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An Interdisciplinary Analysis of the Concept of Percent

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Cognitive and Information Sciences

by

Ayme Tomson

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ABSTRACT OF THE DISSERTATION

An Interdisciplinary Analysis of the Concept of Percent

By

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Doctor of Philosophy in Cognitive and Information Sciences

University of California, Merced, 2021

Professor Teenie Matlock, Chair

The concept of percent is ubiquitous in modern society. Percent is a common component across all forms of communication and occurs along a continuum from small scale (e.g. the tip after a meal) to large scale (e.g. deciding to evacuate from a natural disaster) decisions. Yet, the concept of percent is not simple. Percent is a functional number and therefore takes its conceptual meaning from its functional context. This dissertation analyses the concept of percent as a complex mathematical tool using interdisciplinary approaches focused on three key aspects: construction, context, and comparison. For construction, this work focuses on the inability of the concept of percent to coalesce as an integration of a network of ideas created from historical commercial practices and abstract numerical structures over the ages. For context, natural language processing applications and machine learning approaches are fine-tuned on a corpus of natural language and show the role of language formality as it pertains to percent as a functional number in comparative situations of many types. For comparison, the concept of percent is shown to be most similar to decimals, over integers and fractions, via a continuous-measures behavioral magnitude comparison study. These three interdisciplinary approaches present qualitative and quantitative insights relevant to addressing the known challenges of the concept of percent.

Chapter 1

Percent as a Quantity and Number

1.1 Introduction

One of the most fundamental characteristics of our world is quantity. Every object exists with quantitative properties such as mass, spatial distance, heat, and temporal distance. Our embodied interactions with the world allow us to understand these objects and properties via relations of comparison and ratio. We extend these properties to create and understand abstract cultural concepts of quantity. We include ideas of quantity in everyday communication that incorporate both innate and culturally constructed aspects of quantity. Quantity communication has a long history and has been found in the earliest modern human archaeological records (Marshack, 1991; R. Núñez, 2009). The universality and innateness of quantity can be seen in that most languages contain special treatment of quantity words (e.g. "1", "2", and "3") for very small quantities (Hurford, 1987).

Quantity is communicated via multiple modalities (e.g. speech, gesture, symbol) and within each modality, quantity has distinct cultural manifestations which also change temporally. However, individual understanding of quantity communication spans a vast range depending upon various factors such as educational background and socio-economic status. Beyond these small quantities, cultures have used reference to body parts for quantities from "4" to "30" such as the Manus society in New Guinea where the word for the quantity six literally means "wrist" (Ifrah, 2000). Over time, linguistic evolutionary processes have allowed for compound references to quantity and to the eventual compression of these number terms. For example, "mboona" in the Ali language refers to the quantity 10 and is a compression of the phrase "moro boona" meaning literally "two hands" (Dehaene, 2011).

The concept of percent is a ubiquitous yet challenging form of quantity. Percent is abundant across many arenas in our daily lives yet persists as a difficult concept for both students and adults to master. These difficulties arise because percent is often considered, and taught, to be a simple mathematical concept, merely associated to fraction and decimal numbers. However, Parker and Leinhardt (1995), in their review of percent in mathematical education, provide an alternative view of percent to that of a simplified view:

> Percent is a multifaceted and complex concept ... rich in relationships, comparisons, and actions. [Percent is] represented by a numeral and symbol [or word] combination that is commonly converted to decimals and fractions while being used in comparative situations of many types.

To address the challenges surrounding the concept of percent, we must begin with an understanding of quantity. In the following sections, first quantity is defined and its use in language and as information is described. Next, the mental representations of quantity are examined from a numeric cognition perspective as well as an embodied cognition perspective. Lastly, the approaches to learning the cultural aspects of quantity are discussed. This cognitive scientific approach to understanding quantity affords the groundwork to understanding the concept of percent. We close this introduction with a summary of percent as quantity and motivate the following chapters of this dissertation.

1.2 Quantity

Quantity is one of the basic classifications of objects and can be thought of as magnitude comparison. Quantity can be symbolic (e.g. number) or non-symbolic (e.g. perceptual area or density) and is often thought of in two ways: intensive and extensive (Lewis & Randall, 1923; Tolman, 1917). Extensive quantities are derived from, and dependent upon, physical properties of an object such as the mass or volume. A change in the amount of an object also changes the extensive quantities associated with that object. Intensive quantities are not dependent upon the size of an object (scale invariant) such as density or concentration. For example, speed (s) is an intensive quantity where s=d/t, and distance (d) and time (t) are extensive quantities.

1.2.1 Magnitude

Magnitude refers to the relative size of an object. Magnitude can be discrete or continuous, can be ordered, and has relationships of equality and inequality. How magnitude is measured depends on the type of the object. The mathematical meaning of magnitude for natural numbers is the size of a set or number of similar objects grouped together (e.g. "four apples"). The meaning of magnitude for real numbers is defined as a distance from zero along a number line. Magnitude measures can be more concrete measures such as brightness of luminescence (light), decibels (sound), and energy (earthquake), or more abstract measures such as area, volume, and density (shapes). Measurements of magnitude can take a numeric form and be exact or approximate (e.g. "four" and "thousands", respectively) or they can take a non-numerical form and be exact for very small numbers and approximate thereafter (e.g. "a couple" and "many" respectively). Physical comparisons of magnitudes are approximate and rely on sensory input such as vision (light), audition (sound), and haptic response (vibration). This comparison only becomes exact when described with numbers. Magnitude can be thought of as a basic unit which can be operated on via comparison relationships such as ratio and proportion.

1.2.2 Ratio and Proportion

Ratio and proportion have been intrinsic to magnitude dating back to Euclid where relations between magnitudes resulted in whole numbers (Heath et al., 1956). These relations have evolved to result in real numbers. Ratio is the concept of comparison between two values or pairs of values and can be written as $a/b = c$ or $a : b = c$. Ratio is required to be positive as it expresses comparison between physical objects and collections of objects. When thought of as a fraction, ratio also cannot have zero in the denominator, however, in a non-numerical sense, ratio can contain a zero in the a or b value. For example, a sports win-loss ratio of $4 : 0$ where the team has won 4 games and lost none. Additionally, ratio is considered to be a relational concept (intensive quantity) which operates on values derived from physical properties (extensive quantities). Ratios can also be irrational values, such as π. In magnitude comparison, the size of the ratio between the two magnitudes determines how easily the comparison is made. According to Weber's law, which states that the ability to compare ratios is determined by the ratios of the magnitudes (Fechner, Boring, Howes, & Adler, 1966), the closer the ratio of two magnitudes is to $1:1$, the harder they are to discriminate.

Proportion is a relationship between two ratios where $a/b = c/d$, where a, b, c, and d are non-zero. Proportion can also be described as a constant ratio k where $y = kx$ and y and x are quantities. There are two main types of proportion, direct and indirect. Direct, or linear, proportion is indicated when k is a constant. Indirect, or inverse linear, proportion occurs when $y = k/x$ and thus y varies inversely with x.

1.3 Quantity in Language

Within language, quantity can be found in many forms: linguistic, pictorial and gestural. Across all of these forms, quantity can be expressed through iconic, indexical, and symbolic signals. From (H. H. Clark, 1996), we can think of icons as descriptions of a thing, indices as indicators of a thing, and symbols as demonstrations of a type of a thing.

Gestures such as hand motions indicating a rise or fall are iconic signals of quantity. We also demonstrate quantity using dual hand motions along a lateral plane from our bodies. Iconic pictorial forms of quantity include images of objects as well as graphs, charts, and figures of numerical data. Numerals, such as "1", often say how many of a kind are being referred to. We gesture with fingers, hands, and even whole body parts to indicate categorical set size (Winter, Perlman, & Matlock, 2013). While ordinal numbers such as "1st", or "2nd" show position of an object. Non-numerical linguistic signals such as quantifiers (e.g. "many", "most") are also used to index quantity.

Numerals and linguistic quantifiers are words or signals that describe the type of, rather than a distinct quantity of, a concrete or abstract object or set of objects. As cultural tools, we learn numerals and quantifiers during our childhood developmental process and are expected to obtain proficiency in communicating and understanding quantity before reaching adulthood. Linguistic quantifiers such as “more” describe the position within a quantity relationship where one object is “greater than” another object.

Communication about quantity is prevalent in everyday decision-making tasks as well as less frequent high-risk decisions. We compare quantities during important life moments such as financial decisions (Kruger & Vargas, 2008; Lusardi, 2012) and health-care choices (Gigerenzer & Edwards, 2003; Peters, Hibbard, Slovic, & Dieckmann, 2007). Each of these decisions requires a basic understanding of quantity. More complex decisions also require an ability to manipulate the quantity relationships. However, when quantity is expressed in numeral form difficulties emerge. Often, quantity is expressed in a simple part-whole relationship which is most easily understood in the form of frequency (e.g. 3 out of ten) but when quantity is expressed as probability or percent (e.g. .3 or 30%), understanding of the quantity is impaired (Gigerenzer, 2015a, 2015b; Hill & Brase, 2012; Peters, 2012). Difficulties exist as probabilities and percentages are the default presentation of quantity for statistics in areas such as health and medicine, weather, economics, sports, academia, politics, and crisis and policy communication.

1.3.1 Quantity as Number

Quantity as a tool has varied across time and cultures. One example of this is numerals. Numerals have evolved from many places in the world from the Far East (China), the Middle East (Mesopotamia), and even Central America (Mayans). Modern base-ten Arabic numerals arrived in Europe by way of the Mediterranean and Middle East. Modern number systems, from the natural numbers to the rational numbers, are thought to exist along one or multiple mental-number-lines which are based on innate proto-numerical abilities but culturally shaped through reading and writing. In western cultures which read left-to-right, the number line moves from smallest numbers (left) to largest numbers (right). This ordering effect is reversed for cultures that read and write from right to left (Shaki, Fischer, & Petrusic, 2009). However, modern number systems are not universally used and cultures exist in the world that have unique forms of quantity signals (Dehaene, 2011, pp. 88-89).

1.3.2 Representations of Quantity

Most adults in cultures that teach numbers have two systems for magnitude representation. The approximate number system (ANS) and a learned cultural-based system of numbers and measurement. Modern whole number representations have spatial correlations and are thought to exist along a mental number line (MNL) with smaller numbers to the left and larger numbers to the right (for cultures that read and write from left to right). This MNL is created as a result of symbolic reference which combines the innate proto-numerical abilities of subitizing and numerosity with culturally created number systems via embodied experiences and enculturation concepts (Dehaene, 2011; R. E. Núñez, 2011).

However, the innateness of a MNL is controversial across fields (Nieder, 2017; R. E. Núñez, 2017a, 2017b). One argument is for a non-abstract representation for number in the brain which stems from the existence of “tuning curves” within the human and monkey brain thought to represent specific numbers (or quantities). However, this argument has difficulties both in the definition of "abstraction" and in the lack of "encapsulation" of these neuronal populations which activate as a part of larger brain networks (Dehaene, 2009; Kadosh & Walsh, 2009; R. Núñez, 2009). Other researchers argue the innate existence of a MNL based on the idea that it is complementary to the ANS and consists of all real numbers (C. Gallistel, Gelman, & Cordes, 2006). This "real number" MNL argument is linked to the "continuous" characteristic of continuous magnitudes. Evidence from animal (Rugani, Vallortigara, Priftis, & Regolin, 2015) and infant (Wynn, 2018) studies are used to argue for an innate MNL. While, cross-cultural numerical cognition studies are often used in support of a culturally instantiated MNL (Cooperrider, Marghetis, & Núñez, 2017). A difficulty in this controversy is the conflation of "number" and "numerosity" across studies. For example, animal and infant studies use non-symbolic stimuli and thus fall into the "numerosity" category while using the term "number" in both results and titles of research.

1.3.3 Mental Number Line

Embodied Cognitivists have suggested the existence of an everyday naturally continuous number line acquired in childhood and elementary school (Lakoff & Núñez, 2000, pp. 278-284). In this view, a naturally occurring continuous MNL might exist upon which whole numbers, integers, rational numbers, and eventually real numbers are placed as locations along a line. Importantly, these numbers merely rest on the line and do not construct the line. Two primary MNLs are posited to exist. One along the horizontal axis and one along the vertical axis. Many studies examine quantity cognition along one of these axes. Additional studies investigating the MNL have looked at both the horizontal or vertical axes together (Sixtus, Lonnemann, Fischer,

& Werner, 2019) as well as in three dimensions (Winter, Matlock, Shaki, & Fischer, 2015).

The horizontal mental number line is thought to arise from cultural practices such as reading and writing and gives rise to the Spatial-Numerical Association of Response Code (SNARC) effect where people respond faster to larger numbers with their right hand and respond faster to smaller numbers with their left hand (Dehaene, Bossini, & Giraux, 1993; Göbel, Shaki, & Fischer, 2011; R. E. Núñez, 2011). This effect has been widely observed (Wood, Willmes, Nuerk, Fischer, et al., 2008) and has also been produced using embodied modalities such as directed pointing (Fischer, 2003) and body movements (Hartmann, Grabherr, & Mast, 2012).

The vertical mental number line is thought to be related to embodied experiences in the world. As we interact with fluids in containers and sets of objects in the world. As we add to them and subtract from them, we learn spatial-numerical associations such as MORE IS UP, and LESS IS DOWN. From these interactions, we create a vertical mental number line where smaller numbers are associated with lower spatial position and larger numbers are associated with higher spatial position. Similar to the horizontal effect, the vertical SNARC effect has been observed in button pushes (Shaki & Fischer, 2012), eye-gaze (Loetscher, Bockisch, Nicholls, & Brugger, 2010), head movements (Winter & Matlock, 2013), and body position (Hartmann et al., 2012) as well as other methodologies.

1.4 Quantity as Information

In language, number terms are often used to assert information. We report quantities of things such as "three people died", "that cost four dollars", "I'm six feet tall", and “six thousand people lost their homes”. We also talk about quantities without explicitly using number terms such as “there were a lot of fireworks”, “there is a small risk of infection”, and “most people loved the movie”. Non-numerical quantity words are called “scalars”. Scalar terms (e.g. “none”, “many”, “all”) are expressions of degree and thought to operate along continuums (Carston, 1998). For example, “none” and “all” would be extreme ends of a continuum with “many” falling closer to “all” than “none” (Van Tiel, Van Miltenburg, Zevakhina, & Geurts, 2016).

Classical views of linguistic quantifiers assign lower boundedness, upper boundedness, or two-sided (exact) boundedness properties to both number terms and scalar quantities in a similar manner according to truth conditions and based on the Gricean Maxims of quantity and quality. Importantly, this view assigns the quantity meaning of an utterance primarily from contextual inference (pragmatics) rather than from any inherent meaning (semantics) of the number term (Kennedy, 2013). Recent studies designed to tease apart the semantic and pragmatic aspects of number terms have found a preference for exact meaning in both adults and developing children using a covered box task (Huang, Spelke, & Snedeker, 2013) and counting and matching tasks (Hurewitz, Papafragou, Gleitman, & Gelman, 2006). A full understanding of quantity in linguistic communication requires further examination of the semantic and pragmatic meanings of number terms as well as theories of communication.

1.4.1 Semantic Meaning

In isolation, a number term has meaning associated with a cardinal magnitude of a set such as the number of objects in a group, an ordinal position, and a spatial location (Lakoff & Johnson, 2008). This semantic meaning provides an exact quantity for each number term and allows for relational operations such as ratio and part-whole and arithmetic operations such as addition and subtraction. Number terms also have roundedness and sharpness properties associated with the idea of significant digits where the placeholder of the rightmost non-zero digit signals the level of exactness (Dehaene, 2011). For example, “120” conveys an exactness for the hundred’s and ten’s placeholder but an approximate value for the one’s placeholder. Due to the placeholder effect, the semantic meaning of number terms can convey either a broad or a narrow two-sided boundedness property.

1.4.2 Pragmatic Interpretation

When number terms are used to convey quantity meaning they often occur within a linguistic context and acquire a pragmatic meaning derived from the given context according to truth conditions such as Grice’s Maxims of Quality and Quantity. Historically, number terms have been considered similar to scalar terms within these utterances (Spector, 2013) such that the contexts convey the quantity information via scalar implicature and result in three types of quantity expression: lower bounded, upper bounded, or two-sided bounded (or exact). It is important to note that the pragmatic meaning is assigned to the utterance which contain number terms and not to the number terms themselves.

Lower boundedness allows for an “at least” quantity meaning for number terms such that “3” can mean “at least 3 and maybe more”. This property is similar to the patterning of scalar terms (e.g. “some”) by leaving open the possibility that higher quantities are true (Horn, 1972). Horn asserts that utterances with number terms first have a meaning of lower boundedness and it is only pragmatic meaning that creates upper or two-sided boundedness. This lower boundedness meaning attribution can be the result of number terms being considered part of a determiner or adjective meaning such as quantification determiners (Barwise & Cooper, 1981), cardinality predicates (Krifka, 1999), or singular terms (Frege, 1980) and thus open to interpretation based on syntactic or sentence level evidence.

Upper boundedness is the opposite of lower boundedness and allows for an "at most" quantity meaning for number terms. The combination of upper and lower boundedness creates a two-sided (or exact) boundedness. Both upper and two-sided boundedness have been thought due to truth conditionals (maxims) or alternatively due to an exhaustivity operator. An exhaustivity operator allows for parallel assessment of all possible utterance meanings for all possible quantities greater ($n+1$, $n+2$, etc.) than the given quantity, n, and returns the most reasonable meaning (and quantity, m, where $m \geq n$) and rejects all quantities greater than m (Chierchia, Fox, & Spector, 2012).

More recently, challenges to this account for pragmatic meaning of number terms have resulted in these same boundedness expressions of quantity due to either underspecification accounts and pragmatic enrichment (Carston, 1998), two-sided semantics resolved via implicature (Breheny, 2008), or semantic polysemy where only upper boundedness is addressed via implicature (Geurts, 2006). Finally, a neo-Fregean challenge rejects the pragmatics approach to number term meaning, and argues that number term meaning resides entirely in semantics properties with scalar (upper/lower boundedness) properties given via scopal interactions with other utterance constituents (Kennedy, 2013).

1.4.3 Communicative Theories

The words we speak always happen within a larger conversational context. Theories looking at speaker's choices in conversation help shed light on the motivations of word choice. Straightforward theories of everyday conversation can be found in The Cooperative Principle and Relevance Theory.

1.4.4 Cooperative Principle

Everyday communication can be described as a cooperative effort where participants interact with each other in a mutually beneficial manner and can be decomposed into four maxims: quality, quantity, relevance, and manner. Grice (1975) proposed this pragmatic principle based on rational theories of expectations in everyday conversation. Grice's framework treats rational language as binary with literal utterances in keeping with the maxims and figurative language as violations of one or more of the four maxims.

Speakers say only what they believe to be true. Quantifiers, especially numerals, play an interesting role in truthfulness as people are predisposed to believe numerical information as true. Thus, numerical quantifiers can be used to relay especially "truthful" information. On the other hand, hyperbole (exaggeration) is ubiquitous in everyday language and is a "simple phenomenon, depending only on a salient quantitative shift toward the extreme end of a scale" (Carston & Wearing, 2015). Quantifiers are commonly used in a hyperbolic manner to express humor (Stewart & Kreuz, 2003), to critique or be persuasive (McCarthy & Carter, 2004), or for irony (Colston & Keller, 1998; Gibbs, 2000). Quantifiers used in hyperbole violate this maxim of quality.

