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Convex subspaces of Lie incidence geometries

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Abstract. We classify the convex subspaces of all hexagonic Lie incidence geometries (among which all long root geometries of spherical Tits-buildings). We perform a similar classification for most other Lie incidence geometries of spherical Tits-buildings, in particular for all projective and polar Grassmannians, and for exceptional Grassmannians of diameter at most 3.

Keywords. Buildings, parapolar spaces, long root geometries, hexagonal Lie incidence geometries

Mathematics Subject Classifications. 51E24

1. Introduction

Spherical Tits-buildings are combinatorial objects geometrically interpreting groups of Lie type, among which the Chevalley groups, semisimple linear algebraic groups, the classical groups, and groups of mixed type. A further "geometrization" of these objects was obtained by the so-called point-line approach to buildings of the last century starting with the seminal work of Buekenhout–Shult (introducing a simple axiom system in terms of points and lines for polar spaces), culminating in the introduction of the notion of "parapolar spaces" by Cooperstein. This theory in particular contains the so-called *long root geometries* which can be obtained from a simple Chevalley group by calling the long root subgroups "points" and the product of two distinct elementwise commuting such subgroups "lines", with natural incidence. These are the main examples of the more abstractly defined *hexagonic geometries*, see below. The interplay between these geometrical objects and the corresponding groups and algebras is a fascinating phenomenon. For instance, [Coh21] points out a connection between inner ideals generated by extremal elements of a simple Lie algebra associated to some simple algebraic groups and flags

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of the corresponding spherical Tits-building. In the proof of Theorem 4.1 of *loc.cit.*, convex subspaces, satisfying some additional properties, of some Lie incidence geometries (more exactly, hexagonic geometries) need to be recognized. This would have been simpler if one had at their disposal a list of all convex subspaces of the Lie incidence geometries in question. Part of our motivation is to fill that gap in the literature and provide such a classification for all interesting Lie incidence geometries, in particular for the long root geometries, or, more generally, for the hexagonic Lie incidence geometries. Briefly, we will show the following result.

Main result 1. *An arbitrary convex subspace of a hexagonic Lie incidence geometry containing polar spaces of rank at least* 3 *either corresponds to a residue of a flag in the associated spherical building (which we assume to be thick and irreducible), or consists of a union of lines pairwise intersecting in the same point* c *such that, for each pair of points from distinct lines,* c *is the unique point collinear to both.*

Pairs of non-collinear points collinear to a unique common point are called *special*. The requirement of containing a polar space of rank at least 3 is necessary. For instance, the geometries arising from the point-hyperplane pairs of projective spaces are long root geometries and contain additional convex subspaces, see Theorem 9.1 for more details.

There is also a cosmetic motivation. One knows that the singular subspaces (which are automatically convex) of a given Lie incidence geometry can be read off the Coxeter diagram of the corresponding building as residues of flags of certain types (this follows from Theorem 10.2.10 in [BCN89]). In the above Main Result 1, a similar thing for convex subspaces is almost true; there is only one type of exceptions. We can ask ourselves for which Lie incidence geometries there are no exceptions. If we call a Lie incidence geometry *grammatical* if all its convex subspaces correspond to residues of the corresponding building (hence if the convex subspaces can be read off the Coxeter diagram), then we will classify all grammatical Lie incidence geometries. This is our Main Result 2.

Main result 2. *The grammatical Lie incidence geometries are precisely the Lie incidence geometries without special pairs.*

We will present a precise detailed list later on, and we note that this list, in the case that the building has rank at least 3, coincides with the list of all Lie incidence geometries whose point graph is distance transitive (or in the finite case, equivalently, distance regular), see Theorem 10.4.6 in [BCN89]. This reflects in a certain sense the observation that the obstruction for the point graph being distance transitive is always already visible at distance 2, where there are special pairs (giving rise to convex subspaces not obtained from a residue) and non-special ones (the convex closure of which conforms to a residue).

Finally, the third motivation for our work is to unify and extend classifications and characterizations of convex subspaces of certain Lie incidence geometries in the literature. For instance, Pankov [Pan12, Corollary 4.2] proves that a certain isomorphism class of convex subspaces of polar Grassmannians arises as a class of residues of the corresponding building; Kasikova [Kas09, Corollary 6.3] roughly proves a similar statement for a class of Lie incidence geometries arising from buildings with simply laced diagram of Y-shape. These two results are now also corollaries of our results, the main and most popular part of which can be stated roughly as follows (here we exclude the geometries already mentioned in Main Results 1 and 2):

Main result 3. *Any convex subspace of a polar Grassmannian not related to the next-to-maximal singular subspaces either corresponds to a residue of the underlying spherical building, or else is the union of singular subspaces pairwise intersecting in a fixed point* p*, such that any pair of points in distinct such subspaces is special.*

The case of a polar Grassmannian related to the next-to-maximal singular subspaces is more complicated and technical, see Theorem 8.4.

While our Main Results characterise subspaces arising from residues of the underlying spherical building in terms of convexity, also characterizations in terms of isomorphism classes can be found in the literature, see e.g. [Kas09, Corollary 6.4] and [Kas13, Corollary 1.1].

Outline of the paper. In the next section, we introduce the Lie incidence geometries and the language of the parapolar spaces, assuming the reader is more or less familiar with some of the theory of spherical buildings. However, we also make an effort to make our results accessible to readers only familiar with the classical geometries, i.e., projective spaces and their Grassmannians, and polar spaces and their Grassmannians; the results about such geometries are proved without reference to parapolar theory.

In Section 3, we state more precise and detailed versions of the main results above, using the language of parapolar spaces and Lie incidence geometries. Section 4 gathers some basic properties of some specific Lie incidence geometries that we will need. In Section 5, we contribute to the proof of Main Result 2 by treating projective Grassmannians. This section only needs knowledge of projective spaces. The result is also used in further proofs. The proof of Main Result 1 is the content of Section 6. Along the way, we also treat some grammatical Lie incidence geometries, and we end by completing the proof of Main Result 2. Section 7 treats the polar Grasmmannians, except for the case related to the next-to-maximal subspaces, which is done in Section 8. Finally, in Section 9, we classify the convex subspaces of the (hexagonic) geometry of point-hyperplane flags of projective spaces. The last three sections, as well as the auxiliary results of the other sections dealing only with classical types (as e.g. Proposition 6.10), i.e., the underlying building corresponds to a projective or polar space, are written in the language of projective and polar spaces; the reader only interested in these classical cases can skip the preliminaries about parapolar spaces and Lie incidence geometries.

2. Preliminaries

A *point-line geometry* $\Gamma = (X, \mathcal{L}, *)$ consists of a point set X together with a set $\mathcal{L}$ of lines and an incidence relation $*$ between X and $\mathcal{L}$. If no two lines are incident with exactly the same points, then we can identify each line with the set of points incident with it. In that case, $\mathcal{L}$ is a set of subsets of X and incidence is containment; we delete the $*$ in the notation and write $\Gamma = (X, \mathcal{L})$. *Collinear* points are points incident with the same line and if two points are always on at most one line, then we say that Γ is a *partial linear space*. A *subspace* Γ' of Γ is a set of points with the property that, as soon as two collinear points belong to Γ', then also all points on all lines incident with both of these two points belong to Γ' (often we will also consider the lines contained in Γ' and consider Γ' as a point-line subgeometry of Γ, whence the notation Γ').

A subspace with the property that every pair of points is collinear is called a *singular subspace*. The *collinearity graph* of Γ is the graph on the points, adjacent when collinear. A point-line geometry is called *connected* if its collinearity graph is connected. A *convex* set of points is a set of points closed under taking shortest paths in the collinearity graph of any pair of points of the set. As usual, the *convex closure* of a set S of points is the intersection of all convex sets containing S. We are interested in sets of points which are both subspaces and convex. The *convex subspace closure* is defined in the obvious way.

We assume the reader to be familiar with the theory of Tits-buildings and polar spaces, see e.g. [Tit74], [AB08], [Wei03]. For our purposes, a polar space is always non-degenerate and every of its lines has at least three points. Let Δ be an irreducible thick spherical building of rank n. We consider Δ as a numbered simplicial complex over the type set S, where the numbering of the vertices corresponds to the Bourbaki labelling ([Bou68]) of the corresponding (connected) Coxeter diagram X_n (for our purpose we do not need Dynkin diagrams). We choose a subset $T \subseteq S$ of the types and declare all flags of type T to be the points of a geometry Δ_T, where the lines are determined by the flags of type $S \setminus \{t\}$, for $t \in T$: the points of Δ_T lying on the line corresponding to such a flag f' are the flags f of type T with the property that $f \cup f'$ is a chamber (a flag of type S). Note that different f' can lead to identical lines in Δ_T. The resulting geometry is called the *T-Grassmannian of* Δ. In general, we write $\tau(\Delta_T) = \mathsf{X}_{n,T}$. If $T = \{t\}$, then we also write $\mathsf{X}_{n,t}$.

Often T is very small, and the interesting examples all have $|T| = 1$, except if $\mathsf{X}_n = \mathsf{A}_n$ and $T = \{1, n\}$. It are precisely these examples that we will refer to as *Lie incidence geometries*:

Definition 2.1. *A* Lie incidence geometry *is a point-line geometry associated with* $\mathsf{X}_{n,T}$, *where* X_n *is an irreducible spherical Coxeter diagram and* $|T| = 1$ *or* $\mathsf{X}_{n,T} = \mathsf{A}_{n,\{1,n\}}$.

Note that we only consider thick buildings, hence diagrams of type H_3 and H_4 do not occur here.

These interesting cases are examples of parapolar spaces.

Definition 2.2. *A* parapolar space *is a connected point-line geometry* $(X, \mathcal{L})$ *such that the convex subspace closure of a pair of points at distance 2 from each other in the collinearity graph, is either a polar space (called a* symplecton*, or* symp *for short; the pair of points is called* symplectic*), or the union of two intersecting lines (then the pair of points is called* special*), and such that each line is contained in a symplecton.*

Since polar spaces are partial linear spaces, the last condition implies that a parapolar space is automatically a partial linear space. One often also assumes that a parapolar space is not a polar space, but this is not important for our purposes. The advantage of working within the framework of parapolar spaces is that we can deal at the same time with a whole family of Lie incidence geometries sharing some properties, see for instance Lemma 6.9. We will need the following terminology.

A parapolar space without special pairs is called *strong*. If the rank of any symplecton is at least r, then we say that the parapolar space has *(symplectic) rank at least* r. The *diameter* of a parapolar space is the diameter of its collinearity graph. A polar space isomorphic to a hyperbolic quadric is simply called *hyperbolic*. We will denote the unique symp containing the symplectic pair of points x, y by $\xi(x, y)$. The parapolar spaces we will encounter will all have the property that each singular subspace is a projective space (this is automatic if the symplectic rank is at least 3, see Theorem 13.4.1(2) of [Shu11]). In this case there is an obvious notion of *point residual*: For an arbitrary point p of such a parapolar space Γ the point residual $\mathrm{Res}_\Gamma(p)$ is the point-line geometry with point set the lines passing through p and line set the (full projective) planes passing through p. In general, a point residual of a Lie incidence geometry is not a Lie incidence geometry in the narrow sense as we defined, but rather a direct product of several ones; it is a Lie incidence geometry if the type of the points of Γ in the underlying building corresponds to an end-node of the Coxeter diagram.

Note that we always mention dimensions pojectively. Hence the points of the $(i+1)$-Grassmannian of a projective space are the i-dimensional subspaces, or i-subspaces for short.

There is an axiomatic notion of "hexagonic" parapolar spaces, see Chapter 17 of [Shu11]. Slightly abusing that definition, we define:

Definition 2.3. *A* hexagonic geometry *is any Lie incidence geometry of type* $\mathsf{A}_{n,\{1,n\}}$, $n \geqslant 2$, $\mathsf{B}_{n,2}$, $n \geqslant 3$, $\mathsf{D}_{n,2}$, $n \geqslant 4$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{8,8}$, $\mathsf{F}_{4,1}$ *or* $\mathsf{G}_{2,1}$.

Note that we deal with Coxeter diagrams, so type $\mathsf{F}_{4,4}$ coincides with type $\mathsf{F}_{4,1}$ by symmetry of the Coxeter diagram. A hexagonic geometry shares important properties with the so-called "long root" geometries, but we do not insist on that. However, see [KS02].

