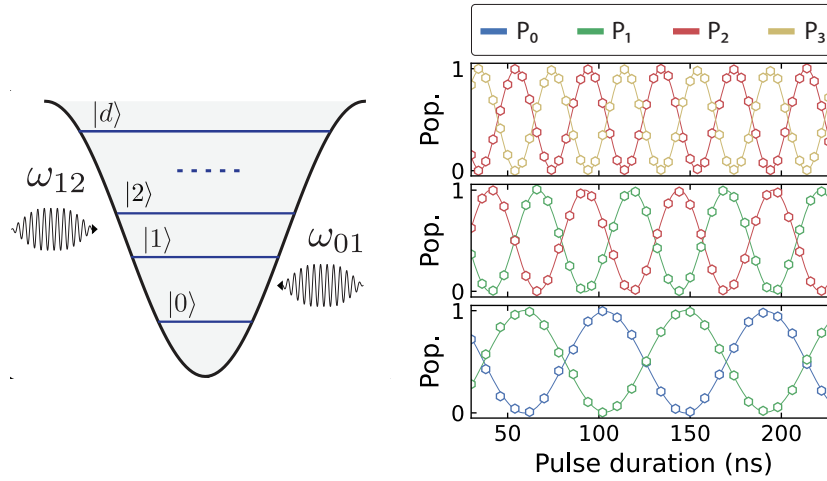


Figure 3.3: **Rabi Driving a qudit.** We display experimental results for driving the spectrally distinct single photon Rabi oscillations of $|k\rangle \leftrightarrow |k+1\rangle$ for a four level superconducting qudit.

Under this transformation (and after making the RWA), the Hamiltonian becomes,

$$\tilde{H} = \Delta a^\dagger a + \frac{\eta}{2} a^\dagger a^\dagger a a + \frac{\Omega(t)}{2} (e^{i\phi} a + e^{-i\phi} a^\dagger), \quad (3.10)$$

where $\Delta = \omega_q - \omega_d$ is the detuning of the drive from the qubit frequency, and we have applied the rotating wave approximation (RWA) to drop the terms oscillating at $2\omega_d$, which are suppressed by a factor of $\Omega/\omega_d \ll 1$. To make the multilevel structure explicit, it is helpful to project this Hamiltonian onto the transmon eigenbasis keeping d qudit states. Recalling that the matrix elements of the ladder operators between transmon levels are $\langle j|a|k\rangle = \sqrt{k} \delta_{j,k-1}$, we can write the full driven Hamiltonian as,

$$\tilde{H} = \sum_{k=0}^{d-1} k\Delta |k\rangle \langle k| + \frac{\eta}{2} \sum_{k=0}^{d-1} k(k-1) |k\rangle \langle k| + \frac{\Omega(t)}{2} \sum_{k=0}^{d-2} \sqrt{k+1} (e^{i\phi} |k\rangle \langle k+1| + e^{-i\phi} |k+1\rangle \langle k|), \quad (3.11)$$

where the first two terms give the energy levels in the rotating frame, and the last term captures the microwave drive that can drive all adjacent transitions in the transmon, with a bosonic enhancement factor of $\sqrt{k+1}$ on the $|k\rangle \leftrightarrow |k+1\rangle$ transition. In Fig. 3.3, we display experimentally the characteristic bosonic enhancement for Rabi oscillations as we drive $|k\rangle \leftrightarrow |k+1\rangle$ single photon transitions in a transmon qudit. The Rabi interactions naturally generate two-level subspace Pauli-X operations. To realize universal single qudit control and perform Z-like operations, we employ virtual-Z gates which update the phase of the relevant carrier waves at $\omega_{k,k+1}$ [165]. In the qudit setting, we work in the rotating

frame,

$$U = \exp \left\{ -it \left[\sum_{j=1}^{d-1} \omega_j |j\rangle \langle j| \right] \right\}. \quad (3.12)$$

This rotating frame is a natural choice because it leads to no accumulation of phase when the qudit is idling. In practice, this corresponds to frame tracking the carrier frequency for each single photon transition of the qudit, which we will ultimately Rabi drive to perform gates. Phase updates to these carrier waves generate generalized virtual Z gates, which we can express for a phase update of ϕ as,

$$Z^{k,k+1}(\phi) = \sum_{l=k+1}^{d-1} e^{i\phi} |l\rangle \langle l| + \sum_{l=0}^k |l\rangle \langle l|, \quad (3.13)$$

where $Z^{k,k+1}$ corresponds to an update to the carrier wave at frequency $\omega_{k,k+1}$. Between these subspace Rabi oscillations and virtual Z gates, we can realize fully universal control of a qudit, as we will demonstrate later in this chapter.

Calibrating Single Qudit Gates

In an experimental setting, it is ideal to calibrate a limited number of pulses to serve in our gateset for running quantum circuits, as each such calibration requires time and effort. Thankfully, in the context of qubits, given the continuous degree of freedom of virtual Z gates, it suffices to compile a single $X_{\pi/2} = e^{-i\sigma_x \pi/4}$ gate, as an arbitrary $SU(2)$ gate can be parameterized via [165],

$$U(\theta, \phi, \lambda) = Z_{\phi-\pi/2} X_{\pi/2} Z_{\theta-\pi} X_{\pi/2} Z_{\lambda-\pi/2}. \quad (3.14)$$

For performing arbitrary qudit gates over $SU(d)$, as shown in Fig. 3.5, we can build up arbitrary qudit gates from the $SU(2)$ blocks using a technique from linear algebra known as Givens rotations. In particular, we construct $SU(d)$ utilizing 3(6) blocks or 6(12) physical pulses per unitary for $d = 3(4)$ [93]. The algorithm for performing this operation in local dimension d is iterative, generally requiring $\mathcal{O}(d^2)$ single qudit gates, and thus for very high qudit dimension becomes impractical. For the case of our work, where we target an encoded local dimension of $d = 2 - 4$, the overhead is reasonable, but we point out to the reader that alternative techniques to treat high dimensional qudits as a large spin and perform “displacements” by driving many levels at once have recently been explored with experimental success [38, 215, 280].

As we aim to calibrate subspace $X_{\pi/2}$ pulses, our calibration procedures can straightforwardly benefit from well trodden qubit gate calibration routines. After spectroscopic calibration of the qudit transition frequencies and a rough calibration of the pulse amplitudes and readout, we calibrate qudit gates for high fidelity operation by performing the calibration sequence shown in Fig. 3.4 first on the $\{|0\rangle, |1\rangle\}$ subspace,

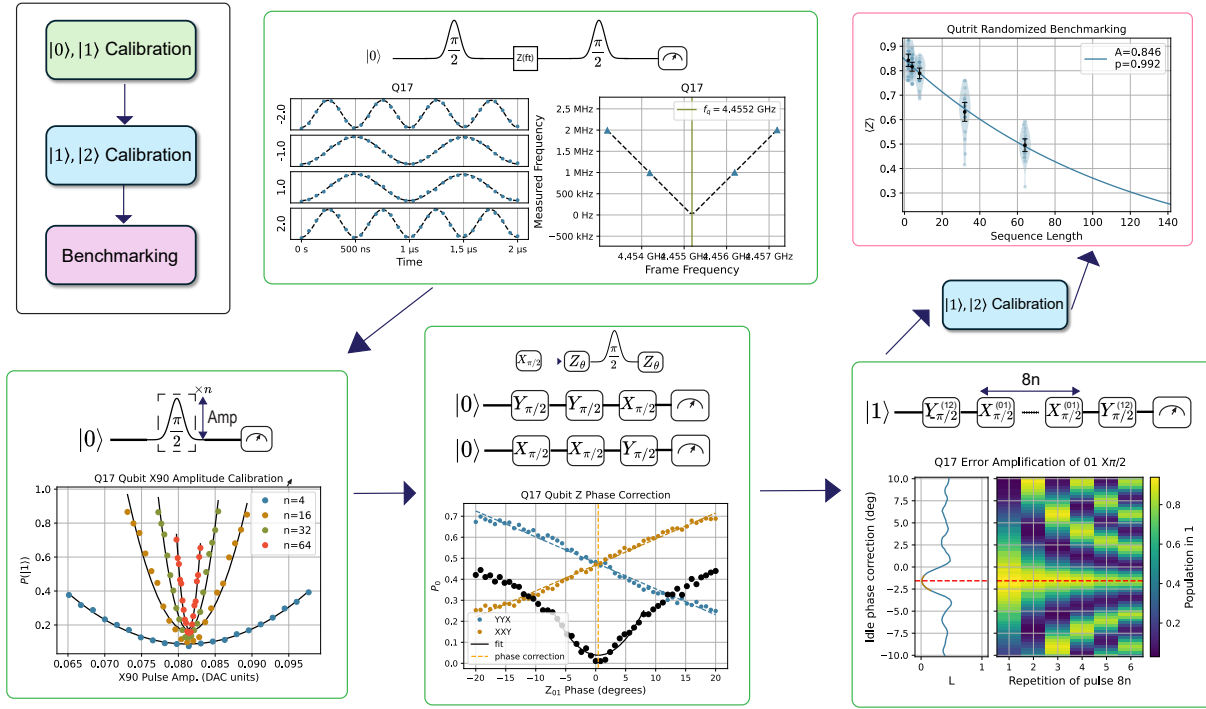


Figure 3.4: **Calibration Protocol of a Qutrit** (black outline) the high level calibration protocol for a qutrit. First the $|0\rangle \leftrightarrow |1\rangle$ transition is calibrated, then the $|1\rangle \leftrightarrow |2\rangle$ transition, and finally the calibration is verified via randomized benchmarking. Note this approach can be bootstrapped to arbitrary local dimension d . (green outline) The calibration graph as described in the text. (purple outline) The qutrit randomized benchmarking results of the pictured calibration confirming high-fidelity gate operation.

- Ramsey-based frequency calibration
- Fine pulse amplitude calibration via error amplification
- Phase correction on pulse due to Stark shift from off-resonant drive on the idling levels
- Phase correction on the idling qudit levels due to the Stark shift

and then performing the same suite of calibrations for the next single photon transition, progressively moving up the qudit ladder. After calibrating the $X_{\pi/2}$ rotation in the $\{|d-2\rangle, |d-1\rangle\}$ subspace, we can verify the quality of our calibration by performing randomized benchmarking (discussed more shortly).

Before describing our basic characterization of single qudit gates, let's quickly summarize these calibrations and their purpose. The Ramsey experiment for calibrating the transition

frequency consists of applying a pulse (either detuned in frequency or artificially detuned with virtual Z gates) to prepare a superposition of $|k\rangle, |k+1\rangle$, waiting a variable amount of time, and then projecting the state back onto $|k\rangle$ and measuring the resulting population as a function of time to infer the frequency detuning of the pulse. By sweeping the real or artificial detuning of the $\pi/2$ pulses, we can extract small detuning errors in our pulses and correct them.

After a careful calibration of the relevant qudit transition frequency, the amplitude of the $\pi/2$ pulse (typically with an envelope of a Gaussian or cosine) is calibrated by error amplification. The basic set up for performing this is preparing the qudit in state $|k\rangle$, then performing integer multiples of $4 X_{\pi/2}^{k,k+1}$ gates to return the qudit to state $|k\rangle$, fitting a cosine to the swept amplitude of each pulse and continuously refining the pulse amplitude.

Next we turn to correcting the phase errors in the $\{|k\rangle, |k+1\rangle\}$ subspace due to the off-resonant drive on the idling states that shifts the energy of the driven transition. This can in principle be fixed by adjusting the drive frequency, but we use virtual Z gates that are prepended and appended to the subspace $X_{\pi/2}$ gate as in Ref. [165].

Finally, we apply virtual Z gates on idling states of the qudit that are Stark shifted by the pulse in the $\{|k\rangle, |k+1\rangle\}$ subspace as outlined in Ref. [176]. In principle, this calibration can be performed for all transitions in the computational space but not addressed by the pulse. For calibrating the Z phase correction on the $\{|1\rangle, |2\rangle\}$ transition while driving $\{|0\rangle, |1\rangle\}$, we can place the qudit in a superposition of $|1\rangle + |2\rangle$, then apply a variable number of $8n X_{\pi/2}^{01}$ pulses (where 8 is chosen to avoid the geometric phase from a single rotation out of the $\{|0\rangle, |1\rangle\}$ space). By sweeping the virtual Z gate on the idling transition, we can then infer the phase required to account for the Stark shift on the idling level.

Qudit Randomized Benchmarking

The randomized benchmarking toolbox has emerged as a powerful suite of experimental measurements to understand the performance of the gateset available in a quantum computer [157, 100]. The core idea of randomized benchmarking (RB) is to extract an average error rate for a gateset by measuring how quickly the fidelity of a random quantum circuit decays as a function of its length. In a standard RB experiment, one constructs sequences of m randomly sampled Clifford gates, appends a final recovery gate (chosen so that the ideal circuit is the identity), and measures the probability of returning to the initial state (almost always $|0\rangle^{\otimes n}$). Averaging over many such random sequences suppresses the dependence on state preparation and measurement (SPAM) errors and on the specific details of individual gates, leaving behind a quantity that depends only on the average gate error. Under the assumption that the noise is gate-independent and time-stationary, the sequence fidelity decays as a simple exponential in the sequence length,

$$F(m) = Ap^m + B, \quad (3.15)$$

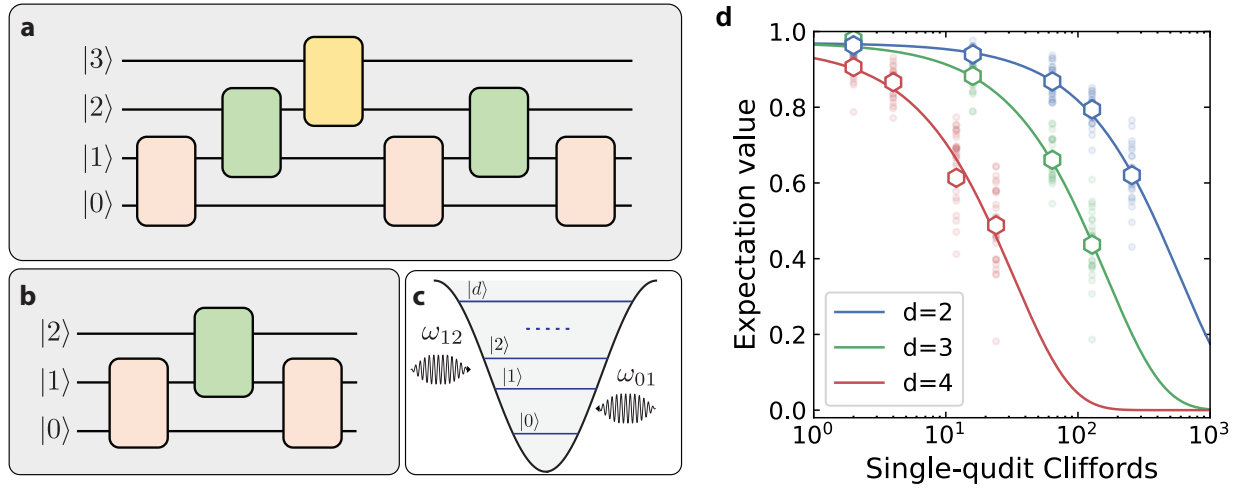


Figure 3.5: **Decomposition of single qudit control pulses and RB results.** (a) An arbitrary $SU(4)$ qudit gate can be decomposed into 6 blocks of $SU(2)$ unitaries in two-level subspaces of the qudit. (b) Similarly, any $SU(3)$ qudit gate can be decomposed using only three subspace $SU(2)$ unitaries. (c) Subspace $SU(2)$ operations are performed via resonant microwave drives at frequency $\omega_{k,k+1}$ and virtual-Z gates. d, Randomized benchmarking results of a transmon with local qudit dimension of $d = 2, 3, 4$.

where p is the depolarizing parameter of the gateset, and A and B are constants that absorb SPAM errors. From the decay parameter p , one can extract the average gate fidelity as,

$$r = 1 - \frac{(d-1)(1-p)}{d}, \quad (3.16)$$

where d is the Hilbert space dimension. One important limitation of this approach is that the exponential decay is largely insensitive to coherent errors that average out under the random twirling, nonetheless the fit to $F(m)$ can be performed with relatively few circuits making RB an efficient and practically robust benchmarking primitive.

Standard RB benchmarks the average performance of the full Clifford group, but provides limited information about the performance of specific gates or error channels. This has motivated the development of a rich family of RB variants. Interleaved RB [158] isolates the fidelity of a specific gate of interest by interleaving it between the random Clifford gates and comparing the resulting decay rate to the reference RB decay, allowing one to bound the error of that individual gate. Purity benchmarking separates coherent from incoherent errors by tracking the decay of the purity of the output state rather than its fidelity [66]. In fact, there are now many benchmarking protocols of interest in the toolkit of a quantum experimentalist, and we point the reader to Ref. [100] for a more exhaustive enumeration.

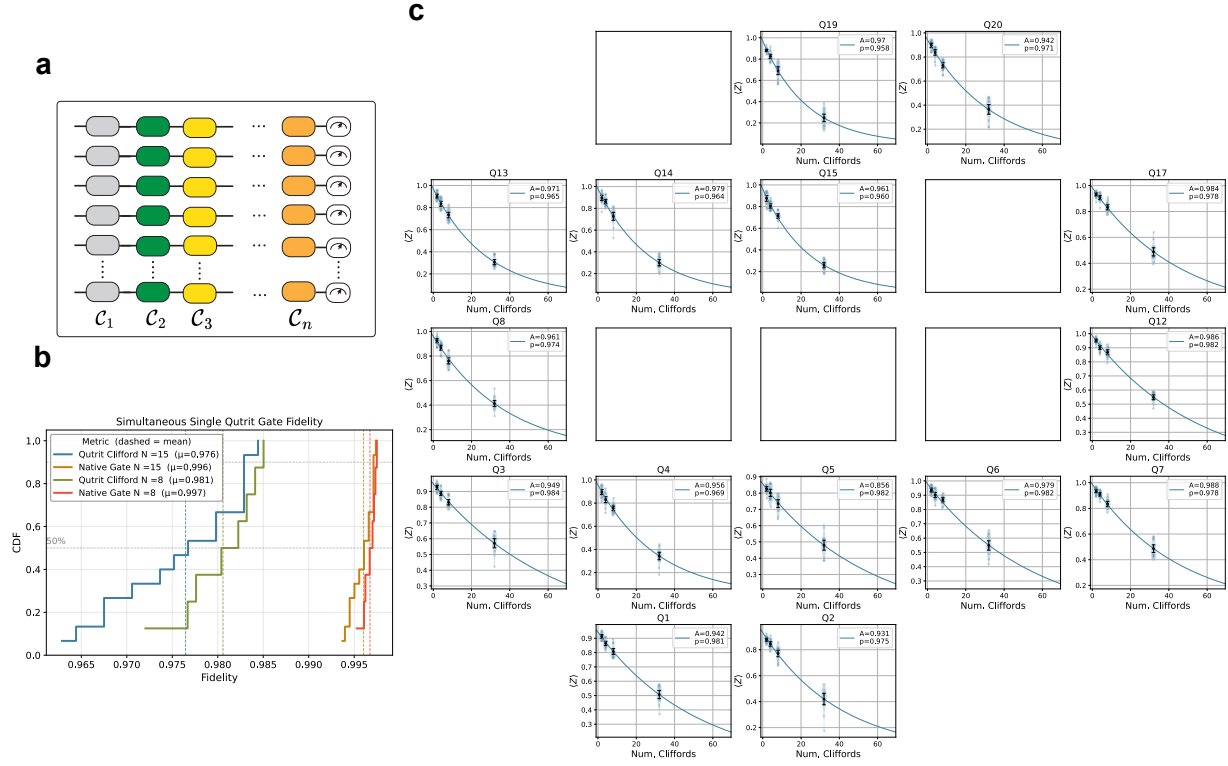


Figure 3.6: **Simultaneous Randomized Benchmarking on a 20 qutrit processor.** (a) Circuit diagram of simultaneous randomized benchmarking circuits. Each C_i consists of a string of randomly chosen qutrit Clifford gates for each qutrit in the chain. $n - 1$ Cliffords are performed before a final cycle of qutrit gates C_n which inverts all the previous Clifford gates. (b) Integrated histogram of simultaneous qutrit randomized benchmarking results on an $N = 15$ and $N = 8$ chain. We compare both the simultaneous single qutrit gate fidelity, as well as the average fidelity of the native gates (recall each Clifford is composed of 6 native pulses). (c) Qutrit randomized benchmarking results on the $N = 15$ subset of the processor.

In the context of qudit systems, RB generalizes naturally but with some important subtleties. The Clifford group for a d -dimensional qudit is well-defined for prime d [157], and a qudit RB experiment proceeds analogously to the qubit case by sampling random qudit Clifford circuits and measuring the sequence fidelity decay [176]. However, the compilation of qudit Cliffords into native gates is generally more complex than in the qubit case, and the larger Hilbert space means that more sequences and longer sequence lengths may be required to faithfully fit the decay curve. Furthermore, in the presence of leakage standard RB can give misleading results, as leakage errors do not necessarily cause a depolarizing-like decay and can cause the sequence fidelity to deviate from a single exponential.

In Fig. 3.5, we present the compilation from $SU(2)$ blocks for arbitrary single-qudit gates and the subsequent randomized benchmarking results with average native gate fidelities of $\mathcal{F}_g = \{0.99936(3), 0.99909(4), 0.9978(2)\}$ for $d = \{2, 3, 4\}$, respectively. The observed Clifford fidelity decreases for higher d due to the increasing number of native qudit rotations required, with compilations of Clifford groups requiring $\{2, 6, 12\}$ native gates for $d = \{2, 3, 4\}$, respectively. One note about this figure is that for $d = 4$, as we do not have a prime dimension qudit, we use the two-qubit Clifford group transpiled into the 4 level qudit.

Simultaneous Randomized Benchmarking

A critical practical question when operating a multi-qudit processor is whether the performance of individual qudit gates degrades when multiple qudits are operated in parallel. This degradation can arise from a number of sources, including microwave crosstalk between control lines, always-on quantum crosstalk from the static ZZ coupling between neighboring qudits (as described in Sec. 2.4), and spurious drive-induced transitions on spectator qudits caused by the finite bandwidth of control pulses, microwave crosstalk, and frequency collisions. Standard RB, performed on one qudit at a time while all others are idle, does not capture these effects. Simultaneous randomized benchmarking addresses this by running independent RB sequences on all qudits in parallel allowing a comparison to the resulting per-qudit error rates to those obtained from isolated single-qudit RB.

The protocol proceeds as follows. At each sequence length m , one draws independent random Clifford sequences for each qudit in the processor and applies them simultaneously, so that every qudit is actively being driven while its neighbors are also undergoing random gates. The sequence fidelity is then measured for each qudit independently.

In the context of qudit processors, simultaneous RB takes on particular importance. As discussed in Sec. 2.4, the always-on ZZ -type quantum crosstalk between transmon qudits grows as g^2 and, crucially, acquires additional terms when higher transmon levels are populated. When a neighboring qudit is being actively driven during a simultaneous gate sequence, the cross-Kerr shifts experienced by a given qudit will fluctuate in a circuit-dependent way, producing coherent errors (and possibly dephasing errors) that would not be captured by isolated RB. In Fig. 3.6, we perform simultaneous qutrit randomized benchmarking on a 20 transmon processor that integrates flux tunable couplers which can critically null the cross-Kerr terms. Nonetheless, as depicted in Fig. 3.6b, the average gate fidelity and outlier gate fidelities both were worse when characterizing the simultaneous benchmark on a subset of 15 qutrits vs a smaller subset of 8 qutrits, showing the cumulative deleterious effects of crosstalk in our qutrit systems.

Error Budgeting Single Qudit Gates

Finally, having calibrated and benchmarked our local gates, it is natural to want to understand what may be limiting their performance. From our master equation model of a single qutrit presented in Chapter 2, we can calculate the coherence limit of a single qutrit gate

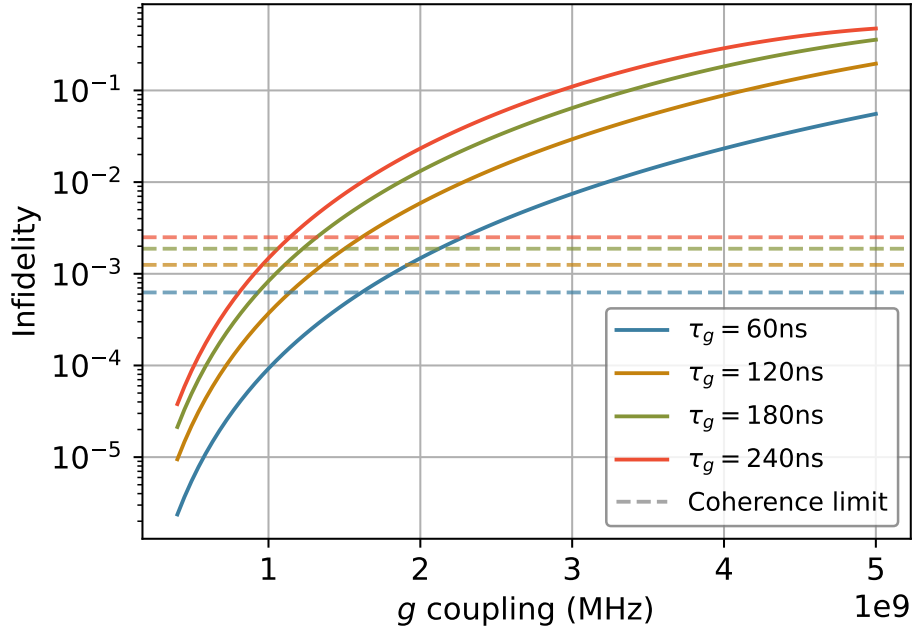


Figure 3.7: **Error Budgeting Single Qutrit Gates.** We plot the estimated fidelity limit to single qudit gates due to the cross-Kerr interaction as a function of the coupling strength g between transmons. The horizontal lines mark the coherence limit for gates of those same time calculated using the highest qutrit coherence presented in this thesis.

following Ref. [176] as,

$$e_{\mathcal{F}} \approx \frac{1}{8}(2\Gamma_2^{01} + 2\Gamma_2^{02} + 2\Gamma_2^{12} + \Gamma_1^{01} + \Gamma_1^{12})\tau_g, \quad (3.17)$$

where $e_{\mathcal{F}}$ is the gate infidelity and τ_g is the gate time. As we have mentioned before, especially in the context of fixed coupling architectures, the more complicated and generally larger quantum crosstalk between qudits can severely limit gate performance. To illustrate this, in Fig.3.7, we estimate the infidelity of simultaneous single qutrit gates due to the cross-Kerr interaction as a function of the direct coupling. We calculate these rates as a function of g using Eqs. 2.57-2.60 fixing $\Delta = 150$ MHz and $\eta = 270$ MHz similar to our design parameters for the fixed coupling device employed often in this thesis. We can estimate the error due to the static cross-Kerr entanglement by calculating,

$$e_{\mathcal{F}} = 1 - \text{Tr}[U_{\text{id}}^\dagger U_{\text{ck}}]/d^2 \quad (3.18)$$

with U_{ck} defined,

$$U_{\text{ck}} = \begin{matrix} & |00\rangle & |01\rangle & |02\rangle & |10\rangle & |11\rangle & |12\rangle & |20\rangle & |21\rangle & |22\rangle \\ \begin{matrix} |00\rangle \\ |01\rangle \\ |02\rangle \\ |10\rangle \\ |11\rangle \\ |12\rangle \\ |20\rangle \\ |21\rangle \\ |22\rangle \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i2\pi\alpha_{11}\tau_g} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i2\pi\alpha_{12}\tau_g} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i2\pi\alpha_{21}\tau_g} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i2\pi\alpha_{22}\tau_g} \end{pmatrix} \end{matrix} \quad (3.19)$$

As can be observed in Fig. 3.7, when we compare the infidelity due to the coherent cross-Kerr interaction to our incoherent loss due to decoherence (where we used the coherences of the longest lived qutrit utilized in this thesis), we expect that even for weak couplings $g \lesssim 2$ MHz, the primary source of infidelity for single qutrit gates can be due to the always on cross-Kerr coupling. This is a recurring problem that will be tackled throughout this thesis. In the next chapter we will introduce a microwave driven approach to tune this cross-Kerr coupling and in later chapters we explore dynamical decoupling approaches to remove the coherent evolution of the cross-Kerr coupling. Finally in the last chapter we explore qutrit operation on a processor with integrated tunable coupler in which all of the cross-Kerr rates on the device are on average below 30 kHz, representing a significant improvement in scalable qutrit operation. This device, a QPU from IQM consisting of a grid of 20 flux tunable transmons with flux tunable couplers [3], was utilized in the simultaneous RB experiment in Fig. 3.6, where the qutrit gate fidelities were no longer limited by cross-Kerr coupling, but rather coherence and uncanceled microwave crosstalk.

Chapter 4

Two-Qutrit CZ Gates via the Differential-AC Stark Shift

In the previous chapters, we have built up the concepts of quantum information, qudit quantum logic, and how individual qudits are encoded in superconducting circuits, manipulated, and readout. In this section we will describe how to move beyond individual qudit operation to explore qudit entanglement. In the previous decade, remarkable progress has been achieved to push two-qubit gate fidelities far beyond the thresholds necessary for logical error suppression in quantum error correcting codes such as the surface code[6]. Two-qudit gates, on the other hand, have received much less experimental focus, and have lagged considerably in fidelity. In this section, we describe the motivation for, and implementation of, a high-fidelity two-qutrit CZ gate based on the differential AC-Stark effect. We begin by describing the physics of the effective driven Hamiltonian that generates the two-qutrit gate, then we move on to describing how we tailor that effective Hamiltonian to compactly realize a maximally entangling two-qutrit CZ gate. Finally, we discuss our careful benchmarking and error budgeting of the gate and provide an outlook for future improvements. The original work to develop and benchmark the gate that is the focus of this chapter can be found in Ref. [85].

4.1 Two-Qutrit Unitaries

In section 1.3, we outlined how the qubit Pauli operators could be naturally generalized to form the Weyl-Heisenberg operators. Here we recall that the qudit X and Z operators are given as,

$$X = \sum_{j \in \mathbb{Z}_d} |j \oplus_d 1\rangle \langle j|, \quad Z = \sum_{j \in \mathbb{Z}_d} \omega^j |j\rangle \langle j|, \quad (4.1)$$

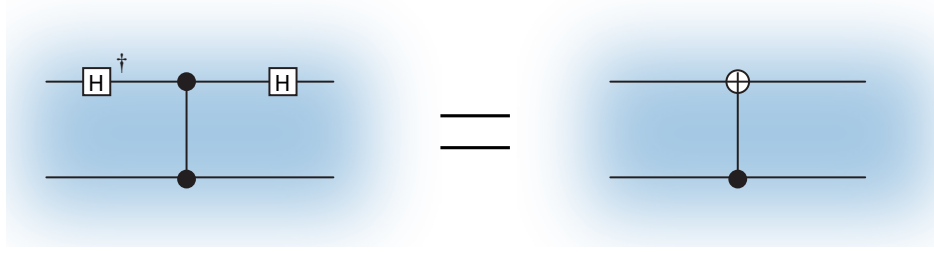


Figure 4.1: **CZ/CSUM local equivalence.** Quantum circuits demonstrating the local equivalence via Hadamard conjugation between a two-quitrit CZ and two-quitrit CSUM gate.

with $\omega = \exp\left(\frac{2\pi i}{d}\right)$, the d -th-root of unity. Additionally, like in the case of qubits, the qudit Hadamard gate acts as a Fourier transformation between the X and Z basis,

$$H = \frac{1}{\sqrt{d}} \sum_{k,j} \omega^{kj} |k\rangle \langle j|. \quad (4.2)$$

From these definitions, it is straightforward to generalize canonical two-qubit gates such as the CNOT and CZ gate. A qudit CZ gate is the unitary that given some control qudit state $|n\rangle$, applies a Z^n to the neighboring qudit,

$$U_{CZ} = \sum_{n=0}^{d-1} |n\rangle \langle n| \otimes Z^n. \quad (4.3)$$

Although previously described in terms of a “control” and “target” qudit, like its qubit counterpart, U_{CZ} forms a symmetric diagonal unitary, a fact which can be more easily seen when writing it in terms of the qudit basis states:

$$U_{CZ} = \sum_{k,j \in \mathbb{Z}_{d-1}} \omega^{kj} |kj\rangle \langle kj|. \quad (4.4)$$

Similarly we can define the generalization of the two-qubit CNOT gate, typically called controlled-sum, or CSUM for short, as the unitary that given some control qudit state $|n\rangle$ applies a X^n to a target qudit,

$$U_{CSUM} = \sum_{n=0}^{d-1} |n\rangle \langle n| \otimes X^n. \quad (4.5)$$

The CSUM unitary is also more intuitively understood by analyzing its action on qudit basis states $|k, j\rangle$:

$$U_{CSUM} |k, j\rangle = |k, k \oplus_d j\rangle. \quad (4.6)$$

We observe, that just like their qubit counterparts, U_{CZ} and U_{CSUM} are locally equivalent to each other via conjugation with single qudit Hadamard gates, i.e. $U_{CSUM} = (I \otimes H^\dagger) U_{CZ} (I \otimes H)$. Finally, the reader should note that these definitions reduce to the familiar CZ and $CNOT$ gates in the limit that $d \rightarrow 2$. In this section, we specifically experimentally realize two-qutrit gates, whose unitary definitions are found from these general qudit definitions by choosing $d = 3$.

4.2 Motivation

Why Off-Resonant Coupling for a Fully Entangling Interaction?

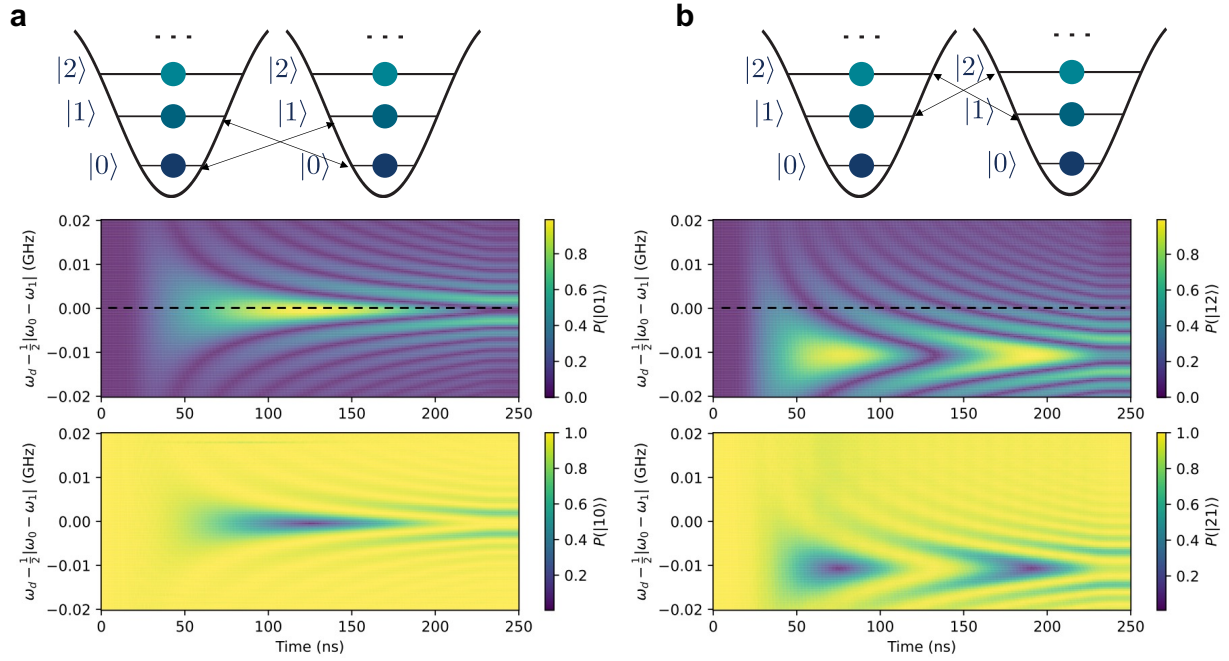


Figure 4.2: **The problem with resonant exchange of two qutrits.** **a** Numerical simulation of a Chevron of parametrically induced $|01\rangle \leftrightarrow |10\rangle$ swap between two statically decoupled transmon qutrits with $|\omega_0 - \omega_1| = 0.485$ GHz and $\Delta\alpha = 0.02$ GHz **b** The population exchange Chevron of the higher levels of the qutrit under the same parametric drive. The reader should note that in the case a parametric iSWAP is induced in the lower qubit manifold, this will lead to off-resonant swaps in the higher levels of the qutrit.

The science of engineering high-fidelity two-qubit gates between superconducting qubits has progressed in leaps and bounds over the last decade, with now numerous examples of

two-qubit gates in the superconducting platform with fidelities in excess of 99.9% fidelity in fluxonia [52, 147, 285] and transmons [145, 6]. Two-qubit gate schemes can be broadly classified into two somewhat distinct classes:

- resonant coupling which leads to coherent population swapping between relevant qubit modes
- off-resonant coupling which leads to state dependent frequency shifts

Often for two-qubit gates, the simplest and most efficient approaches rely on resonant coupling; this is quite sensible as two-qubit gate calibration schemes can then straightforwardly port methods from single qubit gates which also rely on resonant energy exchange. Additionally, resonant approaches saturate the speed limit set by the coupling strength g between the two qubit modes present in the two-qubit system. In the typical setting of a resonant two qubit gate, flux-tunable transmon qubits are brought into resonance, allowed to exchange excitations for a period of time to engineer entanglement, and then brought back to their statically detuned frequencies. It would therefore seem that the most straightforward approach to engineering a high-fidelity two-qutrit would be to attempt to bring two transmon qutrits into resonance in a similar fashion.

Unfortunately, when one wants to encode in the higher levels of the transmon, gate schemes that rely on this resonant exchange coupling become significantly more complicated. The first of these complications is due to bosonic enhancement, which dictates that the swap rates between different modes are not equivalent. This can be seen immediately from the matrix elements that couple these levels:

$$\langle 01 | g(a_1^\dagger a_2 + a_1 a_2^\dagger) | 10 \rangle = g, \quad (4.7)$$

$$\langle 12 | g(a_1^\dagger a_2 + a_1 a_2^\dagger) | 21 \rangle = 2g. \quad (4.8)$$

In the case of two transmons with identical anharmonicities α , this can be largely mitigated as the doubled rate of the $|12\rangle \leftrightarrow |21\rangle$ swap means that when engineering a subspace $|01\rangle \leftrightarrow |10\rangle$ swap, the higher level population will return to its initial value after a full population transfer in the lower qubit manifold (although deviations from the Kerr oscillator approximation will already lead to an error). Unfortunately, the transmon anharmonicity is largely set by the charging energy of the circuit $\alpha \approx -E_C$, and therefore is limited by typical fabrication dispersion and fixed at the time a device is created. Parametric approaches can also be explored, but in the realistic case that the anharmonicities are detuned, as can be seen in the numerics in Fig. 4.2b, this leads to an off-resonant swap at a different rate than double the $|01\rangle \leftrightarrow |10\rangle$ period, which will induce large errors in a two-qutrit gate. One way out would be to tile large differences in anharmonicity, $\Delta\alpha \gg g$ to suppress this off-resonant swapping, but another more straightforward approach is to explore state dependent frequency shifts via off-resonant coupling. As we will show in the rest of this chapter, this turns out to be quite a natural choice for engineering a fully entangling two-qutrit effective Hamiltonian between

statically decoupled qutrits, with no harsh constraints on individual transmon frequencies and anharmonicities. Additionally, it has been shown numerically that gate mechanisms that do not rely on fixed resonant frequencies have better scaling behavior in terms of device yield and frequency collision in scaled up quantum processors [177].

And why do we need CZ?

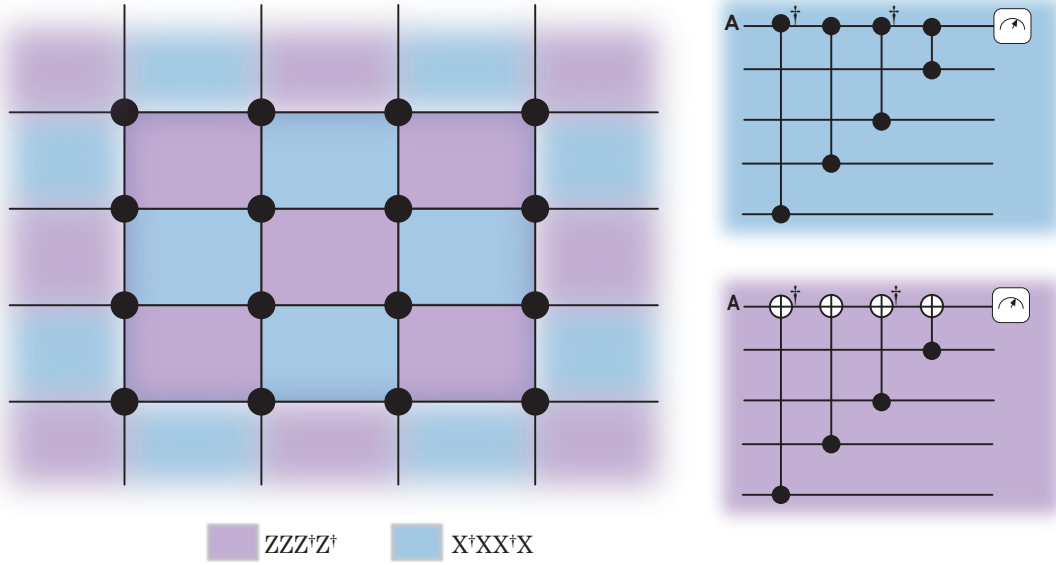


Figure 4.3: **Qudit Surface Code.** (left) A two-dimensional lattice of qudits, with data qudits residing on the edges and ancilla qudits used to measure plaquette (Z -type) and vertex (X -type) stabilizers in the center. Each stabilizer measurement extracts syndrome information without collapsing the logical state, enabling repeated error detection across many code cycles. (right) The syndrome extraction circuits for measuring qudit stabilizers on the ancilla qudits where $\dagger$ represents the Hermitian conjugate of the 2 qudit unitary.

Having established our gate mechanism, we now turn to motivating the specific choice of the CZ gate as a target unitary. No matter how high the fidelity of a quantum gate, some residual error rate will always remain in any realistic physical implementation. The best current error rates across any qubit modality are around 10^{-4} , whereas it is known that this must ultimately improve by around 8 orders of magnitude to execute algorithms such as Shor’s factoring algorithm or quantum chemistry simulations [242]. This enormous chasm represents a seemingly insurmountable obstacle to reliably executing quantum algorithms, and yet there is hope. Quantum error correction (QEC) provides the solution to this challenge: by encoding a single *logical* qudit redundantly across many *physical* qudits, errors

on individual physical qudits can be detected and corrected without disturbing the encoded logical information [88, 87]. The central result in QEC, the so-called threshold theorem, guarantees that arbitrarily long quantum computations are possible provided the physical error rate falls below a code-specific threshold value, with the logical error rate suppressed exponentially as more physical qudits are added [5].

The dominant framework for constructing QEC codes is the *stabilizer formalism*, in which a code is defined as the simultaneous +1 eigenspace of a set of commuting multi-qudit Pauli operators, called stabilizers [88]. Errors are identified by measuring these stabilizer operators without disturbing the encoded logical state and using the resulting error syndrome to infer and correct the underlying fault. This framework generalizes naturally to qudits: the qudit stabilizer formalism employs the Weyl-Heisenberg operators X and Z of dimension d , and admits a straightforward extension of CSS codes, surface codes, and the associated threshold theory to arbitrary prime dimension [87, 23].

Among the most promising stabilizer codes for near-term fault-tolerant quantum computation is the surface code, owing to its high threshold, local stabilizer structure, and compatibility with planar qubit or qudit arrays [73, 23]. As depicted in Fig. 4.3, the qudit surface code is defined on a two-dimensional lattice, with data qudits residing on the edges and ancilla qudits used to measure plaquette (Z -type) and vertex (X -type) stabilizers. Each stabilizer measurement extracts syndrome information without collapsing the logical state, enabling repeated error detection across many code cycles.

Crucially, the syndrome measurement circuits for both qubit and qudit surface codes require entangling two-qudit gates between each ancilla and its neighboring data qudits. Specifically, measuring a Z -type stabilizer of the form $ZZZ^\dagger Z^\dagger$ requires U_{CZ} and $U_{CZ}^\dagger$ gates, while measuring an X -type stabilizer of the form $X^\dagger X X^\dagger X$ is most naturally performed via U_{CSUM} and $U_{CSUM}^\dagger$ gates. As we showed in Fig. 4.1, U_{CZ} and U_{CSUM} are locally equivalent via Hadamard conjugate, and therefore the full family $\{U_{CZ}, U_{CZ}^\dagger, U_{CSUM}, U_{CSUM}^\dagger\}$ are all locally equivalent to one another as U_{CZ} and $U_{CZ}^\dagger$ are related to each other by local rotations as well. Therefore, implementing any one of these gates provides access to all of the entangling operations required for complete syndrome extraction in a qudit stabilizer code. The two-qutrit CZ gate demonstrated in this chapter thus represents a concrete step toward the full syndrome extraction primitive required for fault-tolerant qutrit quantum computation.

4.3 Two-Qutrit Driven Dispersive Coupling

In Sec. 2.4, we described how the bare coupling between two qudits manifests as a ZZ like quantum crosstalk with quadratically many phases in qudit dimension d . The first attempt at realizing a two-qutrit CZ gate in the superconducting platform relied on the static dispersive coupling between the transmons and employed dynamical decoupling to tailor the entangling phases [19]; this represented a significant leap forward in qutrit operation but the gate schematic has significant drawbacks. Chiefly among these drawbacks is that the only way to increase the speed of the two-qutrit gate was to engineer a stronger static coupling

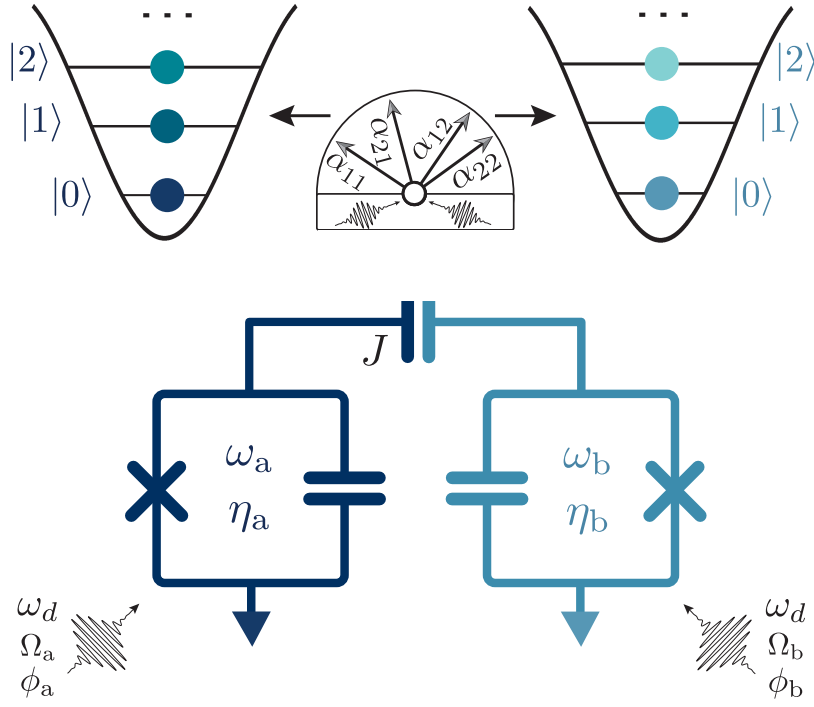


Figure 4.4: **Microwave-activated cross-Kerr entanglement.** Two transmon qutrits with qubit frequency ω_i , anharmonicity η_i , and coupling J , experience a dynamical cross-Kerr (ZZ -like) entanglement when simultaneously driven by an off-resonant microwave drive. The strength of the cross-Kerr entangling terms ($\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$) is tuned by the parameters of the microwave drive (ω_d, Ω, ϕ)

J , which comes at the expense of more quantum crosstalk and therefore lower fidelity single qutrit gates. In this section, we describe how to add in simultaneous off-resonant, or Stark, drives that up to local phases, still map to an effective ZZ -like Hamiltonian, only now those ZZ terms become tunable with the drive conditions (see Fig. 4.4). This gate schematic can critically realize a much larger on-off ratio than the static approach.

Recent works by Refs. [172, 267, 276, 189] demonstrated that the architecture employed in the Cross-resonance (CR) entangling gate can also realize a two-qubit CZ gate by leveraging the conditional Stark shifts from simultaneously driving a pair of coupled qubits off-resonantly. The advantages of this method are two-fold: firstly, the CZ gate commutes with ZZ errors from the always-on dispersive coupling between the transmons. Secondly, unlike in the CR gate, the frequency of the microwave drive can take a range of values. The generalization of conditional Stark shifts to qutrits and its application to tailoring qutrit controlled-phase gates is the subject of the following sections.

Perturbation Theory

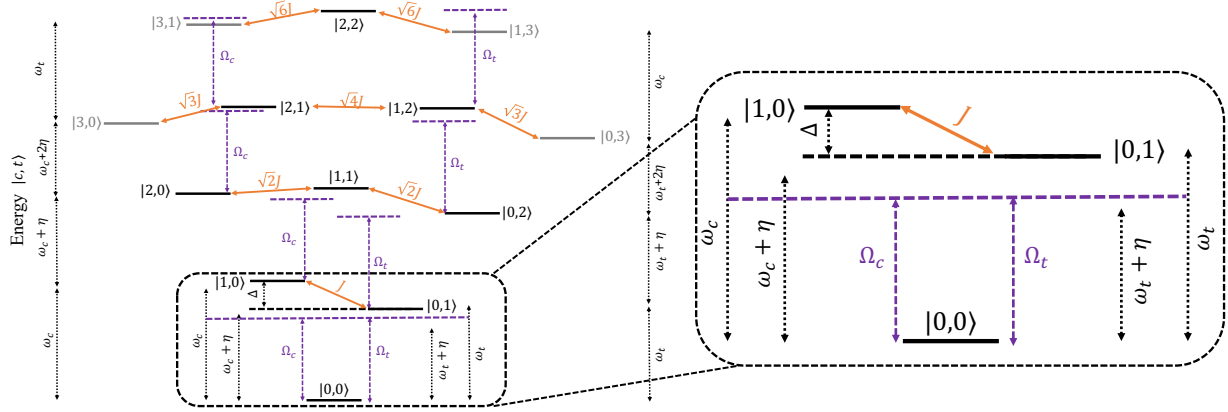


Figure 4.5: **Levels Diagram for Conditional Stark-induced Cross-Kerr Hamiltonian**. We present an example energy level diagram in the rotating frame of the drive where we place the off resonant microwave drive on both qutrits between their two respective $|1\rangle \rightarrow |2\rangle$ transitions. The simultaneous off resonant microwave drives induce conditional Stark shifts that generate entangling phases on four states in our two qutrit Hilbert space.

In the frame of the drive at frequency ω_d and after making a rotating wave approximation (RWA), the system Hamiltonian is

$$H = \sum_{i=c,t} \left[(\omega_i - \omega_d) a_i^\dagger a_i + \frac{\eta_i}{2} a_i^\dagger a_i^\dagger a_i a_i + \Omega_i \left(e^{i\varphi_i} a_i + e^{-i\varphi_i} a_i^\dagger \right) \right] + J \left(a_c^\dagger a_t + a_c a_t^\dagger \right), \quad (4.9)$$

with $\hbar = 1$ and the transmons approximated as Duffing oscillators with qubit frequency ω_i , anharmonicity η_i , capacitive coupling J , and where we define a_i as the bosonic annihilation operator. The parameters of the drive are given as amplitude Ω_i , and phase φ_i . Since only the relative drive phase is physical, we choose a basis where $\varphi_c = 0$ and $\varphi_d = \varphi_t - \varphi_c$. The detuning of transmon i from the drive is $\Delta_i = \omega_i - \omega_d$. We analyze the system perturbatively in the limit $\Omega_i, J \ll |\eta_i|, |\Delta_i|$. In this limit the bare transmon Hamiltonians serve as the unperturbed system, H_0 , and the perturbation, V , is composed of the single qubit drive terms and the coupling term. Time independent perturbation theory applied to Eq. 4.9 will yield energies E_{ij} , the approximate diagonal elements of the Hamiltonian in the basis labelled by the transmon occupation numbers $|ij\rangle$ with $i, j = 0, 1, 2, \dots, d-1$ the state of the “control” (c) and “target” (t) transmons respectively. The dimension where we truncate the Hamiltonian is d . In the present case of qutrits $d = 3$ and $H' \approx \sum_{i,j=0}^2 \tilde{E}_{ij} |ij\rangle \langle ij|$ where we define $\tilde{E}_{ij} = E_{ij} - E_{00}$, performing a global shift of the energies to set the energy of the $|00\rangle$ state to zero.

We isolate the entangling cross-Kerr terms by transforming H' according to the unitary

$$U = \exp \left\{ -it \left[\tilde{E}_{01} I \otimes |1\rangle \langle 1| + \tilde{E}_{10} |1\rangle \langle 1| \otimes I + \tilde{E}_{02} I \otimes |2\rangle \langle 2| + \tilde{E}_{20} |2\rangle \langle 2| \otimes I \right] \right\}, \quad (4.10)$$

where $I = |0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|$ is the single qutrit identity operator. The transformation eliminates the single qutrit energies $\tilde{E}_{i0}$ and $\tilde{E}_{0j}$, $i, j = 1, 2$ which simply result in local phases that can be eliminated by virtual single qutrit phase gates. The transformed Hamiltonian

$$H'' = \alpha_{11} |11\rangle \langle 11| + \alpha_{21} |21\rangle \langle 21| + \alpha_{12} |12\rangle \langle 12| + \alpha_{22} |22\rangle \langle 22|, \quad (4.11)$$

is written in terms of the cross-Kerr rates α_{ij} which describe the rates at which the entangling phases on the states $|ij\rangle$ are accumulated. These α_{ij} , explicitly given by

$$\alpha_{ij} = E_{ij} + E_{00} - E_{0j} - E_{i0}, \quad (4.12)$$

generalize the qubit ZZ rate ζ to the case of qutrits [172].

In perturbation theory we expand $\alpha_{ij} = \alpha_{ij}^{(0)} + \alpha_{ij}^{(1)} + \dots$ where $\alpha_{ij}^{(n)} = E_{ij}^{(n)} + E_{00}^{(n)} - (E_{i0}^{(n)} + E_{0j}^{(n)})$ and $E_{ij}^{(n)}$ is the n th order correction to the energy of state $|ij\rangle$. The first two terms $\alpha_{ij}^{(0)}$ and $\alpha_{ij}^{(1)}$ are zero because in the unperturbed system, H_0 , the transmons are not interacting and the perturbation V only maps states $|ij\rangle$ to states $|i'j'\rangle$ that are orthogonal to $|ij\rangle$. The first contribution comes at second order where the static coupling of strength J between the transmons generates a static cross-Kerr interaction with rates

$$\alpha_{11}^{(2)} = \frac{4\eta J^2}{(\eta - \Delta)(\eta + \Delta)} \quad (4.13)$$

$$\alpha_{21}^{(2)} = \frac{2\eta J^2(5\eta - 4\Delta)}{\Delta(\eta - \Delta)(2\eta - \Delta)} \quad (4.14)$$

$$\alpha_{12}^{(2)} = -\frac{2\eta J^2(5\eta + 4\Delta)}{\Delta(\eta + \Delta)(2\eta + \Delta)} \quad (4.15)$$

$$\alpha_{22}^{(2)} = \frac{16\eta J^2}{(\eta + \Delta)(\eta - \Delta)} \quad (4.16)$$

where $\Delta = \omega_c - \omega_t$ and we have set $\eta_c = \eta_t = \eta$ in order to arrive at more compact expressions. In most systems this is a reasonable approximation and in particular the pair of transmons used in this work have anharmonicities $\eta_c = 272$ MHz and $\eta_t = 270$ MHz.

As in the qubit case the driven cross-Kerr interaction contributes starting at third order when both transmons are driven (see Appendix A) and we find cross-Kerr contributions with both the static and driven contributions $\alpha_{ij} \approx \alpha_{ij}^{(2)} + \alpha_{ij}^d$, where

$$\alpha_{11}^d = \frac{8 \cos \phi \eta^2 \Omega_C \Omega_T J}{\Delta_C \Delta_T (\eta - \Delta_C)(\eta - \Delta_T)} \quad (4.17)$$

$$\alpha_{12}^d / \alpha_{11}^d = \frac{\Delta_C^2 (\eta - 2\Delta_T) + \eta (2\eta^2 - 2\eta \Delta_T - \Delta_T^2) + \Delta_C (-3\eta^2 + 2\eta \Delta_T + 2\Delta_T^2)}{(2\eta - \Delta_C)(2\eta - \Delta_T)} \quad (4.18)$$

$$\alpha_{21}^d/\alpha_{11}^d = \frac{\Delta_T^2(\eta - 2\Delta_C) + \eta(2\eta^2 - 2\eta\Delta_C - \Delta_C^2) + \Delta_T(-3\eta^2 + 2\eta\Delta_C + 2\Delta_C^2)}{(2\eta - \Delta_C)(2\eta - \Delta_T)} \quad (4.19)$$

$$\alpha_{22}^d/\alpha_{11}^d = \frac{(\eta - 2\Delta_C)(\eta - 2\Delta_T)}{(2\eta - \Delta_C)(2\eta - \Delta_T)} \quad (4.20)$$

The expressions are complicated but we now point out some important features. In typical systems the drive strengths Ω_i are larger than the coupling strength J and as a result $\alpha_{ij}^{(3)} > \alpha_{ij}^{(2)}$. For example, the CZ gate described in the following section is performed with $\Omega_c, \Omega_t \approx 11$ MHz on a pair of transmons with estimated coupling $J = 2.7$ MHz. Therefore, the static cross-Kerr can in principle be canceled by the driven cross-Kerr. The result also gives the leading order linear dependence of the rates α_{ij} on the drive strengths Ω_i and the coupling J as well as their sinusoidal dependence on the relative drive phase φ . In the next section, we will show how we experimentally verify this perturbative model for our effective Hamiltonian.

4.4 Characterizing the Driven Qutrit Cross-Kerr

In the last section, we found that according to our perturbative model, the α_{ij} terms in the driven cross-Kerr Hamiltonian take the form,

$$\alpha_{ij} = A_{ij}(\omega_d)\Omega_a\Omega_b\cos(\phi_a - \phi_b), \quad (4.21)$$

where Ω_i and ϕ_i are respectively the amplitude and phase of the drive on transmon i . The coefficients A_{ij} are functions of the proximity of the microwave drive frequency (ω_d) to nearby transitions. We note that this Hamiltonian generates entanglement between the entire two-qutrit Hilbert space, contrary to the CR case, where the entanglement is mostly restrained to a subspace of the qutrit [19, 37]. It is important to note that while this effective Hamiltonian natively generates qutrit controlled-phase unitaries, the number of degrees of freedom in the interaction are not generally sufficient to realize a Clifford two-qutrit gate like the CZ gate with a single round of cross-Kerr entanglement, a difficulty discussed in further detail in the next section.

Measuring the ZZ interaction in the qubit case can be performed by a conditional Ramsey experiment or through a dynamically decoupled JAZZ (Joint-Amplification-of- ZZ) sequence that removes the low frequency drift [245, 78]. In the larger Hilbert space of two qutrits, we need to measure four of these entangling phases with a rate of accumulation set by α_{ij} in Eq. 4.21. To simplify the measurement and reduce the number of experiments needed, we generalize the controlled-Ramsey experiment to the full qutrit space with a pulse sequence presented in Fig. 4.6a. In this sequence, we apply simultaneous ternary Hadamard gates on both qutrits, execute the microwave drive, and subsequently perform the full two qutrit state tomography. Doing so for several durations of the Stark driving allow us to fully characterize the conditional and unconditional Stark shifts.

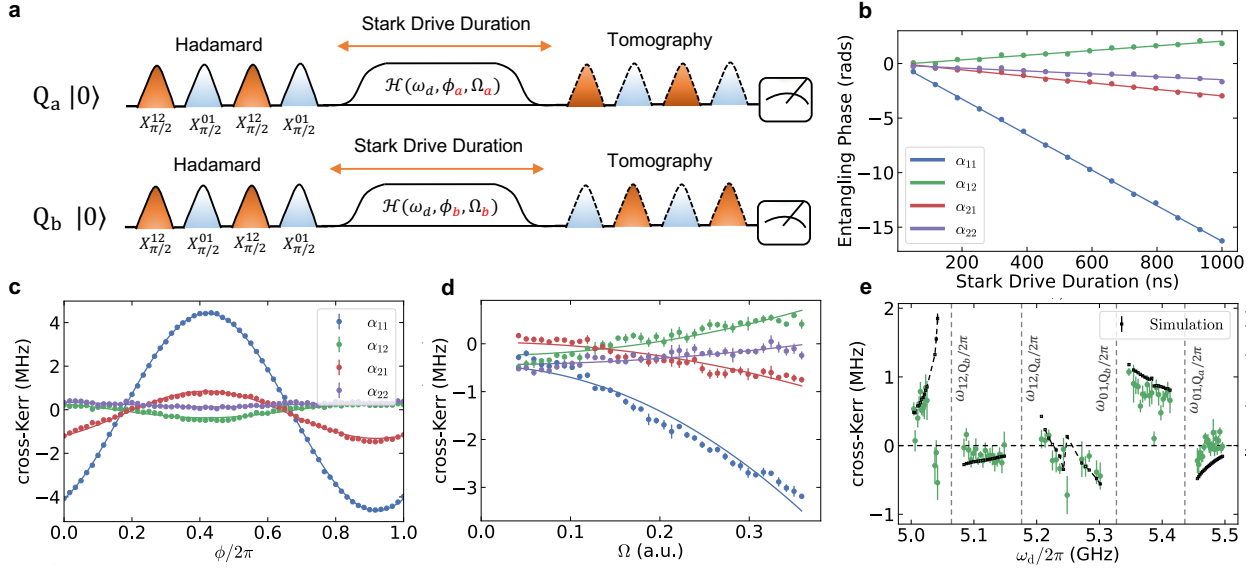


Figure 4.6: **Characterizing the dynamical cross-Kerr entanglement.** **a**, To study the accumulation of entangling phases under the driven cross-Kerr interaction, we place two qutrits in a full superposition using ternary Hadamard gates (virtual Z gates omitted in diagram), then study the evolution under the Stark drive scheme by performing state tomography. **b**, We demonstrate fitting the accumulation of entangling phase found by tomography to our linear, driven cross-Kerr model, where α_{ij} is the slope of the line and the uncertainty is from the linear fit. **c-d**, We match the behavior of the cross-Kerr entanglement given relevant experimental parameters in our system to our Hamiltonian model for the relative phase of the driving, ϕ , and amplitude of the driving, fixing $\Omega = \Omega_a = \Omega_b$. **e**, We additionally compare the dependence of α_{12} on the frequency of the drive ω_d using an *ab-initio* master equation simulation in QuTiP [123, 122].