Speakers only provide the amount of information necessary for the given conversation, no more or no less. The type of quantifier used in a speech act provides information about specificity. General scalar quantifiers such as “most”, “many”, or “few” give an approximate quantity while numbers provide more concise quantity information. Thus, speakers can communicate with either a strong (precision) or weak (approximation) focus on quantity.

Speakers strive to add relevant discourse to a conversation. Numbers often provide relevant quantity information such as size or ratio. Numbers can describe relations between objects or events in an objective manner. An interesting use of quantifiers is scalar hyperbole. In the statement, “I believe you 200 percent,” the speaker infuses emotional intensity into the statement without losing credibility through the use of numbers.

The same quantity can be expressed in different ways. Statements like “About 30 percent of women will go on a date just for a free meal” downplay the numerical quantity involved and allow for focus on non-quantity information (Collisson, Howell, & Harig, 2020). In contrast, a common approach in communication is to up-play numerical quantity with emotionally charged (or risk related) issues such as “3 out of 10 Norwegian women experience violence” which focuses on the numerical quantity (Pieter Wijnen, 2018).

1.4.5 Relevance Theory

Sperber and Wilson (1986) provide a pared-down pragmatic theory of verbal communication based on Grice’s relevance maxim. Relevance Theory (RT) posits that the maxim of relevance and the human cognition ability to attend to relevance are sufficient to describe everyday conversation (Wilson & Sperber, 2012). While Grice's cooperative principle describes everyday communication in a binary manner, RT uses a continuous paradigm with degrees of relevance thus eliminating some violations of truth conditionals. In RT, a quantity communication contains the intent of the speaker both to inform the listener of some quantity knowledge and also to draw the listener's attention to this intent to inform. While RT is based on a generative model of language, its focus on how linguistic knowledge (semantics) and non-linguistic knowledge (pragmatics) interact provides a more embodied cognitive understanding of everyday quantity communication (Evans & Green, 2018).

1.5 Understanding Quantity

Quantity cognition presents as a unified system that seamlessly operates over an analogue magnitude representation. The idea of an analogue (versus digital for example) representation for quantity follows from magnitude comparison studies showing inherent distance effects (Moyer & Landauer, 1967) and size effects (Meck & Church, 1983). These effects are thought to exist because the underlying representations for quantity are distributions with the number associated with the quantity as the mean value of the distribution. These distributions are ordered in some fashion with larger numbers corresponding to wider distributions. Distance effects result from overlap between quantity representations. Thus, comparing more similar quantities (e.g. 2 and 3) takes longer time and is less accurate than comparing less similar quantities (e.g. 2 and 9). The size effect posits that if two comparisons have the same distance between quantities, the comparison of the larger pair of quantities will take longer and be less accurate due to the larger overlap of the distributional representations. Since this early foundational work, hundreds of studies have replicated these effects in both humans and animals alike (Dehaene, 2011; C. R. Gallistel & Gelman, 2000). However, not all researchers agree with the dominant view of these effects (Van Opstal, Gevers, De Moor, & Verguts, 2008).

Walsh (2003) proposes A Theory of Magnitude (ATOM) where number, time, and space are part of a generalized magnitude system. Analog Magnitude Representations (AMR), are primary instantiations of spatial, temporal, numerical, and related magnitudes, which require no basic unit of measure, and exhibit a sensitivity to ratio. AMRs are found across species. For example, AMRs are found in the foraging patterns of ducks (Harper, 1982) and fish (Godin & Keenleyside, 1984). AMRs are thought to underlie learned magnitude discrimination in rats (Meck & Church, 1984), as well as magnitude habituation and duration studies in infants (VanMarle & Wynn, 2006; Xu & Spelke, 2000). The exact neural realizations of AMRs are unknown, such as whether they are realized via more neurons firing or a fixed number of neurons firing more rapidly. However, specific areas of the brain have been found to contribute to AMR, namely, the intraparietal sulcus of the left and right hemispheres in the brain.

While it is intuitive to associate continuous representations, such as a line, to AMRs, an important aspect of AMRs is that they need not be continuous or dense (such that between any two numbers, another number exists). This follows from the ability to have sensitivity to ratio using only comparison of discrete sets of objects such as different size networks of neurons firing. Rather than "analogue" meaning "continuous", it is suggested that "analogue" refers to the correspondence between parts of a referent and parts of the object represented (Beck, 2015; Carey, 2000; Maley, 2011).

Suggestions for what AMRs represent have run the gamut of precise integers, real numbers (C. R. Gallistel & Gelman, 2000), approximate cardinal values (Carey, 2000), pure magnitudes (Burge et al., 2010), and numerosity (Dehaene, 2011). Finally, the idea of AMRs representing numerosity seems most acceptable due to its inherent approximation and lack of appeal to cultural number systems (Beck, 2015).

1.5.1 Numerosity

Numerosity is the innate ability to approximately estimate collections of discrete magnitude. This ability is based on the approximate number system (ANS), a cognitive system, which does not rely on symbols such as language or numerals to support the estimation of magnitude. The ANS plays a dominant role in the development of numerical knowledge; individual differences in the ANS have been shown to predict later mathematical abilities (Holloway & Ansari, 2009). In humans, the neural corelates for the ANS have been linked to the intraparietal sulcus with the right IPS linked to digit recognition and the left IPS to both digit and numeral recognition.

An open question is the structure of this underlying representation system. Outside of the subitizing range (1-4), a link between symbolic processing and non-symbolic processing appears to be unidirectional, with symbolic processing influencing non-symbolic processing (Hutchison, Ansari, Zheng, De Jesus, & Lyons, 2020). Some argue that representations of numbers are spatial and shared by both the symbolic and non-symbolic processing systems with bidirectional mappings. Others argue that the ANS is only affected by non-symbolic quantities and there are no bidirectional mappings within the system (Buijsman & Tirado, 2019).

1.5.2 Subitizing

The ability to subitize is considered necessary for quantity cognition, although less integral than numerosity. Humans and other animals are innately able to distinguish visually between up to 3 or 4 objects automatically (Kaufman, Lord, Reese, & Volkmann, 1949). This subitizing ability is supported by the parallel individuation system, or object tracking system; a precise non-symbolic system, which allows for exact discrimination of up to 3 or 4 objects at a time (Feigenson, Dehaene, & Spelke, 2004). However, numerosity processing also plays a role in quantity cognition within this range (1-4) (Dehaene, 2011). The ability to subitize is also present across sensory modalities and has been found in the visual domain as well as in the auditory and haptic domains (Meck & Church, 1983). Recent developmental studies have found a bidirectional link between symbolic and non-symbolic quantity processing within the subitizing range (Hutchison et al., 2020).

1.5.3 Ratio Comparison

Recent work has also posited that ratio comparison is another core process of quantity cognition (Jacob, Vallentin, & Nieder, 2012). Ratio comparison (also proportions) is thought to have a coding scheme remarkably similar to representations of absolute number (magnitude), with the non-symbolic ratio congruity effect suggesting primitive architectures for fractional number processing (Matthews & Lewis, 2017). Neuroimaging habituation studies show similar brain regions activated for both numerosity and ratio comparison processing, namely the intraparietal sulcus. Additionally, distributional activation patterns for ratio comparison suggest similar analogue coding to that of numerosity (Jacob et al., 2012). Viewing ratio comparison as a "more basic, low-level sensory-driven process" (Jacob et al., 2012, p. 159) provides a framework for relational (fractional) magnitude comparisons and a new approach to impacting symbolic math performance (Matthews, Lewis, & Hubbard, 2016).

A view of ratio comparison as an innate process parallels known sensitivity to ratio (Weber's Law). There is a developmental aspect to ratio comparison processing; it is limited in early life but becomes more discerning through development. In habituation tasks, infants have been shown to discern ratios of 1 : 2, but fail for ratios of 2 : 3 (McCrink & Wynn, 2007). Adults are capable of distinguishing ratios up to 7 : 8 for multimodal continuous stimuli (Barth, Kanwisher, & Spelke, 2003) with the capability increasing to ratios up to 11 : 12 for stimuli using sequential magnitudes (Siegler, 2016). Crucially, ratio comparison operates on both sets of discrete objects and continuous magnitudes thus providing an account for a perceived "innate real number system".

1.6 Quantity as Embodied Cognition

How speakers interact with the world creates associations between concepts and shapes cognition. In Conceptual Metaphor Theory and Conceptual Blending Theory these everyday interactions allow for mappings of characteristics of more concrete concepts onto more abstract concepts to facilitate understanding. Through these embodied cognition theories, a numerical system of quantity can be constructed with a focus on the grounded aspects of cognition through conceptual metaphors and the emergent structure of new ideas through conceptual blending.

1.6.1 Conceptual Metaphor Theory

Conceptual Metaphor (CM) theorists argue that embodied interactions with the world create our mental conceptions, such as quantity (Lakoff & Johnson, 2008). Lakoff and Núñez (2000), in their book "Where Mathematics Comes From", use CMs to describe how numerical knowledge is acquired throughout a lifetime and throughout a culture. From an early age, we associate words like "more" with things like liquid in a container increasing in height. We create a map between these two different domains of QUANTITY and SPATIAL LOCATION where associations such as the word "more" and the act of observing an increase in liquid create combine to create CMs such as MORE IS UP and LESS IS DOWN. These embodied experiences help create the individual's cultural learning of number systems which reinforce our conceptions of quantity.

According to Lakoff and Núñez, there are four structurally similar grounding CMs upon which basic ideas of arithmetic exist: ARITHMETIC IS OBJECT COLLECTION, ARITHMETIC IS OBJECT CONSTRUCTION, THE MEASURING STICK METAPHOR, and ARITHMETIC AS MOTION ALONG A PATH. From these four grounding metaphors, more abstract ideas of modern number systems and higher mathematics are built via linking metaphors. CMs have also been used to describe cognition at many levels of quantity cognition from the Spatial-Numerical Association of Response Code (SNARC) effect (R. Núñez & Marghetis, 2015), which shows that people response faster to relatively larger numbers on the right and faster to relatively smaller numbers on the left, to higher level mathematics (R. E. Núñez, 2005) such as the Calculus and the idea of finite limits and actual infinity.

Contrary to common assumptions, there might not be just one conceptualization of the mental number line. Lakoff and Núñez (2000, pp. 278-284) describe both an everyday naturally continuous number line acquired in elementary and middle school and a discretized version of a number line acquired through learning higher mathematics. This duality of number-line-concepts has direct impact on behavioral studies of quantity cognition and research on numeracy and risk literacy. The concept of a naturally continuous number line (the Number-Line blend) arises from a blending of the domains of the CM NUMBERS ARE POINTS ON A LINE (FOR NATURALLY CONTINUOUS SPACE) where the domain of points on a line maps to the domain of a collection of numbers. On this number line, the origin maps to the quantity zero with points to the right being "greater than" points to the left ("less than"). Points on the line left of the origin map to negative numbers; points at the same location on the line equal the same number. The absolute value of a number is defined by its distance from the origin. In this blend, entities are numbers and points at the same time. It is important to note that numbers on this naturally continuous number line do not take up any space. They merely occur on the line and do not construct it.

In contrast, multiple CMs (and blends) are needed to conceptualize the discretized number line: THE STATE-SET BLEND which blends the domain of the special case of a line for naturally continuous space with point-locations and the domain of a set of elements. In THE STATE-SET BLEND, properties of naturally continuous space are associated to relations on elements of the set. To arrive at the blend NUMBERS ARE POINTS ON A LINE (FOR DISCRETIZED SPACE), THE STATE-SET BLEND is mapped to the domain of numbers and all three domains are blended together. A critical distinction of this blend from the one for naturally continuous space is that the "points" on this line take up "space", construct the line, and are elements of a set.

1.6.2 Conceptual Blending

A view related to Conceptual Metaphor Theory is that of Conceptual Blending (CB) Theory (also called Conceptual Integration or Blending Theory) which includes CMs as instances of blends. Fauconnier and Turner (2008a) define CBs as networks of mental spaces that begin with known mental concepts as inputs and result in an integrated blended mental space where the emergent structure is more than the sum of its parts. Thus, CBs are crucial for creativity and cultural evolution. In Conceptual Blending Theory, blended spaces can also serve as input spaces to other blends. This recurrence of blends creates a way to build increasingly more abstract networks of concepts. CBs is considered to be central to human thought and imagination and has been applied to research in various fields such as linguistics, mathematics, computer science, genetics, and anthropology (Evans & Green, 2018, pp. 400-440). Within the domains of numerical cognition and mathematics, CB allows for a description of thinking about mathematical problems in terms of small spatial stories consisting of actors and events which provides insight into areas such as proof-writing (Bou et al., 2015; Marghetis & Núñez, 2013). Another area of numerical cognition, closely tied to quantity cognition, that CBs help clarify is the construction of the modern number system.

The modern number system (along a mental number line) can be thought of as a recursive conceptual blending network where each new blended space of number takes as one of its inputs the old mental space of number. For each new blend of number, the old numbers and the category number is taken from the old number input space. From the other input space, elements that cross-map to the old numbers are fused with that number in the blended space. Elements that do not have a cross-mapping with old numbers, become new numbers in the blend and thus fill in gaps between the old numbers. Both input spaces contribute structure to the blend, thus maintaining previous number properties and also extending those properties for the new numbers (Fauconnier & Turner, 2008a, pp. 242-244). The network begins with the input mental spaces of points on a line, whole numbers, and containers of objects, similar to the CMs of ARITHMETIC IS OBJECT COLLECTION and ARITHMETIC IS MOTION ALONG A PATH, which result in a blend for the integer numbers. The input space of integers numbers blends together with proportions and leads to rational numbers. Additional recursive blends result in the real numbers which blend with two-dimensional geometric space to arrive at the complex numbers (Turner, 2005).

Two important aspects of quantity cognition are the part-whole relationship and the proportional relationship. These concepts emerge in the rational number blend. This particular blend has two drastic consequences where properties of whole numbers are different from properties of all following number blends. First, whole numbers have successors such that the number 1 is followed by the number 2 and there are no numbers in between them, but beginning with rational numbers, there are an infinity of numbers between any two numbers. Second, in whole numbers each number relates to a unique magnitude however the proportional input space creates an emergent structure in the blend that maps an infinite number of rational numbers (i.e.1 = 2/2 = 3/3 = ...) to each unique magnitude (Fauconnier & Turner, 2008a, pp. 242-244).

1.7 Learning Quantity

How do children acquire an understanding of quantity that is linked to modern numbers? Different theories of numerical development have attempted to answer this difficult question (Rips, Bloomfield, & Asmuth, 2008). Research on whole number development has dominated the field. Tasks such as magnitude comparison provide insight into ways that children might learn the relationships between quantity and positive integers. Research on rational number development often extends knowledge of whole numbers with a "whole number bias" affecting rational number development. Not surprisingly, little work has been done to incorporate the full real numbers into quantity cognition. However, research on the cognition of higher mathematics might shed light on how the real numbers are understood (Marghetis & Núñez, 2013).

While it is unclear if all modern numbers are mapped to a single MNL or to different MNLs, most current research theorizes that the MNL begins with whole number understanding (Feigenson et al., 2004; Geary, 2006; Gelman & Williams, 1998; Siegler, 2016). However, some researchers contend that real number understanding underlies all later developmental learning and that symbolic learning limitations restrict access to this "innate" real number system (C. R. Gallistel & Gelman, 2000).

Various approaches to understanding the developmental learning of modern systems of number have been suggested. Privileged domain approaches posit a more "natural" learning of whole numbers than for other types of numbers, such as rational numbers (Gelman & Williams, 1998). Evolutionary and conceptual change approaches also place a "primacy" on whole numbers over rational numbers to varying degrees (Geary, 2006; Vamvakoussi & Vosniadou, 2010). Integrated approaches focus on the similarities and differences of whole numbers and rational numbers (Siegler, Thompson, & Schneider, 2011). More recently, a ratio-processing approach suggests that, like whole numbers, some rational numbers have a form of "primacy" as well (Matthews & Lewis, 2017). These, often conflicting, theoretical approaches to numerical development influence study design and analysis and impact the overall direction of numerical cognition research.

1.7.1 Whole Numbers

Whole numbers (also called the natural or counting numbers) are the set of all numbers which include all positive integers and zero. A complete understanding of whole numbers includes properties of closure and discreteness. For closure, the operations of addition and multiplication on whole numbers result in other whole numbers. Example of non-closure would be the operations of subtraction and division on whole numbers which might not result in other whole numbers. Discreteness describes the non-continuity of whole numbers such that for each whole number n, there is a direct successor which can be found by adding the basic unit of one to it to get $n + 1$ (e.g. $2 + 1 = 3$, $3 + 1 = 4$, etc.). Another aspect of whole numbers is that they are infinite in size which allows for an understanding of potential infinity.

Studies looking at single-digit whole number comparison (e.g. "Which is larger, 3 or 8?") have shown distance and size effects across different experimental designs (Cordes, Gelman, Gallistel, & Whalen, 2001; Feigenson et al., 2004; Moyer & Landauer, 1967). However, studies extending to double-digit comparison have to deal with effects of perceptual and placeholder differences. Multi-digit notation for magnitude has more than one numeral and the placement of the numeral conveys quantity meaning. This change in quantity notation impacts the logarithmic nature of the MNL where larger numbers are harder (take longer and are less accurate) to compare. Double-digit magnitude comparisons such as "Which is larger, 32 or 47?" show distance effects occurring within a decade (e.g. comparing 51 to 53 or 51 to 57) but not across decades (e.g. comparing 49 to 51 or 47 to 51). Any size effect seems mitigated by place congruency which states that it is harder to determine distance if the larger number has a smaller value in the unit place (e.g. comparing 37 to 42 is harder than comparing 32 to 47). These studies have found a more componential than holistic processing of double-digit whole numbers and are not restricted to magnitude comparison tasks (Nuerk, Moeller, Klein, Willmes, & Fischer, 2015).

An, often overlooked, aspect of numerical cognition is the integers which extend the whole numbers to include negative whole numbers. In terms of quantity cognition, negative numbers might conceptualize a "need" or "desire" for a certain quantity of objects. Children that have no knowledge of negative numbers appear to use a rule-based approach in comparing the magnitudes of mixed (positive and negative) numbers, while adults display an inverse distance effect when comparing the magnitude of mixed numbers. This effect provides support to the idea that learning new symbolic systems restructures magnitude representations (Varma & Schwartz, 2011).

1.7.2 Rational Numbers

Once knowledge of integer numbers has been acquired, the next step is to learn rational numbers. Like whole numbers, magnitude comparisons of decimals and fractions display distance effects. Thus, an analogue representation is thought to exist for rational numbers (DeWolf et al., 2014). Ratio-dependent responses to magnitude comparison tasks posit an integrated mental number line for rational numbers across whole numbers, fractions, and decimals (Hurst & Cordes, 2016). Rational number learning begins around the 3rd grade for most students and consists mainly of learning first about fractions and then about decimals. When learning about rational numbers, children often have conceptual and operational difficulty because they try to use their knowledge about whole numbers when dealing with rational numbers (Resnick et al., 1989). For example, the conceptual belief that "the more digits a number has the bigger it is" (Vamvakoussi & Vosniadou, 2004) makes sense with whole numbers but not with rational numbers such as decimals (e.g. 23 and .203 where the fewer digits corresponds to the bigger number). An example of an operational difficulty would be when "multiplication always makes numbers bigger" which is true for whole numbers but not for proper fractions (Fischbein, Deri, Nello, & Marino, 1985).

