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EXISTENCE OF CONTRACTIVE PROJECTIONS ON PREDUALS OF JBW^*-TRIPLES

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ABSTRACT

In 1965, Ron Douglas proved that if X is a closed subspace of an L^1-space and X is isometric to another L^1-space, then X is the range of a contractive projection on the containing L^1-space. In 1977 Arazy–Friedman showed that if a subspace X of C_1 is isometric to another C_1-space (possibly finite dimensional), then there is a contractive projection of C_1 onto X. In 1993 Kirchberg proved that if a subspace X of the predual of a von Neumann algebra M is isometric to the predual of another von Neumann algebra, then there is a contractive projection of the predual of M onto X.

We widen significantly the scope of these results by showing that if a subspace X of the predual of a JBW^*-triple A is isometric to the predual of another JBW^*-triple B, then there is a contractive projection on the predual of A with range X, as long as B does not have a direct summand which is isometric to a space of the form $L^\infty(\Omega, H)$, where H is a Hilbert space of dimension at least two. The result is false without this restriction on B.

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1. Introduction and background

1.1. Introduction. In 1965, Douglas [10] proved that the range of a contractive projection on an L^1-space is isometric to another L^1-space. At the same time, he showed the converse: if X is a closed subspace of an L^1-space and X is isometric to another L^1-space, then X is the range of a contractive projection. Both of these results were shortly thereafter extended to L^p-spaces, $1 < p < \infty$ by Ando [2] and Bernau–Lacey [7]. The first result fails for L^∞-spaces as shown by work of Lindenstrauss–Wulbert [31] in the real case and Friedman–Russo [17] in the complex case. But not by much—the image of a contractive projection on L^∞ is a C_σ-space.

Moving to the non-commutative situation, it was already known in 1978 through the work of Arazy–Friedman [4], which gave a complete description of the range of a contractive projection on the Schatten class C_1, that the range of such a projection is isometric to a direct sum of C_1 spaces. Moreover, in 1977, Arazy–Friedman [3] showed that if a subspace X of C_p $1 \le p < \infty, p \neq 2$ is isometric to another C_p-space (possibly finite dimensional), then there is a contractive projection of C_p onto X. In 1992, Arazy–Friedman [5] extended and expanded their earlier results on C_1 to C_p, $1 < p < \infty, p \neq 2$.

Generalizing the 1978 work of Arazy–Friedman on C_1 to an arbitrary noncommutative L^1-space, namely the predual of a von Neumann algebra, Friedman–Russo [19] showed in 1985 that the range of a contractive projection on such a predual is isometric to the predual of a JW^*-triple, that is, a weak*-closed subspace of $B(H,K)$ closed under the triple product $xy^*z + zy^*x$. Important examples of JW^*-triples besides von Neumann algebras and Hilbert spaces ($H = B(H,\mathbf{C})$) are the subspaces of $B(H)$ of symmetric (or anti-symmetric) operators with respect to an involution, and spin factors. Actually, the Friedman–Russo result was valid for projections acting on the predual of a JW^*-triple, not just on the predual of a von Neumann algebra.

A far reaching generalization of both the 1977 work of Arazy–Friedman (in the case $p = 1$) and the 1965 work of Douglas was given by Kirchberg [28] in 1993 in connection with his work on extension properties of C^*-algebras. Kirchberg proved that if a subspace X of the predual of a von Neumann algebra M is isometric to the predual of another von Neumann algebra, then there is a contractive projection of the predual of M onto X.