We demonstrate in Fig. 4.6b the result of this measurement scheme: the entangling phases increase linearly with the duration of the Stark drive, where the proportionality constant is set by α_{ij} as predicted by our cross-Kerr model in Eq. 4.11. In Fig. 4.6c-d, we present how the driven cross-Kerr interaction depends on the parameters of our entanglement scheme, specifically the phase and the amplitude of the Stark drive. We note that the qualitative behavior is properly captured by the perturbation theory in Eq. 4.21; all four cross-Kerr terms display a sinusoidal dependence on the relative phase of the two drives ϕ , and a quadratic dependence on the drive strength $\Omega = \Omega_a = \Omega_b$. We also explore the behavior as a function of the drive frequency in Fig. 4.6e. In this case, the perturbation theory fails, but an *ab-initio* master equation simulation captures some of the response. We expect the unaccounted features can be attributed to an inaccurate characterization of the transfer function of the chip, higher-order terms, frequency dependent classical crosstalk, or parasitic two level systems

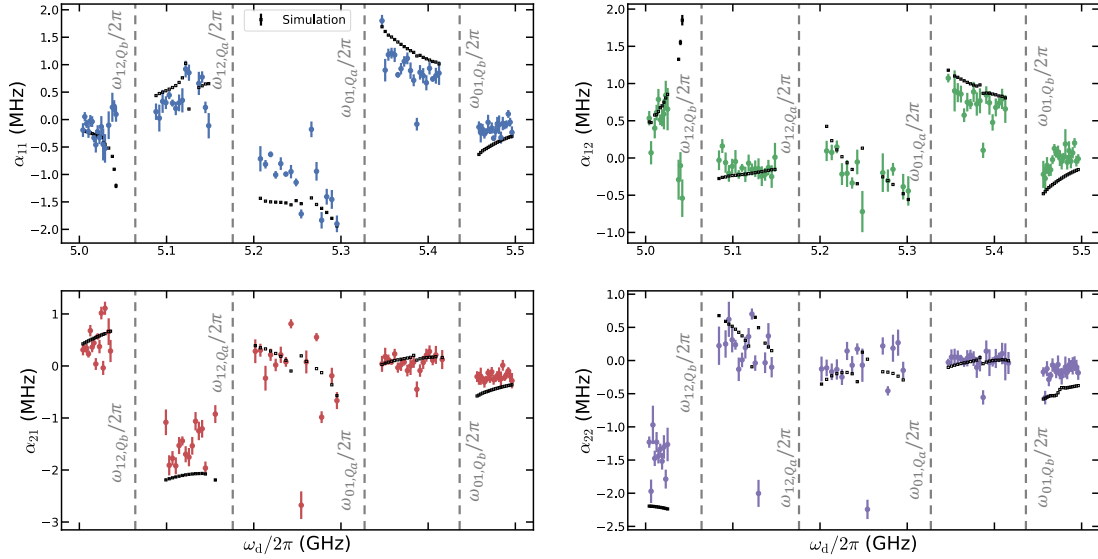


Figure 4.7: **Frequency dependence of the full microwave-activated cross-Kerr Hamiltonian.** We compare the dependence of all of the α_{ij} in the driven cross-Kerr Hamiltonian on the frequency of the drive ω_d using an *ab-initio* master equation simulation in QuTiP.

(TLS) in our device. The full cross-Kerr frequency dependent drive behavior can be seen in Fig. 4.7. In an experimental setting, the flexibility of this entanglement allows us the freedom to set the drive frequency far from any particularly deleterious features such as TLS or multi-photon resonances.

4.5 Tuning up a Two-Quitrit CZ Gate

We next construct quitrit controlled-phase gates utilizing this entangling interaction. Under simultaneous Stark drives, the two-quitrit Hilbert space follows the unitary evolution $U = \exp(-i(\mathcal{H} + \phi_1 I \otimes Z^{01} + \phi_2 I \otimes Z^{12} + \phi_3 Z^{01} \otimes I + \phi_4 Z^{12} \otimes I)\tau)$ where $\mathcal{H}$ is given in Eq. 4.11. To perform a CZ gate with a single round of cross-Kerr driving, for example, one would need to find driving parameters meeting the conditions: $\{\alpha_{11} = \alpha_{22} = 2\alpha_{21} = 2\alpha_{12}\}$, a task that is not broadly feasible, at least in the monochromatic approach presented in this chapter. Practically speaking, we desire a compromise between the most general and robust gate scheme and this “fine-tuned” approach, while still taking advantage of the dynamical nature of our cross-Kerr interaction. By employing the pulse scheme in Fig. 4.8, where echo pulses in the $\{|1\rangle, |2\rangle\}$ subspace shuffle entangling phases, we have the modified unitary evolution

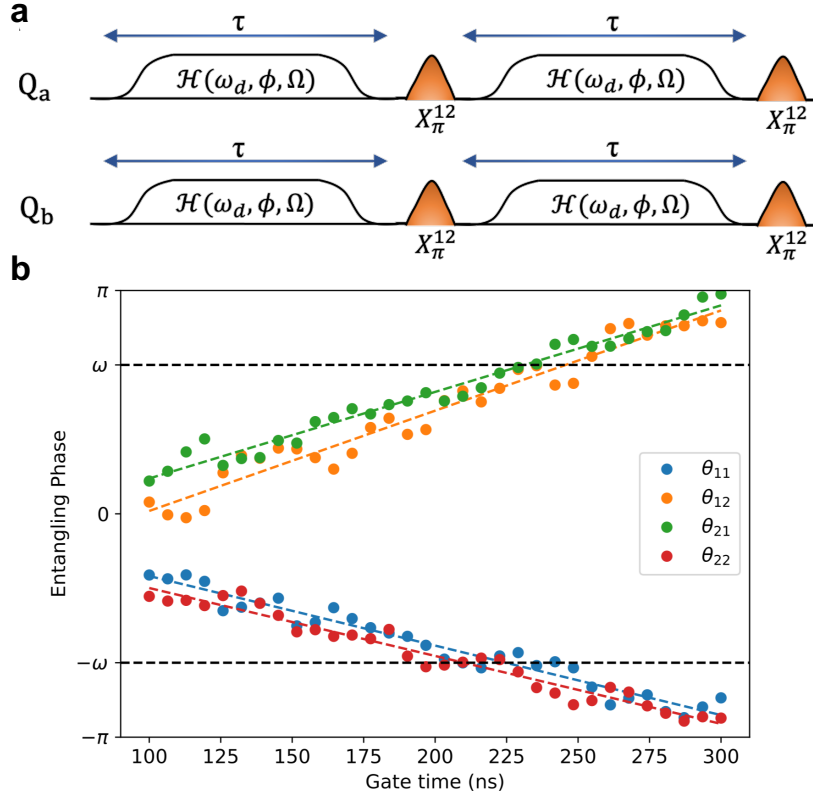


Figure 4.8: **Gate schematic.** **a**, For the CZ and $CZ^\dagger$ gate, we perform two rounds of cross-Kerr entanglement for duration τ with interleaved echo pulses in the $\{|1\rangle, |2\rangle\}$ subspace which shuffle the entangling phases. For proper conditions on the α_{ij} terms in Eq. 4.11, the $CZ^\dagger(CZ)$ is compiled with a total gate time of 580(783) ns. The local Z terms in both two level subspaces of the qutrit are then undone using virtual Z gates. **b**, Time evolution of conditional phases when the conditions on the cross-Kerr terms for a $U_{CZ}^\dagger$ gate are approximately met. Here gate time refers to τ in the pulse diagram.

(omitting the single-qutrit phases for brevity):

$$U = \exp(-i[(\alpha_{11} + \alpha_{22})\tau(|11\rangle\langle 11| + |22\rangle\langle 22|) + (\alpha_{12} + \alpha_{21})\tau(|12\rangle\langle 12| + |21\rangle\langle 21|)]), \quad (4.22)$$

Using the experimental knobs demonstrated in Fig. 4.6, we are able to satisfy approximate conditions on our cross-Kerr evolution that are ideal for compactly generating qutrit controlled phase gates. Specifically, we find drive parameters that satisfy $(\alpha_{11} + \alpha_{22}) \approx -(\alpha_{12} + \alpha_{21})$ for performing the CZ gate and $(\alpha_{11} + \alpha_{22}) \approx 2(\alpha_{12} + \alpha_{21})$ for performing the $CZ^\dagger$ gate. The Stark drive parameters that meet these conditions are provided in Supple-

mentary Note 1. Under these conditions on the cross-Kerr, and with the unitary evolution provided by our gate scheme in Eq. 4.22, at some drive time τ , we will have approximately acquired the desired entangling phases to synthesize respectively a CZ or $\text{CZ}^\dagger$ unitary. To ensure adiabaticity and limit leakage, we perform the Stark drive via flat-top cosine pulses with ramp up and down features. This ramp leads to effective offsets on the accumulation of the entangling phases, as we only expect the linear accumulation of entangling phase given by α_{ij} in Eq. 4.11 to correspond to the flat-top of the Stark drive. When tuning up the two-qutrit gates, this means that we first perform parameters sweeps to find regions where the previously mentioned conditions on the α_{ij} s are met, then perform the actual pulse scheme in Fig. 4.8 and adjust the Stark drive parameters until the target entangling phases are most accurately acquired. For an example of a time sweep of the Stark drives in such a region, see Fig. 4.8b. As shown in Fig. 4.9, fine grained inspection of conditional phase errors on both gate-involved and spectator qutrits can also be performed using modified JAZZ-like circuits. These circuits act as useful diagnostic tools to improve the calibration of the gates by adjusting drive parameters and adding additional cross-Kerr cancelling drives on spectator qutrits. Finally, one can undo the local Z phases (found via tomography) in both the $\{|0\rangle, |1\rangle\}$ and $\{|1\rangle, |2\rangle\}$ subspaces with virtual Z gates [165]. In this chapter, we demonstrate the CZ and $\text{CZ}^\dagger$ on two different pairs of transmon qutrits, demonstrating the flexible nature of generating two-qutrit gates from this driven cross-Kerr scheme.

4.6 Gate Benchmarking

Device Parameters

Both gates and cross-Kerr characterization data was taken using an 8 transmon ring with fixed-frequency transmons and fixed-frequency coupling mediated by a coplanar waveguide resonator (as described in detail in Chapter 2). The CZ and $\text{CZ}^\dagger$ gate are performed on two different chips, using two different pairs of transmon qutrits. We give the relevant single-qutrit parameters and coherences for the CZ and $\text{CZ}^\dagger$ gate in Table 4.1 and Table 4.2 respectively. Here, T_{2r} denotes coherence statistics taken using a Ramsey experiment and T_{2e} using an echo pulse. We additionally provide the approximate gate parameters for the two gates performed in the work, where ω_d denotes the drive frequency, Ω denotes the approximate drive strength, and τ_g denotes the total gate time including the π pulses in the $\{|1\rangle |2\rangle\}$ subspace.

Summary of Benchmarking Results

We first provide an overview of the benchmarking results of our two-qutrit gates before describing the details of the benchmarking routines more thoroughly in the following sections. The first benchmark we employ for our two-qutrit gates is cycle benchmarking (CB) [62]. While originally written in terms of qubits, CB naturally generalizes to qutrits [176]. We

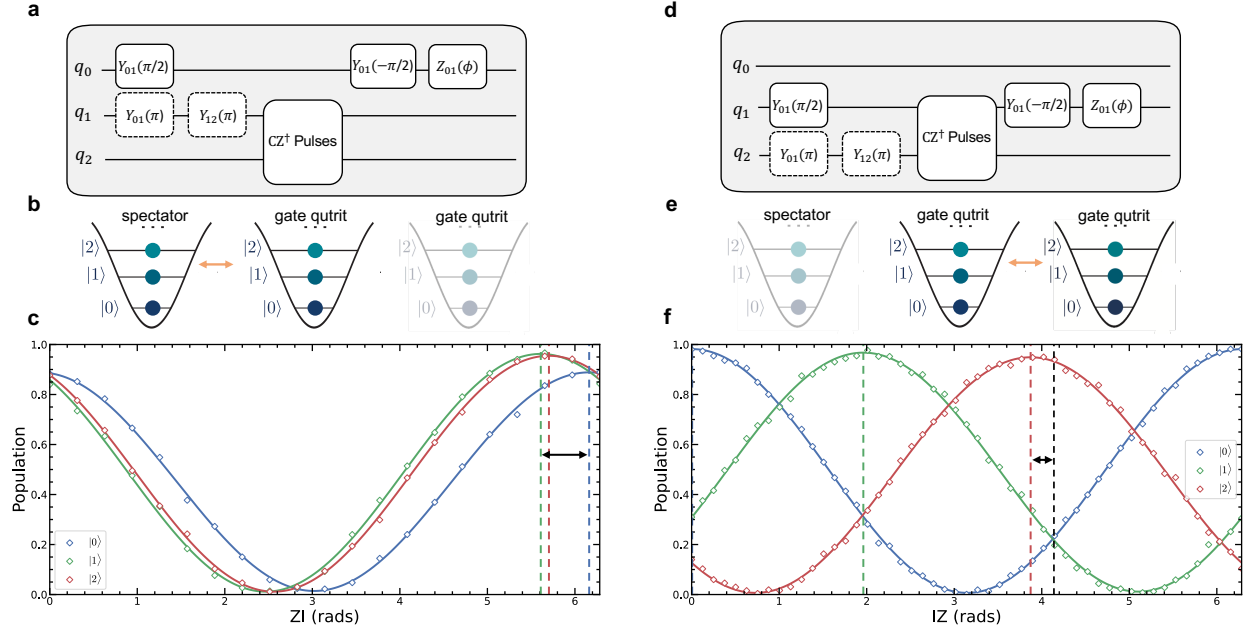


Figure 4.9: **Entangling Phase Characterization.** (a) The quantum circuit for characterizing the entangling phase (in a single subspace of the qutrit) between the spectator and gate qutrit given different control states of the gate qutrit. (b) A schematic drawing of the qutrits in the system and their involvement in the experiment. (c) The characterized entangling phase given different control states of the gate qutrit. Note the spread in phase of the sinusoids demarcated by the black arrow visualizes the coherent error present. (d) The quantum circuit for characterizing the entangling phase (in a single subspace of the qutrit) between the two gate qutrits. (e) A schematic drawing of the qutrits in the system. (f) The characterized entangling phase between the two gate qutrits. Note the uneven spacing in the phase of the sinusoids demarcated by the black arrow visualizes the coherent error present.

use CB instead of, e.g., interleaved randomized benchmarking [157, 158], because it requires significantly fewer multi-qutrit gates per circuit. We describe the generalization in the supplementary material. With this technique, we estimate the Weyl (generalized-Pauli) error rate of the $CZ^\dagger$ and CZ gate to be 2.7(1)% and 4.8(3)% respectively. By contrast, the highest fidelity, two-qutrit gate performed previously with transmons had an error rate of 11.1% [19]. CB also allows us to construct the Weyl-twirled error per channel [176] of the unitary in Fig. 4.10. This provides us with an estimate of the worst case scenario of less than 8% and demonstrates a relatively low dispersion of our error channels.

As an added confirmation of the fidelity of the $CZ^\dagger$ gate, we generalize the cross-entropy benchmarking (XEB) routine [20, 178] to tailor all gate errors into a depolarizing channel, for work with qutrit unitaries. In the qutrit case, we find sufficient tailoring of our noise

	Q3	Q4
Qubit freq. (GHz)	5.436	5.327
Anharm. (MHz)	-260.20	-262.94
T_1^{01} (μ s)	125(37)	78(16)
T_1^{12} (μ s)	63(9)	47(5)
T_{2e}^{01} (μ s)	190(28)	138(25)
T_{2e}^{12} (μ s)	61(13)	45(7)
T_{2e}^{02} (μ s)	75(19)	62(6)
T_{2r}^{01} (μ s)	114(47)	99(24)
T_{2r}^{12} (μ s)	17(8)	17(9)
T_{2r}^{02} (μ s)	20(16)	21(9)
ω_d (GHz)	5.287	5.287
Ω (MHz)	≈ 13	≈ 13
τ_g (ns)	783	

Table 4.1: Single-qutrit parameters for the pair of transmons used to perform the CZ gate.

	Q5	Q6
Qubit freq. (GHz)	5.362	5.523
Anharm. (MHz)	-275	-271.35
T_1^{01} (μ s)	45(7)	58(7)
T_1^{12} (μ s)	33(3)	28(3)
T_{2e}^{01} (μ s)	63(7)	84(6)
T_{2e}^{12} (μ s)	28(3)	30(3)
T_{2e}^{02} (μ s)	37(3)	35(3)
T_{2r}^{01} (μ s)	36(9)	76(8)
T_{2r}^{12} (μ s)	10(6)	18(6)
T_{2r}^{02} (μ s)	11(6)	21(8)
ω_d (GHz)	5.191	5.191
Ω (MHz)	≈ 11	≈ 11
τ_g (ns)	580	

 Table 4.2: Single-qutrit parameters for the pair of transmons used to perform the $CZ^\dagger$ gate.

can be performed by interleaving random $SU(3)$ gates around our target gate. The circuit diagram for qutrit XEB is in Fig. 4.10b and the results can be found in Fig. 4.10e. We find that the depolarized fidelity of the $CZ^\dagger$ dressed with random $SU(3)$ gates agrees with the estimate of the process fidelity from our Weyl twirled CB results within a standard error. Additional discussion of the qutrit XEB method is provided in the supplementary materials.

Finally, we would like to be able to characterize what fraction of the errors present in our two qutrit unitaries are coherent on the time scale of multiple experiments, and thus could be removed by improved calibration. As we show in supplementary note 6, under the depolarizing unitary noise model, the variance of CB and XEB circuits both provide a robust method of estimating the purity limit [261, 10]. The corresponding estimates are shown in Fig. 4.10c with the CB estimate of 97.3(9)% exceeding the speckle-purity limit of 96.1(3)% for the dressed $CZ^\dagger$ gate. This disagreement is likely due to the fact that the CB data reveals that the noise is dominated by single-qutrit phase errors. As these errors are likely to fluctuate around a mean, they will add dephasing errors that are not captured by

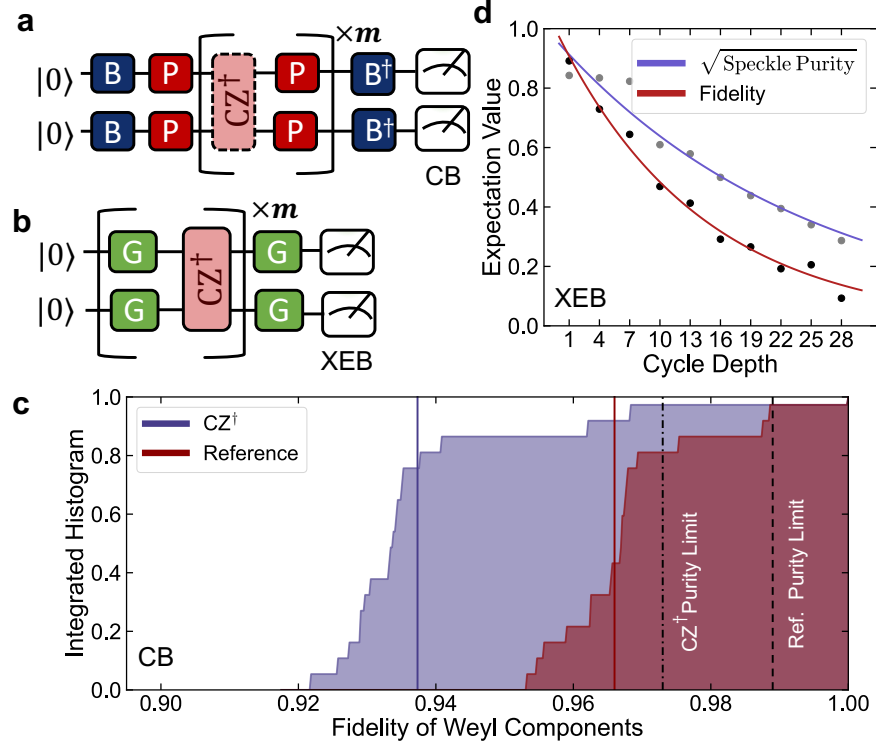


Figure 4.10: **Benchmarking of Qutrit $CZ^\dagger$.** **a**, Circuit schematic of cycle benchmarking (CB). The errors of the $CZ^\dagger$ are twirled via random Weyl gates (red) to tailor errors into stochastic Weyl channels. The initial state and measurement basis (blue) are selected to pick out the decay associated with specific Weyl channels. **b**, Circuit schematic of cross-entropy benchmarking (XEB). The errors of the $CZ^\dagger$ are twirled via random $SU(3)$ gates (green) to tailor the noise to a simple depolarizing channel. **c**, An integrated histogram of CB for both the $CZ^\dagger$ gate and a reference cycle, with the solid vertical lines giving the fidelities 0.936(1) and 0.966(1) respectively, yielding an estimated process fidelity of 97.3(1)%. We extract an error budget directly from CB, estimating a purity limited fidelity of 0.973(9) and 0.989 (with negligible error) for the dressed $CZ^\dagger$ and reference cycles, yielding a purity limit 0.986(9) for the isolated $CZ^\dagger$ gate. **d**, From XEB we estimate the depolarized fidelity of the dressed cycle as 0.933(3). Additionally, we estimate the speckle-purity limited fidelity of the $CZ^\dagger$ dressed with random $SU(3)$ gates to be 0.961(3).

the depolarizing unitary model used to analyze XEB. Another possible contributing factor is that the noise fluctuated between the XEB and CB experiments, which were performed in separate batches.

Cycle Benchmarking

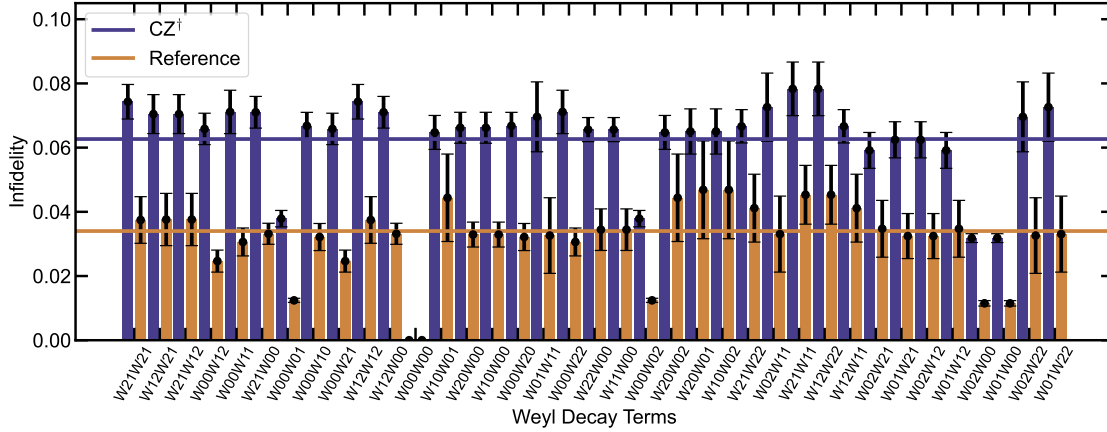


Figure 4.11: **Cycle benchmarking results with Weyl decay terms of $CZ^\dagger$ gate.** The cycle benchmarking results shown analyze a subset of 53 Weyl channels at depths $m \in \{0, 3, 6, 15\}$. In order to provide a SPAM free estimation of the error, we compare our CB results to a reference CB experiment without the two qutrit gate to isolate and subtract the error due to the Weyl twirling itself. Here we plot only the shared decay channels analyzed for both the $CZ^\dagger$ and reference CB experiments.

Cycle Benchmarking (CB) is useful for qudit-based QCVV, specifically in the context of multi-qudit gates, as performing randomized benchmarking on a multi-qudit gate requires sampling and decomposing multi-qudit Clifford gates, often requiring many native multi-qudit gates. In contrast, CB can be performed with significantly fewer multi-qudit gates per circuit and can be scaled to studying larger systems. For qudit CB, the eigenstates and twirling gates are chosen from the Weyl-Heisenberg group. Notably, each $W \in W_n$ commutes with its Hermitian conjugate $W^\dagger = W^{d-1}$, which also belongs to the Weyl-Heisenberg group. This implies that these operators share the same eigenbasis, such that $\langle W^\dagger \rangle = \langle \bar{W} \rangle$ which allow us to reduce the number of required measurements needed to characterize the Weyl decays.

For Markovian noise, the expectation values for the CB protocol become factorized in terms of the initial Weyl basis state chosen, and from this one can extract an estimate of both the Pauli noise model (see Fig. 4.11) and the purity limited fidelity (see additional details in the supplementary of Ref. [85]). Another particularly useful aspect of the CB protocol is one can easily include idling spectator qudits into the cycle being benchmarked, which can provide very useful insight into spectator errors during gate operations.

Cross-Entropy Benchmarking

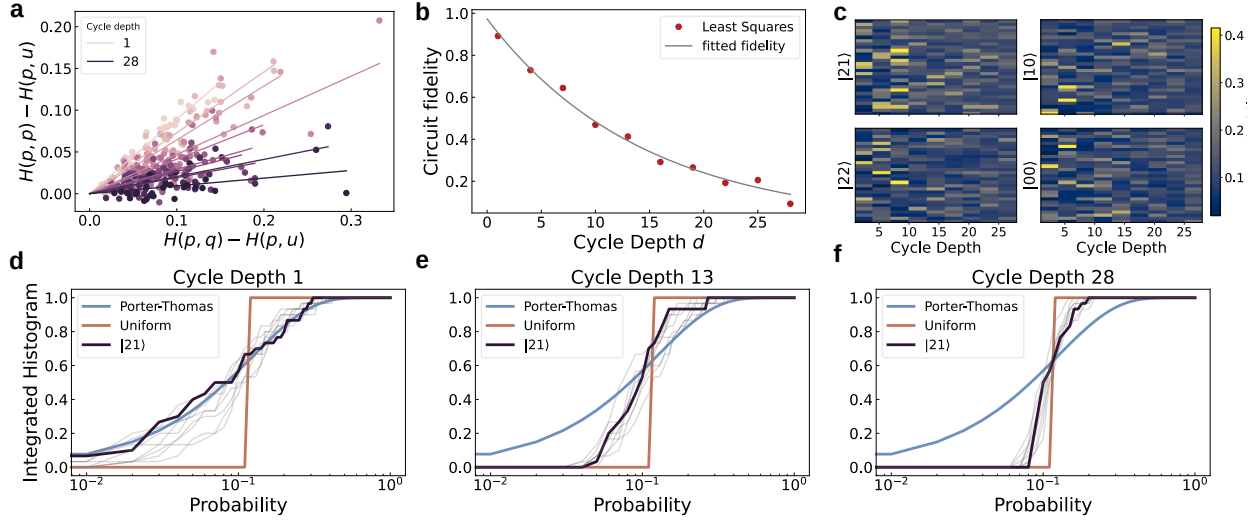


Figure 4.12: **Cross entropy benchmarking (XEB) for qutrits**, **a**, For the XEB circuits, the entropy difference $H(p, q) - H(p, u)$ is plotted against the ideal $H(p, p) - H(p, u)$ at each cycle depth, with 30 randomizations at each depth. The linear fit gives the fidelity at that particular depth. **b**, We show the exponential decay of the depolarized $\text{CZ}^\dagger$ fidelity obtained from the linear fits in **a**. **c**, We plot an example of the speckle purity decay as a function of cycle depth for a representative set of four states in the two-qutrit Hilbert space. The probabilities of measuring the given tritstring are shown as a function of cycle depth across all 30 randomizations. The bright “speckle” pattern characteristic of the Porter-Thomas distribution at low circuit depths is smoothed out at larger depths. **d-f**, We show that the distribution of tritstrings transitions from the Porter-Thomas distribution at low depth, to a uniform distribution at deeper depths. The CDF is emphasized for a representative tritstring, $|21\rangle$, while the CDFs of the other tritstrings are shown in grey to show typical variation of the CDF across different states.

As discussed in the previous section, performing randomized benchmarking (RB) on a two qutrit gate is not ideal as it can require many CZ gates to compile an arbitrary $\text{SU}(9)$ Clifford gate [176]. We utilize cross entropy benchmarking (XEB) as a second SPAM (state preparation and measurement) free benchmarking protocol to corroborate the fidelity obtained from cycle benchmarking (CB). In 2019 XEB played a central role in the quantum supremacy experiment in [10] and was used at Berkeley to benchmark the non-Clifford, three qubit i Toffoli gate in [133]. In the context of qudits, XEB may prove particularly useful because unlike the majority of the randomized benchmarking suite of protocols, it does not rely on the two-qudit gate being a Clifford which, as has been seen in this chapter, is generally a more difficult condition for an experimentalist to realize. XEB theory has

been discussed in detail in previous works so we present only a brief review of the method before outlining more explicitly how it can be used to benchmark the two-qutrit CZ gate demonstrated here.

For XEB purposes a convenient definition of a random quantum circuit (RQC) is a quantum circuit randomly selected from an ensemble of circuits such that the distribution of probabilities (across the ensemble of circuits) of observing a particular ditstring, x , follows the Porter-Thomas distribution (for all possible ditstrings x) [178]. In the present case of quantum circuits involving two qutrits we consider the probabilities of tritstrings 00, 01, $\dots$, 21, 22. After averaging over an ensemble of random circuits errors are tailored to be purely depolarizing so an error corresponds to the circuit outputting a fully mixed state. In a mixed state the tritstring distribution is uniform with each outcome x_i having equal probability of $1/3^n$ for an n qutrit system. Thus, intuitively the circuit fidelity (under this error model) can be thought of as the deviation of the measured tritstring distribution from the uniform distribution. The XEB fidelity makes this relationship precise as we now outline.

Denote the possible tritstrings x_i for $i = 1, \dots, 3^n$ and let $p(x_i)$ be the ideal tritstring distribution for the output of a particular quantum circuit. Then $q(x_i)$ is the measured distribution. The linear cross entropy of two probability distributions $p_1(x)$ and $p_2(x)$ with the same support is defined as

$$H(p_1, p_2) = \sum_x p_1(x)p_2(x), \quad (4.23)$$

where the sum runs over the full support of the probability distributions and the self entropy is written compactly as $H(p_1) \equiv H(p_1, p_1)$. Under the depolarizing error model it can be shown straightforwardly that the circuit fidelity is

$$F_{\text{XEB}} = \frac{H(p, q) - H(p, u)}{H(p, p) - H(p, u)}, \quad (4.24)$$

where $u(x_i)$ is the uniform probability distribution [10]. This is ultimately the difference in the ideal to measured and ideal to uniform cross entropies, normalized by the difference if the measured distribution were to perfectly match the ideal distribution ($p(x_i) = q(x_i)$ for all i).

We follow the XEB protocol for benchmarking gates/cycles outlined in Ref. [10]. This consists of measuring two qutrit circuits of varying cycle depths M where cycle i consists of two single qutrit gates $G_{i,j}$ where $i = 1, \dots, M$ labels the cycle and $j = 1, 2$ labels the qutrit, followed by the two qutrit CZ gate[†] (see Figure 4.12a). The single qutrit gates are randomly selected unitaries from $SU(3)$ and decomposed according to the decomposition given in [53]. For each cycle depth M we generate N random circuits to be run and compute the ideal probability distributions of the output tritstrings for each circuit. The XEB fidelity at cycle depth M , $F_{\text{XEB},M}$ is determined by performing a least squares fit of the linear relationship between $H(p_i, q_i) - H(p_i, u)$ and $H(p_i, p_i) - H(p_i, u)$ with $i = 1, \dots, N$ labeling the random circuit at the given depth (see Figure). The cycle infidelity, ϵ_{cycle} , is estimated from the

exponential decay of $F_{\text{XEB},M}$ as a function of cycle depth M . The error rate extracted in this way agrees well with CB (see Benchmarking section in the main text).

Next we validate our implementation of XEB by verifying that the distribution of probabilities of a given tritstring across the ensemble of random circuits does indeed approach the Porter-Thomas distribution,

$$\mathcal{P}(p) = (D-1)(1-p)^{D-2} \approx De^{-Dp} \quad (4.25)$$

where again D is the Hilbert space dimension, $D = 3^2$ for two qutrits and the approximate equality holds in the limit of large Hilbert space dimension, $D \gg 1$. Next we consider the evolution of this distribution as a function of cycle depth. At short to intermediate cycle depths, the circuit is sufficiently random and the distribution approaches Porter-Thomas. At larger cycle depths, the distribution begins to converge to $\mathcal{P}(p) \rightarrow \delta(p - 1/D)$ since the depolarizing errors dominate and the tritstring distributions approach the uniform distribution for all circuits. We plot our experimental observation of this behavior in Fig. 4.12d-f.

The method of Speckle Purity Benchmarking (SPB) is based on this observation. Denoting the state purity as γ , it can be estimated at depth M from the raw results of the XEB protocol outlined above by the relationship

$$\gamma(M) = \text{Var}(p_M) \frac{D^2(D+1)}{(D-1)}, \quad (4.26)$$

where p_M is the set of measured probabilities of a given tritstring x across the N random circuits at cycle depth M [10]. Thus, once we have demonstrated that the distribution does indeed converge to the Porter-Thomas distribution we can estimate the decay of the state purity from just the variance of the distribution, without the need for state tomography which requires an exponential number of measurements, $3^{n(d-1)}$ where n is the number of qudits and d the dimension of each qudit. For our present case of two qutrits we would need to perform 81 measurements per circuit to determine the state purity by state tomography.

Process Tomography

We analyze the Pauli transfer matrix (PTM) of the $\text{CZ}^\dagger$ gate in the Gell-Mann basis. To construct the PTM in this basis, we prepare 81 two-qutrit input states by applying an informationally complete set of native gates on each qutrit:

$$\{I, X_{\pi/2}^{(01)}, Y_{\pi/2}^{(01)}, X_{\pi}^{(01)}, X_{\pi}^{(12)}X_{\pi}^{(01)}, Y_{\pi}^{(12)}X_{\pi}^{(01)}, X_{\pi/2}^{(12)}X_{\pi}^{(01)}, Y_{\pi/2}^{(12)}X_{\pi}^{(01)}, X_{\pi}^{(12)}X_{\pi/2}^{(01)}\}. \quad (4.27)$$

We measure the output tomography using the same set of native gates, and reconstruct the PTM from the data using the maximum likelihood estimation method. Figure 4.13 shows the reconstructed PTM of the $\text{CZ}^\dagger$ gate $\mathcal{E}_{\text{exp}}$ in the Gell-Mann basis, and in Figure 4.14 we plot $\mathcal{E}_{\text{ideal}}^\dagger \mathcal{E}_{\text{exp}}$. The process fidelity is calculated from the PTM to be $\mathcal{F}_{\text{PTM}} = \text{Tr}[\mathcal{E}_{\text{ideal}}^\dagger \mathcal{E}_{\text{exp}}]/D^2 = 93.2\%$. We note that as process tomography does not decouple our characterization of our gate errors from state preparation and measurement errors, it is disfavored as a benchmarking technique when compared to CB or XEB; this also can explain the discrepancy to our other methods for benchmarking the gate.

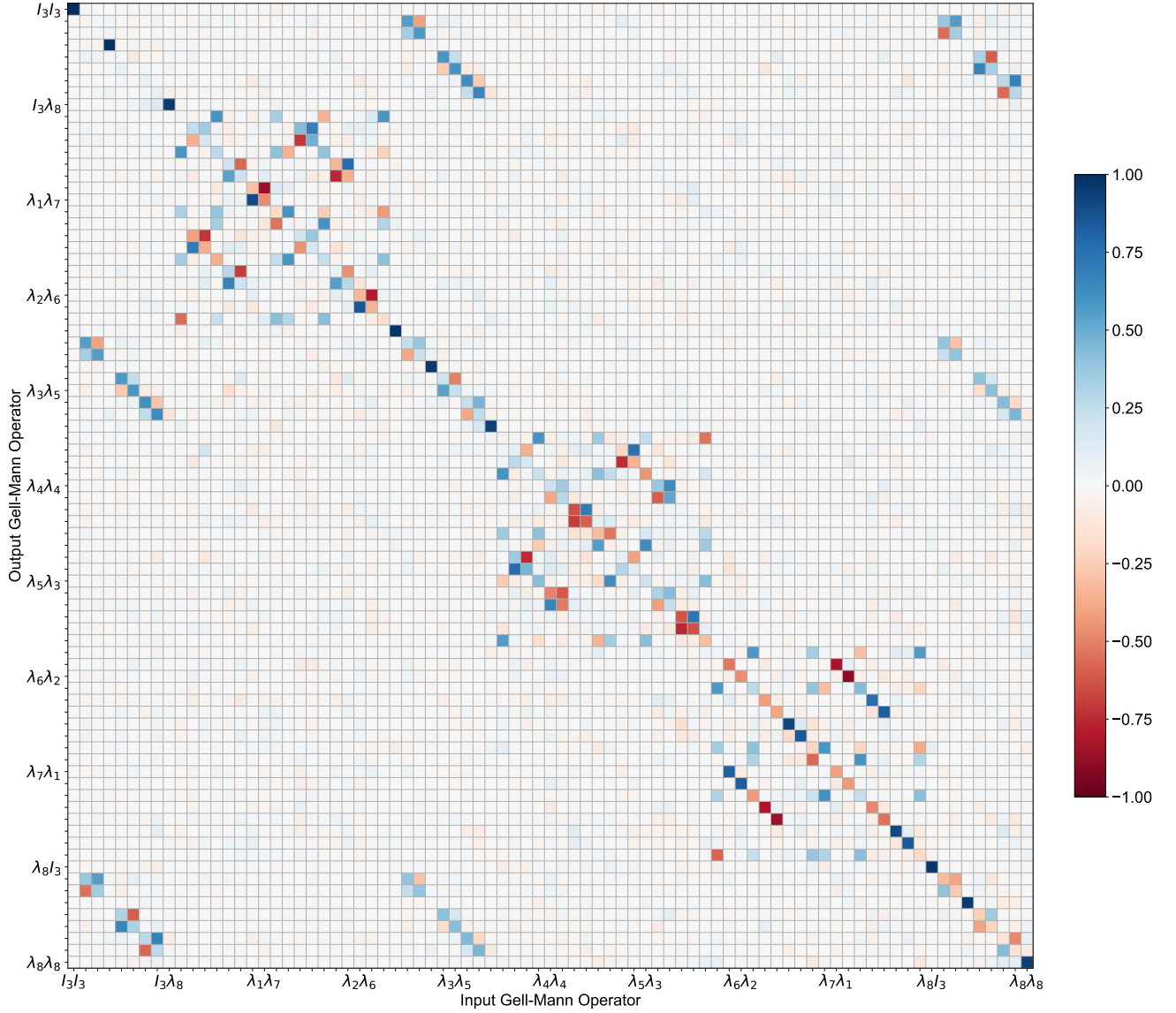


Figure 4.13: **$\text{CZ}^\dagger$ process matrix.** Experimentally reconstructed Process Matrix of Qutrit $\text{CZ}^\dagger(\mathcal{E}_{\text{exp}})$ gate with process fidelity of 93.2%.

Gate Expressibility: Comparing to subspace two-qutrit gates

To study the expressibility of the two-qutrit gates in this work, we numerically explore the ability of the ternary $\text{CZ}/\text{CZ}^\dagger$ (locally equivalent to each other and the CSUM gate) to synthesize other two-qutrit gates, and compare them to two-qutrit entangling gates that

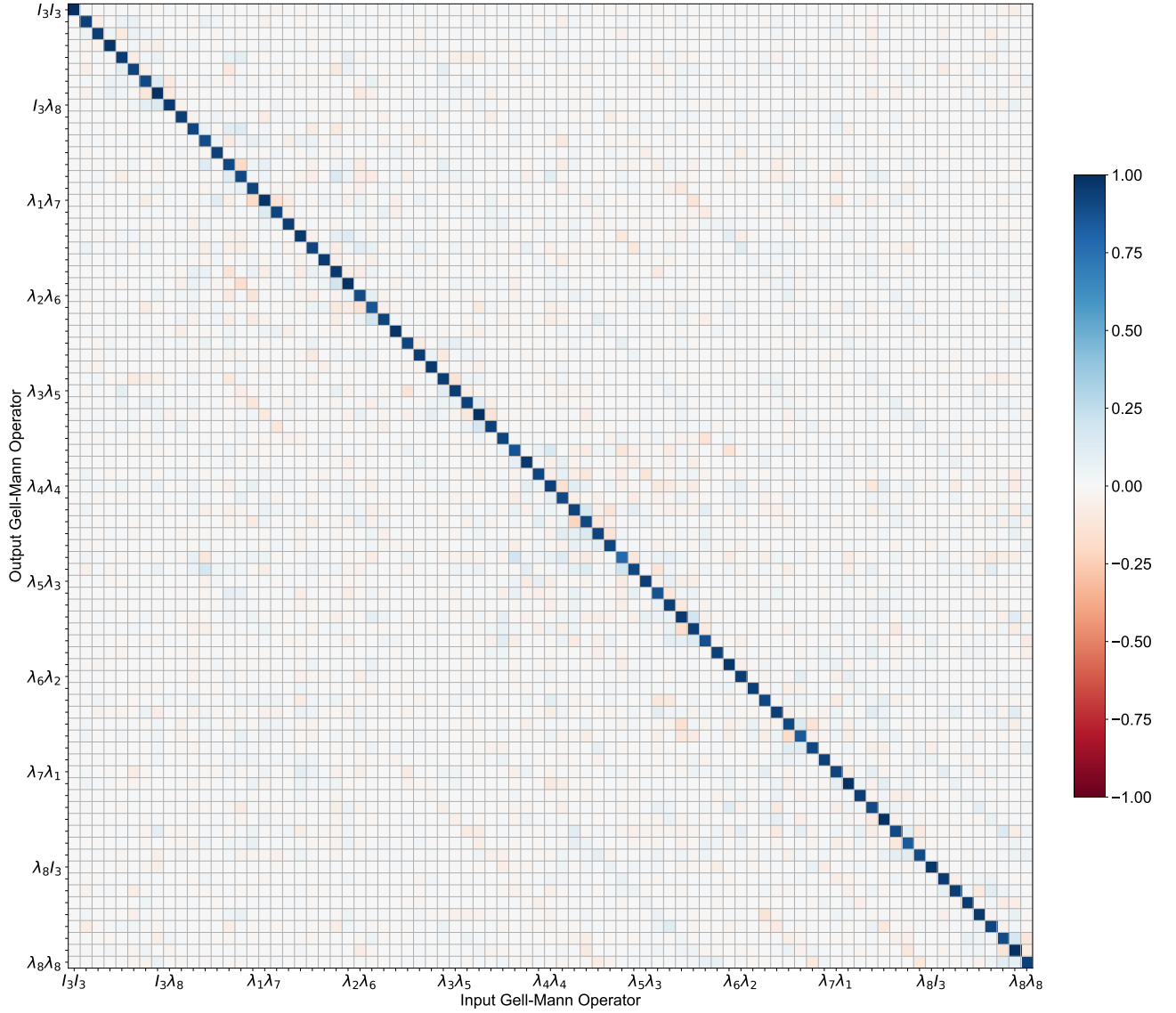


Figure 4.14: **Comparing the experimentally reconstructed process matrix to the ideal case.** We plot $\mathcal{E}_{\text{ideal}}^\dagger \mathcal{E}_{\text{exp}}$, from which we can estimate the process fidelity of the $\text{CZ}^\dagger$ gate.

only entangle a subspace of the qutrit, such as the controlled-exchange (C_{ex}) and controlled-increment (C_{inc}) gates performed on a trapped ion system in Ref. [211] (these comparisons should also be kept in mind in the next chapter where we will realize similar subspace qudit entangling gates on a superconducting device). The relevant subspace entangling gates that

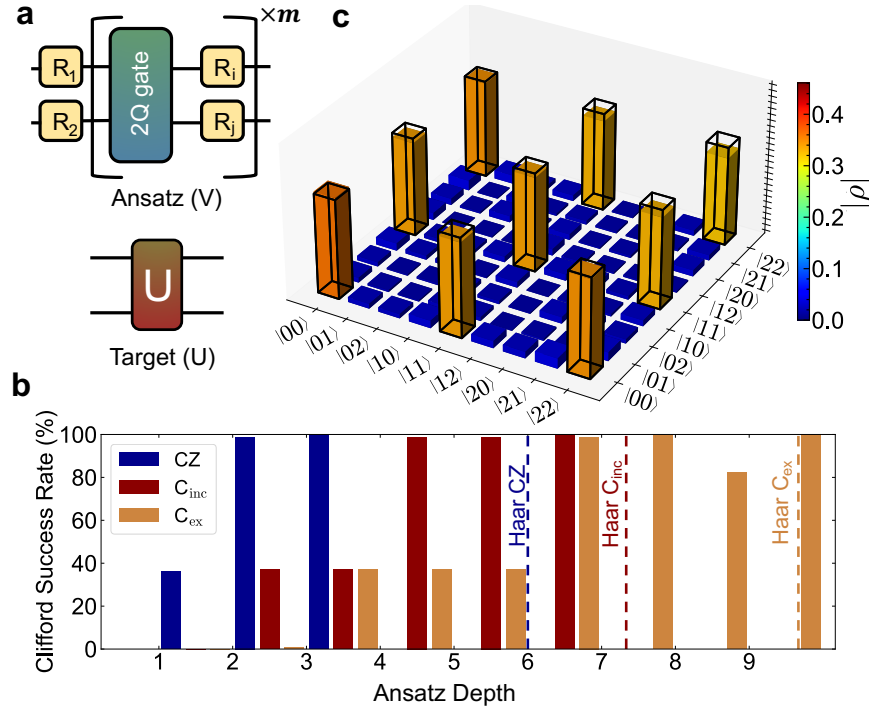


Figure 4.15: **Demonstration of gate expressibility.** **a**, A parameterized Ansatz circuit (V) is used to synthesize a target unitary (U), given some 2-qutrit gate and arbitrary $SU(3)$ gates. **b**, We study the Ansatz circuit (V) for the different two-qutrit gates discussed in the text for 1000 Haar random and Clifford gates, minimizing the infidelity $1 - \mathcal{F}(V, U)$. The dashed lines represent 100% numerical success for synthesizing our set of Haar random gates, and the bars display the success rate for synthesizing Clifford gates. We perform the minimization until we find a 100% success rate for each two-qutrit gate between depths $0 \leq m \leq 9$. **c**, An experimentally reconstructed density matrix of the two qutrit Bell state $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ formed using a single CZ gate with state fidelity $\mathcal{F} = 0.952$. The black outline is the target density matrix.

we use for numerical comparison have the unitaries:

$$C_{ex, t_1, t_2} : \begin{cases} |c, t_1\rangle \leftrightarrow |c, t_2\rangle \\ |j, k\rangle \rightarrow |j, k\rangle & \text{for } j \neq c, k \neq t_1, t_2. \end{cases} \quad (4.28)$$

$$C_{inc} : \begin{cases} |j, k\rangle \leftrightarrow |j, k\rangle & \text{if } j < d - 1 \\ |j, k\rangle \rightarrow |j, k \oplus_3 1\rangle & \text{if } j = d - 1. \end{cases} \quad (4.29)$$

To compare, we consider an Ansatz circuit V as in Fig. 4.15a, with depth m , which we use to synthesize target circuits belonging to either the two qutrit Clifford group or set of Haar random gates. The gate synthesis is performed by optimizing the ansatz parameters to minimize the distance between V and U , i.e. the infidelity $1 - \mathcal{F}(V, U)$.

We perform this numerical investigation on 1000 Haar random gates and 1000 Clifford gates. We find that all 1000 Haar random gates can be synthesized at depth 6 for $CZ/CZ^\dagger$, 7 for C_{inc} , and 9 for C_{ex} . The synthesis success rate for all 3 unitaries in terms of target Clifford circuits are shown in Fig. 4.10b. Notably, almost all two qutrit Clifford gates were successfully compiled at depth 2 in $CZ/CZ^\dagger$, with 100% success at depth 3. By contrast, C_{ex} and C_{inc} did not demonstrate as much improvement at synthesizing Clifford gates over Haar random gates, achieving 100% success for target Clifford gates at depth 6 for C_{inc} and 9 for C_{ex} . Additionally, unique amongst these gates, the $CZ/CZ^\dagger$ can generate maximally entangled two qutrit states with a single iteration of the gate. We demonstrate the power of this feature in the experimentally reconstructed qutrit Bell-state density matrix in Fig. 4.15c. In summary, the maximally entangling $CZ/CZ^\dagger$ gates have low intrinsic errors and can also synthesize the qutrit Cliffords at much lower depths than the two-qutrit gates which only entangle a subspace of the qutrit and are not themselves native members of the Clifford group.

4.7 Outlook

In this chapter, we covered the realization of a microwave-activated, dynamic cross-Kerr entangling interaction that can be employed to engineer qutrit entangling phases with high precision. Leveraging this interaction, we generated two maximally entangling and high-fidelity two-qutrit gates on two separate pairs of fixed-frequency transmon qutrits. We demonstrated numerically that these two qutrit gates are efficient for producing additional two-qutrit unitaries, especially other Clifford gates. These improvements in gate fidelity represent a large step towards useful qudit based quantum computing with superconducting circuits. As we will show in following chapters, we have employed these gates to explore interesting quantum simulations and as the basis gate for demonstrations of error suppression techniques in qutrit circuits; longer term, as motivated at the start of this chapter, even higher fidelity implementations of two-qutrit CZ gates may unlock powerful applications in quantum error correction with qudits. Nonetheless, there are a few existing limitations with this approach to two qutrit gates:

- the two-qutrit on-off ratio
- coherent (or unitary) errors.

The second issue can largely be addressed with black box optimization procedures that use the gate-fidelity as a cost function. However, the two-qutrit on-off ratio remains a problem even now with the addition of the Stark drives. The current gate approach limited us to

performing CZ gates in about 500ns, about an order of magnitude slower than state of the art two-qubit gates. As we described in Chapter 3, in the event that we want to speed up the gate by increasing the bare coupling between qudits, the qudit based quantum crosstalk sets a harsh limit on simultaneous single qudit gate fidelity. This strongly suggests that the best approach to bypass this tradeoff and perform faster two-qudit gates will involve integrating a tunable-coupler element which can enable a much larger on-off ratio. We will touch on this further in section 7 where we perform a large scale qudit quantum simulation with a fully flux tunable square lattice of qudits.

Chapter 5

Subspace Qudit Entangling Gates From Two-Photon Raman Driving

The previous chapter demonstrated that off-resonant microwave drives can engineer a high-fidelity two-qudit CZ gate via a dynamical cross-Kerr interaction. While powerful, the driven cross-Kerr physics becomes complicated when scaling to higher qudit dimension — the number of terms in the cross-Kerr Hamiltonian will grow as $\mathcal{O}(d^2)$ when encoding in higher qudit dimension d , whereas the number of knobs in the entangling interaction will remain fixed. In this chapter we describe a complementary and extensible interaction, *Raman-assisted two-photon hopping*, that enables coherent population exchange between bipartite qudit states of the form $|k, l\rangle \leftrightarrow |k+1, l+1\rangle$. This process, which we describe as *traversing the dual bosonic ladder*, opens a qualitatively new door for qudit operations. We demonstrate its reach by realizing hardware-efficient multi-qubit CCZ and CCCZ gates, preparing a rich suite of high-dimensional entangled states — qudit Bell states, NOON states, and atomic Schrödinger cat states — and programmably distributing entanglement across a four-qudit chain. Together with the cross-Kerr gate of Chapter 4, these two primitives constitute a hardware-efficient and expressive interaction toolkit for high-dimensional quantum computing in superconducting circuits, specifically in the platform of fixed-frequency, fixed-coupling transmons. The original work described in this chapter can be found in Ref. [183].

5.1 Motivation: Why Subspace Entangling Gates for Large d

When climbing the qudit ladder and encoding many computational states within the transmon potential, how many ZZ -like terms appear. Consider an arbitrary two-qudit Hamiltonian,

$$H = \sum_{i,j \in \mathbb{Z}_d} E_{ij} |ij\rangle \langle ij|. \quad (5.1)$$

When working in the doubly rotating frame of a two-qudit system that removes all the local phases, i.e., going into the rotating frame of the unitary,

$$U = \exp \left\{ -it \left[\sum_{j=1}^{d-1} E_{0j} I \otimes |j\rangle \langle j| + \sum_{l=1}^{d-1} E_{l0} |l\rangle \langle l| \otimes I \right] \right\}. \quad (5.2)$$

We are left with the d -level generalized cross-Kerr or ZZ quantum crosstalk Hamiltonian,

$$H = \sum_{i,j=1}^{d-1} \alpha_{ij} |ij\rangle \langle ij|, \quad (5.3)$$

where the cross-Kerr terms are given by $\alpha_{ij} = (E_{ij} - E_{i0}) - (E_{0j} - E_{00})$ (with $\hbar = 1$ everywhere). Now let us quickly turn our attention to a qudit CZ gate in $d = 5$. In such a ququint system, there are 16 cross-Kerr terms, and equivalently, a ququint CZ gate requires careful tuning of 16 controlled phases. Unfortunately, at least in the monochromatic case, the number of knobs in our entangling interaction remains fixed. This then becomes an over-constrained control problem that can only be solved with methods like optimal control which has found some success in single qudit gates [223, 224] and has been proposed for two-qudit gates [193] or with dynamical decoupling pulses that will ultimately lead to much longer gate times [19]. As we will demonstrate experimentally in this chapter, if one allows a parameterization of the swap angle of the two-photon gates we employ in this section, then a linear number $(d-1)$ two-qudit gates can be employed to realize a maximally entangled two-qudit state and the control problem becomes considerably more streamlined — essentially boiling down to calibrating two-qubit gates within one double-excitation rung of the the larger qudit ladder of states.

Turning to subspace entangling gates that significantly simplify the control problem for two-qudit gates is therefore a natural choice. Specifically, engineering coherent swapping between just a subspace of the multi-qudit state space is desirable, but an immediate concern may be the generic issue with resonant coupling described in Sec. 4.2. As described in further detail in the following sections, the scheme employed in this chapter of strongly driven two-photon hopping avoids the generic problems of resonant coupling by employing microwave drives that only excite coherent $|k, l\rangle \leftrightarrow |k+1, l+1\rangle$ hopping. The interaction is ultimately frequency selective, and as we show in detail, does not induce deleterious swapping on other idling states in the two-qudit Hilbert space during the gate.

5.2 Two-Photon Entangling Gates: Concept

The transmon inherently behaves as a nonlinear harmonic oscillator with d eigenlevels forming a bosonic ladder within the potential well. The circuit's eigenstates can be accessed using microwave drives to induce energy excitation, one level at a time. The qudits can thus be naturally visualized using noncyclical gears, as depicted in Fig. 5.1. A high-dimensional

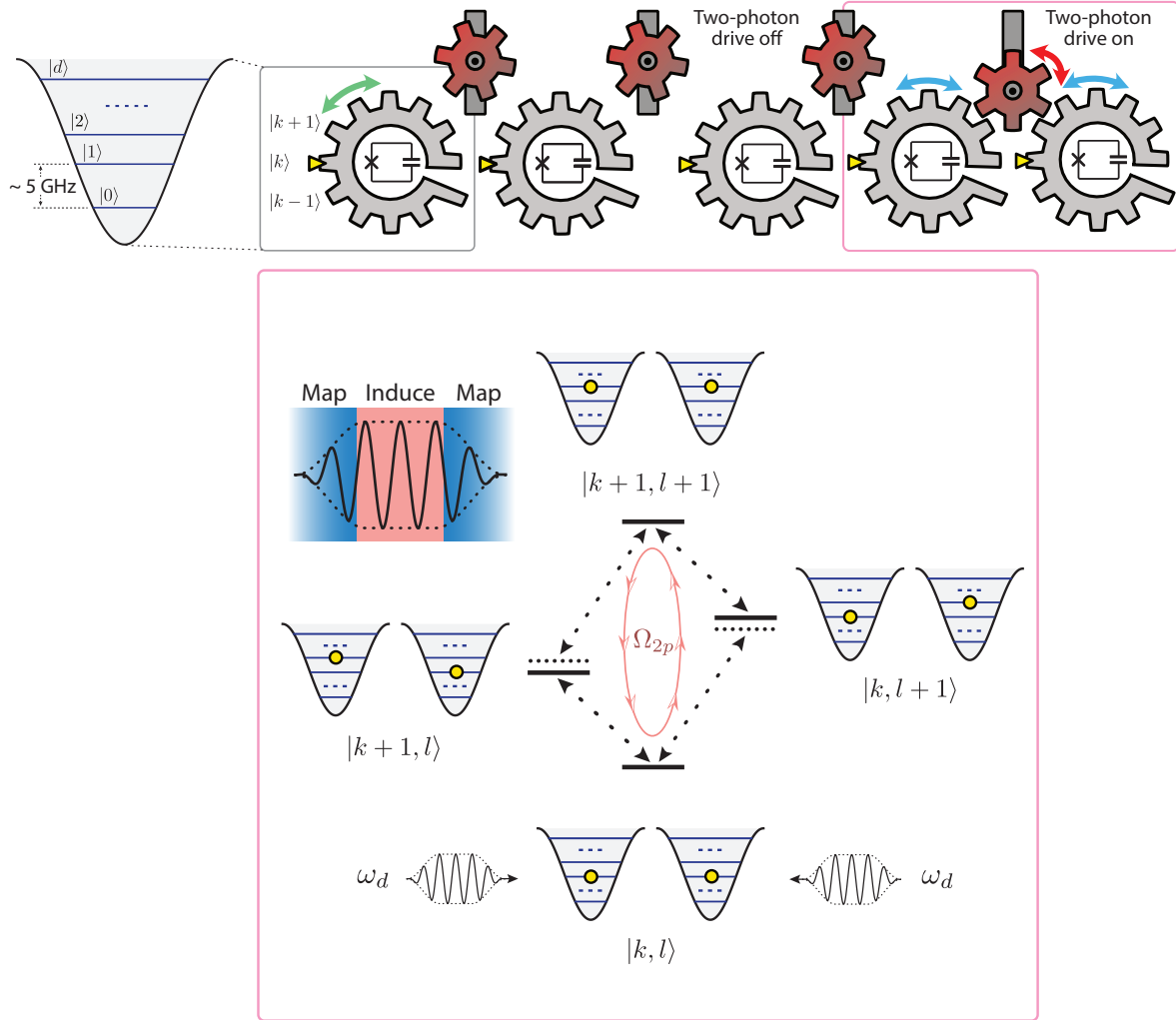


Figure 5.1: **Schematic of Two-Photon Qudit Entangling Gates.** (**top**) Schematic of the high-dimensional system which is constructed by linking individual qudits into an array. Each qudit in the chain is a nonlinear harmonic oscillator comprising of a Josephson junction and a capacitor in parallel, and its eigenstates form a bosonic ladder of d levels within the cosine potential. They are depicted as non-cyclical gears that can be rotated either individually using local controls or simultaneously using two-photon drives. (**bottom**) Simplified depiction of the Raman-assisted two-photon-driven dynamics. The coupled-qudits pair form a set of eigenstates, visualized as a dual bosonic ladder. The entangling dynamic takes place in a four-level manifold within this structure. The effect of microwave drives applied to both qudits at a frequency close to the average of the single-excitation frequencies in this subspace is twofold: they first map the relevant qudit states to the dressed frame, then induce a Raman-assisted transition between the bipartite states $|k, l\rangle$ and $|k+1, l+1\rangle$ (inset), resulting in an interaction at a rate Ω_{2p} .

quantum device can be built by capacitively connecting the gears into an array, each one serving as a link in the chain with individual control and readout circuitry

A natural approach to perform multi-qudit operations on these gears is to engineer their co-rotation which involves the simultaneous rotation of both qudits. This process embodies a non-energy-conserving two-photon dynamic characteristic of bosonic systems, depicted using the red idler gears in Fig. 5.1. We construct the desired interactions for all the high-dimensional subspace manifolds without additional circuitry or instrumentation as follows.

First, a subspace spanned by the bipartite eigenstate $|k, l\rangle$ and its single- and double-excitation with negligible static cross-Kerr can be represented using the dual bosonic ladder as illustrated in Fig. 5.1. While transitions involving single-photon hopping are permitted, the direct two-photon hopping process $|k, l\rangle \rightarrow |k+1, l+1\rangle$ is prohibited due to the lack of parity conservation under coherent drives. Notably, within the realm of quantum mechanics, a double excitation of this nature can take place through intermediate states. If these states are solely virtually occupied, the dynamics can be viewed as the multi-partite counterpart of the well-studied Raman process. In multi-qubit systems, this process has been utilized to engineer two-qubit gates in trapped ions with vibrational modes by employing two laser beams at frequencies that combine to match the double-excitation frequency [174]. Here, we harness these dynamics in solid-state systems without the need for ancillary components or additional degrees of freedom.

To simplify the description, we examine the monochromatic driving scenario where two pulses are administered to both qudits at a single frequency ω_d . This frequency is detuned by $\Delta_{1(2)} = \omega_d - \omega_{q,1(2)}$ from those of the relevant single-photon transitions $\omega_{q,1(2)}$, i.e. between states $|k, l\rangle$ and $|k+1, l\rangle$ ($|k, l+1\rangle$) for qudit 1(2). In this context, the initial effect of the drives involves dressing the qudit states, contingent upon the fulfillment of the adiabaticity condition [185] (Fig. 5.1, inset). With a finite static coupling amplitude g_{kl} , the drives subsequently induce coherent two-photon hopping in the interaction frame via sequential single-photon virtual transitions through the hybridized states. Coherent transfer of the population between $|k, l\rangle$ and $|k+1, l+1\rangle$ ensues, with an oscillation rate Ω_{2p} given as (see Sec. 5.3)

$$\Omega_{2p} = \frac{g_{kl}}{2} \left[(1 + \cos \theta_1) (1 + \cos \theta_2) + (1 - \cos \theta_1) (1 - \cos \theta_2) \right], \quad (5.4)$$

where the angles $\{\theta_i\}$ are defined through the relation $\cos \theta_i = \Delta_i / \sqrt{\Delta_i^2 + \Omega_i^2}$. Here, Ω_i denotes the driving amplitude on qudit i , expressed in terms of the single-photon Rabi oscillation rate between the respective states.