For more background on parapolar spaces, hexagonic geometries and proofs of the above mentioned facts we refer to [Shu11], Chapters 13 and 17. In Section 4, we have collected many properties of parapolar spaces and some specific Lie incidence geometries that we will need in our proofs.

3. Main results

We now make the brief versions of our Main Results more explicit. We start with the hexagonic geometries. These were our main target. However, our methods require to also classify the convex subspaces of some other Lie incidence geometries and so we deal with all classical cases and some other exceptional ones.

More precisely, we will prove the following classifications. Note that, since the full space and the empty subspace are always convex subspaces corresponding to the residue of the empty flag and a chamber, respectively, we can restrict to *proper* convex subspaces, meaning exactly those distinct from the whole space and the empty space.

Theorem 3.1. *Let* $\Gamma = (X, \mathcal{L})$ *be a hexagonic Lie incidence geometry with no rank* 2 *symplecta and* $\Gamma' = (X', \mathcal{L}')$ *a proper convex subspace, viewed as a subgeometry. Then either* Γ' *corresponds to a residue in the underlying building geometry, or there exists a unique point* $c \in X'$ *such that* $\mathcal{L}'$ *is a set of lines of cardinality at least* 2*, each line contains* c *and every point pair in* X' *on distinct such lines is a special pair.*

The proof of Theorem 3.1 will be achieved in Section 6.4. The theorem also holds for the Lie incidence geometries of type $\mathsf{G}_{2,1}$ and $\mathsf{A}_{2,\{1,2\}}$ if we consider, in these geometries, any pair of points at distance 2 in the collinearity graph as special. The hexagonic Lie incidence geometries with rank 2 symplecta are the ones of type $\mathsf{A}_{n,\{1,n\}}$, $n \geqslant 3$, and $\mathsf{B}_{3,2}$. The statements for these cases require some more preliminaries and we simply refer to Lemma 6.6 and Theorem 9.1 for the precise results.

Remark 3.2. *It may be of some interest to know precisely what the subspaces Γ' of a hexagonic Lie incidence geometry Γ with no rank 2 symplecta corresponding to a residue of the underlying spherical building are. For the convenience of the reader, and for further reference, we enumerate these here:*

(1) Γ' *is a singular subspace;*
(2) Γ' *is a symp;*
(3) $\tau(\Gamma) = \mathsf{B}_{n,2}$ *and* $\tau(\Gamma') = \mathsf{A}_{i,2}$, $4 \leqslant i \leqslant n-1$. *The subgeometry* Γ' *is obtained from the underlying building geometry by taking a residue of type* $\{i+1,\ldots,n\}$*;*
(4) $\tau(\Gamma) = \mathsf{D}_{n,2}$ *and* $\tau(\Gamma') = \mathsf{A}_{i,2}$, $4 \leqslant i \leqslant n-1$. *The subgeometry* Γ' *is obtained from the underlying building geometry by taking a residue of type* $\{i+1,\ldots,n\}$*, or, for* $i = n-1$*, also a residue of type* $n-1$ *works;*
(5) $\tau(\Gamma) = \mathsf{E}_{6,2}$ *and* $\tau(\Gamma') = \mathsf{D}_{5,5}$. *The subgeometry* Γ' *is obtained from the underlying building geometry by taking a residue of type either* 1 *or* 6*;*
(6) $\tau(\Gamma) = \mathsf{E}_{7,1}$ *and* $\tau(\Gamma') = \mathsf{E}_{6,1}$. *The subgeometry* Γ' *is obtained from the underlying building geometry by taking a residue of type* 7*;*

We now state our results for other Lie incidence geometries.

Theorem 3.3. *Let* $\Gamma = (X, \mathcal{L})$ *be a projective Grassmannian, that is, a Lie incidence geometry of type* $\mathsf{A}_{n,j}$, $1 \leqslant j \leqslant n$, $n \geqslant 2$, *and let* $\Gamma' = (X', \mathcal{L}')$ *be a convex subspace containing at least one point. Then* Γ' *corresponds to a residue in the underlying projective space. In other words,* Γ' *arises from the set of all* $(j-1)$*-spaces of the corresponding projective space (over a skew field), containing a given* i_1*-space,* $-1 \leqslant i_1 \leqslant j-1$*, and contained in a given* i_2*-space,* $j-1 \leqslant i_2 \leqslant n$*.*

Theorem 3.4. *Let* $\Gamma = (X, \mathcal{L})$ *be a polar Grassmannian, more precisely, a Lie incidence geometry of type* $\mathsf{B}_{n,j}$ *or* $\mathsf{D}_{n,j}$, $3 \leqslant j \leqslant n-2$, *and let* $\Gamma' = (X', \mathcal{L}')$ *a convex subspace containing at least one point. Then either* Γ' *corresponds to a residue in the underlying building, or it consists of the union of a set* $\mathfrak{S}$ *of singular subspaces pairwise intersecting in a common point* c *such that every point of* $U \setminus \{c\}$ *is special to every point point of* $U' \setminus \{c\}$*, for every pair of distinct members* U *and* U' *of* $\mathfrak{S}$*.*

The previous theorem excludes the types $\mathsf{B}_{n,j}$ and $\mathsf{D}_{n,j}$, $j = 1, 2, n-1, n$. The case $j = 1$ is straightforward (only singular subspaces and the whole space), the case $j = 2$ is covered by Theorem 3.1. The case $j = n$ is the case of dual polar spaces or half spin geometries, see below. Finally, the case $j = n-1$ for type B is special in that there are many more exceptional examples not arising from residues in the underlying building (once again demonstrating the exceptional behaviour of these Lie incidence geometries, which do not turn up in most characterization theorems of parapolar spaces, see Chapters 14–17 in [Shu11]). The statement for type $\mathsf{B}_{n,n-1}$ is cumbersome and we refer to Theorem 8.4. The statement for type $\mathsf{B}_{n,n}$, commonly known as *dual polar spaces*, is simple, and we can take it together with type $\mathsf{D}_{n,n}$ (which coincides with type $\mathsf{D}_{n,n-1}$ by symmetry of the diagram), $n \geqslant 5$, commonly known as *half spin geometries*.

Theorem 3.5. *A convex subspace of a dual polar space or half spin geometry, that is, a Lie incidence geometry of type* $\mathsf{B}_{n,n}$*,* $n \geqslant 2$*, or* $\mathsf{D}_{n,n}$*,* $n \geqslant 3$*, always corresponds to a residue of the underlying spherical building.*

A special case of both Theorem 3.3 and Theorem 3.5 can be deduced from Lemma 5.2 in [CGP18]. It follows from that lemma that the convex subspace closure of two points in the relevant geometries always corresponds to a residue of the underlying spherical building.

The exceptional cases that we will handle and which are not covered by Theorem 3.1 are the following.

Theorem 3.6. *A convex subspace of a Lie incidence geometry of type* $\mathsf{E}_{6,1}$ *or* $\mathsf{E}_{7,7}$ *always corresponds to a residue of the underlying spherical building.*

For the other Lie incidence geometries of exceptional type (like type $\mathsf{E}_{6,3}$ and so on), we expect the classification of convex subspaces to be as complicated as for type $\mathsf{B}_{n,n-1}$, and the proofs to be case-by-case.

It is noteworthy to consider the special case of strong parapolar spaces. It just follows from our classification, without a unified proof, that Lie incidence geometries which are strong parapolar spaces have only convex subspaces coming from residues in the underlying spherical building.

Corollary 3.7. *A convex subspace of a Lie incidence geometry which is a strong parapolar space (including polar spaces themselves) always corresponds to a residue of the underlying spherical building. Conversely, every Lie incidence geometry containing special pairs of points contains convex subspaces which do not correspond to any residue of the associated spherical building.*

The second assertion of the corollary follows from the simple observation that the convex subspace closure of a special pair never corresponds to a residue of the associated building. For the first assertion, we need the classification of strong Lie incidence geometries. This can be accomplished as follows. Firstly, the classical cases (types A_n up to D_n) can be easily checked using the axioms of a projective and polar space (or, for the polar Grassmannians of symplectic rank at least 3, the following arguments will also work). This already provides types $\mathsf{A}_{n,i}$, $1 \leqslant i \leqslant n$, $\mathsf{B}_{n,1}$ and $\mathsf{B}_{n,n}$, $n \geqslant 2$, and $\mathsf{D}_{n,1}$ and $\mathsf{D}_{n,n}$, $n \geqslant 4$. Secondly, Lemma 3.2(3) of [Shu17] implies that, if the symplectic rank is at least 3, then the point residuals have diameter 2. Now, these point residuals can be read off the Coxeter diagram as residues in the building. It is easy to see that the diameter of a direct product space of three projective spaces is 3; the diameter of the direct product space of two nontrivial geometries at least one of which has diameter 2 has diameter at least 3; dual polar spaces of rank at least 3 have diameter at least 3; half spin geometries have diameter 2 only for types $\mathsf{D}_{n,n}$, with $n = 4, 5$. For exceptional types, only the Lie incidence geometries of type $\mathsf{E}_{6,1}$ (and $\mathsf{E}_{6,6}$) have diameter 2, see Table 10.5 in [BCN89]. With this information, the only remaining candidates for strong Lie incidence geometries of exceptional type are, up to a "duality" of the diagram, the ones of type $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$. But these are all strong, as for instance already remarked in [CC83] (in particular Theorem 1); it is also straightforward to see that the converse of Lemma 3.2(3) of [Shu17] is true, that is to say, a parapolar space of symplectic rank at least 3 with all point residuals of diameter 2 is strong. (Perhaps only the case $\mathsf{F}_{4,2}$ needs some more explanation: a residue of type $\{1,2,3\}$ is a convex subspace of type $\mathsf{B}_{3,2}$, and it contains special pairs.)

Remark 3.8. *If we consider (thick irreducible spherical) building Grassmannians of type $\mathsf{X}_{n,T}$, with arbitrary T, then no further geometries are strong. This can be proved by considering suitable residues, e.g., if T contains two adjacent types i,j (adjacency in the Coxeter graph), then the residue of a flag of cotype $\{i,j\}$ is a non-thick generalized polygon of diameter at least 3, hence contains special pairs. We do not insist further on this since we do not need this in the sequel of this paper.*

We now prove all the preceding theorems. We start with gathering some basic properties in Section 4. Then we prove Theorem 3.3 and treat the bulk of the other theorems in Section 6. It appears to be most efficient and insightful to combine the proof of Theorem 3.1 with several other classifications, since some arguments appear to be uniform across several types. After that, we treat the types $\mathsf{B}_{n,n-1}$ and $\mathsf{A}_{n,\{1,n\}}$ separately.

It may be noted that the assumptions of some lemmas are merely geometric or even axiomatic, leaving room to be applied on potentially other parapolar spaces than the Lie incidence geometries we are aiming at. Such lemmas may be useful for future characterizations or classification results. Indeed, it is for instance still an open question whether there exist strong parapolar spaces with symplectic rank at least 3 not arising from spherical buildings. To make this clear in the sequel, we will refer to such results as *axiomatic results*. Their proofs do not require knowledge of specific properties of the Lie incidence geometries but only use the given properties assumed.

4. Basic properties of some Lie incidence geometries

Here we state a few facts that we will need in our proofs. The first one confirms that subspaces corresponding to residues of the corresponding building are convex and follows immediately from Corollary 3.14 of [Tit74].

Fact 4.1. *Let $\Gamma=(X,\mathcal{L})$ be a Lie incidence geometry associated with $\mathsf{X}_{n,T}$. Let F be a flag of the corresponding building. Then the set Σ of points P of Γ such that $P\cup F$ is a flag is a convex subspace of Γ.*

With Σ and F as in the previous fact, we say that Σ corresponds to the residue of F. As explained above, the converse is not always true; however for certain types of convex subspaces, it is true. We record it for further reference.