In view of the result of Friedman–Russo mentioned above, it is natural to ask if the result of Kirchberg could be extended to preduals of JBW^*-triples (the axiomatic version of JW^*-triples), that is, if a subspace X of the predual of a JBW^*-triple M is isometric to the predual of another JBW^*-triple N, then is there a contractive projection of the predual of M onto X? We show that the answer is yes as long as the predual of N does not have a direct summand which is isometric to $L^1(\Omega, H)$ where H is a Hilbert space of dimension at least two (Theorem 1 in Subsection 1.2). To see that this restriction is necessary, one has only to consider a subspace of L^1 spanned by two or more independent standard normal random variables. Such a space is isometric to L^2 but cannot be the range of a contractive projection on L^1 since by the result of Douglas it would also be isometric to an L^1-space, and therefore one dimensional (consider the extreme points of its unit ball).

1.2. Projective rigidity. The main result. A well-known and useful result in the structure theory of operator triple systems is the "contractive projection principle," that is, the fact that the range of a contractive projection on a JB^*-triple is linearly isometric in a natural way to another JB^*-triple (Kaup, Friedman–Russo). Thus, the category of JB^*-triples and contractions is stable under contractive projections.

To put this result, and this paper, in proper prospective, let $\mathcal{B}$ be the category of Banach spaces and contractions. We shall say that a sub-category $\mathcal{S}$ of $\mathcal{B}$ is **projectively stable** if it has the property that whenever A is an object of $\mathcal{S}$ and X is the range of a morphism of $\mathcal{S}$ on A which is a projection, then X is isometric (that is, isomorphic in $\mathcal{S}$) to an object in $\mathcal{S}$. Examples of projectively stable categories (some mentioned already) are, in chronological order,

(1) L_1; contractions (Grothendieck 1955 [21]),
(2) L^p, $1 \leq p < \infty$; contractions (Douglas 1965 [10], Ando 1966 [2], Bernau–Lacey 1974 [7], Tzafriri 1969 [38]),
(3) C^*-algebras; completely positive unital maps (Choi–Effros 1977 [9]),
(4) ℓ_p, $1 \leq p < \infty$; contractions (Lindenstrauss–Tzafriri 1978 [30]),
(5) ℓ^p-direct sums of C_p (Schatten classes), $1 \leq p < \infty, p \neq 2$; contractions (Arazy–Friedman 1978 [4] 1992 [5]),
(6) JC^*-algebras; positive unital maps (Effros–Stormer 1979 [14]),
(7) *TROs* (ternary rings of operators); complete contractions (Youngson 1983 [41]),

(8) JB^*-triples; contractions (Kaup 1984 [27], Friedman–Russo 1985 [19]),
(9) ℓ^p-direct sums of $L^p(\Omega, H)$, $1 \leq p < \infty$, H Hilbert space; contractions (Raynaud 2004 [35]).

For a survey of results about contractive projections and their ranges in Köthe function spaces and Banach sequence spaces, see [34].

It follows immediately that if $\mathcal{S}$ is projectively stable, then so is the category $\mathcal{S}_*$ of spaces whose dual spaces belong to $\mathcal{S}$. It should be noted that $TROs$, C^*-algebras and JC^*-algebras are not stable under contractive projections and JB^*-triples are not stable under bounded projections.

By considering the converse of the above property, one is lead to the following definition which is the focus of the present paper. A sub-category $\mathcal{S}$ of $\mathcal{B}$ is **projectively rigid** if it has the property that whenever A is an object of $\mathcal{S}$ and X is a subspace of A which is isometric to an object in $\mathcal{S}$, then X is the range of a morphism of $\mathcal{S}$ on A which is a projection. Examples of projectively rigid categories are fewer in number (all are projectively stable), namely,

(1) ℓ_p, $1 < p < \infty$, contractions (Pelczynski 1960 [33]),
(2) L^p, $1 \leq p < \infty$, contractions (Douglas 1965 [10], Ando 1966 [2], Bernau–Lacey 1974 [7]),
(3) $C_p, 1 \leq p < \infty$, contractions (Arazy–Friedman 1977 [3]),
(4) Preduals of von Neumann algebras, contractions (Kirchberg 1993 [28]),
(5) Preduals of $TROs$, complete contractions (Ng–Ozawa 2002 [32]),
(6) C_p, $1 \leq p < \infty, p \neq 2$; complete contractions (LeMerdy, Ricard, Roydor 2009 [29]).