The monochromatic driving scheme is particularly well-suited for the current high-dimensional system, as superconducting qudits inherently possess finite detunings between their energy levels. The drive frequency can be adjusted by varying the ratio between the drive amplitudes, enabling the adjustment of the dressed qudit frequencies. Employing drives detuned

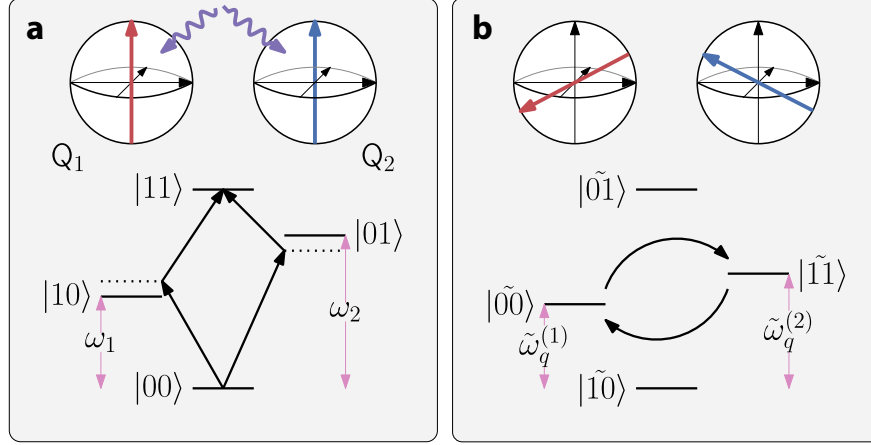


Figure 5.2: **Levels Diagram of Two-Photon Interaction.** (a) Level diagram of the two-qubit subspace in the laboratory frame. The two-photon transition between $|00\rangle$ and $|11\rangle$ is enabled by a Raman-like mechanism that avoids transitions outside of the computational subspace. (b) Level diagram of the two-qubit subspace in the rotating frame. Although in the laboratory frame, the bare states $|00\rangle$ and $|11\rangle$ appear far from resonance, in the rotating frame, the energies of the corresponding *dressed* states can be tuned into resonance.

similarly from the pertinent transitions also simplifies pulse calibration to ensure adiabaticity. This approach is readily achievable given the existence of intermediate states that facilitate virtual transitions. The requirements for level detuning are relatively flexible, as a greater detuning permits stronger driving without inducing non-adiabatic effects. Therefore, the presented process is broadly feasible and adaptable across various platforms. We now spend some time to elucidate the theory of this two-photon driven entanglement in our transmon systems. We begin by building up intuition from two-level systems, before handling the anharmonic corrections to the system.

5.3 Theoretical Analysis

We describe here the theoretical framework for the two-photon transitions demonstrated in this chapter. The two-qudit two-photon transition between states $|k, l\rangle$ and $|k + 1, l + 1\rangle$ can be understood as a two-qudit generalization of the standard single-atom Raman transition in a “ladder” configuration in which a single-photon transition from the ground state to the second excited state is not possible, but a two-photon transition mediated by a “virtual” energy level is possible. Crucially, this transition avoids populating the first excited state while achieving full population transfer between the ground and second excited states.

Theory of coupled two-level systems

We begin by considering two coupled transmons within the two-level system approximation, i.e. treating them as qubits. We presently derive resonance conditions and transition rates for coherent oscillation between the $|00\rangle$ and $|11\rangle$ states. The static Hamiltonian $\hat{H}_{\text{ss}}$ for this system is given as

$$\begin{aligned}\hat{H}_{\text{ss}} &= \sum_{i=1}^2 \hat{H}_{\text{q}}^{(i)} + \hat{H}_{\text{c}}, \\ \hat{H}_{\text{q}}^{(i)} / \hbar &= -\frac{1}{2} \omega_q^{(i)} \hat{\sigma}_z^{(i)}, \\ \hat{H}_{\text{c}} / \hbar &= g \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)},\end{aligned}\tag{5.5}$$

where $\omega_q^{(i)}$ (used interchangeably with $\omega_{q,i}$) is the transition frequency between the ground state and the first excited state of the i -th qubit and g is the direct qubit-qubit coupling. Without loss of generality, we will work with the assumption $\omega_q^{(1)} < \omega_q^{(2)}$.

We first consider driving both qubits at the same frequency ω_d , generating the Hamiltonian in the laboratory frame as

$$\begin{aligned}\hat{H}_{\text{full}} &= \hat{H}_{\text{ss}} + \sum_{i=1}^2 \hat{H}_{\text{d}}^{(i)}, \\ \hat{H}_{\text{d}}^{(i)} / \hbar &= \Omega_i \left(\hat{\sigma}_+^{(i)} + \hat{\sigma}_-^{(i)} \right) \cos(\omega_d t + \phi_i),\end{aligned}\tag{5.6}$$

where Ω_i is the amplitude of the drive on qubit i (expressed in terms of the resonant Rabi rate) and ϕ is the relative phase difference between the drives. We also find it convenient to express the results in terms of the “drive ratio” $\lambda \equiv \Omega_1 / \Omega_2$, which models crosstalk. The setup in the lab frame is sketched in Figure 5.2a, forming a ladder-like Raman configuration in which the direct transition between the $|00\rangle$ and $|11\rangle$ states is forbidden but can be mediated by virtual levels [174, 181].

The formation of the virtual energy levels is better understood in the rotating frame in which the *dressed* energy levels can be tuned to achieve the desired resonance. In the rotating frame, the Hamiltonians read

$$\begin{aligned}\hat{H}_{\text{rot,full}} &= \sum_{i=1}^2 \hat{H}_{\text{rot,qd}}^{(i)} + \hat{H}_{\text{rot,c}}, \\ \hat{H}_{\text{rot,qd}}^{(i)} / \hbar &= -\frac{1}{2} \left(\Delta_i \hat{\sigma}_z^{(i)} + \Omega_i (\hat{\sigma}_+^{(i)} e^{-i\phi_i} + \hat{\sigma}_-^{(i)} e^{i\phi_i}) \right), \\ \hat{H}_{\text{rot,c}} / \hbar &= g (\hat{\sigma}_+^{(1)} \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \hat{\sigma}_+^{(2)}),\end{aligned}\tag{5.7}$$

where the fast-rotating terms have been omitted. Suppose that at $t = 0$, $\Omega_i(t = 0) = 0$ and is slowly ramped up to some final value $\Omega_i(t = t_{\text{ramp}}) = \Omega_i$. As in Ref. [185], the eigenstates of the uncoupled, undriven Hamiltonians evolve into those of $\hat{H}_{\text{rot,qd}}^{(i)}$ and are tilted with respect to the z -axis of the Bloch spheres. Defining angles θ_i via

$$\begin{aligned}\cos \theta_i &= \frac{\Delta_i}{\sqrt{\Delta_i^2 + \Omega_i^2}}, \\ \sin \theta_i &= \frac{\Omega_i}{\sqrt{\Delta_i^2 + \Omega_i^2}},\end{aligned}\tag{5.8}$$

the eigenstates of $\hat{H}_{\text{rot,qd}}$ are given by

$$\begin{aligned}|\theta_i\rangle &= \cos\left(\frac{\theta_i}{2}\right)|0\rangle + e^{i\phi_i}\sin\left(\frac{\theta_i}{2}\right)|1\rangle \\ |\bar{\theta}_i\rangle &= -e^{-i\phi_i}\sin\left(\frac{\theta_i}{2}\right)|0\rangle + \cos\left(\frac{\theta_i}{2}\right)|1\rangle\end{aligned}\tag{5.9}$$

When $\Delta_i > 0$ ($\Delta_i < 0$) and $\Omega_i(t)$ is ramped up adiabatically from zero, the state $|0\rangle$ evolves into $|\theta\rangle$ ($|\bar{\theta}\rangle$). The dressed qubit frequency evolves to $\tilde{\omega}_q^{(i)}$, which is given by

$$\tilde{\omega}_q^{(i)} = \text{sgn}(\Delta_i)\sqrt{\Delta_i^2 + \Omega_i^2}.\tag{5.10}$$

By choosing $\omega_q^{(1)} < \omega_d < \omega_q^{(2)}$, we achieve the level structure shown in Figure 5.2b. Then, by adjusting Ω_i , we seek degeneracy between the levels for states $|\bar{\theta}\theta\rangle$ and $|\theta\bar{\theta}\rangle$. Solving for this resonance condition, we obtain the following relationship between the resonant drive frequency, the drive amplitude $\Omega \equiv \Omega_2$ to one qudit, the drive ratio $\lambda = \Omega_1/\Omega_2$, and the qudit frequencies,

$$\omega_d = \frac{\omega_q^{(12)}}{2} + \frac{(\lambda^2 - 1)\Omega^2}{2\Delta_q^{(12)}},\tag{5.11}$$

where $\omega_q^{(12)} = \omega_q^{(1)} + \omega_q^{(2)}$ and $\Delta_q^{(12)} = \omega_q^{(1)} - \omega_q^{(2)}$. Note that the sign of the shift $\omega_d - \omega_q^{(12)}/2$ is determined by whether the higher or lower frequency qubit is driven more strongly.

Having identified the resonant condition, we enter the frame of $\hat{H}_{\text{rot,qd}}$. The interaction term $\hat{H}_{\text{rot,c}}$ is transformed into

$$\begin{aligned}\tilde{H}_c/\hbar &= J_I \left(\tilde{\sigma}_x^{(1)} \tilde{\sigma}_x^{(2)} - \tilde{\sigma}_y^{(1)} \tilde{\sigma}_y^{(2)} \right) + J_Q \left(\tilde{\sigma}_x^{(1)} \tilde{\sigma}_y^{(2)} + \tilde{\sigma}_y^{(1)} \tilde{\sigma}_x^{(2)} \right) + J_{ZZ} \tilde{\sigma}_z^{(1)} \tilde{\sigma}_z^{(2)}, \\ J_I &= \frac{g}{8} \left[(1 + \cos \theta_1)(1 + \cos \theta_2) \cos 2\phi_1 + (1 - \cos \theta_1)(1 - \cos \theta_2) \cos 2\phi_2 \right], \\ J_Q &= -\frac{g}{8} \left[(1 + \cos \theta_1)(1 + \cos \theta_2) \sin 2\phi_1 + (1 - \cos \theta_1)(1 - \cos \theta_2) \sin 2\phi_2 \right], \\ J_{ZZ} &= -\frac{g}{2} \sin(\theta_1) \sin(\theta_2) \cos(\phi_1 - \phi_2).\end{aligned}\tag{5.12}$$

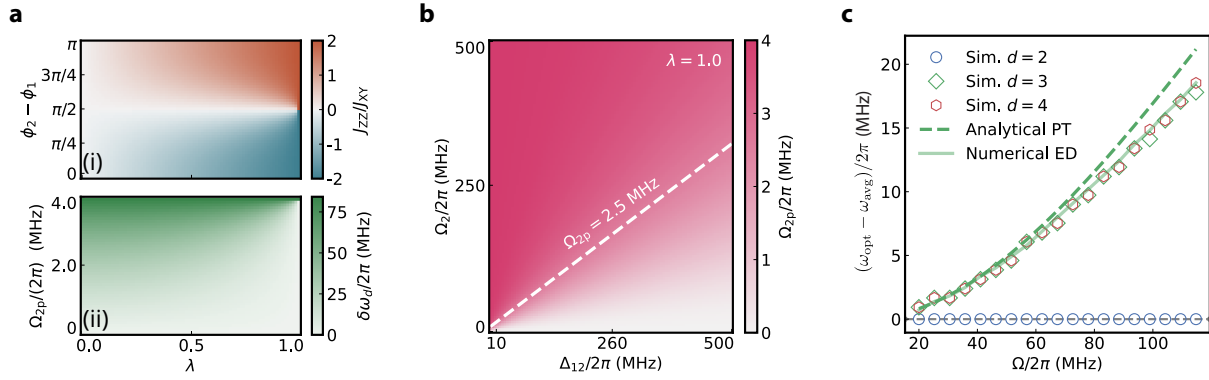


Figure 5.3: Theoretical results. (a) Tunability of the Hamiltonian through drive ratio and phase. (i) The drive ratio $\lambda = \Omega_1/\Omega_2$ and relative microwave phase $\phi_2 - \phi_1$ of the single qubit drives can be tuned to apply a two-qubit entangling interaction with a variety of relative interaction strengths J_{ZZ}/J_{XY} (ii) The two-photon interaction through monochromatic driving can be activated across the range of frequencies between $\omega_{q,1}$ and $\omega_{q,2}$. For simplicity, we have assumed $\Omega_1 \leq \Omega_2$, so $\omega_d \geq (\omega_{q,1} + \omega_{q,2})/2$, but the opposite detuning can be achieved by driving qudit 1 harder than qudit 2. (b) Two-photon swap rate Ω_{2p} as a function of the drive amplitude and qudit detuning Δ_{12} , computed in the symmetrically driven regime $\lambda = 1.0$ such that $\omega_d = (\omega_{q,1} + \omega_{q,2})/2$. Even when the energy levels $|k+1, l\rangle$ and $|k, l+1\rangle$ are far-detuned, it is possible to activate the two-photon swap, albeit requiring more drive power. (c) Effect of finite anharmonicity on optimal drive frequency. Stark effects from higher levels of the transmon, particularly the third level, shift the dressed qubit frequency and consequently the optimal drive frequency to achieve two-photon transitions. The optimal drive frequency is shown with respect to the average qubit frequency, $\omega_{\text{avg}} = (\omega_{q,1} + \omega_{q,2})/2$. Numerical simulations (labeled as “Sim.”) were carried out with $d = \{2, 3, 4\}$ levels for each transmon to identify the optimal drive frequencies and compared to those found by the exact diagonalization (labeled as “Numerical ED”) of the rotating frame Hamiltonian and analytical perturbation theory (labeled as “Analytical PT”).

As expected, the maximal oscillation rate Ω_{2p} in the $00 - 11$ subspace, $4\sqrt{J_I^2 + J_Q^2}$, is set by the bare coupling g . This asymptotic behavior is well-captured by this theoretical approach, which, in contrast to prior work, treats the coupling term $\hat{H}_c$ *after* exactly diagonalizing the full uncoupled rotating Hamiltonian $\hat{H}_{\text{rot,qd}}$ rather than diagonalizing the full, coupled static Hamiltonian first and then treating the driving terms perturbatively, such as in Ref. [200]. In that work, the swap rate Ω_{2p} acquires significant corrections due to the close frequency spacing of the $|11\rangle$ and $|02\rangle$ states, and only the limit of low drive power is considered. However, standard expressions for Ω_{2p} in the low-power limit, including the pure qubit limit of Ref. [200], can be recovered by Taylor expansion of Equations (5.12). This alternative approach is justified due to the unique hierarchy of energy scales that arises from entangling qubits via strong off-resonant drives: the parameter Ω_i/Δ_i is no longer “small”, so corrections to the perturbation theory are significant.

To build intuition for the expressions in Equations (5.11) and (5.12), we consider two limiting cases. First, when the qubits are driven symmetrically ($\lambda = 1$) and in-phase with the same amplitude Ω , the rates are simplified to

$$\begin{aligned}\omega_d &= \frac{\omega_q^{(12)}}{2} \\ J_I &= \frac{g}{4} \frac{\Omega^2}{\Omega^2 + (\Delta_q^{(12)})^2/4}, \\ J_Q &= 0, \\ J_{ZZ} &= -\frac{g}{2} \frac{\Omega^2}{\Omega^2 + (\Delta_q^{(12)})^2/4}.\end{aligned}\tag{5.13}$$

On the other hand, when only one qubit is driven (e.g. $\lambda = 0$) as done in Ref. [268], we obtain

$$\begin{aligned}\omega_d &= \frac{\omega_q^{(12)}}{2} - \frac{\Omega^2}{2\Delta_q^{(12)}}, \\ J_I &= \frac{g}{4} \left(1 - \frac{\Delta_q^{(12)}/2 - \Omega^2/\Delta_q^{(12)}}{\sqrt{\Omega^2 + (\Delta_q^{(12)}/2 - \Omega^2/\Delta_q^{(12)})^2}} \right) \\ J_Q &= 0, \\ J_{ZZ} &= 0.\end{aligned}\tag{5.14}$$

Equations (5.11) and (5.12) suggest that by controlling the drive ratio λ and the relative phase of the microwave drives on each transmon, $\phi = \phi_2 - \phi_1$, one can control (i) the ratio of the two-photon excitation term $J_{XY} = \sqrt{J_I^2 + J_Q^2}$ to the ZZ term J_{ZZ} , as well as (ii) the optimal drive frequency ω_d . We demonstrate the flexibility of this control in Figure 5.3a. We speculate that this control could be useful for performing quantum simulations of many-body

systems with nearest-neighbor couplings using fewer two-qubit gates, as one can tune ratios of the interaction terms in the Hamiltonian easily.

Our analysis has thus far concentrated on the two-photon transition between the $|00\rangle$ and $|11\rangle$ states of a system of two coupled qubits. However, we can apply a similar analysis to subspaces consisting of higher levels to obtain rates for general high-dimensional systems, assuming that all other levels are far enough apart that they do not contribute significantly to the dynamics. To obtain the expressions for the $|k, l\rangle \rightarrow |k+1, l+1\rangle$ transition, the detunings Δ_i and $\Delta_q^{(12)}$ are now defined relative to the $k \rightarrow (k+1)$ and $l \rightarrow (l+1)$ transition frequencies of the qudit pair. The drive amplitude Ω_1 (Ω_2) is defined with respect to the Rabi rate in the $|k\rangle, |k+1\rangle$ ($|l\rangle, |l+1\rangle$) subspace when this transition is driven resonantly, and g is replaced by the bare coupling between states $|k, l+1\rangle$ and $|k+1, l\rangle$. The latter typically scales as $\sqrt{(k+1)(l+1)}$ due to the matrix elements of the bosonic ladder operator. We therefore fit the experimental data for the measured resonance frequencies ω_d and swap rates Ω_{2p} in Fig. 2c-d using Equations (5.11) and (5.12) as ansätze with an additional parameter to convert voltage output on drive lines to Rabi rates $\{\Omega_1, \Omega_2\}$. The extracted drive amplitude ratio is $\lambda = \{0.64, 0.63, 0.66\}$ for the $\{00-11, 11-22, 22-33\}$ transitions.

Finally, we note that even when the single-photon transition energies from $|k, l\rangle$ to $|k+1, l\rangle$ and $|k, l+1\rangle$ are far-detuned, it is still possible to activate the two-photon transition from $|k, l\rangle$ to $|k+1, l+1\rangle$, although a larger drive amplitude (in the relevant qudit subspaces) is probably needed. We illustrate this point in Figure 5.3b, which shows the two-photon swap rate Ω_{2p} as a function of the drive amplitude Ω_2 and the difference in single photon transition energies Δ_{12} in the symmetrically driven regime.

Finite anharmonic correction

The preceding analysis of the $|00\rangle \rightarrow |11\rangle$ two-photon transition assumes that states outside the computational subspace are sufficiently off-resonant such that they do not contribute significantly to the dynamics, which is a good assumption for systems with high anharmonicity [181, 182]. However, deviations from this theory manifest when higher levels of the transmon are taken into account. The strong off-resonant drive induces a Stark shift of the dressed qubit frequency in Equation (5.10), which must be taken into account when solving for the optimal drive frequency ω_d at a given drive amplitude Ω .

We examine this effect by performing numerical simulation of two symmetrically driven transmons using QuTiP [123], retaining up to four energy levels per transmon as shown in Figure 5.3c. As predicted by Equation (5.11), when only two levels are retained, the optimal frequency ω_d is independent of the drive strength Ω . However, when higher levels are taken into account, a nontrivial dependence manifests. This dependence can be understood by considering the Hamiltonian of the three-level Duffing Oscillator in the rotating frame of the drive for each transmon separately,

$$\hat{H}_i/\hbar = \left(\Delta_i - \frac{\alpha_i}{2}\right) \hat{a}_i^\dagger \hat{a}_i + \frac{\alpha_i}{2} (\hat{a}_i^\dagger \hat{a}_i)^2 + \frac{\Omega_i}{2} (\hat{a}_i + \hat{a}_i^\dagger), \quad (5.15)$$

where α_i is the anharmonicity of qudit i , a_i is the bosonic annihilation operator for the modes of qudit i , and the fast-rotating terms have been dropped. By considering Ω_i as a parameter which is slowly turned on from 0 at $t = 0$ to their final values Ω_i at $t = t_{\text{ramp}}$, one can follow the eigenstates of the rotating frame Hamiltonian from $t = 0$ to $t = t_{\text{ramp}}$ and compute the energy difference $\tilde{\omega}_q^{(i)}$ between the states $|0\rangle$ and $|1\rangle$. This generalizes the dressed qubit frequency introduced in Equation (5.10). Similarly, the resonance between the $|00\rangle$ and $|11\rangle$ states is achieved at a given drive power when ω_d is tuned such that $\tilde{\omega}_q^{(1)} + \tilde{\omega}_q^{(2)} = 0$. The entire procedure of diagonalization and solution for ω_d can be carried out through exact diagonalization of Equation (5.15) using linear algebra software (Numerical ED) or by standard Schrieffer-Wolff perturbation theory (Analytical PT) to obtain expressions for $\{\tilde{\omega}_q^{(i)}\}$ and then again using numerics to solve for resonance. We compare these two approaches to the optimal frequency obtained by simulations of spectroscopy in QuTiP in Figure 5.3c. The agreement between the numerical simulation and the semi-analytical approaches for $d > 2$ suggests that the nontrivial dependence of ω_d on Ω can indeed be understood through the framework of effective transition frequencies used in this work.

Adiabaticity requirements

The adiabaticity condition can be understood as arising from the need to transform from the laboratory frame to the dressed frame. Specifically, consider the Hamiltonian of a single, off-resonantly driven qubit in the rotating frame of the drive: $H(t) = \Delta_q \sigma^z + \Omega_R(t) \sigma^x$. The drive has an envelope $\Omega_R(t)$. Due to the quantum adiabatic theorem, if Ω_R is slowly ramped up or down, the eigenstates of σ^z will evolve into the instantaneous eigenstates of $H(t)$. In practice, it is important to change $H(t)$ sufficiently slowly. Failure to do so leads to coherent errors which can be understood as Landau-Zener transitions. The relevant parameter in such a transition is the rate at which the spectral gap of H is changing, which is, to low order in Ω_R/Δ_q , $\sim \Omega_R \dot{\Omega}_R/\Delta_q$. This tells us that at fixed Δ_q and Ω_R , we must decrease $\dot{\Omega}_R$ to fulfill adiabaticity as much as possible, e.g. by slowly ramping up and down the drives. However, as with other gate schemes, a slower gate generally results in greater incoherent error, so the ramps should be chosen to balance these two types of error. Notably, Ref. [185] showed that shortcut-to-adiabaticity techniques can be used to effectively reduce the ramping time. Future works exploring the diabatic regime are also of immense interest, and such advances can be applied to operate fast gates using two-photon interactions in fixed-frequency superconducting circuits.

5.4 Experimentally Characterizing the Two-Photon Driving

Now having developed the theoretical framework to describe our two-photon Raman-assisted driving, we turn to implementing these gates experimentally. At first we characterize the

physics of two-photon drives as we climb the dual-bosonic, or double excitation hopping, ladder of our coupled qudit systems. We begin by elucidating the physics of our strongly driven system as we coherently couple higher and higher-level states in our qudit manifold. Finally, after demonstrating agreement with our theoretically developed model, we employ these two-qudit operations to generate programmable high-dimensional entanglement and efficiently synthesize N-body Toffoli gates.

Experimental Device

Our experiments for the rest of this chapter are performed on a superconducting quantum processor consisting of eight connected transmon circuits forming a ring, where each transmon serves as the qudit. This device has already been described in detail in Chapter 2, and was also employed in some of the experiments in the previous chapter. It is therefore not necessary to go into significant detail into the device at this moment, other than to mention that in the following experiments, we access and coherently control up to 4 levels in each qudit, with the relevant coherences and frequencies that characterize the full $d = 4$ qudit

Parameters	Q ₁	Q ₂	Q ₃	Q ₄
$\omega_{01}/2\pi$ (GHz)	5.333	5.396	5.572	5.745
$\omega_{12}/2\pi$ (GHz)	5.061	5.124	5.303	5.465
$\omega_{23}/2\pi$ (GHz)	4.757	4.821	5.005	5.171
$\omega_r/2\pi$ (GHz)	6.563	6.622	6.678	6.736
$\kappa/2\pi$ (MHz)	1.44	1.45	1.83	1.73
$\chi_{01}/2\pi$ (MHz)	1.09	1.09	1.26	1.44
Avg. T_1^{01} (μ s)	50(4)	49(4)	60(5)	61(9)
Avg. T_1^{12} (μ s)	35(2)	35(4)	31(8)	34(2)
Avg. T_1^{23} (μ s)	24(4)	26(3)	23(4)	15(1)
Avg. T_{2e}^{01} (μ s)	78(5)	85(9)	90(6)	74(5)
Avg. T_{2e}^{12} (μ s)	57(4)	57(4)	56(9)	34(2)
Avg. T_{2e}^{23} (μ s)	26(2)	27(2)	24(3)	20(2)
$J_{01}/2\pi$ (MHz)	2.5	2.5	-	-
	-	2.7	2.7	-
	-	-	3.1	3.1

Table 5.1: Qudit parameters of the experimental device employed in these $d = 4$ entangling gates. Here, g is the coupling strength between the qudits and their respective readout resonators, and J_{01} refers to the effective coupling strength between the qudits' 0–1 subspace. The coherence statistics (average values and standard deviations) are each obtained from an ensemble of 100 measurements.

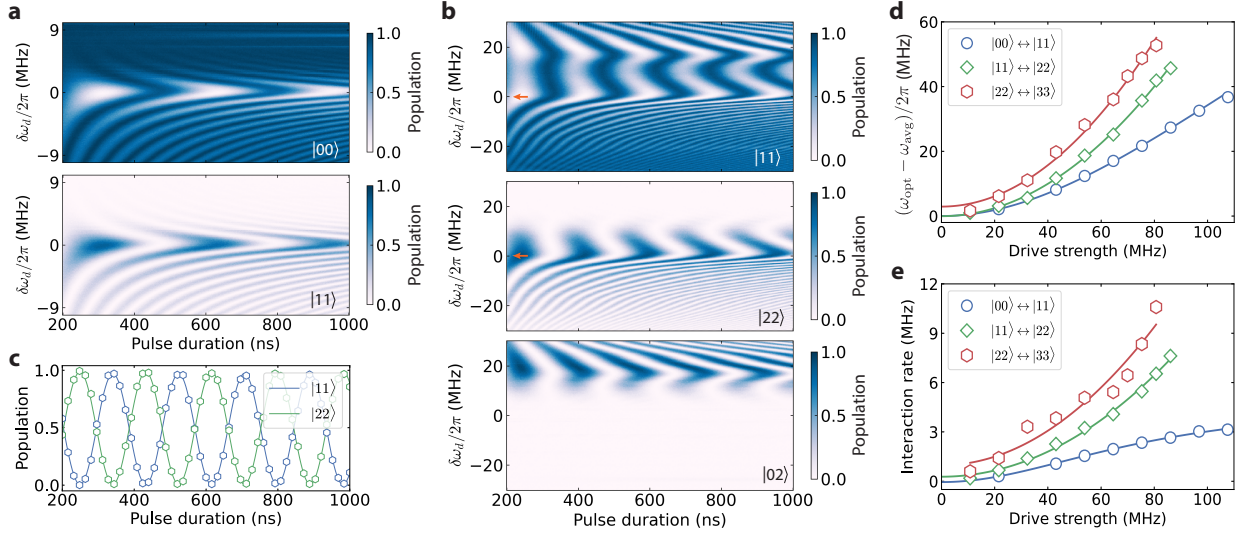


Figure 5.4: **Traversing the dual bosonic ladder.** (a) Coherent $|00\rangle \leftrightarrow |11\rangle$ population exchange between two coupled qudits. The interaction is induced by monochromatic microwave drives. Here, the drive frequency is shown with respect to the optimal point where the oscillation is the most coherent. The chevron is also quasi-symmetric about this frequency point. (b) Coherent $|11\rangle \leftrightarrow |22\rangle$ and $|11\rangle \leftrightarrow |02\rangle$ population exchange between two coupled qudits. The former interaction is induced by the two-photon Raman process, while the latter originates from the resonant condition between $|11\rangle$ and $|02\rangle$ in the driven frame. (c) Coherent flip-flop oscillation between $|11\rangle$ and $|22\rangle$ states obtained at the optimal drive frequency, indicated by the red arrows in panel b. The solid lines represent cosine fits. (d) Optimal driving frequencies at various drive amplitudes. The data are shown with respect to $\omega_{\text{avg}} = (\omega_{k,l+1} + \omega_{k+1,l})/2$ in the y-axis. The x-axis is expressed in terms of the single-photon Rabi rate in the 0-1 subspace of the first qudit. The solid lines are analytical results. (e) Interaction rates of the dual bosonic transitions up to $d = 4$ at various drive amplitudes. All data points are measured at the optimal driving frequencies. Analytical results are shown as solid lines.

system at the time of these experiments available in Table 5.1.

Traversing the dual Bosonic Ladder

We commence the experiment from the bottom of the ladder, starting with the $|00\rangle$ state. As the frequency ω_d of the microwave drives applied to the qudit pair is swept across the region around the median qudit frequency in the dressed frame, a coherent oscillation between $|00\rangle$ and $|11\rangle$ appears and forms a chevron pattern with respect to the drive frequency and pulse duration (Fig. 5.4a). A pulse ramping time of $\tau_r = 100$ ns is used to ensure adiabaticity [185],

which is verified by the absence of qudit population in the intermediate states following the drive in all of the following measurements. We note that (i) shorter ramping times can be used, depending on the drive parameters, and (ii) shortcut-to-adiabaticity techniques may help reduce the total interaction times further [185].

The procedure is then extended to induce coherent energy exchange between other bipartite qudit states. Notably, within the $|11\rangle$ and $|22\rangle$ chevron, we observe an energy-conserving exchange interaction between $|11\rangle$ and $|02\rangle$ at certain drive frequencies, as shown in Fig. 5.4b. This interaction is due to the resonance condition in the dressed frame between these states, previously reported in Ref. [185], further validating our understanding of the strong drive effects in the system. We can use our chevron measurements as a detection of such a frequency collision and avoid the unwanted resonance by changing the optimal two-photon drive frequency through adjusting the ratio of drive strengths. Having done so, we are able to realize the desired interactions without unwanted effects. As Fig. 5.4c shows, the fast oscillation between $|11\rangle$ and $|22\rangle$ at the optimal driving frequency (red arrows in Fig. 5.4b) remains highly coherent for longer than a microsecond without any spurious effect, indicating the robustness of the interaction and lack of unwanted swapping to the nearby spurious transition to $|02\rangle$.

We then repeat the measurement at different drive amplitudes to thoroughly explore the two-photon drive dynamics. We note that microwave line crosstalk and the presence of other energy levels lead to a deviation of the optimal driving frequency from the median frequency between the pertinent multi-qudit levels, which can be accounted for by our analytical model (Fig. 5.4d). Notably, this reveals that the optimal drive frequency can be modified by changing the ratio of the drives, which we already verified numerically in our theoretical treatment of the two-photon drives.

The population oscillation observed at the optimal drive frequency is then used to extract the interaction rate, as shown in Fig. 5.4e. Our analytical model (Eq. (5.4)) again exhibits excellent agreement with the experimental data. Notably, the higher we climb up the ladder, the faster oscillations we observe. We attribute this to the larger matrix elements of the exchange interaction at higher qudit levels. The accelerated interaction crucially allows for shorter gate times that may compensate for the reduced coherence times of these transitions. The small deviation of the obtained data from the analytical solution for $|22\rangle \leftrightarrow |33\rangle$ interaction rates prompts a more detailed theoretical investigation in the future.

We additionally emphasize that while the previous discussion focused on the two-photon dynamics in the $|k, l\rangle$ -spanned subspace for equal excitation quanta in each qudit, $k = l$, the process is also applicable for the general case, $k \neq l$, provided there is coupling between the intermediate states $|k + 1, l\rangle$ and $|k, l + 1\rangle$. For example, it can be leveraged to induce the energy exchange between $|01\rangle$ and $|12\rangle$ as we showcase in Fig. 5.5. The first evidence appears in a spectroscopy measurement in which we prepare the qudits to be in $|01\rangle$ and then sweep the drive frequency across a wide range, as shown in Figure 5.5a. Using the single-shot readout capability, we can deterministically compute the probability of the qudits in specific states after applying the drives at each frequency point.

The coherent population transfer is then further verified by performing a time-domain

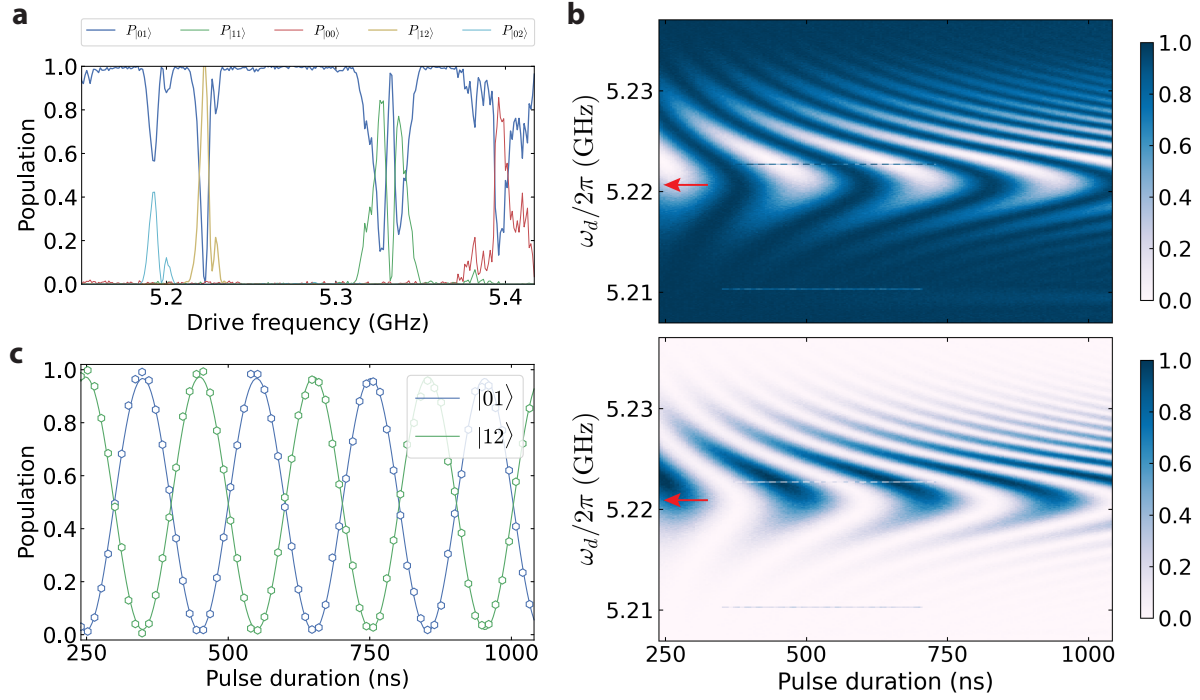


Figure 5.5: $|01\rangle \leftrightarrow |12\rangle$ **interaction**. (a) Transition spectrum of a pair of qudits prepared in $|01\rangle$, revealing the sign of the transition. (b) Chevron pattern of the transition with varying drive frequency and pulse duration. (c) Coherent driven oscillation between $|01\rangle$ and $|12\rangle$ originating from the two-photon dynamics. The drive frequency is indicated by the red arrows in panel b.

measurement of the qudit population after applying pulses with varying durations. As the drive frequency and duration are swept across the preliminary data point in the spectral domain, the exchange interaction manifests as a chevron pattern in Figure 5.5b. Fixing the drive frequency at the optimal (symmetric) point, we can inspect the microwave-induced oscillation between $|01\rangle$ and $|12\rangle$, as shown in Figure 5.5c.

5.5 High-Dimensional Entanglement

The capacity of a quantum system is often associated with how efficiently entanglement can be generated within it. The two-photon interaction is particularly well-suited to propagate entanglement to the higher levels in coupled qudit systems. By traversing the dual bosonic ladder, we proceed to create qudit Bell states with $d = \{2, 3, 4\}$. By leveraging the continuous nature of the two-photon $|k, l\rangle \leftrightarrow |k + 1, l + 1\rangle$ rotations, we can generate arbitrary qudit Bell states with a linear number of two-photon gates. First, note that for qudit-dimension

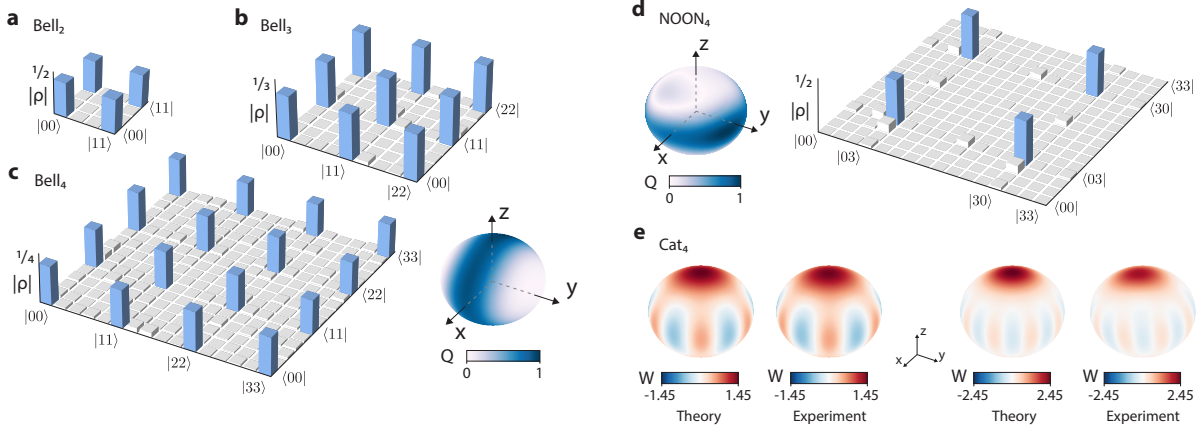


Figure 5.6: **High-dimensional entanglement.** (a) Density matrix of the qubit ($d = 2$) Bell state with a raw (purified) fidelity of 99.2% (99.9%). (b) Density matrix of the qutrit ($d = 3$) Bell state with $\mathcal{F} = 97.7\%$ (99.6%). (c) Density matrix (left) and Husimi-Q distribution (right) of the ququart ($d = 4$) Bell state with $\mathcal{F} = 94.3\%$ (99.3%). (d) Husimi-Q distribution (left) and density matrix (right) of the ququart ($d = 4$) NOON state $|30\rangle + e^{i\alpha}|03\rangle)/\sqrt{2}$ with $\mathcal{F} = 94.6\%$ (97.3%). (e) Wigner functions of the high-dimensional atomic cat states, $(|00\rangle + e^{i\alpha}|33\rangle)/\sqrt{2}$ with $\mathcal{F} = 98.6\%$ (99.3%) (left) and $(|000\rangle + e^{i\alpha}|333\rangle)/\sqrt{2}$ with $\mathcal{F} = 80.1\%$ (90.9%) (right).

d , we define a *Bell* state to be the pure and maximally entangled state given as

$$|\text{Bell}_d\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i\phi_k} |kk\rangle, \quad (5.16)$$

where we allow $\phi_k \in [0, 2\pi]$ to be an arbitrary phase on each state $|kk\rangle$ for $k = \{0, 1, \dots, d-1\}$.

The Bell_d states can be compactly generated from the vacuum (ground) state $|00\rangle$ with selective two-photon drives between $|k, k\rangle \leftrightarrow |k+1, k+1\rangle$. In particular, a Bell_2 state is realized via a single π rotation of the $|00\rangle \leftrightarrow |11\rangle$ transition. A Bell_3 state is formed by a $2\pi/3$ rotation between $|00\rangle \leftrightarrow |11\rangle$ followed by a π rotation between $|11\rangle \leftrightarrow |22\rangle$, where the state progresses (up to arbitrary phases ϕ_k) as $|00\rangle \rightarrow \sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle \rightarrow \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. Similarly, Bell_4 requires a $3\pi/4$ rotation between $|00\rangle \leftrightarrow |11\rangle$, followed by a $2\pi/3$ rotation between $|11\rangle \leftrightarrow |22\rangle$, and finally a π rotation between $|22\rangle \leftrightarrow |33\rangle$.

To verify our ability to generate these high-dimensional entangled states, we perform quantum state tomography on the output using $\text{SU}(4)$ local rotations. The results are shown in Fig. 5.6a-c. To estimate the effects of decoherence on the reconstructed states, we

compute the fidelity with and without McWeeny purification [166], which finds the nearest idempotent matrix. This purification leads to a significant improvement of the quantum state fidelities compared to the unpurified results, highlighting that decoherence is a primary factor affecting the results. This can serve as a first-order benchmarking of our control scheme. Notably, we observe a minor decrease in fidelity for purified states of larger dimension d , which is attributable to coherent unitary errors that grow as we perform more subspace entangling gates.

The qudit quantum state tomography protocol we implement has been previously applied to isolated [15, 150, 33] as well as multipartite qudit states [84, 19, 37] and was also utilized in the previous chapter. To review, to reconstruct an arbitrary single qudit state, we perform projections over an informationally complete set of local rotations. To limit the effects of the state preparation and measurement (SPAM) errors, we choose these to be closest to the native gate set of our device. We recall that the Gell-Mann operators, given by $X^{k,l} = |k\rangle\langle l| + |l\rangle\langle k|$ and $Y^{k,l} = i(|k\rangle\langle l| - |l\rangle\langle k|)$, correspond to the native rotations induced by Rabi oscillations at $\omega_{k,k+1}$ for the case of $l = k + 1$. We thus choose the natural set of local projections for state tomography as projections onto all the computational states $|0\rangle, \dots, |d-1\rangle$ as well as two-level $\sqrt{X^{k,k+1}}$ and $\sqrt{Y^{k,k+1}}$ projections within each subspace, or explicitly, the set of projectors M given by

$$\begin{aligned} M = \{ & I, X^{0,1}, X^{1,2}X^{0,1}, X^{2,3}X^{1,2}X^{0,1}, \sqrt{X^{0,1}}, \sqrt{Y^{0,1}}, \\ & \sqrt{X^{1,2}}X^{0,1}, \sqrt{Y^{1,2}}X^{0,1}, X^{1,2}\sqrt{X^{0,1}}, X^{1,2}\sqrt{Y^{0,1}}, \\ & X^{2,3}X^{1,2}\sqrt{X^{0,1}}, X^{2,3}X^{1,2}\sqrt{Y^{0,1}}, X^{2,3}\sqrt{X^{1,2}}X^{0,1}, \\ & X^{2,3}\sqrt{Y^{1,2}}X^{0,1}, \sqrt{X^{2,3}}X^{1,2}X^{0,1}, \sqrt{Y^{2,3}}X^{1,2}X^{0,1} \}. \end{aligned} \quad (5.17)$$

For the projections of an n -qudit state, we simply apply the projections in $M_n = M^{\otimes n}$.

To further mitigate the effects of SPAM measurements, we characterize the readout confusion matrix over the relevant n qudit states $\{|0\rangle \dots |d-1\rangle\}^{\otimes n}$. Given the fidelity of our native operations characterized by RB, we can assume that the readout assignment errors are dominant. This allows us to account for the measurement errors by applying the inverse of the confusion matrix to our averaged ditstring results. Then, the qudit density matrices are reconstructed via maximum likelihood estimation (MLE). The state fidelity is computed following the relation

$$\mathcal{F}(\rho, \sigma) = \text{Tr}(\sqrt{\rho}\sigma\sqrt{\rho}) \quad (5.18)$$

where ρ is the experimentally reconstructed density matrix, and σ is the ideal density matrix. Finally, we note that the diagonal terms in the density matrix can be measured directly to reduce fluctuation effects arising during the long tomography measurement.

Next, it is worth pointing out that the cascaded two-photon interactions are reminiscent of the two-mode squeezing effect in quantum optics, which is described by the operator $\hat{S}(\zeta) = \exp(\zeta\hat{a}^\dagger\hat{b}^\dagger - \zeta^*\hat{a}\hat{b})$, where ζ is the squeezing strength, and $\hat{a}$ ($\hat{b}$) is the annihilation operator on the first (second) mode. This results in a resemblance between the high-dimensional Bell states and photonic two-mode squeezed states, $|\Psi\rangle_{2\text{ms}} \propto \sum_{N=0}^{\infty} c_N |NN\rangle$, where c_N is a

coefficient dependent on ζ and N is the quanta number [36]. In the present system, N effectively ranges from 0 to $d - 1$. With this motivation in mind, we explore the visualization of our high-dimensional qudit system using methods typically associated with bosonic systems.

We verify the squeezing effect by analyzing the Bell_4 state using the Husimi-Q quasiprobability distribution (QPD). For more details on how we transform the standard qudit state tomography protocol to visualizing the qudit in terms of QPDs (both Wigner and Husimi-Q), we direct the reader to Appendix B. In this section, we use the collective spin coherent state as the basis state with isotropic QPD [9]. In contrast, a squeezed state would manifest as an elongated strip with a shrunk QPD along a geodesic on the sphere [135]. As shown in Fig. 5.6c(right), the measured state indeed displays squeezing along the y-axis, confirming the expected outcome. Interestingly, the observed QPD resembles the results associated with atomic squeezed states generated by the two-axis countertwisting Hamiltonian [135].

The NOON state is another important entangled state in quantum metrology, recognized for its exceptional sensitivity in measuring phase differences in interferometric experiments. It is defined as $|\Psi\rangle_{\text{NOON}} = (|N0\rangle + e^{i\alpha}|0N\rangle)/\sqrt{2}$, where N is equal to $d - 1$ in qudits. We proceed to utilize a combination of a two-photon process and single-qudit rotations to construct a high-fidelity NOON_4 state, as depicted in Fig. 5.6d. Notably, the Husimi-Q QPD verifies that the NOON state exhibits squeezing along the geodesic orthogonal to that of the Bell states.

Atomic Schrödinger cat states, which are superpositions of macroscopically distinct quantum states, belong to another category of entangled states that attract considerable interest [4]. In an n -qudit system where each qudit has dimension $d + 1$, the coherent states of interest are $|00\dots 0\rangle$ and $|dd\dots d\rangle$ [135]. Using the two-photon interaction, we examine two different high-dimensional atomic cat states, $|00\rangle + e^{i\alpha}|33\rangle$ and $|000\rangle + e^{i\alpha}|333\rangle$. To characterize these highly entangled states, we employ the Wigner function, which is known to allow the visualization of cat states due to its inherent capacity to accommodate negative values [216]. The experimental results in Fig. 5.6e show the fringes characteristic of cat states, verifying our analytical prediction. Importantly, qudit cat states have been proposed as the building block for novel quantum error correction schemes [192], so this experimental implementation may serve as the primitive for future resource-efficient fault-tolerant architectures.

Distribution of qudit entanglement

The process of swapping populations between the qudit states $|k, l\rangle$ and $|k + 1, l + 1\rangle$ is functionally equivalent to a subspace energy-conserving swap process $|k + 1, l\rangle \leftrightarrow |k, l + 1\rangle$. This mechanism is crucial in constructing high-dimensional quantum circuits and is particularly important for distributing entanglement across a qudit array with limited connectivity. To examine this feature, our approach begins with the preparation of a two-qudit Bell_3 state (Fig. 5.7a). Following this, we engage in a two-pronged approach: firstly, we focus on redistributing entanglement among distant qudits, and secondly, we aim to extend entanglement throughout the entire qudit array. This dual strategy explores the breadth of entanglement

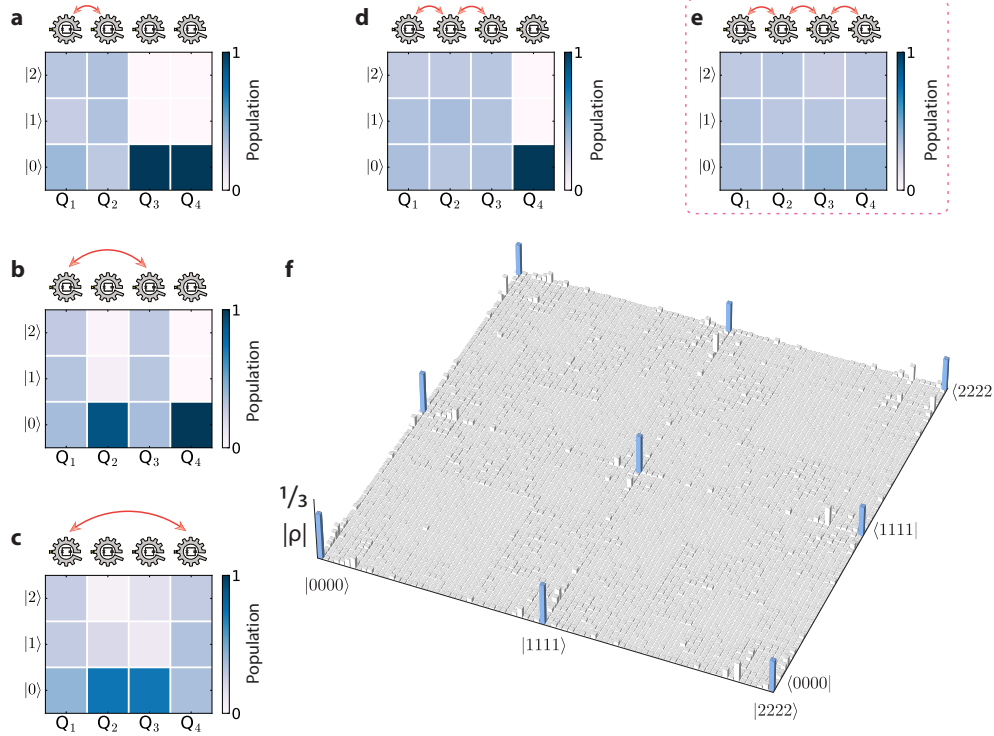


Figure 5.7: **Distribution of qudit entanglement.** Measured populations of (a) Q_1 - Q_2 Bell₃ state, (b) Q_1 - Q_3 Bell₃ state, (c) Q_1 - Q_4 Bell₃ state, (d) Q_1 - Q_2 - Q_3 GHZ₃ state, (e) Q_1 - Q_2 - Q_3 - Q_4 GHZ₃ state. (f) Density matrix of the four-qudit GHZ₃ state. The raw and purified fidelities are 70.6% and 82.7%, respectively.

distribution possibilities in high-dimensional quantum systems with larger Hilbert space compared to architectures based on qubits.

To approximate the effectiveness of the operations, we directly measure the populations of the participating qudit states after implementing the sequences. This focuses on examining the diagonal elements of the multi-qudit density matrices and provides a straightforward approach to assess the impact of the swap operation. The observed transfer of population to adjacent qudits, as depicted in Fig. 5.7b,c, confirms the successful redistribution of populations across the qudit array.

Additionally, we verify their entanglement by measuring the fidelities of the resulting Bell₃ states between Q_1 - Q_3 and Q_1 - Q_4 (see Fig. 5.8). The results show quantum state fidelities of 75.7% for the unpurified density matrices and 96.4% for the purified ones in the case of Q_1 - Q_3 , and 53.0% (unpurified) versus 95.7% (purified) for Q_1 - Q_4 . The substantial discrepancy between the unpurified and purified results indicates that decoherence is the primary source of error in the operations.

Notably, we find that expanding the entanglement space to encompass other qudits re-

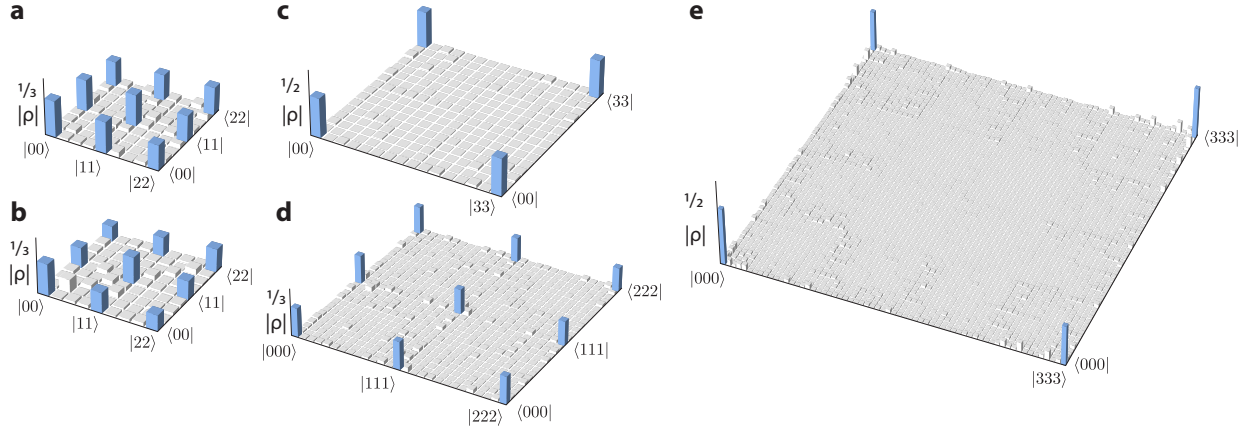


Figure 5.8: **Extended tomography results.** (a) Q_1 - Q_3 qutrit Bell state. The raw (purified) fidelity is 75.7% (96.3%). (b) Q_1 - Q_4 qutrit Bell state with $\mathcal{F} = 53.0\%$ (95.7%). (c) Schrödinger cat state $(|00\rangle + |33\rangle)/\sqrt{2}$ with $\mathcal{F} = 98.6\%$ (99.3%). (d) High-dimensional GHZ state with $n = d = 3$ and $\mathcal{F} = 90.8\%$ (96.4%). (e) Schrödinger cat state $(|000\rangle + |333\rangle)/\sqrt{2}$ with $\mathcal{F} = 80.1\%$ (90.9%).

quires fewer subspace swaps, in comparison with entanglement redistribution. This leads to the efficient generation of multi-qudit GHZ_3 states. The population measurement results show minimal imperfections, as illustrated in Fig. 5.7d,e.

We proceed to measure the four-qudit GHZ_3 state achieving a fidelity of 70.6% (82.7%) for unpurified (purified) density matrix, which is shown in Fig. 5.7f. While the difference between these metrics originates from unavoidable decoherence, we attribute the infidelity of the purified result to SPAM errors occurring during the long tomography sequences. In comparison, the three-qudit GHZ_3 unpurified (purified) density matrix has a fidelity of 90.8% (96.4%) (see Fig. 5.8). This implies the need for more efficient and SPAM-free verification methods and benchmarking tools in the future, which we expect to provide a more accurate assessment of the robustness of our procedure.

5.6 Efficient N-Body Qubit Gates with Higher Level Ancilla

As motivated in the introduction to this chapter, the higher levels of qudits can be utilized for shelving (or use as ancillary states), allowing access to otherwise resource-expensive operations [64, 45, 185, 188, 111]. Here, we present a new method to synthesize high-fidelity multi-qubit gates using the two-photon interactions. The gate sequence, shown in Fig. 5.9a, involves sandwiching a cross-Kerr two-qutrit gate [85] (generated using the methods described in the previous chapter) on Q_1 - Q_2 between two $|11\rangle \leftrightarrow |22\rangle$ swap gates

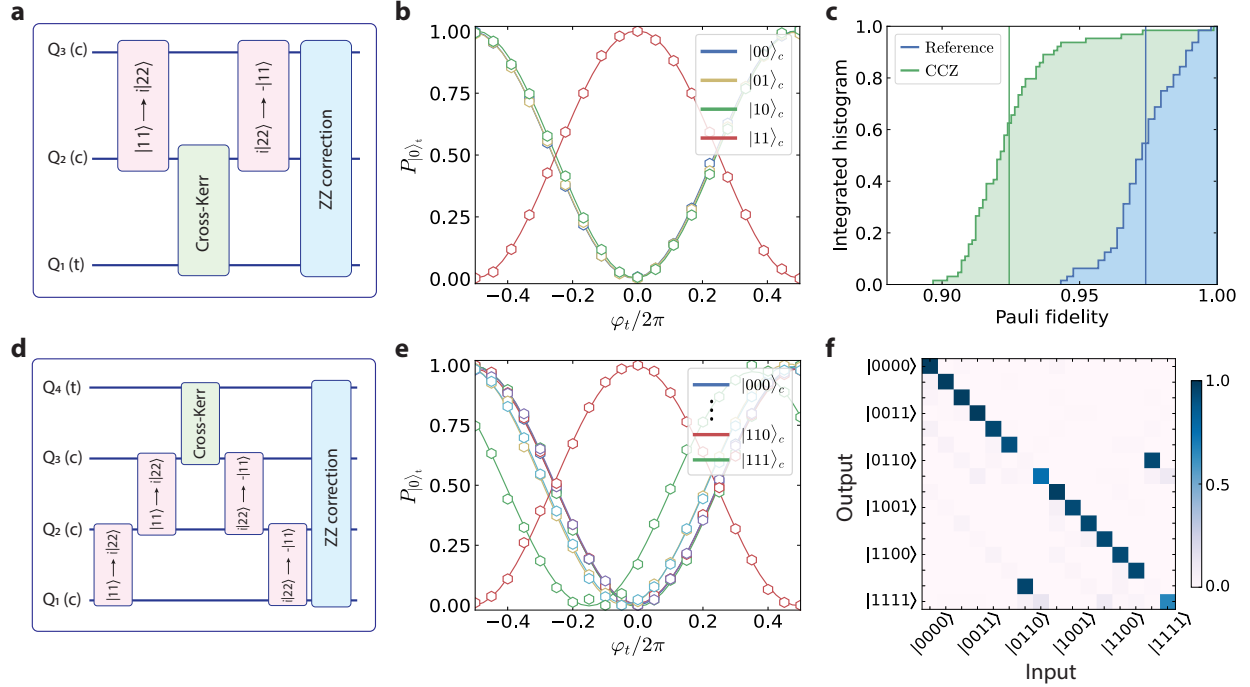


Figure 5.9: **Implementation and verification of multi-qubit gates.** (a) Gate sequence to implement a CCZ unitary. The qutrit swap gates (pink) are used to shelve and then retrieve the control state $|11\rangle_c$. They sandwich a cross-Kerr gate (green) that induces a Z gate on Q_1 if and only if Q_2 is in $|2\rangle$. The final stage (blue) is used to correct the residual ZZ phases between the qubits. (b) The three-body operation manifests as the phase shift of the target qubit (Q_1) when the control qubits (Q_2 and Q_3) are in $|11\rangle_c$. The solid lines represent cosine fits. (c) Pauli fidelities of the dressed cycle and the reference cycle from cycle benchmarking. The resulting gate fidelity is $\mathcal{F}_{\text{CCZ}} = 96.0(3)\%$. (d) Gate sequence to implement a CCCZ unitary. A cascade of qutrit swap interactions (pink) is used to shelve and retrieve the respective $|11\rangle_c$ states. In the middle of the sequence is a cross-Kerr gate (green) that induces a Z gate on Q_4 if and only if Q_3 is in $|2\rangle$. All the residual correlated phases are corrected in the final stage (blue). (e) The four-body operation is effectively revealed through a π -phase-shift of the target qubit (Q_4) for the control state $|110\rangle_c$. The solid lines represent cosine fits. (f) Truth table of the four-qubit Toffoli gate. The target state is shown to be flipped when the control state is $|Q_3 Q_2 Q_1\rangle = |110\rangle_c$ ($|0110\rangle \leftrightarrow |1110\rangle$). The corresponding truth table fidelity is $\mathcal{F}_{\text{CCCZ}} = 92(1)\%$.

on Q_2 - Q_3 . First, the two-photon swap is used to substitute and then retrieve the control state $|11\rangle_c$. Then, the implemented cross-Kerr unitary between Q_1 and Q_2 applies a π -phase twist to Q_1 's $0-1$ subspace if and only if Q_2 is in its $|2\rangle$ state, effectively leading to a phase shift of the tripartite input state $|111\rangle$. This is ultimately achieved by using the Stark drives

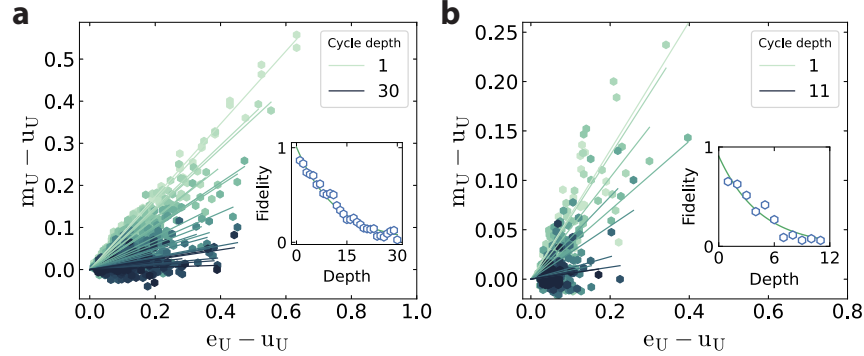


Figure 5.10: **Cross-entropy benchmarking of n-body gates.** (a) CCZ result. The differences in the normalized linear cross-entropies for ideal ($m_U - u_U$) and measured ($e_U - u_U$) are shown, which can be used to extract the cycle fidelity. Varying the cycle depth results in an exponential decay (inset) that embodies the dressed cycle fidelity. (b) CCCZ result. The increased cycle error cuts down the maximum cycle depth required.

method from Chapter 4 to find an effective Hamiltonian such that in Eq. 4.11 $\alpha_{11} = 2\alpha_{12}$ such that for some drive time τ , $\alpha_{11} \times \tau = 2\pi$ and $\alpha_{12} \times \tau = \pi$. In addition, the off-resonant drives introduce spurious ZZ coupling in the qubit subspace, which we take an additional step to negate using dynamical cross-Kerr pulses.

This method is in essence similar to the sequence utilizing $|11\rangle \leftrightarrow |02\rangle$ energy-exchange interaction performed in Refs. [64, 210, 45, 185], with the notable exception of exchanging Q_2 's $|1\rangle$ state with $|2\rangle$ instead of $|0\rangle$. This switching thus necessitates a cross-Kerr interaction that induces an entangling phase in the qutrit subspace spanned by $|0\rangle$ and $|2\rangle$ instead of the regular qubit ZZ gate. To actually perform this, the lessons learned from the full two-qutrit gates in the previous section proved instrumental. Notably, this gate is in essence a mixed-radix, or qubit-qutrit gate, which actually significantly simplifies the control problem over the two-qutrit gate performed in the last chapter.