Fact 4.2. *Let $\Gamma=(X,\mathcal{L})$ be a Lie incidence geometry associated with $\mathsf{X}_{n,T}$. Let Γ' be a convex subspace of Γ' isomorphic to either a projective space, or a polar space. Then there exists a flag F of the corresponding building such that Γ' corresponds to the residue of F.*

For singular subspaces, this follows from Theorem 10.2.10 in [BCN89] (as mentioned above); for polar spaces, this follows immediately from the fact that the symps of a Lie incidence geometry are convex subspaces, they are determined by any pair of its non-collinear points, and every convex subspace of a polar space is either singular or the whole space.

In the theory of spherical buildings, the notion of *opposition* is an important one. It expresses that two elements are at maximal distance from each other. In the standard realization of an apartment on a sphere, such elements are really diametrically opposite. Opposite elements need not necessarily have the same type, but in any hexagonic geometry, the elements opposite a point are points, and the elements opposite a line are lines.

Fact 4.3. *Let $\Gamma = (X, \mathcal{L})$ be a hexagonic Lie incidence geometry. Then the following hold.*

(i) *Points that are at distance* 3 *in the collinearity graph are opposite; consequently the diameter of* Γ *is* 3*.*

(ii) *If $p \in X$ is opposite some point of $L \in \mathcal{L}$, then there is a unique point q on L not opposite p; the pair $\{p, q\}$ is special.*

(iii) *If $p \perp q \perp r \perp s$ for four points of Γ such that $\{p, r\}$ and $\{q, s\}$ are special pairs, then p is opposite s.*

(iv) *If Γ has symplectic rank at least* 3 *then the following two properties hold.*

$$\text{If } \xi \text{ is a symplecton and } x \text{ a point not on } \xi, \text{ then } \dim(x^\perp \cap \xi) \neq 0. \tag{A}$$

$$\begin{gathered}\text{If } \xi \text{ is a symplecton and } x \text{ a point not on } \xi \text{ with } \dim(x^\perp \cap \xi) \geqslant 2, \\ \text{then } x^\perp \cap \xi \text{ is a maximal singular subspace of } \xi.\end{gathered} \tag{B}$$

Noting that hexagonic geometries are non-degenerate root filtration spaces in the sense of [CI07], (i) follows from Axiom (F) of [CI07], (ii) follows from Axiom (F) and Lemma $2(iv)$ in [CI07], and (iii) follow from Lemma $2(v)$ of [CI07].

Regarding (iv), (A) is the same as (H1) in Theorem 17.1.1 of [Shu11]; (B) is straightforward to check for polar line Grassmannians; for the exceptional types (B) it is a reformulation of (F4) in [CC83] (taking care of types E_i, $i = 6, 7, 8$), and of (F5) in [Coh82] (taking care of type F_4).

Our proof will also use the point residuals of hexagonic geometries. By Theorems 17.1.2, 17.2.6 and 17.2.7 of [Shu11], these have properties (i), (ii) and (iii) below. Property (iv) follows from Fact 3.18 of [DSSVM22] in the exceptional case; in the classical cases this is straightforward to verify.

Fact 4.4. *Let $\Gamma = (X, \mathcal{L})$ be a parapolar space isomorphic to the point residual of a hexagonic Lie incidence geometry of symplectic rank at least* 3*, Then the following assertions hold (where opposition is relative to the spherical building underlying* Γ*).*

(i) *Points that are at distance* 3 *in the collinearity graph are opposite; consequently the diameter of* Γ *is* 3*.*

(ii) *If $p \in X$ is opposite some point of $L \in \mathcal{L}$, then there is a unique point q on L not opposite p; the pair $\{p, q\}$ is symplectic.*

(iii) *A point $p \in X$ not contained in a symp ξ is collinear to at least one point of it; if p is collinear to exactly one point q of ξ, then it is opposite every point of ξ not collinear to q.*

(iv) *Two symps are opposite if and only if collinearity defines a bijection between their point sets.*

Finally we shall need a general property, called (CL) below, of some specific classes of classical hexagonic geomeries $\Gamma = (X, \mathcal{L})$.

(CL) The parapolar space Γ contains a symp ξ with the property that, whenever a point $x \in X \setminus \xi$ is collinear to a point of ξ, then it is precisely collinear to a line L of ξ. Also, x is special to every point of ξ not collinear to L.

Lemma 4.5. *The parapolar spaces of types $\mathsf{B}_{n,i}$, $n \geqslant 3$, $2 \leqslant i \leqslant n-1$ and $\mathsf{D}_{n,2}$, $n \geqslant 4$, $2 \leqslant i \leqslant n-2$, satisfy Property* (CL) *with ξ a symp corresponding to a residue of a flag of type $\{1, 2, \ldots, i-1\}$ in the underlying spherical building.*

Proof. We view the points of the parapolar space as the $(i-1)$-dimensional singular subspaces of the corresponding polar space. Let ξ be a symp consisting of the singular subspaces of dimension $i-1$ containing a fixed singular subspace U of dimension $i-2$. Let $W \supseteq U$ be a point of ξ. Let $W' \notin \xi$ be collinear to W, that is, W' is an $(i-1)$-dimensional singular subspace spanning an i-dimensional subspace Y together with W and $U \not\subseteq W'$. Then the points of ξ collinear with Y are precisely the $(i-1)$-dimensional spaces containing U and contained in Y, which proves the assertion, because they form a unique line of the parapolar space. $\square$

5. Proof of Theorem 3.3

We prove this theorem by induction on n. Translated to the projective space, the union of the j-spaces corresponding to the points of Γ' spans a subspace U' and the intersection of these subspaces is a subspace U. Let i_1 be the dimension of U and i_2 be the dimension of U'. If $i_1 \neq -1$ or $i_2 \neq n$, then we can consider either the upper residue of U (all singular subspaces strictly containing U) or the lower residue of U' (all subspaces strictly contained in U) and apply the induction hypothesis to conclude the proof. So we may assume $U = \varnothing$ and U' is the whole projective space. By duality we may also assume $j \leqslant \frac{n}{2}$.

We now have to show that all j-spaces belong to Γ'. Since U' is generated by all j-spaces that are points of Γ', it suffices to show that for any two $U_1, U_2 \in \Gamma'$ all j-spaces in $\langle U_1, U_2\rangle$ are contained in Γ'. So let $U_1, U_2 \in \Gamma'$ be arbitrary. If $U_1 \cap U_2 \neq \varnothing$ or $\dim(\langle U_1, U_2\rangle) \neq n$ we can again look at the residues of $U_1 \cap U_2$ or $\langle U_1, U_2\rangle$, respectively, and use the induction hypothesis. Hence we may assume $U_1 \cap U_2 = \varnothing$ and $\dim(\langle U_1, U_2\rangle) = n$.

Let W be a j-space intersecting U_1 in a subspace of dimension $j-1$. Then U_1 and W span a $(j+1)$-dimensional subspace which intersects U_2 in a subspace of dimension 0. Let V be the span of this 0-dimensional subspace and $U_1 \cap W$. Then by Proposition 6.6.2 of [BC13], V is lying on a shortest path between U_1 and U_2, so $V \in \Gamma'$ and moreover $W \in \Gamma'$. Hence all neighbours of U_1 in Γ are contained in Γ'. By applying the same argument to U_2 we can find $V' \in \Gamma'$ such that $W \cap V' = \varnothing$. This shows that we can repeat the previous argument to W in place of U_1 and by connectedness we get that all j-spaces are contained in Γ'.

6. Proof of Theorem 3.1

6.1. Short outline of the proof

In all cases where the convex subspace corresponds to a residue of the underlying spherical building, Γ' contains neither special nor opposite pairs (throughout this section, opposition will refer to Γ and not to Γ'). We will hence first reduce to that case by showing that Γ' cannot contain opposite pairs of points and, if Γ' contains at least one special pair (but no opposite pair), then it consists of a union of lines passing through the same point such that each pair of points from distinct lines is special. We refer to this as the *Special Case* and we call a convex subspace of that type a *special star*. If there are only collinear pairs in Γ', then clearly Γ' is a singular subspace. Hence the only remaining case to deal with is then when Γ' contains symplectic pairs, but no special or opposite ones. We refer to this as the *Symplectic Case*.

For the rest of this section, Γ is a hexagonic Lie incidence geometry, and except for Lemma 6.1, we assume it is not of type $\mathsf{A}_{n,\{1,n\}}$, $n \geqslant 2$. But we explicitly do allow type $\mathsf{B}_{3,2}$.

6.2. The Special Case

We first show that Γ' never contains opposite pairs of points. The following lemma also holds when $\tau(\Gamma) = \mathsf{A}_{n,\{1,n\}}$, $n \geqslant 2$.

Lemma 6.1. *A proper convex subspace $\Gamma' = (X', \mathcal{L}')$ of the hexagonic Lie incidence geometry $\Gamma = (X, \mathcal{L})$ never contains any pair of opposite points.*

Proof. We claim that whenever Γ' contains a point x which has an opposite in X', then every point of Γ collinear to x belongs to X' and has an opposite in X'. A connectivity argument then concludes the proof.

So let $x, y \in X'$ with x and y opposite. Let z be an arbitrary point of X collinear to x. Let L be the line containing x and z. Then, by Fact 4.3(ii), there is a unique point x_L on L not opposite y and x_L belongs to a shortest path joining x and y. Hence $L \subseteq X'$ and so $z \in X'$. Consider a line M through y opposite L. Then likewise $M \subseteq X'$. But every point of M except for one is opposite z, showing our claim. □

We need a preparatory lemma which we will apply later on to a point residual. It is an axiomatic result.

Lemma 6.2. *Let Δ be a strong parapolar space of diameter 3 such that every point is collinear with at least one point of any symplecton. Then each subspace Δ' containing a point x and a symplecton S, such that $\dim(x^\perp \cap S) = 0$, closed under taking the symplecton through two symplectic points, coincides with Δ. In particular, each such convex subspace coincides with Δ.*

Proof. Let z be the unique neighbour of x in S. We first show $x^\perp \subseteq \Delta'$. Consider y collinear with x. First assume that y is not collinear to z. Then y is collinear with a point $s \in S$ different from z. Then x and s are symplectic and both in Δ', hence $y \in \Delta'$. Assume now that y is collinear to z. We can find a symplecton T containing yx. There exist two lines through x generating this symplecton and containing points not collinear to z. By the previous case, $T \subseteq \Delta'$ and hence $y \in \Delta'$.

Now consider a point y symplectic with x. Since the symplecton through x and y is generated by two lines through x, which are contained in Δ', we get $y \in \Delta'$. Finally consider a point y at distance 3 from x. Consider a symplecton T through y, then x is collinear with a unique point t of T and symplectic with all lines through t in T. Since these lines generate T and are contained in Δ', we get $y \in \Delta'$. □

Corollary 6.3. *A proper convex subspace Δ' of a Lie incidence geometry of type $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$ or $\mathsf{E}_{7,7}$ does not contain a pair of opposite points.*

Proof. Although a proof similar to that of Lemma 6.1 could be given, we can also use Lemma 6.2. To that end, recall first that $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$ and $\mathsf{E}_{7,7}$ are the types of the point residuals of the Lie incidence geometries of type $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ and $\mathsf{E}_{8,8}$, respectively, so that we can use Fact 4.4. Now let $x, y \in \Delta'$ with x opposite y. By Fact 4.4(i), there is a path $x \perp p \perp q \perp y$ of length 3 (which is contained in Δ' by convexity). By Fact 4.4(ii), there is a unique symp $\xi(p, y)$. Now the conditions of Lemma 6.2 are met and the assertion follows. □

Now we treat the case where the convex subspace Γ' contains special pairs of points, but no opposite ones, and Γ does not contain rank 2 symplecta (the latter is equivalent to Γ not having type $\mathsf{B}_{3,2}$). If two points x and y are special, we denote by $x \bowtie y$ the unique point collinear with both x and y.

Lemma 6.4. *Suppose the convex subspace $\Gamma' = (X', \mathcal{L}')$ of the hexagonic Lie incidence geometry $\Gamma = (X, \mathcal{L})$ not containing rank 2 symplecta contains a special pair $\{x, y\}$ of points, and no opposite pair. Set $z = x \bowtie y$. Then Γ' does not contain a point u collinear to both x and z not on xz.*

Proof. Suppose for a contradiction that it does. Then Γ does not have type $\mathsf{G}_{2,1}$ and hence the assumptions imply that Γ has symplectic rank at least 3.