Fractions

Fractions are rational numbers, a/b, which contain integers in both the numerator and denominator position and where b cannot equal zero. Fractions have a unique notation consisting of multiple integers with a vinculum (fraction bar) separating the numerator from the denominator. A forward slash is also optional in place of the vinculum. This added perceptual processing is thought to contribute to the longer reaction time and lower accuracy of fractional magnitude comparison tasks compared to whole numbers and decimal magnitude comparison tasks (DeWolf et al., 2014). Additionally, fractions convey relational information based on the ratio of the numerator and denominator. In relational reasoning problems, involving discrete or discretized concepts, fraction results have greater accuracy than decimals. However, this 2-dimensional nature of fraction notation encourages whole number bias through componential processing of fraction quantity (DeWolf, Bassok, & Holyoak, 2015). In ratio-dependent responses, fractional responses show whole number bias via eye-tracking methods (Hurst & Cordes, 2016).

A ratio-processing approach, using non-symbolic fractional magnitudes, found a congruity effect for overall fraction size (Matthews & Lewis, 2017). This effect as well as distance effects during magnitude comparisons of fractions suggests a holistic approach to fraction processing. However, as fraction magnitudes get closer together, a whole number bias emerges within the magnitude comparison tasks (DeWolf & Vosniadou, 2015). To help combat this, developmental studies have shown that learning fractions along a number line helps mitigate the whole number bias often found in rational number comparisons and also outperforms fractional learning based on area models (e.g. pie charts) (Hamdan & Gunderson, 2017).

Decimals

Decimals reside in the real-number realm and are considered to be continuous in nature. Decimals are thought to be easier to learn than fractions because we can conceptualize them as fractions where the denominator increases by a factor of 10 with each place value, for example $.5 = 5/10$, $.05 = 5/100$, and $.005 = 5/1000$. Children, when learning decimals, are taught to focus on the structural aspects of decimals. In additive operations, emphasis is put on learning place-value positions and structural consistency of the decimal point. In magnitude comparison studies, decimals are processed holistically and are processed faster than fractions: closer to the speed of processing whole numbers (DeWolf et al., 2014).

1.8 Percent as Quantity

Percent is a quantity because it is a number that represents ratio comparison. The concept of percent has been assigned the status of a number because it is a ratio comparison (Usiskin & Bell, 1983) - not to be confused with count-type numbers such as fractions and decimals. As a quantity, percent has magnitude (the numeral assigns distance from zero or set size). Percent inherently represents ratio and is often considered to be a "privileged proportion" (Parker & Leinhardt, 1995). As a ratio and proportion, percent is an intensive quantity that represents relationships between extensive quantities. Percent is an abstract quantity used to describe relationships between other abstract or concrete quantities.

In language, percent quantity occurs as numerals and words or numerals and a symbol. Thus percent acts as indexical and symbolic signals in language. Percent numerals are affected by numeric cognition effects such as roundedness and in language by semantic and pragmatic effects such as boundedness and relevance. Percent is a generalized mathematical tool for comparison used extensively in descriptive statistical reporting and is problematic in decision making. Percent depends on its context to explain its functional operation (to help construct the percent equation as seen in (1)). This formulaic approach helps to describe the relationships expressed through the concept of percent.

$$percent \text{ of } base = percentage \tag{1.1}$$

Within education, percent is a fundamental building block necessary for success in higher mathematics. The concept of percent is taught at a pivotal point in middle-school education during the time of rational number learning and this can lead to conflation of the concept of percent with fractions and decimals. Percent instruction often emphasizes the similarities between percent and other rational numbers to assist teachers and students in the learning process. However, this simplified approach to learning a complex mathematical tool creates challenges to understanding the concept of percent.

1.8.1 Delving into the Challenge of Percent

This dissertation approaches the concept of percent from an interdisciplinary cognitive science perspective. The difficulties surrounding the concept of percent are evaluated in the following chapters using three different methodologies each looking at a different aspect of the concept of percent.

In chapter 2, the evolution of the concept of percent is examined using historical data and conceptual integration (blending). This work emphasizes the construction of the concept of percent over time as a mathematical tool with many usages and how challenges have emerged from this construction. This chapter discusses qualitative issues involving percent as a number and symbol, percent as an abstract (intensive) quantity, and percent as a shifting signal in language. Additionally, challenges in using percent in varied contexts are analyzed from a view of conceptual integration.

Chapter 3 addresses the concept of percent in context of language usage and as information. Using natural language processing (NLP) techniques and a large natural language corpus supporting varied types of language formality to quantitatively show how context impacts the functional alternations of the concept of percent and thus how percent is used in a given context. This chapter focuses on the concept of percent as a tool - whose functionality is dependent on its context.

In chapter 4, a computer-mouse tracking magnitude comparison study introduces percent magnitude representations to the rational number space. This behavioral methods approach highlights the comparison aspect of the concept of percent and compares and contrasts percent with other rational numbers: integers, fractions, and decimals. This chapter gives a first cognitive look at quantitative comparisons between percent and rational numbers.

The concluding chapter summarizes the concept of percent and current challenges in understanding this concept. It briefly describes how the above listed chapters address the concept of percent using a cognitive scientific approach of varied methodologies and from the different perspectives of construction, context, and comparison. Final thoughts conclude with suggestions for future work on the concept of percent.

Chapter 2

Formal Conceptual Blending in the Concept of Percent

2.1 Introduction

2.1.1 The Concept of Percent

Percent is a common mathematical tool that is taught early and encountered daily. A quick look at the back of most food products will reveal a mass of information compressed into percent form. The concept of percent is used to understand acceptable quantities in health issues, consumer purchases, and even to make planning decisions in our daily lives. Percentages also compress information about probability in weather forecasts and newscasts about political polling or the economy (see Figure 2.1 for a summary of percent in public discourse). Young children are often exposed to common percentages (e.g. 100%, and 50%) and have a basic fractional understanding of the concept of these percentages before exposure to them in elementary and middle-school education. During rational number learning, children are taught the concept of percent – often associated as a "type of fraction" or "like a decimal" number (Parker & Leinhardt, 1995).

Although percent is ubiquitous in everyday life, students and adults have been shown to have difficulty in understanding the concept of percent (Carpenter, Coburn, Reys, & Wilson, 1975; Chen & Rao, 2007; Edwards, 1930; Kruger & Vargas, 2008; Scribner Guiler, 1946). This difficulty may arise from an inability to understand the complexity of the construction of the concept of percent as it evolved over time to meet different societal needs. The concept of percent was constructed over many years, influenced by various sources, and assigned a multitude of tasks. Parker and Leinhardt (1995) provide an excellent review of the concept of percent and its complexities. Some of the major difficulties surrounding the concept of percent that they describe are: the competing proto-origins of the concept of percent from the fields of commerce and mathematics, its definition as a number, and the many usages expected from percent in everyday tasks. The current study utilizes Parker and Leinhardt's review, supplemented with additional sources, to build an investigation into the concept of percent and its evolution over time. We use the approach of conceptual integration networks to analyze why this commonly used mathematical concept is in some cases so difficult to understand.

2.1.2 Conceptual Blending

The theory of conceptual blending posits that human thought emerges from the dynamic integration of networks of knowledge, memory, and perception (Fauconnier & Turner, 2008b). Such integration networks can be constructed spontaneously at the

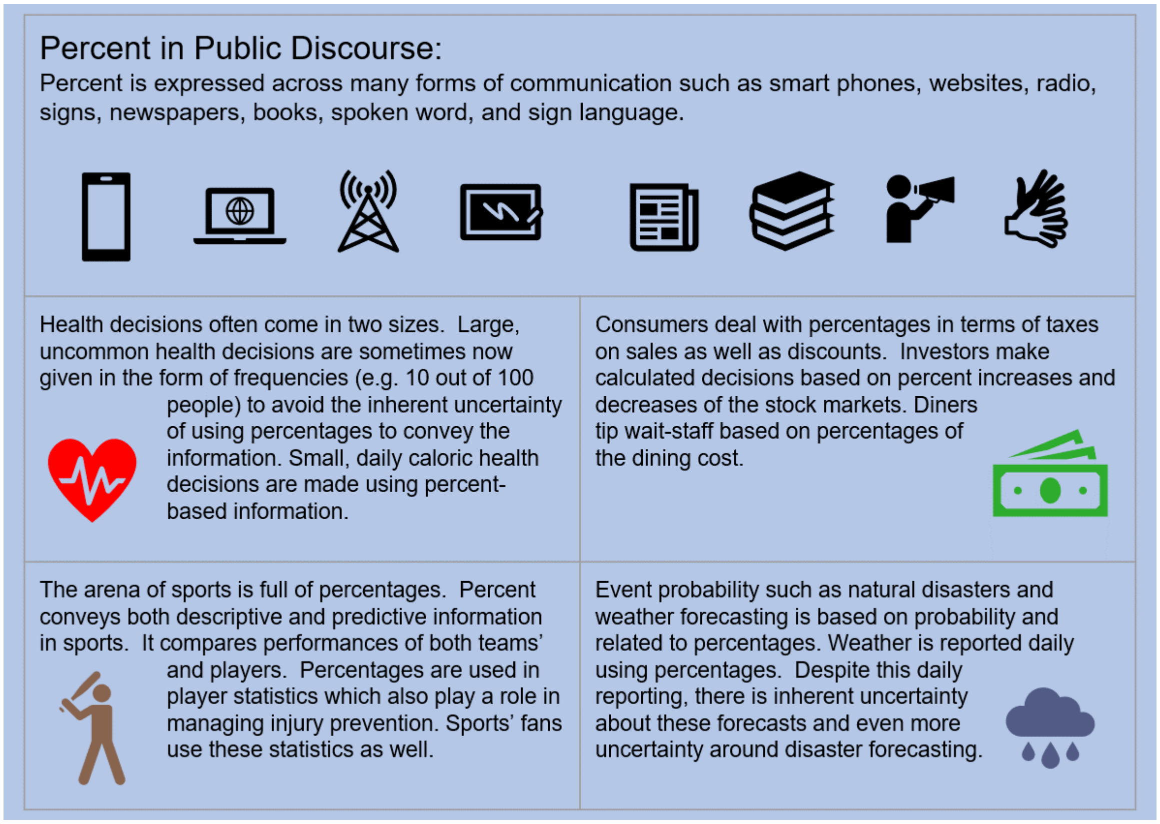


Figure 2.1: Percent is used to convey information in many areas of modern life across various forms of communication.

cognitive level. For instance, the verbal agility of people occupied in improvisational language play during conversation arises through the dynamic construction of conceptual integration networks. Other instances involve the engagement of integration networks that have been developed and disseminated over time through cultural rituals and artefacts. Examples of these types of culturally shared integration networks include rituals such as baptisms, weddings, and graduations as well as artefacts such as compasses, clocks, and computer interfaces.

An example of an integration network from (Fauconnier & Turner, 2008b) is the solution to the riddle of the Buddhist monk using the conceptual blending approach. Arthur Koestler presents the riddle in (Koestler, 1964) and attributes its origin to the Gestalt psychologist Karl Duncker. In the riddle, the monk begins the morning meditating at the foot of a mountain. He then journeys along a path to the mountain peak where he spends several days in further meditation. After these several days, he descends the mountain along the same path along which he had previously ascended.

The riddle poses the following question: Is there a location on the path that the monk occupies at the same time of day on the ascending and descending journeys? Koestler (1964, p. 183) quotes an individual's intuitive solution of the riddle.

> I tried this and that, until I got fed up with the whole thing, but the image of the monk in his saffron robe walking up the hill kept persisting in my mind. Then a moment came when, superimposed on this image, I saw another, more transparent one, of the monk walking down the hill and I realized in a flash that the two figures must meet at some point some time - regardless at what speed they walk and how often each of them stops. Then I reasoned out what I already knew: whether the monk descends two or three days later comes to the same; so I was quite justified in letting him descend on the same day, in duplicate so to speak.

Fauconnier and Turner (Fauconnier & Turner, 2008b) analyze the intuitive solution in terms of structure that emerges through a conceptual blending network. Figure 2.2 shows a diagram of the network involved in the solution. The network integrates four conceptual spaces. The generic space brings the framework of our general knowledge of mountain treks, including the involvement of a moving individual ascending or descending along a path over an extended period of time. The two input spaces bring the specific knowledge given in the riddle itself, such as a unique individual ascending the mountain on a unique day and the identical individual descending the mountain on a unique day not identical to the day of ascent. Within these input spaces, the

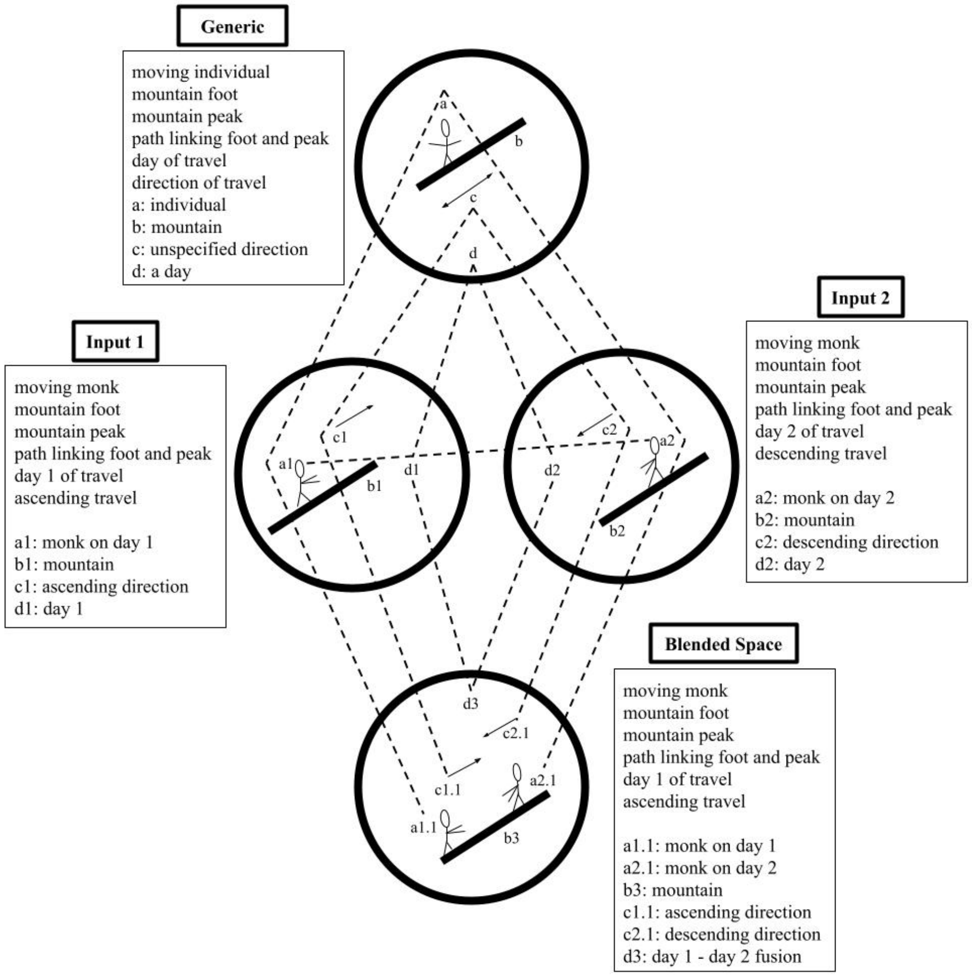


Figure 2.2: Conceptual integration networks can be diagrammed as a series of nodes, each of which represents a conceptual frame and contains the elements that participate in the frame. Relations such as IDENTITY, as well as properties such as DIRECTION, can be linked and projected across frames. These links are represented by lines.

monk in input space one is linked to the monk in input space two via an IDENTITY and a UNIQUENESS relation, and the day in input space one is linked to the day in input space two via an ANALOGY relation. By decompression of the UNIQUENESS relation linking the monk across the two input spaces, the monk in the blended space becomes two separate individuals who nevertheless still share an identity; and by compression of the ANALOGY relation linking the days across the two input spaces, the day in the blended space becomes a single day characterized within this blended space by

IDENTITY and UNIQUENESS relation. This allows us to see that two travellers, one ascending and the other descending the mountain on the same day must meet each other along the path at some point in time; and by analogy, the monk ascending and descending the mountain on separate days must still occupy the same location along the path at some point in time.

With this example, we have attempted to succinctly introduce the general features of a conceptual blending analysis, whereby elements with distinct properties are linked throughout a network of conceptual spaces via a series of vital relations which can change dynamically through operations such as compression and decompression.

2.1.3 Structure of the Paper

In the following sections, the contributing factors to the construction of the concept of percent will be discussed in greater detail. The historical origins of the concept of percent from commercial and mathematical perspectives are analyzed in section 2. The concept of percent as a tool, both as a number and as a symbol are examined in section 3. Section 3 also contains descriptions of the various usages that percent can have in everyday tasks. After showing how the concept of percent has been constructed, we highlight where this structure is flawed in section 4. We conclude with suggestions on how to improve both communicative and educational aspects of the concept of percent. We believe this paper contributes to the current research at the intersection of math education and numeric cognition and to the study of the concept of percent more specifically.

2.2 A History of Percent

The origins of the concept of percent can be traced back to around 300 B.C.E. and can be divided into two main pathways: commerce and mathematics (Parker & Leinhardt, 1995). Historically, commerce and mathematics were separate arenas of life that did not overlap. Commerce was an applied domain dealing with interest and taxation and used in daily commercial enterprise. Mathematics was a theoretical field studied by the wealthy and privileged. Over the past two millennia, these two arenas have periodically interwoven and through this intermingling, the concept of percent has evolved.