The entangling effect can be verified using a modified Ramsey sequence to probe the phase φ_t of Q_1 for different Q_2 and Q_3 states [185]. As we apply the CCZ gate sequence, the target phase φ_t corresponding to the control state $|11\rangle_c$ displays a π shift relative to the others, signifying the three-body entanglement (Fig. 5.9b). We then employ cycle benchmarking [62] (CB) to characterize the precision of the realized operation. By comparing the dressed cycle fidelity with the reference cycle fidelity, as shown in Fig. 5.9c, we estimate a three-qubit gate fidelity of 96.0(3)%.

We further add a microwave-induced interaction at the end of the sequence to completely cancel residual correlated ZZ errors arising from static couplings and microwave crosstalks. The realized unitary then amounts to a control-control-Z, where the phase of the target qubit is flipped only when the control qubits are in their excited states, hence the designated

control state is $|11\rangle$. This is in contrast with earlier works where the designated control state is $|10\rangle$ [64, 210, 45, 185]. We proceed to benchmark the performance of the gate using cycle benchmarking [62]. Comparing the fidelity of the dressed cycle to that of the reference cycle (Fig. 5.9(c)), we extract a gate fidelity of 96.0(3)%. Besides the decoherence errors during the gate sequence, we attribute 1 – 2% error to leakage out of the qutrit subspace. We believe that it is rooted from the strong microwave crosstalk in the test device, through which the cross-Kerr gate can induce $|2\rangle \rightarrow |3\rangle$ transitions in neighboring transmons.

To further demonstrate the extensibility of this approach, we implement a four-qubit entangling gate by cascading the shelving steps as shown in Fig. 5.9d. To showcase its versatility, we presently place the cross-Kerr gate between Q_3 - Q_4 and perform the shelve-retrieve operations on the other pairs. Together with the ZZ correction at the end, the sequence effectively applies a π -phase-shift to the input state $|Q_4Q_3Q_2Q_1\rangle = |1110\rangle$. We again verified this through a conditionality measurement, as shown in Fig. 5.9e. We note that there now exists a coherent error that manifests as a small phase shift in φ_t when the control qubits are in $|111\rangle_c$ state, which is most likely caused by uncanceled microwave crosstalk generating spurious cross-Kerr terms.

Since finding the inverse unitary is challenging for a four-qubit circuit, CB cannot be used to determine the gate fidelity efficiently. Alternatively, we validate the operation by sandwiching it between two Hadamard gates on Q_4 and measuring the qubit states for different combinations of input states. The truth table summarizing the outcomes in Fig. 5.9f shows the flipping of the target qubit (Q_4) when the control state is $|Q_3Q_2Q_1\rangle = |110\rangle$. The corresponding truth table fidelity is $\mathcal{F}_{\text{CCZ}} = 92(1)\%$, on-par with the best results previously reported in superconducting qubits [45] and trapped-ions [125]. A subsequent XEB measurement of the CCCZ gate yields a fidelity of 88(2)% (see Fig. 5.10), implying small phase errors not present in the truth table.

5.7 Outlook

By combining the ease of quantum control and measurement characteristic of qubit architectures with the concept of an expansive Hilbert space inherent to bosonic systems, in this chapter we systematically showcased the operational principles of a multi-qudit device based on Raman-assisted two-photon interactions. In contrast with other dynamical processes in strongly driven systems, the reported quantum dynamics remain largely independent of non-linearity [144, 91] or level-matching [200]. In principle, as we showed in Fig. 5.4d, the optimal drive frequency can be tuned *in situ* by adjusting the drive amplitudes and their relative ratio, which can critically aid in avoiding frequency collisions which plague fixed frequency quantum systems. These advantages enhance the extensibility and adaptability of the process, providing a crucial interaction repertoire to high-dimensional solid-state systems.

As the simplicity and robustness of fixed-frequency superconducting circuits with fixed coupling have allowed them to be scaled up to devices integrating hundreds of qubits, our demonstrations using the same type of device without additional circuitry or instrumentation

represent an avenue to advance the toolkit for existing quantum technologies through high-dimensional quantum operations. On the one hand, this work opens the door for exploring quantum sensing and quantum information processing with high-dimensional systems, and on the other hand, the results encourage the future development of bosonic encodings, such as spin-cats [192], using superconducting qudits.

Nonetheless, there remain a few downsides to this two-photon driving approach. For one, unlike in the last chapter, the two-qudit gates in this chapter do not commute with the quantum crosstalk, which we expect to limit their fidelity in a generic qudit circuit. Moreover, the reason these gates could not be benchmarked as two-qudit unitaries at the time of this experiment is that the exchange of energy between qudits means that they are not compatible with virtual Z gates unless proper frame tracking is implemented [268] or novel compilation for arbitrary single qudit gates is developed that removes trailing virtual Z gates [40]. We do note that quite a bit of benchmarking still exists in this chapter for these interactions, both in the form of their use in generating multi-qubit gates which sidestep this virtual Z gate problem, and in the tomography of the entangled states generated via these two-photon gates. Also, fortunately the virtual Z gate compatibility issue can be addressed at the level of software and compilation. We also expect that the quantum crosstalk terms that do not commute with the $|k, l\rangle \leftrightarrow |k + 1, l + 1\rangle$ swaps could be absorbed into the definition of the two-qudit unitaries for NISQ applications like two-qubit f-sim gates in the literature [74]. In principle, with these challenges addressed, the two-qudit gate framework presented in this chapter could serve as a high-fidelity, continuously parameterizable, and expressive gate set for universal qudit quantum computation.

Chapter 6

Qudit Error Suppression

So far in this thesis, we have presented how to encode, readout, and manipulate individual and coupled qudit systems. In the process, we have developed a toolkit that enables us to calibrate and operate superconducting quantum processors as qudit-based systems, unlocking, in principle, the ability to execute arbitrary qudit quantum algorithms. But readers should also have a clear sense, at this point, that significant errors persist in all of our operations. In our motivation for developing a qutrit CZ gate, we already discussed one way out: quantum error correction. Indeed, for applications requiring extraordinary numbers of operations (think 10^8 gates and beyond), there really seems to be, at least at present, no other meaningful route to their realization.

Nonetheless, comparatively “shallow” quantum circuits on large arrays of qubits have already stretched or exceeded the limits of classic supercomputers for certain well-tailored tasks [10, 134, 175]. It is very natural then to ask, within the context of existing noisy quantum devices, how to most effectively improve the signal to noise ratio we can extract from running quantum algorithms in the pre fault-tolerant era. One can broadly categorize techniques that perform these signal-to-noise improvements, without the heavy qubit-count overhead of quantum error correction, under the umbrella of “error suppression.” Within this category, multiple complementary approaches have been developed, including techniques based on tailoring and extrapolating beyond noise — often called “error mitigation” [26] — and circuit-level techniques that attempt to lower or remove noise such as robust pulse shaping and dynamical decoupling [63].

Within the context of qudit-based quantum computing, the noise becomes significantly more complicated. As we discussed previously, with increasing qudit dimension d there are now quadratically more quantum crosstalk terms as well as significantly increased sensitivity for noise and channels for decoherence. A pressing question, then, in the context of the near-term viability of these devices for similar “shallow” quantum circuit tasks, is if they can still benefit and leverage similar error suppression techniques which have shown promise in qubit applications?

In this chapter, we explore the development and implementation of error suppression techniques for our superconducting qudit systems. In particular, in the first part of the

chapter we cover our work in Ref. [248] to suppress qudit quantum crosstalk and system-bath decoherence via dynamical decoupling. In the second part of the chapter we cover our work in Ref. [84] to develop and implement noise-tailoring and post-processing based qudit quantum error mitigation schemes.

6.1 Qudit Dynamical Decoupling

To begin our exploration of error suppression for qudit systems, we start with a review of the simplest DD experiment: the Hahn-echo. Then we will present a general DD framework tailored for high dimensional quantum systems and experimentally demonstrate its effectiveness using coupled superconducting transmon circuits operated as qutrits ($d = 3$) and ququarts ($d = 4$). Our framework employs the Weyl-Heisenberg group, a group with which readers should now be quite familiar. We present DD sequences for universal noise suppression and also introduce a single-axis DD sequence designed to suppress the prevalent $1/f$ dephasing noise that plagues superconducting qudits. We then introduce a multi-qudit DD sequence designed to suppress unwanted cross-Kerr interactions between coupled qudits (see Fig. 6.1), which stand in the way of scaling superconducting qudit systems. Using our DD sequences, we additionally report a significant enhancement in preserving the fidelity of a qutrit Bell state over time.

Introduction to Dynamical Decoupling

Dynamical decoupling (DD) [257, 284, 256, 55, 259, 244] is a powerful technique designed to enhance the fidelity of quantum states by employing carefully timed control pulses. Its effectiveness has already been demonstrated in the context of decoupling superconducting qubits from environmental noise [199, 63, 25] and unwanted crosstalk [249, 291, 24, 222, 229]. DD has been studied across a broad spectrum of qubit-based systems, but until recently, its experimental application to qudits has been limited primarily to trapped ions and nitrogen-vacancy ensembles [281, 290, 180, 260], and to enhance the fidelity of a qutrit-assisted (similar to the n -qubit gate we developed in Chapter 5) three-qubit Toffoli gate on an IBM transmon device [111].

The principle underlying DD has its origins in the spin echo experiment introduced by Hahn in 1950 in the context of nuclear magnetic resonance [96]. The spin echo remains the simplest and most instructive example of dynamical decoupling, and its logic extends directly to the multi-pulse sequences used in modern superconducting qubit experiments.

The Hahn Echo

Consider a transmon qubit initialized in the ground state $|0\rangle$, whose Bloch vector points along $+z$. A resonant $\pi/2$ microwave pulse rotates the Bloch vector into the equatorial

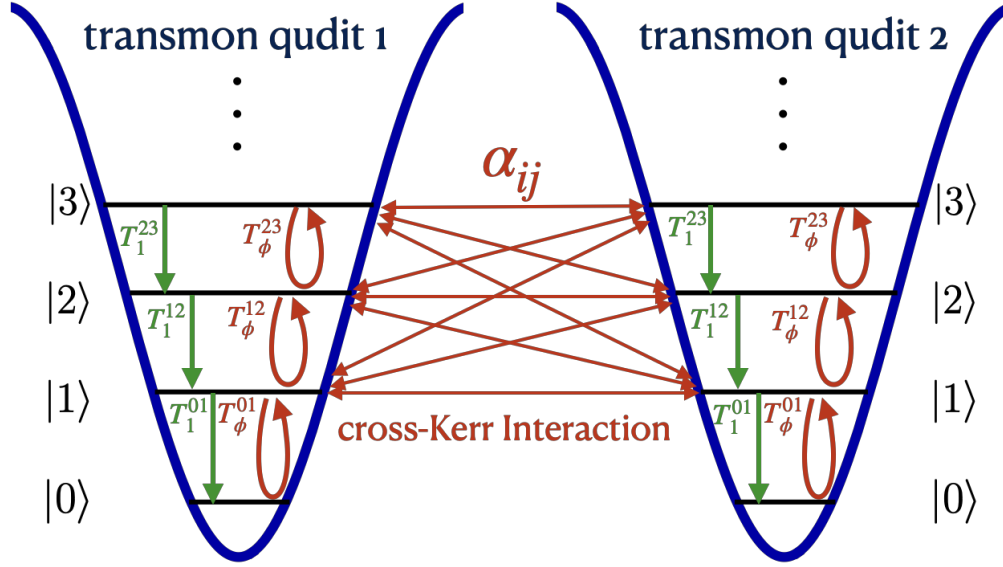


Figure 6.1: Schematic illustration of two transmon qudits with quantized energy levels affected by relaxation and dephasing errors, along with the qudit-qudit cross-Kerr couplings α_{ij} .

plane of the Bloch sphere, placing the qubit in the superposition

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \quad (6.1)$$

Once in the equatorial plane, the qubit precesses about the z -axis at its transition frequency ω_{01} . In the rotating frame this precession ideally vanishes, but in practice low-frequency noise causes ω_{01} to deviate from its nominal value by a small, slowly-varying amount $\delta\omega(t)$. Over a free evolution time τ , the qubit accumulates a stochastic phase

$$\phi = \int_0^\tau \delta\omega(t) dt, \quad (6.2)$$

and the Bloch vector drifts away from its initial direction. In an ensemble, or equivalently, across repeated experimental shots, different realizations of $\delta\omega(t)$ produce different values of ϕ , causing the Bloch vectors to fan out across the equatorial plane. This dephasing is characterized by the timescale T_2^* or T_2 Ramsey time, discussed earlier in Chapter 2.

The key insight of the spin echo is that low-frequency dephasing can be reversed. After the free evolution period τ , a resonant π pulse is applied. This pulse acts as a mirror: it flips the Bloch vector across the equatorial plane, exchanging the roles of faster and slower precessing components. Spins that had accumulated a positive phase now lag behind the reference frame, and those that lagged now lead. Provided the noise $\delta\omega(t)$ is approximately

constant over the total sequence duration 2τ , which holds when the noise correlation time is long compared to 2τ , as is the case for $1/f$ noise in transmons, the dephasing accumulated during the second free evolution period $[\tau, 2\tau]$ exactly cancels that of the first. At $t = 2\tau$ all Bloch vectors refocus coherently, forming the spin echo.

To measure the qubit state, a second $\pi/2$ projection pulse is applied immediately after the refocusing, rotating the recovered superposition state back to the $\pm z$ axis for dispersive readout. The full pulse sequence therefore reads

$$\frac{\pi}{2} - \tau - \pi - \tau - \frac{\pi}{2} - \text{readout}, \quad (6.3)$$

and is illustrated in Fig. 6.2. The population measured in $-z$ should in principle follow an amplitude decay of $e^{-2\tau/T_{2e}}$ rather than $e^{-2\tau/T_2^*}$, reflecting the fact that static and quasi-static inhomogeneity is refocused while only true irreversible relaxation, i.e. energy decay at rate $\Gamma_1 = 1/T_1$ and high-frequency dephasing, contributes to the measured decay. The ratio T_{2e}/T_2^* thus quantifies how much of the dephasing is low-frequency and therefore recoverable by refocusing pulses.

Dynamical Decoupling Theory

Before jumping into qudit based dynamical decoupling, let's review the machinery of qubit based dynamical decoupling following Ref. [63]. We begin by considering a time-independent Hamiltonian,

$$H = H_S + H_B + H_{SB}, \quad (6.4)$$

where H_S and H_B contain terms that act only on the system and bath respectively, and H_{SB} contains the system-bath interactions. We can expand $H_S = H_S^0 + H_S^1$ where H_S^1 represents undesired, always-on terms such as crosstalk, such that,

$$H_{err} = H_S^1 + H_{SB} \quad (6.5)$$

is now our “error Hamiltonian” that will be addressed with our DD routines and H_S^0 represents all the terms we wish to keep. We can define a corresponding free unitary evolution for a duration τ as,

$$f_\tau = U(\tau) = e^{-i\tau H}. \quad (6.6)$$

We can perform DD by an additional time-dependent control Hamiltonian $H_c(t)$, so our new total Hamiltonian in the presence of this DD control is,

$$H(t) = H_S^0 + H_{err} + H_B + H_c(t). \quad (6.7)$$

In an idealized or perfect pulse sequence picture, we can consider a control Hamiltonian that is a sum of error-free instantaneous Hamiltonians $\{\Omega_0 H_{P_k}\}_{k=1}^n$ that generate the pulses at intervals $\{\tau_k\}_{k=1}^n$:

$$\hat{H}_c(t) = \Omega_0 \sum_{k=1}^n \delta(t - t_k) H_{P_k}, \quad t_k = \sum_{j=1}^k \tau_j, \quad (6.8)$$

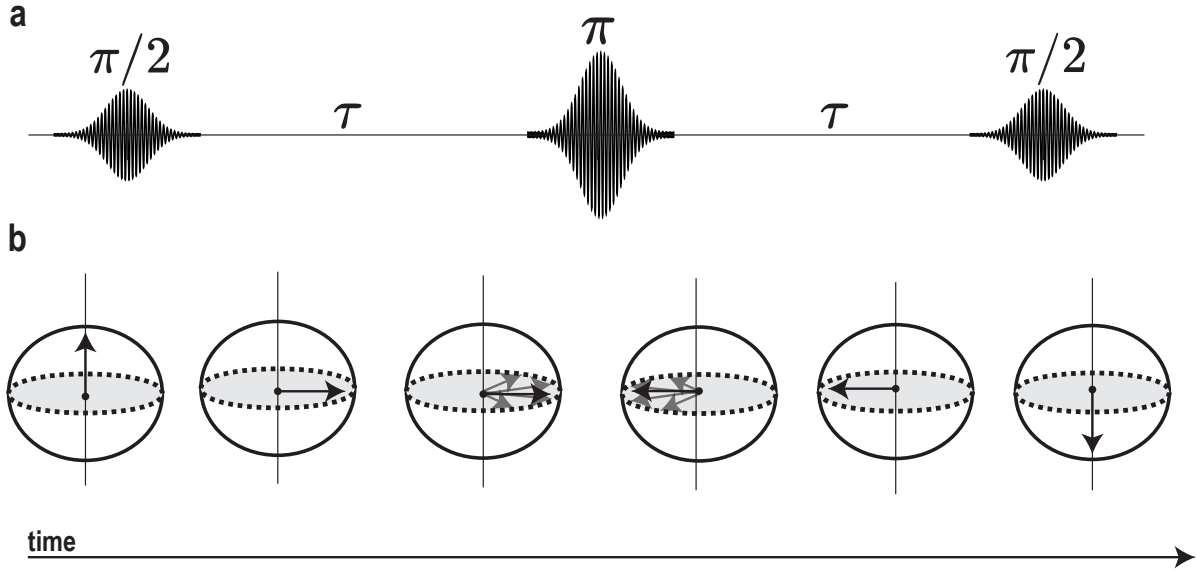


Figure 6.2: **Schematic of the Hahn echo pulse sequence.** **(Top)** Microwave pulse sequence: an initial $\pi/2$ pulse tips the Bloch vector onto the equatorial plane; a π refocusing pulse at $t = \tau$ mirrors the dephased spin components; and a final $\pi/2$ projection pulse at $t = 2\tau$ rotates the refocused state to the $\pm z$ axis for readout. **(Bottom)** Corresponding Bloch sphere snapshots: (i) equilibrium along $+z$; (ii) after the first $\pi/2$, tipped to the equator; (iii) dephasing causes the Bloch vectors to fan out; (iv) the π pulse mirrors the fan to the opposite side; (v) after another time evolution of τ , the spins refocus; (vi) a final $\pi/2$ pulse projects the qubit towards $-z$;

where the $\hat{\cdot}$ denotes ideal conditions and Ω_0 has units of energy. We can set $\Omega_0\Delta = \pi/2$ which gives rise to instantaneous unitaries,

$$\hat{P}_k = e^{-i\tau H_{P_k}/2}, \quad (6.9)$$

with a full time evolution under this idealized instantaneous pulse picture given by,

$$\tilde{U}(T) = f_{\tau_n} \hat{P}_n \dots f_{\tau_2} \hat{P}_2 f_{\tau_1} \hat{P}_1, \quad (6.10)$$

with $T = t_n = \sum_{j=1}^n \tau_j$ being the total sequence time. The resulting unitary evolution $\tilde{U}(T) = U_0(T)B(T)$ can be decomposed into the desired error-free evolution $U_0(T) = \exp(-iT H_S^0) \otimes I_B$ and the unitary error $B(T)$. If we have that $B(T) = I_S \otimes e^{-iT\tilde{B}}$ where $\tilde{B}$ is an arbitrary Hermitian bath operator, then by applying N repetitions of an ideal DD sequence of duration T , the system will stroboscopically decouple from the bath at uniform intervals of $T_j = jT$. In a real experimental setting with imperfect pulses and less idealized noise, we only ever can achieve approximate decoupling such that $B(T) = I_S \otimes e^{-iT\tilde{B}} + \text{err}$,

and therefore the goal of DD is to design pulse techniques to make this error as small as possible under realistic experimental parameters.

General theory of qudit dynamical decoupling

Let's begin by introducing a general framework for dynamical decoupling in the context of qudit quantum systems. Again, we can consider the total time-independent Hamiltonian of a system coupled to a bath $H = H_S \otimes I_B + I_S \otimes H_B + H_{SB}$ where H_S , H_B , and H_{SB} are the Hamiltonian terms associated with the system, the bath, and the system-bath interaction, respectively. Here, H_S represents undesired system terms such as crosstalk and stray local fields. Pulses are applied to the system via an additional, time-dependent control Hamiltonian. Correspondingly, the *decoupling group* $\mathcal{G}_d$ is defined by a set of unitary transformations g_j acting purely on the system: $\mathcal{G}_d = \{g_0, \dots, g_K\}$, where g_0 is the d -dimensional identity operator I . Under the instantaneous (zero width) and ideal (error-free) pulse assumptions, cycling over all elements of the group yields the following DD pulse sequence [284, 256]:

$$U(T) = \prod_{j=0}^{|\mathcal{G}_d|-1} g_j^\dagger f_\tau g_j = e^{-iTH'} + \mathcal{O}(T^2). \quad (6.11)$$

Here, τ is the pulse interval (the time between consecutive pulses), $T = |\mathcal{G}_d|\tau$ is the total time taken by the sequence, and $f_\tau = e^{-i\tau H}$ is the free-evolution unitary. The effective Hamiltonian at the end of the sequence is

$$H' = H'_{SB} + H'_S + H_B, \quad (6.12)$$

where

$$H'_{SB} = \mathcal{P}_{\mathcal{G}_d}(H_{SB}), \quad H'_S = \mathcal{P}_{\mathcal{G}_d}(H_S). \quad (6.13)$$

Here

$$\mathcal{P}_{\mathcal{G}_d}(\Omega) = \frac{1}{|\mathcal{G}_d|} \sum_{j=0}^{|\mathcal{G}_d|-1} g_j^\dagger \Omega g_j \quad (6.14)$$

is the projection of the operator Ω into the commutant of $\mathcal{G}_d$, i.e., the set of operators that commute with every element of $\mathcal{G}_d$. Crucially, this projection can be made proportional to I or even vanish via a proper choice of $\mathcal{G}_d$. When $H'_{SB} = H'_S = 0$, we call $\mathcal{G}_d$ and the corresponding DD sequence universal. For example, a universal DD sequence for a qubit ($d = 2$) is obtained by choosing the decoupling group as the Pauli group $\mathcal{G}_2 = \mathcal{P}$, which leads to the well-known XY4 sequence $U = Y f_\tau X f_\tau Y f_\tau X f_\tau$ [164].

For $d \geq 2$, we instead employ the Weyl-Heisenberg group of order d^2 , which generalizes the Pauli group. We define shift and phase operators as elsewhere in this thesis, repeated here for convenience:

$$X_d \equiv \sum_{k=0}^{d-1} |(k+1) \bmod d\rangle\langle k|, \quad Z_d \equiv \sum_{k=0}^{d-1} \gamma_d^k |k\rangle\langle k|, \quad (6.15)$$

where $\gamma_d = e^{2\pi i/d}$ is the d 'th root of unity.

Note that $X_d^d = Z_d^d = I$, and that X_d and Z_d are generally non-Hermitian but are both unitary for all d and hence satisfy

$$(X_d^\dagger)^\alpha X_d^\beta = X_d^{\beta-\alpha}, \quad (Z_d^\dagger)^\alpha Z_d^\beta = Z_d^{\beta-\alpha}. \quad (6.16)$$

In particular,

$$X_d^\dagger = X_d^{-1} = X_d^{d-1}, \quad Z_d^\dagger = Z_d^{-1} = Z_d^{d-1}. \quad (6.17)$$

X_d and Z_d are the generators of the HWG, whose elements are

$$\Lambda_{\alpha\beta} = (-\sqrt{\gamma_d})^{\alpha\beta} X_d^\alpha Z_d^\beta, \quad (6.18)$$

where $\alpha, \beta \in \mathbb{Z}_d$. For $d = 2$, the HWG trivially reduces to the Pauli group.

For the two generators X_d and Z_d , we have

$$X_d Z_d = \sum_{k=0}^{d-1} e^{2\pi i k/d} |k+1 \bmod d\rangle \langle k| \quad (6.19a)$$

$$Z_d X_d = \sum_{k=0}^{d-1} e^{2\pi i (k+1)/d} |k+1 \bmod d\rangle \langle k|, \quad (6.19b)$$

i.e.,

$$Z_d X_d = \gamma_d X_d Z_d. \quad (6.20)$$

Similarly, we find:

$$Z_d^\dagger X_d = \gamma_d^{-1} X_d Z_d^\dagger. \quad (6.21)$$

Using Eq. 6.20 and Eq. 6.21, we can show that

$$Z_d^\beta X_d^\alpha = \gamma_d^{\alpha\beta} X_d^\alpha Z_d^\beta \quad (6.22a)$$

$$(Z_d^\dagger)^\beta X_d^\alpha = \gamma_d^{-\alpha\beta} X_d^\alpha (Z_d^\dagger)^\beta. \quad (6.22b)$$

Proof. For $\alpha = \beta = 1$, Eq. 6.22 reduces to Eq. 6.20. Consider $\beta \geq 2$:

$$Z_d^\beta X_d = \gamma_d Z_d^{\beta-1} X_d Z_d = \cdots = \gamma_d^\beta X_d Z_d^\beta. \quad (6.23)$$

When $\alpha \geq 2$:

$$Z_d^\beta X_d^\alpha = (Z_d^\beta X_d) X_d^{\alpha-1} = \gamma_d^\beta X_d (Z_d^\beta X_d) X_d^{\alpha-2} \quad (6.24a)$$

$$= \gamma_d^{2\beta} X_d^2 Z_d X_d^{\alpha-2} = \cdots = \gamma_d^{\alpha\beta} X_d^\alpha Z_d^\beta. \quad (6.24b)$$

Eq. 6.22b follows analogously. $\square$

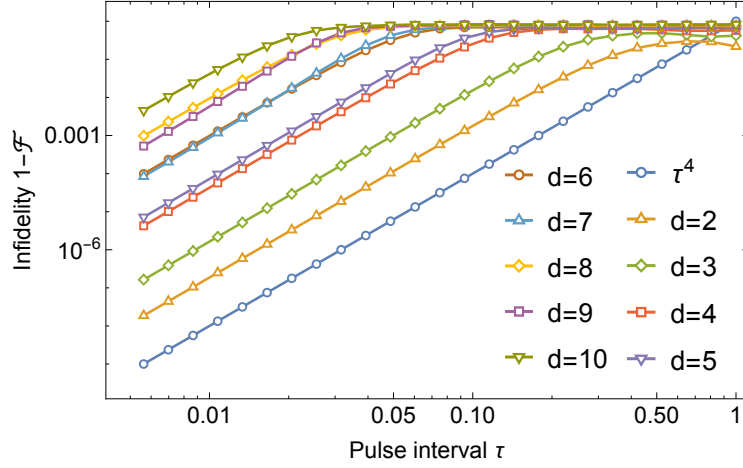


Figure 6.3: Numerical simulation of the the universality of the DD sequence generated by cycling over the Weyl-Heisenberg Group for $2 \leq d \leq 10$. Here we plot the infidelity of the resulting unitary evolution as a function of the pulse interval time τ . Since the sequence is expected to cancel the errors to the first order $\mathcal{O}(\tau^2)$, the infidelity should scale as $\mathcal{O}(\tau^4)$, as confirmed by our simulations.

This means that the commutation relations for two arbitrary HW operators are

$$\Lambda_{\alpha\beta}\Lambda_{\mu\nu} = \gamma_d^{\beta\mu-\alpha\nu}\Lambda_{\mu\nu}\Lambda_{\alpha\beta} \quad (6.25a)$$

$$\Lambda_{\alpha\beta}^\dagger\Lambda_{\mu\nu} = \gamma_d^{\alpha\nu-\beta\mu}\Lambda_{\mu\nu}\Lambda_{\alpha\beta}^\dagger. \quad (6.25b)$$

Unless $\alpha = \beta = 0$, the HW operators are non-Hermitian for $d \geq 3$

$$\Lambda_{\alpha\beta}^\dagger = e^{-i\pi\alpha\beta\frac{d+1}{d}}(X_d^\alpha Z_d^\beta)^\dagger \quad (6.26a)$$

$$= (-\sqrt{\gamma_d})^{-\alpha\beta}(Z_d^\dagger)^\beta(X_d^\dagger)^\alpha \neq \Lambda_{\alpha\beta}, \quad (6.26b)$$

but unitary for all d :

$$\Lambda_{\alpha\beta}^\dagger\Lambda_{\alpha\beta} = (Z_d^\dagger)^\beta(X_d^\dagger)^\alpha X_d^\alpha Z_d^\beta = I, \quad (6.27)$$

where we used the unitarity of X_d and Z_d . Combining unitarity with Eq. 6.25a, we obtain the identity

$$\Lambda_{\alpha\beta}^\dagger\Lambda_{\mu\nu}\Lambda_{\alpha\beta} = \gamma_d^{\alpha\nu-\beta\mu}\Lambda_{\mu\nu}, \quad (6.28)$$

which will prove to be crucial below for demonstrating that the Weyl-Heisenberg group is a universal DD group.

The operators $\{\Lambda_{\alpha\beta}\}_{\alpha,\beta \in \mathbb{Z}_d}$ form an irreducible, unitary, and projective representation of the HWG over the d -dimensional system Hilbert space when d is a prime power ($d = p^k$ for prime number p and positive integer k). This implies, by Schur's Lemma, that both H'_{SB}

and H'_S are proportional to I or vanish. Therefore, the unitary $U(T)$ defined in Eq. 6.11 reduces (up to a global phase) to the identity operation on the system, i.e., the condition for first-order decoupling is satisfied. The latter (or an equivalent one using group character tables) was the argument used in Refs. [272, 213, 217, 21]; going beyond the case of prime powers, we now show that, in fact, first-order decoupling holds for arbitrary d .

The Weyl Heisenberg operators also form an operator basis for the d -dimensional system Hilbert space. Thus, we can expand $H = H_S \otimes I_B + H_{SB} + I_S \otimes H_B$ as

$$H = \sum_{\mu, \nu=0}^{d^2-1} \Lambda_{\mu\nu} \otimes B_{\mu\nu}, \quad (6.29)$$

where $B_{\mu\nu}$ is either zero, proportional to I_B (to account for $H_S \otimes I_B$), or is a non-identity bath operator. The term with $\mu = \nu = 0$ corresponds to the pure-bath term $I_S \otimes H_B$.

Now recall that cycling over the decoupling group $\mathcal{G}_d$ yields Eq. 6.11. Choosing the decoupling group as the HWG $\{\Lambda_{\alpha\beta}\}$ means that the effective Hamiltonian becomes

$$H' = \mathcal{P}_{\mathcal{G}_d}(H) \quad (6.30a)$$

$$= \frac{1}{d^2} \sum_{\alpha, \beta=0}^{d^2-1} \Lambda_{\alpha\beta}^\dagger \sum_{\mu, \nu=0}^{d^2-1} \Lambda_{\mu\nu} \Lambda_{\alpha\beta} \otimes B_{\mu\nu} \quad (6.30b)$$

$$= \frac{1}{d^2} \sum_{\mu, \nu=0}^{d^2-1} f_{\mu\nu} \Lambda_{\mu\nu} \otimes B_{\mu\nu}, \quad (6.30c)$$

where, using Eq. 6.28,

$$f_{\mu\nu} = \sum_{\alpha, \beta=0}^{d^2-1} \gamma_d^{\alpha\nu - \beta\mu}. \quad (6.31)$$

Let us now show that

$$f_{\mu\nu} = d^4 \delta_{\mu 0} \delta_{\nu 0}. \quad (6.32)$$

Intuitively, this follows from the zero-sum property of the roots of unity: $\sum_{k=0}^{d-1} \gamma_d^k = 0$.

Proof. Note that $f_{\mu\nu} = h_\mu^* h_\nu$, and

$$h_\nu = \sum_{\alpha=0}^{d^2-1} \gamma_d^{\nu\alpha} = \sum_{k=0}^{d-1} \sum_{j=0}^{d-1} e^{2\pi i(kd+j)\nu/d} \quad (6.33a)$$

$$= \sum_{k=0}^{d-1} \sum_{j=0}^{d-1} (e^{2\pi i\nu/d})^j = dS, \quad (6.33b)$$

where $S = \sum_{j=0}^{d-1} \omega^j$ and $\omega = e^{2\pi i\nu/d}$. If $\nu \neq 0$ then $\omega \neq 1$ is a d 'th root of unity (since $\omega^d = 1$). Multiplying both sides by $\omega - 1$ yields $(\omega - 1)S = \sum_{j=0}^{d-1} \omega^{j+1} - \sum_{j=0}^{d-1} \omega^j$. The

terms $\omega, \omega^2, \dots, \omega^{d-1}$ cancel out, leaving $(\omega - 1)S = \omega^d - 1 = 0$. Since $\omega \neq 1$ we can divide both sides by $\omega - 1$, giving $S = 0$. If $\nu = 0$ then $S = d$. $\square$

Combining Eq. 6.32 and Eq. 6.30, we finally have

$$H' = I_S \otimes H_B, \quad (6.34)$$

i.e., $H'_{SB} = H'_S = 0$, leaving only the pure-bath term. This proves that the HWG is a universal decoupling group.

We numerically confirm this universality in Fig. 6.3. For various dimensions $2 \leq d \leq 10$, we consider a system-bath Hamiltonian containing all the HW operators with randomized coefficients (i.e., a classical bath). We then apply the corresponding universal sequence and compare the fidelity of the resulting unitary to the identity operator I in each case. Given that the errors in the unitary evolution under DD are suppressed to the first order [Eq. 6.11], i.e., leaving the leading order term $\mathcal{O}(T^2)$ where $T \propto \tau$, we expect the fidelity to scale as $\mathcal{O}(\tau^4)$. This is confirmed in Fig. 6.3.

Application to single-axis, pure dephasing noise

We utilize the formalism developed above to analyze the single-axis noise problem (i.e., pure dephasing) as a special case. This is the basis for the transmon-based qutrit and ququart experiments we present in the following sections, where dephasing is the dominant source of decoherence. In this single-axis scenario, the system-bath interaction component of Eq. 6.29 reduces to

$$H_{SB}^Z = \sum_{\nu=1}^{d-1} \Lambda_{0\nu} \otimes B_\nu, \quad (6.35)$$

where $\Lambda_{0\nu} = Z_d^\nu$ [Eq. 6.18].

We could choose the full Weyl Heisenberg group as a decoupling group, but since the Z_d -type WH operators commute with H_{SB}^Z , we need only consider the pure X_d -type decoupling operators $\Lambda_{\alpha 0} = X_d^\alpha$, which satisfy the commutation relations Eq. 6.25 non-trivially. I.e., for pure qudit dephasing the relevant decoupling group is the order- d HWG subgroup $\mathcal{G}_d^X = \{\Lambda_{\alpha 0}\}_{\alpha=0}^{d-1} = \{I, X_d, X_d^2, \dots, X_d^{d-1}\}$.

To show that $\mathcal{G}_d^X$ decouples H_{SB}^Z , we observe, using Eq. 6.28, that the effective Hamiltonian is

$$(H_{SB}^Z)' = \mathcal{P}_{\mathcal{G}_d^X}(H_{SB}^Z) \quad (6.36a)$$

$$= \frac{1}{d} \sum_{\alpha=0}^{d-1} \Lambda_{\alpha 0}^\dagger \sum_{\nu=1}^{d-1} \Lambda_{0\nu} \Lambda_{\alpha 0} \otimes B_\nu \quad (6.36b)$$

$$= \frac{1}{d} \sum_{\nu=1}^{d-1} f_\nu \Lambda_{0\nu} \otimes B_\nu, \quad (6.36c)$$

where

$$f_\nu = \sum_{\alpha=0}^{d-1} \gamma_d^{\nu\alpha} = d\delta_{\nu 0}, \quad (6.37)$$

and the last equality is again due to zero-sum property of the d 'th root of unity. It follows that $(H_{SB}^Z)' = 0$, i.e., $\mathcal{G}_d^X$ is a decoupling group for qudit dephasing.

Qutrit ($d = 3$) dynamical decoupling and $3X_3$

For concreteness, we now illustrate the results above by giving the explicit form of the HW subgroup $\mathcal{G}_d^X$ for the case of qutrit dephasing. This is the simplest non-trivial example going beyond qubits.

The generators of the qutrit Weyl Heisenberg group are X_3 and Z_3 . The reader should recall their commutation properties follow from Eq. 6.22:

$$X_3 Z_3 = \omega^2 Z_3 X_3, \quad X_3^2 Z_3 = \omega Z_3 X_3^2, \quad (6.38a)$$

$$X_3 Z_3^2 = \omega Z_3^2 X_3, \quad X_3^2 Z_3^2 = \omega^2 Z_3^2 X_3^2. \quad (6.38b)$$

Since $X_3^\dagger = X_3^2$, we can write

$$(X_3^2)^\dagger Z_3 X_3^2 = \omega^2 Z_3, \quad X_3^\dagger Z_3 X_3 = \omega Z_3, \quad (6.39a)$$

$$(X_3^2)^\dagger Z_3^2 X_3^2 = \omega Z_3^2, \quad X_3^\dagger Z_3^2 X_3 = \omega^2 Z_3^2, \quad (6.39b)$$

which is a special case of Eq. 6.28. The decoupling group is $\mathcal{G}_3^{X_3} = \{I, X_3, X_3^2\}$, which suppresses dephasing due to the system-bath interaction H_{SB}^Z with $d = 3$ [Eq. 6.35]. For example, consider free evolution subject to a term of the form $Z_3 \otimes B$; writing out the DD sequence Eq. 6.11 explicitly, we have, for $T = 3\tau$:

$$U(T) = (I f_\tau I) ((X_3^2)^\dagger f_\tau X_3^2) (X_3^\dagger f_\tau X_3) \quad (6.40a)$$

$$= e^{-i\tau Z_3 \otimes B} e^{-i\tau (X_3^2)^\dagger Z_3 X_3^2 \otimes B} e^{-i\tau X_3^\dagger Z_3 X_3 \otimes B} \quad (6.40b)$$

$$= e^{-i\tau (Z_3 \otimes B)} e^{-i\tau (\omega^2 Z_3 \otimes B)} e^{-i\tau (\omega Z_3 \otimes B)} \quad (6.40c)$$

$$= e^{-i\tau (1 + \omega^2 + \omega) Z_3 \otimes B} + \mathcal{O}(T^2) \quad (6.40d)$$

$$= I + \mathcal{O}(T^2), \quad (6.40e)$$

i.e., suppression to second order of dephasing due to $Z_3 \otimes B$. Replacing $Z_3 \otimes B$ with $Z_3^2 \otimes B'$ simply rearranges the order of the roots of unity, yielding $1 + \omega + \omega^2 = 0$ instead of $1 + \omega^2 + \omega = 0$ in Eq. 6.40d, with the same outcome.

Note that since $(X_3^2)^\dagger = X_3 = X_3^2 X_3^\dagger$, this DD sequence in fact reduces to three equidistant pulses of type X_3 :

$$U(T) = X_3 f_\tau X_3 f_\tau X_3 f_\tau \quad (6.41)$$

which is the reason we called it $3X_3$ in the following sections. We could have equivalently used the sequence consisting of three X_3^2 pulses. It is straightforward from this analysis to see how the $d = 4$ sequence generalizes as

$$U(T) = X_4 f_\tau X_4 f_\tau X_4 f_\tau X_4 f_\tau \quad (6.42)$$

Application to Suppression of Quantum Crosstalk

Now having introduced methods for single-qudit DD meant to combat against decoherence, we now deploy DD to mitigate qudit crosstalk. Recent work has demonstrated the efficacy of DD in suppressing coherent crosstalk errors in qubit systems. In our fixed linear coupling qudit processor, single-qudit operations suffer from always-on crosstalk, which is a generalization of the ZZ interaction between transmon qubits [247] and has been discussed at length in Chapters 3 and 4 (see Eq. 4.11).

Although these cross-Kerr interactions, along with off-resonantly applied microwave drives, have been shown to facilitate entangling operations, even minor cross-Kerr interactions during idle periods can introduce significant coherent errors. Building on the qudit DD formalism developed above, we now propose a DD sequence for coupled qudits. This DD sequence effectively suppresses all the cross-Kerr interactions, thereby enhancing system stability and operational fidelity.

The evolution operator due to dX_d applied only to the first qudit is given by $U_{d\tau}^{(1)} \equiv dX_d \otimes I$. By concatenating this sequence with the same dX_d sequence applied to the second qudit, we obtain the total evolution $U_{d^2\tau} \equiv U_{d\tau}^{(2)} \circ U_{d\tau}^{(1)}$, i.e.,

$$U_{d^2\tau} = (I \otimes X_d) U_{d\tau}^{(1)} (I \otimes X_d) \cdots U_{d\tau}^{(1)} (I \otimes X_d) U_{d\tau}^{(1)}. \quad (6.43)$$

Concatenation of DD sequences was originally introduced in order to obtain high-order suppression [130]; here, it serves the purpose of staggering the sequences on the two qudits, thus generalizing the idea of robust qubit-crosstalk suppression via staggering [291].

We now show that $U_{d^2\tau} = e^{i\theta} I \otimes I + O(T^2)$, where θ is a global phase, and $T = d^2\tau$. Since the Weyl-Heisenberg group is an operator basis, and in particular its phase elements $\{Z_d^\alpha\}_{\alpha=0}^{d-1}$ are a basis for diagonal operators, we can expand each diagonal term in H_{CK} as

$$|i\rangle\langle i| = \sum_{k=0}^{d-1} c_{ik} Z_d^k, \quad (6.44)$$

where

$$c_{ik} = \frac{1}{d} \text{Tr}[(Z_d^\dagger)^k |i\rangle\langle i|] = \frac{1}{d} \gamma_d^{-ik}. \quad (6.45)$$

This is a consequence of Eq. 6.16 and the zero-sum property: $\text{Tr}[(Z_d^\dagger)^k Z_d^l] = \text{Tr}(Z_d^{l-k}) = \sum_{m=0}^{d-1} \gamma_d^{m(l-k)} = d\delta_{kl}$. Thus, we can rewrite the cross-Kerr Hamiltonian as

$$H_{CK} = \sum_{k,l=0}^{d-1} \zeta_{kl} Z_d^k \otimes Z_d^l, \quad (6.46)$$

where

$$\zeta_{kl} = \sum_{i,j=1}^{d-1} c_{ik} c_{jl} \alpha_{ij} = \frac{1}{d^2} \sum_{i,j=1}^{d-1} \gamma_d^{-(ik+jl)} \alpha_{ij}. \quad (6.47)$$

The CKDD sequence $U_{d^2\tau}$ [Eq. 6.43] consists of an inner X_d -type sequence applied to the first qudit and an outer X_d -type sequence applied to the second qudit. Namely, $U_{d^2\tau} \equiv U_{d\tau}^{(2)} \circ U_{d\tau}^{(1)} = (I_d \otimes X_d) U_{d\tau}^{(1)} (I_d \otimes X_d) \cdots U_{d\tau}^{(1)} (I_d \otimes X_d) U_{d\tau}^{(1)}$, where $U_{d\tau}^{(1)} \equiv dX_d \otimes I_d$ and $U_{d\tau}^{(2)} \equiv I_d \otimes dX_d$.

Generalizing from the single qudit dephasing case, the effect of the inner $U_{d\tau}^{(1)}$ sequence is to project H_{CK} into the commutant of $\mathcal{G}_d^X \otimes I_d$, i.e.,

$$H'_{\text{CK}} = \mathcal{P}_{\mathcal{G}_d^X \otimes I_d}(H_{\text{CK}}) \quad (6.48a)$$

$$= \frac{1}{d} \sum_{\alpha=0}^{d-1} X_d^{\alpha\dagger} \otimes I_d \sum_{k,l=0}^{d-1} \zeta_{kl} Z_d^k X_d^\alpha \otimes Z_d^l \quad (6.48b)$$

$$= \frac{1}{d} \sum_{k,l=0}^{d-1} f_k \zeta_{kl} Z_d^k \otimes Z_d^l, \quad (6.48c)$$

where $f_k = \sum_{\alpha=0}^{d-1} \gamma_d^{k\alpha} = d\delta_{k0}$, just as in Eq. 6.37. Thus, the effect of the inner sequence is to leave just the identity term on the first qudit:

$$H'_{\text{CK}} = \sum_{l=0}^{d-1} \zeta_{0l} I_d \otimes Z_d^l. \quad (6.49)$$

The outer sequence then removes the remaining dephasing terms:

$$H''_{\text{CK}} = \mathcal{P}_{I_d \otimes \mathcal{G}_d^X}(H'_{\text{CK}}) \quad (6.50a)$$

$$= \frac{1}{d} \sum_{\alpha=0}^{d-1} I_d \otimes X_d^{\alpha\dagger} \sum_{l=0}^{d-1} \zeta_{0l} I_d \otimes Z_d^l X_d^\alpha \quad (6.50b)$$

$$= \frac{1}{d} \sum_{l=0}^{d-1} f_l \zeta_{0l} I_d \otimes Z_d^l = \zeta_{00} I_{d^2}. \quad (6.50c)$$

Consequently, it follows from Eq. 6.11 that $U_{d^2\tau} = e^{-i\zeta_{00}T} I_d \otimes I_d + \mathcal{O}(T^2)$, where $T = d^2\tau$ and $\zeta_{00} = \frac{1}{d^2} \sum_{i,j=1}^{d-1} \alpha_{ij}$. This proves that the CKDD sequence achieves first-order decoupling of the cross-Kerr interaction.

Note that applying simultaneous X_d -type sequences to both qudits does not work. I.e., using $\mathcal{G}_d^X \otimes \mathcal{G}_d^X$ as the decoupling group results instead of Eq. 6.48 in the projection

$$H'_{\text{CK}} = \mathcal{P}_{\mathcal{G}_d^X \otimes \mathcal{G}_d^X}(H_{\text{CK}}) \quad (6.51a)$$

$$= \frac{1}{d^2} \sum_{\alpha=0}^{d-1} \sum_{k,l=0}^{d-1} \zeta_{kl} X_d^{\alpha\dagger} Z_d^k X_d^\alpha \otimes X_d^{\alpha\dagger} Z_d^l X_d^\alpha \quad (6.51b)$$

$$= \frac{1}{d^2} \sum_{k,l=0}^{d-1} \zeta_{kl} Z_d^k \otimes Z_d^l \sum_{\alpha=0}^{d-1} \gamma_d^{(k+l)\alpha}, \quad (6.51c)$$

and $\sum_{\alpha=0}^{d-1} \gamma_d^{(k+l)\alpha} = d\delta_{k+l,d}$, i.e., terms of the form $Z_d^k \otimes Z_d^{d-k}$ are not suppressed. This is a generalization of the qubit case, where it is well known that simultaneous X -type sequences do not cancel crosstalk [249]. Thus, we expect that, to first order in the pulse interval, the $d^2\tau$ -long *cross-Kerr DD* (CKDD) sequence in Eq. 6.43 suppresses all crosstalk between two coupled qudits.

Experimental Implementation of Qudit DD

Having now worked out the formalism and developed DD pulse schemes for fixing common errors on qudit systems, we now turn our focus to experimentally testing these DD sequences. We will begin by looking at fixing system-bath decoherence for $d = 3$ and $d = 4$ qudits, before tackling the quantum crosstalk in those same systems. We conduct all our experiments on the same 8 transmon qudit processor with $d = 3$ and 4 which has been discussed in previous chapters. The specific relevant metrics of the device at the time of the experiment is presented in Table. 6.1.

Decoupling from System-Bath Noise

Since DD sequences are particularly effective against low-frequency noise, and superconducting circuits are especially susceptible to such noise when higher excited states are targeted due to bosonic enhancement, we focus primarily on the dX_d sequence family. The underlying cycle operator X_d is compiled using $2(d-1)$ native $\sqrt{\sigma_s^x}$ subspace rotations where $s \in \{(0,1), (1,2), \dots, (d-1,d)\}$, and $\sigma_{(i,j)}^x = |i\rangle\langle j| + |j\rangle\langle i|$ is the Pauli- x operator between levels i and j . Fig. 6.4 presents our experimental single-qudit dX_d results. Free evolution (no DD) corresponds to the preparation of a uniform qudit superposition state $|+\rangle_d \equiv (|0\rangle + \dots + |d-1\rangle)/\sqrt{d}$, waiting for a specified delay time, unpreparing the state, and finally measuring the qudit. Assuming ideal preparation, unpreparation, and measurement, the fidelity of the superposition state $|+\rangle_d$ is the probability of finding the qudit back in the $|0\rangle$ state. We then repeat the experiment with the dX_d sequence applied during the delay time and study its impact on the state fidelity.

Fig. 6.4(a) presents the results for the qutrit experiments. Crucially, all the DD curves exhibit an improvement over the free evolution (No DD) experiment, confirming the effec-

Parameters	Q ₁	Q ₂	Q ₃
$\omega_{01}/2\pi$ (GHz)	5.333	5.396	5.572
$\omega_{12}/2\pi$ (GHz)	5.061	5.124	5.303
$\omega_{23}/2\pi$ (GHz)	4.757	4.821	5.005
Avg. T_1^{01} (μ s)	50(4)	49(4)	60(5)
Avg. T_1^{12} (μ s)	35(2)	35(4)	31(8)
Avg. T_1^{23} (μ s)	24(4)	26(3)	23(4)
Avg. T_{2e}^{01} (μ s)	78(5)	85(9)	90(6)
Avg. T_{2e}^{12} (μ s)	57(4)	57(4)	56(9)
Avg. T_{2e}^{23} (μ s)	26(2)	27(2)	24(3)

Table 6.1: Transition frequencies $\omega_{ij} = (E_j - E_i)/\hbar$ up to $d = 4$ of the qudits employed in our DD experiments. Additionally, we provide the two-level subspace mean T_1 and T_2 echo times of the device calculated from 100 repetitions of each coherence experiment.

tiveness of our DD sequences in suppressing decoherence. In more detail, for each total time $T = 9\tau$ we conducted a free evolution experiment and four DD experiments: 1, 2, or 3 repetitions of $3X_3$, and universal qutrit DD (the full order-9 HWG), with respective pulse intervals of $\tau_1 = 3\tau$, $\tau_2 = \frac{3}{2}\tau$, $\tau_3 = \tau$, and $\tau_{\text{univ.}} = \tau$. DD theory predicts that for instantaneous, ideal pulses, state preservation fidelity increases monotonically as the pulse interval decreases for a fixed total evolution time [131, 253]. Moreover, universal DD is expected to outperform single-axis DD. Our results exhibit the opposite of both expectations: the single repetition $1 \times 3X_3$ experiment, with the longest pulse interval $\tau_1 = 3\tau$, yields the highest fidelity, while the universal sequence, with the shortest interval τ (as for $3 \times 3X_3$) yields the lowest DD fidelity. The reason for these results is likely to be the presence of coherent pulse errors, which accumulate more detrimentally the longer the pulse sequence, and whose effect overwhelms the benefit of shorter pulse intervals [235, 198]. While X_3 gates can be decomposed in terms of four native $\sqrt{\sigma_s^x}$ gates, the remaining Weyl-Heisenberg gates require six native $\sqrt{\sigma_s^x}$ gates using the decomposition presented in Chapter 2, so that $\tau_{\text{min}} = 180$ ns for universal DD compared to $\tau_{\text{min}} = 120$ ns for the $3X_3$ sequences. This additional opportunity for the accumulation of coherent errors explains why the universal sequence underperforms the $3 \times 3X_3$ sequence. The superior performance of the $3X_3$ sequences also confirms that the dominant source of noise is dephasing.

Similar improvements are observed with DD for ququarts, as shown in Fig. 6.4(b). Since ququarts are more susceptible to charge noise due to the involvement of the third excited state [136], the free evolution fidelity is significantly lower than in the qutrit case, and the improvement with DD is even more pronounced. Note that the difference between the DD sequences is much smaller than in the qutrit case, except for the $4 \times 4X_4$ case at times $< 25 \mu$ s. This could be attributed to the stronger $1/f$ dephasing suppression effect of the

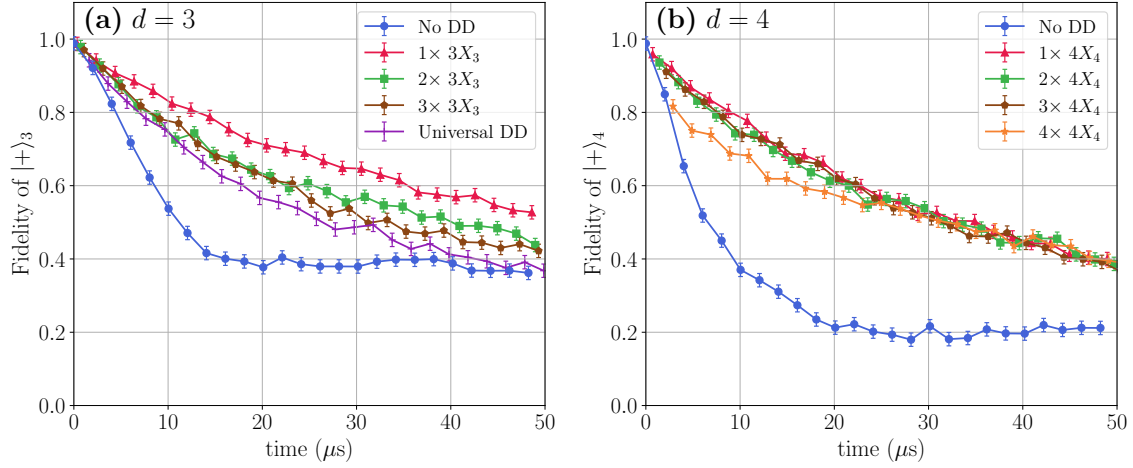


Figure 6.4: Experimental results showing the fidelity of the qudit uniform superposition state $|+\rangle_d$ as a function of total time under free evolution (No DD) and dX_D , for (a) qutrits and (b) ququarts. The minimum pulse interval is $\tau_{\min} = 120$ ns. For qutrits, we implement 1, 2, and 3 repetitions of the $3X_3$ sequence, with corresponding pulse intervals 3τ , $3\tau/2$, and τ , respectively. The total evolution time is always 9τ . Universal DD is a sequence of 9 pulses formed by cycling over the entire HWG, applied once with a pulse interval of τ and $\tau_{\min} = 180$ ns. For ququarts, we similarly implement 1, 2, 3, and 4 repetitions of the $4X_4$ sequence, with corresponding pulse intervals of 4τ , 2τ , $\frac{4}{3}\tau$, and τ respectively, where the total time is always 16τ .

$4X_4$ sequences relative to the $3X_3$ sequences, which outweighs the accumulation of pulse errors. These ququart results highlight the effectiveness of DD in higher dimensions and its critical role in suppressing noise in more complex systems.

Decoupling from Qudit Quantum Crosstalk

We have already discussed how quantum crosstalk sets a harsh limit on the fidelity of local operations, especially so in the context of high-dimensional quantum systems. We note that the relevant quantum crosstalk rates for our experiments in both the coupled qutrit and ququart experiments are in Table 6.2 and Table. 6.3.

We now validate our approach to improving this performance limit via the CKDD sequence. In particular, we conduct experiments on coupled transmon qutrits and ququarts. Fig. 6.5(a) shows the results for a linear chain of three coupled qutrits, where we prepare the set of nine initial states $|i\rangle \otimes |+\rangle_3 \otimes |j\rangle$, $i, j \in \mathbb{Z}_3$. We trace out the states of the left and right “spectator” qutrits, and display the fidelity of the middle (main) qutrit’s state with respect to $|+\rangle_3$ as a function of delay time. As depicted by the dashed light-color curves, different initial spectator states exhibit distinct curves that oscillate at different frequencies.

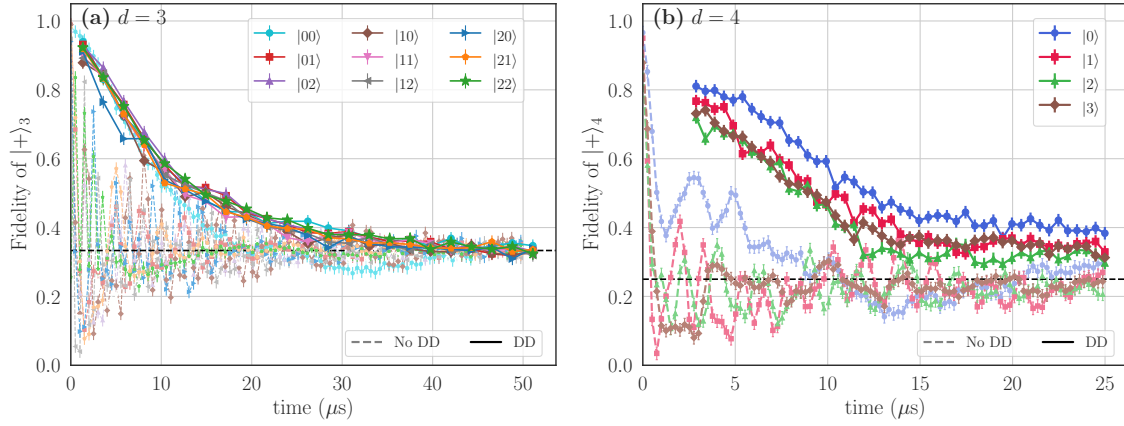


Figure 6.5: Experimental results showing suppression of cross-Kerr interactions using the CKDD sequence. (a) Fidelity of the qutrit superposition state $|+\rangle_3$, with CKDD (solid) and without (dashed). CKDD is a 9τ -long sequence, with $\tau_{\min} = 120$ ns. (b) Fidelity of the ququart superposition state $|+\rangle_4$, with CKDD (solid) and without (dashed). CKDD is a 16τ -long sequence, with $\tau_{\min} = 180$ ns. As in Fig. 6.4, longer evolution times correspond to a single repetition of CKDD with increased τ . CKDD removes the cross-Kerr oscillations and improves the fidelity in both cases, converging to the fully mixed state fidelity baseline (dashed horizontal line) more slowly than the free evolution (No DD) curves.

Parameters/ 2π	Q_1 - Q_2	Q_2 - Q_3
α_{11} (MHz)	0.112	0.212
α_{12} (MHz)	0.623	0.465
α_{21} (MHz)	-0.515	-0.162
α_{22} (MHz)	0.341	0.615

Table 6.2: The qutrit cross-Kerr crosstalk rates present in the spectator DD experiment in Fig. 6.5(a).

The high frequency oscillations and the differences between the curves are attributable to the cross-Kerr interactions between the main qutrit and the two spectators. Next, we apply the CKDD sequence as $U_{d\tau}^{(2)} \circ (U_{d\tau}^{(1)} \otimes U_{d\tau}^{(3)})$, that is, an inner sequence where dX_d is applied simultaneously to the spectators, and an outer sequence where dX_d is applied to the main qutrit. The spectators' pulses are synchronized, so the sequence still takes a total time of $d^2\tau$. The resulting solid, bold-color curves exhibit a higher fidelity and none of variation of the free evolution curves, highlighting the efficacy of the CKDD sequence in suppressing cross-Kerr interactions and stabilizing the system dynamics for different initial spectator

Parameters/ 2π	Q_1 - Q_2
α_{11} (MHz)	0.112
α_{12} (MHz)	0.623
α_{13} (MHz)	0.021
α_{21} (MHz)	-0.515
α_{22} (MHz)	0.341
α_{23} (MHz)	0.730
α_{31} (MHz)	0.226
α_{32} (MHz)	-0.442
α_{33} (MHz)	0.345

Table 6.3: The ququart cross-Kerr crosstalk rates present in the spectator DD experiment in Fig. 6.5(b).

states.

In Fig. 6.5(b), we present the fidelity results for two coupled ququarts, for the set of four initial states $|i\rangle \otimes |+\rangle_4$, $i \in \mathbb{Z}_4$ for the spectator. We again trace out the spectator state. Similarly to the qutrit case, under free evolution we observe different curves corresponding to different states of the spectator, but when CKDD is applied, all four curves exhibit a similar exponential decay, indicating suppression of the cross-Kerr interactions. The CKDD sequence consists of the ququart shift operators X_4 comprising three $\sigma_{i,i+1}^x$ gates in the two-level subspaces spanned by $\{|i\rangle, |i+1\rangle\}$, $i \in \mathbb{Z}_4$. In the presence of large cross-Kerr interactions, driving the two-level subspaces is prone to large detuning errors. This results in somewhat lower CKDD fidelities for the ququart experiments compared to qutrits. Despite this, the suppression of the cross-Kerr interactions is clearly evident from the results, thus providing a proof-of-principle demonstration of the scalability of the CKDD protocol to higher dimensions.