In the residue $\mathrm{Res}_\Gamma(z)$, the points zx and zy have distance 3 (by Fact 4.4(i)) and there is a unique line in the plane on x, z and u, which we may assume to be zu, at distance 2 from zy in this residue (by Fact 4.4(ii)). It follows that $\delta_\Gamma(y, u) = 2$ and $\{y, u\}$ is a symplectic pair. Hence Γ' contains the symp $\xi(y, u)$. In $\mathrm{Res}_\Gamma(u)$, the symp $\xi(y, u)$ and the point ux are such that a unique point of the symp is collinear to the point (because the symp $\xi_\Gamma(y, u)$ contains a point at distance 3 from the point ux). By Lemma 6.2 all points of $\mathrm{Res}_\Gamma(u)$ belong to Γ', since if there is a symp ξ through two intersecting but non-coplanar lines, then ξ belongs to the convex subspace closure of these two lines. Note that it follows from Fact 4.4(iii) that $\mathrm{Res}_\Gamma(u)$ satisfies the conditions of this lemma. Hence Γ' contains all planes through the line uz. Likewise, if v is a point collinear to z, u and y and distinct from z, then all planes through vz are contained in Γ'.

Let π be the plane through z, u, v and let α be an arbitrary plane through z intersecting π in a line K. If $K \in \{uz, vz\}$, then $\alpha \subseteq \Gamma'$. If $K \notin \{uz, vz\}$, and α is in some 3-space together with π, then clearly also $\alpha \subseteq \Gamma'$. If $K \notin \{uz, vz\}$, and α is not in any 3-space together with π, then α and π are contained in a unique symp, which also contains some plane through uz not contained in a 3-space together with π, and hence belongs to Γ'. But then also α belongs to Γ'.

Now let N be an arbitrary line through z. Then N is contained in at least one symp together with some line K through z in π. But that symp is also generated by two planes through K, which both belong to Γ', and so we see that N belongs to Γ'. Hence all lines through z belong to Γ' and so all points symplectic to z do, too. We claim that may choose two of these that are opposite, and then we contradict the previous lemma. Indeed, let ξ_1 and ξ_2 be two locally opposite symps through z (hence opposite in the point residual at z). Select now points $z_i \in \xi_i$ not collinear to z, $i = 1, 2$. Assume for a contradiction that $\delta(z_1, z_2) = 2$ and select $u \in z_1^\perp \cap z_2^\perp$. By Fact 4.3$(iii)$Equation (A), $u^\perp \cap \xi_i$ is either a line or a maximal singular subspace of ξ_i. It follows that $\{z, u\}$ is a symplectic pair; say the determine the symp ζ. If for some $i \in \{1, 2\}$, $u^\perp \cap \xi_i$ is a line, then Fact 4.4(iii) implies that either z is collinear to that line, and hence to z_i, or $\{z, u\}$ is a special pair, both contradictions. Hence $u^\perp \cap \xi_i$ is a maximal singular subspace U_i, for $i = 1, 2$. Let W_i be the maximal singular subspace of ξ_i generated by z and $z^\perp \cap U_i$. Then $W_i \subseteq \zeta$ and hence each point of $W_1 \setminus \{z\}$ is collinear to at least a plane of W_2, contradicting Fact 4.4(iv). The claim follows. □

We now conclude the special case.

Proposition 6.5. *Suppose the proper convex subspace $\Gamma' = (X', \mathcal{L}')$ of the hexagonic Lie incidence geometry $\Gamma = (X, \mathcal{L})$ not containing rank 2 symplecta contains a special pair $\{x, y\}$ of points. Then Γ' is a special star.*

Proof. If Γ has type $\mathsf{G}_{2,1}$, then the assertion is an easy exercise. So we may assume that Γ is a (non-strong) parapolar space of symplectic rank at least 3. Suppose Γ' contains a special pair $\{x, y\}$. Set $z = x \bowtie y$. Let u be a further point of Γ', not on the line xz or yz.

We claim that u is collinear to z. Indeed, by Lemma 6.1, u is not opposite z. If $\{u, z\}$ is special, then $u \bowtie z$ is not on any of the lines xz or yz, as, by Fact 4.3(iii), we also obtain in this case an opposite pair $\{u, y\}$ or $\{u, x\}$, respectively, in Γ'. By the same token, $\{x, u \bowtie z\}$ is either collinear or symplectic as otherwise x and u are opposite. But then there is a plane containing the line zx, contradicting Lemma 6.4. Hence the only remaining possibility to rule out is when u and z are contained in a symp. Then there is a plane π containing z. No point of π is special to x, by Lemma 6.4. Hence all points of π are either collinear or symplectic; but then this yields a plane on xz, again contradicting Lemma 6.4. The claim follows.

Now u is not symplectic to x as otherwise we again obtain a plane through xz, contradicting Lemma 6.4. Similarly, u is not symplectic to y. It follows that X' consists of lines through z such that points on different lines are special pairs. □

The following lemma shows that the condition in Lemma 6.4 that there are no symplecta of rank 2 is necessary. Indeed, the last type of convex subspaces in the following lemma contains a special pair x, y and a point u collinear to both x and $x \bowtie y$.

Lemma 6.6. *The proper convex subspaces of a Lie incidence geometry of type $\mathsf{B}_{3,2}$ are singular subspaces, symplecta, special stars, or there exist a plane π and a point $p \in \pi$ in the underlying polar space of rank 3 such that this convex subspace consists of all lines through p (a symplecton) and all lines contained in π (a maximal singular subspace).*

Proof. Let X' be a proper convex subspace. First assume that X' contains only identical, collinear and special pairs, let $\{x, y\}$ be such a special pair. Set $z = x \bowtie y$. First assume that there exists a point u not on xz collinear with a point on the line xz (or yz) distinct from z. If u is not collinear with z, then by assumption $\{u, z\}$ is a special pair and hence u and y are opposite, so X' would not be proper by Lemma 6.1. Hence u, x and z span a plane, contained in X'. For any $r \in X$, let U_r be the corresponding line in the polar space of rank 3. There exists a plane π of the polar space such that U_x, U_u and U_z are contained in π. Note that $\{r \in X \mid U_r \subseteq \pi\} \subseteq X'$. Since z and y are collinear, U_y is a line intersecting π in a point p. Since the rank of the polar space is 3, there exists a line in π through p not collinear with U_y. Then X' contains a symplectic pair, contradicting our assumption. Now assume that u is a point not collinear with z; by assumption it is then special to z. By the previous considerations $u \bowtie z$ is collinear with z and not with x (nor y), so it is special to x. Hence u and x are opposite, again contradicting our assumption. So X' is a special star.

If X' contains only identical, collinear or symplectic pairs then Proposition 6.8 applies (and note that the proof of Proposition 6.8 does not rely on the current lemma). So we may assume that X' contains a symplecton and a special pair. Note that a symplecton of $\mathsf{B}_{3,2}$ consists of all lines through a fixed point of the polar space. In particular, if X' contains two distinct symplecta, it would contain an opposite pair. So X' contains a unique symplecton ζ, consisting of all the lines through the point p. By assumption there exists $r \in X' \backslash \zeta$, i.e., U_r is a line not containing p. If U_r is not collinear to p, then we find a line through p opposite U_r, and X' thus contains an opposite pair. So U_r is collinear with p and X' contains all lines contained in the plane π on p and U_r. Hence X' contains the subspace consisting of all lines through p and all lines contained in π. One checks that this is indeed a convex subspace.

Assume that X' strictly contains this convex subspace, then the argument in the previous paragraph shows that there exists another plane π' through p such that all lines of π' are contained in X'. If π and π' intersect in a line L, then consider two non-collinear lines through a point on L distinct from p. These are both contained in X' and hence X' would contain another symplecton, quod non. If π and π' intersect only in p, then X' contains an opposite pair. □

6.3. The Symplectic Case

Type $\mathsf{G}_{2,1}$ is trivial. So it remains to deal with hexagonic parapolar spaces of symplectic rank at least 3. We prove an auxiliary axiomatic result independent of that assumption.

Lemma 6.7. *Let $\Gamma = (X, \mathcal{L})$ be a non-strong parapolar space containing a convex subspace Γ' properly containing a symp ξ satisfying Property* (CL). *Then Γ' contains special pairs. If moreover $\Gamma = (X, \mathcal{L})$ is a hexagonic parapolar space of symplectic rank at least* 3*, then $\Gamma' = \Gamma$.*

Proof. Consider a point x in Γ' not contained in ξ. By replacing x with the second point of a shortest path joining a point of ξ and x, we may assume that x is collinear to at least one point of ξ. By (CL), x is collinear to a line L of ξ. Also by (CL), there are points y of ξ special to x. Hence Γ' contains special pairs $\{x, y\}$. The last assertion follows from Proposition 6.5. □

We can now handle the types $\mathsf{B}_{n,2}$, $\mathsf{D}_{n,2}$ and $\mathsf{F}_{4,1}$.

Proposition 6.8. *Let $\Gamma = (X, \mathcal{L})$ be a Lie incidence geometry of type $\mathsf{B}_{n,i}$, $n \geqslant 3$ and $1 < i < n$, $\mathsf{D}_{n,i}$, $n \geqslant 4$ and $1 < i < n-1$, or $\mathsf{F}_{4,1}$. Let $\Gamma' = (X', \mathcal{L}')$ be a proper convex subspace such that every pair $\{x, y\} \subseteq X'$ is either identical, collinear or symplectic. Then either*

- Γ' *is a singular subspace;*
- Γ' *is a symp;*
- Γ *has type* $\mathsf{B}_{n,i}$ *or* $\mathsf{D}_{n,i}$ *and there exist singular subspaces* U *and* V *of the corresponding polar space of type* $\mathsf{B}_{n,1}$ *or* $\mathsf{D}_{n,1}$*, respectively, of projective dimension* $i_1 - 1$*, with* $0 \leqslant i_1 < i-1$*, and of projective dimension* $i_2 - 1$*, with* $i+1 < i_2 \leqslant n$*, respectively, and* $(i_1, i_2) \neq (i-2, i+2)$*, such that* X' *is the set of all* $(i-1)$*-spaces contained in* V *and containing* U*. (The case* $(i_1, i_2) = (i-2, i+2)$ *corresponds to a symp.)*

Proof. We may assume that Γ' is not a singular subspace, that is, Γ' contains a symp ξ. For the types $\mathsf{B}_{3,2}$, $\mathsf{B}_{4,2}$, $\mathsf{D}_{4,2}$ and $\mathsf{F}_{4,1}$, the assertion now follows immediately from Lemma 6.7, as these Lie incidence geometries have Property (CL) by Lemma 4.5 for the classical types, and by Lemma 4.2(ii) in [Coh82] for type F_4.

Now suppose that Γ has type $\mathsf{B}_{n,i}$ or $\mathsf{D}_{n,i}$, with $n \geqslant 5$. If ξ is a polar space of type $\mathsf{B}_{m,1}$ or $\mathsf{D}_{m,1}$ with $m = n - i + 1$, respectively, then, by Lemma 4.5, ξ possesses Property (CL), and the assertion follows. Hence all symps of Γ' consist of all $(i-1)$-spaces contained in a (fixed) singular subspace of dimension $i+1$ and containing a (fixed) singular subspace of dimension $i-3$. Consequently, X' consists of a set of $(i-1)$-spaces of the corresponding polar space which are pairwise contained in a common singular subspace. It follows that the union of these $(i-1)$-spaces is contained in a singular subspace U and so X' is the point set of a convex subspace of the i-Grassmannian of U. The assertion now follows from Theorem 3.3. □

In order to unify some proofs for the exceptional types, we now prove another axiomatic result. We note that, by Theorems 1 and 2 in [CC83], each symp of any Lie incidence geometry of type $\mathsf{E}_{6,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{7,7}$ or $\mathsf{E}_{8,8}$ satisfies (HC). Moreover, all symps of these parapolar spaces have respective rank 5, 4, 5, 6, 7 (this can be read off the respective Coxeter diagrams).