2.2.1 Percent and Commerce

The concept of percent within commerce can be traced back to ancient forms of calculation such as the "Rule of Three" - a computational tool for quick calculation of interest and tax on any base amount. The "Rule of Three" was a procedural method, often told in verse and varying across cultures, for finding proportion based on three given amounts. Importantly, these calculations were additive and part-whole in nature, where a fixed interest or tax was taken on (or out of) a given base amount (e.g. 12 sheep were taken out of every 100 sheep). The earliest records of taxation and interest rates are from India (circa 300 B.C.E.) and were performed per months per hundreds (Kautilya, 1967). Shortly thereafter in China (200-100 B.C.E.), the *Arithmetic in Nine Sections* contained the "Rule of Three" as it related to partnership and shares (Boyer & Merzbach, 1989, p.222). Moving to C.E. times, Roman usage of percent precursors, instituted during the time of Augustus, were written in the *centesima rerum venalium*, regarding taxation $\frac{1}{100}$ on goods sold at

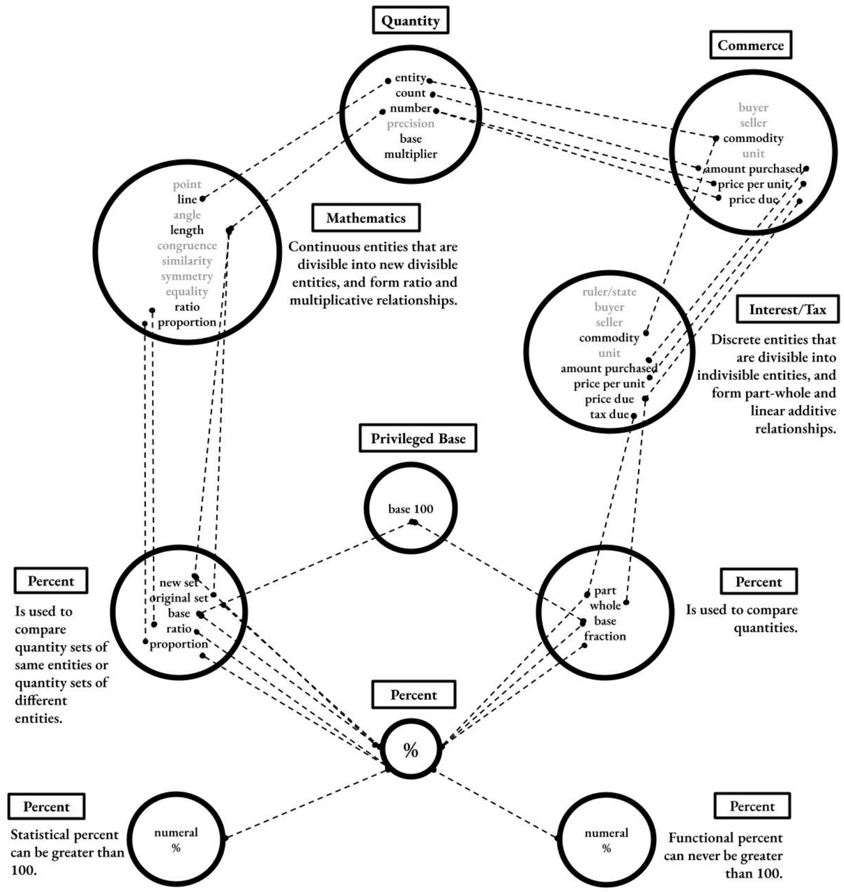


Figure 2.3: Historically, the concept of percent integrates mathematical concepts such as the idea of proportional equivalence in geometry and socio-economic practices such as interest and taxation. Inherent in this integration network is a comparison between quantities – e.g., the length of lines or the quantity of goods bought or sold – to determine a third quantity. The network evolves to include the privileged base of 100, which is further compressed into the percent symbol (%).

auction (Smith, 1923, p.247). Continued use of the "Rule of Three" was found in India dating around 499 (W. E. Clark et al., 1930, p.38-39), 628 (Smith, 1958, p.483), and 1150 C.E. (Datta & Singh, 1962, p.213). As commerce became more complex, compound interest calculations with a base of 100 were found in India dating back to circa 850 C.E. (Datta & Singh, 1962, p.220-225). By the end of the Middle Ages, larger quantities of goods were in appearance and 100 was considered a standard base number as seen in ledgers in Italy (circa 1200 C.E.). During the Renaissance period, the concept of percent was heavily influenced by the standardization of 100 as a base amount and also the use of the "Rule of Three" as a computational tool. It is at this time that mathematicians pointed out the proportional nature of the "Rule of Three" and the concept of percent within commerce started to become multiplicative.

2.2.2 Percent and Mathematics

In mathematics, the concept of percent was multiplicative in nature based on ratios of geometric and algebraic relationships. In early Greece, Euclid included in the *Elements*, theories of proportions for numbers and measures - the mathematical roots of percent (Heath et al., 1956, p. 114). While around the same time in China, the *Arithmetic in Nine Sections* also discussed ratios and proportions (Smith, 1923, 1958). A key moment in the history of percent occurred in the Renaissance period around 1500 C.E. when Niccolo Tartaglia published the first printed edition of the *Elements* in a modern language. Included in the translation were many comments and notes by Tartaglia revealing a focus on practical application. During this time Tartaglia also published his own treatise, the *Trattato generate di numeri e misure*, in which he successfully created an encyclopedia of practical mathematics to include measurement and proportions (Wikisource, 2019). Between these two works, Tartaglia crossed traditional boundaries and imbued the mathematical world with concern for the practical applications (including taxation) of everyday mathematics (Gavagna, 2012).

Around the 17th century there is evidence of compression of the percent terms as well as the use of a percent symbol (per cento $\rightarrow$ per 100 $\rightarrow$ p 100$\rightarrow$ p cento $\rightarrow$ per c$^\circ$ $\rightarrow$ pc$^\circ$ $\rightarrow$ per$\frac{\circ}{\circ}$ $\rightarrow$ $\frac{\circ}{\circ}$ $\rightarrow$ %) (Parker & Leinhardt, 1995, p. 430). This compression helped shift the meaning of the concept of percent from a more concrete quantity to a more abstract relationship. The concept of percent as a general concept of normalization has been linked to the birth of statistics in the late 18th and early 19th centuries (Parker & Leinhardt, 1995, p. 433). Within statistics, large amounts of data of varying types needed to be compared. A tool was needed for generalized comparison - percent fit the bill. Within statistics, percent took on several roles to fill both a need for functional operations as well as descriptive statistics.

2.2.3 A Conceptual Blending Analysis of Percent

Figure 2.3 shows the conceptual integration network associated with the evolution of the concept of percent. In the network, the conceptual spaces of mathematics and interest/tax contain elements from the generic space of quantity and, in the latter instance, the generic space of commerce. The mathematics and interest/tax spaces contain elements – ratio and proportion in the former and interest and tax in the latter – that are linked by the vital relation of ROLE. The role linking these elements can be characterized as "ascertaining" because their function is to learn or discover a new magnitude or quantity (or sets of magnitudes or quantities) with certainty. For example, using proportion to find all equivalences classes of a set or using a tax rule to determine how much wheat a farmer owes to a lord.

When elements are linked across conceptual spaces through the relation of ROLE, they are also necessarily linked through the relation of ANALOGY. For instance, the Queens of England and Denmark perform an analogous role in their respective countries, but they do not share an identity nor are they unique in the sense that they are not a unified entity. Thus, proportion/ratio and interest/tax are linked across conceptual spaces not only through the relation of ROLE but also through ANALOGY. However, when properties of these elements are projected into the spaces associated with the concept of percent, the ANALOGY relation is compressed into one of IDENTITY since they are all referenced by the same term or symbol.

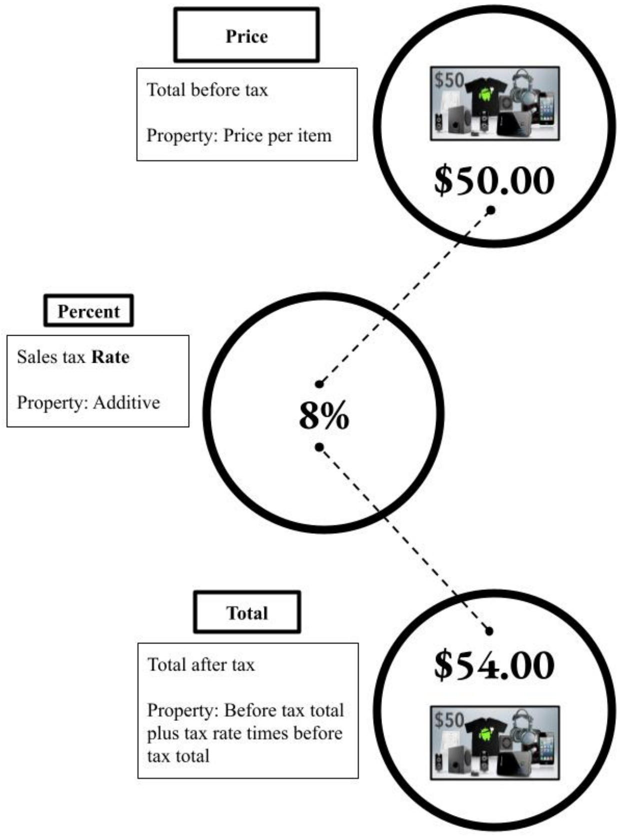


Figure 2.4: Percent can be a tool to obtain an increase or decrease in the size of a single quantity or set. Percent in this case functions as a rate to find the quantity to be added to the original quantity. The diagram depicts an example found in the sentence: *Items on sale for $50 (with 8% sales tax).*

2.3 Percent as a Tool

The concept of percent as a general concept of normalization has been linked to the birth of statistics in the late 18th and early 19th centuries (Parker & Leinhardt, 1995, p. 433). According to Parker and Leinhardt, percent broadly always describes proportional relationship between two quantities and can be categorized in two ways: as a descriptive statistic or as a functional operator. The functional use of percent is considered to be the more important mathematical use of the concept of percent (Davis, 1988) and dates back earlier than its descriptive statistical use. The descriptive use of percent dates back to commercial Italy during the Renaissance and became more widespread after the birth of statistics (Parker & Leinhardt, 1995).

2.3.1 Functional Operations

The functional use of percent involves the creation of a new quantity or set from an original referent and a given percent. As a functional operator, percent will already exist within a given context and then be used to create the relationship between the original set or quantity and the new set or quantity. This relationship can be increasing or decreasing in nature and can be an additive or multiplicative relationship. In this context, percent represents a uniform rate (such as tax, interest, discounts, etc.) that is used to quantify "the magnitude of a *functional operator*", usually taking the form of a fraction or decimal for multiplicative operations (Davis, 1988; Risacher, 1992). In the example in Figure 2.4, percent is multiplied by the original amount to find the tax amount, which is then added to the original cost of the item to create the total cost including tax. An example where the percent rate is multiplicative can be seen in Figure 2.5. Here the original cost of a prescription drug is multiplied by the rate increase to create the new cost of the drug.

2.3.2 Descriptive Statistics

A major use of percent is as a descriptive statistic. Often in reporting data, it is beneficial to highlight changes in the data. When percent is used as a descriptive

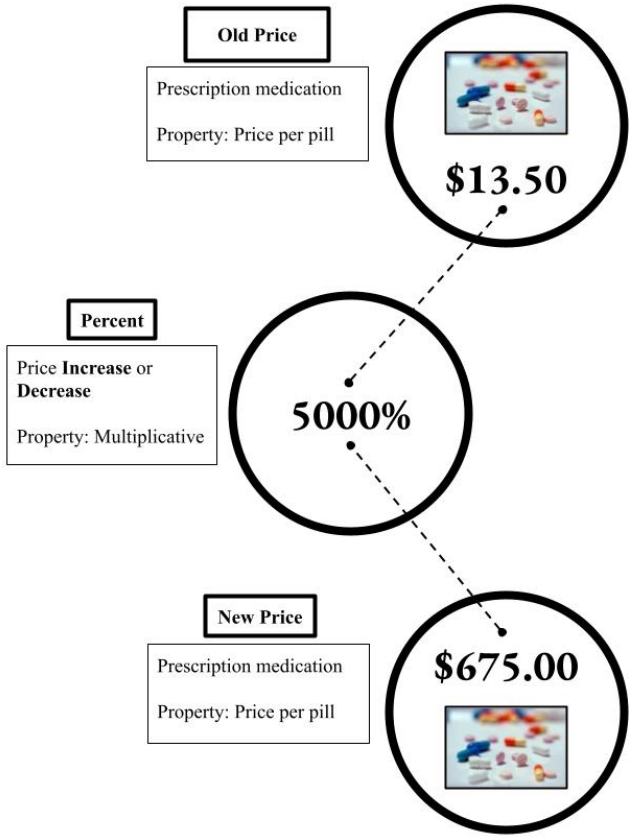


Figure 2.5: Percent can be a tool to compute an increase or decrease in the size of a single quantity or set. Percent in this case functions as a rate of increase to be multiplied with the original quantity. The diagram depicts an example found in the sentence: *The company increased the drug price of $13.50 per tablet by 5,000 percent.*

statistic it often conveys the relationship of one set size to another. The sets can be of the same object (set and subset) or different objects (set and set). This use of percent is based on pre-existing referent quantities. In reporting, both referents need not always be present. It is often the case that one referent quantity must be inferred from the context, given the referent quantity and the percent. When only the percent is given, both referent quantities must be inferred. There are two main forms of percent usage as a descriptive statistic: one percent given to convey a part-whole description and the comparison of two percent statistics. A use of a percent statistic to convey a part-whole description can be seen in Figure 2.6 where the subset of a larger set is highlighted in a specific way. Figure 2.7 shows a comparison relationship between two distinct sets and highlights the multiplicative difference. The percent reveals the proportional relationship of the size of the target set to the size of the reference set. This example focuses on a decreasing aspect of the proportional relationship. Due to the flexibility in reporting referents when using percent for descriptive statistics, it is possible that referent information can be hidden or altered.

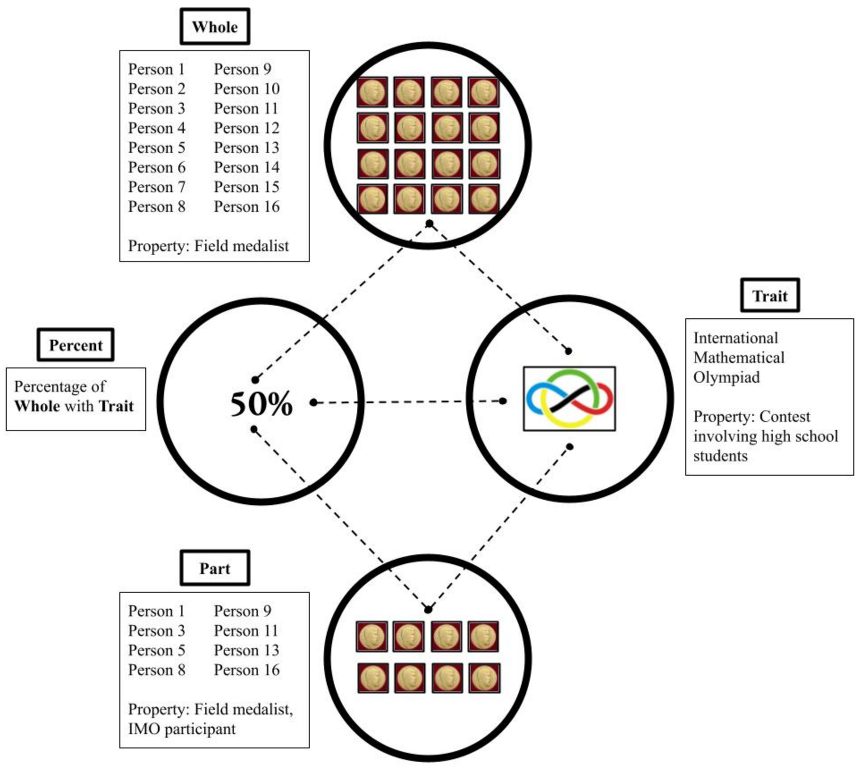


Figure 2.6: Percent can be a tool to describe a partial quantity. In this case, a whole set and a partial set possess a shared quality. The partial set possesses an additional quality not possessed by the whole set. Percent gives the size of the partial set. The diagram depicts an example found in the sentence: *Fifty percent of the past 16 Field medalists were IMO participants.*

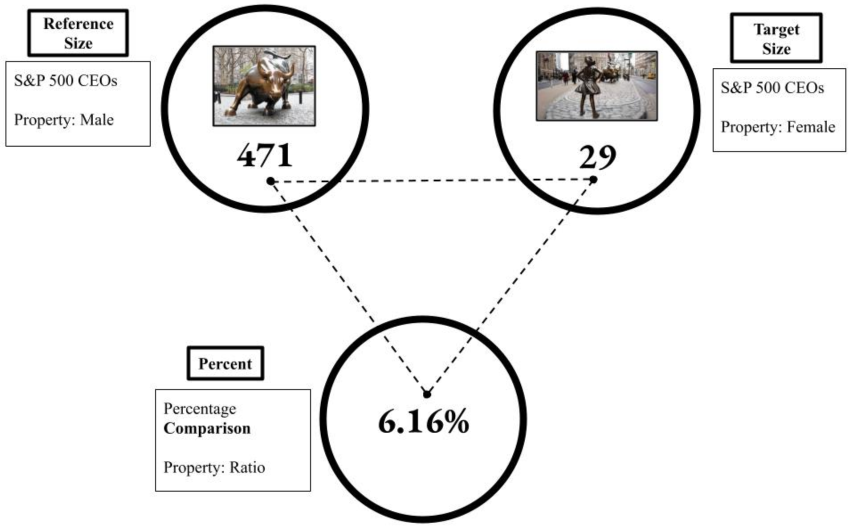


Figure 2.7: Percent can be a tool to obtain a comparison of the sizes of two entities or sets. Percent again functions as a comparison operator. The diagram depicts an example found in the sentence: *The number of female CEOs is 6.16% the number of male CEOs.*

2.4 Where the Tool Breaks

Percent is used every day in many arenas of life. Depending on the context, percent might be working as a descriptive statistic or a functional operator. It could represent a part-whole relationship or a ratio relationship. Additionally, percent is used in these situations to compare quantities across many different comparison types. Unfortunately, percent in these contexts is often misunderstood. Below we list several possible reasons for this misunderstanding relating to: processing the symbol as a compressed mathematical operator, difficulties between part-whole and ratio relationships of percent, and the lack of ability to decompress the many aspects of the concept of percent in a given situation to correctly determine the appropriate application of percent within a context.

2.4.1 Percent Symbol Difficulties

Percent has been defined as a number because it is a ratio comparison concept. In the 1980's, Usiskin and Bell (1983) assigned number status to 6 types of numerals: counts, measures, locations, ratio comparison, codes, and formula constants. A problem exists when percent is taught within rational numbers and a link is formed between count number types like fractions and decimals. The concept of percent is then thought to be similar to count numbers rather than ratio comparison numbers. This number-type confusion can affect how the percent symbol (%) is interpreted and how percent calculations are performed. For example, a common error in solving percent problems is when students ignore the percent symbol or treat the percent symbol as a label, such as the dollar sign (Brueckner, 1930; Edwards, 1930; Kircher, 1926).

2.4.2 Percent Part-Whole and Proper Fractions

The concept of percent is commonly taught following fractions and decimals in middle school education. During this instruction, an emphasis is often placed on magnitudes less than one: proper fractions, fractional decimals, and percent equal to or less than 100. This emphasis highlights the part-whole aspect of the concept of percent for students and causes difficulties when solving percent problems related to percents greater than 100 (Parker & Leinhardt, 1995). While historically, the part-whole aspect of percent was important to commercial calculations, this use of percent related to concrete amounts and objects. Since the birth of percent as a descriptive statistic, an abstract application of percent, the ratio aspect of percent is the more apt usage for generalized comparisons. Simply put, students are primarily taught percent as a part-whole concept which does not sufficiently prepare them to understand percent in its ratio contexts.

2.4.3 Decompression Difficulties

In the above sections, we've listed the different ways that percent is used as a tool. Since around the birth of statistics, the concept of percent has changed from a concept with a simple application to a concept with multiple applications, dependent upon context. This shift from a local calculation tool to a global mathematical comparison tool could only be accomplished through compression of many applications into the percent concept and the symbol. This is how we can have either a part-whole or a ratio understanding of percent, depending on context. As a global mathematical tool, percent is used either functionally or descriptively and across varied types of comparisons. However, as discussed above, this compression is not taught during the instruction of the concept of percent. An emphasis on the part-whole aspect of percent is complemented by a concentration on the percent equation (1) which leads to formulaic memorization of percent calculations. Rote memorization of the percent equation requires the ability to parse a given context to find the needed referents (percent, base, or percentage) which are sometimes not given. This limited understanding of the concept of percent prevents the student (or future adult) from being able to properly decompress how a percent is used in context.

$$percent \text{ of } base = percentage \tag{2.1}$$

2.5 Concluding Remarks

In this study, we have shown how the concept of percent has evolved over time to form an exceptionally complex network of blended identities and roles being compressed into a single symbol (%) or a pair of words (percent or percentage). Through an analysis of the integration network underpinning the concept of percent, as well as a review of the difficulties people exhibit in handling the concept of percent, we have described how problems with correct decompression of this complex network lead to such difficulties. We believe it to be likely that these problems with correct decompression stem from a lack of resources (linguistic or symbolic) for prompting such decompression. As illustrated in Figure 2.3, a number of distinct conceptual spaces are referenced by the same term or symbol. With nothing to indicate which conceptual space we should be operating in, our navigation through this complex conceptual network becomes obscured.