The result by Ng and Ozawa fails in the category of operator spaces with complete contractions. Referring to Kirchberg's paper, Ng and Ozawa conjectured that "a similar statement holds for JC^*-triples." While we found that this is not true in general, we have been able to prove the following which, in view of the counterexample mentioned earlier, is the best possible.

THEOREM 1: *Let X be a subspace of the predual A_* of a JBW^*-triple A. If X is isometric to the predual of another JBW^*-triple, then there is a contractive projection P on A_* such that $X = P(A_*) \oplus^{\ell^1} Z$, where Z is isometric to a direct sum of spaces of the form $L^1(\Omega, H)$ where H is a Hilbert space of dimension at least two, $P(A_*)$ is isometric to the predual of some JBW^*-triple with no such $L^1(\Omega, H)$-summand, and $P(Z) = 0$.*

In particular, the category of preduals of JBW^*-triples with no summands of the above type is projectively rigid. The proof of Theorem 1 will be achieved in Corollaries 6.8 and 6.9. Theorem 2 in Subsection 4.3 and Theorem 3 in Section 6, are key steps in the proof of Theorem 1.

As has been made clear, JB^*-triples are the most natural category for the study of contractive projections. It is important to note that JB^*-triples are also justified as a natural generalization of operator algebras as well as because of their connections with complex geometry. Indeed, Kaup showed in [26] that JB^*-triples are exactly those Banach spaces whose open unit ball is a bounded symmetric domain. Kaup's holomorphic characterization of JB^*-triples directly led to the proof of the projective stability of JB^*-triples in [27] mentioned above. Many authors since have studied the interplay between JB^*-triples and infinite dimensional holomorphy (see [15], [39], [40] for surveys).

Preduals of JBW^*-triples have been called pre-symmetric spaces ([11]) and have been proposed as mathematical models of physical systems ([16]). In this model the operations on the physical system are represented by contractive projections on the pre-symmetric space.

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2. Preliminaries

2.1. JBW^*-TRIPLES. A **Jordan triple system** is a complex vector space V with a **triple product** $\{\cdot,\cdot,\cdot\} : V \times V \times V \longrightarrow V$ which is symmetric and linear in the outer variables, conjugate linear in the middle variable and satisfies the Jordan triple identity (also called the main identity),

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

The triple product is also written $\{xyz\}$. A complex Banach space A is called a JB^*-*triple* if it is a Jordan triple system such that for each $z \in A$, the linear map

$$D(z) : v \in A \mapsto \{z, z, v\} \in A$$

is Hermitian, that is, $\|e^{itD(z)}\| = 1$ for all $t \in \mathbb{R}$, with non-negative spectrum in the Banach algebra of operators generated by $D(z)$, and $\|D(z)\| = \|z\|^2$. A summary of the basic facts about JB^*-triples can be found in [37] and some of the references therein, such as [26], [18], and [20]. The operators $D(x,y)$ and $Q(x,y)$ are defined by $D(x,y)z = \{xyz\}$ and $Q(x,y)z = \{xzy\}$, so that $D(x,x) = D(x)$ and we define $Q(x)$ to be $Q(x,x)$. We use the notation x^3 for $\{xxx\}$.

A JB^*-triple A is called a *JBW^*-triple* if it is a dual Banach space, in which case its predual, denoted by A_*, is unique (see [6] and [22]), and the triple product is separately weak* continuous. Elements of the predual are referred to as normal functionals. It follows from the uniqueness of preduals that an isomorphism from a JBW^*-triple onto another JBW^*-triple is automatically normal, that is, weak*-continuous. We will use this fact repeatedly in the paper. The second dual A^{**} of a JB^*-triple is a JBW^*-triple.