Lemma 6.9. *Let* $\Gamma = (X, \mathcal{L})$ *be a parapolar space containing a hyperbolic symp* ξ *of rank* $r \geqslant 3$ *possessing the following property.*

(HC) *If a point* $x \in X \setminus \xi$ *is collinear to a submaximal singular subspace of* ξ*, then it is collinear to a maximal singular subspace of* ξ*.*

Let Ω *be the convex subspace spanned by* ξ *and a point* $p \notin \xi$ *collinear to some submaximal singular subspace of* ξ*. Suppose that every symp of* Γ *has rank* r*. Then*

(i) *For every maximal singular subspace* U *of one of the two natural systems of* ξ*, there is a singular subspace of* Γ *properly containing* U *and belonging to* Ω*.*

(ii) *Every maximal singular subspace of the other system of* ξ *is contained in a symp distinct from* ξ *and belonging to* Ω*.*

(iii) *If* r *is odd and* $r \geqslant 5$*, then* Ω *contains a point* q *and a symp* ζ *such that no point of* ζ *is collinear to* q*.*

(iv) *If* r *is even and* $r \geqslant 6$*, then* Ω *contains a point* q *and a symp* ζ *such that* $q^\perp \cap \zeta$ *is a singleton.*

Proof. Let us denote the two different systems of maximal singular subspaces of ξ by Υ_1 and Υ_2. We may assume that p is collinear to a member $U_1 \in \Upsilon_1$. Let $U_1' \in \Upsilon_1$ be adjacent to U_1, that is, $\dim(U_1 \cap U_1') = r - 3$. Pick a point $x \in U_1' \setminus U_1$ and set $\zeta := \xi(p, x)$. Then ζ contains the $(r-2)$-space S spanned by x and $U_1 \cap U_1'$, and also the $(r-1)$-space $U_2 \in \Upsilon_2$ spanned by x and $x^\perp \cap U_1$. Hence an arbitrary $(r-1)$-space U of ζ containing S and distinct from U_2 is not contained in ξ. Pick a point $q \in U \setminus S$, then q is collinear to all points of S. Hence, by (HC), q is collinear to either U_1' or U_2. Since the latter cannot happen, $q \perp U_1'$. Hence U_1' is properly contained in a singular subspace $\langle q, U_1' \rangle$ of dimension r contained in Ω. By connectivity of the graph on Υ_1 with above adjacency, we conclude that every member of Υ_1 is properly contained in a singular r-subspace contained in Ω. This shows (i).

Note that the member U_2 of Υ_2 above is essentially arbitrary so that our arguments above also imply that every member of Υ_2 is contained in a second symp belonging to Ω. This shows (ii).

Now let $r \geqslant 5$ be odd. Select $U_1 \in \Upsilon_1$ and $U_2 \in \Upsilon_2$ such that $U_1 \cap U_2 = \varnothing$. Let $\zeta \subseteq \Omega$ be a symp distinct from ξ containing U_2 and let $y \in \Omega$ be a point not in ξ collinear to U_1. Suppose y is collinear to some point $z \in \zeta$. Then $z \notin U_2$. Set $W := z^\perp \cap U_2$. We have $W \neq U_2$ since U_2 is maximal in ζ and $z \notin U_2$. Our assumptions then imply that $z^\perp \cap \xi = U$ is a maximal singular subspace intersecting U_1 in a point u. Pick $w \in W$. Then $\xi(y, w)$ contains z, and also it contains $w^\perp \cap U_1$. Hence, for each $u' \in (w^\perp \cap U_1) \setminus \{u\}$, we have $\xi(y, w) = \xi(z, u')$. Since for arbitrary $w' \in W \setminus \{w\}$, $\dim(w'^\perp \cap w^\perp \cap U_1) = r - 3 > 0$, every point w' of W is collinear to some point u' of $(w^\perp \cap U_1) \setminus \{u\}$. We deduce $w' \in \xi(z, u')$ and so $W \subseteq \xi(y, w)$. Likewise $y^\perp \cap U_1 \subseteq \xi(y, w)$, which would imply that $\xi(y, w) = \xi$, a contradiction. Hence no point of ζ is collinear to y. This shows (iii).

Now let $r \geqslant 6$ be even. Select $U_1 \in \Upsilon_1$ and $U_2 \in \Upsilon_2$ such that $U_1 \cap U_2$ is a point x. Let $\zeta \subseteq \Omega$ be a symp distinct from ξ containing U_2. In the residue of x, we can apply (iii) and (iv) follows. $\square$

In order to handle the cases of types $\mathsf{E}_{6,2}$ and $\mathsf{E}_{7,1}$, we need the classification of convex subspaces of the parapolar spaces of types $\mathsf{D}_{5,5}$ and $\mathsf{E}_{6,1}$, respectively. We consider the former as a special case of type $\mathsf{D}_{n,n}$.

Proposition 6.10. *A proper convex subspace of a Lie incidence geometry Γ of type $\mathsf{D}_{n,n}$, $n \geqslant 5$, is either singular or corresponds to the set of all maximal singular subspaces of one natural system of the corresponding polar space containing a fixed singular subspace of projective dimension j, with $0 \leqslant j \leqslant n-5$. The case $j = n-5$ corresponds to a symp.*

Proof. We prove this proposition by induction on n. To that aim, we allow $n = 4$, in which case we just have a polar space and the assertion follows from the fact that nonsingular convex subspaces of a polar space are improper.

Now let $n \geqslant 5$. Let Υ be the natural system of maximal singular subspaces, sometimes called generators, of the corresponding polar space of type $\mathsf{D}_{n,1}$ corresponding to the points of Γ (that is, Υ is the set of vertices of the corresponding building of type D_n of type n). Let $X' \subseteq \Upsilon$ be the set of generators corresponding to the points of an arbitrary proper convex subspace Γ' of Γ. Let S be the intersection of all members of X'. If S is nonempty, then the assertion follows from the induction hypothesis applied in the residue of S. If $S = \varnothing$, then we claim that X' contains two disjoint members (in particular, n is even).

Indeed, if not, set $\epsilon = 0, 1$ according to whether n is even or odd, respectively, and let $\frac{n-\epsilon}{2} - j$, $0 \leqslant j < \frac{n-\epsilon}{2}$, be the diameter of Γ'. Then there exist two members U, U' of X' with

$\dim(U \cap U') = 2j - 1 + \epsilon$. If $j = \epsilon = 0$, then U and U' are disjoint, so $\ell := 2j - 1 + \epsilon \geqslant 0$. In the (upper) residue of $U \cap U'$, the generators U and U' become disjoint generators and so the induction hypothesis implies that all members of Υ containing $U \cap U'$ belong to X'. By hypothesis, there exists $U'' \in X'$ not containing $U \cap U'$. We may choose U'' so that $m := \dim(U \cap U' \cap U'')$ is minimal. If $m \leqslant \ell - 2$, then we can find a generator $W \in \Upsilon$ containing $U \cap U'$ and intersecting U'' in $U \cap U' \cap U''$, contradicting the value of the diameter of Γ'. Hence $m = \ell - 1$. Suppose first $\ell \geqslant 1$. Then we can find two members $W, W' \in X'$ with $\dim(U \cap U' \cap W \cap W') = m - 1$. Applying induction in the residue of $U \cap U' \cap W$ yields that all members of Υ containing $U \cap U' \cap W$ belong to X'. Likewise, all members of Υ containing $U \cap U' \cap W'$ belong to X'. This now implies that we can find two members of X' intersecting in just $U \cap U' \cap W \cap W'$, contradicting the value of the diameter of Γ' again. Hence $\ell = 0$. Set $\{x\} = U \cap U'$. It is easy to see that we can select a member of Υ containing x and intersecting U'' in just a point, say y. Induction implies that all members of Υ containing y belong to X'. Similarly, every point z of the corresponding polar space of type $\mathsf{D}_{n,1}$ collinear to both x and y, but not on the line xy is the intersection of a member of X' through x and one though y and hence all members of Υ through z belong to X'. Interchanging the role of y with z, we also obtain this property for points of the line xy. Since there is a member U^* of Υ containing xy, and since $\ell = 0$ implies $\epsilon = 1$, every member of Υ intersects $U^* \subseteq (xy)^\perp$ and hence belongs to X'. But then Γ' is improper, a contradiction. The claim follows.

The previous claim implies that we may assume that n is even and that X' contains two opposite members U and U'. Each line L in U can be obtained as the intersection of U with a member of Υ in a shortest path between U and U', and hence induction implies that all members of Υ intersecting U (or U') nontrivially, belong to X'. Since each such member clearly has an opposite in X', the same argument now readily implies that $\Upsilon = X'$. $\square$

This now implies for the case $n = 5$ the following corollary.

Corollary 6.11. *A proper convex subspace of a Lie incidence geometry of type* $\mathsf{D}_{5,5}$ *is either singular or coincides with a symp.*

The same is true for type $\mathsf{E}_{6,1}$, as the following proposition claims. We use the fact that any two symps of an incidence geometry of type $\mathsf{E}_{6,1}$ intersect nontrivially (see Exercise 15.5 in [Shu11]).

Proposition 6.12. *Each proper convex subspace of a Lie incidence geometry* $\Gamma = (X, \mathcal{L})$ *of type* $\mathsf{E}_{6,1}$ *is either singular or coincides with a symp.*

Proof. Let $\Gamma' = (X', \mathcal{L}')$ be a convex subspace of Γ. We may assume that Γ' is not singular. Then it contains at least one symp ξ (as Γ is a strong parapolar space of diameter 2; see for instance Exercise 15.4 in [Shu11]). We may assume that there exists $x \in X' \setminus \xi$. If x is opposite ξ, then, since no two symps are disjoint, every point of Γ is contained in a symp determined by x and some point of ξ, hence Γ' is not proper. If x is not opposite ξ, then, noting that ξ has Property (HC), there exist, by Lemma 6.9(iii), a point p and a symp ζ in Γ' such that $p^\perp \cap \zeta = \varnothing$. But then p and ζ are opposite and we again conclude that Γ' is not proper. $\square$

Now we can finish the exceptional cases of type E. Note that the cases in the next proposition also follow from the proof of Theorem 4.1 of [Coh21], using different arguments.

Proposition 6.13. *Each proper convex subspace Γ' of a Lie incidence geometry Γ of type $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ or $\mathsf{E}_{8,8}$ containing no special pairs of points is obtained from a residue in the underlying spherical building.*

Proof. We may assume that Γ' contains a symp ξ and some point p outside ξ. Since Γ' does not contain special point pairs, Fact 4.4(iii) and Property (B) imply that $p^\perp \cap \xi$ is a maximal singular subspace U of ξ. Note that the symp ξ satisfies the hypotheses of Lemma 6.9. We now analyse the three distinct cases.

$\mathsf{E}_{6,2}$ We observe first that ξ and p are contained in a unique convex subspace Ω of type $\mathsf{D}_{5,5}$ obtained from the residue of a vertex of type 1 or 6 in the corresponding building of type E_6. By Corollary 6.11 we have $\Omega \subseteq \Gamma'$. Assume that Γ' contains a further point $p' \notin \Omega$. Then we may assume that $p'^\perp \cap \xi = U'$ is a maximal singular subspace of the natural system not containing U. Without loss of generality we may assume that $U \cap U'$ is a point u. By properties (easily verified in an apartment of the underlying building) of Lie incidence geometries of type $\mathsf{E}_{6,2}$, the point pair $\{p, p'\}$ is a special pair (and $p \bowtie p' = u$), a contradiction.

$\mathsf{E}_{7,1}$ We observe first that ξ and p are contained in a unique convex subspace Ω of type $\mathsf{E}_{6,1}$ obtained from the residue of a vertex of type 7 in the corresponding building of type E_7. Now Proposition 6.12 implies $\Omega \subseteq \Gamma'$. Assume now there exists a point $x \in \Gamma' \setminus \Omega$. Then by properties of Lie incidence geometries of type $\mathsf{E}_{7,1}$ (see for instance Fact 3.13 of [DSSVM22] which can easily be derived from the ($\mathsf{E}_{6,1} - \mathsf{E}_{6,2} - \mathsf{E}_{6,1}$)-representation of the thin parapolar space of type $\mathsf{E}_{7,2}$ in Section 7.2 of [VMV19]), there exists some point of Ω special to x, a contradiction.

$\mathsf{E}_{8,8}$ Lemma 6.9(iii) yields a point x and a symp ζ, both belonging to Γ', such that $x^\perp \cap \zeta = \varnothing$. Now the second diagram of Section 7.3 of [VMV19] implies that some point of ζ is special to x, a contradiction.