We hope that our analysis has revealed some of the beauty and complexity of percent as an abstract global mathematical tool for comparison. We believe it is possible to master this tool through an understanding of the compression involved in its creation. Successful attempts to teach the concept of percent have included a focus on percent as a proportional operator and a reordering of rational number instruction (Moss & Case, 1999). The reordering of instruction helps to prevent the part-whole bias imparted from fractional instruction to percent instruction. By starting with percent instruction, the ratio aspect can be properly taught before introducing the part-whole aspect. We believe this study also illustrates the importance of contextual cues when communicating the concept of percent to help clarify which usage of percent is required. This study complements previously existing work on the concept of percent as a proportional operator. Additionally, this interdisciplinary study extends work in the fields of mathematical education and cognitive linguistics for the concept of percent.

Chapter 3

Percent as a Functional Number in the Wild

3.1 Introduction

In modern society, percent is a ubiquitous tool of quantity communication. From an early age and throughout life, the average person comes in frequent contact with percent across most forms of communication and in many areas of life (See Figure 3.1). While children often have some exposure to the concept of percent before formal learning, percent is a historically difficult concept for students to master (Carpenter et al., 1975; Kouba et al., 1988). Percent is difficult within this formal setting because it is a functional number that always requires some form of calculation with the degree of difficulty (or type) of calculation being informed by the linguistic context surrounding the specific usage of percent. This difficulty is further compounded in that percent in more casual settings can perform a demonstrative function through hyperbole or other types of figurative language. Context in these cases again informs how the concept of percent is to be understood, and the usual calculations associated with percent in more formal settings may be unhelpful or even nonsensical. In light of this dependence on context, applied linguistics has the opportunity to play an important role in developing greater understanding of the concept of percent, thus contributing to the fields of education, numeric cognition, and numeric discourse.

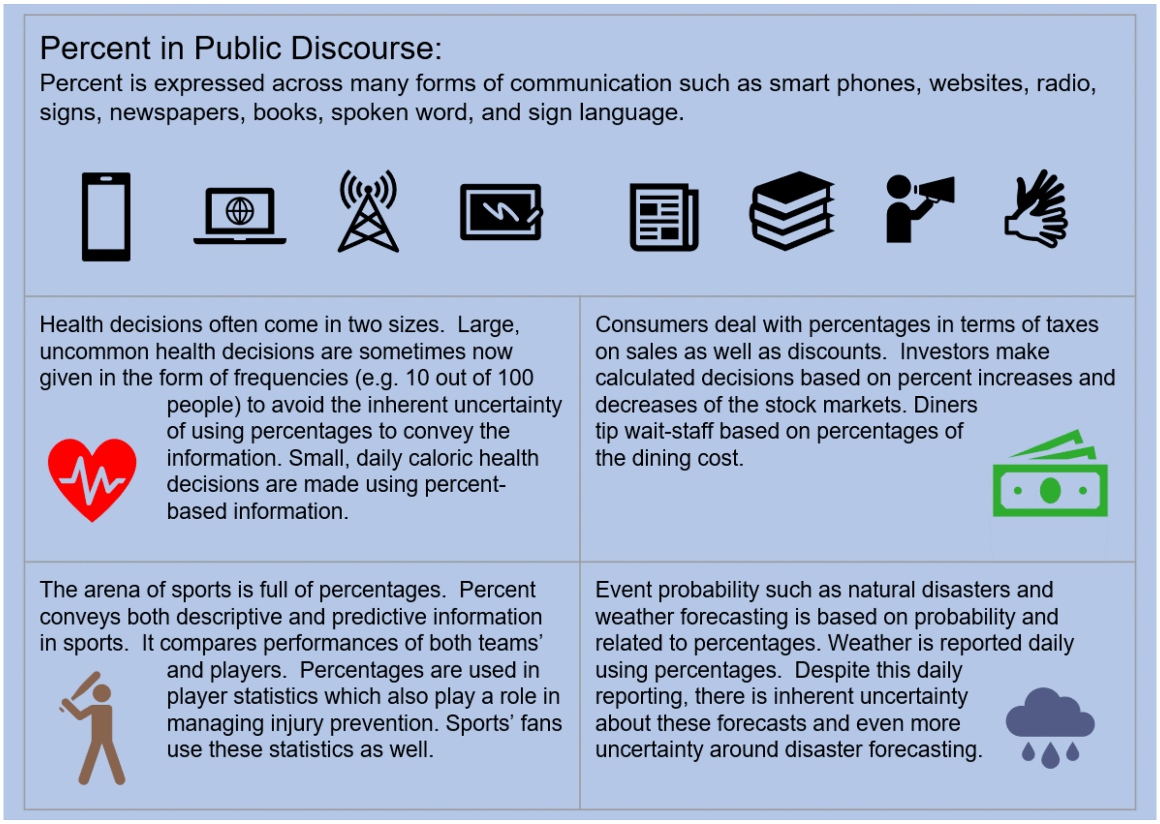


Figure 3.1: Percent in Public Discourse

$$\textit{percent} \text{ of } \textit{base} = \textit{percentage} \tag{3.1}$$

In the field of mathematical education, Parker and Leinhardt (1995) extensively reviewed the concept of percent and provided several ways in which percent operates as a functional number. At the broadest level, percent alternates into a literal and non-literal usage. Within the literal usage, percent further alternates among finer-grained categorizations. Percent can represent a part-whole quantity or a comparison between quantities. Percent can also represent a description of a quantity that already exists or it can represent a newly created quantity. Within mathematics, percent problems can often be represented as an equation as in (1) where the missing piece of the equation (either percent, base, or percentage) determines the functional operations needed to solve the percent problem. In (2), the percent is given as 20% and the base is given as 250. The task is to solve for the percentage using the information that is given. Word problems are a more difficult form of percent problems as students must find the givens and figure out what needs to be solved. To solve percent word problems, students must use the words provided in the context of the problem description to conceptually understand how to construct the percent equation from the information provided. In order to construct the percent equation, the correct "missing" piece of the equation must be identified and the correct other parts of the equation must be found. Solving percent word problems is similar to percent "in the wild" – where percent is used to convey information and individuals are asked to make sometimes serious life decisions based on the information given within the context.

$$20\% \text{ of } 250 = \_\_ \tag{3.2}$$

Examples of percent in context can be seen in (1)-(4) taken from the COCA corpus. Example (1) shows a part-whole use of percent with the reference to 12 to 15 percent of a group – here archers of a certain type. According to Parker and Leinhardt (1995) this is a descriptive usage of percent. In example (2), $5,000$ percent refers to a literal number. That it refers to a literal number is not immediately apparent and can only be confirmed through additional context (not provided here) around the given sentence. In this example, $5,000$ percent indicates a rate of increase from the original cost of the drug to its new cost. This is again a descriptive use of percent. The percent in (3) is rather unique. One of the many ways that percent is used mathematically is in likelihoods. As such, understanding likelihoods assumes an understanding of percent. In this example, the 1.25 percent is being reported as an additive increase to an original likelihood. Here percent is used as a comparison and is descriptive. In example (4), the number 100% refers to an abstract concept ("love") that resists objective quantification. Thus, the percent in this context is non-literal and is used to convey the sense of "completeness" or "wholeness". Percent in this case is demonstrative.

If you're among the 12 *to* 15 *percent of archers who shoot carbon arrows, you're pointed in the right direction. (magazine)* (1)

I get that, but if you raise it 5,000 *percent, it seems a bit like a hostage situation. (spoken)* (2)

Having a good teacher in the fourth grade alone increases by 1.25 *percent the likelihood that a student will go to college. (magazine)* (3)

Part black, part white. It doesn't matter, Mrs. Ryan. This child will be 100% *loved. (movies/TV)* (4)

Across all examples (1)-(4), the percent numbers are all numbers in context. As numbers, each of these examples is influenced by number characteristics (Dehaene, 2011; Lakoff & Núñez, 2000). Additionally, distance and size effects, impact how these numbers are understood, where comparisons between quantities are affected depending on how large the two quantities are and how different in size they are (Moyer & Landauer, 1967). Other numerical effects are rounding effects (Dehaene,

2011) and the pragmatic halo effect (Kao, Wu, Bergen, & Goodman, 2014) which impact how a number is understood with round numbers being less precise. Moving from numbers to context, now brings things like syntax and semantics into the picture.

This study examines the functional alternations of percent in terms of percent operational roles and the context associated with such roles. Here, the term "functional" hearkens to functional theories of syntax and grammar, which propose that functional constraints inform the well-formedness of sentences and the phrases constituting those sentences (Kuno, 1987). The term "alternations" refers to theories of lexical knowledge and semantics, which propose that the context surrounding a word, and the context affording the use of the word, effects subtle changes in the meaning of the word (Levin, 1993). In this study, we propose that functional alternations can account for changes in the meaning of percent. As noted above, how we understand the concept of percent and what we are able (or not able) to do with it shifts when we use it in a non-literal, demonstrative sense as opposed to using it in a literal descriptive sense. We expect that contexts that allow for frequent alternations between demonstrative and descriptive functions, perhaps in some cases requiring frequent alternations, will be more resistant to predictive modeling than contexts that restrict such alternations.

This article takes a natural language processing approach to begin answering these questions by showcasing how well the recent transformer model BERT performs on classification percent context tasks. The model section describes the BERT model used to classify the percent contexts for our different studies. The data section provides meta data for the corpus used in the studies as well as the preprocessing necessary for data preparation. In study 1, we address the broad question of percent functional alternations between literal and non-literal usage and look for clusters within the classifications of the model output along the spectrum of language formality. In study 2, the linguistic cues associated with percent computations within the context are analyzed as possible ways to improve the computational model. Finally, in study 3, we address deeper functional alternations of percent by looking at percent contexts which include two percent references. A full description of the analysis and results is presented in the general discussion section of the paper. Additionally, the article suggests that percent context "in the wild" might be best analyzed through the functional alternations of percent usage. In addition, we demonstrate the effectiveness of the BERT model for classification of number specific language tasks and show that adding additional linguistic cues does not significantly improve the model performance. We believe this paper contributes to the current research at the intersection of applied linguistics and numeric cognition and to the study of the concept of percent more specifically.

3.2 Models

In recent years, pretrained language models have been shown to be effective across many natural language processing tasks (Devlin, Chang, Lee, & Toutanova, 2019). One such model, BERT, or Bidirectional Encoder Representations from Transformers, has achieved state-of-the-art performance on multiple natural language processing tasks including dependency parsing, general language understanding, natural language inference, question answering, and sentence-pair completion (Merchant, Rahimtoroghi, Pavlick, & Tenney, 2020). Further studies indicate that BERT models obtain substantially improved results on tasks such as identification of metaphors (Choi et al., 2021) and classification of scientific articles (García-Silva & Gómez-Pérez, 2021).

BERT is a neural language model that is pretrained on a large general purpose corpus by jointly conditioning on left and right context. As a Transformer model, BERT consists of multiple layers that each contain multiple attention heads. The attention heads assign weights to describe the relative importance of all other tokens in producing the next representation of the current token (K. Clark, Khandelwal, Levy, & Manning, 2019). The BERT pretrained model can be fine-tuned to perform specific downstream tasks through the addition of a single subsequent layer. In studies 1 and 3, we employed the Hugging Face implementations of BERT (BERT-BASE-UNCASED and BertForSequenceClassification) with a subsequent tanh layer. Tanh is the hyberbolic tangent function which as an activation function overcomes the vanishing gradient problem in a computationally efficient manner and often converges faster than other activation functions often employed for multilabel classification.

3.3 Data

Our data was collected from the Corpus of Contemporary American English (COCA) Davies (2015). COCA is a well cited monitor corpus that contains over 1 billion words across 30 years (1990-2019). COCA is balanced across eight genres: spoken, fiction, popular magazines, newspapers, academic texts, fiction, television and movies, blogs, and web pages. These eight genres provide a variety of linguistic contexts that represent a spectrum of language formality. On one end of the spectrum, the spoken and movies/TV genres group together to represent the less formal language genres. Theses genres tend to contain less structure and formatting and have fewer numbers overall. The unplanned dialogues and more everyday content in these genres may contribute to this informal structure. On the opposite end of the spectrum, the academic genre contains the most structure of all the genres with the fiction genre being the second most structured genre within the corpus. The nature of academic research and fiction writing styles as well as publishing requirements narrows the structure and vocabulary of these genres. Within the center of the spectrum are the other genres: magazine, web, blog, and news. To varying degrees each of these center genres contain both some structure and some everyday conversation as well as the usage of many numbers. COCA provides interactive websites for the corpus as well as search-able lists for the top 60,000 most frequent words in the corpus. "Percent" can be found in that list and so COCA provides extensive corpus information about the word "percent". However, COCA does not provide any information for the percent symbol "%" which is of equal interest in our studies and to percent understanding "in the wild". Therefore, we expand our interest in {percent} to include {percent, %, percentage, percentile}, and any variants of those words within the COCA data.

3.3.1 Data Preprocessing

We separated out all sections from COCA that contained the percent terms of interest listed above. We then looked at the sub-section level within COCA for instances of the search terms. On average, the sub-section lengths were about 1000 characters long and the terms of interest were allowed to be located anywhere within the sub-sections. Importantly, punctuation was preserved within all data. However, punctuation that had been added to the corpus as markers (within the spoken genre) was removed, specifically the "@!" symbols before each speaker as well as the "@" symbol usage overall within the spoken genre.

3.4 Study 1

For study 1, we evaluated the use of percent in different contexts through a text classification task in which a language model was trained to predict genre as labeled in the COCA corpus. As noted above, the COCA genres represent a spectrum of language formality with academic and fiction texts comprised of more formal, more highly structured examples of the use of percent; spoken and movie/TV texts comprised of less formal, less highly structured examples; and news, magazine, blog, and web texts comprised of a mixture of more formal and less formal, more highly structured and less highly structured examples. We hypothesized that the model would perform better at classifying genres at the ends of the spectrum – i.e., academic, fiction, spoken, and movies/TV – than it would at classifying genres in the middle of the spectrum – i.e., news, magazine, blog, and web. We included a random model as a baseline model and a decimal model as a control model to compare with the percent model which was our model of interest.

3.4.1 Data Processing

For study 1, we created control data by separating out all sections from COCA that contained decimals to the ones and to the hundredths place (e.g. 3.47), as decimal is often defined in this way. We further cleaned the data to contain only single instances of a decimal term in each sub-section of context. The data consisted of randomly sampled sub−sections from the COCA data for percent and for decimal, evenly distributed across the genres. Each sub−section was labeled with one of the eight genres. Table 3.1 shows the dataset size by genre for study 1.

Model	Academic	Fiction	News	Magazine	Blog	Web	Spoken	Movies/TV	TOTAL
BERT Percent	12685	12750	12282	12459	12255	12280	12672	12744	100127
BERT Decimal	12704	12750	12188	12645	12282	12366	12583	12750	100268

Table 3.1: Study 1 dataset size by genre.

3.4.2 Methods

We used the Hugging Face (Wolf et al., 2019) implementations of BERT (BERT-BASE-UNCASED and BertForSequenceClassification) (as described above) within a TensorFlow pipeline. For tokenization, we set a maximum sequence length of 256, padding shorter sequences with zeros and truncating longer sequences. We used a base model of 12 layers with 768 units and 12 attention heads per layer. For fine-tuning, we input the last layer embeddings of the classification token '[CLS]' into an 8-dimensional tanh layer (the number of classification labels). We trained the models for 3 epochs with batch size 32, the Adam algorithm with weight decay (AdamW), a learning rate of 3e-5, and a cross-entropy loss function. We used 10-fold cross-validation for training and validation and report the average of the results. We performed our computations via a Google Colab Research notebook for high-RAM GPU processing.

3.4.3 Results and Discussion

Using the percent and decimal singles sub-sections data described above, our models performed the F1-scores seen in Table 3.2. First, we point out in Table 3.2 that both BERT models perform significantly better than the random baseline model. The advantages of transfer learning, transformer learning, and bidirectional attention help the BERT models to perform much better than chance. It is worth noting that the "control" decimal model performed much better than the percent model overall. This was to be expected as there is more variation, or noise, within the percent terms of interest. The decimal model only has one format for its term of interest (e.g. 1.34), the percent model has several more (e.g. "%", "percent", "percentage") and introduces issues like changes in part-of-speech for the term of interest.

Model	**F1 Score**	**Accuracy**	**Precision**	**Recall**
Baseline	0.13	0.13	0.13	0.13
BERT Percent	**0.87**	0.88	0.88	0.88
BERT Decimal	0.96	0.96	0.96	0.96

Table 3.2: Study 1 averaged results across 10-fold cross validation.

Looking more closely into model performance across genres (see Table 2), we see that the BERT percent model performs very well on the genres at the ends of the language formality spectrum described above: academic and fiction genres at the more formal end and spoken and movies/TV genres at the less formal end, which is in keeping with our hypothesis for the study. The percent model classifies the genres at the center of the language formality spectrum reasonably well, performing best on the news genre followed by the magazine genre. The model performs substantially worse on the blog and web genres with respect to all other genres. One reason that the percent context in the news genre might stand out is the scripted nature of any functional alternations – i.e., the descriptive function may be required for more serious and informative stories and programs whereas the demonstrative function may play a larger role in human interest and opinion-based stories and programs. The BERT percent model achieved its worst performance on the web and blog genres. Examining the COCA blog data, it appears that blogs contain information about news events, health and fitness regimes, instructional posts, social issues, personal stories, and so on – all of which create extra noise within the data. Similarly, the web genre is a general web scrape of the internet which would include content from all of the other genres and make it difficult for the model to clearly predict anything unique about this genre. With respect to the BERT decimal performance across genres, we see similar trends but improved performance.

Model	**Academic**	**Fiction**	**News**	**Magazine**	**Blog**	**Web**	**Spoken**	**Movies/TV**
BERT Percent	0.95	1.0	0.91	**0.83**	**0.63**	0.63	1.0	1.0
BERT Decimal	0.97	1.0	0.95	0.90	0.87	0.79	1.0	1.0

Table 3.3: Study 1 accuracy by genre averaged across 10-fold cross-validation.

3.5 Study 2

In this study we turned to the numeric cognition, math education, and linguistic literature for theoretical guidance and constructed four context features that could be created computationally from the data. Our aim of study 2 was to see how well we could improve on the output of a BERT model using theoretically motivated but computationally constructed context features. Based on the ability of the BERT model attention heads to capture semantic and syntactic information, we hypothesized minimal improvement with the addition of the context features. To test this hypothesis, we constructed a 3 phase approach to the study. Across all 3 phases, we again predict on genre to investigate the relationship between the concept of percent and the context of percent. In phase 1, we use the constructed context features to predict genre; in phase 2, we use the '[CLS]' token BERT embeddings from study 1; and, in phase 3, we use a combination of the context features and the BERT embeddings.