Finally, to demonstrate the effectiveness of CKDD beyond the preservation of product states, we prepare the qutrit Bell state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$ and measure its fidelity over time with and without CKDD. The preparation of this state involves a qutrit controlled-phase (CZ) gate (see Chapter 4). We employ quantum state tomography to compute the fidelity of the Bell state. As depicted in Fig. 6.6, the experimental results for free evolution (No DD) and with CKDD applied to both qutrits contrast sharply. In the absence of DD, the state fidelity suffers significantly due to strong coherent errors arising from large cross-Kerr interactions. The fidelity drops to near zero in $\sim 1 \mu\text{s}$, meaning that the state evolves to an orthogonal qutrit Bell state, then oscillates around the fully mixed state fidelity baseline of $1/9$ (dashed horizontal line), highlighting the importance of suppressing cross-Kerr interactions. In contrast, with CKDD we observe a marked improvement; the oscillations are nearly eliminated, and the fidelity remains $> 50\%$ even after $10 \mu\text{s}$. The state tomography

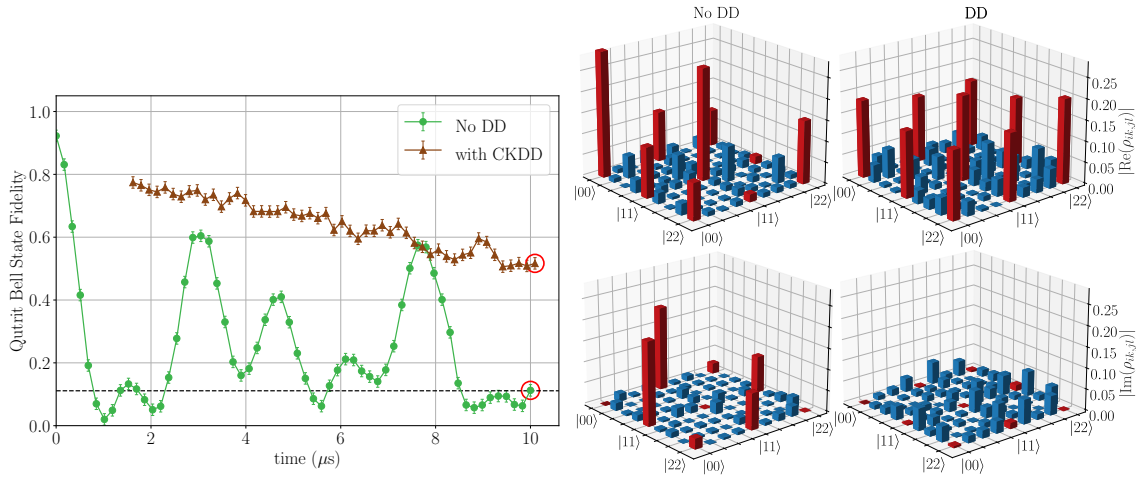


Figure 6.6: Left: Fidelity of the qutrit Bell state over time with and without CKDD. See text for details. Right: real (upper) and imaginary (lower) components of quantum state tomography results for the final time point in the top plot (red circles). Ideally, $\text{Re}(\rho_{ik,jl}) = \frac{1}{3}\delta_{ik}\delta_{jl}$ and $\text{Im}(\rho_{ik,jl}) = 0$ for $i, j, k, l \in \mathbb{Z}_3$, where ρ is the density matrix. Left: free evolution. The nine red-colored bars at the ideal positions are of varying magnitude and some contain large imaginary components, indicating deviations from the ideal qutrit Bell state. Right: with CKDD. In contrast, the nine red-colored bars are nearly uniform in height have negligible imaginary components, indicating proximity to the ideal qutrit Bell state. Note that here CKDD is a 9τ -long sequence with $\tau_{\min} = 180$ ns, the X_3 gate duration; this differs from the previous figures where the X_3 gate duration is 120 ns due to a different calibration.

histograms in Fig. 6.6 further highlight CKDD’s ability to maintain the integrity of qutrit Bell states.

6.2 Gate Based Qudit Error Mitigation

Having demonstrated DD techniques to improve qudit circuit results via pulsed-based methods, we now turn our attention to a distinct but complementary approach to improving expectation values extracted from qudit circuits using gate-based techniques broadly characterized as *error mitigation*. While running extremely deep quantum circuits will ultimately require quantum error correction, alternative near-term efforts in qubit devices to mitigate and extrapolate expectation values beyond the noise present in the system have garnered interest lately due to the lack of an increase in hardware requirements and overall feasibility [146, 246, 60, 124, 241, 80, 143, 61, 132, 137, 102, 108, 12, 234, 288, 50]. Typically, these methods require carefully learning the noise in the system and have a large sampling overhead for extrapolating beyond the noise present in the system. Notably, recent works have

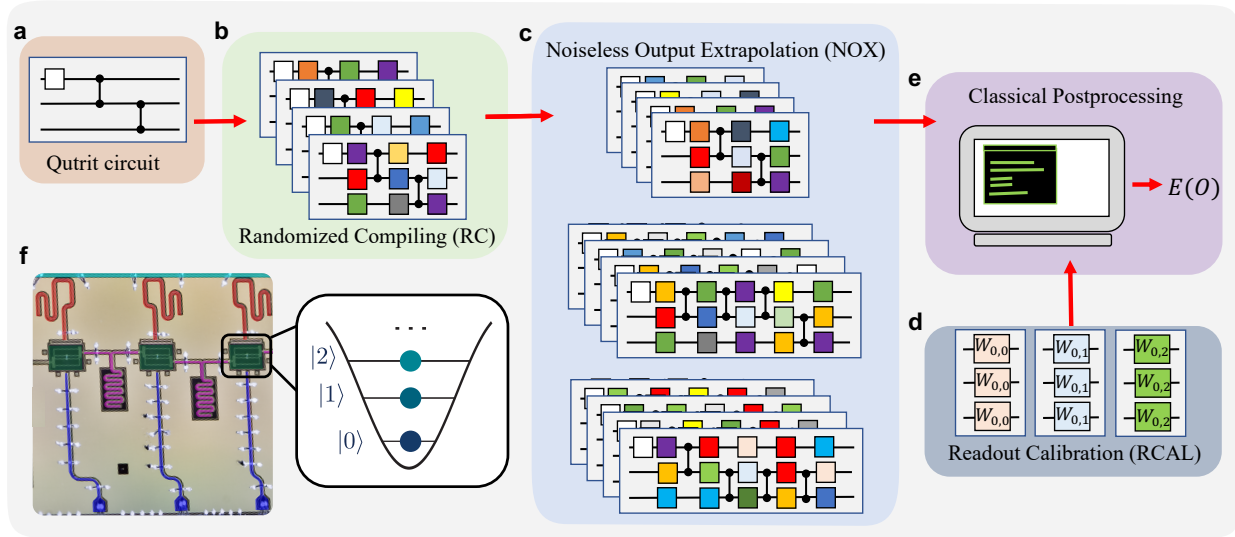


Figure 6.7: Schematic of error mitigation in qutrit circuits. (a) An arbitrary qutrit circuit is (b) randomly compiled into many logically equivalent copies, effectively twirling the noise of the multi-qutrit gate cycles into stochastic channels. (c) The noise present in the multi-qutrit gates in the circuit is amplified by insertions of the identity. (d) The readout assignment errors are efficiently characterized by measuring the single qutrit confusion matrices. (e) Via classical post-processing, noiseless expectation values are achieved from the combination of RC, NOX, and RCAL. (f) A false colored micrograph of the system of three capacitively coupled, fixed-frequency transmon qutrits used in the experiment.

demonstrated that error mitigation protocols can prove effective for large scale qubit experiments, allowing an exciting pathway to high-fidelity results in the near term for problems of interest [60, 124, 234, 288, 241, 50, 102, 12, 132, 254, 69].

In the following section, we introduce techniques to tailor arbitrary qudit Markovian noise to stochastic Weyl-Heisenberg channels and mitigate noise which commutes with our Clifford and universal two-qudit gate (introduced in Chapter 4) in generic qudit circuits (see Fig. 6.7). We experimentally demonstrate these methods on qutrits, and benchmark their effectiveness for multipartite entanglement and random circuit sampling, obtaining up to 3x improvement in our results. These results suggest that despite the intrinsic complexity of manipulating higher-dimensional quantum systems, noise tailoring and error mitigation can significantly extend the computational reach of today’s qudit processors.

Qudit Randomized Compiling

Being a unitary 1-design (see Chapter 1), Weyl operators can be used to *twirl* noise [90, 101, 67, 68]—that is, to transform arbitrary Markovian noise processes into stochastic channels

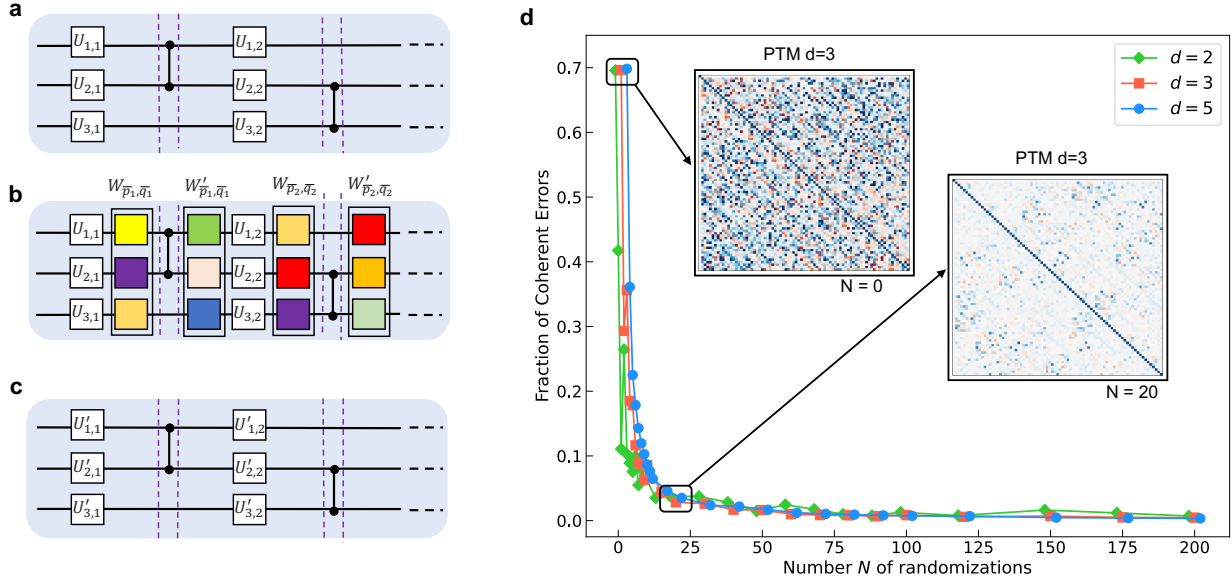


Figure 6.8: Randomized Compiling (RC) for qudit circuits. (a) The input circuit, which alternates cycles of one- and multi-qudit gates. (b) Random Weyl gates ($W_{\bar{p}_j, \bar{q}_j}$) and their inverses ($W'_{\bar{p}_j, \bar{q}_j} = W_{\bar{p}_j, \bar{q}_j}^\dagger H_j W_{\bar{p}_j, \bar{q}_j}$) are added before and after every cycle of multi-qudit gates H_j . (c) Before executing the circuit, the extra gates are recompiled into the native one-qudit gates. In this way, the returned circuit has the same depth as the input one. (d) Numerical study of the fraction of coherent errors under RC randomizations in randomly generated two-qudit Pauli Transfer Matrices (PTM) in $d \in \{2, 3, 5\}$. The formula used to quantify the fraction of coherent errors is derived in the text of this section. All PTMs are chosen to initially have 70% of their errors be coherent. As it can be seen, the coherent errors decrease exponentially with N , confirming the validity of Eq. 6.56. Furthermore, its value does not depend on d . This demonstrates that for an equal suppression of coherent errors, RC does not require additional twirls in higher dimensions. The inset PTMs visualize the suppression of off-diagonal terms as a function of N .

of the form

$$\mathcal{W}(\rho) = \sum_{\vec{p}, \vec{q}}^{4^n} \text{prob}(W_{\vec{p}, \vec{q}}) W_{\vec{p}, \vec{q}} \rho W_{\vec{p}, \vec{q}}^\dagger, \quad (6.52)$$

where ρ is an n -qutrit state, $W_{\vec{p}, \vec{q}} = \otimes_{i=1}^n W_{q_{k_i}, p_{k_i}}$ is a tensor product of one-qutrit Weyl operators, and $\text{prob}(W_{\vec{p}, \vec{q}})$ is the probability that a Weyl error $W_{\vec{p}, \vec{q}}$ occurs. Twirling noise processes can significantly improve the performance of noisy devices [101]. Indeed, while coherent errors can accumulate quadratically in the number of noisy gates, stochastic channel accumulate linearly and dramatically lower worst-case error rates [99].

In our experiments, we twirl the noise using the RC protocol from Ref. [262] generalized to qutrits. To illustrate this procedure, let us consider a “target” circuit of the type in Fig. 6.8a, i.e., a circuit that alternates between cycles of single-qutrit gates (represented by Completely Positive Trace-Preserving, or CPTP, maps $\mathcal{U}_j$) and cycles of multi-qutrit gates (represented by CPTP maps $\mathcal{H}_j$), implementing the operation

$$\mathcal{C} = \mathcal{U}_{m+1} \mathcal{H}_m \mathcal{U}_m \cdots \mathcal{H}_1 \mathcal{U}_1. \quad (6.53)$$

Expressing a noisy implementation of $\mathcal{U}_j$ (respectively $\mathcal{H}_j$) as $\mathcal{D}_j \mathcal{U}_j$ (respectively $\mathcal{F}_j \mathcal{H}_j$), a noisy implementation of this circuit performs the operation

$$\tilde{\mathcal{C}} = \mathcal{D}_{m+1} \mathcal{U}_{m+1} \mathcal{E}_m \mathcal{H}_m \mathcal{U}_m \cdots \mathcal{E}_1 \mathcal{H}_1 \mathcal{U}_1, \quad (6.54)$$

where $\mathcal{E}_j = \mathcal{F}_j \mathcal{H}_j \mathcal{D}_j \mathcal{H}_j^{-1}$ is the combined noise of $\mathcal{U}_j$ and $\mathcal{H}_j$. Analogous to the original protocol, to perform RC we recompile randomly-chosen Weyl gates and their inverses into the cycles of one-qubit gates (Figs 6.8b, 6.8c). Implementing the circuit $N > 1$ times with different choices of random Weyl gates and averaging over the various implementations is equivalent to implementing a circuit

$$\tilde{\mathcal{C}}_{\text{RC}} = \mathcal{U}_{m+1} \mathcal{W}_m \mathcal{H}_m \mathcal{U}_m \cdots \mathcal{W}_1 \mathcal{H}_1 \mathcal{U}_1, \quad (6.55)$$

afflicted by stochastic noise processes $\mathcal{W}_j$ of the type in Eq. 6.52, up to statistical fluctuations that decrease exponentially with N . Specifically, denoting by $\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_N$ the N circuits with random Weyl operators, Hoeffding’s inequality [105] ensures that for every state ρ , observable O , and positive number $\epsilon < 1$, we have

$$\begin{aligned} & \text{prob} \left(\left| E(O|\tilde{\mathcal{C}}_{\text{RC}}) - \frac{1}{N} \sum_{k=1}^N E(O|\tilde{\mathcal{C}}_k) \right| < \epsilon \right) \\ & \geq 1 - \exp(-2\epsilon^2 N), \end{aligned} \quad (6.56)$$

where $E(O|\mathcal{C}) = \text{Tr}[O\mathcal{C}(\rho)]$ is the expectation value of O at the end of a circuit $\mathcal{C}$. Importantly, the r.h.s. of the above inequality does not depend on the dimension of the Hilbert space. Hence, achieving the desired approximation level requires implementing the same number of circuits for qudits of arbitrary dimension as for qubits (Fig. 6.8d). Stated another

way, this means that despite the larger Hilbert space of a two-qudit system, one does not need to employ more twirling randomizations to achieve the same suppression of coherent errors.

To numerically demonstrate this feature of twirling in different qudit dimensions, we generate random two-qudit PTMs with equivalent proportion of off-diagonal elements, and then apply twirling. In Fig. 6.8d, we show the numerical results of twirling away (using qudit RC) off-diagonal elements (or coherent errors) in qudit PTMs. Here, we briefly comment on how one calculates the proportion of coherent errors present in a qudit PTM, following Ref. [35].

Consider a quantum error channel with PTM denoted as $\mathcal{E}$. The process fidelity is defined as the normalized trace of the $(d^2 \times d^2)$ PTM:

$$F(\mathcal{E}) := \text{tr } \mathcal{E} / d^2. \quad (6.57)$$

Let's define the decoherent process fidelity of $\mathcal{E}$ as

$$F_{\text{decoh.}}(\mathcal{E}) := \sqrt{\text{tr } \mathcal{E}^\dagger \mathcal{E} / d^2} = \|\mathcal{E}\|_F / d. \quad (6.58)$$

In Ref. [35], it is shown that as long as the error channel is reasonably close to the identity, that is if $F(\mathcal{E}) > 1/2$ and $F_{\text{decoh.}}(\mathcal{E}) > 1/\sqrt{2}$, the error channel has a well-defined coherent-decoherent polar decomposition:

$$\mathcal{E} = \mathcal{U}\mathcal{D}, \quad (6.59)$$

where $\mathcal{U}$ is a coherent (i.e. unitary) error process and $\mathcal{D}$ is a purely decoherent process. The precise definition of a decoherent channel is elaborated and justified in Ref. [35].

In realistic physical scenarios, the process fidelity can be expressed as a simple product:

$$F(\mathcal{E}) = F(\mathcal{U}) F(\mathcal{D}) + h.o., \quad (6.60)$$

where the higher order term (*h.o.*) is of second order in the infidelity, $O((1 - F(\mathcal{E}))^2)$. For reference, Eq. 6.60 does not hold in pathological cases where a significant part of the error process originates from specially crafted high-body interactions (e.g. Hamiltonian/Lindbladian terms that are made of tensor products of a large number of subsystems). A discussion of these pathological cases is included in Ref. [35].

Notice that by combining Eq. 6.58 and Eq. 6.59, we get:

$$\begin{aligned} F_{\text{decoh.}}(\mathcal{E}) &= F_{\text{decoh.}}(\mathcal{U}\mathcal{D}) \\ &= \|\mathcal{U}\mathcal{D}\|_F / d \\ &= \|\mathcal{D}\|_F / d \\ &= F_{\text{decoh.}}(\mathcal{D}) \\ &= F(\mathcal{D}) + O((1 - F(\mathcal{D}))^2) \end{aligned} \quad (6.61)$$

where the last line comes from the fact that non-pathological decoherent errors obey ([35])

$$\text{tr } \mathcal{D}^\dagger \mathcal{D} / d^2 = (\text{tr } \mathcal{D} / d^2)^2 + O((1 - F(\mathcal{D}))^2) . \quad (6.62)$$

The above is essentially a corollary from the fact that decoherent errors build up according to a multiplicative decay.

By substituting Eq. 6.61 in Eq. 6.60, we get, up to second order in the infidelity,

$$F(\mathcal{E}) = F(\mathcal{U}) F_{\text{decoh.}}(\mathcal{E}) , \quad (6.63)$$

or in terms of infidelity (up to second order in the infidelity)

$$\underbrace{1 - F(\mathcal{E})}_{\text{Tot. infid.}} = \underbrace{1 - F(\mathcal{U})}_{\text{Coh. infid.}} + \underbrace{1 - F_{\text{decoh.}}(\mathcal{E})}_{\text{Decoh. infid.}} . \quad (6.64)$$

As such, the contribution of coherence to the process infidelity is given by (up to the second order in the infidelity):

$$1 - F(\mathcal{U}) = F_{\text{decoh.}}(\mathcal{E}) - F(\mathcal{E}) . \quad (6.65)$$

The relative coherent contribution to the infidelity is therefore obtained via

$$\frac{F_{\text{decoh.}}(\mathcal{E}) - F(\mathcal{E})}{1 - F(\mathcal{E})} = \frac{\text{tr } \mathcal{E} / d^2 - d \|\mathcal{E}\|_F / d}{1 - \text{tr } \mathcal{E} / d^2} . \quad (6.66)$$

The above is the formula used in Fig. 6.8d to quantify the effect of twirling on the error channel.

Noiseless Output Extrapolation

While implementing RC alone leads to significant performance gains, employing it in tandem with other error-mitigation protocols leads to even larger gains. In our experiments later in this chapter, we utilize RC in combination with NOX, a protocol designed to mitigate stochastic errors that commute with cycles of gates. In its simplest version, NOX requires implementing the target circuit $\tilde{\mathcal{C}}_{\text{RC}}$ alongside several copies of it. Each copy implements the same computation as $\tilde{\mathcal{C}}_{\text{RC}}$, but the noise of one of its cycles is amplified in a controlled way, by specifically inserting compilations of the identity into cycles via repeated native gates. Specifically, the j -th copy implements the operation

$$\begin{aligned} \tilde{\mathcal{C}}_{\text{RC}}^{(j)} := \\ \mathcal{U}_{m+1} \mathcal{W}_m \mathcal{H}_m \Upsilon_m \cdots (\mathcal{W}_j)^{\alpha_j} \mathcal{H}_j \mathcal{U}_j \cdots \mathcal{W}_1 \mathcal{H}_1 \mathcal{U}_1 , \end{aligned} \quad (6.67)$$

where $\alpha_j > 1$ is an integer specified by the user. By combining the outputs of the target circuit and those of the various copies, NOX returns the quantity

$$E_{\text{NOX}}(O) = E(O|\tilde{\mathcal{C}}_{\text{RC}}) - \sum_{j=1}^m \frac{E(O|\tilde{\mathcal{C}}_{\text{RC}}) - E(O|\tilde{\mathcal{C}}_{\text{RC}}^{(j)})}{\alpha_j - 1} . \quad (6.68)$$

$E_{\text{NOX}}(O)$ is an estimator of the correct result, $E(O|\mathcal{C})$, and its bias is quadratically smaller than that of $E(O|\tilde{\mathcal{C}})$ [69].

In order to perform noise amplification, in our experiments we utilize Unitary Folding (UF) [80]. That is, given a noisy cycle $\mathcal{W}\eta$ and a number α such that $\eta^{\alpha+1} = \eta$, we replace $\mathcal{W}\eta$ with $(\mathcal{W}\eta)^{\alpha+1}$. In the instance where the error channel $\mathcal{W}$ commutes with the cycle η , i.e. $\mathcal{W}\mathcal{H} = \mathcal{H}\mathcal{W}$, UF leads to $(\mathcal{W}\eta)^{\alpha+1} = \mathcal{W}^{\alpha+1}\eta$, which provides the desired noise amplification. In the more general situation, UF leaves a remainder term:

$$(\mathcal{W}\eta)^{\alpha+1} = \mathcal{W}^{\alpha+1}\mathcal{H} + \mathcal{R}\eta . \quad (6.69)$$

We briefly argue why the term $\mathcal{R}\mathcal{H}$ may have little impact on mitigated results in realistic scenarios. Given a cyclicity c representing the smallest positive integer such that $\mathcal{H}^c = I$, we can express the error channel $\mathcal{W}$ as

$$\mathcal{W} = e^{\sum_{k=0}^{c-1} L_k} , \quad (6.70)$$

where the dissipator terms $\{L_0, L_1, \dots, L_{c-1}\}$ obey:

$$L_k\mathcal{H} = \omega^k\mathcal{H}L_k , \quad (6.71)$$

where $\omega = e^{2i\pi/c}$. In our experiments, the cycle $\mathcal{H}$ is a tensor product of $\text{CZ}^\dagger$ gates and has a cyclicity of $c = 3$. This yields $\mathcal{W} = e^{L_0+L_1+L_2}$ with $L_k\mathcal{H} = \omega^k\mathcal{H}L_k$ where $\omega = e^{2i\pi/3}$. In this case, UF yields:

$$\begin{aligned} (\mathcal{W}\eta)^{\alpha+1} &= (e^{L_0+L_1+L_2}\mathcal{H})^{\alpha+1} \\ &= \left(\prod_{k=0}^{\alpha} e^{\mathcal{H}^k(L_0+L_1+L_2)\mathcal{H}^{-k}} \right) \mathcal{H} \\ &= \left(\prod_{k=0}^{\alpha} e^{L_0+\omega^k L_1+\omega^{2k} L_2} \right) \mathcal{H} \\ &\approx e^{(\alpha+1)L_0+L_1+L_2}\mathcal{H} , \end{aligned} \quad (6.72)$$

where in the last line we truncated the second-order terms from the BCH expansion for simplicity. The purpose behind the derivation of Eq. 6.72 is to show that, up to a second-order approximation, only the dissipator term L_0 that commutes with the cycle $\mathcal{H}$ gets properly amplified by UF. One reason that makes UF an effective error amplification strategy despite this limitation is that L_0 is often a dominating source of error compared to the other dissipators because dephasing noise which couples to Z, Z^2 commutes with our gate and contributes heavily to our errors. Indeed, following the derivation of [104], given an noisy process $\mathcal{W}\mathcal{H}$ obtained from integrating a Lindbladian,

$$\mathcal{W}\mathcal{H} = \int_0^T -iH(t) + L(t) dt , \quad (6.73)$$

where $H(t)$ and $L(t)$ correspond to the super-operator versions of the ideal and noisy dynamics respectively, we can approximate $\mathcal{W}$ as

$$\mathcal{W} \approx \mathcal{H} \exp \left(\int_0^T dt \mathcal{U}^\dagger(t) L(t) \mathcal{U}(t) \right) \mathcal{H}^{-1}, \quad (6.74)$$

where $\mathcal{U}(t)$ is the ideal evolution at time t . This approximation corresponds to the truncation of the Magnus expansion to first order. The integral in Eq. 6.74 is close to a twirl operation, and has the effect of partially suppressing the matrix components that do not commute with $\mathcal{W}$. To see this more clearly, let's assume that $H(t)$ and $L(t)$ are time-independent and let's perform column-vectorization on the integral in Eq. 6.74:

$$\begin{aligned} \text{vec} \left(\int_0^T dt \mathcal{U}^\dagger(t) L \mathcal{U}(t) \right) = \\ \underbrace{\left(\int_0^T dt \mathcal{U}(t)^T \otimes \mathcal{U}^\dagger(t) \right)}_{\text{Partial twirl operator.}} \text{vec}(L) \end{aligned} \quad (6.75)$$

where we used the identity $\text{vec}(ABC) = C^T \otimes A \text{vec}(B)$. We denote the operator acting on $\text{vec}(L)$ as a “partial twirl”; a full twirl would involve the self-inverting evolution from 0 to τ where $\mathcal{U}(0) = \mathcal{U}(\tau)$. It is straightforward to show that the full twirl projects $\text{vec}(L)$ unto the subspace of operators that commute with $\mathcal{U}(t) = \exp(iHt)$ for any time t . The partial twirl operator instead dampens the components of L that do not commute with the unitary evolution.

This argument is reflected in our results since we observe significant improvements when resorting to UF amplification despite its possible imperfections. That being said, we acknowledge that more sophisticated techniques can be used to amplify noise when $\mathcal{W}\mathcal{H} \not\approx \mathcal{H}\mathcal{W}$; some methods rely on learning the errors by using cycle-based noise reconstruction techniques [69, 34], some others rely on pulsed-based inverse evolution [104]. We leave the utilization of those methods for future work.

In addition to RC and NOX, we employ Readout Calibration (RCAL) to mitigate measurement noise. RCAL requires implementing three simple circuits (the first with a $W_{0,0}$ gate on every qutrit, the second with a $W_{0,1}$ gate, and the third one with a $W_{0,2}$ gate) to estimate the probabilities of state-dependent readout errors—i.e., the probabilities that an output $s \in \{0, 1, 2\}$ is incorrectly reported as $s' \neq s$. It then uses this information to efficiently suppress readout errors by inverting confusion matrices.

Purification

Another strategy after performing randomized compiling to mitigate coherent noise is to learn and numerically purify the incoherent noise in our system. Formally, incoherent noise can be defined in terms of error rates associated with Weyl channels. A Weyl channel can

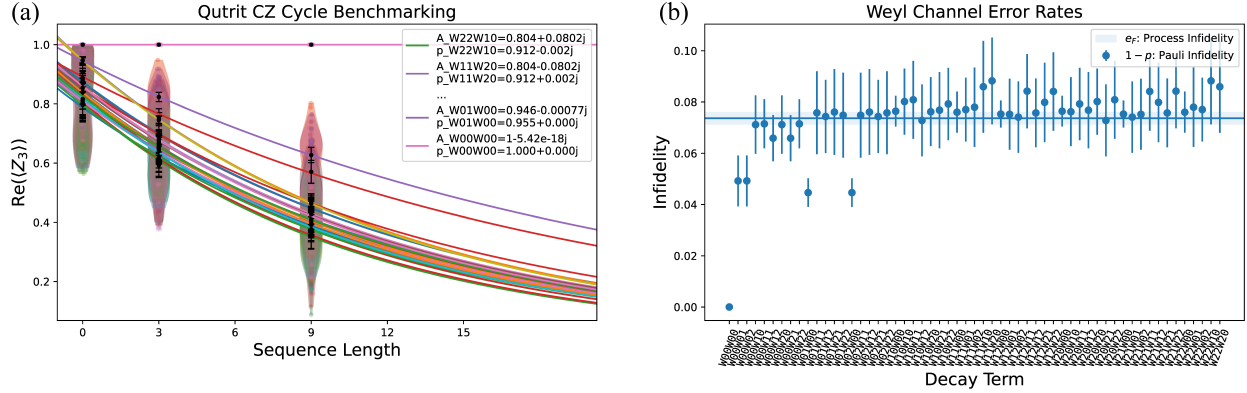


Figure 6.9: Cycle Benchmarking results for a two-qutrit CZ gate employed as the native entangling gate in the digital quantum simulations of the Abelian Higgs model (see Chapter 7). (a) The violin plots of individual Weyl decay exponentials and their associated uncertainty. Both the SPAM parameter (A) and the decay term (p) are fit for each prepared Weyl eigenstate. (b) The infidelity and uncertainty associated with each decay term extracted from fitting the Weyl eigenstate decays from our CB circuits.

be expressed as

$$\begin{aligned}
 \mathcal{E}(\rho) &= \sum_{p,q=0}^2 w_{p,q} W_{p,q} \rho W_{p,q}^\dagger \\
 &= w_{0,0} \rho + \sum_{p,q,p+q \neq 0} w_{p,q} W_{p,q} \rho W_{p,q}^\dagger.
 \end{aligned} \tag{6.76}$$

We efficiently learn the error rate $w_{p,q}$ of the Weyl channels $W_{p,q}$ using cycle benchmarking [62]. Recall from Chapter 4 that cycle benchmarking is performed by executing the following steps

- Prepare eigenstates of a randomly chosen Weyl channel.
- Select a cycle (e.g. our native two-qudit gate) and repeat it N -times. In between each cycle perform Weyl twirling to turn the channel into a stochastic one.
- Measure the expectation value of the same Weyl operator after N times.

Repeating the steps for several layers (N) and initial choice of Weyl eigenstates, the decay rate of each associated channel can be measured by fitting its corresponding error curve. The cycle benchmarking results for an example two-qutrit CZ gate can be found in Fig. 6.9.

We can estimate an effective depolarizing noise parameter from the average of our Weyl decay terms measured by cycle benchmarking. We note this simplified depolarizing noise

model for our system takes the form,

$$\epsilon(\rho) = \lambda\rho + (1 - \lambda)\frac{I_3}{3^N} \quad (6.77)$$

As can be observed in Fig. 6.9, the fairly even distribution of Weyl decay terms makes this a reasonable approximation. To mitigate our incoherent noise, the estimated depolarization strength is used to rescale the Z expectation values measured from our twirled circuits, providing a more accurate estimate of the observables. Additionally, if the circuit employed performs tomography to experimentally reproduce a density matrix, one can directly numerically purify the density matrix using the McWeeny purification formula,

$$\rho_{n+1} = 3\rho_n^2 - 2\rho_n^3. \quad (6.78)$$

where one quickly obtains an idempotent density matrix after iterating for a number of n .

Experimental Investigation of Error Mitigation Methods

We test our ability to effectively twirl and extrapolate noiseless results in two sets of multi-qutrit experiments. In the first experiment, we use state tomography to reconstruct a three-qutrit GHZ state. In the second experiment, we perform random circuit sampling (RCS) for two and three qutrits at a variety of depths. In both experiments, we find significant improvements in our results via the combination of RC and NOX.

Multipartite Qutrit Entanglement

We generate a maximally entangled state on a $D = 27$ dimensional Hilbert space using only three transmon qutrits. Specifically, we realize the qutrit GHZ state $|\Psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$.

To characterize this highly-entangled state, we perform tomography on the full three-qutrit Hilbert space [15], requiring a total of 729 circuits. We measure an informationally-complete set of projections to experimentally reconstruct the density matrix for $\rho = |\Psi\rangle\langle\Psi|_{\text{GHZ}}$, finding a state fidelity of $\mathcal{F} = \text{Tr}(\sqrt{\sigma}\rho\sqrt{\sigma}) = 0.818$, where σ is the ideal density matrix. Even in the unmitigated case, our work represents, to the best of our knowledge, the highest fidelity exploration of multipartite qutrit entanglement to date.

Next, we perform the same experiment using RC and NOX, requiring a total of 43740 ($729 \times 20 \times 3$) circuits, where we use 20 logically-equivalent compilations of the bare circuit and for each of the identity insertion for the two different two-qutrit gate cycles. We find a mitigated state fidelity of $\mathcal{F} = 0.951$, resulting in a greater than 3x reduction in infidelity compared to the unmitigated case. The circuits and the experimentally-reconstructed density matrices can be found in Fig. 6.10. Finally, we purify the reconstructed state measured with RC (no NOX) [255] by numerically finding the nearest density matrix that is idempotent ($\rho^2 = \rho$) [251], which improved the state fidelity from 0.912 to 0.998 (see Fig. 6.11). These

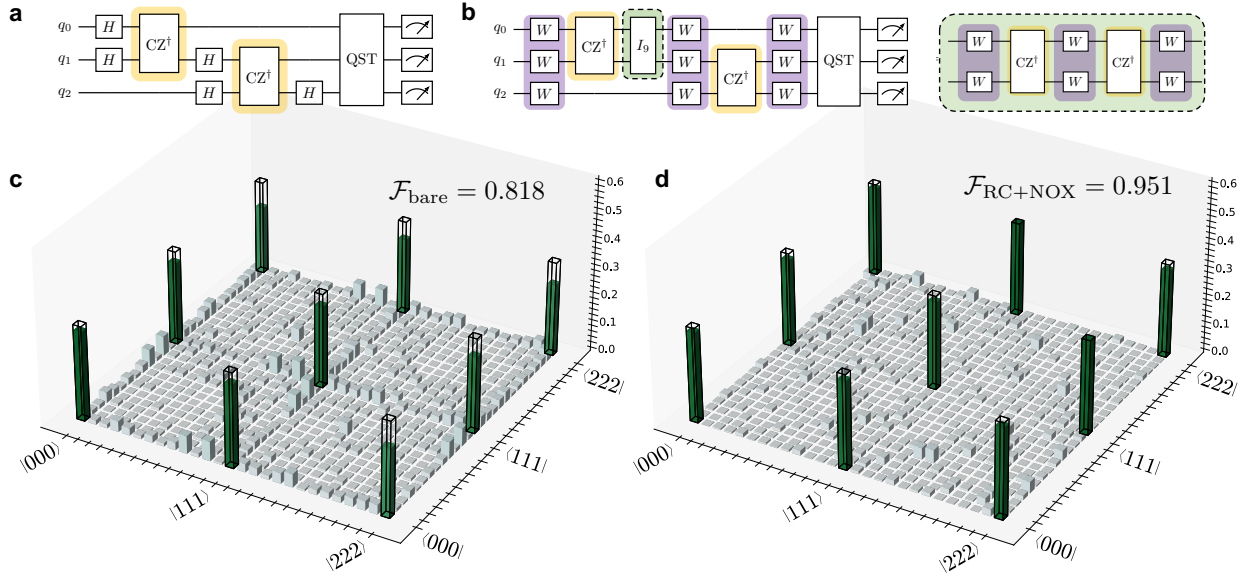


Figure 6.10: Experimentally reconstructed qutrit GHZ density matrices. (a) The circuit diagram for experimentally reconstructing the density matrix of a 3-qutrit GHZ state. The state can be efficiently generated with two-qutrit $CZ^\dagger$ gates (yellow) and single qutrit Hadamard gates (H). (b) An example circuit diagram with RC+NOX, including Weyl twirling (purple), where the identity insertion (green) is placed after the $CZ^\dagger$ (yellow) between q_0 and q_1 . An example of a twirled decomposition of the identity (green) is provided to the right. (c) The experimentally reconstructed density matrix (plotting $|\rho|$) of the 3-qutrit GHZ state with only readout assignment errors corrected. The ideal density matrix is shown with a black outline. (d) The density matrix generated by the noiseless expectation values produced by RC+NOX.

results demonstrate the power of RC to tailor coherent errors to purely stochastic channels, i.e. by employing RC, we were able to tailor the errors in our circuit to look like bare decoherence which was then fixable via simple purification protocol. A summary of the results of all the state tomography experiments and mitigation strategies can be found in Table 6.4.

Qutrit Random Circuit Sampling

We employ random circuit sampling (RCS) as another way to benchmark the performance of our qudit error mitigation protocols. Whereas GHZ circuits are highly structured, the random nature of the RCS experiments provides a useful diagnostic counterpoint for the effectiveness of these methods in a more general circuit setting. RCS with qubits has recently garnered significant interest due to its role in demonstrating quantum advantage [10, 274],

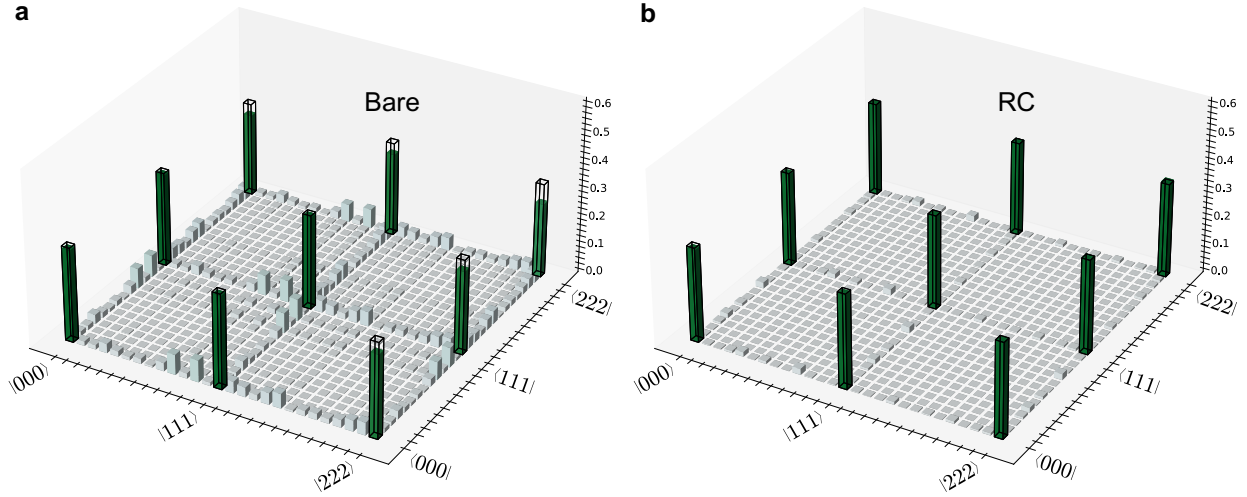


Figure 6.11: Purified GHZ Density Matrices with and without randomized compiling. (a) The purified density matrix in the bare case, with a state fidelity of $\mathcal{F} = 0.912$. (b) The purified density matrix in the 20 randomizations RC case, with $\mathcal{F} = 0.998$.

Mitigation Strategy	State Fidelity
Bare	0.818
RC + NOX	0.951
Bare + Purified	0.912
RC + Purified	0.998
RC + NOX + Purified	0.995

Table 6.4: A summary of the results for the three-qutrit state tomography experiment employing the various mitigation strategies outlined in the text. We note that the reported state fidelity for each experiment employs RCAL.

and it has been speculated that RCS may also have pragmatic use cases [1, 81]. In this work, we extend the task of RCS to qutrits, and study the two and three-qutrit experiments at a number of depths. Notably, as mentioned in Chapter 1, leveraging qutrits for RCS in principle could allow one to probe the regime of quantum supremacy with significantly fewer qutrits (~ 35) than qubits (~ 55).

In both the two and three-qutrit RCS experiments, Haar-random single qutrit gates are interleaved with $\text{CZ}^\dagger$ gates (hard cycles). We study 20 separate RCS instances for each circuit depth. In Fig. 6.12, the bare RCS results are compared to the mitigated results measured using RC (20 randomizations) and NOX. To quantify circuit performance, we calculate the

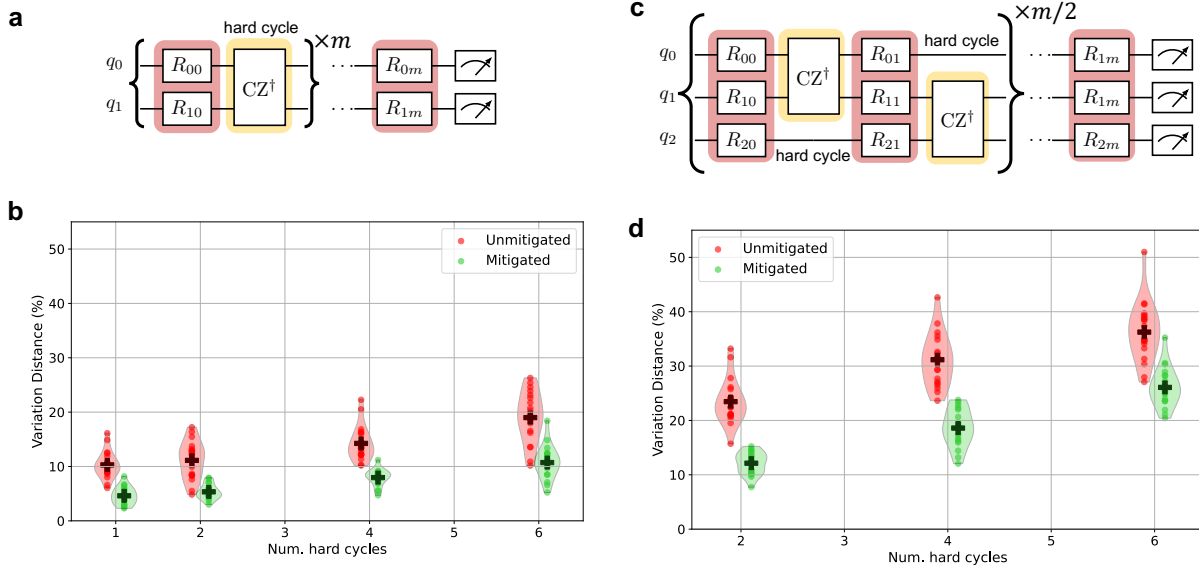


Figure 6.12: Random circuit sampling with qutrits. (a) The circuits for two-qutrit random circuit sampling. $CZ^\dagger$ (yellow) hard cycles are interleaved with Haar random $SU(3)$ gates, R_{ij} (red). (b) Violin plots showing the distribution of the variation distances for the 20 RCS instances with (green) and without (red) RC+NOX at depths $m \in \{1, 2, 3, 6\}$, with mean values marked by crosses. (c) Circuits for three-qutrit RCS. (d) Results for three-qutrit RCS at $m \in \{2, 4, 6\}$.

variation distance from the ideal trit string distributions, defined as

$$VD := \frac{1}{2} \sum_{\bar{s}} |p_{\text{id}}(\bar{s}) - p_{\text{exp}}(\bar{s})| \quad (6.79)$$

where $\bar{s} \in (0, 1, 2)^{\otimes n}$ and p_{id} and p_{exp} are the ideal and experimentally measured trit string distributions, respectively. The best results for the mitigated case were found with 3 insertions of the identity (see Fig. 6.13). Notably, when employing RC+NOX in both the two and three-qutrit RCS experiments, the variation distances at depth 6 were comparable to the unmitigated case at depth 2, and at all depths we found at least a 30% fractional improvement in our results with RC+NOX.

To summarize thus far, we have introduced generalized versions of multiple powerful methods for tailoring and mitigating noise in contemporary qudit systems: randomized compiling, noiseless output extrapolation, and purification. We have so far tested the efficacy of these methods at generating noiseless expectation values on a system of three coupled

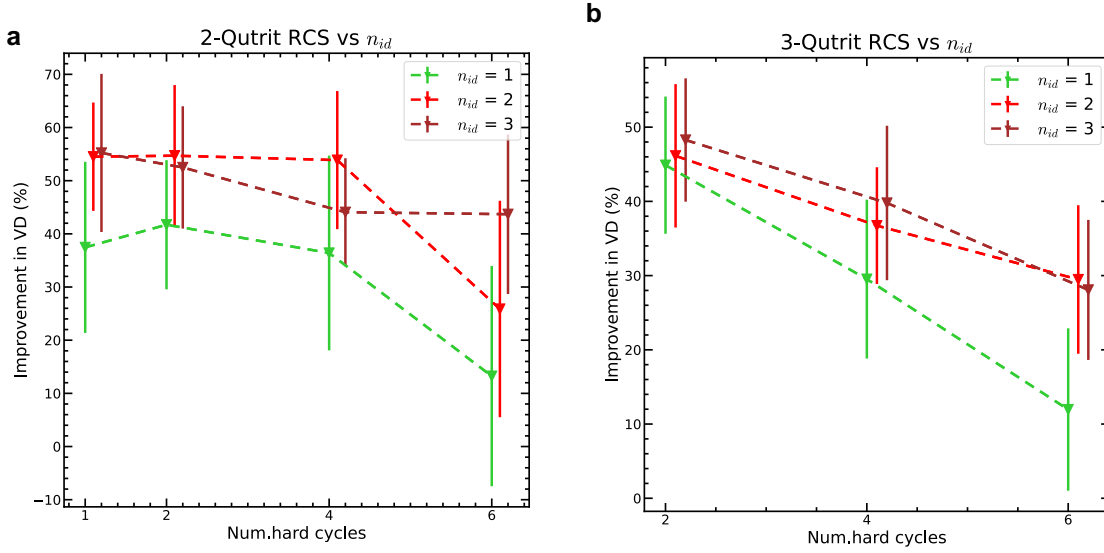


Figure 6.13: Qutrit RCS Improvements with Variable Insertions of the Identity. (a) 2-qutrit RCS data showing improvements in the variation distance as a function of the number of identity insertions (n_{id}) as a function of circuit depth. (b) 3-qutrit RCS data.

transmon qutrits. Specifically, we explored the experimentally reconstructed density matrix of a 3-qutrit GHZ state and the problem of multi-qutrit random circuit sampling. We found that despite the more complex noise environment, added noise sensitivity, and more difficult control requirements, our protocols proved to be a powerful tool for significantly improving our results for all of the aforementioned experiments. Specifically, we effectively tailored coherent errors to stochastic errors on both gate and spectator qutrits via the first demonstration of randomized compiling for qudit dimension $d \geq 3$, and extrapolated beyond the noise in our system via the first demonstration of noiseless output extrapolation in $d \geq 3$.

As higher-dimensional quantum devices begin to mature and compete with qubit systems, the ability to perform longer-depth algorithms without significant errors will be critical to convincing the community of their feasibility and scalability. To this end, this work, in conjunction with the DD developed earlier in the chapter, opens the door to explore many of the advantages leveraged by qudit devices in both quantum algorithms and gate based quantum simulation in the near term on contemporary devices. For the remaining portion of this chapter, we briefly turn our attention to employing some of these methods in the setting of our digital quantum simulation, specifically on problems where qutrit systems very naturally emulate the native problem of interest.

6.3 Outlook

The results presented in this chapter establish that the error suppression toolbox developed for qubit systems can be meaningfully extended to higher-dimensional quantum systems, and that doing so is necessary to make near-term qudit processors competitive with their qubit counterparts for shallow-circuit tasks.

Several near-term improvements to the DD framework are worth pursuing. The central limitation we encountered is that coherent pulse errors in the X_d and Weyl-Heisenberg gates accumulate faster than the benefit of shorter pulse intervals, causing the universal DD sequence to underperform the simpler dX_d family. This is essentially a gate fidelity problem: as the native $\sqrt{\sigma_s^x}$ subspace rotations improve, the crossover point at which more pulses becomes beneficial will shift toward shorter intervals, and higher-order cancellations will become accessible in practice. In the meantime, shortcut-to-adiabaticity, optimal control methods, and multitone approaches like in Ref. [38] for compiling the X_d operator more compactly would directly improve the performance ceiling of both the dX_d and universal sequences. On the crosstalk side, the CKDD sequence as demonstrated here adds a $d^2\tau$ overhead per protected idle window, which at $d = 4$ represents a 16-pulse sequence. Interleaving CKDD with active gate operations, rather than treating it as an idle-period insert, would be a natural and practically important extension, analogous to the dynamical decoupling techniques recently demonstrated in large-scale qubit circuits [222].

For the error mitigation side, the most pressing limitation is sampling overhead. The RC+NOX protocol demonstrated here required up to 43,740 circuits to characterize the three-qutrit GHZ state — a cost that grows rapidly with system size and circuit depth (although much of this is due to the exponential nature of tomography itself). One direction stands out as a natural extension: the demonstration here was limited to circuits whose native two-qutrit gate is a Clifford (the $CZ^\dagger$); extending RC and NOX to non-Clifford gates (such as the two-photon gates of Chapter 5) would require developing twirling schemes for non-Clifford qudit gates.

Looking further ahead, the logical next step is to integrate the error suppression methods of this chapter with the entanglement generation primitives of Chapters 4 and 5 towards studying near-term quantum applications, such as quantum simulation. In the next chapter, we show some results doing just this.

Ultimately, the tools developed in this chapter represent an intermediate milestone on the path toward fault-tolerant qudit quantum computation. Error mitigation and dynamical decoupling cannot replace quantum error correction for deep circuits, but they significantly extend the regime in which near-term qudit processors can extract useful scientific and computational results. As two-qutrit gate fidelities approach and exceed 99%, the balance will shift: the threshold theorem guarantees that error correction becomes exponentially more powerful than mitigation once the physical error rate falls below the code threshold, and the qudit surface code analyzed in Chapter 4 provides a concrete target architecture. Nonetheless, we expect error mitigation protocols, even on top of logical quantum computation, to be here as a useful tool for quantum scientists for the foreseeable future.

Chapter 7

Digital Quantum Simulation with Qudits

In this chapter, we synthesize many of the methods of the preceding chapters of this thesis, specifically leveraging our Clifford two-qutrit gate, arbitrary local gates, and error mitigation strategies, and apply them towards relevant physical problems of interest. In particular, we will employ our gate-based universal qutrit processor to study scientific problems that are well tailored to such a Hilbert space structure via digital quantum simulation. In such quantum simulations, typically, a quantum many body problem is realized in a gate-based manner using Trotterization.

The details of the underlying models we will study, namely collective neutrino oscillations and the spin-1 truncated abelian Higgs model, are both briefly introduced in this chapter, before focusing on hardware results and specifically highlighting the importance of error mitigation to recover useful signals from our device. Significantly more information on both physical systems and the relevant experiments can be found in the original works in Ref. [236] and Ref. [11]. Section 7.1 covers the neutrino oscillation simulations, and Section 7.2 covers the Abelian Higgs model.

7.1 Collective Neutrino Oscillations

Collective neutrino flavor oscillations are of primary importance in understanding the dynamic evolution of core-collapse supernovae and subsequent terrestrial detection, but also among the most challenging aspects of numerical simulations. This situation is complicated by the quantum many-body nature of the problem due to neutrino-neutrino interactions which demands a quantum treatment. An additional complication is the presence of three flavors, which often is approximated by the electron flavor and a heavy lepton flavor. In this section, we provide both qubit and qutrit encodings for all three flavors, and develop optimized quantum circuits for the time evolution and analyze the Trotter error. We conclude our study with a hardware experiment of a system of two neutrinos with superconducting

hardware: the IBM Torino device for qubits and on the qutrit processor employed throughout this chapter. We find that error mitigation greatly helps in obtaining a signal consistent with simulations. While hardware results are comparable at this stage, we expect the qutrit setup to be more convenient for large-scale simulations since it does not suffer from probability leakage into nonphysical qubit space, unlike the qubit setup.

Several efforts have been made to simulate collective neutrino oscillations on a quantum computer in the two-flavor setting [98, 278, 114, 7, 115, 233] and, more recently, also with the full three flavors [252, 43]. Here, we seek to develop optimized circuits for both qubit and qutrit representations based upon the specific properties of the Hamiltonian and the two-body interactions. We represent the Hamiltonian and the resulting SU(3) flavor algebra properties, irrespective of encoding. Then, we proceed to analyze the Trotter error and outline the qubit and qutrit representations of our model. Notably, we apply the computed circuits to study a system of two neutrinos on the IBM Torino superconducting quantum computing device for the qubit setup, and on the same qutrit device used throughout this chapter.

Algebra of collective three-flavor oscillations

Here, we introduce the Hamiltonian model for collective neutrino oscillations in three flavors. Accounting only for forward scattering, the Hamiltonian governing the flavor evolution for a system of N neutrinos can be expressed as $H = H_\nu + H_{\nu\nu}$, where H_ν encompasses the one-body neutrino interaction terms, including vacuum oscillations and interactions with the matter background. The full two-body term including momentum exchanges is beyond the scope of this work.

Ignoring neutrino-matter interactions for simplicity (as they can be nearly rotated away with an appropriate change-of-basis transformation), the contribution from vacuum oscillations can be written as

$$H_\nu = \sum_{q=1}^N \omega_q \vec{B} \cdot \vec{\lambda}_q . \quad (7.1)$$

Within the Hilbert subspace of each neutrino indexed by q , we denote by $\vec{\lambda}$ the vector of Gell-Mann matrices, and $\vec{B}$ is likewise a constant, eight-component unit vector. Defining the neutrino mass-squared differences, $\Delta_{ij} = m_i^2 - m_j^2$, we express the vacuum oscillation frequencies ω_q and the non-zero entries of $\vec{B}$ in the mass basis as follows:

$$\begin{aligned} \omega_q &= \frac{\sqrt{\Delta_{12}^2 + (\Delta_{13} + \Delta_{23})^2/3}}{4p_q} \\ B_3 &= \frac{\Delta_{12}}{\sqrt{\Delta_{12}^2 + (\Delta_{13} + \Delta_{23})^2/3}} \\ B_8 &= \frac{\Delta_{13} + \Delta_{23}}{\sqrt{3\Delta_{12}^2 + (\Delta_{13} + \Delta_{23})^2}} . \end{aligned} \quad (7.2)$$

Here we denote p_q the momentum of the neutrino indexed by the label q . All other elements of $\vec{B}$ are zero.

Similarly, we may write the two-body term

$$H_{\nu\nu} = \frac{\mu}{2N} \sum_{q < k}^N (1 - \cos(\theta_{qk})) \vec{\lambda}_q \cdot \vec{\lambda}_k := \sum_{q < k}^N J_{qk} \vec{\lambda}_q \cdot \vec{\lambda}_k, \quad (7.3)$$

where coupling constant $\mu = \sqrt{2}G_F n_\nu$ depends on both Fermi's constant G_F and the local neutrino number density n_ν ($= N/V$ for N neutrinos quantized in a box of volume V), and $\theta_{qk} = \hat{p}_q \cdot \hat{p}_k$ is the angle between the momenta $\vec{p}_q$ and $\vec{p}_k$. A full derivation of the Hamiltonian can be found in the appendix of Ref. [236].

The Gell-Mann matrices satisfy the following algebra,

$$\begin{aligned} [\lambda_i, \lambda_j] &= 2if_{ijk}\lambda_k, \\ f^{147} &= -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\ f^{458} &= f^{678} = \frac{\sqrt{3}}{2}, \quad f^{123} = 1 \end{aligned} \quad (7.4)$$

We note that there are three embedded SU(2) sub-algebras,

$$\begin{aligned} i &\rightarrow \{\lambda_1, \lambda_2, \lambda_3\} \\ j &\rightarrow \{\lambda_4, \lambda_5, \lambda_+\} \\ k &\rightarrow \{\lambda_6, \lambda_7, \lambda_-\}, \end{aligned} \quad (7.5)$$

where $\lambda_\pm = \frac{1}{2}(\pm\lambda_3 + \sqrt{3}\lambda_8)$ and have denoted them by (i, j, k) . The Casimir operators $(\vec{\lambda} \cdot \vec{\lambda})_{i,j,k}$ of these sub-algebras mutually commute. Another important property of the SU(n) quadratic Casimir operator is its relation to the SWAP operator,

$$\frac{1}{n} \mathbb{1}_n + \frac{1}{2} \vec{g}_q \cdot \vec{g}_k = \text{SWAP}_{qk}, \quad (7.6)$$

where $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any pair vectors in the respective n -dimensional qudit spaces, and $\vec{g}$ are the generators of the algebra, i.e., Pauli matrices for SU(2) and Gell-Mann matrices for SU(3).

Trotter Error

Given the Hamiltonian model for oscillations outlined in the previous section, we are ready to describe how we simulate real-time evolution of a neutrino system. In order to implement the time evolution operator e^{-iHt} we will use a simple Trotterization method.

First, we can approximate the evolution operator by breaking it into the product of the vacuum interaction and the 2-body term as follows:

$$e^{-iHt} \approx e^{-iH_\nu t} e^{-iH_{\nu\nu} t} \quad (7.7)$$

and the error in doing this approximation is

$$\|e^{-iHt} - e^{-iH_\nu t} e^{-iH_{\nu\nu} t}\| \leq \frac{t^2}{2} \| [H_\nu, H_{\nu\nu}] \| . \quad (7.8)$$

More precisely, in the appendix of Ref. [236] we calculate the bound

$$\|e^{-iHt} - e^{-iH_\nu t} e^{-iH_{\nu\nu} t}\| \leq t^2 \mu N \max_{q,k} |\omega_k - \omega_q| . \quad (7.9)$$

The one-body term H_ν is itself a sum of commuting single-body operators, so we may simply factor

$$e^{-iH_\nu t} = \prod_{q=1}^N e^{-it\omega_q \vec{B} \cdot \vec{\lambda}_q}, \quad (7.10)$$

without involving any further approximation. Moreover, for the two-body interaction, we can use again the first order Trotter formula, factoring individual interaction terms;

$$\begin{aligned} & \left\| e^{iH_{\nu\nu} t} - \prod_{q < k} e^{-itJ_{qk} \vec{\lambda}_q \cdot \vec{\lambda}_k} \right\| \leq \\ & \frac{t^2}{2} \sum_{K=1}^{\binom{N}{2}} J_K \left\| \left[\vec{\lambda}_{q_K} \cdot \vec{\lambda}_{k_K}, \sum_{L > K} \vec{\lambda}_{q_L} \cdot \vec{\lambda}_{k_L} \right] \right\| =: \frac{t^2}{2} \|C_{22}\| \end{aligned} \quad (7.11)$$

where J_K is the coupling strength for the pair $K = 1, \dots, \binom{N}{2}$. As shown in detail in the full derivation presented in Appendix of Ref. [11], we have

$$\|C_{22}\| < \sqrt{3} \mu^2 N \max_{q,k,l} |\cos(\theta_{lq}) - \cos(\theta_{lk})| . \quad (7.12)$$

In general, by doing r steps with time t/r , the full Trotter error will be bounded by

$$\frac{t^2}{2r} (2\mu N \max_{q,k} |\omega_k - \omega_q| + \|C_{22}\|). \quad (7.13)$$

By increasing the number of steps r , and thus the number of operations, we can reduce the approximation error. Notably, our above analysis is independent of the representation of neutrino flavor on hardware, applying equally well to simulations on qubit and qutrit hardware.

Qutrit and Qubit Representations

Having outlined the Hamiltonian for collective oscillations and how to decompose the evolution operator into its individual terms, it remains to be shown how they are simulated

on quantum hardware. Before showing the operations on qubits or qutrits to simulate our system, let us introduce how to encode our problem on each kind of hardware.

When considering qutrit-based hardware, we can straightforwardly encode each neutrino with a single qutrit, by associating each flavor of a neutrino with a different state of the qutrit. Our convention is as follows:

$$|\nu_e\rangle = |0\rangle, |\nu_\mu\rangle = |1\rangle, |\nu_\tau\rangle = |2\rangle. \quad (7.14)$$

This encoding is particularly easy to use, since it preserves the locality of the Hamiltonian: one-body terms become single-qutrit gates, and two-body terms becomes two-qutrit gates. Consequently, the depth and gate count will be much lower here than in the qubit case.

Next, we explain the mapping from neutrinos onto qubits. A three-flavor neutrino has three possible states, which means that we need two qubits for each neutrino, entailing one extra ‘unphysical’ state, which ideally will never be used. Considering the computational basis of two qubits, we choose the following encoding:

$$\begin{aligned} |\nu_e\rangle &= |01\rangle, |\nu_\mu\rangle = |10\rangle \\ |\nu_\tau\rangle &= |11\rangle, |\tilde{\nu}\rangle = |00\rangle \end{aligned} \quad (7.15)$$

Since the state $|\tilde{\nu}\rangle$ is non-physical, no wave-function amplitude should propagate into this state throughout the entire time evolution.

Likewise, we map the generators of the $SU(3)$ algebra, the Gell-Mann matrices that we denoted with $\vec{\lambda}$ above, into a two-qubit operator space spanned by Pauli strings of the form $\{\sigma_i \otimes \sigma_j\}_{i,j=0,1,2,3}$. Note that this mapping must be consistent with the mapping of states above, so that the new two-qubit operators reduce correctly to the Gell-Mann matrices if we delete the $|00\rangle$ state. This mapping is given by the following definitions,

$$\begin{aligned} 2Q_1 &= X_0X_1 + Y_0Y_1 & 2Q_2 &= Y_0X_1 - X_0Y_1 \\ 2Q_3 &= Z_0\mathbb{1}_1 - \mathbb{1}_0Z_1 & 2Q_4 &= X_0\mathbb{1}_1 - X_0Z_1 \\ 2Q_5 &= Y_0\mathbb{1}_1 - Y_0Z_1 & 2Q_6 &= \mathbb{1}_0X_1 - Z_0X_1 \\ 2Q_7 &= \mathbb{1}_0Y_1 - Z_0Y_1 \\ 2Q_8 &= \frac{\mathbb{1}_0Z_1 + Z_0\mathbb{1}_1 - 2Z_0Z_1}{\sqrt{3}}. \end{aligned} \quad (7.16)$$

Here we denote the 2×2 identity by $\mathbb{1}$ and the Pauli matrices $X, Y, Z = \sigma_{\{x,y,z\}}$. The subscript on the Pauli operators denotes the qubit on which operator acts, and we omitted the tensor product for brevity. In short, we have $Q_i = 0 \oplus \lambda_i$ (there the 0 entry is meant to be on the $|00\rangle\langle 00|$ entry of the matrix). With this definition, the Q_i matrices satisfy the same algebra of the Gell-Mann matrices. Thus we encode the $SU(3)$ algebra in a larger two-qubit space containing the non-physical $|\tilde{\nu}\rangle$ state. The qubit-encoded Hamiltonian behaves in the same way as the original one in the subspace generated by $\{|01\rangle, |10\rangle, |11\rangle\}$ and does not involve the non-physical $|00\rangle$ state.

Hardware Implementation

We describe now the quantum circuit implementation of the time evolution operator, providing versions for both an all-to-all and a nearest-neighbors connectivity in the qubit case.

Qubit Circuit

Since we want to observe the flavor oscillations of neutrinos, we would like to encode and measure neutrinos in the flavor basis. However, the Hamiltonian is more concisely expressed in the mass basis, so we start our quantum circuit computations in the flavor basis, transform to the mass basis through the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, apply time evolution, and finally transform back to the flavor basis before performing measurements. To be able to switch from one basis to the other we need to implement the PMNS matrix as a gate. This is a two-qubit gate that directly depends on the mixing angles, as explained in the Appendix of Ref. [11], where we also provide its circuit. Our results are similar to those found in Ref. [173], which used a different qubit mapping for the three flavors, and we also provide a version in terms of cross resonance gates. Having computed the error associated with the Trotter approximation, here we focus on the gate implementation of the unitary operations $e^{-it\omega_q \vec{B} \cdot \vec{Q}_q}$ and $e^{-itJ_{qk} \vec{Q}_q \cdot \vec{Q}_k}$.