The proof of the proposition is complete. □

Finally, to complete the cases of diameter at most 3 in the exceptional case, we have the following classification of convex subspaces for Lie incidence geometries of type $\mathsf{E}_{7,7}$.

Proposition 6.14. *Each proper convex subspace Γ' of a Lie incidence geometry Γ of type $\mathsf{E}_{7,7}$ either is singular or coincides with a symp.*

Proof. Let Γ' be a proper convex subspace of Γ. By Corollary 6.3, Γ' does not contain opposite pairs of points. Hence, if Γ' is not singular and does not coincide with a symp, then it contains a symp ξ and a point p such that $\dim(p^\perp \cap \xi) \neq 0$. Then automatically $p^\perp \cap \xi$ is a maximal singular subspace of ξ (by Theorem 1 in [CC83]). Then Lemma 6.9(iv) yields a symp ζ and a point x in Γ' such that $x^\perp \cap \zeta$ is a point. Consequently Γ' contains pairs of opposite points after all, a contradiction. □

We covered all strong parapolar spaces which are Lie incidence geometries, except for the dual polar spaces. This is the content of the next proposition. In fact, the result is well known and is contained in Theorem 8.5.15 of [Shu11], which is noted there along the way of proving Cameron's characterisation [Cam82] of dual polar spaces. We include a proof for completeness's sake; it differs from the one in [Shu11] in that it is entirely written in terms of the underlying polar space.

Proposition 6.15. *Each proper convex subspace of a Lie incidence geometry Γ of type $\mathsf{B}_{n,n}$ corresponds to a residue of the underlying building. In other words, each convex subspace corresponds to the set of maximal singular subspaces of the corresponding thick polar space containing a fixed singular subspace (which can be empty or maximal, or anything in between).*

Proof. In a similar way to the last paragraph of the proof of Proposition 6.10 one shows that no proper convex subspace contains two points corresponding to disjoint maximal singular subspaces of the corresponding polar space Δ. We denote by U_p the maximal singular subspace of Δ corresponding to the point p of Γ. Now let Γ' be a proper convex subspace of Γ and let d be its diameter, $0 \leqslant d \leqslant n-1$. Then there exist two points p, q in Γ' such that $\dim(U_p \cap U_q) = n - d - 1$. Considering the (upper) residue of $U_p \cap U_q$, we see that Γ' contains all points r such that $U_p \cap U_q \subseteq U_r$. But for any other maximal singular subspace U_x, there exists a maximal singular subspace $U_y \supseteq U_p \cap U_q$ with $\delta(x, y) > d$, a contradiction. This shows the proposition. □

6.4. Conclusion of the proof of Theorem 3.1

We wrap up everything we proved in this section: We proved Theorem 3.1 in

Lemma 6.1: opposite pairs of points in the proper convex subspace cannot occur,

Proposition 6.5: only special stars occur when at least one pair of points is special,

Proposition 6.8: the remaining cases for the classical types and type $\mathsf{F}_{4,1}$,

Proposition 6.13: in the remaining cases for the exceptional E-types.

Moreover, we classified the convex subspaces of the hexagonal geometries of type $\mathsf{B}_{3,2}$ in Lemma 6.6. Also, we completed the proof of Main Result 2 in Proposition 6.10 for type $\mathsf{D}_{n,n}$, in Proposition 6.12 for type $\mathsf{E}_{6,1}$, in Proposition 6.14 for type $\mathsf{E}_{7,7}$, and in Proposition 6.15 for dual polar spaces (the case of projective Grassmannians was done in Section 5).

7. Proof of Theorem 3.4

Let $\Gamma = (X, \mathcal{L})$ be a Lie incidence geometry of type $\mathsf{B}_{n,j}$ or $\mathsf{D}_{n,j}$, with $3 \leqslant j \leqslant n-2$. Let Δ be the corresponding polar space. Without danger of confusion, we will denote with $\perp$ the collinearity in both Γ and Δ (it will always be clear from context). For a point p of Γ, we denote by U_p the singular subspace of dimension $j-1$ of Δ corresponding to p. For a subset $Y \subseteq X$, we denote by $\Delta(Y)$ the corresponding set of $(j-1)$-spaces of Δ, that is, $\Delta(Y) = \{U_y \mid y \in Y\}$.

We first establish the distance between two points of Γ. Two distinct singular subspaces U, U' of the same dimension of a polar space are called *locally opposite* if $U \cap U' = U^\perp \cap U'$. Hence being opposite is equivalent with being locally opposite and disjoint at the same time.

Lemma 7.1. *Let $p, q \in X$. Let $i = \dim(U_p \cap U_q)$. Then $\delta(p,q) = j - i$, if U_p and U_q are locally opposite, and $\delta(p,q) = j - i - 1$ otherwise.*

Proof. Performing induction on i, it suffices to show the assertion for $i = -1$. Suppose first that U_p and U_q are not opposite. Since $(j-1)$-spaces corresponding to collinear points of Γ (and we refer to such $(j-1)$-spaces as being *adjacent*) share a $(j-2)$-space, we easily see that $\delta(p,q) \geqslant j$. Let a be any point in $U_p^\perp \cap U_q$ and let W be a hyperplane of U_p with either $W \subseteq U_p \cap U_q^\perp$ or $W \supseteq U_p \cap U_q^\perp$. Then a and W generate a $(j-1)$-space U_r, for some point $r \in X$, with $p \perp r$. Also, U_r and U_q are not locally opposite and the induction hypothesis implies $\delta(U_r, U_q) = j - 1$. Hence $\delta(p,q) \leqslant j$ and the assertion follows.

If U_p and U_q are opposite, then every $(j-1)$-space adjacent to U_p is disjoint from U_q. It follows that $\delta(p,q) \geqslant j+1$. But we can certainly find a $(j-1)$-space adjacent to U_p and not opposite U_q, proving $\delta(p,q) = j+1$ (using the previous paragraph). $\square$

We now show that the convex subspace closure of certain pairs of points is the whole space Γ.

Lemma 7.2. *If for two points $p, q \in X$ we have $U_p \cap U_q = \varnothing$ and $U_p \cap U_q^\perp \neq U_p$, then the only convex subspace containing p and q is Γ itself.*

Proof. Let $\Gamma' = (X', \mathcal{L}')$ be the convex subspace generated by p and q.

First suppose that U_p and U_q are opposite. Let $r \perp p$, with $r \in X$ arbitrary. Then U_r and U_p are contained in a common singular j-space W. Set $\{a\} = U_q^\perp \cap W$, note that $a \notin U_p$. Then we may assume that $a \in U_r$ (by possibly varying r on the line pr). But then $\delta(p,q) = 1 + \delta(r,q)$ by Lemma 7.1. Hence $r \in X'$. Hence all points of Γ collinear with either p or q belong to X'. Noting that every point in $p^\perp$ has an opposite in $q^\perp$, an obvious induction argument yields $X' = X$.

Now assume that $i := \dim(U_p \cap U_q^\perp) \geqslant 1$. Let $r \in X$ be such that U_r intersects U_p in a(n arbitrary) $(j-2)$-space and $U_q \cap U_p^\perp$ in a(n arbitrary) point. Then, since $i \geqslant 1$, we have $U_r \cap U_p \cap U_q^\perp \neq \varnothing$, and thus U_r and U_q are not locally opposite; hence Lemma 7.1 implies $\delta(r,q) = \delta(p,q) - 1$, and so $r \in X'$. It follows from Theorem 3.3 that all points r such that U_r is contained in the singular subspace W spanned by U_p and $U_q \cap U_p^\perp$ belong to X'. Now let $r \in X'$ be such that $U_r \leqslant W$ contains $U_q \cap U_p^\perp$ and $U_r \cap (U_p \cap U_q^\perp) = \varnothing$. Then U_r and U_q are locally opposite and the first part of the proof implies that all points $s \in X$ such that $U_q \cap U_p^\perp \subseteq U_s$ belong to X'. Similarly, all points $s \in X$ such that $U_p \cap U_q^\perp \subseteq U_s$ belong to X'. It is now easy to see that we can select $U_s \supseteq U_q \cap U_p^\perp$ and $U_{s'} \supseteq U_p \cap U_q^\perp$ such that $\dim(U_s \cap U_{s'}^\perp) < i$.

Hence we may assume that $\dim(U_p \cap U_q^\perp) = 0$. Set $\{x\} = U_p \cap U_q^\perp$ and $\{y\} = U_q \cap U_p^\perp$. Let W_p and W_q be the j-spaces generated by U_p and y, and by U_q and x, respectively. As in the previous paragraph, X' contains all points r such that $xy \subseteq U_r \subseteq W_p \cup W_q$. The first part of the proof (or Theorem 3.1 if $j = 3, 4$), applied to the upper residue of xy, implies that all $r \in X$ for which $xy \subseteq U_r$ belong to X'. Also, Theorem 3.3 implies that all $r \in X$ for which $x \in U_r \subseteq W_p$ or $y \in U_r \subseteq W_q$ belong to X'.

Now let V_p be any $(j-2)$-space in U_p containing x. Then V_p is collinear with a line L of U_q and we can select a $(j-2)$-space V_q in U_q containing y but not L. It follows that $V_p^\perp \cap V_q = \{y\}$

and $V_q^\perp \cap V_p = \{x\}$. Now select a $(j-1)$-space $U_{p'}$, $p' \in X$, containing V_p but not collinear with y. Also, select a $(j-1)$-space $U_{q'}$, $q' \in X$, containing V_q but not collinear with x. Then it is easily checked that $U_{p'}$ and $U_{q'}$ are opposite.

Now we show $p', q' \in X'$. Consider any $(j-3)$-space X_p of V_p containing x. Since all points of X corresponding to $(j-1)$-spaces through $\langle X_p, y\rangle$ are contained in X', and $p \in X'$, we can apply Theorem 3.1 to the (upper) residue of X_p to obtain that all $(j-1)$-spaces through X_p are contained in X'. In particular $p' \in X'$. Similarly $q' \in X'$. Now the first paragraph of the proof implies $X' = X$. □

Lemma 7.3. *Let $\Gamma' = (X', \mathcal{L}')$ be a convex subspace of Γ, with $\operatorname{diam}\Gamma' \geqslant 3$. Then Γ' corresponds to a residue in the corresponding building.*

Proof. If the U_p, $p \in X'$, are pairwise collinear, then they are contained in a common singular subspace and the result follows from Theorem 3.3. So we may assume that not all U_p, $p \in X'$, are collinear. Let $\Delta(X')$ be the set of U_p with $p \in X'$.

Now suppose that there exist $p, q \in X'$ such that $\delta(p,q) = 3$, with U_p and U_q not collinear. Then there are two possibilities.

(1) *U_p and U_q are locally opposite.* In this case $\dim(U_p \cap U_q) = j-3$. Applying Theorem 3.1 to the upper residue of $U_p \cap U_q$, we deduce that all $(j-1)$-dimensional singular subspaces of Δ containing $U_p \cap U_q$ belong to $\Delta(X')$.

(2) *U_p and U_q are not locally opposite.* In this case $\dim(U_p \cap U_q) = j-4$. Applying Lemma 7.2 to the upper residue of $U_p \cap U_q$, we again deduce that all $(j-1)$-dimensional singular subspaces of Δ containing $U_p \cap U_q$ belong to $\Delta(X')$.

Now let $W \subseteq U_p \cap U_q$ have minimal dimension with respect to the property that all $(j-1)$-spaces of Δ through W belong to $\Delta(X')$ (this is well-defined and of dimension smaller than or equal to $j-3$ since $U_p \cap U_q$ satisfies the given property). If $X' = \{x \in X \mid W \subseteq U_x\}$, then the assertion follows. So assume that there is some point $r \in X'$ with $Z = W \cap U_r \neq W$. Then there exists a $(j-1)$-space U_s, for $s \in X'$, containing W and intersecting U_r in Z. Applying Lemma 7.2 to the upper residue of Z, we deduce that all $(j-1)$-dimensional singular subspaces of Δ containing Z belong to $\Delta(X')$, contradicting the minimality of $\dim W$.