3.5.1 Context Features

We annotated the percent data from study 1 according to 4 context features: percent number range, percent number roundness, other number in context, and directionality in context. In order to create the context features we performed additional preprocessing of the percent data from study 1. We searched for all number words in front of percent terms of interest and converted them to numeral form. The two features: percent number range and percent number roundness correspond to a number existing directly in front of the percent term of interest (which we will call the percent number). The other two features: other number in context and directionality in context search over the entire sub-sections to create the feature sets.

The percent number range feature was divided into 4 labels {within, above, below, none}. This feature was based on the common definition of "percent" as "of a hundred". The within label categorizes all values of the percent number n, where $0 \leq n \leq 100$, the above label is for all $n > 100$, the below label for all $n < 0$. The none label was used to categorize all sub-sections where no percent number exists. For instances when the percent term of interest occurred as the first word in the sub-section, the label was none.

The percent number roundness feature was based on the pragmatic halo effect which states that round numbers are more approximate and thus have a fuzzy value where the intended meaning is not an exact value as opposed to non-round numbers (Kao et al., 2014). The feature was divided into 4 labels {round, unround, decimal, none}. We distinguish between integer and decimal percent numbers and assume no rounding for the decimal label. The round label categorizes all integer percent numbers that end with either 5 or 0. The unround label is for all integer percent numbers that end in numbers other an 5 or 0. The none label was used to categorize all sub-sections were no percent number exists. Similarly to the percent number range feature, for instances when the percent term of interest occurred as the first word in the sub-section, the label was none.

The other number in context feature was motivated by the question of number frequency across genre. Our section-level rational number survey showed that numbers occurred frequently throughout COCA and often occurred together. However, the survey did not scale to the sub-section level of the data. We thought it relevant to label at the sub-sections level. The other number in context feature was labeled {1, 0}, 1 for other numbers being present in the context, 0 if absent.

The directionality in context feature was labeled {up, down, neutral, none}. This feature was based on the ratio comparison aspect of percent − the increase or decrease between compared percents. We created a list of upward and downward direction terms via linguistic dictionaries and synonym lists. We finalized the upward and downward list and also ranking on proximity to percent term of interest by committee consensus. Examples of the up label category words are "increase", "climb", "up", "hike", "spike", and "rise". The down category included words such as "decrease", "down", "drop", "fall", and "loss". If a word from the down category occurred closer to the percent term of interest than the label would be down. Likewise, if a word from the up category was closer than the label would be up. However, on occasion, a word from the up category and a word from the down category would be equidistant from the percent term of interest and then the label would be neutral. If none of the words in either of the direction categories were found in the sub-section then the label was none.

That statement is supported by the rapid ***increase*** *-* ***15*** *to* ***20 percent*** *per year - in the use of herbal remedies and medicinal plant derivatives worldwide (Consumer Reports* ***1995****).* (5)

In the above example 5 the bold text in the part of the context found by our computation methods for the feature creation. For the feature labels, this sub-section would be labeled: {within, round, 1, up}. We processed all percent data from study 1 and obtained feature sets for our four context feature sets. The context feature set data were used for study 2.

3.5.2 Data Processing

The data for this study used the preprocessed percent data as described in Section 3.1. We constructed the context feature vectors for phase 1 as described above. For phase 2, we constructed BERT embedding vectors using the 768-by-N-dimensional vectors from the CLS token in the last hidden layer of the fine-tuned model from study 1. The CLS token is often used for classification tasks because it is a good representation of the raw sequence input (Rogers, Kovaleva, & Rumshisky, 2020). We scaled the vector values via a minimum-maximum algorithm. For phase 3, we concatenated the context feature and the scaled BERT embedding vectors to fit the implemented models. The dataset size by genre is the same as that of the study 1 BERT percent dataset as shown in Table 3.1.

3.5.3 Methods

For phase 1, we used the SCIKIT-LEARN Python package to implement two traditional machine learning models – i.e, Categorical Naïve-Bayes, and Multi-Layer Perceptron – using the context feature vectors as described in Section 5.1. We used an exhaustive grid search to optimize the hyperparameters for the Multi-layer Perceptron model and found best activation (ReLU), alpha (0.05), hidden layer sizes (50, 100, 50), learning rate method (adaptive), and solver (Adam). For phase 2, we implemented Multinomial Naïve-Bayes and Multi-Layer Perceptron models using the BERT embedding vectors. We again used an exhaustive grid search to optimize the hyperparameters for the Multi-layer Perceptron model and found best activation (tanh), alpha (0.0001), hidden layer sizes (100,), learning rate method (adaptive), and solver (stochastic gradient descent). For phase 3, we implemented Mixed Naïve-Bayes and Multi-Layer Perceptron models using the concatenated vectors. We again used an exhaustive grid search to optimize the hyperparameters for the Multi-layer Perceptron model and found best activation (tanh), alpha (0.0001), hidden layer sizes (50, 50, 50), learning rate method (adaptive), and solver (stochastic gradient descent).

3.5.4 Results and Discussion

The overall results across the different phases of study 2 can be seen in Tables 3.4 and 3.6. For both the Naïve-Bayes and Multi-Layer Perceptron models, context features alone do not perform well. Overall, the Naïve-Bayes models outperformed the Multi-Layer Perceptron models. Using the Naïve-Bayes, the BERT embeddings performed best but their performance was impaired when concatenated with the context features. This impairment might reflect the assumption of independence within the Naïve-Bayes model. There was implicit dependence between the BERT embeddings and the context features as both were created from the same raw data sub-sections. With the Multi-layer Perceptron model, the addition of the context features to the BERT embeddings also impaired the overall F1-score, perhaps also due to overlap in the information stored in the BERT embeddings and the context features. Breaking down the results by genre, we see in Tables 3.5 and 3.7 a similar trend to that of study 1 by the BERT model. Across all models, the genres at the center of the language formality spectrum perform the worst. For both Naïve-Bayes and Multi-Layer Perceptron models, the addition of the context features to the BERT embeddings did not improve the classification across the language formality spectrum.

Model	**F1 Score**	**Accuracy**	**Precision**	**Recall**
Context Features	0.12	0.17	0.15	0.17
BERT Embeddings	**0.85**	0.86	0.86	0.86
Combined	0.78	0.82	0.81	0.80

Table 3.4: Study 2 Naïve-Bayes results averaged across 10-fold cross-validation.

Model	**Academic**	**Fiction**	**News**	**Magazine**	**Blog**	**Web**	**Spoken**	**Movies/TV**
Features	0.22	0.14	0.00	0.24	0.00	0.00	0.09	0.27
BERT	**0.92**	0.99	0.85	0.83	0.64	0.60	0.98	0.98
Combined	0.93	0.95	0.87	0.76	0.55	0.58	0.87	0.76

Table 3.5: Study 2 Naïve-Bayes F1-scores by genre averaged across 10-fold cross-validation.

Model	**F1 Score**	**Accuracy**	**Precision**	**Recall**
Context Features	0.14	0.18	0.15	0.18
BERT Embeddings	**0.78**	0.79	0.79	0.79
Combined	0.65	0.70	0.69	0.67

Table 3.6: Study 2 Multi-Layer Perceptron results averaged across 10-fold cross-validation.

Model	**Academic**	**Fiction**	**News**	**Magazine**	**Blog**	**Web**	**Spoken**	**Movies/TV**
Features	0.24	0.16	0.12	0.22	0.00	0.00	0.11	0.23
BERT	**0.91**	0.92	0.78	0.74	0.60	0.59	0.88	0.78
Combined	0.74	0.83	0.68	0.57	0.46	0.46	0.77	0.69

Table 3.7: Study 2 Multi-Layer Perceptron F1-scores by genre averaged across 10-fold cross-validation.

3.6 Study 3

Study 1 addressed a broad question of functional alternations between literal and non-literal usage of percent in context. For study 3, we dove deeper into the concept of percent and extended the context to include two instances of percent in order to examine the functional alternations of percent at the comparison level. In this study, we again compared performance of BertForSequenceClassification models on a genre classification task. Our data for this study was the single instance of percent data from study 1 and a new set of double instance of percent data for comparison. We hypothesized that both models would perform equally well at both ends of the language formality spectrum but that the double model would perform better in the center of the spectrum due to the additional percent input in the context.

3.6.1 Data Processing

The single instance data is the same data used in study 1. For the double instance data we processed the sub-sectioned COCA data to remove sub-sections that contain any combination of exactly two percent terms of interest. We further cleaned the data in exactly the same manner as the single percent data. After cleaning the percent data, there was extreme disparity in the sample sizes across the genres (smallest genre size was fiction) for the double percent data. To address this issue, we chose to create only one stratified percent double dataset and run the model once without cross validation as the sample size was too small. Table 3.8 shows the dataset size by genre for study 3.

Model	Academic	Fiction	News	Magazine	Blog	Web	Spoken	Movies/TV	TOTAL
BERT Singles	1267	1275	1224	1241	1231	1232	1271	1275	10016
BERT Doubles	1585	506	1437	1390	1650	1640	1343	1524	11075

Table 3.8: Study 3 dataset size by genre.

3.6.2 Methods

Our methods for study 3 replicated those of study 1 in terms of the implementations of BERT and model hyper-parameters. Due to the limited size of the double instance of percent dataset, instead of 10-fold cross-validation we partitioned the data into training, validation, and test sets at a ratio of 6:2:2.

3.6.3 Results and Discussion

The percent singles model and percent doubles model performance is shown in Table 3.9. Both models performed equally well on the classification task for the data described above with little overall difference to report. Examining Table 3.11 for a closer look across genres, we see that the percent singles model performed very well on the most formal and least formal language genres and performed worst on the center genres of the language formality spectrum, similar to study 1. We see that the percent doubles model performed is a similar manner. The drop in the fiction genre for the doubles model links back to the size of the fiction data sample so it is not surprising for this to be lower than the singles model. Overall, the inclusion of a second percent into a context does not seem to improve the models performance.

These results did not fully support our hypothesis that added percent complexity to the context would improve the model's ability to classify across the entire language formality spectrum. The doubles model did perform equally well to the singles model across the spectrum but its improved performance in the center came at the cost of its performance on the external genres. The added complexity of the percent context seemed to have shifted some of the attention of the doubles model.

Model	**F1 Score**	**Accuracy**	**Precision**	**Recall**
BERT Percent Singles	**0.76**	0.77	0.76	0.77
BERT Percent Doubles	**0.76**	0.76	0.76	0.76

Table 3.9: Study 3 results averaged across stratified data.

Model	**Academic**	**Fiction**	**News**	**Magazine**	**Blog**	**Web**	**Spoken**	**Movies/TV**
BERT Percent Singles	0.90	0.87	0.82	0.61	0.68	0.18	0.99	0.95
BERT Percent Doubles	0.89	0.80	**0.81**	**0.58**	**0.70**	**0.31**	0.97	0.95

Table 3.11: Study 3 accuracy by genre averaged across stratified data.

3.7 General Discussion

This article has looked at the concept of percent as it changes "in the wild" from a natural language processing perspective. In study 1, we computationally confirmed our intuitions that percent contexts which on the ends of the language formality spectrum are easier to categorize then contexts in the center of the spectrum. We tried to further enhance the model in study 2 by creating a custom feature set – specific to percent problem contexts. Study 2 further showcased the effectiveness of the BERT model as the addition of the context features to the BERT embedding only marginally improved the model performance. As BERT has been known to implicitly code semantic and syntactic information in its embeddings, this outcome is not entirely surprising. In study 3 we took a slightly more exploratory approach and delved deeper into the concept of percent to analyze the BERT model performance on contexts containing a single percent instance versus two percent instances. Study 3 showed similar results for both models across the differing levels of percent comparison within contexts. However, for study 3, we were not able to gather sufficient data for all the genres and this limited the scope of the study.

Through this work, we found that navigating the concept of percent requires something like code-switching in that functional alternations involve variances in language and meaning. Genres with more frequent and spontaneous alternations were less predictable and in a sense more confusing for the model. Genres with less frequent or more scripted alternations were more predictable and in a sense easier for the model to understand. Considering that the texts we used across our studies was produced by people "in the wild", our findings could have implications for understanding why the concept of percent can be so difficult to master and for identifying how mastering the code-switching aspects of the concept of percent might contribute to better outcomes for students, teachers, and the general public.

For future work, we would like to look more closely into additional BERT model embeddings as the percent token might offer additional insight to enhance or even replace that of the '[CLS]' token (this would be relevant only for the single percent context data). We also want to further annotate the data using human annotation to further mark each sub-section with additional features of interest: literal, figurative, ratio, comparison. Lastly, we would like to look more closely into the percent doubles data. Future work would need to enhance certain genres or find additional data sets to further explore percent comparison for contexts containing more than one percent instance.

Chapter 4

An Introduction of Percent into Magnitude Comparison Tasks Using Computer-Mouse Tracking Methods

4.1 Introduction

Percent is ubiquitous in daily life. From social media and news articles to health and consumer decisions, percent is often used to express quantity and comparisons. The concept of percent is a mathematical tool used for generalized comparisons. According to Parker and Leinhardt (1995), percent is "represented by a numeral and symbol combination that is commonly converted to decimals and fractions while being used in comparative situations of many types." The concept of percent is taught during rational number learning and is often conflated with fraction and decimal number concepts. Although taught early and used often, understanding the concept of percent still remains a challenge to both students and adults (Carpenter et al., 1975; Chen & Rao, 2007; Edwards, 1930; Kouba et al., 1988; Kruger & Vargas, 2008; Scribner Guiler, 1946). To address this challenge, we take a numeric cognition approach to exploring the relationships between percent and other number types within rational number space.

The field of numeric cognition has well established the importance of understanding the nature of mental representations of numerical magnitudes. One such way this has been studied is the distance effect in magnitude comparison tasks, which states that distinguishing between two numbers is easier the greater the magnitude distance between them. Distance effects with single digit comparisons have been found to follow a logarithmic function with comparisons between larger magnitudes being more difficult, when holding the distance between the numbers constant (Moyer & Landauer, 1967). This distance effect has been observed in children as well as adults (Dehaene, Dupoux, & Mehler, 1990; C. R. Gallistel & Gelman, 1992; Xu & Spelke, 2000).

For the majority of numeric cognition research, distance effects have been inferred based on discrete measures such as reaction time and accuracy from button presses. However, in recent years, related studies have used continuous measures by tracking hand and eye movements, thus allowing for a rich and dynamic measure of processing over time during “cognition-as-competition” style tasks. Spivey, Grosjean, and Knoblich (2005) described the cognition-as-competition approach in a language comprehension task where the participants were asked to use a computer mouse to choose a picture the represented the spoken word that was presented to them. The (x, y) coordinates of the computer mouse-trajectories were used as the continuous measures for the study. Spivey et al. found that when the spoken words were phonetically similar to each other (e.g. CANDY versus CANDLE), the trajectories were pulled towards the incorrect response early in the process. Thus showing a sensitivity to influence and dynamic competition during the decision process. This cognition-as-competition approach is related to embodied cognition where bodily affordances contribute to the process of cognition.

A similar bodily affordance in numeric processing is the SNARC effect (Spatial-Numerical Association of Response Codes; (Dehaene et al., 1993)). A SNARC effect occurs in tasks with numeric stimuli along a continuum. The traditional SNARC effect is horizontal where participants tend to select smaller numbers faster on the left and larger numbers faster on the right. However, this is culturally dependent (see Shaki et al. (2009)). The SNARC effect has also been observed along both the horizontal and vertical axes together (Sixtus et al., 2019) as well as in three dimensions (Winter et al., 2015). In two recent single-digit magnitude comparison studies, directed hand trajectories were used as continuous response measures - both showed distance effects (Santens, Goossens, and Verguts (2011); Song and Nakayama (2008)). In their numeric comparison task for single-digit integers, Song and Nakayama (2008) used a visually-guided manual reaching task to provide direct evidence for analogue representation of (integer) numbers. Song and Nakayama (2008) reported their results with an emphasis on the SNARC effect. Whereas Santens et al. (2011) emphasized their results in terms of competitive processing. Santens et al. used an additional congruent and incongruent manipulation to show that the distance effect remains independent of congruency.

4.2 Magnitude Representations

Numeral representation plays a role in the processing of a numeric magnitude. As integers move from single to multi-digit numerals, the question of holistic versus componential processing is raised. Overall multi-digit integers seem to be processed componentially (see (DeWolf et al., 2014) for a brief review of these tasks). DeWolf et al. (2014) look at magnitude comparison tasks across the rational number realm to include multi-digit integers, fractions, and decimals and find distance effects for all three number types.

The numeral representation of fractions, decimals, and percents are different than integers. Researchers have examined how the numeral representations of fractions relate to the distance effect. Bonato, Fabbri, Umilta, and Zorzi (2007) found no distance effect with respect to fraction magnitude comparison. While Schneider and Siegler (2010) argue that the fraction must be sufficiently difficulty enough to require an integrated magnitude to get distance effects. Another aspect of fractions is the dual structure of the format with a numerator and a denominator. DeWolf et al. (2014) find that fractions are harder to process in a magnitude comparison task than decimals and multi-digit integers and suggest it is due to this bipartite ($\frac{a}{b}$) structure of fractions.

Decimal numeral representation is very similar to that of integer. The decimal point is the key difference between these two number formats and often leads to processing difficulties both in children and adults (Givvin, Stigler, & Thompson, 2011; Rittle-Johnson, Siegler, & Alibali, 2001). Often, conceptual knowledge from integers (whole numbers) is applied when processing fraction or decimal numeral representations - this is referred to as the whole number bias (Hurst & Cordes, 2016; Vamvakoussi &

Vosniadou, 2004).

Percent is represented more similar to decimal than to fraction. However, there is often an additional symbol, the "%" symbol (or the word "percent") that perhaps adds difficulty to processing a percent representation. Additionally, a percent could also contain a decimal or fraction numeral (Parker & Leinhardt, 1995).

4.3 Goals of the Present Study

In the current study, we introduced the numeric magnitude for percent. We examined the similarities and differences of the magnitude representations for percent compared with the rational number magnitude representations of single digit integers, decimals, and fractions. We focused on adult performance as each of these number types should be firmly acquired by adulthood. For the single-digit integers, we sought to replicate the Song and Nakayama (2008) reaction time and accuracy hand-reaching results. Across the integers, decimals, and fractions, we sought to conceptually replicate the DeWolf et al. (2014) reaction time and percent error results using values evenly spaced between 0 and 1. Additionally, we added the magnitudes for quarter and three-quarter to evaluate the different numeric representations across these commonly used magnitudes. We assumed that, based on previous work, decimals would be processed fastest and with the least amount of error. Fractions would have the most errors and would be processed the slowest due the the bipartite structure. We hypothesized, based on the similarity of numeral representation, that percents would be processed similarly to decimals. We also hypothesized an increase in processing speed across all number types for the privileged quarter and three-quarter magnitudes listed above. We also expected to see a distance effect for the percent as well as for single-digit integer, decimal, and fraction number type as this is considered indicative of magnitude comparisons based on an internal mental number line.

4.4 Study Design

4.4.1 Participants

75 undergraduate students at the University of California, Merced (mean age 19.8, 49 females), participated in the study for extra credit. All participants were right-handed and had normal or correct-to-normal vision. All experimental protocols were approved by the University of California, Merced, IRB.