The exponent of the one-body term can be written as a linear combination of Z_0, Z_1 and $Z_0 Z_1$, which are mutually commuting operators. In the physical subspace the unitary operator is equivalent to a product of single qubit rotations $R_{Z_0}(\alpha)R_{Z_1}(\beta)$ up to a global phase, for an appropriate choice of the angles α and β . We can then implement the evolution under the one-body term without using entangling operations, a strategy also employed recently in Ref. [252].

Using the commutation relations given by the subalgebras in Eq. (7.5), one finds that $\vec{Q}_q \cdot \vec{Q}_k$ can be broken into the following commuting pieces (denoting $Q_i \otimes Q_i = Q_{ii}$):

$$\begin{aligned} \sum_i Q_{ii} &= (Q_{11} + Q_{22}) + (Q_{44} + Q_{55}) \\ &+ (Q_{66} + Q_{77}) + (Q_{33}) + (Q_{88}) \end{aligned} \quad (7.17)$$

where each parenthetical term commutes with every other. Therefore we can break the exponential *exactly*, as follows:

$$\begin{aligned} e^{-itJ_{qk} \vec{Q}_q \cdot \vec{Q}_k} &= e^{-itJ_{qk}(Q_{11}+Q_{22})_{qk}} \times \\ &e^{-itJ_{qk}(Q_{44}+Q_{55})_{qk}} \times \\ &e^{-itJ_{qk}(Q_{66}+Q_{77})_{qk}} \times \\ &e^{-itJ_{qk}(Q_{33})_{qk}} e^{-itJ_{qk}(Q_{88})_{qk}}, \end{aligned} \quad (7.18)$$

where we used the subscript qk in the parenthesis surrounding the operator to clarify on which qubits the operators actually act. For instance with the notation $(Q_{33})_{qk}$ we intend

the tensor product between Q_3 acting on the pair of qubits representing neutrino k and Q_3 on the pair for neutrino q .

Now, for example let us consider in detail the second exponential $e^{-itJ_{qk}(Q_{44}+Q_{55})_{qk}}$. We can write $Q_{44} + Q_{55}$ in terms of Pauli matrices using the mapping from Eq. (7.16)

$$\begin{aligned} Q_4 \otimes Q_4 &= \frac{X_1 X_3}{4} (1 - Z_4 - Z_2 + Z_2 Z_4), \\ Q_5 \otimes Q_5 &= \frac{Y_1 Y_3}{4} (1 - Z_4 - Z_2 + Z_2 Z_4). \end{aligned} \quad (7.19)$$

We note that the first two indices (1, 2) refer to the first two qubits and the first neutrino, and the second two indices (3, 4) refer to the second two qubits and second neutrino, respectively. Then, we define the following operator:

$$\begin{array}{c} q_a \\ q_b \end{array} \begin{array}{|c|} \hline G_{ab} \\ \hline \end{array} := \begin{array}{c} \boxed{H} \\ \bullet \\ \oplus \end{array} \quad (7.20)$$

It is straightforward to prove that G_{13} diagonalizes the operator $Q_{44} + Q_{55}$ (see Appendix in Ref. [236] for details), and thus

$$G_{13} (Q_{44} + Q_{55}) G_{13}^\dagger = \frac{Z_1 - Z_1 Z_3}{4} \times (1 - Z_4 - Z_2 + Z_2 Z_4) \quad (7.21)$$

Using this unitary transformation, the exponential term $e^{-itJ_{qk}(Q_{44}+Q_{55})_{qk}}$ can then be implemented as,

$$G_{13}^\dagger e^{-itJ_{qk} \left(\frac{Z_1 - Z_1 Z_3}{4} (1 - Z_4 - Z_2 + Z_2 Z_4) \right)_{qk}} G_{13}. \quad (7.22)$$

We can implement $e^{-itJ_{qk}(Q_{66}+Q_{77})_{qk}}$ and $e^{-itJ_{qk}(Q_{11}+Q_{22})_{qk}}$ in a similar fashion. The diagonal unitary operations $e^{-itJ_{qk}Q_{33}}$ and $e^{-itJ_{qk}Q_{88}}$ are four-qubit operators whose weight can be reduced by introducing gadgets similar to the diagonalizing operator G_{ab} . Implementation details (i.e., in terms of elementary gates) for all these terms can be found in the Appendices of Ref. [236]. Our circuit implementation requires 34 CNOT gates when qubits have all-to-all connectivity and 39 CNOT gates for the T connectivity present on IBM devices. It is likely that further optimization could be done since the authors of Ref. [252] obtain circuits with only 18 CNOT with all-to-all connectivity and 30 CNOT for qubits on a chain, even though the mapping of the flavor space is slightly different from ours.

Qutrit circuit

By considering qutrits instead of qubits, each neutrino can be encoded with a single qutrit. The PMNS matrix is then a single-qutrit gate, which is in general easy to implement, as are the single-body Hamiltonian terms. So, in the following we will only focus on the two-body term, which requires the use of two-qutrit gates.

We recall the CSUM gate (also referred to here as $C\tilde{X}$ gate), defined as follows:

$$C\tilde{X} |x\rangle |y\rangle = |x\rangle |-x - y\rangle \quad (7.23)$$

This gate is self-inverse and can be decomposed into three two-level controlled gates as

$$\begin{array}{c} \bullet \\ | \\ \boxed{C\tilde{X}} \end{array} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ | \\ \boxed{X^{12}} \quad \boxed{X^{02}} \quad \boxed{X^{01}} \end{array} \quad (7.24)$$

It can be implemented with a qutrit controlled- Z gate and a single-qutrit quantum Fourier transform, which is nothing but the generalization of the Hadamard gate. By using this gate, the two-qutrit SWAP becomes the standard circuit if we replace the CNOTs with $C\tilde{X}$:

$$\text{SWAP} = \begin{array}{c} \bullet \\ | \\ \boxed{C\tilde{X}} \end{array} \begin{array}{c} \bullet \\ | \\ \boxed{C\tilde{X}} \end{array} \begin{array}{c} \bullet \\ | \\ \boxed{C\tilde{X}} \end{array} \quad (7.25)$$

Since we need a partial swap to simulate the interaction, we can generalize the procedure of qubits, and replace the middle CX with a CR . Here, CR is a controlled- $SU(3)$ gate where the $SU(3)$ operation is the product of a two-level phase gate P_z gate and a two-level rotation R_x gate,

$$\begin{aligned} R^{12}(\theta) &= \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & \cos(\theta) & -i \sin(\theta) \\ 0 & -i \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= P_z^{01}(\theta) R_x^{12}(2\theta) \\ R^{02}(\theta) &= P_z^{12}(\theta) R_x^{02}(2\theta) \\ R^{01}(\theta) &= P_z^{21}(\theta) R_x^{01}(2\theta) \end{aligned} \quad (7.26)$$

The resulting circuit, which applies $e^{-2i\theta\text{SWAP}}$, is:

$$\begin{array}{c} \bullet \\ | \\ \boxed{R^{12}(2\theta)} \quad \boxed{R^{02}(2\theta)} \quad \boxed{R^{01}(2\theta)} \\ | \\ \boxed{C\tilde{X}} \quad 0 \quad 1 \quad 2 \quad \boxed{C\tilde{X}} \end{array} \quad (7.27)$$

Given this quantum circuit, there are many ways one can implement it on hardware. We use the BQSket compiler [279] to optimize the quantum circuit in terms of CZ gates and single-qutrit rotations. By leveraging our compiler with a numerical synthesis tolerance of 10^{-6} (significantly lower than any gate infidelities on the device), we construct circuits that require four CZ gates plus local $SU(3)$ rotations per Trotter step. Our $SU(3)$ rotations were implemented by embedded $SU(2)$ rotations with subspace Rabi oscillations [176, 93]. We note that the gate cost of our qutrit circuit is the same as the corresponding construction from Ref. [252].

Hardware experiments

We can now proceed to study the circuits developed in the previous section on the IBM Torino device, an IBM Quantum Heron r1 processor, and the qutrit device presented previously in the earlier chapters. Based on the past success with similar algorithms, we implement an error mitigation technique based on noise renormalization. We first apply Pauli twirling (RC) to the compiled circuits in order to approximate the error channel of the device as global depolarizing noise

$$\Lambda[\rho] = (1 - p)\rho + \frac{p}{\text{Tr}[\mathbb{1}]} \mathbb{1} , \quad (7.28)$$

with $\mathbb{1}$ the global identity matrix, ρ the noiseless quantum state and p the noise strength. We added 30 twirled circuits to the original circuit, and took 5120 shots per circuit. Strictly speaking, Pauli twirling does not convert a generic quantum channel into a depolarizing channel, but into a stochastic Pauli channel, which in practice may be approximated by the depolarizing channel rather well.

We perform simulations of the time-evolution using the first-order product formula from Eq. (7.7) with a fixed time-step $dt = 0.5\mu^{-1}$. With this strategy to reach a total evolution time $t = Ldt$, we need to perform L layers of single steps involving evolution with both the one-body and two-body Hamiltonians. Since we expect the noise strength to depend on the circuit depth, we expect to have the following noisy state

$$\Lambda[\rho(t = Ldt)] = (1 - p_L)\rho(t = Ldt) + \frac{p_L}{\text{Tr}[\mathbb{1}]} \mathbb{1} , \quad (7.29)$$

with $p_L \rightarrow 1$ as $L \rightarrow \infty$. We estimate p_L , for a given depth, using a calibration circuit involving only Clifford gates and use the results to remove the error contribution. The Clifford circuit has the same gate structure as the target circuit, but performs a SWAP operation by leveraging Eq. (7.6). For instance, if we are interested in expectation values $\langle \Pi_k(t) \rangle$ of some projector operator Π_k at time $t = Ldt$, we can use the calibration circuit to get

$$\langle \Pi_c(L) \rangle_{\text{calib}} = (1 - p_L) + \frac{p_L}{\text{Tr}[\mathbb{1}]} , \quad (7.30)$$

where Π_c is a rank-1 projector on the state we expect to observe in the absence of noise. The resulting expectation value can be used to determine empirically the circuit fidelity as a function of depth. The results for two three-flavor neutrinos on both devices is shown in Fig. 7.1.

Further, we model the results obtained with an exponential fit of the form

$$\langle \Pi_c(L) \rangle_{\text{calib}} = sF^L , \quad (7.31)$$

where the base F represents a fidelity per step and the coefficient s accounts for state-preparation and measurement (SPAM) errors. The best fit parameters for both devices are reported inside the plot in Fig. 7.1. The much larger SPAM error observed in the qubit case—even when no time-steps are performed (i.e. $L = 0$)—is in large part a consequence

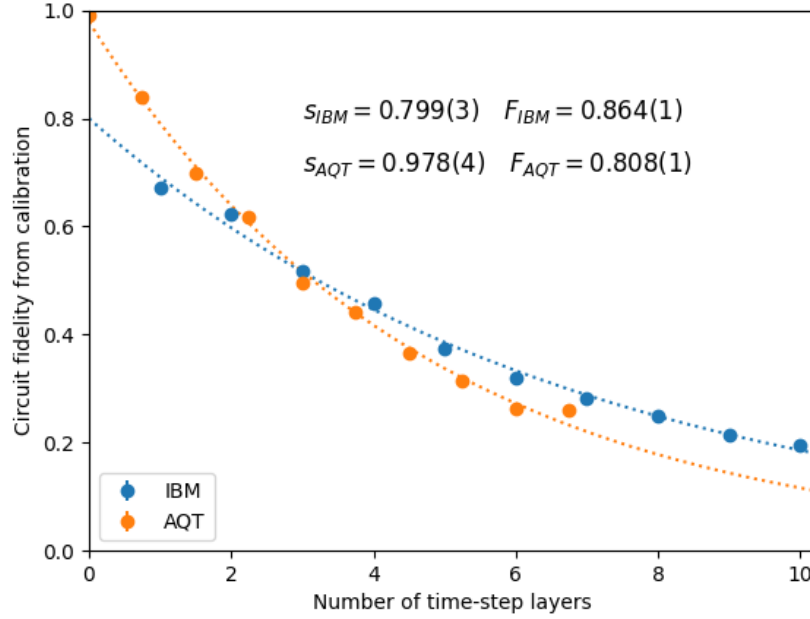


Figure 7.1: Fidelity normalization factor estimation by Clifford version of the circuit assuming depolarizing noise. Blue dotted points depict results from the IBM device, and orange points from the AQT device. The curves plotted are fits of the aforementioned data to an exponential function, as in Eq. (7.31). We took over 150,000 shots per point in the plot for IBM device and over 30,000 for the AQT device. We show uncertainties by including the standard error with vertical lines, which are rather small. Since the AQT device used three CZ gates for calibration and four CZ gates for time evolution, the time steps have been rescaled by a factor of $3/4$. For $t = 0$ we show the initial state preparation results for both devices. As state preparation in the qubit case requires entangling gates, SPAM error is higher than the qutrit case.

of the transformations between the flavor and mass bases, which involve entangling operations. In contrast, on the qutrit implementation, state preparation involves only local $SU(3)$ transformations instead.

The error mitigation proceeds then as follows: we first estimate the depolarizing noise strength p_L using Eq. (7.30) obtaining

$$p_L = \frac{\text{Tr}[\mathbb{1}]}{\text{Tr}[\mathbb{1}] - 1} (1 - \langle \Pi_c(L) \rangle_{\text{calib}}) . \quad (7.32)$$

For the qubit simulations we estimate $\langle \Pi_c(L) \rangle_{\text{calib}}$ directly from the measured calibration circuits. For the qutrit case, owing to the difference in gate counts per layer between the simulation circuits and the calibration ones, we use instead the fit from Eq. (7.31) which provides a good model for the measured data. Using this estimator for the noise strength we can then estimate the noise mitigated expectation value of a rank-1 projector Π_k as follows

$$\langle \Pi_k(L) \rangle_{\text{mit}} = \frac{\langle \Pi_k(L) \rangle - \frac{p_L}{\text{Tr}[\mathbb{1}]}}{1 - p_L} . \quad (7.33)$$

In the qubit case we can express the mitigated expectation value directly in terms of the bare expectation value and the result of calibration at step L as

$$\langle \Pi_k(L) \rangle_{\text{mit}} = \frac{15 \langle \Pi_k(L) \rangle + \langle \Pi_c(L) \rangle_{\text{calib}} - 1}{16 \langle \Pi_c(L) \rangle_{\text{calib}} - 1} , \quad (7.34)$$

where we used $\text{Tr}[\mathbb{1}] = 16$ for our four qubits. We have checked that the mitigated values obtained from the fit in the qubit case agree quite well with the extrapolation from the bare Clifford circuits.

We present a selection of results from our simulations in Fig. 7.2. A complete set of results is provided in the Fig. 7.5 and Fig. 7.4. The system is initially prepared in the state $|\nu_e \nu_\mu\rangle$ and evolved under the full flavor Hamiltonian $H = H_\nu + H_{\nu\nu}$ using a first order Trotter formula as discussed in Sec. 7.1. The neutrino frequencies are $\omega = \{2, 2.5\} \mu$. The mass dependent parameters are $\{B_3, B_8\} = \{0.025483, 0.999567\}$. The mixing angles are $\theta_{12} = 33.44^\circ$, $\theta_{13} = 8.57^\circ$, $\theta_{23} = 49.2^\circ$, $\delta_{cp} = 0$. Here we show the survival probabilities of $|\nu_e \nu_\mu\rangle$ and $|\nu_\mu \nu_e\rangle$ evolved for up to ten time steps, before and after performing error mitigation techniques described earlier. We can observe that bare results do not match expected theoretical values beyond the first couple of Trotter steps, and as time increases resembles an equipartition of probabilities among all states. The match to the theoretical results at later times is more likely coincidence. Mitigation improves results rather drastically, with both devices agreeing with theoretical expectations for up to five Trotter steps, particularly for the flavor probability of the $|\nu_\mu \nu_e\rangle$ state, which is very small throughout the time evolution and therefore more challenging to resolve.

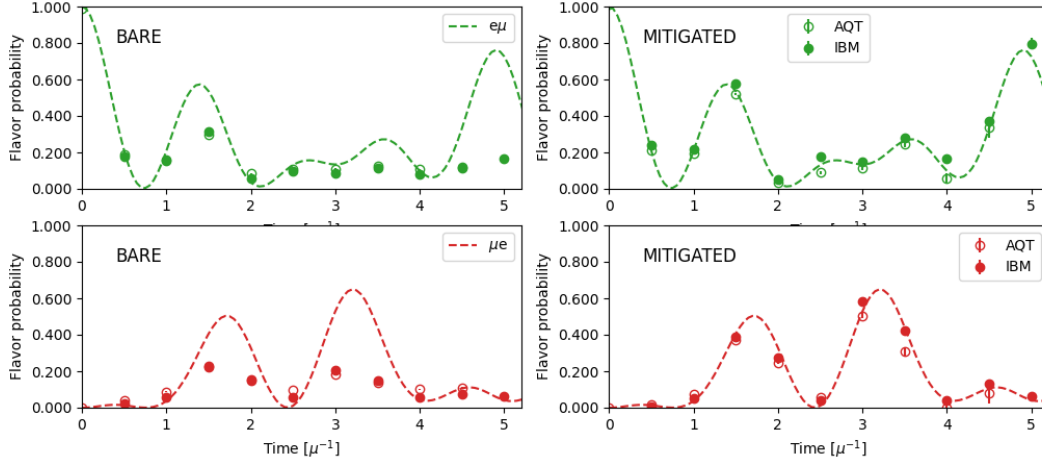


Figure 7.2: Survival probability for the initial state $|\nu_e \nu_\mu\rangle$ (top) and its swapped counterpart $|\nu_\mu \nu_e\rangle$ (bottom) as function of time evolution. We show the results from measurements both directly on the device (left) and after error mitigation (right) for up to ten time steps. Solid dots are the results from the (qubit-based) IBM Torino device and empty dots are the results from the (qutrit-based) AQT device. Uncertainty bars are estimated by the standard error from each simulation on hardware. Exact results are plotted as curves for reference. The improvement due to error mitigation is quite drastic, and depolarizing noise, after Pauli twirling has been performed, seems to be a significant fraction of the noise.

Extended hardware results

In this section we present the results for all states of the two neutrino systems. We start by presenting in Fig. 7.3 the total variation distance (TVD)

$$\delta(P_{ref}, P_{meas}) = \frac{1}{2} \sum_x |P_{ref}(x) - P_{meas}(x)|, \quad (7.35)$$

between the expected distribution P_{ref} over the nine flavor states of two neutrinos and the measured distribution P_{meas} obtained from the hardware runs, with and without error mitigation. The bands are 90% confidence intervals obtained by resampling the experimental results assuming they follow a normal distribution with the estimated means and standard deviations. From these results, the beneficial effect of error mitigation is clear, consistently lowering the TVD in all cases. This analysis also shows that the qubit simulations, both with and without error mitigation, achieve a smaller TVD than the respective calculations on qutrit hardware even though after mitigation the spread appears reduced.

In Fig. 7.4, data before error mitigation are shown. Each plot depicts the evolving probability of a given definite-flavor state, except the first panel which is a superposition

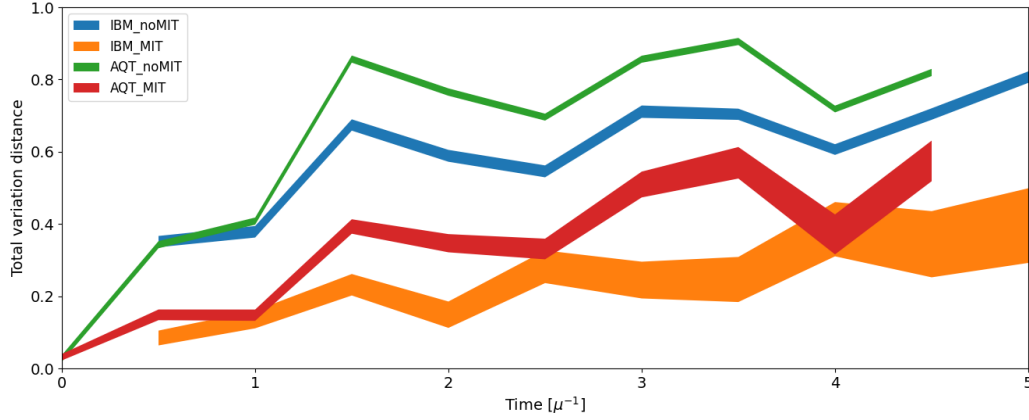


Figure 7.3: Total variation distances between the expected flavor probability distribution over flavor states and the distribution estimated from quantum computations. We show in blue and orange the qubit result without (IBM_noMIT) and with (IBM_MIT) error mitigation respectively. Similarly, we report the qutrit result without error mitigation (AQT_noMIT) in green and with error mitigation (AQT_MIT) in red. The bands correspond to 90% confidence intervals.

of all other plots to underline the relative magnitude. In each case we compare qubit and qutrit with the exact result. Without any error mitigation, it is clear that hardware noise dominates our results, and data are far from the exact result after a couple of Trotter steps.

However, one difference can be found in the probabilities on hardware for states that should have a very small theoretical probability during runtime. The time evolution operator plays almost no role in the evolution of these states, and the major contribution is the depolarizing noise, which should converge to $1/\text{Tr}[\mathbb{1}]$ where $\text{Tr}[\mathbb{1}]$ is the total number of global states for (16 or 9 for qubits and qutrits, respectively). Since for qubits we have the presence of a nonphysical state, the depolarizing noise will converge to $1/16 \approx 0.06$. In the case of qutrits instead, the noise will converge to $1/9 \approx 0.1$. Indeed, unmitigated hardware results converge close to these values as function of time. This difference is taken into account during error mitigation, as explained in more detail in Sec. 7.1.

As already mentioned, this distinction can give us some intuition about the impact of depolarizing noise with system scaling. Considering the qubit encoding for N neutrinos, the fraction of physical states over the total number of states is $(3/4)^N$, which vanishes as the number of neutrinos increases. Since probability amplitudes will increasingly populate unphysical states due to the depolarizing noise in the qubit case, we can expect the qutrit encoding to scale better when considering only error-mitigation strategies. In practice, the problem with unphysical state might not be so impactful if one is only interested in local, small-weight observables.

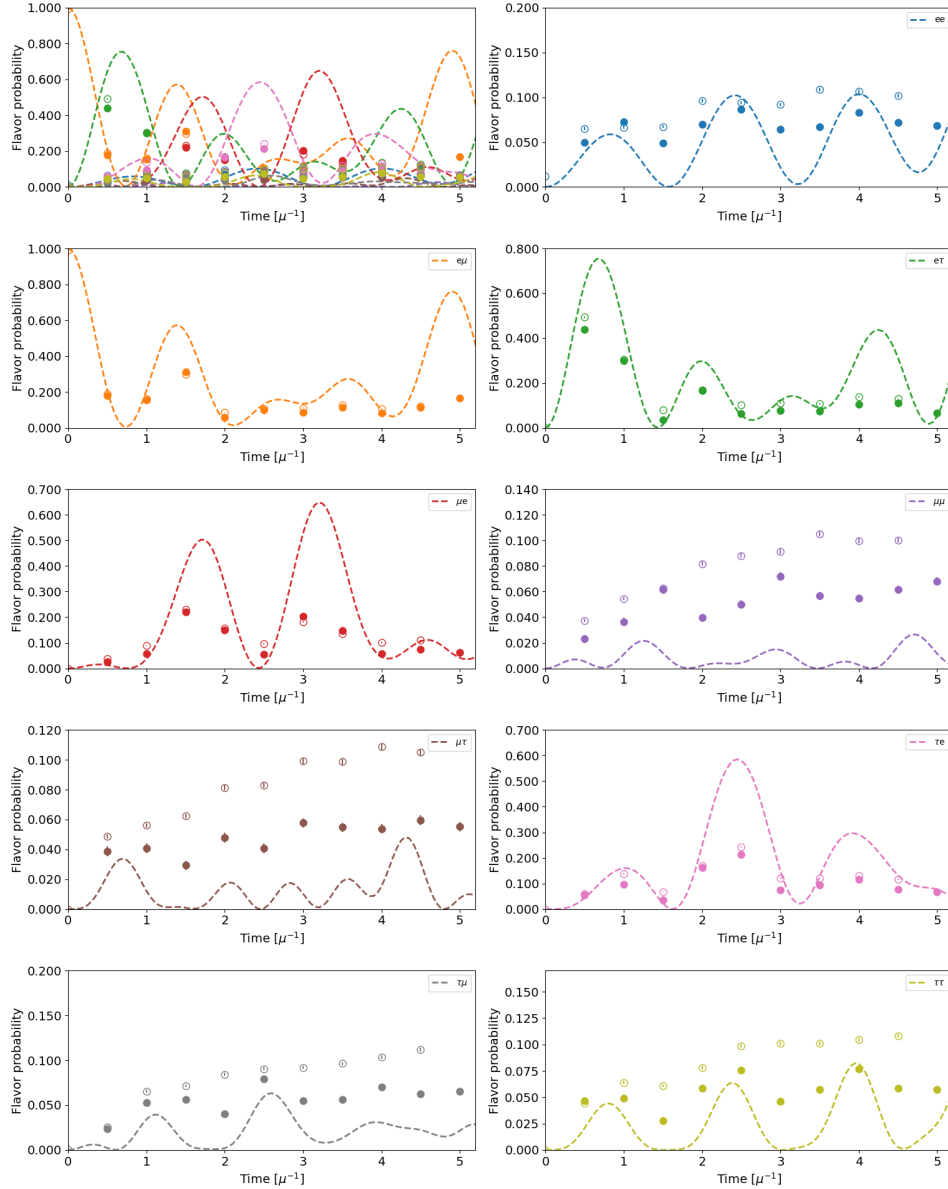


Figure 7.4: The vertical axis shows the probability of occupying a given final state. We present one plot for each possible two-flavor neutrino state, while the first panel shows a summary, highlighting the relative magnitudes. We also quantify the uncertainty using standard error through the vertical error bars. The dashed line is the exact calculation, while solid and empty dots represent the qubit (IBM) and qutrit (AQT) data respectively, before error mitigation.

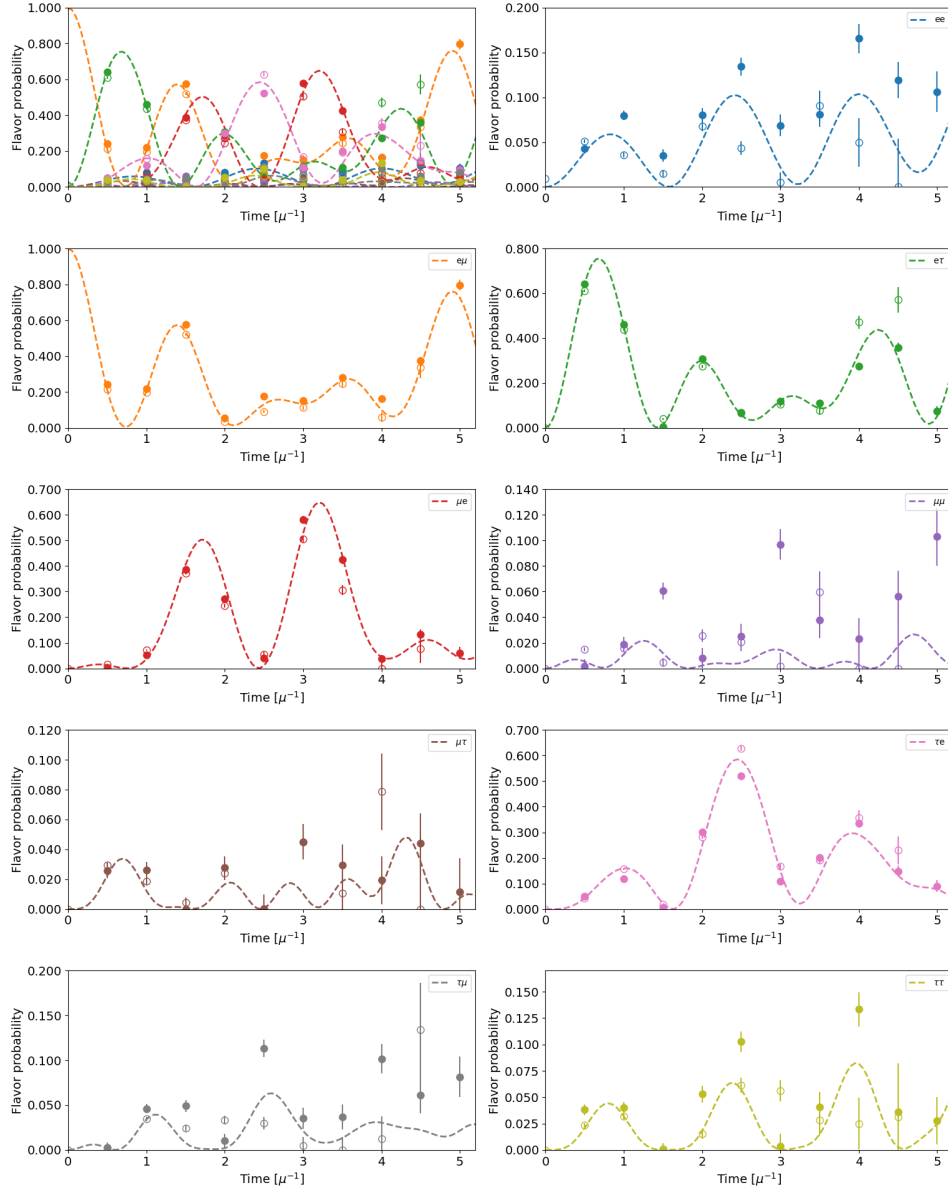


Figure 7.5: The vertical axis shows the probability of occupying a given final state. We present one plot for each possible two-flavor neutrino state, while the first panel shows a summary, highlighting the relative magnitudes. The dashed line is the exact calculation, while solid and empty dots represent the qubit (IBM) and qutrit (AQT) data respectively, after error mitigation. We also quantify the uncertainty using standard error through the vertical error bars. As expected error mitigation increases the variance observed.

We show data after error-mitigation in Fig. 7.5. As before, every plot displays the probability of being in a particular final state, while the plot in the first panel shows the relative magnitude between different states. Both qubit and qutrit results are close to the exact evolution, and even information in data that looked like fully depolarized have been recovered with error mitigation. There is no clear advantage between the two methods for this small system, but scaling up the particle number should lead to the differences outlined above.

Conclusion

In this section, we have developed and compared quantum circuits to perform collective neutrino flavor oscillations on both qubit and qutrit quantum hardware. Time evolution was computed through the first-order Trotter approximation and we find that, for a fixed total time, the error scales linearly with the number of neutrinos and inversely with the number of Trotter steps. In the qubit encoding, the one-body term becomes a two-qubit operator as does the PMNS matrix, and as such even initial product state preparation requires entangling gates. Meanwhile, in the qutrit case no such need arises. For noisy, intermediate-scale quantum (NISQ) hardware, this translates to additional noise for state preparation on qubits. Similarly, the time evolution will have more entangling gates for the qubit setup, as two-body interactions become four-qubit gates, which are decomposed into a sequence of one- and two-qubit gates. Even with the optimized circuits we construct through various gadgets, there is significant overhead for a neutrino pair time evolution step. For both hardware, in larger systems, SWAP operations will be required to represent the all-to-all two-body interactions, which fortunately we can absorb in the two-body operation due to Eq. 7.6. On the other hand, qutrit gates require higher control capabilities and can be prone to higher infidelities as a result. For both setups we need to perform error mitigation on hardware to achieve the quality of results shown in Fig. 7.2. We note that there are still large margins of improvement for fidelity of qutrit gates, and it is remarkable to see the level of control on these more complex architectures. Limited hardware connectivity issues will start to play an important role for larger systems, so the qutrit-based design is expected to have advantages in terms of overall fidelity.

One main difference between the two encodings is the mapping of the physical Hilbert space to the qudit space on device. In the case of qubits, there is one neutrino to two qubits, with one of the amplitudes being non-physical and decoupled from the rest during simulations. We take advantage of this fact to further improve our circuits and reduce the gate count. For a system of N neutrinos, the fraction of nonphysical states on the qubit device is $1 - (3/4)^N$, which becomes the vast majority of states as the number of particles increases. The presence of depolarizing noise will inevitably result in leakage to the nonphysical space. On the qutrit device there is a one-to-one mapping between physical amplitudes and device amplitudes. Thus, despite the presence of noise, such leakage cannot occur. In supernovae simulations, the main observables are single-neutrino flavor measurements, and the overall flavor composition might be robust against leakage. However, correlations and higher order

operators might be more sensitive to this type of noise. One could take into consideration hardware noise in devising encodings with noise-resistant observables, provided the noise is well-categorized. For now, we leave such endeavors for future research.

7.2 Spin-1 Truncated Abelian Higgs Model

The prospect of simulating quantum field theories with quantum simulators has attracted considerable interest in recent years. While classical numerical methods based on importance sampling have successfully established lattice quantum chromodynamics (QCD) with imaginary time as a reliable nonperturbative formulation of strong interactions, they face significant challenges in addressing the *sign problem* that appears in the presence of large chemical potential [250, 79] and the difficulty of observing real-time non-equilibrium dynamics [293].

In light of these difficulties, there have been several efforts to simulate lattice gauge theories with various quantum computing platforms. These studies mark essential intermediate steps towards simulating QCD in (3+1) dimensions, and they address diverse physical phenomena including phase transitions, scattering of particles, real-time confining dynamics, quench dynamics, the role of entanglement in thermalization, and quantum chaos.

So far, most studies have been performed on platforms composed of quantum two-level systems, or qubits. Platforms made of qudits, however, have the potential to reduce the gate depths and non-local quantum operations required for quantum simulations of the naturally d -level systems that are common in lattice field theory models [120, 140, 94, 112]. And yet, despite recent successes in the development of qudit processors across various physical platforms, and despite a number of theoretical proposals to study field theories with qudits [47, 95, 168, 83, 282, 113, 203, 112, 27, 28, 120], to our knowledge, there has been only a single experimental realization of a qudit-based field theory simulation, which was demonstrated on trapped-ion quantum hardware [167].

In the original work in Ref. [11], we consider a particular lattice field theory model—the spin-1 truncated Abelian Higgs model (AHM) in two spacetime dimensions—and perform qudit-based quantum simulations of the model in superconducting circuit platforms with two separate but complementary approaches. In the first, we leverage Hamiltonian engineering techniques to perform an analog-digital hybrid simulation, while in the second, we perform a fully digital simulation, taking advantage of state-of-the-art error mitigation methods. The analog methods were performed by colleagues at University of Rochester and are beyond the scope of this thesis. The AHM is a useful testbed for proof-of-principle simulators, not least because of its shared features with QCD, including confinement, and string breaking, making it a valuable step towards quantum simulation of QCD. Here we consider the model on two lattice sites, but the methods may be applied more generally to longer chains. Owing to the spin-1 truncation of the model, each site can be naturally mapped to a qutrit ($d = 3$), which in both simulation methods we encode in a truncated subspace spanned by the lowest three energy eigenstates of a transmon. In the context of the digital quantum simulation, compared

to a qubit-based gate decomposition of the model, we observe a four-fold reduction in resource requirements when decomposing the desired entangling operation with qutrit-based quantum gates.

Abelian Higgs Model

The Abelian Higgs model is a toy model of scalar quantum electrodynamics that describes the interaction of a photon field with spin zero particles. We begin with a brief description of the model and its approximate representation as a spin-1 Hamiltonian on the lattice. The action for the model is described by a $U(1)$ gauge field and a complex matter field Φ [168]

$$S = \int d^2x \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + |D_\mu \Phi|^2 + \frac{\mu^2}{2} \Phi^\dagger \Phi - \frac{\lambda}{4} (\Phi^\dagger \Phi)^2, \quad (7.36)$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor and $D_\mu = \partial_\mu + iqA_\mu$ is the covariant derivative with matter charge q . The model is invariant under the $U(1)$ gauge transformation $\Phi \rightarrow e^{i\alpha(x)}\Phi$, $A_\mu \rightarrow A_\mu - \frac{1}{q}\partial_\mu\alpha(x)$. We work in the limit where the Higgs mode is decoupled and the complex scalar field is reduced to a constant multiplied by a complex phase (the Nambu-Goldstone mode). In (1+1) dimensions, this model has no deconfining phase; in higher dimensions, however, it does exhibit a similar confining feature of a charge and anti-charge pair [75, 56]. Confinement is marked by a linear growth in potential due to static charge separation below a critical distance scale and string breaking phenomena is observed at large distances signifying charge screening [201]. In real-time quantum simulators, both the confining feature and string breaking phenomena can be observed from the local charge density [97]. The shape of the light cone can identify confining features in the model and string breaking can be identified from the integrated local charge density by counting the number of charges present.

The action is translated into a discretized form by placing the matter fields on the vertices and gauge fields on the links of a 2D lattice, which can also be used for classical numerical simulation of the model using the path integral formulation. Taking the time continuum limit of the lattice action following the steps outlined in Ref. [13], we obtain the lattice Hamiltonian on N_s sites, assuming open boundary conditions,

$$\begin{aligned} H_{\text{AHM}} = & \frac{\kappa}{2} \sum_{k=1}^{N_s} (L_z^{(k)})^2 - \chi \sum_{k=1}^{N_s} F_x^{(k)} \\ & + \frac{\beta}{2} \sum_{k=1}^{N_s-1} (L_z^{(k+1)} - L_z^{(k)})^2. \end{aligned} \quad (7.37)$$

We define $\hbar = 1$ throughout this work. The L_z operator appearing in the Hamiltonian formulation is the canonical conjugate of the gauge field and is analogous to electric field operator in quantum electrodynamics, whereas charge corresponds to the conjugate of the phase operator of the Higgs field. In this (1+1) dimensional model, we do not need to

account for the matter fields explicitly, as the charge can be computed as the difference of the electric field operators $(L_z^{k+1} - L_z^k)$, due to Gauss's Law. The terms in the Hamiltonian can be interpreted as the local energy of the electric field; an injection of energy that drives local evolution of the field; and the cost in energy due to creation of charges, respectively. The operators $L_z^{(k)}$, where k indexes the site that the operator acts on, are identical to the angular momentum operator defined by

$$L_z|m\rangle = m|m\rangle. \quad (7.38)$$

The operators $F_x^{(k)}$ are defined by the following equations:

$$F_\pm|m\rangle = |m \pm 1\rangle, \quad F_x = \frac{1}{2}(F_+ + F_-). \quad (7.39)$$

To capture the model exactly, the operators $L_z^{(k)}$ and $F_x^{(k)}$ are required to act on infinite levels: $m \in \{0, \pm 1, \pm 2, \dots, \pm \infty\}$, but in practice, some cutoff $|m| \leq l$ is required. In this work we take $l = 1$, and in this case we may identify $\sqrt{2}F_x$ with the spin-1 L_x operator. We may map spin L_z eigenstates to transmon eigenstates with

$$\begin{aligned} |m = 1\rangle &\rightarrow |n = 0\rangle, & |m = 0\rangle &\rightarrow |n = 1\rangle, \\ |m = -1\rangle &\rightarrow |n = 2\rangle. \end{aligned} \quad (7.40)$$

This spin-1 truncation does not recover the continuum limit of the theory, but it allows for the development of new tools for simulating various gauge theory models with a more efficient encoding than is available on qubit platforms. In addition, the aspect of triality in a spin-1 model is intriguing for its connection to QCD, where physical states transform under the center of the gauge group $SU(3)$.

Mapping of the unitary operations to basis gates

For an N_s -qutrit system, the time evolution operator under the model Hamiltonian in Eq. (7.37) can be approximated using a first-order Trotter approximation,

$$\begin{aligned} U(t) &= \exp(-iH_{\text{AHM}}t) \\ &\simeq \left(\prod_{k=1}^{N_s} M_z^k(\kappa, \delta t) \prod_{k=1}^{N_s-1} M_{zz}^{k,k+1}(\beta, \delta t) \prod_{k=1}^{N_s} M_x^k(\chi, \delta t) \right)^n, \end{aligned} \quad (7.41)$$

where the evolution time t is discretized into uniform steps δt such that $t = n\delta t$. Here, $M_z^k = \exp(-i\theta(L_z^{(k)})^2)$ and $M_x^k = \exp(-i\phi L_x^{(k)})$ denote on-site operations that act on an individual qutrit k , while $M_{zz}^{k,l} = \exp(-i\gamma L_z^{(k)} L_z^{(l)})$ represents an entangling operation between a pair of qutrits k and l . The angles ϕ , θ , and γ are determined by the Hamiltonian

parameters χ , κ , and β , as well as the Trotter step duration δt . The error in the approximation is bounded by $\mathcal{O}(t \delta t)$. An improved error scaling can be achieved with a higher order Trotter approximation with an increased cost in the number of entangling gates.

A decomposition for the entangling operation M_{zz} can be found by expressing the two-qutrit Hermitian operator in terms of tensor products of Gell-Mann matrices. This results in four non-zero coefficients. A much simpler decomposition, however, is found by inspection: We decompose the entangling operation M_{zz} into three controlled-SUM gates interleaved with local phase gates,

$$M_{zz}(\gamma) = U_{\text{CSUM}} D(\gamma) U_{\text{CSUM}} G(\gamma) U_{\text{CSUM}} \quad (7.42)$$

where $\gamma = \delta t \beta$. D and G are diagonal matrices constructed from the elementary phase gates,

$$D(\gamma) = R_z^{01}\left(\frac{4\gamma}{3}\right) R_z^{12}\left(\frac{2\gamma}{3}\right), \quad (7.43)$$

$$G(\gamma) = R_z^{01}\left(\frac{2\gamma}{3}\right) R_z^{12}\left(\frac{4\gamma}{3}\right). \quad (7.44)$$

In addition to our circuit decomposition analysis above, we tested whether a more efficient compilation was possible with numerical synthesis methods provided by the TrueQ software [14]. We found comparable costs in entangling operations from numerical synthesis to that of our analytical decomposition.

We find that, for the most efficient decomposition, a single Trotter step of the AHM requires three native entangling gates (Eq. (7.42)) representing a fourfold reduction compared with the twelve entangling operations needed in the qubit-based implementation with all-to-all connectivity. In addition, a restrictive qubit connectivity, like IBM's heavy-hex topology, would roughly increase the cost tenfold (see Ref. [11] appendix for more details). Thus, if the gate times for the native entangling operations, relaxation times and dephasing times of transmon qutrit processors become comparable to state-of-the-art qubit processors, qutrit-based implementations can be favorable for the implementation for certain gauge theory models that inherit $SU(3)$ group structure. This is particularly relevant since $SU(3)$ appears in different limits of QCD. Thus the phase structure of QCD and variants of QCD with various degrees of approximation can likely be studied more efficiently with digital qutrit-based quantum processors [47, 95, 120].

Device results

We now present hardware results of our gate-based simulation for the single-site and two-site lattice Abelian Higgs model. In the single-site experiment, the fast high-fidelity single-qutrit gates allow us to observe dynamics for longer times compared to the two-site experiment.

Fig. 7.6 shows error-mitigated hardware results for $\langle L_z^{(k)} \rangle$ and $\langle (L_z^{(k)})^2 \rangle$ expectation values, which are calculated from the measured transmon eigenstate populations, compared with the exact diagonalization and the hardware results with randomized compilation. We

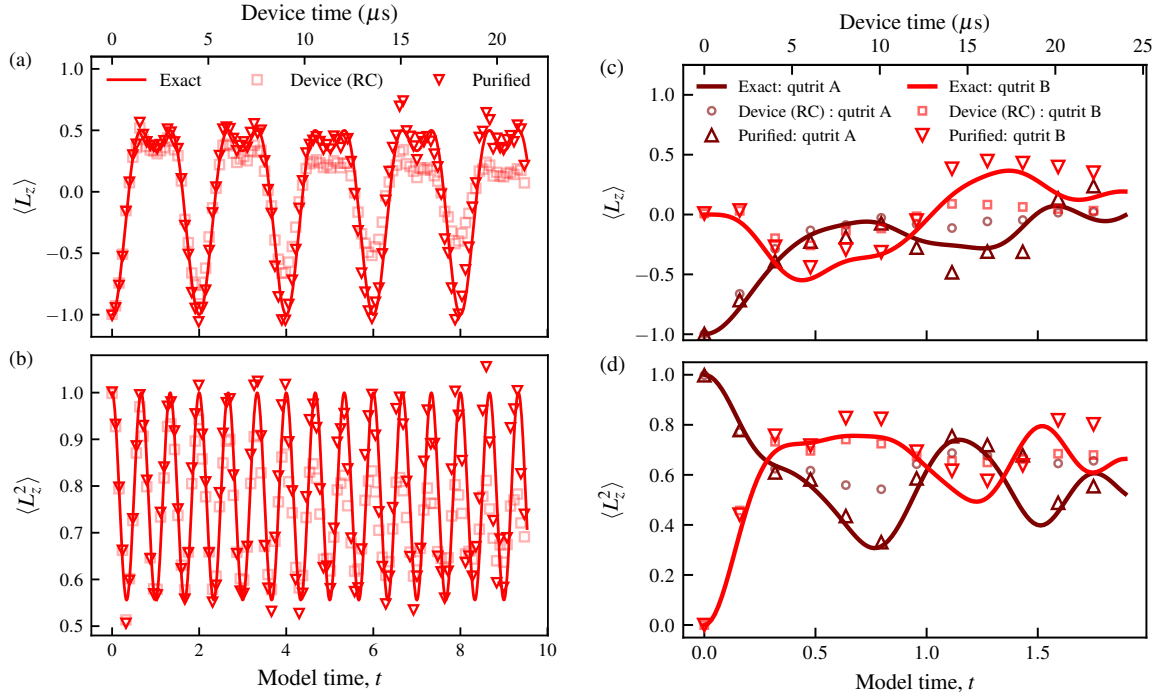


Figure 7.6: Exact diagonalization (Exact), device data with randomized compilation (Device (RC)) and error-mitigated device results (Purified) of field operators $\langle L_z \rangle$ and $\langle L_z^2 \rangle$ at (a,b) $\kappa/2\pi = \chi/2\pi = 1.0$ for the single qutrit experiment and (c,d) $\kappa/2\pi = \beta/2\pi = \chi/2\pi = 1.0$ for the two-qutrit experiment. The dynamics start from the initial state $|\psi_{\text{init}}\rangle_{AB} = |21\rangle$.

find improved fidelity of the circuit operation as we apply qudit randomized compiling to mitigate coherent errors and rescale our expectation values with noise measured from cycle benchmarking.

We use 30 distinct twirled versions of the Trotterized circuit where we use $N_{\text{shots}} = 1024$ shots per twirled circuit. Twirling refers to applying local random single-qutrit Pauli operations for each qutrit before and after each cycle of entangling operation. We also employ cycle benchmarking to learn the error rates of different Pauli and SPAM channels associated with the native gates in our circuit, allowing us to correct the observables by post-processing the estimate with a numerical correction factor. Additionally, readout-error mitigation (RCAL) is performed by confusion matrix inversion.

Conclusion

In summary, we have realized qutrit-based digital quantum simulation of the Abelian Higgs model on a transmon device (in the original work, collaborators also performed a hybrid digital analog protocol) and we have demonstrated that it provides a scalable trajectory

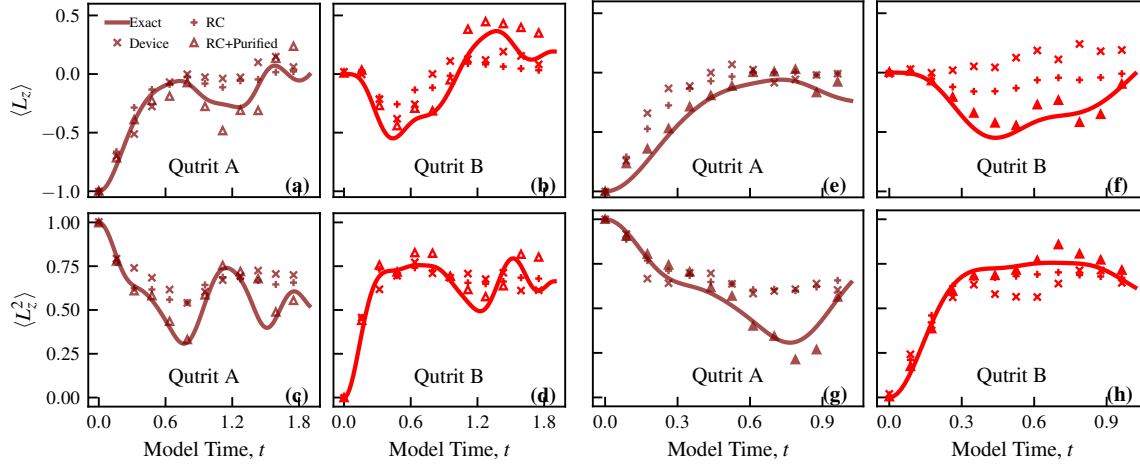


Figure 7.7: Impact of error mitigation scheme is investigated in two experiments with different Trotter steps. A larger Trotter step ($2\pi\delta t = 1.0$) allows us to observe real-time dynamics longer times (a-d), whereas a smaller Trotter step ($2\pi\delta t = 0.55$) reduces Trotter error but limits the accessible timescale (e-h). In both experiments, we find error mitigation schemes to be effective in improving fidelity of our quantum operations on average. For both experiments, we present here raw hardware data (Device), error-mitigated device data with randomized compilation (RC), data with randomized compilation and numerical purification (RC+purified), and compare them with exact diagonalization (Exact) for qutrit A (Maroon) and qutrit B (Red). Expectation values are computed for an initial state $|21\rangle$. We note that statistical errors in our experiments are insignificant compared to other sources of error in the experiment. Statistical error scales as the inverse of the square root of the number of shots and thus statistical errors in our experiment are of the order $\sim \frac{1}{\sqrt{1024 \times 30}} \sim 0.006$.

for simulating lattice gauge theories and spin-1 quantum systems. Our results show that our protocol can extract physical observables relevant to confinement within the coherence limit of present-day transmon processors. Despite the additional challenges associated with controlling more levels and more complicated interaction Hamiltonians, the qutrit-based experiments performed in this work allowed for a more hardware-efficient encoding of the spin-1 model than a qubit-based simulation, requiring fewer entangling operations.

7.3 Outlook

The experiments presented in this chapter demonstrate that superconducting qutrit processors can already engage meaningfully with problems in nuclear and high-energy physics, providing a proof-of-principle for the scientific reach of qudit-based quantum simulation. In both the neutrino oscillation and the Abelian Higgs model experiments, error mitigation

was essential to extract physically meaningful signals from circuits whose raw output was dominated by noise. This reliance on mitigation reflects the current trajectory of NISQ approaches within the field: the combination of hardware improvements and software-level error suppression is the most realistic near-term path to useful quantum simulation.

Several near-term directions emerge naturally from this work. In the neutrino oscillation setting, the two-neutrino experiment presented here is a first step; extending larger systems of coupled neutrinos would probe the onset of genuinely many-body collective behavior and could be a prime ground to search for meaningful quantum advantage on a scientifically relevant problem. The qutrit encoding is expected to pull ahead of the qubit encoding in resource requirements as N grows, since the fraction of physical states in the qubit Hilbert space decreases as $(3/4)^N$ while the qutrit encoding maintains a one-to-one correspondence between physical sites in the model and qutrits employed in the experiment. The immediate hardware bottleneck is gate fidelity: the experiments reported here showed comparable TVD between qutrit and qubit implementations at $N = 2$, with the balance expected to shift once two-qutrit gate fidelities on our devices improve to compete more with the current state of the art in two-qubit gates.

In the Abelian Higgs model, we demonstrated a fourfold reduction in entangling gate requirements relative to a qubit implementation with all-to-all connectivity. Longer chains of $N_s > 2$ sites would provide access to string breaking phenomena and confinement dynamics that are the primary scientific targets of this model. Given the linear connectivity of our qutrit ring processor, extending to larger chains requires no architectural changes.

Both sets of experiments point toward the same conclusion: the primary limiting factor is not circuit depth or encoding overhead, but two-qutrit gate fidelity. The gate-based simulations performed here used the differential-AC Stark shift CZ gate of Chapter 4 and relied on fixed linear coupling and suffered from always-on quantum crosstalk. While the error mitigation methods of Chapter 6 successfully recovered several Trotter steps' worth of signal, the purification methods employed are not particularly scalable, and more complicated methods to extrapolate noise such as NOX (chapter 6) can suffer from difficult sampling overhead. The path to deeper circuits therefore runs through hardware: specifically, faster and more accurate two-qutrit entangling gates and lower quantum crosstalk, which would benefit directly from tunable coupler architectures with higher on-off ratios, as discussed in the outlook of Chapter 4. In the next chapter, we will move to performing experiments on qudit systems with tunable couplers integrated into the processor, significantly lowering quantum crosstalk and speeding up two-qudit gates.

In a broader sense, this chapter illustrates the thesis's central point: that high-dimensional quantum systems can already compete with extremely advanced qubit systems (such as the IBM Torino processor) for well tailored problems, motivating further research into their hardware development. The resource reductions demonstrated here are quantitative and arise directly from the natural alignment between the d -level structure of the physical Hilbert space and the d -level structure of the encoded transmons. As qudit processors mature and gate fidelities improve, the arguments sketched in this chapter for the neutrino oscillation and gauge theory settings will generalize to the broader class of quantum simulation problems,

spin models, open quantum systems, quantum chemistry, in which the natural degree-of-freedom count is three or higher. In the next and final chapter of this thesis, we tackle another natural setting for qudit experimentation: out-of-equilibrium spin 1 models.

Chapter 8

Many-Body Physics With Qudits: A Qutrit Discrete Time Crystal

In the preceding chapters we have worked to scale up qudit operation with the end of ultimately unlocking more interesting scientific applications. Thus far, we have built out the machinery of single and two-qudit operation, including entangling gates and readout. We then built out machinery to perform useful error suppression techniques in qudit circuits. Finally, in Chapter 7, we synthesized all of these directions to perform small scale demonstrations of digital quantum simulation problems that were particularly well tailored to qudits, in the process comparing these methods to qubit and analog techniques.

Now, in this final chapter in this thesis, we perform a significantly scaled up experiment with qudits. In particular, we investigate driven, or Floquet, dynamics in a native chiral clock model on a 20 transmon processor, performing experiments with up to 15 qutrits. Periodically driven quantum many-body systems can spontaneously break discrete time-translation symmetry, realizing discrete time crystals (DTCs), an exotic non-equilibrium phase of matter with no equilibrium counterpart. Experimental realizations have thus far been confined almost exclusively to qubit platforms implementing Ising-like $\mathbb{Z}_2$ models, leaving the richer physics of higher discrete symmetries largely unexplored. Here, we realize a $\mathbb{Z}_3$ discrete time crystal in a chain of up to 15 superconducting qutrits natively implementing a driven chiral clock model — a system that, unlike the Ising model, harbors a tunable chiral angle that governs domain-wall dynamics, spectral degeneracies, and the stability of time-crystalline order. Using disordered nearest-neighbor interactions engineered via fast flux control of tunable couplers, we observe robust subharmonic period tripling that persists across a wide range of drive strengths and is independent of initial state. By continuously tuning the chiral angle through degeneracy-inducing values, we directly demonstrate chirality-controlled destabilization of the time crystal, a mechanism with no analog in $\mathbb{Z}_2$ systems. We map the thermal-to-DTC phase transition using Floquet analysis and a generalized Edwards-Anderson order parameter, extracting results consistent across finite-size scaling of both experiment and numerics. Finally, we demonstrate a continuously tunable interpolation between $\mathbb{Z}_2$ and $\mathbb{Z}_3$ time-crystalline order within the native qutrit state space.

Briefly summarized, the main takeaways for the results presented in this chapter are that:

- They demonstrate the scalability of many of the developed qudit operations we have presented throughout this thesis.
- After many decades of theory, it is, to the best of our knowledge, the first experimental implementation of the chiral clock model with short range interactions and only implementation with tunable chiral angle.
- $\mathbb{Z}_3$ discrete time crystal is observed for the first time in a short range coupled experiment.
- The special role of chirality in the stability of dynamical phases in the chiral clock model is explored for the first time experimentally.
- As discussed and demonstrated in the chapter, unlocking native qudit methods for exploring driven higher spin models can open the door to a much broader exploration of dynamical phases rooted in higher discrete symmetries beyond the $\mathbb{Z}_2$ Ising paradigm.

8.1 Qubit Time Crystals in the Kicked Ising Model

Out-of-equilibrium many-body physics has emerged as a fertile arena for discovering phenomena with no equilibrium counterpart. Periodically driven quantum many body systems exhibit one such remarkable non-equilibrium phenomenon of discrete time-translation symmetry breaking, giving rise to DTCs [129, 58, 277, 283, 59, 128]. To-date, most experimental realizations of such phases have been confined to short range Ising-like $\mathbb{Z}_2$ models that are natural to qubit based platforms [170, 76, 169, 275, 289, 121, 270] or models with long range interactions [286, 196, 214, 103, 141] which have also explored a $\mathbb{Z}_3$ DTC [44].

A discrete time crystal (DTC) is a non-equilibrium phase of matter that spontaneously breaks the discrete time-translation symmetry of a periodically driven (Floquet) system. In the disordered $\mathbb{Z}_2$ Ising model, we consider the Floquet unitary operator

$$\hat{U}_F = e^{-\frac{i}{2} \sum_i h_i \hat{Z}_i} e^{-\frac{i}{4} \sum_i \phi_i \hat{Z}_i \hat{Z}_{i+1}} e^{-\frac{i}{2} \pi g \sum_i \hat{X}_i}, \quad (8.1)$$

which describes one full period of the evolution of the driven system. Each Floquet cycle consists of three sequential operations: a disordered longitudinal field h_i acting on each spin, a nearest-neighbor Ising interaction with coupling ϕ_i , and finally a global rotation of all spins by πg about the x -axis. In the trivial case at $g = 1$, this last term is a perfect π pulse that flips every spin, and the system returns to its original state after two Floquet cycles rather than one, giving a subharmonic response in applications of U_F (Fig. 8.1b).

The robustness of this period-doubling against perturbations is what distinguishes a true DTC as a dynamical phase of matter. When g is slightly detuned from unity, one might not expect the subharmonic period doubling response to persist. However, in the many-body

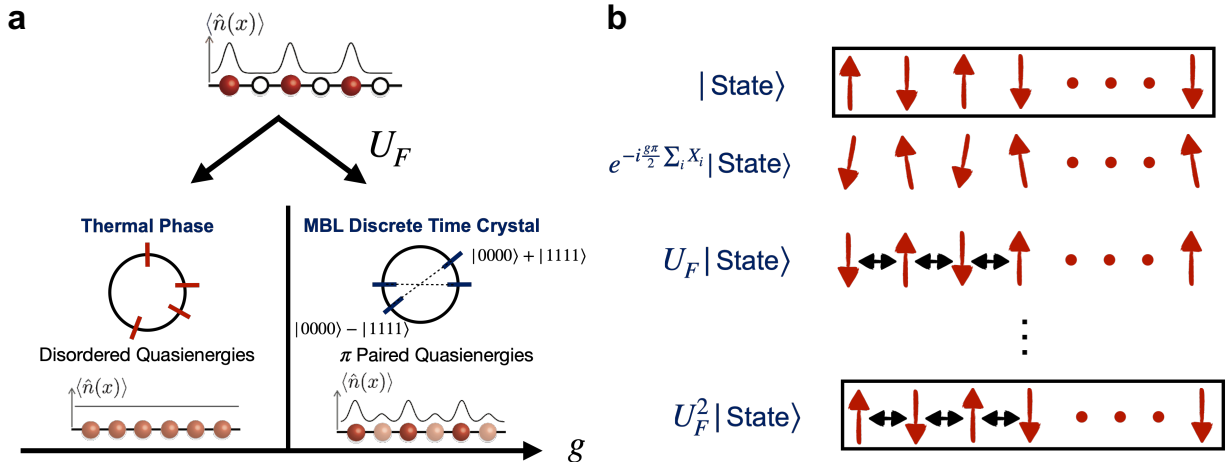


Figure 8.1: **Schematic of a qubit DTC in the $\mathbb{Z}_2$ Ising Model** **a**, The system response under applications of the Floquet unitary U_F depending on the kick parameter g . We can have 2 phases: on the left for $g < g_c$ the system is in a thermal phase where the quasienergies are randomly spread out. On the right side for $g > g_c$ ergodicity is broken and the system enters the MBL DTC phase, where all quasienergies are π paired. Furthermore, their corresponding eigenstate are long range cat states, which stabilize their time evolution against local perturbations. **b**, In the MBL DTC phase, an arbitrary initial state will demonstrate subharmonic period doubling even as g is tuned away from realizing a perfect π flip of the spins.

localized (MBL) regime, enabled by the disorder in both h_i and ϕ_i , ergodicity is broken and the system is protected against thermalization, yielding a persistent period-doubling for long times even for $g \neq 1$. As seen in Fig. 8.1a, there is a phase transition as the kicking strength g is tuned away from the trivial point $g = 1$, where as the kick weakens, eventually the system loses its subharmonic response and melts into the thermal phase. Another way to observe this is that the Floquet quasienergies in the MBL DTC phase are all paired off into groups of two and represent long range cat states, whereas in the thermal regime they are disordered. The result is the DTC is a dynamical phase whose long-range spatio-temporal order is a genuine emergent property of the driven, disordered many-body system.

8.2 $\mathbb{Z}_3$ DTC Results in the Driven Chiral Clock Model

Beyond Ising models, $\mathbb{Z}_n$ chiral clock models (CCMs) have a long and rich history for hosting phenomena outside the Ising paradigm. Unlike the Ising model, they feature a tunable chiral angle that introduces directionality and frustration in domain wall dynamics [65, 119], stabilize incommensurate phases [106, 109, 194, 292, 287], and drive critical behavior outside

the Ising universality class [218, 190, 269, 77, 126]. Despite these remarkable properties and potential to uncover richer phases in far less explored disordered [77] and driven [243, 238] settings, experimental realizations of CCMs have remained largely elusive due to the challenges associated with their physical implementation.

Moving from qubits to native multilevel systems fundamentally enlarges the landscape of accessible quantum matter and offers a natural and powerful route to access higher spin models [44, 29]. By encoding information directly in qudits, such experimental platforms can efficiently realize higher spin models such as the CCM without the added connectivity and compilation overhead associated with qubit-based hardware demonstrations [29, 219, 191]. In particular, as we have throughout this thesis, we employ superconducting transmon circuits which combine coherence, controllability, and scalability with intrinsic multi-level structure, enabling direct implementations of $\mathbb{Z}_3$ dynamics using existing processor technology.