So we may assume that U_p and U_q are collinear whenever $\delta(p,q) = 3$, for $p, q \in X'$. We show that this leads to a contradiction. Select such $p, q \in X'$. Then by Theorem 3.3 there exist singular subspaces $W \subseteq U_p \cap U_q$ and $V \supseteq U_p \cup U_q$ such that the set $S(W,V)$ of all points $r \in X$ with $W \subseteq U_r \subseteq V$ is contained in X', and such that $\dim V - \dim W$ is maximal with that property. Note that $\dim W \leqslant j-4$ and $\dim V \geqslant j+2$. Let $s \in X' \setminus S(W,V)$ be collinear to some point $r \in S(W,V)$ (such s exists since we assume that not all U_x, $x \in X'$, are pairwise collinear). By Theorem 3.3, U_s being collinear to all U_x, $x \in S(W,V)$ contradicts the maximality of $\dim V - \dim W$. So the $(j-2)$-dimensional subspace $U_r \cap U_s$ coincides with $W_s := V \cap U_s$. Also, $V_s := U_s^\perp \cap V$ has dimension at most $\dim V - 1$. Consequently there exists $y \in S(W,V)$ such that U_y is not contained in V_s, and such that $\dim(U_y \cap W_s) = j-4$. But then, since U_s and U_y are not locally opposite, we see that $\delta(y,s) = 3$ and U_y and U_s are not contained in a singular subspace. This contradicts our assumption.

The lemma is proved. □

So it remains to classify the convex subspaces with diameter 2.

We first prove a lemma.

Lemma 7.4. *If a point $p \in X$ is special to every point of some line L, then either*
(i) *$p \bowtie x = c$, for all points $x \in L$ and some point c; or*
(ii) *there is a symp ξ such that $p^\perp \cap \xi$ is a line M and $L \subseteq \xi$ is ξ-opposite M.*

Proof. Let W_L and V_L be such that $L = \{r \in X \mid W_L \subseteq U_r \subseteq V_L\}$. Note that $W_L \not\subseteq U_p$, as otherwise p is collinear to some point of L. There are two possibilities.

(i) Suppose $\dim(W_L \cap U_p) = j-4$ and $\dim(V_L \cap U_p) = j-2$. Note that $W_L \cap U_p^\perp$ then has dimension $j-3$. Letting $c \in X$ be the point corresponding to the $(j-1)$-space $\langle U_p \cap V_L, W_L \cap U_p^\perp \rangle$, we obtain Conclusion (i).

(ii) Suppose $\dim(W_L \cap U_p) = j-3$. We may argue in the upper residue of $W_L \cap U_p$. Then V_L is a plane disjoint from the line U_p, and since p is special to every point of L, there are two possibilities. (1) The point W_L belongs to $U_p^\perp$; in fact $W_L = U_p^\perp \cap V_L$. This leads to Conclusion (ii), with ξ the symp defined by W_L. (2) The line U_p is collinear to a line of the plane V_L not containing the point W_L, but not to the entire plane. In this case, there is a unique point $W_L^\perp \cap U_p$ and Conclusion (ii) holds with ξ the hyperbolic quadric defined by (the line Grassmannian of) the 3-space generated by $W_L^\perp \cap U_p$ and V_L. In Δ, this symp is defined by all $(j-1)$-spaces containing $U_p \cap W_L$ and contained in $\langle W_L^\perp \cap U_p, V_L \rangle$.

The lemma is proved. □

Proposition 7.5. *Let $\Gamma' = (X', \mathcal{L}')$ be a convex subspace of Γ, with $\operatorname{diam}\Gamma' = 2$. Then exactly one of the following occurs.*
(i) *If X' contains a pair of symplectic points p, q with U_p not collinear to U_q, then Γ' is a symplecton.*
(ii) *If X' contains a symplectic pair, and each such pair $\{p, q\}$ has the property that U_p and U_q are collinear, then either*
  (a) *there is a unique $(j-3)$-space W, a unique number ℓ, $j+1 \leqslant \ell \leqslant n-1$, and a unique ℓ-space V such that $X' = \{r \in X \mid W \subseteq U_r \subseteq V\}$, or*
  (b) *there is a unique $(j+1)$-space V, a unique number ℓ, $-1 \leqslant \ell \leqslant j-3$, and a unique ℓ-space W such that $X' = \{r \in X \mid W \subseteq U_r \subseteq V\}$.*
(iii) *If X' does not contain symplectic pairs, then there is a unique point $c \in X'$ such that $c = p \bowtie q$, for every special pair of points p, q in $X' \setminus \{c\}$. Moreover, X' is the union of singular subspaces pairwise intersecting in exactly $\{c\}$.*

Proof. First suppose that Γ' contains a symp ξ determined by two points p, q such that $\dim(U_p \cap U_q) = j-2$ (and U_p and U_q are locally opposite). Set $W = U_p \cap U_q$. If we are not in Case (i), then there exists a point $r \in X' \setminus \xi$. Then also $\dim(W \cap U_r) = j-3$, as otherwise we can easily find a point $x \in \xi$ with $\delta(x, r) \geqslant 3$. But considering the upper residue in Δ of $W \cap U_r$, it follows from Theorem 3.1 that Γ' contains points at distance 3. This concludes Case (i).

Now suppose that X' contains a symplectic pair $\{p, q\}$ such that U_p and U_q are collinear. Set $W = U_p \cap U_q$ and $V = \langle U_p, U_q \rangle$. Assume that there is a point $r \in X'$ with $U_r \not\subseteq V$. If $W \not\subseteq U_r$, then it is easy to find $s \in \xi(p,q)$ with $\delta(r,s) = 3$, a contradiction. Hence $W \subseteq U_r$. Now Theorem 3.1 applied to the upper residue of W yields Conclusion $(ii)(a)$. Hence every point $x \in X'$ satisfies $U_x \subseteq V$. Then clearly Conclusion $(ii)(b)$ holds.

Finally suppose that X' does not contain symplectic pairs. Since $\operatorname{diam}\Gamma' = 2$, there is at least one special pair $\{p,q\}$ in X'. Set $c = p \bowtie q$. We claim that c is collinear to every point of Γ'. Indeed, if not there exists $r \in X'$ special to c; set $s = c \bowtie r$. Since p and q are special, at least one of p, q is special to s, say p. Then p is special to every point of rs as otherwise $\{p,s\}$ would be symplectic. Lemma 7.4 implies that c is collinear to r after all. Thus, Conclusion (iii) holds. □

8. Convex subspaces of Lie incidence geometries of type $\mathsf{B}_{n,n-1}$

Let $\Gamma = (X, \mathcal{L})$ be a Lie incidence geometry of type $\mathsf{B}_{n,n-1}$, with $n > 3$. The case $n = 3$ is handled in Lemma 6.6, which we sometimes (implicitly) use in this section when we look at certain residues. Let Δ be the corresponding polar space. For a point p of Γ, we denote by U_p the singular subspace of dimension $n-2$ of Δ corresponding to p.

First note that the distance between two points is determined as in Lemma 7.1.

Let U be a singular subspace of Δ of dimension smaller then or equal to $n-3$ and let P be a maximal singular subspace of Δ. Set $\Pi(U) = \{x \in X \mid U \subseteq U_x\}$, $\Pi(P) = \{x \in X \mid U_x \subseteq P\}$, $\Pi(U,P) = \Pi(U) \cap \Pi(P)$ and $\Pi(U;P) = \Pi(U) \cup \Pi(P)$.

Let M be a singular subspace of Δ of dimension at most $n-2$, let H be a set of hyperplanes of M and, for each $h \in H$, let P_h be a maximal singular subspace containing M, such that the following holds:

$$\begin{array}{l}\text{If } P_{h_1} = P_{h_2}, \text{ for } h_1, h_2 \in H, \text{ then for all hyperplanes } h_3 \text{ of } M \\ \qquad \text{such that } h_1 \cap h_2 \subseteq h_3 : h_3 \in H \text{ and } P_{h_3} = P_{h_1}.\end{array} \tag{C}$$

Now set $\Pi(M, (P_h)_{h\in H}) = \Pi(M) \cup \bigcup_{h\in H} \Pi(h, P_h)$.

We consider $\Pi(p;P)$, with p a point and P a maximal singular subspace, as a special case of this construction, it is the same as $\Pi(p, P_\varnothing)$, with $H = \{\varnothing\}$ and $P_\varnothing = P$. (We consider the empty set as the unique hyperplane of a point.) Moreover, $\Pi(U)$ is obtained by setting $H = \varnothing$, and $\Pi(U,P)$ equals $\Pi(M, (P_h)_{h\in H})$, with M a singular subspace of dimension $n-2$ of P containing U, H the set of hyperplanes of M containing U and $P_h = P$ for all $h \in H$. Also, note that $\Pi(P)$ is the same as $\Pi(\varnothing, P)$.

Lemma 8.1. *For any point p and maximal singular subpace P such that $p \in P$, $\Pi(p;P)$ is a maximal proper convex subspace. For any line L, any subset H of the point set of L and any maximal singular subspace P_h for every $h \in H$ such that $P_{h_1} = P_{h_2}$ implies $h_1 = h_2$, $\Pi(L, (P_h)_{h\in H})$ is a convex subspace.*

Proof. We first prove that $\Pi(p;P)$ is a convex subspace. Clearly $\Pi(p)$ and $\Pi(P)$ are convex subspaces so consider x and y in $\Pi(p;P)$ such that $p \in U_x \nsubseteq P$ and $p \notin U_y \subseteq P$. Note that $U_y^\perp \cap U_x = \langle p, U_x \cap U_y\rangle$, so any neighbour z of y on a shortest path between x and y satisfies $p \in U_z$, showing that $\Pi(p;P)$ is convex. Similarly $\Pi(L, (P_h)_{h\in H})$ is convex.

Assume that X' is a proper convex subspace properly containing $\Pi(p; P)$ and consider $z \in X' \backslash \Pi(p; P)$. By definition $p \notin U_z$. If U_z is not collinear to p we can find a subspace of dimension $n-2$ through p opposite to U_z and as in the proof of Lemma 7.2 we find that X' is not proper. So we may assume that p is collinear with U_z. By assumption $U_z \cap P$ has dimension at most $n-3$. So we can find $r \in X'$ such that $U_r \subseteq P$, $U_r \cap U_z = U_r \cap P$ and U_r and U_z are locally opposite. By considering the upper residue of $U_z \cap U_r$ we see that $\{x \in X \mid U_z \cap U_r \subseteq U_x\}$ is contained in X'. Since $p \notin U_z \cap U_r$ we can then find $y \in X'$ such that p is not collinear with U_y and hence X' would not be proper. □

Lemma 8.2. *Consider a convex subspace X' containing p and q such that $U_p \cap U_q = \varnothing$. Then it equals X itself, $\Pi(x; P)$ or $\Pi(L, (P_h)_{h \in H})$, where x is a point, P and P_h are maximal singular subspaces of Δ and H is a subset of the point set of the line L with $|H| \geqslant 2$. Moreover, $P_{h_1} = P_{h_2}$ implies $h_1 = h_2$.*

Proof. If U_p and U_q are opposite, as in Lemma 7.2, we get that $X' = X$. Since the maximal singular subspaces are of dimension $n-1$, we may assume $U_p \cap U_q^{\perp} = \{x\}$ and $U_q \cap U_p^{\perp} = \{y\}$. Let L be the line through x and y. Let W_p be the maximal singular subspace containing U_p and y, and W_q the maximal singular subspace containing x and U_q. By the same argument as in Lemma 7.2 we get that $\{z \in X \mid L \subseteq U_z\}$, $\{z \in X \mid x \in U_z \subseteq W_p\}$ and $\{z \in X \mid y \in U_z \subseteq W_q\}$ are subsets of X'.

By applying Lemma 8.1 to the upper residue of x we see that if X' contains r such that $x \in U_r$, $y \notin U_r$ and $U_r \nsubseteq W_p$, then X' contains all $(n-2)$-spaces through x. Applying Theorem 3.3 to W_q, we see that X' then contains all $(n-2)$-spaces contained in W_q. Hence X' equals X or $\Pi(x; W_q)$ by Lemma 8.1. Similarly if X' contains r such that $x \notin U_r$, $y \in U_r$ and $U_r \nsubseteq W_q$.