4.4.2 Interface

The visual interface consisted of three, equally sized (128x400 pixel) grey rectangles equally spaced across the top of the 23-inch screen against a black background. The middle rectangle was centrally located with the other two rectangles equal distance from the center. The black Arabic digit was presented on the central rectangle (see Fig 4.1).

4.4.3 Task

Participants were seated 70 cm from the front of the monitor. Seat-height was adjusted for appropriate eye-level with monitor. For simplicity, the Integer task will be described. Before each block, participants were instructed that the reference number

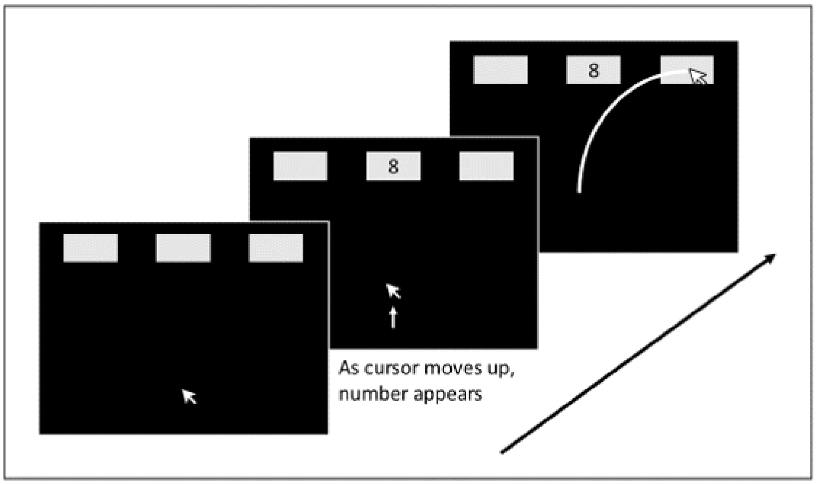
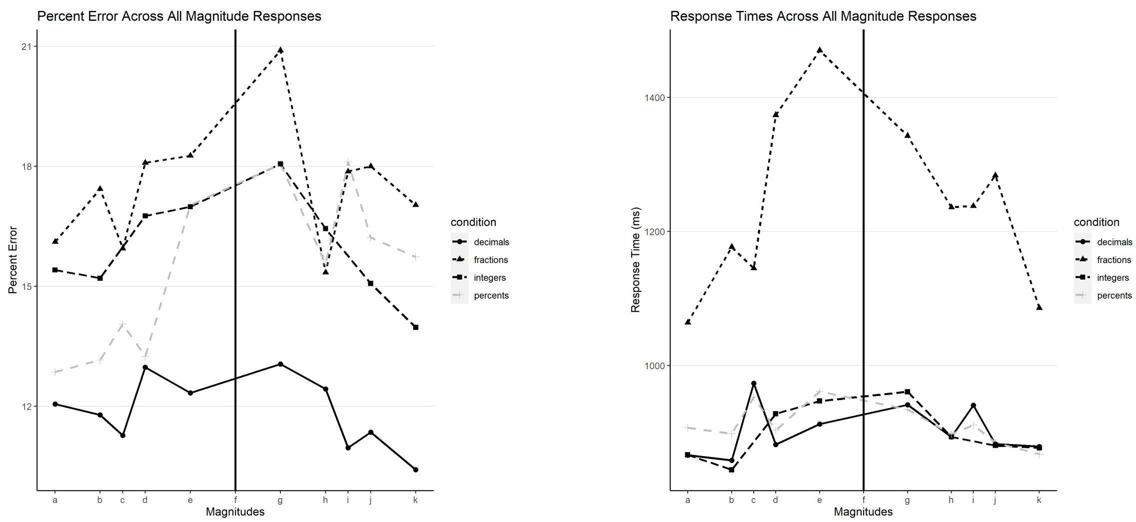


Figure 4.1: Magnitude comparison task. Participants were asked to use the computer mouse to select the left square for quantities smaller than the reference number, the right square for quantities larger than the reference number, and the center square for quantities equal to the reference number.

for the task would be the number 5. Participants were provided a visual aid at any time during the task to remind themselves of the reference number and also given a break between trials with a reminder of the reference number. The participants clicked out of the break to begin each new trial. For each trial, three rectangular buttons were presented across the top of the screen. Upon upward movement of the computer-mouse, an Arabic digit between 1 and 9 appeared in the center rectangular button. Participants were instructed to click on the left rectangular button for numbers that were smaller than the reference number (5 for this task), to click on the right rectangular button for numbers that were larger than the reference number, and to click on the middle rectangular button when the reference number appeared. The left-right computer-mouse clicks were not counterbalanced due to interference with the typical number line arrangement. Participants performed one block of 45 trials with each Arabic digit appearing 5 times. Participants practiced with alphabetic letters “a” to “i” with the reference letter being “e”, each letter was practiced one time. Participants received feedback at the end of the practice block indicating accuracy and reaction time. The task was similar for fraction, decimal, and percent tasks with reference numbers of $\frac{1}{2}$, 0.5, and 50%, respectively.



(a) (b)

Figure 4.2: Conceptual replications of the distance effect from study 3 in DeWolf et al. (2014) (a) Percent error distributions for magnitudes for percents, fractions, decimals, and integers. (b) Response time distributions for magnitudes for all number type tasks. The solid vertical line marks the reference value. See Table A.1 in the appendix for numerical values related to magnitudes along x-axis.

4.4.4 Data Treatment

All data was assigned distance and magnitude features following Song and Nakayama (2008) and DeWolf et al. (2014) respectively. The distance feature describes the absolute distance from the reference number. The magnitude feature provides an alphabet letter to represent a numeric magnitude (distance from zero).

Discrete Measures

Accuracy data was centered. For reaction time data, incorrect responses were removed. Reaction time data was log transformed and centered. The log transformation normalized the time-dependent skew of the data. The centering allowed for a 0 average reaction time for interpretation.

Continuous Measures

For trajectory analysis, we included only correct responses. For each participant, we temporally normalized the computer-mouse tracking trajectories by resampling 101 equally spaced time points during the trajectory movement and spatially normalized the trajectories by setting common start and end points following Spivey et al. (2005) using the MouseTrap software (Kieslich, Henninger, Wulff, Haslbeck, & Schulte-Mecklenbeck, 2019) in R.

4.5 Results

Following DeWolf et al. (2014), we collapsed over ordering of number types for our statistical analyses. We analyzed the data along two separate tracks: discrete and continuous measures. For discrete measures, we performed separate linear mixed effects models for accuracy and reaction time across the magnitude data and report the results in the following subsection. For both models, the intercept was set to the percent condition to allow for description of the concept of percent within the rational number space along these discrete measures. For continuous measures, for each rational number type, we calculated the maximum absolute deviation (MAD) of the trajectories for each distance using the MouseTrap software in R (Kieslich et al., 2019). We analyzed the MAD results using one-way repeated measures ANOVAs. For ANOVAs giving significant results, we performed post-hoc TukeyHSD tests to find significant differences between pairs of distances. For each significant pair found, we further explored significance along the temporal nature of the trajectories following Dale, Kehoe, and Spivey (2007) where 8 or more consecutive t-tests along the normalized time-path show significance at $p < 0.01$.

Table 4.1: Mean reaction time (non-transformed) and accuracy as a function of target-standard distances (SE). The integer data (column 1) is a conceptual replication of the Song and Nakayama (2008) study. Total time is reported as stimuli was displayed after computer-mouse movement began.

	Integer		Fraction		Percent		Decimal	
	Total time (ms)	Accuracy (%)	Total time (ms)	Accuracy (%)	Total time (ms)	Accuracy (%)	Total time (ms)	Accuracy (%)
Distance 1	977(26)	82(4)	1570(73)	81(3)	977(23)	83(4)	943(26)	87(4)
Distance 2	918(22)	84(4)	1444(75)	83(4)	956(26)	85(4)	885(21)	87(4)
Distance 2.5	-	-	1309(56)	83(4)	958(25)	84(4)	961(25)	89(3)
Distance 3	882(20)	85(4)	1309(61)	82(4)	917(23)	85(4)	882(20)	88(4)
Distance 4	892(24)	85(4)	1236(63)	84(4)	910(22)	86(4)	893(23)	89(3)
Standard	869(29)	97(1)	985(29)	97(2)	838(25)	94(2)	834(24)	97(2)

4.5.1 Discrete Measures

We analyzed the discrete measures of reaction time and accuracy from our data using R (RStudio Team, 2020) and the *lme4* software (Bates, Mächler, Bolker, & Walker, 2015). We performed separate linear mixed effects models to analyze the relationships between reaction time and conditions (integer/fraction/decimal/percent) and between accuracy and conditions. As fixed effects for both models, we entered the conditions. As random effects for both models, we had intercepts for participants and magnitudes, as well as by-participant random slopes for the effect of condition. Residual plots did not reveal obvious deviations from assumptions from normality or homoscedasticity upon visual inspection. We obtained p-values by comparing the full model with the effect in question against the model without the effect in question using likelihood ratio tests.

Accuracy

The error distributions (inverse accuracy) at the magnitude level can be seen in Figure 4.2a. As the magnitude comparisons get farther away from the reference, error goes down - which shows the numeric distance effect similar to the DeWolf et al. (2014) study. These results also replicated when averaging across magnitudes where fraction response time were the highest ($M = 16\%$, $SD = 13$) compared with percents ($M = 15\%$, $SD = 9$), integers ($M = 15\%$, $SD = 8$) and decimals ($M = 11\%$, $SD = 7$) as the lowest. However, decimals errors were marginally higher than integers in the original study.

The accuracy distributions (inverse error) at the distance level can be seen in Table 4.1. Similar to the Song and Nakayama (2008) study, accuracy increases as distance from the target-standard increases. The integer column in Table 4.1 contains the results for the replication study. We see this effect extended across all rational number types - to include percent. The linear mixed effects model did not provide significant results in errors based on number type. These results failed to replicate the DeWolf et al. (2014) study 3 error which showed a significant difference between fraction errors (higher) and other number type errors (lower). The error model coefficient estimates with 95% confidence intervals are shown in 4.2.

Table 4.2: Results for linear mixed model on error

	Coefficient Estimate	**Lower 95% CI Boundary**	**Higher 95% CI Boundary**
Percent (Intercept)	0.005	-0.066	0.076
Decimals	-0.033	-0.090	0.023
Fractions	0.016	-0.034	0.066
Integers	-0.001	-0.056	0.053

Response Time

The distribution of response times for correct decisions at the magnitude level can be seen in Figure 4.2b which again shows the numeric distance effect similar to the DeWolf et al. (2014) study. Magnitudes closer to the reference took longer in the comparison task than magnitudes further from the reference number. These results also replicate the ordering trend when averaging across magnitudes, fraction response time were the highest ($M = 1201$ ms, $SD = 317$) compared with percents ($M = 902$ ms, $SD = 153$), decimals ($M = 896$ ms, $SD = 147$) and integers ($M = 894$ ms, $SD = 153$) as the lowest. At the distance level, the response distributions (as shown in Table 4.1) replicate Song and Nakayama (2008) by showing a decreasing linear trend in the integer column as the distance from the standard (reference) increased. This effect extends across all rational number types.

For our statistical analysis, the linear mixed effects model showed interesting results on response time based on number type (see Table 4.3). A likelihood ratio test of the model with the fixed effects against the model without the effects revealed a significant difference between the models ($\chi^2(12) = 975.61, p < 0.001$). The model shows a strong difference in the response times between fractions and the other number types. This result replicates the DeWolf et al. (2014) work for fractions. The model results also weakly suggest that the percent number type response times are faster than the average for rational numbers overall.

Table 4.3: Results for linear mixed model on response time (in log time)

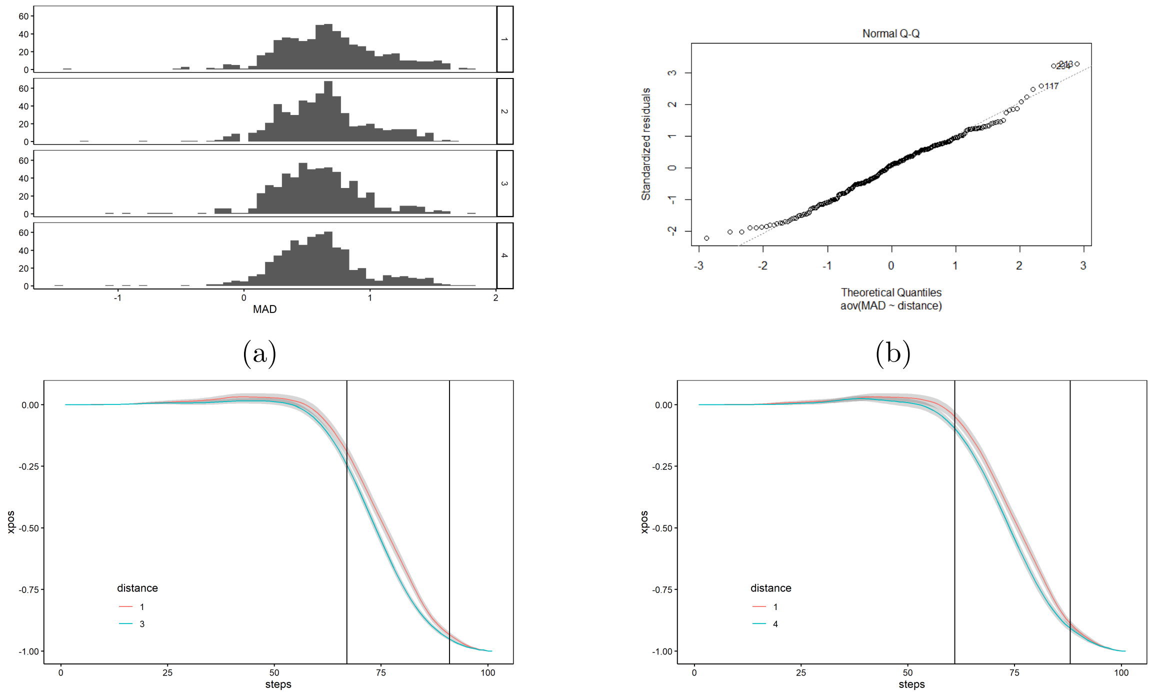
	Coefficient Estimate	**Lower 95% CI Boundary**	**Higher 95% CI Boundary**
Percent (Intercept)	-0.058	-0.112	-0.004
Decimals	-0.009	-0.038	0.019
Fractions	0.280	0.214	0.344
Integers	0.002	-0.031	0.034

4.5.2 Continuous Measures

For continuous measures, following Song and Nakayama (2008), we averaged the computer-mouse trajectories across distances from the reference number. The following results refer to this distanced trajectory data. For the continuous measures results, we reported each number type results in a separate section below. For the number types: fraction, decimal, and percent, we also include results for planned comparisons of the quarter/three-quarter distance.

Integers

The MAD distributions for the integer trajectories are shown in Figure 4.3a. The mean and standard deviations of the MAD distributions for each distance were for distance 1 ($M = 0.69$, $SD = 0.19$), distance 2 ($M = 0.64$, $SD = 0.18$), distance 3 ($M = 0.59$, $SD = 0.16$) and finally for distance 4 ($M = 0.60$, $SD = 0.18$). Results for a one-way repeated ANOVA for the integer trajectories showed a significant difference in MAD based on distance $F(3, 253) = 4.09, p < 0.01$.. A visualization of the Q-Q plot for the residual can be seen in Figure 4.3b. A post-hoc TukeyHSD test showed significant differences between distance-from-reference 3 vs. 1 ($p < 0.01$) and 4 vs. 1



(a) (b)

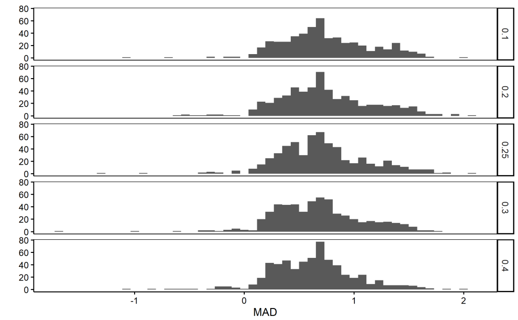
(c) (d)

Figure 4.3: Computer-mouse tracking analysis for integers

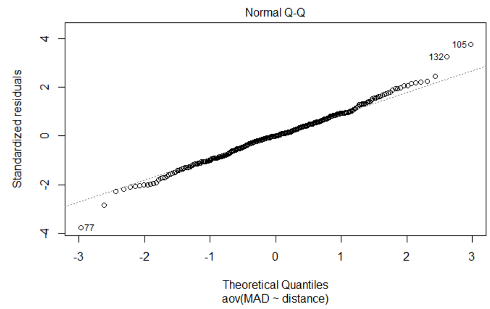
($p < 0.05$). To assess each pair of trajectories for significance, we followed the above mentioned approach by Dale et al. (2007) looking for 8 or more consecutive significant t-tests along the normalized time-path. For distance trajectories 3 vs. 1 (Figure 4.3c), and 4 vs. 1 (Figure 4.3d), we found significant lengths along the latter-end of the time-paths of size 17 and 25, respectively.

Fractions

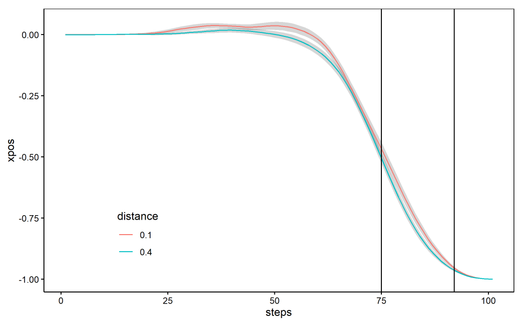
The fraction MAD distributions can be seen in Figure 4.4a. With mean and standard deviations of the MAD distributions for each distance as distance 1 ($M = 0.75$, $SD = 0.17$), distance 2 ($M = 0.75$, $SD = 0.21$), distance 2.5 ($M = 0.71$, $SD = 0.22$), distance 3 ($M = 0.69$, $SD = 0.23$) and finally for distance 4 ($M = 0.64$, $SD = 0.22$). The results from the fraction trajectories showed a significant difference in the one-



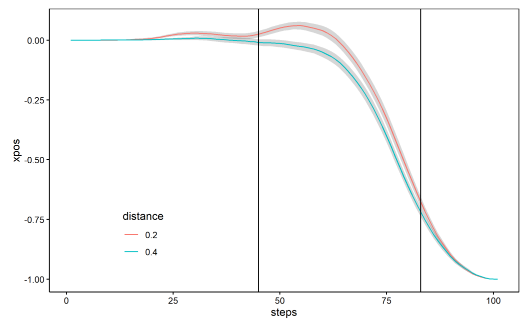
(a)



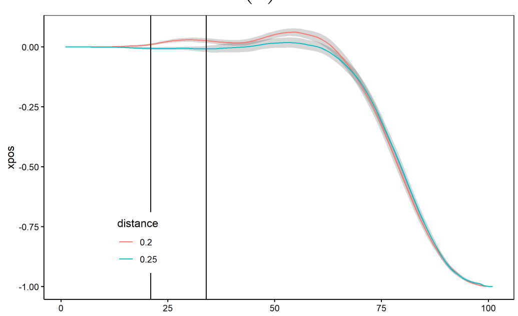
(b)



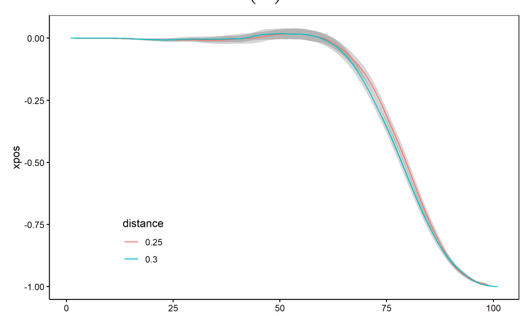
(c)



(d)



(e)



(f)

Figure 4.4: Computer-mouse tracking analysis for fractions

way repeated ANOVA in MAD $F(4, 332) = 3.21, p < 0.05$.. The Q-Q plot of the residuals can be found in Figure 4.4b. A post-hoc TukeyHSD test showed significant differences between distances 4 vs. 1 and 4 vs. 2 ($p < 0.05$) with significant lengths along the latter-end of the 4 vs. 1 (Figure 4.4c) distance trajectory pair of size 17. The significant length along the 4 vs. 2 (Figure 4.4d) distance trajectory pair was from the middle to the latter-end and was of size 42. Additionally, we look at the planned comparisons for the quarter distances for fractions. There is a length of interest early on in the trajectory path for the 2 vs. 2.5 distance trajectory pair of size 13 (Figure 4.4e) distance trajectory pair. However the trajectory path for the 2.5 vs. 3 distance trajectory pair showed no interesting deviations (Figure 4.4f).