We natively realize a short-range driven $\mathbb{Z}_3$ CCM and directly access model behavior that has been previously unexplored in qubit platforms. In particular, we implement qutrit entangling operations via fast-flux operation of tunable couplers to obtain nearest-neighbor CCM chiral spin interactions and employ microwave driven qutrit transitions to realize system kicking, driving the CCM out of equilibrium to access the rich physics of out-of-equilibrium systems. Our platform enables precise engineering of coupling disorder that induces many-body localization, giving rise to the $\mathbb{Z}_3$ DTC phase. We observe robust subharmonic period tripling response that persists across a wide range of drive strengths and is independent of initial state; and proceed to characterize its melting to the thermal phase. We leverage the intrinsic tunability of the superconducting platform to enable continuous control of the chiral interaction angle, which has no analog in Ising physics, and reveal a rich phase structure spanning stable and unstable DTC regimes. Our results establish native qudit platforms as a compelling setting for exploring non-equilibrium phases beyond $\mathbb{Z}_2$, and point toward a broader landscape of dynamical orders accessible with multilevel quantum hardware.

Model Hamiltonian and Experimental Realization Overview

We realize a spin-1 driven chiral clock model on a chain of L superconducting qutrits (Fig. 8.2a), where the spin $|s\rangle \in \{|0\rangle, |1\rangle, |2\rangle\}$ is encoded in the three lowest transmon levels. Our device consists of a square grid of 20 flux-tunable transmon qutrits. Microwave drives and virtual frame updates are employed to implement the unitaries representing, respectively, the global kick $\bar{X}^g$ and the onsite fields e^{-iH_0}

$$\bar{X}^g = \prod_j X_j^g, \quad H_0 = \sum_j h_j e^{i\varphi_j} Z_j + \text{h.c.}, \quad (8.2)$$

composed of the qutrit clock Z and shift X operators, where $Z|s\rangle = \omega_3^s|s\rangle$, $X|s\rangle = |s \oplus_3 1\rangle$, and $\omega_3 = e^{2\pi i/3}$ is the cubed root of unity (here we define ω_3 to disambiguate it with the often quoted angular velocity of the Floquet system ω). Nearest neighbor interactions are mediated by flux-tunable couplers [3, 163]. Fast flux pulsing of the couplers is used to

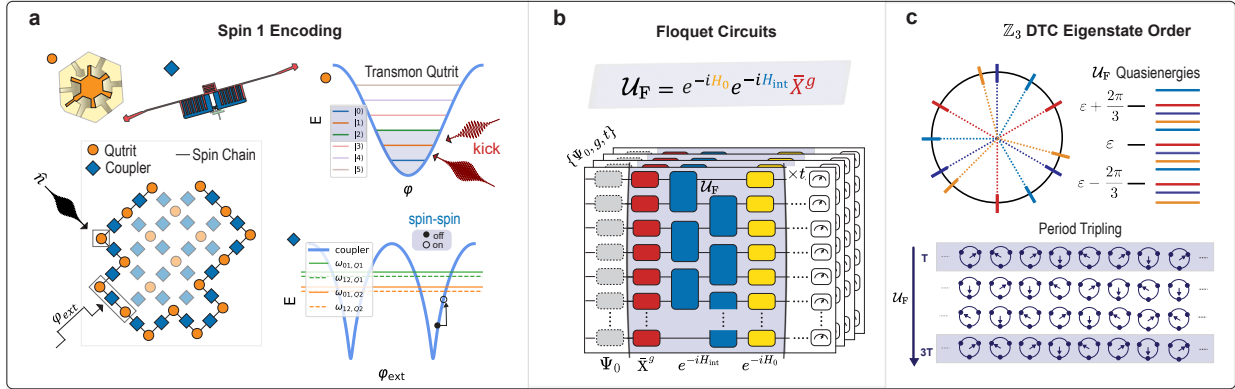


Figure 8.2: **Experimental concepts.** **a**, We engineer a spin-1 chain on a superconducting transmon quantum processor composed of 20 flux-tunable qutrits (orange) and tunable couplers (blue). Local “kicking” (red) is performed by charge driving ($\hat{n}$) coherent Rabi oscillations at the $|0\rangle \leftrightarrow |1\rangle$ and $|1\rangle \leftrightarrow |2\rangle$ transition frequencies. The disordered spin-spin interaction Hamiltonian H_{int} (light-blue) is realized from fast flux pulses on the tunable couplers (φ_{ext}) which turns on a large dispersive coupling between the qutrits. **b**, A schematic illustration of the experimental circuits used to implement the floquet $\mathbb{Z}_3$ chiral clock model. In addition to the kicking and spin-spin interaction described in the previous panel, we also engineer disordered on-site fields (H_0) via virtual frame updates to our microwave pulses. To study the eigenstate ordering of our system across the thermal-DTC phase transition, we vary the choice of initial trit-string state Ψ_0 , kicking strength g , and time steps applying the Floquet unitary $\mathcal{U}_F$. **c**, (top) In the $\mathbb{Z}_3$ -DTC phase, the entire eigenspectrum is paired off into triplets, where each quasienergy is related to its cousins by $\pm 2\pi/3$. (bottom) The characteristic time domain response in the $\mathbb{Z}_3$ -DTC phase is defined by subharmonic period tripling, where all initial clock sites return to their original configuration after 3 applications of $\mathcal{U}_F$.

activate strong dispersive coupling [154] and produces a native chiral clock interaction

$$H_{\text{int}} = \sum_j \underbrace{J_j e^{i\theta_j} Z_j Z_{j+1}^\dagger + J'_j e^{i\theta'_j} Z_j Z_{j+1}}_{\mathbb{Z}_3 \text{ preserving}} + \text{h.c.} \quad (8.3)$$

The spin-spin interaction Hamiltonian admits two terms, the first of which is the typical chiral clock interaction. This term preserves the $\mathbb{Z}_3$ symmetry by commuting with $\bar{X} = \prod_j X_j$ and generates unique phenomena in CCMs, motivating decades of theoretical research, especially in the static and uniform coupling limit [65, 119, 106, 109, 194, 269]. Our experimental system Hamiltonian also admits a second term that breaks this symmetry, and has, to the best of our knowledge, not appeared before in the literature. Since Z is non-Hermitian for systems with more than two levels, the couplings acquire a directional, or *chiral*, phase

θ_j, θ'_j . Combining all these operations, we realize a short-range driven chiral clock model by implementing the time-periodic Floquet unitary $\mathcal{U}_F$

$$\mathcal{U}_F = e^{-iH_0} e^{-iH_{\text{int}}} \bar{X}^g, \quad (8.4)$$

as shown in Fig. 8.2b.

A Qutrit Time Crystal in the Chiral Clock Model

Now having discussed both the theoretical treatment of our driven CCM and its experimental realization (for more details see the experimental methods section of this chapter), we turn our attention to realizing dynamical phases in the model. Let's recall that we are implementing the time-periodic Floquet unitary $\mathcal{U}_F$

$$\mathcal{U}_F = e^{-iH_0} e^{-iH_{\text{int}}} \bar{X}^g.$$

At $g = 1$, the drive $\bar{X}$ performs a perfect global kick, rotating all spins by $2\pi/3$ to the next clock state. As displayed in Fig. 8.2c, after three kicks, the spins return to their initial configuration, yielding a subharmonic response with period tripling. The entire quasienergy spectrum of $\mathcal{U}_F$ is ordered into triplets $\{\varepsilon, \varepsilon + 2\pi/3, \varepsilon - 2\pi/3\}$ with each quasienergy [243]

$$\varepsilon = 2 \sum_j J_j \cos(2\pi(s_j - s_{j+1})/3 + \theta_j). \quad (8.5)$$

Every such eigenstate is a long-range ordered cat state. Notably, the Floquet dynamics near $g \sim 1$ are determined by the quasienergies, which depend on the $\mathbb{Z}_3$ preserving term in H_{int} . This term depends on differences of spins (or domain walls) with energies dictated by the strength J_j and the chiral angle θ_j which tunes directionality in the interaction. This is in contrast to the other Z dependent terms that rotate by integer multiples of ω at each Floquet cycle, effectively dynamically decoupling at three cycle intervals thereby not contributing to the quasienergies. Critically therefore, while the $(J'_j, \theta'_j, h_j, \varphi_j)$ terms are always present, the Floquet dynamics in our model are determined by the couplings (J_j, θ_j) , which we explicitly focus on throughout the chapter.

Crucially, demonstrating true time-crystalline behavior requires robust period tripling extending beyond the trivial $g = 1$ point. Deviations in g act as a local perturbation. In the presence of strongly disordered couplings (J_j, θ_j) the system is in the many-body localized (MBL) regime, such that local perturbations do not lead to hybridization of the macroscopically distinct cat states. Additionally, as will be discussed in detail later in this chapter, avoiding degeneracies in the Floquet quasienergies requires the chiral angles $\{\theta_j\}$ to be detuned from integer multiples of $\pi/3$. With such criteria met, even away from $g = 1$, the quasienergy triplet pairing and long-range cat states can remain stable up to exponentially small corrections with system size N , satisfying the criteria for a stable discrete time-crystalline phase.

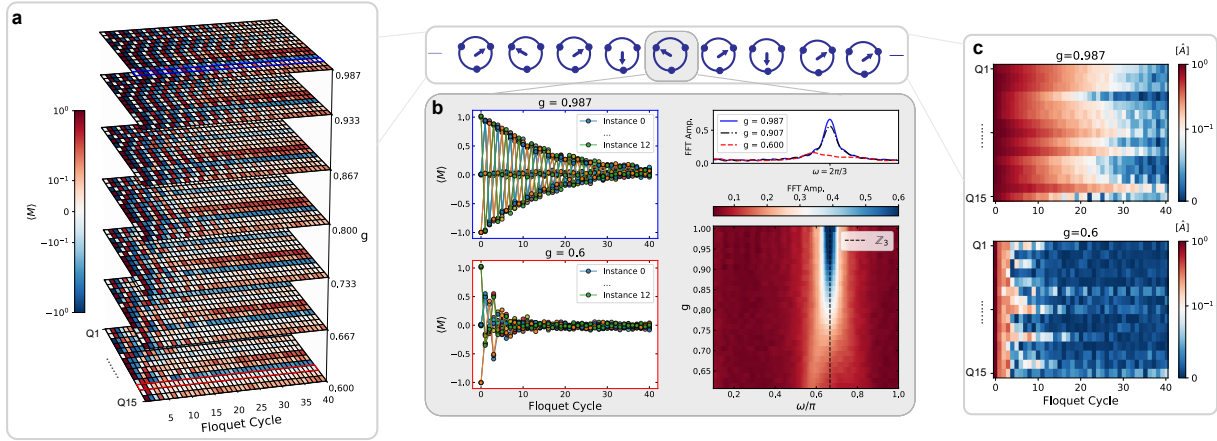


Figure 8.3: **$\mathbb{Z}_3$ DTC in a 15 qutrit chain.** **a**, Spin-1 magnetization of a chain of 15 qutrits as a function of Floquet cycle for multiple values of kicking strength g . **b**, (left) Single qutrit marginalized spin-1 magnetization of Q13 in the DTC-phase $g = 0.987$ (blue) and thermal phase $g = 0.6$ (red). (right) Fourier transform of the spin-1 magnetization as a function of kicking strength g . On top, change in the Fourier peak as the kicking strength g is varied. **c**, Instance-averaged autocorrelator $\langle \hat{A}(t) \rangle$, where $\hat{A}(t) = \omega_3^{-t} \langle Z(t) Z^\dagger(0) \rangle$ for the entire chain (blue) deep in the DTC and thermal phases.

Leveraging our native control over all three critical elements of the $\mathbb{Z}_3$ -DTC, we realize this phase of matter in a chain of 15 superconducting qutrits as displayed in Fig. 8.3. We engineer disorder by choosing flux pulse parameters that randomly select spin-spin couplings with $J_j \in [0.08, 0.25]$ and $\theta_j \bmod \pi/3 \in [0.125, 0.9]$. We demonstrate a continual tuning of our system from a thermal phase ($g < g_c$) to the $\mathbb{Z}_3$ -DTC phase ($g > g_c$). In Fig. 8.3a, for a random initial trit-string state, we display the full chain spin-1 magnetization $M = |0\rangle\langle 0| - |2\rangle\langle 2|$ response (which, unlike the clock Z , is hermitian and therefore observable) across our qutrit chain as the kicking strength is continuously tuned away from $g = 1$. In total, we prepare 13 different initial trit string states, including the 3 ferromagnetic states $|\Psi\rangle_j = |j\rangle^{\otimes n}$, $j \in \mathbb{Z}_3$ and 10 randomly chosen n -qutrit trit string states. Our deepest circuits (up to 40 Floquet cycles) correspond to 80 cycles of two-qutrit gates, utilizing 600 total two-qutrit gates. Remarkably, as displayed in Fig. 8.3b, the robust nature of this phase of matter can be observed in the fast Fourier transform (FFT) of the M response to $\mathcal{U}_F$ at an individual bulk site, where the spin responds robustly at $\omega = 2\pi/3$ even as the kicking strength g is tuned significantly away from the special point ($g = 1$) until eventually melting into the thermal phase. The characteristic time domain response of the two phases can be observed in the left side of Fig. 8.3b, where in the $\mathbb{Z}_3$ -DTC phase robust period tripling is observed for all initial states, whereas in the thermal phase all initial states rapidly thermalize. Finally, as depicted in the autocorrelator expectation values in Fig. 8.3c, this continuous tuning between

a thermal and $\mathbb{Z}_3$ -DTC phase can be observed across all bulk-spin sites, further confirming the lock-step behavior of bulk sites up to expected fluctuations in the individual transmon lifetimes. To certify the stability of our DTC behavior, we employ dynamical decoupling to invert the time evolution from our two-qutrit gates [248], and confirm that the dampening of oscillations in our DTC is the product of intrinsic system decoherence.

Extended Qutrit DTC Data and Characterization

Interactions Off

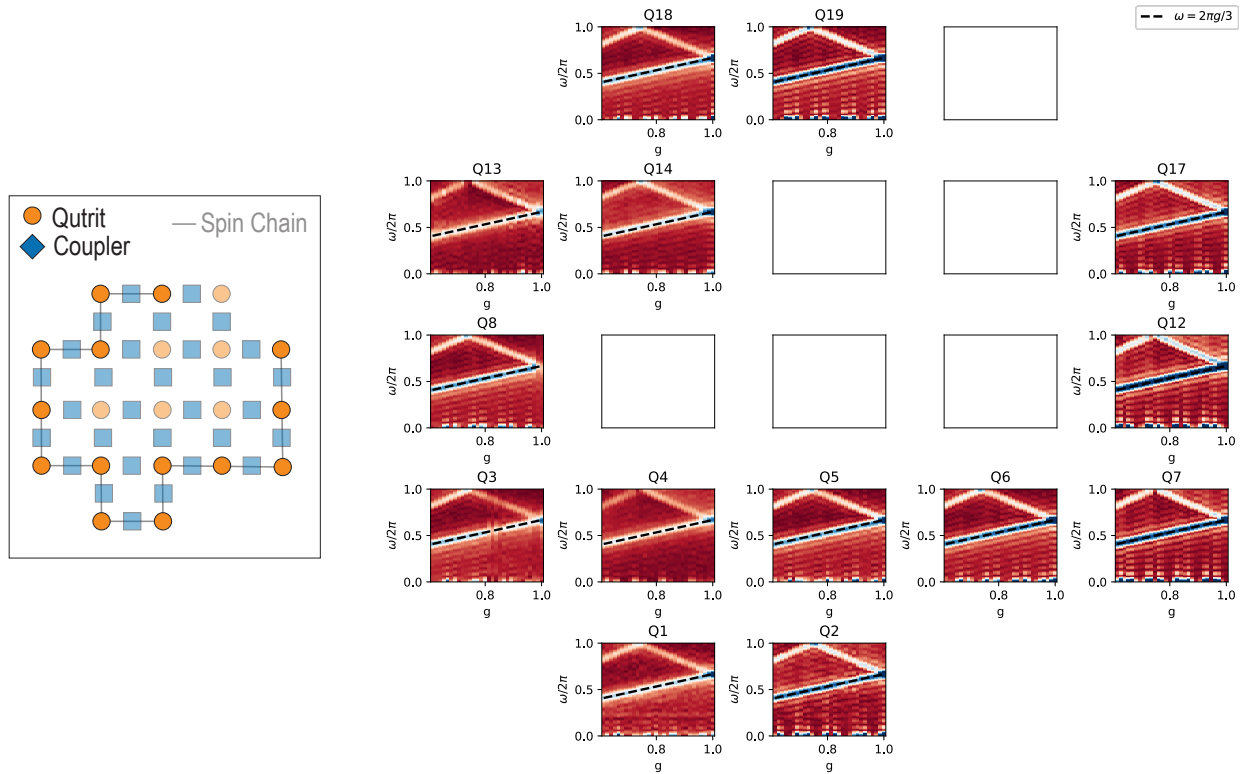


Figure 8.4: **Magnetization Spectroscopy with Interactions Turned Off.** (left) The chosen 14 qutrit spin chain on our 20 qutrit processor where the couplers are opaque to indicate the interaction Hamiltonian has been turned off. (right) The $\langle M \rangle$ FFT response across each site of the chain now displays trivial single particle responses to the kick $\tilde{X}^g$ with all sites responding at $\omega = 2\pi g/3$.

In Fig. 8.3 we showed how the FFT of the averaged spin-1 magnetization $\langle M \rangle$ could serve as a useful probe of the transition between the thermal and $\mathbb{Z}_3$ DTC regimes of our Floquet model. Here we provide an effective “null” experiment for this FFT by analyzing

a chain of 14 qutrits with the couplings turned off. Notably, as depicted in Fig. 8.4, as the kicking strength g in $\tilde{X}^g$ is tuned, the $\langle M \rangle$ FFT response across all sites follows the expected trivial single particle behavior at $\omega = 2\pi g/3$. As should be expected with no spin-spin interactions, the system here does not show any signatures of fast thermalization or robust period tripling. This $\langle M \rangle$ FFT response across the chain without interactions can now serve as a useful contrast to the experiments demonstrating $\mathbb{Z}_3$ DTC behavior.

Full 15 Qutrit Chain Spectroscopy

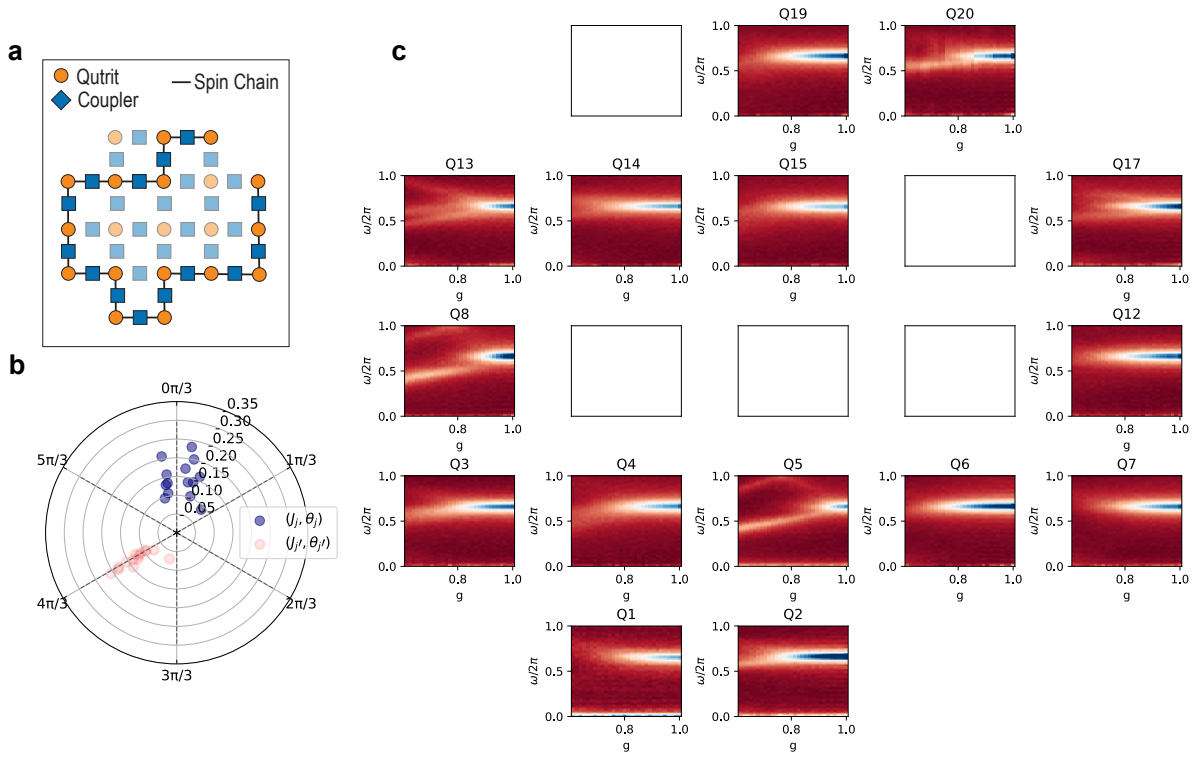


Figure 8.5: **Full 15 Qutrit Spin Chain Spectroscopy.** Extended results from the 15 qutrit $\mathbb{Z}_3$ DTC experiment performed in Fig. 8.3. **a**, The 15 qutrit spin chain embedded in the lattice of our processor. **b**, The couplings $(J_j, J'_j, \theta_j, \theta'_j)$ across the qutrit chain. **c**, The $\langle M \rangle$ FFT response across the whole 15 qutrit chain.

We now provide extended data from the 15 qutrit $\mathbb{Z}_3$ DTC experiment depicted Fig. 8.3. In that experiment, we showed a transition from thermalization at $g < g_c$ to robust period tripling at $g > g_c$ by displaying the marginalized spin-1 magnetization $\langle M \rangle$ FFT response at a single site in our chain. Here we show the specific qutrit chain within our lattice (Fig. 8.5a), couplings across the 14 spin-spin interactions (Fig. 8.5b), and average spin-1

magnetization $\langle M \rangle$ FFT across the entire chain (Fig. 8.5c). Notably, we can observe that for a weak enough kicking strength g , every single FFT shows the characteristic response of rapid thermalization, whereas for a strong enough g we recover robust period tripling. This $\mathbb{Z}_3$ DTC period tripling behavior can be observed to be robust when tuning the kicking strength away from the trivial point at $g = 1$ across all sites, confirming the many body nature of the realized $\mathbb{Z}_3$ DTC state presented previously.

Dynamical Decoupling

We expect that in the $\mathbb{Z}_3$ DTC regime (i.e. for a strong kick $g > g_c$) the subharmonic period tripling response of our qutrit chain should be robust, with oscillations that are principally limited by system decoherence. In prior works study qubit time crystals, such as Ref. [170], the authors confirmed the stability of their $\mathbb{Z}_2$ time crystal by directly reversing the time evolution of their circuits and comparing the resulting envelope to their DTC oscillation envelope. To do this, the authors of that work engineered gates that exactly reversed the ZZ or Ising couplings in their model. In principle, such a method could also be employed in our experiment, but it poses a very serious experimental challenge due to our spin-spin interactions now tuning 4 entangling phases (in contrast to the single phase in the qubit experiments).

Instead, to estimate the role of decoherence in our circuits and compare it to our DTC experiments, we employ the cross-Kerr dynamical decoupling routine first introduced in Ref. [248] (see Fig. 8.6b). To probe average case noise at each qutrit site, we prepare $|+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$ at each site in the chain. We then perform circuits with the same structure as the Floquet circuits, but with the kicking gates replaced by full $\bar{X}$ rotations at each site. After 9 cycles of single and two-qutrit gates, the cross-Kerr(or spin-spin) time evolution is removed and we can project the prepared states back onto $|0\rangle$ to assess the survival probability. In general, consistent with our expectations, we find that the DTC oscillation decay rates closely match the CKDD experiments (Fig. 8.6c) whereas in the thermal regime the spin-1 magnetization decay is considerably faster than the bare system decoherence.

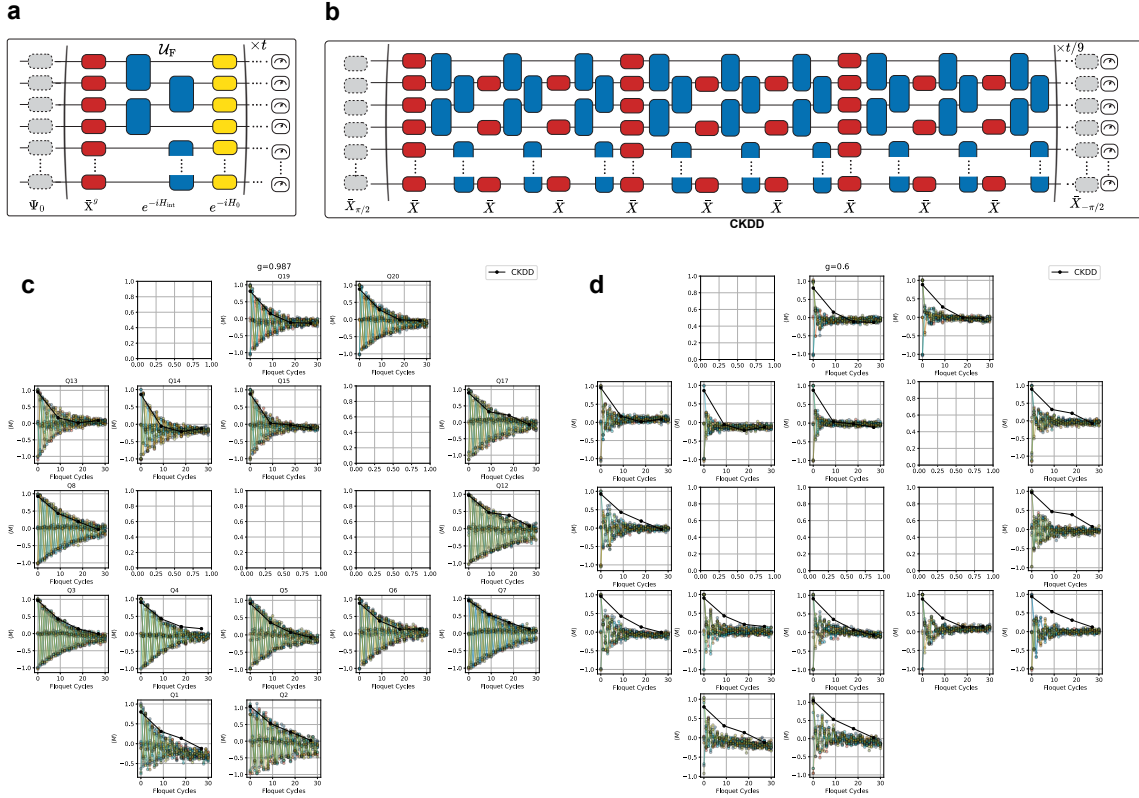


Figure 8.6: Using Dynamical Decoupling to Demonstrate the DTC Oscillations are Primarily Limited by Decoherence. **a**, The Floquet circuits employed to study the driven CCM physics. **b**, The cross-Kerr dynamical decoupling circuits we use to probe the role of decoherence in our DTC oscillations. $|+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$ is prepared at each qutrit in the chain. We perform circuits with the same structure as the Floquet circuits, but now the kicking gates replaced by full $\bar{X}$ rotations at each site. After 9 cycles of single and two-qutrit gates, the cross-Kerr evolution is removed and we can project the prepared states back onto $|0\rangle$ to assess the survival probability. **c**, Comparing the DTC oscillations for a strong kick of $g = 0.987$ in a 15 qutrit chain with the CKDD experiment survival probability. **d**, The same comparison in the thermal regime with $g = 0.6$.

Chirality Determined Stability of Time Crystal

A crucial component for the stability of time crystals is that the eigenstates of $\mathcal{U}_F$ are long range ordered cat states [58]. If a degeneracy is introduced into the Floquet spectrum, these cat states can hybridize to a short-range correlated eigenstate, which in turn will not show stable time crystalline behavior.

In our Floquet model, the chiral angle couplings θ_j determine stability by imposing

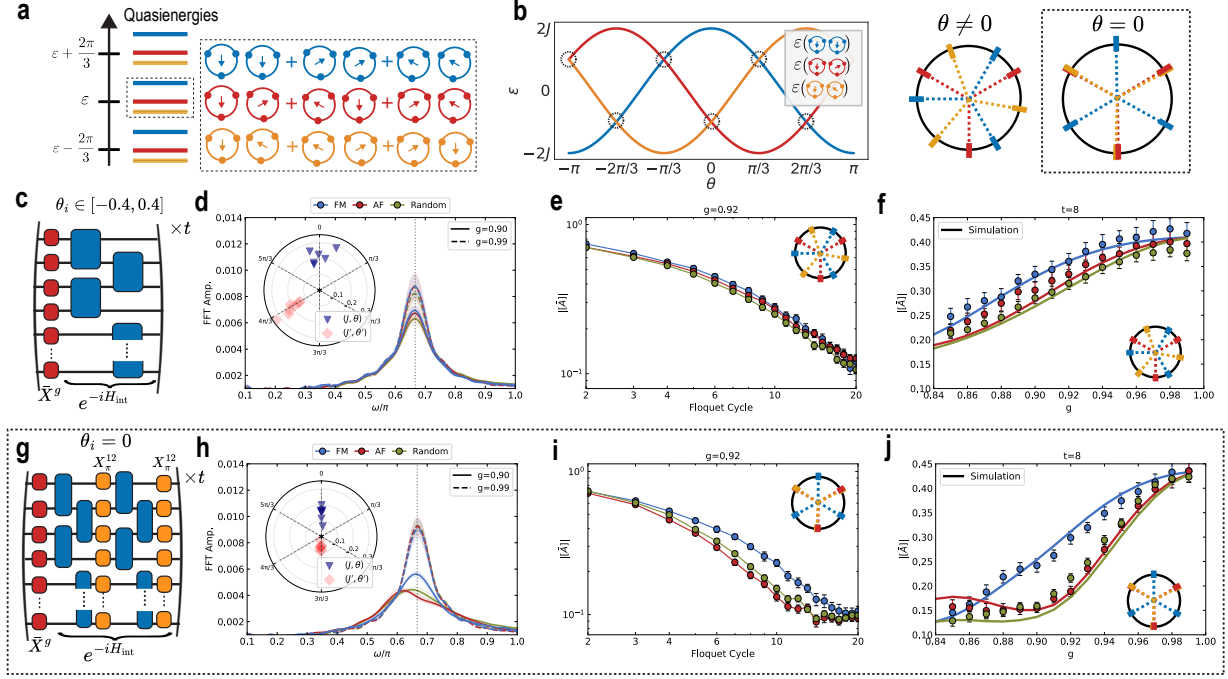


Figure 8.7: **Demonstrating the importance of chirality in DTC stability.** **a**, Quasienergy (ϵ) spectrum and corresponding eigenstates of two clock qutrits. The FM cat state is shown with blue clock spins, and the two kinds of AF states are shown with the red and orange clock spins. **b**, Quasienergies for the displayed eigenstates as a function of chiral angle θ with regions of degeneracy circled. **c** Circuit implementation of $\mathcal{U}_F$ for $e^{-iH_{\text{int}}}$ with disordered θ_j . **d**, Site and initial state averaged $\langle M \rangle$ FFT for FM, AF, and randomly prepared tritstring states with the experimental couplings corresponding to H_{int} displayed on an inset radial plot for an 8 qutrit chain. **e**, Site and initial state averaged autocorrelator $|\langle \bar{A} \rangle|$ response as a function of time for a kick of $g = 0.92$. **f**, $|\langle \bar{A} \rangle|$ response at $t = 8$ as a function of kicking strength g . **g** Circuit implementation for $\mathcal{U}_F$ for $e^{-iH_{\text{int}}}$ with $\theta_j \rightarrow 0$ via local dynamical decoupling pulses (orange). **h-j**, the same analysis as d-f, with chiral couplings turned off. The stable nature of the FM states relative to AF and randomly prepared trit string states can be observed across the FFT and autocorrelator responses of the sites.

different energy costs on domain walls with different magnetic order. This can be observed by analyzing the system of two interacting clock spins (s_1, s_2) . There exist three different types of domain walls corresponding to spin differences $s_2 - s_1 \in \{0, 1, -1\}$, where 0 represents ferromagnetic (FM) ordering and ± 1 represent different anti-ferromagnetic (AF) orderings. We plot their quasienergies (see Eq. 8.5) as a function of chiral angle θ in Fig 8.7(b). At $\theta = 0$, the two AF domains become degenerate and are energetically distinct from the FM domain. As a result, for uniform $\theta_j = 0$, two eigenstates with different flavors of AF domains become equal in energy and only the FM states remain energetically distinct. We therefore expect that all non-FM ordered states will experience short-range hybridization leading to unstable DTC behavior [243].

We highlight this special role of chirality in determining the $\mathbb{Z}_3$ DTC stability by experimentally comparing the disordered $\theta_j \neq 0$ (Fig. 8.7c-f) with the uniform $\theta_j = 0$ case (Fig. 8.7g-j) in an 8 qutrit chain. To probe the contrasting behavior, we analyze the Floquet dynamics of 3 FM, 6 AF, and 10 random initial trit string states. For the $\theta_j \neq 0$ case, we implement $\mathcal{U}_F$ as shown in Fig. 8.7c, with the parameters plotted in panel d. With these chiral couplings, we observe a robust period tripling response for a wide range of kicking strengths g across all initial states, confirmed by the chain averaged magnetization $\langle M \rangle$ spectroscopy in Fig. 8.7d. We further confirm that the site and state averaged autocorrelator $|\langle \bar{A} \rangle|$ (Fig. 8.7e-f) shows minimal initial state dependence across a range of Floquet cycles and kicking strengths. By contrast, when we remove chirality by applying local DD pulses to set $\theta_j \rightarrow 0$ across the chain (Fig. 8.7g), we directly observe the preferred stability of the FM ordered states compared to AF and random trit string states (Fig. 8.7h-j). In Fig. 8.7h, we observe that at $g = 0.9$, only the FM ordered states respond with robust period tripling. Moreover, this difference in stability between the FM states and different magnetic orders is observable in $|\langle \bar{A} \rangle|$ even at early time steps (Fig. 8.7i) and persists across a wide range of kicking strengths (Fig. 8.7j), in close agreement with numerical simulation. This dependence on chirality to determine DTC stability highlights the richness of qudit clock model physics with no analogous counterpart in widely studied $\mathbb{Z}_2$ Ising models.

System Probes of Phase Transition

Phase transitions in non-equilibrium many-body systems are characterized by stark differences in the dynamics of the system and are typically defined in the thermodynamic limit as the chain length increases to infinity, $N \rightarrow \infty$. Therefore, to probe the true nature of these phases in an experimental setting, we employ qutrit tomography to study entanglement growth and investigate the finite system scaling behavior to confirm a genuine phase transition.

State tomography is a powerful tool for probing and contrasting the bulk entanglement growth in the thermal and the DTC phase. The MBL nature of the DTC phase prevents ergodicity, giving rise to extensive local integrals of motion that are dressed localized clock operators called l -bits [110, 2]. The interaction between two l -bits falls off exponentially with their separation, leading to a slow growth of entanglement ($\sim \log(t)$). By

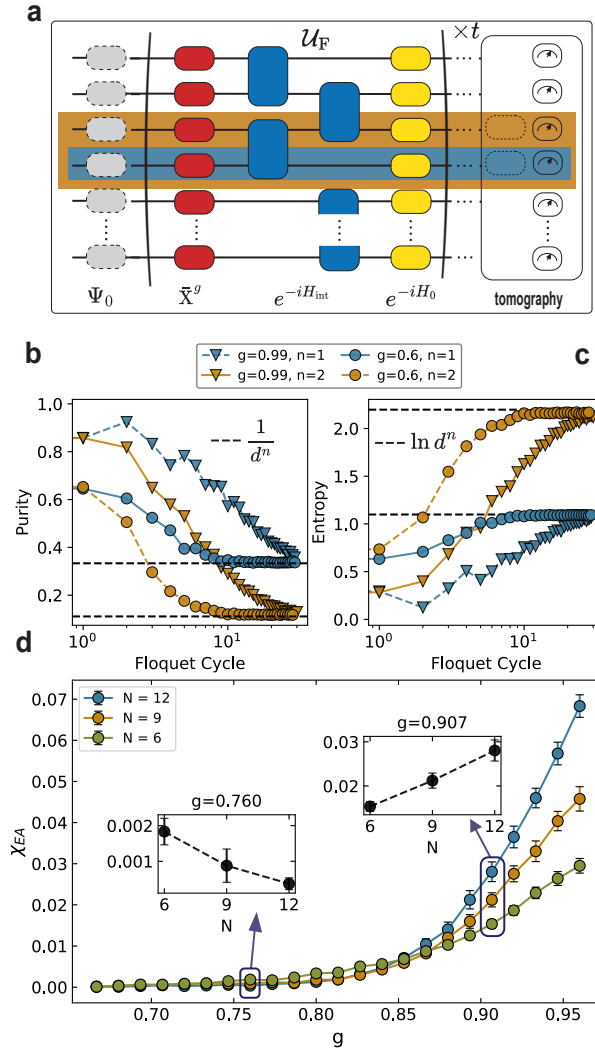


Figure 8.8: **System level probes of thermal- $\mathbb{Z}_3$ DTC phase transition.** **a**, Schematic of Floquet circuits $\mathcal{U}_F$ where at each Floquet cycle t , the bulk spin site response is probed with 1 or 2 qutrit state tomography. **b-c** For a subset of $n = 1$ and $n = 2$ qutrits from the chain, respectively the tomographically reconstructed density matrix state purity ($\text{Tr}[\rho^2]$) and entropy ($-\text{Tr}[\rho \log \rho]$) response as a function of Floquet cycle in both the $\mathbb{Z}_3$ -DTC ($g = 0.99$) and thermal ($g = 0.6$) regimes. Dashed horizontal lines depict the expected values for maximally mixed states $\rho = \frac{1}{d^n} \text{I}_d$. **c**, Spin glass order parameter (χ_{EA}) as a function of kicking strength g and qutrit chain length N . We observe a crossing at $g_c \approx 0.86$, consistent with our numerical modeling of our system. The data points and error bars represent respectively the mean and standard deviation of χ_{EA} calculated across all 13 instances of initial trit-string state (Ψ_0).

contrast, these l -bits delocalize in the thermal case causing a fast entanglement growth ($\sim t$).

To demonstrate this contrast in the behavior of the entanglement, we perform full one and two qutrit state tomography of bulk sites for all times t (Fig. 8.8a). As can be observed in Figs. 8.8b,c, the volume-like entanglement in the thermal phase leads to a significantly faster decay in the subsystem purity and faster growth in the Von-Neumann entropy. Despite the intrinsic decoherence present in the system, the pronounced nature of this difference in entanglement growth between the two phases is still clearly observed on experimentally relevant time scales.

Finally, we move to finite size scaling analysis to confirm the phase transition point (g_c) extracted from the generalized Edward-Anderson order parameter [77]

$$\chi_{EA} = \frac{1}{(N-2)} \sum'_{i \neq j} |\langle \hat{Z}_i^\dagger \hat{Z}_j \rangle|^2, \quad (8.6)$$

(where the primed sum implies that endpoints of the chain are excluded). This parameter measures glassiness from spin correlations across the chain and as a result, increases with N in the MBL regime and decreases with N in the thermal regime until it vanishes. Consequently, we can experimentally identify a crossing that defines the phase transition point between the two phases. We observe this expected behavior by performing our Floquet circuits across a range of kicking strength g , and floquet cycles. In Fig. 8.8c, we present the results in spin chains of length $N = 6, 9, 12$ from averaging χ_{EA} across 5-10 Floquet cycles with 13 initial trit string instances. We observe a critical kicking strength between the DTC and thermal phases in the region of $g_c \approx 0.86$, consistent with our numerical simulations.

Subspace Time Crystal

Finally, we demonstrate that native qutrit operation also offers a platform to investigate time crystalline behavior within a subsystem of our model — in our case a *qubit* time crystal oscillating within the larger qutrit state space. In particular, we modified our local kicking gates in our Floquet circuits from $\bar{X}^g \rightarrow \bar{X}_{12}^{1-g}$, where X_{12} represents the qubit Pauli X operator embedded in the $\{|1\rangle, |2\rangle\}$ subspace of the qutrit. We study these modified Floquet circuits on an 8 qutrit chain. As our subspace kicking gates now only rotate the $\{|1\rangle, |2\rangle\}$ subspace, we prepare the initial trit string state $|12121121\rangle$ such that for small kicking strength g , the qutrit population at each site in the spin chain should ideally oscillate within this subspace.

As shown in Fig. 8.9a, for a weak but non-trivial kick (i.e. $g \neq 0$), we observe the characteristic subharmonic period doubling oscillations of a $\mathbb{Z}_2$ DTC in the $\{|1\rangle, |2\rangle\}$ subspace across the entire chain. Similarly, as shown in Fig. 8.9b, for strong kicking strengths g , we recover robust period tripling behavior consistent with a $\mathbb{Z}_3$ DTC. By observing the $\langle M \rangle$ FFT spectroscopy in Fig. 8.9c, we can observe a line at $\omega = \pi$ visible at each site for weak g corresponding to subharmonic period doubling, before observing the expected thermal melting in the intermediate kicking regime, and finally recovering the period tripling response

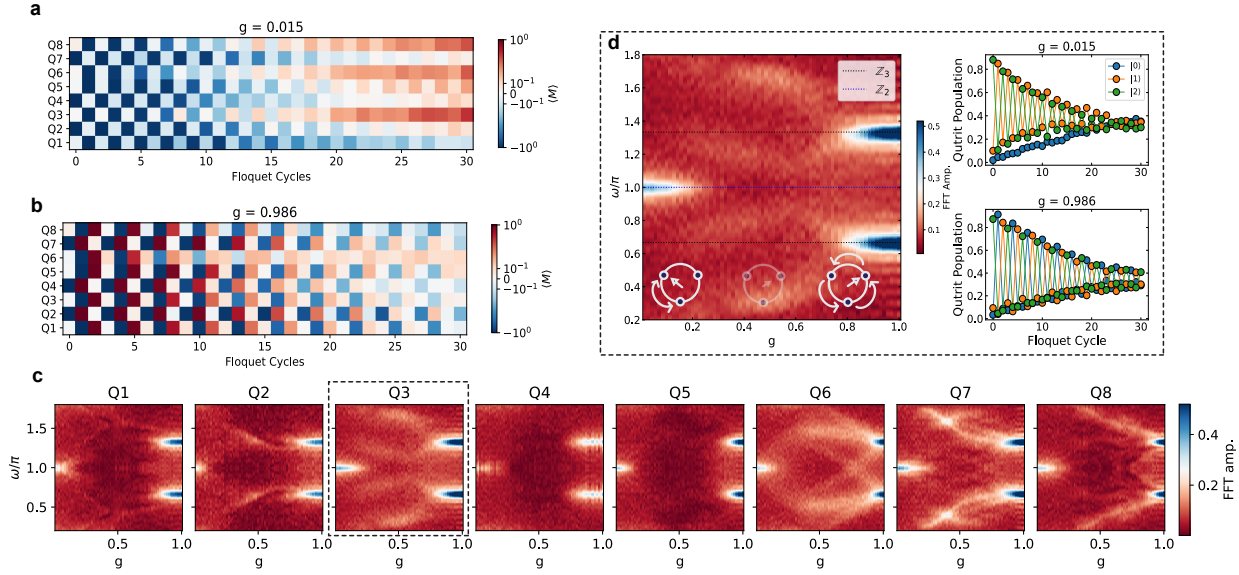


Figure 8.9: **Tuning from a $\mathbb{Z}_2$ to $\mathbb{Z}_3$ DTC.** **a**, $\langle M \rangle$ response for the modified Floquet circuits across an 8 qutrit chain with $g = 0.015$ where our system is now driven with the modified global kicking $\bar{X}^g = \bar{X}_{12}^{1-g} \bar{X}^g$. **b**, $\langle M \rangle$ response for the modified Floquet circuits with $g = 0.986$. **c**, The FFT of the $\langle M \rangle$ response as a function of g across the entire 8 qutrit chain employed in the experiment. **d**, Q3 FFT enlarged with the qutrit state populations in respectively the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ DTC regions.

at $\omega = 2\pi/3$ for large g . Notably, as can be seen in this spectroscopy, the period doubling, thermal melting, and period tripling responses were observable across all spin sites, signaling the many-body nature of the observed behavior. Similar to the role of chirality in our driven CCM, this subspace time crystal behavior has no analog in qubit Ising systems.

8.3 Extended Experimental Methods

We now provide extended experimental methods to accompany the experiments that were demonstrated earlier in this chapter. In particular, we describe device operation, the native implementation of the CCM interaction Hamiltonian, and discuss some aspects of how the experiment was calibrated.

Device Coherences and Frequency Allocation

The quantum processor employed in our DTC experiments consists of a 2D grid of 20 superconducting flux-tunable transmon qubits with interactions mediated by 30 flux tunable

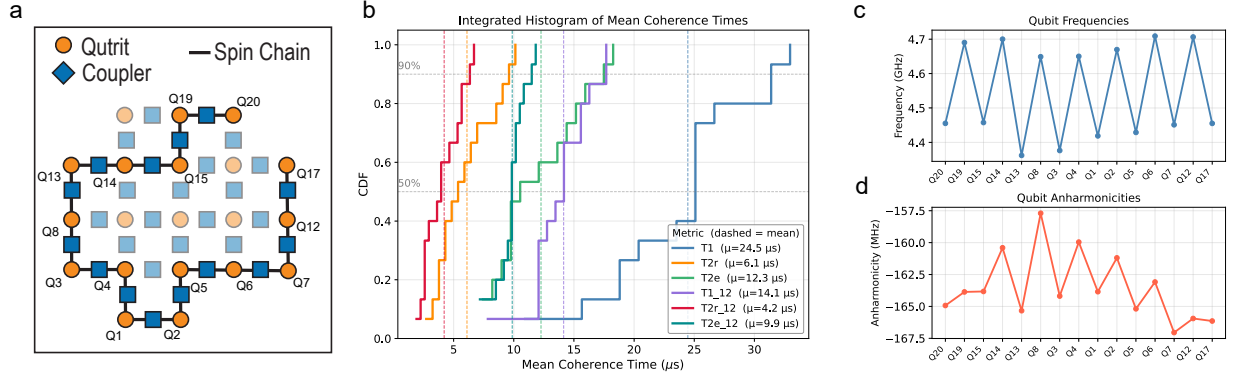


Figure 8.10: **Qutrit Device Specs** (a) Layout of 20 qubit quantum processor utilized in our DTC experiments. 1D spin-chains are selected from the square lattice topology. Actively pulsed qubits and couplers are shown in bold, while idling qubits and couplers are faded. (b) Qutrit coherence statistics across the 15 qutrit spin-chain where all means are collected from averaging over 100 separate coherence measurements. (c) Qubit ($\omega_{01}/2\pi$) frequencies across the spin-chain. (d) Qubit anharmonicities across the spin-chain.

couplers designed, fabricated, and packaged by IQM [3]. The processor is housed on premise at UC Berkeley in a Blue-Fors XLD dilution refrigerator, employing custom wiring with special integrated components such as cryogenic bias-tees for the qubit flux lines supplied by YQuantum and TWPAs provided by Silent Waves as well as HERD (High Energy Radiation Drain) filters on the readout lines. The RF control hardware used in our experiments is the open-source qubic system developed at Lawrence Berkeley National Laboratory [107].

We perform qutrit based computation on our 20-qubit processor; for our DTC experiments, we select sub-chains within the lattice, such as the 15 qutrit chain chain presented in Fig. 8.10a. The transmons are flux-biased to the top of their tuning curves (unless there is a parasitic two-level system spectrally nearby) to maximize coherence during circuit execution. Statistics on the relevant T1, T2 Ramsey, and T2 echo coherence times between the $|i\rangle, |i+1\rangle$ levels of our transmon qutrits are shown in Fig. 8.10b. Finally, in Fig. 8.10c,d we present the the frequency and anharmonicity allocation along the 15 qutrit chain used in our largest DTC experiment in this chapter. We next provide further information on single qutrit gate and readout operation on the device, with information on engineering the CCM interaction provided later in this supplement.

Qutrit Spin-Spin Interactions

As discussed, by encoding qutrits in our transmon lattice, we are able to natively realize a driven CCM. In this section, we provide more information on our tunable coupler method for realizing the interaction terms in the CCM. Specifically, we describe how these interactions

are generated from tunable coupler pulses and show the freedom to tune the strength J chiral angle θ in the CCM interaction.

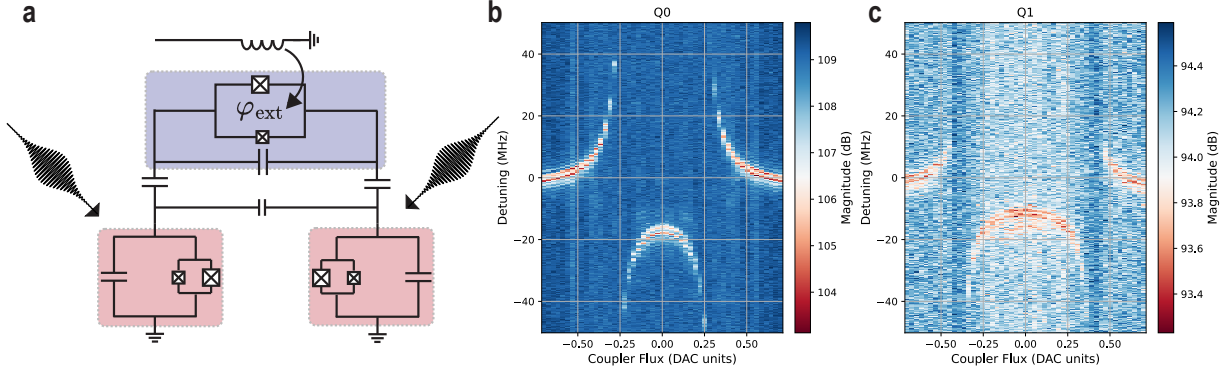


Figure 8.11: **Qubit two-tone spectroscopy vs. coupler flux** (a) Circuit diagram with two transmon qubits (red) coupled to a third flux tunable transmon coupling qubit (blue). The qubits are driven with microwave pulses and the coupler's frequency is adjusted with flux. (b-c) Q0/Q1 qubit two-tone spectroscopy vs coupler flux. Note that, as can be observed from the avoided crossing, the coupler tunes from above the qubits at the integer flux to below the qubits at its half flux point.

We realize a driven effective Hamiltonian that locally maps to the spin-spin interaction term in the CCM by applying fast flux pulses on the tunable couplers which mediate hybridization between neighboring transmons on our chip. In practice, this works by choosing an idling flux point where the tunable coupler mediated hybridization is small, and then pulsing the coupler to a regime where the dispersive coupling between qutrits is large by decreasing the detuning between the coupler and the qutrit transition energies. As shown via the qubit two-tone vs coupler flux experiment in Fig. 8.11, the coupler at 0 flux idles above the two qubit frequencies, before being tuned through them (see the avoided crossing).

Our unit-cell (for more information see Ref. [163]) consists of two transmons modeled as duffing oscillators, coupled via a third transmon acting as a tunable coupler. The data transmons have some direct capacitive coupling (g_{12}) and are coupled to the tunable coupler (g_{ic}). This realizes the system Hamiltonian,

$$H_{\text{sys}} = \sum_{i=0,1} \omega_i a_i^\dagger a_i + \frac{\eta_i}{2} a_i^\dagger a_i^\dagger a_i a_i + \omega_c(\Phi_{\text{ext}}) a_c^\dagger a_c + g_{12}(a_1^\dagger a_2 + a_1 a_2^\dagger) + \sum_{i=1,2} g_{i,c}(a_i^\dagger a_c + a_i a_c^\dagger) \quad (8.7)$$

where a_i , ω_i , and η_i are respectively the qubit bosonic annihilation operator, frequency, and anharmonicity of transmon i and we have neglected the nonlinearity of the coupler for simplicity. As in Refs. [154, 48, 39], we engineer a tunable ZZ interaction between our transmons tuned by the coupler flux φ_{ext} , that when truncated to the coupled qutrit Hilbert

space, realizes (up to local phases), a cross-Kerr Hamiltonian,

$$H_{\text{CK}} = \sum_{i,j \in \{1,2\}} \alpha_{i,j}(\varphi_{\text{ext}}) |i0j\rangle \langle i0j|. \quad (8.8)$$

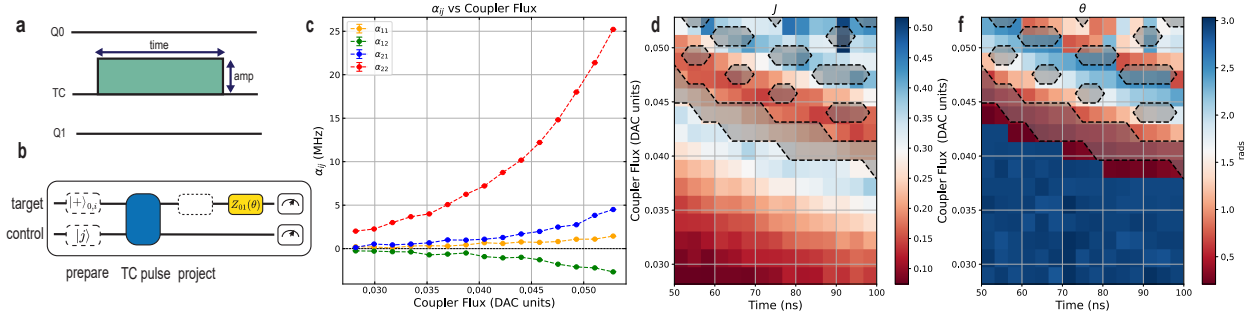


Figure 8.12: **Characterizing the qutrit spin-spin interactions** (a) Pulse diagram for implementing the spin-spin gates. The two qubits (Q0,Q1) are idle and a square baseband pulse is applied to the tunable coupler (TC). (b) A circuit diagram of the qutrit generalized JAZZ experiment used to measure the cross-Kerr rates and entangling phases. (c) The extracted cross-Kerr rates as a function of coupler flux. (d-f) The extracted spin-spin gate parameters expressed in terms of the relevant CCM coefficients J and θ for our $\mathbb{Z}_3$ -DTC. For example, we highlight regions in flux pulse amplitude and time where the spin-spin gates realize chiral angles are greater than 0.3 rads detuned from degeneracy inducing points (i.e. $n\pi/3$).

In the dispersive limit where the qubit-coupler detunings are larger than their coupling g_{ic} the coupler remains unpopulated and we can use perturbation theory to compute the cross-Kerr (ZZ -like) terms as $\alpha_{i,j} = (E_{i,0,j} - E_{i,0,0}) - (E_{0,0,j} - E_{0,0,0})$, employing the energy labeling $\{\text{transmon 1, coupler, transmon 2}\}$. In practice, analytically capturing the behavior of the full qutrit dispersive coupling requires careful attention to higher-order perturbative corrections and is beyond the scope of this text [72].

We measure these cross-Kerr term as a function of flux applied to the coupler in Fig. 8.12 using a generalized JAZZ sequence [245] (see Fig. 8.12b), and display the α_{ij} rates in Fig. 8.12c. Experimentally we find the statically decoupled point where ZZ is minimized corresponds to when the coupler is significantly below the qubits in frequency. Notably, as the qubits are negatively anharmonic and the coupler frequency is below the qubits, the α_{22} rate increases dramatically faster as a function of coupler flux than the other cross-Kerr terms.

We can subsequently extract the qutrit conditional phases as a function of flux and time from the JAZZ experiments. Upon doing so, we map the spin-spin gates $U = e^{-iH_{\text{ck}}t}$ to the two-body interaction term in the CCM where the angles $\theta_{ij} = 2\pi \times \alpha_{ij}t$ are used to

calculate $\{J, J', \theta, \theta'\}$ (see the following section). Via this mapping, as discussed previously, we can generate spin-spin gates that either lead to stable or unstable DTC behavior via the tuning of the chiral phase. In Fig 8.12d,f we show regimes in the space of coupler flux pulse amplitude and duration which generate two-body spin-spin gates that lead to stable DTC behavior. Such characterization can be performed across all coupling sites in the spin chain to generate either stable or unstable DTC behavior.

Mapping the cross-Kerr Hamiltonian to the CCM

In this section, we explicitly map the native cross-Kerr interaction to the chiral clock model $U_{\text{CK}} = e^{iH_{\text{CK}}} \rightarrow e^{iH}$, where $H = H_{\text{int}} + H_0$, contains both, the qutrit-qutrit interaction H_{int} and the single site H_0 term. We perform baseband fast flux pulses on our tunable coupler to implement a cross-Kerr interaction between qutrits. As our experiment works in a Floquet implementation with gates rather than as an analog quantum simulation, to make our mapping between the experimentally realized unitary and the generators of our theory CCM, we define angles $\theta_{ij} = (2\pi\alpha_{ij} \times t) \in (-\pi, \pi]$. We can express the cross-Kerr Hamiltonian in the spin basis

$$H_{\text{CK}} = \sum_{m,n=0}^2 c_{mn} \sigma^m \otimes \sigma^n, \quad (8.9)$$

where the projections

$$c_{mn} = \frac{1}{9} \text{Tr}[(\sigma^m \otimes \sigma^n)^\dagger H] = \frac{1}{9} \sum_{ij} \omega_3^{-im-jn} \langle ij | H_{\text{CK}} | ij \rangle \quad (8.10)$$

$$= \frac{1}{9} (\theta_{11} \omega_3^{-m-n} + \theta_{12} \omega_3^{-m-2n} + \theta_{21} \omega_3^{-2m-n} + \theta_{22} \omega_3^{-2m-2n}). \quad (8.11)$$

Writing all independent terms

$$c_{00} = \frac{1}{9} (\theta_{11} + \theta_{12} + \theta_{21} + \theta_{22}), \quad (8.12)$$

$$c_{01} = \frac{1}{9} (\omega_3^2 (\theta_{11} + \theta_{21}) + \omega_3 (\theta_{12} + \theta_{22})), \quad (8.13)$$

$$c_{10} = \frac{1}{9} (\omega_3^2 (\theta_{11} + \theta_{12}) + \omega_3 (\theta_{21} + \theta_{22})), \quad (8.14)$$

$$c_{11} = \frac{1}{9} (\omega_3 \theta_{11} + \theta_{12} + \theta_{21} + \omega_3^2 \theta_{22}), \quad (8.15)$$

$$c_{12} = \frac{1}{9} (\theta_{11} + \theta_{12} \omega_3 + \theta_{21} \omega_3^2 + \theta_{22}), \quad (8.16)$$

and comparing with the theory ansatz

$$H = J e^{i\theta} \sigma_1 \otimes \sigma_2^\dagger + J' e^{i\theta'} \sigma_1 \otimes \sigma_2 + h_1 e^{i\phi_1} \sigma_1 \otimes I + h_2 e^{i\phi_2} I \otimes \sigma_2 + \text{h.c.} + h_I I \otimes I, \quad (8.17)$$

we obtain

$$J = |c_{12}|, \quad \theta = \arg(c_{12}), \quad (8.18)$$

$$J' = |c_{11}|, \quad \theta' = \arg(c_{11}), \quad (8.19)$$

$$h_1 = |c_{10}|, \quad \phi_1 = \arg(c_{10}), \quad (8.20)$$

$$h_2 = |c_{01}|, \quad \phi_2 = \arg(c_{01}), \quad (8.21)$$

and $h_I = c_{00}$ is the coefficient of Identity which can be ignored.

Calibration of Tunable Coupler for Isolated Single Qutrit Operation

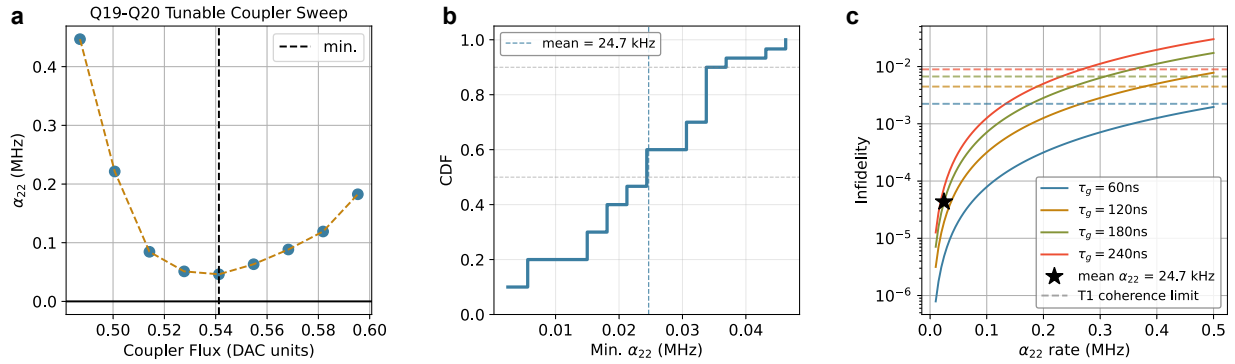


Figure 8.13: **Calibrating the Tunable Coupler Idling Flux Point** (a) An example α_{22} vs coupler flux biasing experiment, where the coupler flux is swept to find a minimal cross-Kerr coupling rate. (b) The extracted minimal α_{22} cross-Kerr rate across all 30 qutrit-qutrit pairs on our 20 transmon processor. We find a mean nulling of 24.7 KHz averaged over all 30 tunable couplers. (c) Error budgeting simultaneous single qutrit gates between two qutrits as a function of gate time (τ_g) for cross-Kerr unitary errors (solid lines) and decoherence (horizontal dashed lines). The coherence limits are calculated using the average T1 times across all qutrits from our statistics displayed in Fig. 8.10.

Previously, we demonstrated that the α_{22} cross-Kerr term grows much faster as a function of flux than the other α_{ij} terms owing to the negatively anharmonic nature of our transmon qutrits and the fact the coupler nulled ZZ point is below the qutrit transitions in frequency. With this in mind, our strategy for finding the best idling coupler flux point for our qutrits is to null the α_{22} cross-Kerr term using the same JAZZ experiment from Fig. 8.12. In Fig. 8.13a, we show the results for a tunable coupler flux calibration experiment, where we sweep the coupler flux to find a point with minimal α_{22} cross-Kerr. As the majority of our qutrits are not straddling, we do not generally find an $\alpha_{22} = 0$ cancellation point, but notably, as demonstrated in Fig. 8.13b, we can achieve an average α_{22} nulling of 24.7 KHz averaged

across all 30 qutrit-qutrit coupled pairs on our 20 data qutrit processor. Finally we suggest a simple error budget for our simultaneous single qutrit gate operation; we estimate the error on our single qutrit operations by defining

$$\mathcal{F} = \text{Tr}[U_{\text{id}}^\dagger U_{\text{err}}]/d^2 \quad (8.22)$$

where we define $U_{\text{err}} = e^{-i(2\pi\alpha_{22} \times t|22\rangle\langle 22|)}$, only keeping the α_{22} term as it dominates and neglecting the other cross-Kerr terms for simplicity. We can then calculate an estimate for the single qutrit gate error as a function of both gate time τ_g and cross-Kerr strength α_{22} .