Now assume that X' contains r such that $L \cap U_r = \varnothing$. Since L can not be collinear to U_r, we may assume that x is not collinear to U_r. Denote by Z the intersection of U_r and W_p. Then we can find $s \in X'$ such that U_s is contained in W_p, U_s and U_r are locally opposite and $U_r \cap U_s = Z$. By looking at the upper residue of Z we see that all $(n-2)$-spaces containing Z are contained in X'. By Theorem 3.3 this implies that all $(n-2)$-spaces in W_p are contained in X'. Hence X' contains r' such that $x \notin U_{r'}$, $y \in U_{r'}$ and $U_{r'} \nsubseteq W_q$ and we can apply the previous paragraph to get that X' is either X or $\Pi(y; W_p)$.

We may now assume that X' contains r such that $L \cap U_r$ is a point z different from x and y. If L is not collinear to U_r, then we can find an $(n-2)$-space containing L (so contained in X'), intersecting U_r in the point z and locally opposite U_r. Hence X' contains all $(n-2)$-spaces through z. Now, since X' contains $\{w \in X \mid x \in U_w \subseteq W_p\}$ and $\{w \in X \mid y \in U_w \subseteq W_q\}$, this implies that all $(n-2)$-spaces contained in $W_p \cup W_q$ are contained in X', by Theorem 3.3. Then Lemma 8.1 implies $X' = X$. Hence L is collinear with U_r; let W_r be the maximal singular subspace containing L and U_r. By considering the upper residue of $Z = U_p \cap U_r$ and applying the argument in the first paragraph of this proof, all $(n-2)$-spaces contained in W_r containing $\langle z, Z \rangle$ belong to X'. Since all $(n-2)$-spaces containing L are contained in X', Theorem 3.3 implies that all $(n-2)$-spaces of W_r containing z are contained in X'. Also note that if $W_r = W_p$, then using Theorem 3.3 all $(n-2)$-spaces of W_p are contained in X'. As in the previous paragraph we obtain that X' equals X or $\Pi(y; W_p)$. Hence we may assume that there exists a subset H of the point set of L and for each $h \in H$ a maximal singular subspace P_h containing L such that $P_{h_1} = P_{h_2}$ implies $h_1 = h_2$, and $\Pi(h, P_h) \subseteq X'$. E.g., $P_x = W_p$, $P_r = W_r$ and $P_y = W_q$. $\square$

Lemma 8.3. *For each singular subspace M of Δ of dimension at most $n-2$, each set H of hyperplanes of M and, for each $h \in H$, P_h a maximal singular subspace containing M such that Equation (C) holds, $\Pi(M, (P_h)_{h\in H})$ is a convex subspace of X.*

Proof. Set $X' = \Pi(M, (P_h)_{h\in H})$. Clearly $\Pi(M)$ and $\Pi(h, P_h)$ are convex subspaces, with $h \in H$. Consider $h \in H$, $x \in \Pi(M)$ and $y \in \Pi(h, P_h)$ arbitrary. We will look at the upper residue of $U_x \cap U_y$. Note that both U_x and U_y are collinear with M, so, looking in this residue, U_x and U_y are contained in the subspace $\Pi(\langle M, U_x \cap U_y\rangle; P_h)$, which is convex by Lemma 8.1. Now $\Pi(\langle M, U_x \cap U_y\rangle)$ and $\Pi(U_x \cap U_y, P_h)$ are contained in X', using $h \subseteq U_x \cap U_y$. Hence all points on a shortest path between x and y are contained in X'.

Now consider $h_1, h_2 \in H$, $x \in \Pi(h_1, P_{h_1})$, $h_1 \neq h_2$, and $y \in \Pi(h_2, P_{h_2})$ arbitrary. In the residue of $U_x \cap U_y$, U_x and U_y are contained in the subspace $\Pi(\langle M, U_x \cap U_y\rangle, (P_h)_{h\in\{h_1,h_2\}})$, which is convex by Lemma 8.1 if $P_{h_1} \neq P_{h_2}$. Note that this subspace is contained in X'. If $P_{h_1} = P_{h_2}$ and $U_x \neq U_y$, then the smallest convex subspace containing U_x and U_y is the line $\Pi(U_x \cap U_y, P_{h_1})$, which is contained in X' by Equation (C) and the fact that every hyperplane of P_{h_1} intersects M in (at least) a hyperplane of M. So in any case, the shortest path between x and y is contained in X', and if x and y are collinear, the line joining them is contained in X'. $\square$

Theorem 8.4. *Let Γ be a Lie incidence geometry of type $B_{n,n-1}$ and Δ be the corresponding polar space. Let $\Gamma' = (X', \mathcal{L}')$ be a proper convex subspace of Γ. Then there exists a singular subspace M of Δ of dimension at most $n-2$, a set H of hyperplanes of M and, for each $h \in H$, a maximal singular subspace P_h of Δ containing M and satisfying Equation (C) such that $\Gamma' = \Pi(M, (P_h)_{h\in H})$.*

Proof. Consider a singular subspace M of Δ of dimension at most $n-2$ such that all singular subspaces of dimension $n-2$ through M are contained in X', and assume that there does not exist a proper subspace of M with the same property.

If all elements of X' contain M, we set $H = \varnothing$. So we may assume that there exists $p \in X'$ such that $M \not\subseteq U_p$. Also note that there exists $q \in X'$ such that $M \subseteq U_q$ and $U_p \cap U_q = U_p \cap M$. Now we can apply Lemma 8.2 in the upper residue of $U_p \cap M$ to obtain a singular subspace M' of M of dimension at most $\dim(U_p \cap M) + 2$ such that all singular subspaces of dimension $n-2$ through M' are contained in X'. By our minimality assumption on M this implies that $\dim(U_p \cap M) \geqslant \dim(M) - 2$. Assume now $\dim(U_p \cap M) = \dim(M) - 2$. Then Lemma 8.2 together with the minimality of M actually implies that any $r \in X'$ such that $U_p \cap M \subseteq U_r$ should intersect M in a hyperplane, contradicting the existence of p. Hence $h := M \cap U_p$ is a hyperplane of M and, again by Lemma 8.2, $P_h := \langle M, U_p\rangle$ is a maximal singular subspace such that $\Pi(h, P_h) \subseteq X'$.

Now, for any hyperplane h of M there is at most one maximal singular subspace P such that $\Pi(h, P) \subseteq X'$, otherwise Lemma 8.1 would show that M is not minimal by looking at the upper residue of h. Now assume that for two distinct hyperplanes h_1 and h_2 of M there exists a maximal singular subspace P such that $\Pi(h_1, P)$ and $\Pi(h_2, P)$ are subspaces of X'. By considering P in the upper residue of $h_1 \cap h_2$ and applying Theorem 3.3 we see that all singular subspaces of dimension $n-2$ contained in P and containing $h_1 \cap h_2$ are contained in X'. So for any hyperplane h_3 of M such that $h_1 \cap h_2 \subseteq h_3$ we have $h_3 \in H$ and $\Pi(h_3, P) \subseteq X'$ $\square$

Remark 8.5. *We did not consider* $\mathsf{D}_{n,\{n-1,n\}}$ *as the type of a Lie incidence geometry, but since in the above proof we never used the fact that the polar space* Δ *is related to a thick building, the conclusion of Theorem 8.4 is also valid for Grassmannians of type* $\mathsf{D}_{n,\{n-1,n\}}$.

9. Convex subspaces of Lie incidence geometries of type $\mathsf{A}_{n,\{1,n\}}$

Let Γ be a Lie incidence geometry of type $\mathsf{A}_{n,\{1,n\}}$ and let Δ be the corresponding projective space over some skew field k. The points of Γ will simply be called *flags* of Δ. Given a flag $\{p, H\}$, with p a point of Δ and H a hyperplane of Δ such that $p \in H$, we call p the *point-component* of $\{p, H\}$ and H the *hyperplane-component* of $\{p, H\}$. For a pair of incident (possibly coinciding) subspaces S, T, $S \subseteq T$, of Δ, we denote by $\Pi(S, T)$ the convex subspace of Γ consisting of all flags $\{p, H\}$, with $p \in S$ and $T \subseteq H$. These are the convex subspaces corresponding to residues of the underlying building. But, as we will see, there are also other convex subspaces. Indeed, we shall prove the following result.

Theorem 9.1. *Let* Γ *and* Δ *be as above. Let* $\Gamma' = (X', \mathcal{L}')$ *be a proper convex subspace of* Γ. *Then either there exist subspaces* S, T *of* Δ, *with* $S \subseteq T$ *and* $0 \leqslant \dim S \leqslant \dim T \leqslant n-1$ *such that* $\Gamma' = \Pi(S, T)$, *or there exist subspaces* S, T *of* Δ, *with* $\dim S = 1 + \dim(S \cap T)$ *and* $1 \leqslant \dim S \leqslant \dim T \leqslant n-2$ *such that* $\Gamma' = \Pi(S, \langle S, T\rangle) \cup \Pi(S \cap T, T)$.

Proof. Let $\Gamma' = (X', \mathcal{L}')$ be a convex subspace of Γ. Let S be the set of the point-components of all members of X' and let $\mathcal{H}$ be the set of all hyperplane-components of all members of X'.

Claim 1. *If* Γ' *contains two opposite flags of* Δ, *then* $\Gamma' = \Gamma$. Indeed, this is Lemma 6.1.

From now on we may assume that Γ' does not contain opposite flags of Δ.

Claim 2. S *is a subspace of* Δ. Indeed, let $p, p' \in S$ and let $\{p, H\}$ and $\{p', H'\}$ be two members of X'. If $p \notin H'$ and $p' \notin H$, then the claim follows from Claim 1. If $p \in H'$ and $p' \in H$, then $\{p, H\}$ and $\{p', H'\}$ are either collinear ($H = H'$) or symplectic. In the former case, the claim follows from the fact that Γ' is a subspace; in the latter case the convex closure of $\{p, H\}$ and $\{p', H'\}$ contains all flags $\{q, G\}$ with $q \in pp'$ and $H \cap H' \subseteq G$, and the claim also follows. Lastly, we may assume $p \in H'$ and $p' \notin H$. Then $\{p, H\}$ is special to $\{p', H'\}$, and $\{p, H\} \bowtie \{p', H'\} = \{p, H'\} \in X'$. The claim then follows from the fact that Γ' is a subspace and $\{p, H'\}, \{p', H'\} \in X'$ are collinear.

Dually, every hyperplane containing the intersection T of the hyperplane-components of all members of X', is a hyperplane-component of some member of X'.

Claim 3. *Let* $p \in S$ *and* $H' \in \mathcal{H}$ *with* $p \in H'$. *Then* $\{p, H'\} \in X'$. Indeed, since $p \in S$, there exists a hyperplane $H \in \mathcal{H}$ such that $\{p, H\} \in X'$. Likewise, there is a point $p' \in S$ such that $\{p', H'\} \in X'$. Then the convex subspace closure of $\{p, H\}$ and $\{p', H'\}$ contains $\{p, H'\}$ (just as in the proof of Claim 2).

Claim 4. $\dim\langle S, T\rangle \leqslant \dim T + 1$. Indeed, suppose $\dim\langle S, T\rangle \geqslant \dim T + 2$ and let $p, p' \in S$ be such that $pp' \cap T = \varnothing$. Then there exist $H, H' \in \mathcal{H}$ with $p \in H \setminus H'$ and $p' \in H' \setminus H$. But then by Claim 3 we have $\{p, H\}, \{p', H'\} \in X'$ and $\{p, H\}$ is opposite $\{p', H'\}$, a contradiction.

Claim 5. *If $S \subseteq T$, then $\Gamma' = \Pi(S, T)$.* Indeed, this is obvious in view of Claim 3. Note that $S = T$ is allowed.

Claim 6. *If $S \not\subseteq T$, then $\Gamma' = \Pi(S, \langle S, T\rangle) \cup \Pi(S \cap T, T)$.* Indeed, this also follows from Claim 3 and the fact that, if $p \in S \setminus T$, and $\{p, H\} \in X'$, then $\langle S, T\rangle \subseteq H$ (because $\langle p, T\rangle = \langle S, T\rangle$ by Claim 4). Moreover, one checks that $\Pi(S, \langle S, T\rangle) \cup \Pi(S \cap T, T)$ is indeed a convex subspace. $\square$

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