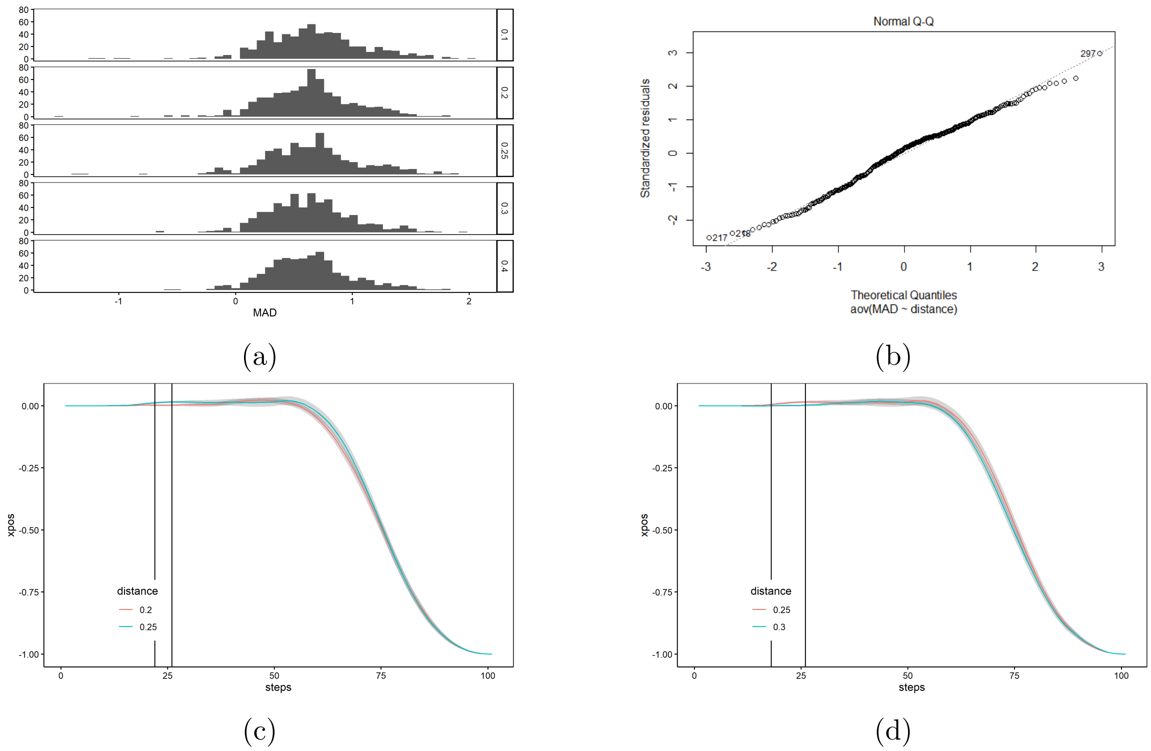


Figure 4.5: Computer-mouse tracking analysis for decimals

Decimals

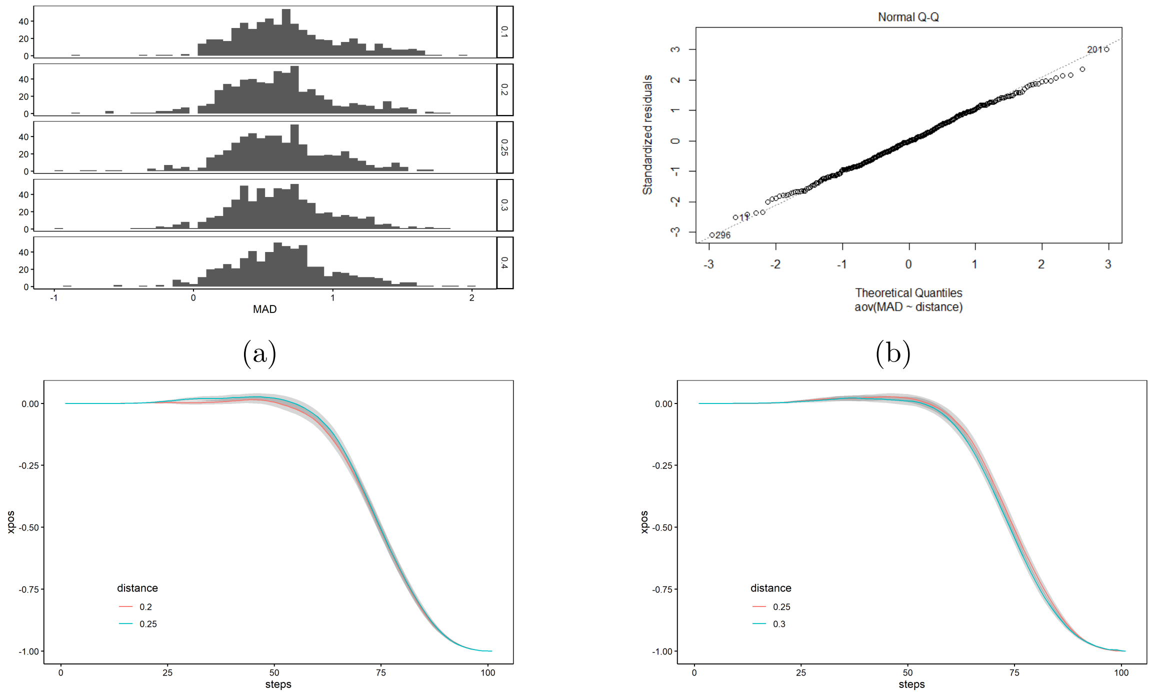
The one-way repeated ANOVA's for decimals for MAD showed no significant differences between trajectories for distances. The decimal MAD distributions and the Q-Q plot for the residuals for the model can be found in Figures 4.5a and 4.5b, respectively. The mean and standard deviations of the MAD distributions results for each distance were distance 1 ($M = 0.68$, $SD = 0.17$), distance 2 ($M = 0.63$, $SD = 0.17$), distance 2.5 ($M = 0.65$, $SD = 0.19$), distance 3 ($M = 0.63$, $SD = 0.19$) and distance 4 ($M = 0.63$, $SD = 0.17$). For the decimal quarter distance planned comparisons, only the 2.5 vs. 3 (Figure 4.5d) distance trajectory pair showed a significant length of size 8 along the trajectory paths. The 2 vs. 2.5 (Figure 4.5c) distance trajectory pair had consecutive t-tests of length size 4 - not long enough to be significant.

Percents

Similar to decimal, the one-way repeated ANOVA's for percents for MAD showed no significant differences between trajectories for distances. The percent MAD distributions can be seen in Figure 4.6a. The mean and standard deviations results of the MAD distributions for each distance were distance 1 ($M = 0.68$, $SD = 0.19$), distance 2 ($M = 0.61$, $SD = 0.20$), distance 2.5 ($M = 0.62$, $SD = 0.19$), distance 3 ($M = 0.62$, $SD = 0.20$) and distance 4 ($M = 0.64$, $SD = 0.18$). Figure 4.6b shows the Q-Q plots for the model residuals. The percent quarter distance planned comparison results showed an interesting lack of significance for both distance trajectory pairs, 2 vs. 2.5 (Figure 4.6c) and 2.5 vs. 3 (Figure 4.6d), respectively. There were no consecutive t-tests within either of these planned comparison trajectory pair paths.

4.6 Discussion

This work had three main goals. Our first goal was to introduce the numeric magnitude for percent within the rational numbers. We situated the concept of percent with respect to the rational number types: single digit integers, decimals, and fractions using a computer-mouse tracking magnitude comparison task that contained both discrete and continuous measures. Using the discrete measures of accuracy and response time, we found that within these measures, the concept of percent is more similar to that of decimals and integers than fractions. Within the response time



(a)

(b)

(c) Distances 2-2.5

(d) Distances 2.5-3

Figure 4.6: Computer-mouse tracking analysis for percents

measure, we found strong indications that the concept of percent is more similar to that of decimals and integers than fractions. Within the accuracy measure, this trend was not as well supported as the fraction, percent, and decimal results intermingled. From the discrete measures, percent representations appear to be processed slightly faster than the average, much faster than fractions, but slower than both decimals and integers. Percent magnitude comparison accuracy is marginally worse than average, much better than fractions, and again worse than integers and decimals.

We further situate percent by looking at the computer-mouse tracking trajectories for a more fine-grained analysis of the rational number magnitude comparison task. We used maximum deviation as a metric to compare how much each of these trajectories departs from a standard output - with deviation in the trajectory interpreted as a pull towards the reference number in the task. These continuous measures reveal hidden differences between the rational numbers that could not be seen in the discrete measures. For integers and fractions, the trajectories exhibit strong deviation between the comparisons closest and farthest from the reference number. These results complement the results from the discrete measures and further show that the deviations between the trajectories occur in the last quarter of the path. The decimal and percent trajectories both had no interesting deviations between pairs of distances based on maximum deviation. From this we see that percent trajectories compare most to the decimal trajectories along the maximum deviation metric. Adding the planned comparison of the quarter distance pairs for fractions, decimals, and percents, we see that for fractions and decimals there are strong pulls towards the reference when compared to nearest neighboring distance trajectories. However, these deviations occur early in the trajectory path, unlike the deviations in pairs found through the MAD metric. The percent distance pair trajectories showed no consecutive deviations along the path. So in these quarter trajectories, fractions and decimals appear most similar while percent trajectories are too like its neighboring distance trajectories.

Our second and third goals were the conceptual replications of the (DeWolf et al., 2014) reaction time and percent error results and the (Song & Nakayama, 2008) reaction time and accuracy hand-reaching results for integers. The discrete measures results successfully replicate the numeric distance effect found in study 3 of (DeWolf et al., 2014). There is some discrepancy in the accuracy replication which we assume is from our fraction errors being closer to those of the other rational numbers than in the previous study. The discrete measures also shows the replicated integer results for the (Song & Nakayama, 2008) study.

4.7 Conclusion and Future Work

In this study, we set out to address the challenge of understanding the concept of percent. We approached this challenge using a rational number comparison task - as percent is a generalized comparison tool. Through this study, we have been able to provide both discrete and continuous measures for evaluative percent within the rational number concept space. For comparison tasks, it appears to be most similar to the decimal concept overall. Additionally, we were able to conceptually replicated previous works related to rational numbers and the magnitude comparison task for integers. From this study, we were able to state several results based on our hypotheses:

- We affirmed previous work showing that decimals were processed the fastest. They were slightly less accurate than integers for this study.
- We confirmed that fractions were slowest to be processed and also had the most errors.
- We showed that percent performed most similarly to decimals. This was shown in both discrete and continuous measures.
- For the quarter and three-quarter magnitudes, we found mixed results. For fractions there was a reduction in processing speed and error. Decimals showed an increase in processing speed but a decrease in errors. The added digit in these decimal magnitudes could account for the slower processing speed. For percent, there was an increase in processing speed and errors. It is not yet clear how these quarter magnitudes are processed in rational number space.
- We found distance effects for all number types.

This work focused on a general introduction of percent to rational number space. This broad approach leaves many unanswered questions regarding similarities and differences between these number types (integer, fraction, decimal, percent). Future work could expand on this single digit $(1 - 9)$ task to include more complex magnitude comparison tasks. Additionally, a more extensive examination of the privileged percents, $25\%, 50\%$, and 75% might reveal specific patterns within the processing for these magnitudes.

Chapter 5

An Integrated View of the Concept of Percent

5.1 Percent as Quantity and Number

This work was motivated by the challenges to understanding the concept of percent - a often thought simple concept. Percent, as a generalized tool for comparison, is used frequently in descriptive statistical reporting and thus plays a problematic role in decision making processes. Within education, percent is a fundamental building block for success in higher mathematics - yet students have historically failed to grasp the concept of percent. Throughout these chapters, the concept of percent has been shown to be much more than a simple concept, but rather a “multifaceted and complex concept” used in various situations as a generalized mathematical tool for comparison (Parker & Leinhardt, 1995).

This body of work began with an introduction of percent as a quantity and number. The concept of percent entails a numeral-symbol or numeral-word combination. Through its numeral, percent expresses quantity magnitude and can be seen as an indexical and symbolic signal in language. This numeral is affected by numeric (semantic) effects such as distance, size, and rounding effects. The percent term within a context is also affected by pragmatic properties of boundedness and relevance with language and communcation. However, percent is primarily a mathematical tool for generalized comparison. As such, it is an abstract (intensive) quantity that represents the relationship between other abstract or concrete quantities. Percent is a type of number that is a ratio comparison and can never directly refer to the count of a set of objects - only to the relationship between sets. This inability to directly refer to the count of a set is one way that percent is markedly different from fractions and decimals. Unfortunately, the concept of percent is taught during rational number instruction and often conflated with fractions and decimals. This is not surprising as percent is "commonly converted to decimals and fractions" for comparative operations (Parker & Leinhardt, 1995).

The difficulties surrounding the concept of percent have been evaluated in chapters 2-4 using three different interdisciplinary methodologies each looking at 3 key aspects of the concept of percent. Chapters 2, 3, and 4 looked at the construction of the concept of percent, percent in context, and percent as a comparison tool, respectively.

5.2 Interdisciplinary Analyses

A qualitative analysis of the evolution of the concept of percent was performed in chapter 2 using historical data and conceptual integration (blending). Percent was addressed as a mathematical tool, a number and symbol, an abstract quantity, and a shifting signal in language. This work showed how the concept of percent evolved over time to form a complex network of blended identities and roles being compressed into a single symbol (%) or a pair of words (percent or percentage). Problems relating to correct decompression of this complex network - due to a number of distinct conceptual spaces referenced by the same term or symbol - could lead to known percent problems.

A quantitative analysis of how language context impacts the functional alternations of the concept of percent was implemented in chapter 3 using natural language processing techniques, machine learning, and a large natural language corpus. Percent was examined as a comparative tool in language and as information. State-of-the-art Transformer language models were fine-tuned on a custom percent datasets and used to classify genres along a language formality spectrum. Results from this work confirmed intuitive hypotheses that percent contexts in very formal and very informal language are most easy to classify due to having fewer percent alternations. Very formal contexts tended to avoid non-literal use of the concept of percent. Very informal contexts made extensive use of non-literal use of the concept of percent. The genres in the middle of the language formality spectrum had overlapping usage of literal and non-literal percent alternations and were harder to classify. This work found that navigating the concept of percent requires something like code-switching in that functional alternations involve variances in language and meaning.

A computer-mouse tracking magnitude comparison study introduced percent magnitude representations to the rational number space in chapter 4. As percent is a generalized comparison tool, this work compared and contrasted percent in a magnitude comparison task with other rational numbers: integers, fractions, and decimals. This task employed the descriptive statistic usage of percent where two percents were present in a given context (the reference percent was held in memory). Percent was found to be most similar to decimal and least similar to fraction for this task. Additionally, this work replicated and extended previous numeric distance effects for rational number magnitude comparison as well as confirmed these same numeric distance effects for the concept of percent. This work provided a first cognitive look at quantitative comparisons between percent and rational numbers.

These three interdisciplinary analyses have provided a strong integrated argument for both the complexity of the concept of percent and also why certain difficulties exist surrounding this concept. This body of work could contribute both qualitative and quantitative results for the concept of percent to many fields: numeric cognition, mathematical education, cognitive linguistics, natural language processing, and decision making sciences and communication.

5.3 Conclusions and Future Work

Hopefully this analysis has revealed some of the beauty and complexity of percent as an abstract global mathematical tool for comparison. Although there are many known difficulties in understanding the concept of percent, it is possible to master this tool through a comprehension of the compression involved in its creation. Based on this work, there are future endeavors that can facilitate overcoming these percent difficulties such as a change in percent instruction and better communicative understanding of percent functional alternations in context.

Intervention studies in the past have shifted instruction of rational numbers and percent to put an emphasis on percent as a comparative tool and proportion. These studies have been successful but small and often in different countries (Moss & Case, 1999). It is an enormous effort to change curriculum, especially as a national level. However, shifting the order of instruction for percent and rational numbers might be the most simple adjustment to affect the biggest outcome in overcoming percent difficulties for students.

The concept of percent in communication is fraught with challenges. Statistical reporting is inundated with different types of comparative situations and since referents are not required to be given in a context, confusion abounds. This plays havoc on decision making processes that must be based on statistical reporting. Considering that the texts for the custom datasets in chapter 3 were produced by people “in the wild”, our findings could have implications for understanding why the concept of percent can be so difficult to master and for identifying how mastering the code-switching aspects of the concept of percent might contribute to better clarity in communication and decision making processes by the general public.

Future work on the concept of percent could involve greater analysis of the comparative aspects of percent. There are many types of comparison such as increasing or decreasing, absolute or relative difference, single set or two set comparison, additive or multiplicative relationships, etc. This dissertation has only begun to scratch the surface of percent as a generalized comparative tool. Additionally, other natural language processing models and technique could be used to analyze percent textual data with an emphasis on what these models are attending to with respect to percent in context. Questions relating to context size and percent comparative situations in context will also need to be addressed. Finally, percent has not been fully situated within rational number space. Further magnitude comparison studies are needed for extensive comparison with each rational number using both discrete and continuous behavioural methods. Percent is more than just magnitude and so studies looking at ratio relationships are also needed to complete our understanding of the concept of percent.

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Appendix A

Magnitude Stimuli for Computer Mouse Tracking Study

Table A.1: Magnitude Stimuli across all rational number types.

Magnitude	**Integer**	**Fraction**	**Percent**	**Decimal**
A	1	1/10	10%	0.1
B	2	1/5	20%	0.2
C	-	1/4	25%	0.25
D	3	3/10	30%	0.3
E	4	2/5	40%	0.4
F	5	1/2	50%	0.5
G	6	3/5	60%	0.6
H	7	7/10	70%	0.7
I	-	3/4	75%	0.75
J	8	4/5	80%	0.8
K	9	9/10	90%	0.9