In Fig. 8.13c, we display the cross-Kerr limited gate fidelities as a function of α_{22} rate. Notably, given our cross-Kerr nulling, for our arbitrary SU(3) gate time of 180 ns, gate fidelities over 99.9% should in principle be achievable. We also plot the associated T1 limited gate fidelities given τ_g , using the decoherence gate infidelity model from Ref.[176]. These results strongly suggest that by integrating tunable couplers, simultaneous single qutrit gate operations are no longer limited by quantum crosstalk, unlike in prior fixed coupling experiments [85, 84, 176]. This now highlights that gate time remains a major bottleneck; in the near-term, future works can explore speeding up the single qutrit gate times by exploring multichromatic and optimal control techniques [38, 224].

8.4 Extended Theory and Numerical Results of $\mathbb{Z}_3$ DTC

Exactly Solvable Limit

To investigate the properties of the driven chiral clock model, it is informative to first consider the exactly solvable limit at $g = 1$. The Floquet unitary is $\mathcal{U}_F = e^{-iH\bar{X}}$, with

$$H = \sum_i \underbrace{J_i e^{i\theta_i} \hat{Z}_i \hat{Z}_{i+1}^\dagger}_{\mathbb{Z}_3 \text{ preserving}} + \sum_i J'_i e^{i\theta'_i} \hat{Z}_i \hat{Z}_{i+1} + h_i e^{i\varphi_i} \hat{Z}_i + \text{h.c.} , \quad (8.23)$$

containing the combined clock spin-spin interaction H_{int} and the single site H_0 term. We represent the spin basis of the entire chain by $|s\rangle = |s_1, s_2, \dots, s_L\rangle$ where $s_i \in \{0, 1, 2\}$ is the spin at site i . The spin basis diagonalizes H with the energy naturally written as the sum of two parts: $H|s\rangle = (\varepsilon(s) + \varepsilon'(s))|s\rangle$ where

$$\varepsilon(s) = 2 \sum_i J_i \cos\left(\frac{2\pi}{3}(s_i - s_{i+1}) + \theta_i\right) , \quad (8.24)$$

$$\varepsilon'(s) = 2 \sum_i \left[J'_i \cos\left(\frac{2\pi}{3}(s_i + s_{i+1}) + \theta'_i\right) + h_i \cos\left(\frac{2\pi}{3}s_i + \varphi_i\right) \right] . \quad (8.25)$$

The first term $\varepsilon(s)$ only depends on the differences of spins and is therefore invariant under a global kick $\bar{X}$. This term arises out of the $\mathbb{Z}_3$ symmetry preserving part of H : $H_{\mathbb{Z}_3} =$

$\sum_i J_i e^{i\theta_i} \hat{Z}_i \hat{Z}_{i+1}^\dagger + \text{h.c.}$, satisfying $[H_{\mathbb{Z}_3}, \bar{X}] = [H_{\mathbb{Z}_3}, \bar{X}^2] = 0$. A consequence is that $H_{\mathbb{Z}_3}$ has the same energy $\varepsilon(s)$ for $\{|s\rangle, \bar{X}|s\rangle, \bar{X}^2|s\rangle\} = \{|s\rangle, |s+1\rangle, |s+2\rangle\}$. The Floquet unitary in this basis is

$$\mathcal{U}_F = e^{-i\varepsilon(s)} \begin{bmatrix} 0 & 0 & e^{-i\varepsilon'(s)} \\ e^{-i\varepsilon'(s+1)} & 0 & 0 \\ 0 & e^{-i\varepsilon'(s+2)} & 0 \end{bmatrix}. \quad (8.26)$$

We diagonalize it to obtain the corresponding quasienergies and vectors $\mathcal{U}_F |\varepsilon_i\rangle = e^{-i\varepsilon_i} |\varepsilon_i\rangle$ where

$$\varepsilon_0(s) = \varepsilon(s), \quad |\varepsilon_0(s)\rangle = \frac{1}{\sqrt{3}}(e^{-i\varepsilon'(s)} |s\rangle + e^{i\varepsilon'(s+2)} |s+1\rangle + |s+2\rangle), \quad (8.27)$$

$$\varepsilon_1(s) = \varepsilon(s) + 2\pi/3, \quad |\varepsilon_1(s)\rangle = \frac{1}{\sqrt{3}}(\omega_3 e^{-i\varepsilon'(s)} |s\rangle + \omega_3^2 e^{i\varepsilon'(s+2)} |s+1\rangle + |s+2\rangle), \quad (8.28)$$

$$\varepsilon_2(s) = \varepsilon(s) - 2\pi/3, \quad |\varepsilon_2(s)\rangle = \frac{1}{\sqrt{3}}(\omega_3^2 e^{-i\varepsilon'(s)} |s\rangle + \omega_3 e^{i\varepsilon'(s+2)} |s+1\rangle + |s+2\rangle). \quad (8.29)$$

We used an additional constraint: $\varepsilon'(s) + \varepsilon'(s+1) + \varepsilon'(s+2) = 0$, following from $H + \bar{X}H\bar{X}^\dagger + \bar{X}^2H\bar{X}^{\dagger 2} = 3H_{\mathbb{Z}_3}$ which we explicitly prove below. We see that the eigenstates are long range ordered cat like states. The quasienergies form triplets separated by $2\pi/3$ and depend only on the $\mathbb{Z}_3$ symmetry preserving part of the Hamiltonian. Our results for the general chiral clock model are consistent with those from Ref. [243] that consider a similar model.

Dynamical decoupling of $\mathbb{Z}_3$ breaking terms — After three Floquet cycles we have the unitary

$$\mathcal{U}_F^3 = e^{-iH} \bar{X} e^{-iH} \bar{X} e^{-iH} \bar{X} = e^{-i(H + \bar{X}H\bar{X}^\dagger + \bar{X}^2H\bar{X}^{\dagger 2})}, \quad (8.30)$$

where we used $\bar{X}^3 = I$ and H is diagonal. Notice that the $\mathbb{Z}_3$ breaking terms rotate by integer multiples of ω : $\bar{X}\hat{Z}_i\bar{X}^\dagger = \omega_3^2\hat{Z}_i$, $\bar{X}\hat{Z}_i\hat{Z}_{i+1}\bar{X}^\dagger = \omega_3\hat{Z}_i\hat{Z}_{i+1}$. Together with the fact that $[H_{\mathbb{Z}_3}, \bar{X}] = 0$, we have

$$H + \bar{X}H\bar{X}^\dagger + \bar{X}^2H\bar{X}^{\dagger 2} = 3H_{\mathbb{Z}_3} + \left((1 + \omega_3 + \omega_3^2) J'_i e^{i\theta'_i} \hat{Z}_i \hat{Z}_{i+1} + (1 + \omega_3^2 + \omega_3) h_i e^{i\phi'_i} \hat{Z}_i + \text{h.c.} \right) \quad (8.31)$$

$$= 3H_{\mathbb{Z}_3}. \quad (8.32)$$

Therefore, near $g \sim 1$, the $\mathbb{Z}_3$ breaking terms dynamically decouple out, leaving the physics to be dictated by the $\mathbb{Z}_3$ preserving part $\mathcal{U}_F^3 = e^{-3iH_{\mathbb{Z}_3}}$ [243]. This is similar to the Ising case when $\mathbb{Z}_2$ breaking terms decouple out [170, 128, 116].

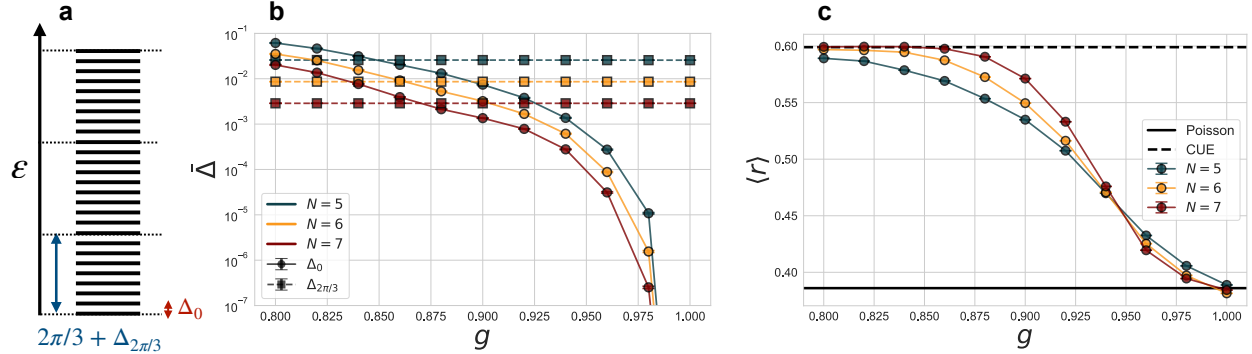


Figure 8.14: (a) Spectrum and definition of quasienergy gaps. (b) The average spectral gap between adjacent quasienergies Δ_0 and the error in triplet pairing $\Delta_{2\pi/3}$ as a function of kick parameter g . (c) Average statistics $\langle r \rangle$ over quasienergies and disorder realization as a function of kick g . The Hamiltonian parameters are chosen to be $J, J' \in [0.08, 0.25]$ and random θ, θ' averaged over 1000 disorder instances.

Eigenstate Order and Many Body Localization

The eigenstate order must be stable beyond the trivial $g = 1$ point for a true time crystalline behavior. To see this, we numerically diagonalize the Floquet unitary $\mathcal{U}_F |\varepsilon_i\rangle = e^{i\varepsilon_i} |\varepsilon_i\rangle$ as a function of the kick parameter g . For each quasienergy $\varepsilon_i \in [-\pi, \pi)$, we denote

$$\Delta_0^i = |\varepsilon_{i+1} - \varepsilon_i|, \quad \Delta_{2\pi/3}^i = |\varepsilon_i - \varepsilon_{i+\mathcal{N}/3} - 2\pi/3|, \quad (8.33)$$

where $\mathcal{N} = 3^N$ the dimension of the Hilbert space. Here Δ_0^i quantifies the gap in adjacent quasienergies and $\Delta_{2\pi/3}^i$ represents the error in triplet pairing as depicted in Fig. 8.14(a). Since eigenstate order is necessary for time crystalline behavior, the error in triplet pairings must be small as compared to the gap between subsequent quasienergies. We average over quasienergies and disorder configurations and respectively show $\bar{\Delta}_0, \bar{\Delta}_{2\pi/3}$ as a function of the kicking strength g in Fig. 8.14(b). We see that the eigenstate order persists beyond $g = 1$ which confirms time-crystalline behavior observed in the experiment. At $g = 1$ the error in triplet pairing approaches machine precision, confirming the analytical derivation above. Furthermore, the triplet pairing error becomes larger than the quasienergy gap around $g_c \sim 0.86$, indicating the transition to the thermal regime. This is consistent with the experimental observation Fig. 8.3, indicated by the Fourier transform losing its sharp peak around g_c .

Next, we study ergodicity breaking due to disorder. We investigate energy level statistics of the quasienergy gap Δ_0^i by calculating ratio

$$r = \frac{\min(\Delta_0^i, \Delta_0^{i+1})}{\max(\Delta_0^i, \Delta_0^{i+1})}. \quad (8.34)$$

This ratio serves as a tool to investigate ergodicity breaking [202, 195]. We plot the average $\langle r \rangle$ over quasienergies and disorder configurations in Fig. 8.14(c) for various chain lengths. Near $g \sim 1$, as the system size increases, the ratio appears to converge to Poisson statistics value $\langle r \rangle_{\text{Poisson}} \approx 0.386$ indicating ergodicity breaking and the system being localized in the MBL phase. On the other extreme at low g , the system is ergodic or thermal as indicated by its convergence to the circular unitary ensemble (CUE) where $\langle r \rangle_{\text{CUE}} \approx 0.599$. For typical disorder range in the experiment, the crossover to MBL region happens after $g \sim 0.93$.

Numerical simulation of the Time Crystal

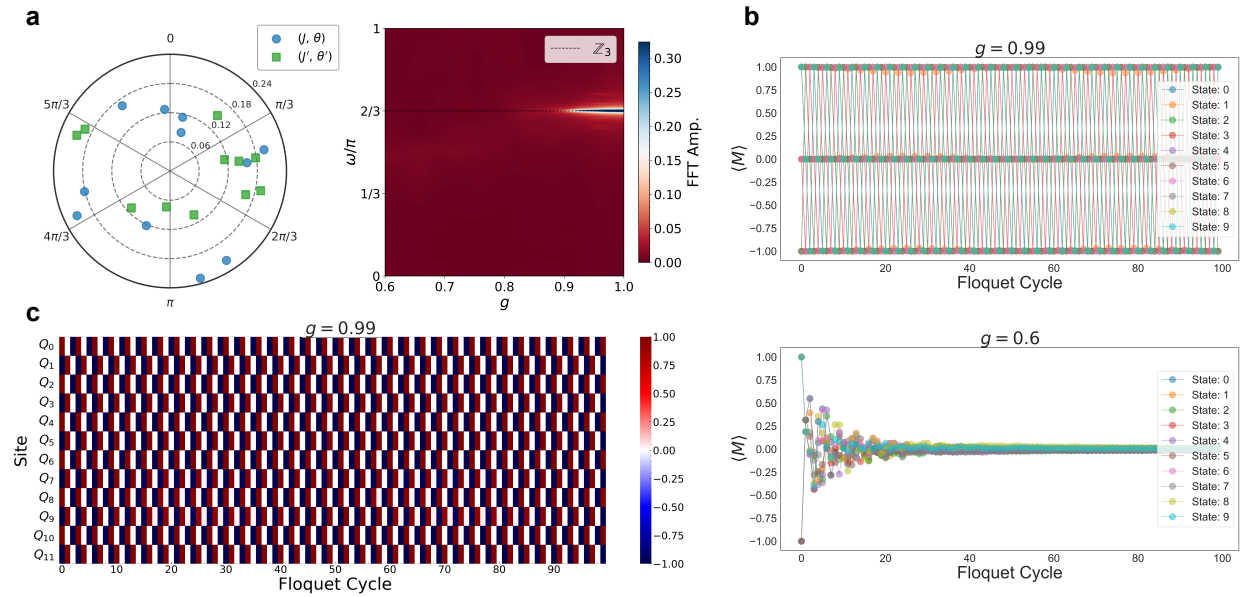


Figure 8.15: **Numerical Simulations of driven CCM Floquet system.** (a) Coupling parameters for a chain of 12 clock spins. FFT response of magnetization $\langle M \rangle$ for the bulk qutrit Q_6 . (b) Magnetization response of the bulk qutrit Q_6 for random initial product states in MBL DTC ($g=0.99$) and thermal regime ($g=0.6$). (c) Bulk response of the entire chain at $g = 0.99$.

Armed with theoretical understanding, we now numerically simulate the driven chiral clock model for a chain of 12 qutrits using tensor network simulation. Figure 8.15(a) depicts the random coupling parameters chosen within the experimentally accessible range. We show the Fourier transform of magnetization $\langle M \rangle$ response for the bulk site Q_6 as a function of g . We confirm that there is a sharp response at $\omega = 2\pi/3$ confirming period tripling and time crystalline behavior. We show that the time crystalline phase is robust to the choice of initial states by considering random initial states in Fig. 8.15(b). Furthermore, in the thermal regime, the oscillations die down approximately within the time scales observed in

the experiment. Finally, we show the response of the entire chain in Fig. 8.15(c) highlighting the collective response of the bulk.

8.5 Outlook

The experiments presented in this chapter represent a convergence of all the technical threads developed throughout this thesis: single and two-qutrit gates, error suppression, and digital quantum simulation, all applied at a scale and to a problem that would be less naturally tackled with qubit hardware. In this chapter, we have realized the first experimental implementation of the chiral clock model with short-range interactions and a tunable chiral angle, observed a $\mathbb{Z}_3$ discrete time crystal for the first time in a short-range coupled system, and directly demonstrated how chirality governs the stability of time-crystalline order.

The $\mathbb{Z}_3$ DTC and related driven CCM physics demonstrated in this chapter establishes native qudit hardware as a qualitatively distinct platform for exploring non equilibrium phases of matter. The chirality-controlled melting of DTC order, with no $\mathbb{Z}_2$ analog, exemplifies this distinction: the chiral angle is an intrinsic feature of the CCM that fundamentally governs domain-wall dynamics, spectral degeneracies, and DTC phase stability. Moreover, our experimental platform allows for this angle to be continuously controlled in-situ, revealing direct experimental observation of its effect on time-crystalline order. In a similar vein, our ability to engineer a $\mathbb{Z}_2$ DTC within our higher spin model points to a wide range of unexplored dynamical orders leveraging fractional — or subspace — kicking schemes.

Looking forward, the driven $\mathbb{Z}_3$ CCM studied here is the simplest member of a family of $\mathbb{Z}_n$ chiral clock models with progressively richer phase diagrams, capable of hosting exotic features such as parafermionic edge modes with braiding statistics richer than Majorana zero modes [65, 119]. As superconducting qudit processors scale to larger system sizes and higher local dimensions and benefit from improved control schemes, these phases become direct experimental targets. More immediately, the native qutrit operations and tunable interactions demonstrated provide a natural architecture for realizing qutrit topological codes [117, 152] and probing non-Abelian anyon physics in a solid-state setting, a frontier that, like the $\mathbb{Z}_3$ DTC itself, can greatly benefit from exploiting the full structure of the qudit Hilbert space.

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Appendix A

Cross-Kerr Perturbation Theory

A.1 Solution Structure

To solve for the cross-Kerr rates between statically coupled qutrits, we start by considering the Hamiltonian of two statically coupled and driven transmons in the Duffing oscillator approximation, with frequencies ω_C, ω_T and anharmonicities α_C, α_T . Qutrit C is driven at frequency ω with strength $2\Omega_C$. Qutrit T driven at frequency ω_d with strength $2\Omega_T$ and relative phase ϕ . Finally the transmons are transversely coupled with strength J . The Hamiltonian can then be written as

$$H = \sum_{i=C,T} \left[\omega_i a_i^\dagger a_i - \frac{\alpha_i}{2} a_i^\dagger a_i (a_i^\dagger a_i - 1) + 2\Omega_i \cos(\omega t + \phi_i) (a_i + a_i^\dagger) \right] + J(a_C a_T^\dagger + a_C^\dagger a_T). \quad (\text{A.1})$$

Note that the transmon anharmonicity is negative so the sign is chosen such that parameters α_i are positive. We can define detunings $\Delta_i = \omega_i - \omega$ and $\Delta = \omega_C - \omega_T$. Moving forward, without loss of generality, we set $\phi_C = 0$ and $\phi_T = \phi$. Then the gate is performed in the regime where $\Omega_C, \Omega_T, J \ll \Delta_i, \Delta$. We may not be strictly in this regime but the results should still hold at least qualitatively.

We now move to the doubly-rotating frame, defined by the unitary transformation

$$U = \exp \left\{ -i \sum_{i=C,T} a_i^\dagger a_i \omega t \right\}. \quad (\text{A.2})$$

Further, we make the RWA and drop terms oscillating at frequency 2ω . The result is the static Hamiltonian,

$$H' = U^\dagger H U + i\dot{U}^\dagger U \quad (\text{A.3})$$

$$= \sum_{i=C,T} \left[\Delta_i a_i^\dagger a_i - \frac{\alpha_i}{2} a_i^\dagger a_i (a_i^\dagger a_i - 1) \right] + \Omega(a_C + a_C^\dagger) + \Omega_T(a_T e^{i\phi} + a_T^\dagger e^{-i\phi}) + J(a_C a_T^\dagger + a_C^\dagger a_T). \quad (\text{A.4})$$

We employ perturbation theory to calculate energies E_{ij} , the diagonal elements of the Hamiltonian in the occupation number basis $|ij\rangle$. Then the Hamiltonian has the diagonal form

$$H = \sum_{i,j=0}^2 E_{ij} |ij\rangle \langle ij|. \quad (\text{A.5})$$

where E_{ij} are the energies up to the n th order of perturbation theory (however high we ultimately decide to go). Now we want to isolate the terms in the Hamiltonian that describe the entangling part of the dynamics. We start by performing a global shift of the energies so that $E_{00} \rightarrow 0$ and all energies are measured relative to this. Defining $\tilde{E}_{ij} = E_{ij} - E_{00}$, the Hamiltonian becomes

$$H' = \sum_{i,j=0}^2 \tilde{E}_{ij} |ij\rangle \langle ij| \quad (\text{A.6})$$

where clearly $\tilde{E}_{00} = 0$. Then we move into a rotating frame to eliminate the single qubit phases/energies $\tilde{E}_{i0}$ and $\tilde{E}_{0j}$, $ij = 1, 2$. The transformation to the rotating frame is given by the unitary operator

$$U = \exp \left\{ -it \left[\tilde{E}_{01} I \otimes |1\rangle \langle 1| + \tilde{E}_{10} |1\rangle \langle 1| \otimes I + \tilde{E}_{02} I \otimes |2\rangle \langle 2| + \tilde{E}_{20} |2\rangle \langle 2| \otimes I \right] \right\} \quad (\text{A.7})$$

where $\hbar = 1$ as before and I is the single qutrit identity operator,

$$I = |0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|. \quad (\text{A.8})$$

The transformed Hamiltonian is

$$H'' = i\dot{U}^\dagger U + U^\dagger H U = H + i\dot{U}^\dagger U \quad (\text{A.9})$$

where the last inequality is due to the fact that $[H, U] = 0$ since H and U are both diagonal and $U^\dagger U = 1$. The second term in this expression is straightforward to compute:

$$\begin{aligned} i\dot{U}^\dagger U &= i \left(it \left[\tilde{E}_{01} I \otimes |1\rangle \langle 1| + \tilde{E}_{10} |1\rangle \langle 1| \otimes I + \tilde{E}_{02} I \otimes |2\rangle \langle 2| + \tilde{E}_{20} |2\rangle \langle 2| \otimes I \right] \right) U^\dagger U \\ &= -\tilde{E}_{01} I \otimes |1\rangle \langle 1| + \tilde{E}_{10} |1\rangle \langle 1| \otimes I + \tilde{E}_{02} I \otimes |2\rangle \langle 2| + \tilde{E}_{20} |2\rangle \langle 2| \otimes I \end{aligned} \quad (\text{A.10})$$

The Hamiltonian now takes the desired form

$$H'' = \alpha_{11} |11\rangle \langle 11| + \alpha_{21} |21\rangle \langle 21| + \alpha_{12} |12\rangle \langle 12| + \alpha_{22} |22\rangle \langle 22| \quad (\text{A.11})$$

where the coefficients are

$$\begin{aligned} \alpha_{11} &= \tilde{E}_{11} - \tilde{E}_{01} + \tilde{E}_{10} \\ &= E_{11} + E_{00} - E_{01} - E_{10} \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned}\alpha_{12} &= \tilde{E}_{12} - \tilde{E}_{10} - \tilde{E}_{02} \\ &= E_{12} + E_{00} - E_{10} - E_{02}\end{aligned}\tag{A.13}$$

$$\begin{aligned}\alpha_{21} &= \tilde{E}_{21} - \tilde{E}_{01} - \tilde{E}_{20} \\ &= E_{21} + E_{00} - E_{01} - E_{20}\end{aligned}\tag{A.14}$$

$$\begin{aligned}\alpha_{22} &= \tilde{E}_{22} - \tilde{E}_{02} - \tilde{E}_{20} \\ &= E_{22} + E_{00} - E_{02} - E_{20}\end{aligned}\tag{A.15}$$

The full nature of the qutrit ZZ interaction will be characterized by these four parameters. One interesting thing to note at this point is that the definition of α_{ij} and choice of which terms in the Hamiltonian to cancel out is not unique. So while four parameters are required to characterize the entangling part of the dynamics, this choice of parameters is driven by our specific choice of the rotating frame in which we operate.

A.2 Second Order Corrections

The first order corrections to the energy are zero, as the perturbation maps all the unperturbed eigenstates onto orthogonal eigenstates or in other words,

$$\langle n_C n_T | V | n_C n_T \rangle = 0 \tag{A.16}$$

for all n_C, n_T . So the fun begins right away at second order. Let's develop a language to describe the different terms that appear. $\Omega_C^\pm$ will refer to a control qubit excitation/de-excitation, with $\Omega_T^\pm$ similarly for the target qubit. Lastly J will refer to a swap. Then the $00 \rightarrow 01 \rightarrow 10 \rightarrow 00$ process described above in the third order corrections will be denoted $\Omega_T^+ - J - \Omega_C^-$. Or more schematically we can omit the super/subscripts and write $\Omega - J - \Omega$. Following the notation and language developed above we have $\Omega - \Omega$ terms and $J - J$ terms at second order. Since the $\Omega - \Omega$ terms involve no coupling between the qutrits they will be ignored as they must cancel out in the calculation of the α_{ij} .

We start with $|00\rangle$. Throughout, tables will be used to show the interaction pathways at each order of perturbation theory. This will help to keep track of the many terms.

Start	I1	End
00	01 10	00

Here I_n refers to the n th intermediate state, of which there are $N - 1$ at N th order. We see there will be two terms contributing to the energy correction,

$$\begin{aligned}E_{00}^{(2)} &= \frac{\langle 00|V|01|00|V|01\rangle \langle 01|V|00|01|V|00\rangle}{E_{00}^{(0)} - E_{01}^{(0)}} + \frac{\langle 00|V|10|00|V|10\rangle \langle 10|V|00|10|V|00\rangle}{E_{00}^{(0)} - E_{10}^{(0)}} \\ &= \frac{\Omega_T^2}{\Delta_T} + \frac{\Omega_C^2}{\Delta_C}\end{aligned}\tag{A.17}$$

Next we consider $|01\rangle$,

Start	I1	End
01	11	01
	10	
	02	
	00	

so there are four terms contributing to the correction.

$$\begin{aligned}
E_{01}^{(2)} &= \frac{\langle 01|V|10|01|V|10\rangle \langle 10|V|01|10|V|01\rangle}{E_{01}^{(0)} - E_{10}^{(0)}} + \frac{\langle 01|V|02|01|V|02\rangle \langle 02|V|01|02|V|01\rangle}{E_{01}^{(0)} - E_{02}^{(0)}} \\
&\quad + \frac{\langle 01|V|00|01|V|00\rangle \langle 00|V|01|00|V|01\rangle}{E_{01}^{(0)} - E_{00}^{(0)}} + \frac{\langle 01|V|11|01|V|11\rangle \langle 11|V|01|11|V|01\rangle}{E_{01}^{(0)} - E_{11}^{(0)}} \\
&= \frac{J^2}{\Delta_T - \Delta_C} + \frac{(\sqrt{2}\Omega_T)^2}{\Delta_T - (2\Delta_T - \alpha_T)} + \frac{(\Omega_C)^2}{\Delta_T - 0} + \frac{\Omega_C^2}{\Delta_T - (\Delta_T + \Delta_C)} \\
&= -\frac{J^2}{\Delta} + \frac{2\Omega_T^2}{\alpha_T - \Delta_T} + \frac{\Omega_T^2}{\Delta_T} - \frac{\Omega_C^2}{\Delta_C}
\end{aligned} \tag{A.18}$$

For concreteness we make the table for $|10\rangle$

Start	I1	End
10	11	10
	01	
	20	
	00	

noting that the result is found by just exchanging the roles of the control and target everywhere. Thus we have the result immediately,

$$E_{10}^{(2)} = \frac{J^2}{\Delta} + \frac{2\Omega_C^2}{\alpha_C - \Delta_C} + \frac{\Omega_C^2}{\Delta_C} - \frac{\Omega_T^2}{\Delta_T} \tag{A.19}$$

Finally we consider $|11\rangle$,

Start	I1	End
11	01	11
	10	
	20	
	02	
	21	
	12	

which has six terms.

$$\begin{aligned}
E_{11}^{(2)} &= \frac{\langle 11|V|01|11|V|01\rangle \langle 01|V|11|01|V|11\rangle}{E_{11}^{(0)} - E_{01}^{(0)}} + \frac{\langle 11|V|10|11|V|10\rangle \langle 10|V|11|10|V|11\rangle}{E_{11}^{(0)} - E_{10}^{(0)}} \\
&+ \frac{\langle 11|V|20|11|V|20\rangle \langle 20|V|11|20|V|11\rangle}{E_{11}^{(0)} - E_{20}^{(0)}} + \frac{\langle 11|V|02|11|V|02\rangle \langle 02|V|11|02|V|11\rangle}{E_{11}^{(0)} - E_{02}^{(0)}} \\
&+ \frac{\langle 11|V|21|11|V|21\rangle \langle 21|V|11|21|V|11\rangle}{E_{11}^{(0)} - E_{21}^{(0)}} + \frac{\langle 11|V|12|11|V|12\rangle \langle 12|V|11|12|V|11\rangle}{E_{11}^{(0)} - E_{12}^{(0)}} \\
&= \frac{\Omega_C^2}{\Delta_T + \Delta_C - \Delta_T} + \frac{\Omega_T^2}{\Delta_T + \Delta_C - \Delta_C} + \frac{(\sqrt{2}J)^2}{\Delta_T + \Delta_C - (2\Delta_C - \alpha_C)} \\
&+ \frac{(\sqrt{2}J)^2}{\Delta_T + \Delta_C - (2\Delta_T - \alpha_T)} + \frac{(\sqrt{2}\Omega_C)^2}{\Delta_T + \Delta_C - (\Delta_T + 2\Delta_C - \alpha_C)} + \\
&+ \frac{(\sqrt{2}\Omega_T)^2}{\Delta_T + \Delta_C - (\Delta_C + 2\Delta_T - \alpha_T)} \tag{A.20}
\end{aligned}$$

For $|02\rangle$,

Start	I1	End
02	11	02

$$\begin{aligned}
E_{02}^{(2)} &= \frac{\langle 02|V|11|02|V|11\rangle \langle 11|V|02|11|V|02\rangle}{E_{02}^{(0)} - E_{11}^{(0)}} \\
&= -\frac{2J^2}{\Delta + \alpha_T} \tag{A.21}
\end{aligned}$$

For $|20\rangle$

Start	I1	End
20	11	20

$$\begin{aligned}
E_{20}^{(2)} &= \frac{\langle 20|V|11|20|V|11\rangle \langle 11|V|20|11|V|20\rangle}{E_{20}^{(0)} - E_{11}^{(0)}} \\
&= \frac{2J^2}{\Delta - \alpha_C} \tag{A.22}
\end{aligned}$$

For $|21\rangle$

Start	I1	End
21	12 30	21

$$\begin{aligned}
E_{21}^{(2)} &= \frac{\langle 21|V|12|21|V|12\rangle \langle 12|V|21|12|V|21\rangle}{E_{21}^{(0)} - E_{12}^{(0)}} + \frac{\langle 21|V|30|21|V|30\rangle \langle 30|V|21|30|V|21\rangle}{E_{21}^{(0)} - E_{30}^{(0)}} \\
&= \frac{4J^2}{\Delta - \alpha_C + \alpha_T} + \frac{3J^2}{2\alpha_C - \Delta}
\end{aligned} \tag{A.23}$$

For $|12\rangle$

Start	I1	End
12	21 03	12

$$\begin{aligned}
E_{12}^{(2)} &= \frac{\langle 12|V|21|12|V|21\rangle \langle 21|V|12|21|V|12\rangle}{E_{12}^{(0)} - E_{21}^{(0)}} + \frac{\langle 12|V|03|12|V|03\rangle \langle 03|V|12|03|V|12\rangle}{E_{12}^{(0)} - E_{03}^{(0)}} \\
&= \frac{4J^2}{-\Delta + \alpha_C - \alpha_T} + \frac{3J^2}{2\alpha_T + \Delta}
\end{aligned} \tag{A.24}$$

Finally, for $|22\rangle$

Start	I1	End
22	13 31	22

$$\begin{aligned}
E_{22}^{(2)} &= \frac{\langle 22|V|13|22|V|13\rangle \langle 13|V|22|13|V|22\rangle}{E_{22}^{(0)} - E_{13}^{(0)}} + \frac{\langle 22|V|31|22|V|31\rangle \langle 31|V|22|31|V|22\rangle}{E_{22}^{(0)} - E_{31}^{(0)}} \\
&= \frac{6J^2}{\Delta + 2\alpha_T - \alpha_C} + \frac{6J^2}{2\alpha_C - \alpha_T - \Delta}
\end{aligned} \tag{A.25}$$

Now the $\alpha_{ij}^{(2)}$ may be computed. Of course, α_{11} is the same as the qubit case

$$\alpha_{11}^{(2)} = \zeta_{ZZ}^{(2)} = 2J^2 \left[\frac{1}{\alpha_C - \Delta} + \frac{1}{\alpha_T + \Delta} \right]. \tag{A.26}$$

The new information is contained in

$$\begin{aligned}
\alpha_{21}^{(2)} &= E_{21}^{(2)} + E_{00}^{(2)} - E_{01}^{(2)} - E_{20}^{(2)} \\
&= J^2 \left[\frac{4}{\Delta - \alpha_C + \alpha_T} + \frac{3}{2\alpha_C - \Delta} + \frac{1}{\Delta} - \frac{2}{\Delta - \alpha_C} \right]
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
\alpha_{12}^{(2)} &= E_{12}^{(2)} + E_{00}^{(2)} - E_{10}^{(2)} - E_{02}^{(2)} \\
&= J^2 \left[\frac{4}{-\Delta - \alpha_T + \alpha_C} + \frac{3}{2\alpha_T + \Delta} - \frac{1}{\Delta} + \frac{2}{\Delta + \alpha_T} \right]
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
\alpha_{22}^{(2)} &= E_{22}^{(2)} + E_{00}^{(2)} - E_{02}^{(2)} - E_{20}^{(2)} \\
&= J^2 \left[\frac{6}{\Delta + 2\alpha_T - \alpha_C} + \frac{6}{2\alpha_C - \alpha_T - \Delta} + \frac{2}{\Delta + \alpha_T} - \frac{2}{\Delta - \alpha_C} \right]
\end{aligned} \tag{A.29}$$

These don't simplify much, so let's see what happens when $\alpha_C = \alpha_T = \alpha$

$$\alpha_{11}^{(2)} = \frac{4\alpha J^2}{(\alpha - \Delta)(\alpha + \Delta)} \tag{A.30}$$

$$\alpha_{21}^{(2)} = J^2 \left[\frac{5}{\Delta} + \frac{3}{2\alpha - \Delta} + \frac{2}{\alpha - \Delta} \right] = \frac{2\alpha J^2(5\alpha - 4\Delta)}{\Delta(\alpha - \Delta)(2\alpha - \Delta)} \tag{A.31}$$

$$\alpha_{12}^{(2)} = J^2 \left[-\frac{5}{\Delta} + \frac{3}{2\alpha + \Delta} + \frac{2}{\alpha + \Delta} \right] = -\frac{2\alpha J^2(5\alpha + 4\Delta)}{\Delta(\alpha + \Delta)(2\alpha + \Delta)} \tag{A.32}$$

$$\alpha_{22}^{(2)} = \frac{16\alpha J^2}{(\alpha + \Delta)(\alpha - \Delta)} \tag{A.33}$$

A.3 Third Order Corrections

We now work on the third order corrections, which again, appear due to the fact that there is a drive on both qutrits. There are $\Omega - J - \Omega$, $\Omega - \Omega - J$ and $J - \Omega - \Omega$ terms as there were in the qubit case.

For $|20\rangle$

Start	I1	I2	End
20	30	21	20
	11	21	
		10	
	21	11	
		30	
	10	11	

$$\begin{aligned}
E_{20}^{(3)} &= \frac{\langle 20|V|30\rangle\langle 20|V|30\rangle\langle 30|V|21\rangle\langle 30|V|21\rangle\langle 21|V|20\rangle\langle 21|V|20\rangle}{(E_{20}^{(0)} - E_{30}^{(0)})(E_{20}^{(0)} - E_{21}^{(0)})} \\
&+ \frac{\langle 20|V|11\rangle\langle 20|V|11\rangle\langle 11|V|21\rangle\langle 11|V|21\rangle\langle 21|V|20\rangle\langle 21|V|20\rangle}{(E_{20}^{(0)} - E_{11}^{(0)})(E_{20}^{(0)} - E_{21}^{(0)})} \\
&+ \frac{\langle 20|V|11\rangle\langle 20|V|11\rangle\langle 11|V|10\rangle\langle 11|V|10\rangle\langle 10|V|20\rangle\langle 10|V|20\rangle}{(E_{20}^{(0)} - E_{10}^{(0)})(E_{20}^{(0)} - E_{11}^{(0)})} + \text{h.c.} \\
&= \Omega_C \Omega_T J \left[\frac{e^{-i\phi} \sqrt{3} \sqrt{3}}{(2\alpha_C - \Delta_C)(-\Delta_T)} + \frac{e^{-i\phi} \sqrt{2} \sqrt{2}}{(\Delta - \alpha_C)(-\Delta_T)} \right. \\
&\quad \left. + \frac{\sqrt{2} e^{-i\phi} \sqrt{2}}{(\Delta_C - \alpha_C)(\Delta - \alpha_C)} + \text{h.c.} \right] \\
&= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{3}{(2\alpha_C - \Delta_C)\Delta_T} + \frac{2}{(\alpha_C - \Delta)\Delta_T} + \frac{2}{(\alpha_C - \Delta_C)(\alpha_C - \Delta)} \right] \quad (\text{A.34})
\end{aligned}$$

Swapping control and target yields

$$E_{02}^{(3)} = 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{3}{(2\alpha_T - \Delta_T)\Delta_C} + \frac{2}{(\alpha_T + \Delta)\Delta_C} + \frac{2}{(\alpha_T - \Delta_T)(\alpha_T + \Delta)} \right] \quad (\text{A.35})$$

Next for $|21\rangle$

Start	I1	I2	End
21	12	22	21
		11	
	30	20	
		31	
	31	22	
		30	
	22	12	
		31	
	20	30	
		11	
	11	20	
		12	

$$\begin{aligned}
E_{21}^{(3)} &= \frac{\langle 21|V|12|21|V|12\rangle \langle 12|V|22|12|V|22\rangle \langle 22|V|21|22|V|21\rangle}{(E_{21}^{(0)} - E_{12}^{(0)})(E_{21}^{(0)} - E_{22}^{(0)})} \\
&+ \frac{\langle 21|V|12|21|V|12\rangle \langle 12|V|11|12|V|11\rangle \langle 11|V|21|11|V|21\rangle}{(E_{21}^{(0)} - E_{12}^{(0)})(E_{21}^{(0)} - E_{11}^{(0)})} \\
&+ \frac{\langle 21|V|30|21|V|30\rangle \langle 30|V|20|30|V|20\rangle \langle 20|V|21|20|V|21\rangle}{(E_{21}^{(0)} - E_{30}^{(0)})(E_{21}^{(0)} - E_{20}^{(0)})} \\
&+ \frac{\langle 21|V|30|21|V|30\rangle \langle 30|V|31|30|V|31\rangle \langle 31|V|21|31|V|21\rangle}{(E_{21}^{(0)} - E_{30}^{(0)})(E_{21}^{(0)} - E_{31}^{(0)})} \\
&+ \frac{\langle 21|V|31|21|V|31\rangle \langle 31|V|22|31|V|22\rangle \langle 22|V|21|22|V|21\rangle}{(E_{21}^{(0)} - E_{31}^{(0)})(E_{21}^{(0)} - E_{22}^{(0)})} \\
&+ \frac{\langle 21|V|20|21|V|20\rangle \langle 20|V|11|20|V|11\rangle \langle 11|V|21|11|V|21\rangle}{(E_{21}^{(0)} - E_{20}^{(0)})(E_{21}^{(0)} - E_{11}^{(0)})} + \text{h.c.} \\
&= \Omega_C \Omega_T J \left[\frac{\sqrt{2}e^{-i\phi}\sqrt{2}2}{(\Delta - \alpha_C + \alpha_T)(\alpha_T - \Delta_T)} + \frac{\sqrt{2}\sqrt{2}e^{-i\phi}2}{(\Delta - \alpha_C + \alpha_T)(\Delta_C - \alpha_C)} \right. \\
&\quad + \frac{e^{i\phi}\sqrt{3}\sqrt{3}}{(2\alpha_C - \Delta)(\Delta_T)} + \frac{\sqrt{3}e^{i\phi}\sqrt{3}}{(2\alpha_C - \Delta)(2\alpha_C - \Delta_C)} \\
&\quad \left. + \frac{\sqrt{2}e^{-\phi}\sqrt{6}\sqrt{3}}{(2\alpha_C - \Delta_C)(\alpha_T - \Delta_T)} + \frac{\sqrt{2}\sqrt{2}e^{-i\phi}}{\Delta_T(\Delta_C - \alpha_C)} \right] + \text{h.c.} \\
&= 2 \cos \phi \Omega_C \Omega_T J \left[\frac{4}{(\Delta - \alpha_C + \alpha_T)(\alpha_T - \Delta_T)} - \frac{4}{(\Delta - \alpha_C + \alpha_T)(\alpha_C - \Delta_C)} \right. \\
&\quad + \frac{3}{(2\alpha_C - \Delta)\Delta_T} + \frac{3}{(2\alpha_C - \Delta)(2\alpha_C - \Delta_C)} \\
&\quad \left. + \frac{6}{(2\alpha_C - \Delta_C)(\alpha_T - \Delta_T)} - \frac{2}{\Delta_T(\alpha_C - \Delta_C)} \right] \tag{A.36}
\end{aligned}$$

Exchanging the control and target yields

$$\begin{aligned}
E_{12}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{4}{(\Delta + \alpha_T - \alpha_C)(\alpha_C - \Delta_C)} + \frac{4}{(\Delta + \alpha_T - \alpha_C)(\alpha_T - \Delta_T)} \right. \\
&\quad + \frac{3}{(2\alpha_T + \Delta)\Delta_C} + \frac{3}{(2\alpha_T + \Delta)(2\alpha_T - \Delta_T)} + \frac{6}{(2\alpha_T - \Delta_T)(\alpha_C - \Delta_C)} - \frac{2}{\Delta_C(\alpha_T - \Delta_T)} \left. \right] \tag{A.37}
\end{aligned}$$

Finally we have $|22\rangle$.

Start	I1	I2	End
22	31	21 32	22
	13	12 23	
	32	23 31	
	23	32 13	
	12	21 13	
	21	12 31	

$$\begin{aligned}
E_{22}^{(3)} &= \frac{\langle 22|V|31|22|V|31\rangle \langle 31|V|21|31|V|21\rangle \langle 21|V|22|21|V|22\rangle}{(E_{22}^{(0)} - E_{31}^{(0)})(E_{22}^{(0)} - E_{21}^{(0)})} \\
&+ \frac{\langle 22|V|31|22|V|31\rangle \langle 31|V|32|31|V|32\rangle \langle 32|V|22|32|V|22\rangle}{(E_{22}^{(0)} - E_{31}^{(0)})(E_{22}^{(0)} - E_{32}^{(0)})} \\
&+ \frac{\langle 22|V|13|22|V|13\rangle \langle 13|V|12|13|V|12\rangle \langle 12|V|22|12|V|22\rangle}{(E_{22}^{(0)} - E_{13}^{(0)})(E_{22}^{(0)} - E_{12}^{(0)})} \\
&+ \frac{\langle 22|V|13|22|V|13\rangle \langle 13|V|23|13|V|23\rangle \langle 23|V|22|23|V|22\rangle}{(E_{22}^{(0)} - E_{13}^{(0)})(E_{22}^{(0)} - E_{23}^{(0)})} \\
&+ \frac{\langle 22|V|32|22|V|32\rangle \langle 32|V|23|32|V|23\rangle \langle 23|V|22|23|V|22\rangle}{(E_{22}^{(0)} - E_{32}^{(0)})(E_{22}^{(0)} - E_{23}^{(0)})} \\
&+ \frac{\langle 22|V|12|22|V|12\rangle \langle 12|V|21|12|V|21\rangle \langle 21|V|22|21|V|22\rangle}{(E_{22}^{(0)} - E_{12}^{(0)})(E_{22}^{(0)} - E_{21}^{(0)})} + \text{h.c.} \\
&= \Omega_C \Omega_T J \left[\frac{\sqrt{2}e^{i\phi}\sqrt{3}\sqrt{6}}{(2\alpha_C - \alpha_T - \Delta)(\Delta_T - \alpha_T)} + \frac{\sqrt{3}\sqrt{2}e^{i\phi}\sqrt{6}}{(2\alpha_C - \alpha_T - \Delta)(2\alpha_C - \Delta_C)} \right. \\
&\quad + \frac{\sqrt{2}\sqrt{3}e^{-i\phi}\sqrt{6}}{(\Delta + 2\alpha_T - \alpha_C)(\Delta_C - \alpha_C)} + \frac{\sqrt{3}e^{-i\phi}\sqrt{2}\sqrt{6}}{(\Delta + 2\alpha_T - \alpha_C)(2\alpha_T - \Delta_T)} \\
&\quad \left. + \frac{\sqrt{3}e^{-i\phi}3\sqrt{3}}{(2\alpha_T - \Delta_T)(2\alpha_C - \Delta_C)} + \frac{\sqrt{2}e^{i\phi}2\sqrt{2}}{(\Delta_T - \alpha_T)(\Delta_C - \alpha_C)} \right] + \text{h.c.} \\
&= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{6}{(2\alpha_C - \alpha_T - \Delta)(\alpha_T - \Delta_T)} + \frac{6}{(2\alpha_C - \alpha_T - \Delta)(2\alpha_C - \Delta_C)} \right. \\
&\quad - \frac{6}{(2\alpha_T - \alpha_C + \Delta)(\alpha_C - \Delta_C)} + \frac{6}{(2\alpha_T - \alpha_C + \Delta)(2\alpha_T - \Delta_T)} \\
&\quad \left. + \frac{9}{(2\alpha_T - \Delta_T)(2\alpha_C - \Delta_C)} + \frac{4}{(\alpha_T - \Delta_T)(\alpha_C - \Delta_C)} \right] \tag{A.38}
\end{aligned}$$

Again we are going to need $\alpha_T = \alpha_C = \alpha$ to get any simplification. The energies become

$$\begin{aligned}
E_{00}^{(3)} &= \frac{2 \cos \phi \Omega_C \Omega_T J}{\Delta_T \Delta_C} \\
E_{01}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{1}{\Delta \Delta_T} + \frac{1}{\Delta \Delta_C} - \frac{2}{\Delta_C(\alpha_T - \Delta_T)} \right] \\
E_{10}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[\frac{1}{\Delta \Delta_C} - \frac{1}{\Delta \Delta_T} - \frac{2}{\Delta_T(\alpha_C - \Delta_C)} \right] \\
E_{11}^{(1)} &= 2 \cos \phi \Omega_C \Omega_T J \left[\frac{2}{(\Delta - \alpha)(\Delta_C - \alpha)} + \frac{4}{(\Delta_T - \alpha)(\Delta_C - \alpha)} + \frac{2}{(\Delta + \alpha)(\alpha - \Delta_T)} \right. \\
&\quad \left. + \frac{2}{\Delta_C(\Delta + \alpha)} + \frac{2}{\Delta_T(\alpha - \Delta)} + \frac{1}{\Delta_T \Delta_C} \right] \\
E_{20}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{3}{(2\alpha - \Delta_C)\Delta_T} + \frac{2}{(\alpha - \Delta)\Delta_T} + \frac{2}{(\alpha - \Delta_C)(\alpha - \Delta)} \right] \\
E_{02}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{3}{(2\alpha - \Delta_T)\Delta_C} + \frac{2}{(\alpha + \Delta)\Delta_C} + \frac{2}{(\alpha - \Delta_T)(\alpha + \Delta)} \right] \\
E_{21}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[\frac{4}{\Delta(\alpha - \Delta_T)} - \frac{4}{\Delta(\alpha - \Delta_C)} + \frac{3}{(2\alpha - \Delta)\Delta_T} + \frac{3}{(2\alpha - \Delta)(2\alpha - \Delta_C)} \right. \\
&\quad \left. + \frac{6}{(2\alpha - \Delta_C)(\alpha - \Delta_T)} - \frac{2}{\Delta_T(\alpha - \Delta_C)} \right] \\
E_{12}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{4}{\Delta(\alpha - \Delta_C)} + \frac{4}{\Delta(\alpha - \Delta_T)} + \frac{3}{(2\alpha + \Delta)\Delta_C} + \frac{3}{(2\alpha + \Delta)(2\alpha - \Delta_T)} \right. \\
&\quad \left. + \frac{6}{(2\alpha - \Delta_T)(\alpha - \Delta_C)} - \frac{2}{\Delta_C(\alpha - \Delta_T)} \right] \\
E_{22}^{(3)} &= 2 \cos \phi \Omega_C \Omega_T J \left[-\frac{6}{(\alpha - \Delta)(\alpha - \Delta_T)} + \frac{6}{(\alpha - \Delta)(2\alpha - \Delta_C)} - \frac{6}{(\alpha + \Delta)(\alpha - \Delta_C)} \right. \\
&\quad \left. + \frac{6}{(\alpha + \Delta)(2\alpha - \Delta_T)} + \frac{9}{(2\alpha - \Delta_T)(2\alpha - \Delta_C)} + \frac{4}{(\alpha - \Delta_T)(\alpha - \Delta_C)} \right]
\end{aligned} \tag{A.39}$$

The α_{ij} driven terms which arise at third order are now calculated and as before we find

$$\alpha_{11}^d = \frac{8 \cos \phi \alpha^2 \Omega_C \Omega_T J}{\Delta_C \Delta_T (\alpha - \Delta_C)(\alpha - \Delta_T)} \quad (\text{A.40})$$

$$\alpha_{12}^d = \frac{8 \cos \phi \alpha \Omega_C \Omega_T J [\Delta_C^2 (\alpha - 2\Delta_T) + \alpha(2\alpha^2 - 2\alpha\Delta_T - \Delta_T^2) + \Delta_C(-3\alpha^2 + 2\alpha\Delta_T + 2\Delta_T^2)]}{\Delta_C \Delta_T (\alpha - \Delta_C)(\alpha - \Delta_T)(2\alpha - \Delta_C)(2\alpha - \Delta_T)} \quad (\text{A.41})$$

$$\alpha_{21}^d = \frac{8 \cos \phi \alpha \Omega_C \Omega_T J [\Delta_T^2 (\alpha - 2\Delta_C) + \alpha(2\alpha^2 - 2\alpha\Delta_C - \Delta_C^2) + \Delta_T(-3\alpha^2 + 2\alpha\Delta_C + 2\Delta_C^2)]}{\Delta_C \Delta_T (\alpha - \Delta_C)(\alpha - \Delta_T)(2\alpha - \Delta_C)(2\alpha - \Delta_T)} \quad (\text{A.42})$$

$$\alpha_{22}^d = \frac{8 \cos \phi \alpha^2 \Omega_C \Omega_T J (\alpha - 2\Delta_C)(\alpha - 2\Delta_T)}{\Delta_C \Delta_T (\alpha - \Delta_C)(\alpha - \Delta_T)(2\alpha - \Delta_C)(2\alpha - \Delta_T)} \quad (\text{A.43})$$

$$\alpha_{11}^d = \frac{8 \cos \phi \alpha^2 \Omega_C \Omega_T J}{\Delta_C \Delta_T (\alpha - \Delta_C)(\alpha - \Delta_T)} \quad (\text{A.44})$$

$$\alpha_{12}^d / \alpha_{11}^d = \frac{[\Delta_C^2 (\alpha - 2\Delta_T) + \alpha(2\alpha^2 - 2\alpha\Delta_T - \Delta_T^2) + \Delta_C(-3\alpha^2 + 2\alpha\Delta_T + 2\Delta_T^2)]}{(2\alpha - \Delta_C)(2\alpha - \Delta_T)} \quad (\text{A.45})$$

$$\alpha_{21}^d / \alpha_{11}^d = \frac{[\Delta_T^2 (\alpha - 2\Delta_C) + \alpha(2\alpha^2 - 2\alpha\Delta_C - \Delta_C^2) + \Delta_T(-3\alpha^2 + 2\alpha\Delta_C + 2\Delta_C^2)]}{(2\alpha - \Delta_C)(2\alpha - \Delta_T)} \quad (\text{A.46})$$

$$\alpha_{22}^d / \alpha_{11}^d = \frac{(\alpha - 2\Delta_C)(\alpha - 2\Delta_T)}{(2\alpha - \Delta_C)(2\alpha - \Delta_T)} \quad (\text{A.47})$$

Appendix B

Quasiprobability Functions

In this Appendix section we provide more context for the quasiprobability functions employed to represent qudit states in Chapter 5. In the realm of quantum mechanics, the choice of a representation is often guided by practical considerations. In this vein, the concept of coherent states has emerged as a fundamental framework for standard descriptions within the field of quantum optics, owing to their attractive characteristics: (i) They are evolved from the vacuum state by a unitary operator, and are minimum-uncertainty states. (ii) They obey a completeness relation and thus form a good set of basis states. (iii) They provide a quantum approach to conceptualize classical fields. Notably, macroscopic quantum states of atomic ensembles can also be described by such a concept. This duality is related to the group contraction of spin operators based on the angular momentum algebra to bosonic operators based on the harmonic-oscillator algebra [9].

A *spin coherent state* (SCS) $|\theta, \phi\rangle$ is defined as an eigenstate of a spin component in the (θ, ϕ) direction, $S_{\theta, \phi} = S_x \sin \theta \cos \phi + S_y \sin \theta \sin \phi + S_z \cos \theta$. Here, θ denotes the polar angle, ϕ denotes the azimuthal angle, and the spin system obeys the cyclic commutation relation $[S_i, S_j] = i\epsilon_{ijk}S_k$, where ϵ_k is the Levi-Civita symbol. The associated uncertainty relation is $\langle \Delta S_i^2 \rangle \langle \Delta S_j^2 \rangle \geq \frac{1}{4} |\langle S_k \rangle|^2$. An SCS satisfies the minimal uncertainty relation with equal uncertainties in both orthogonal components, corresponding to an isotropic Husimi-Q quasiprobability distribution in the spherical phase space [9, 135], as shown in Figure B.1a. In the present framework, we project a high-dimensional spin- j state $|j\rangle$, where $j = \frac{1}{2}(d-1)$, onto an SCS for useful visualization using the generalized Bloch sphere,

$$|\theta, \phi\rangle = \exp \left[\frac{1}{2} \theta (e^{i\phi} J_- - e^{-i\phi} J_+) \right] |j\rangle, \quad (\text{B.1})$$

where $J_{\pm} = J_x \pm iJ_y$ are the raising/lowering angular momentum operators, and we denote $|j\rangle \equiv |j, m=j\rangle$ for a general spin- j system [9].

An ensemble of n pure SCS's simply form a product state,

$$\prod_{k=0}^{n-1} |\theta_k, \phi_k\rangle_k = \prod_{k=0}^{n-1} \left[\cos \frac{\theta_k}{2} |0\rangle_k + \sin \frac{\theta_k}{2} e^{i\phi_k} |1\rangle_k \right], \quad (\text{B.2})$$

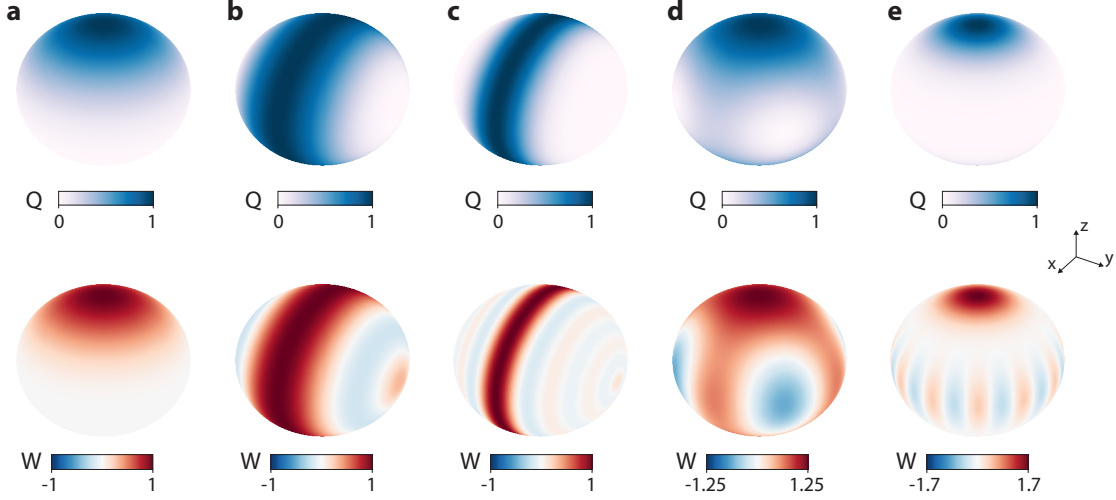


Figure B.1: **Quasiprobability distributions of high-dimensional states.** The top row shows the Husimi-Q distributions. The bottom row shows the Wigner functions. (a) Coherent state $|00\rangle$ with $d = 3$. (b) Bell_3 state. (c) Bell_7 state. (d) Cat_3 state with $n = 2$. (e) Cat_7 state with $n = 2$.

where $|0\rangle_k$ and $|1\rangle_k$ are eigenstates of S_z in the k th SCS. If all the individual constituents are aligned along the same direction (θ, ϕ) , the resulting state becomes the collective SCS,

$$|\theta, \phi\rangle = \left[\cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle \right]^{\otimes n}. \quad (\text{B.3})$$

This alignment can be conveniently realized via initialization of all the qudits to the same state. For a collection of spin- j system with dimension $d+1$, if the basis SCS is projected from $|jj\dots j\rangle$ with $j = d/2$, then the high-dimensional states with isotropic QPD are $|00\dots 0\rangle$ and $|dd\dots d\rangle$.

Starting from the optical coherent state, various interesting states of light can be realized. We presently explore the duality of these states in qudit systems starting from the SCS. An important concept in quantum optics is *squeezing*, which is paradigmatically generated by the operator $\hat{\mathcal{S}}(\zeta) = \exp \left[\frac{1}{2}(\zeta \hat{a}^{\dagger 2} - \zeta^* \hat{a}^2) \right]$, where ζ is the squeezing parameter and $\hat{a}$ ($\hat{a}^\dagger$) is the bosonic annihilation (creation) operator. Specifically, the photonic two-mode squeezing process reads $\hat{\mathcal{S}}(\zeta) = \exp(\zeta \hat{a}^\dagger \hat{b}^\dagger - \zeta^* \hat{a} \hat{b})$, where ζ is the squeezing strength, and $\hat{a}$ ($\hat{b}$) is the annihilation operator of the first (second) mode.

The process results in the two-mode squeezed state, $|\Psi\rangle_{2\text{ms}} \propto \sum_{N=0}^{\infty} c_N |NN\rangle$, where c_N is a coefficient dependent on ζ and N is the quanta number [36]. In a coupled system comprising of two qudits with dimension d , if we draw the duality of N ranging from 0 to $d-1$, the high-dimensional Bell_d state would be close to spin squeezed state (SSS). As opposed to the SCS, the SSS is signified by the shrinking of its Husimi-Q distribution along

a geodesic, which is further enhanced by increasing the dimension of the system, as shown in Figure B.1b,c.

Another important bosonic state of immense interest is the Schrödinger cat state. In atomic systems, this represents a superposition of macroscopic SCS's. The structure of high-dimensional systems and the two-photon dynamics suggest that Schrödinger cat states can be realized efficiently in qudits. We demonstrate this by creating the qudit Schrödinger cat states, which are superpositions of the qudit coherent states, as discussed in the main text. Importantly, the size of the realized cat depends equally on the dimension d and the number of parties n . We provide a gallery of Wigner functions for different cat states in Figure B.1d,e.

Quasiprobability distributions

The concepts of squeezing and macroscopic superpositions are more conveniently represented using quasiprobability distribution functions in phase space, such as the Husimi-Q function or the Wigner function. In particular, the Husimi-Q distribution is frequently employed to display the macroscopic aspect of entangled quantum systems, due to its relatively straightforward calculation procedure via the formula

$$Q(\theta, \phi) \propto \langle \theta, \phi | \hat{\rho} | \theta, \phi \rangle. \quad (\text{B.4})$$

This involves computing the overlap probability between the target state specified by $\hat{\rho}$ and the SCS $|\theta, \phi\rangle$ as the basis state. The Husimi-Q function is nevertheless not suitable to exhibit the non-classical feature of macroscopic superposition states such as the Schrödinger cats, as showcased in Figure B.1d,e.

The Wigner function's ability to take on negative values, on the other hand, has proven to be transformative in the visualization of quantum correlations. It has thereby become an indispensable tool in the analysis of cat states. The Wigner function can be viewed as the expectation value of a normalized parity operator, which renders it possible to be represented using different bases. Here, we follow the convention using tensor product kernels prescribed by Ref. [216],

$$W_{\hat{\rho}}(\theta, \phi) = \langle \hat{U}(\theta, \phi) \hat{\Pi} \hat{U}^\dagger(\theta, \phi) \rangle_{\hat{\rho}}. \quad (\text{B.5})$$

Here, $\hat{\rho}$ is the density matrix, $\hat{U}$ is the general displacement or rotation operator, and $\hat{\Pi}$ can be viewed as the spin parity operator, analogous to the bosonic case. These operators are not unique, and they only have to obey the Stratonovich-Weyl correspondence. Our capability to measure high-dimensional systems motivates the use of a tensor product of spins. This simply involves laying out the $\text{SU}(d)$ rotation of a single-qudit, using $\hat{\Pi}_{\otimes^n \text{SU}(d)} = \otimes^n \hat{\Pi}_{\text{SU}(d)}$ to achieve

$$W_{\otimes^n \text{SU}(d)}(\theta, \phi) = \text{Tr} \left[\hat{U}(\theta, \phi) \hat{\rho} \hat{U}^\dagger(\theta, \phi) \hat{\Pi}_{\otimes^n \text{SU}(d)} \right]. \quad (\text{B.6})$$

Following this convention, we can use the predefined functions in QuTiP [122] to compute the QPD for density matrices of arbitrary spin dimensions. Our analysis involves the normalization of the Husimi-Q QPD values such that the highest probability amplitude is unity.

The computed Wigner function is kept as-is due to the physical significance of the values. A gallery of both functions for two-mode qudit states is displayed in Figure B.1.