The JB^*-triples form a large class of Banach spaces which include C^*-algebras, Hilbert spaces, spaces of rectangular matrices, and JB^*-algebras. The triple product in a C*-algebra $\mathcal{A}$ is given by

$$\{x,y,z\} = \frac{1}{2}\,(xy^*z + zy^*x).$$

In a JB^*-algebra with product $x \circ y$, the triple product making it into a JB^*-triple is given by $\{x,y,z\} = (x \circ y^*) \circ z + z \circ (y^* \circ x) - (x \circ z) \circ y^*$.

An element e in a JB^*-triple A is called a **tripotent** if $\{e,e,e\} = e$ in which case the map $D(e) : A \longrightarrow A$ has eigenvalues 0, 1/2 and 1, and we have the following decomposition in terms of eigenspaces,

$$A = A_2(e) \oplus A_1(e) \oplus A_0(e),$$

which is called the *Peirce decomposition* of A. The $k/2$-eigenspace $A_k(e)$ is called the **Peirce k-space**. The **Peirce projections** from A onto the Peirce k-spaces are given by

$$P_2(e) = Q^2(e), \quad P_1(e) = 2(D(e) - Q^2(e)), \quad P_0(e) = I - 2D(e) + Q^2(e)$$

where, as noted above, $Q(e)z = \{e,z,e\}$ for $z \in A$. The Peirce projections are contractive, and weak*-continuous if A is a JBW^*-triple. In the latter case, we denote their action on A_* by $P_j(e)_*$, that is, $(P_j(e)_*)^* = P_j(e)$.

A powerful computational tool connected with Peirce decompositons is the so-called **Peirce calculus**, which states that

$$\{A_k(u), A_j(u), A_i(u)\} \subset A_{k-j+i}(u),$$

$$\{A_0(u), A_2(u), A\} = \{A_2(u), A_0(u), A\} = 0,$$

where it is understood that $A_j(u) = 0$ if $j \notin \{0, 1, 2\}$.

For any tripotent v, the space $A_2(v)$ is a JB^*-algebra under the product $x \cdot y = \{xvy\}$ and involution $x^\sharp = \{vxv\}$. We use implicitly in Lemmas 3.7 and 3.8 the correspondence between projections in $A_2(v)$ and tripotents of A majorized by v and the fact that the order on such tripotents (defined below) coincides with the order in the JB^*-algebra $A_2(v)$ ([12, Lemma 2.4]).

Tripotents u and v are **compatible** if $\{P_k(u), P_j(v) : k, j = 0, 1, 2\}$ is a commuting family. This holds for example if $u \in A_k(v)$ for some k. Tripotents u, v are **collinear** if $u \in A_1(v)$ and $v \in A_1(u)$, notation $v \top u$, and **rigidly collinear** if $A_2(u) \subset A_1(v)$ and $A_2(v) \subset A_1(u)$.

Tripotents u, v are **orthogonal** if $u \in A_0(v)$, that is, $\{uvv\} = 0$. More generally, arbitrary elements x, y are orthogonal if $D(x, y) = 0$, and we write $x \perp y$ if this is the case. Since $\|x\|^3 = \|\{xxx\}\|$ holds in a JB^*-triple and for orthogonal elements x and y we have $\{x + y, x + y, x + y\} = \{xxx\} + \{yyy\}$, it follows that $\|x + y\| \le 2^{1/3} \max(\|x\|, \|y\|)$ and by iteration that $\|x + y\| \le 2^{3^{-n}} \max(\|x\|, \|y\|)$, so that $\|x + y\| = \max(\|x\|, \|y\|)$ for orthogonal elements x, y. The converse is false in general, but is true in case one of x, y is a tripotent (as pointed out to us independently by R. Hügli and A. Peralta, [25, Th. 4.1]). This latter fact is needed in Lemma 6.1.

For tripotents u, v, the following four statements are equivalent: $D(u, v) = 0$, $D(v, u) = 0$, $\{uuv\} = 0$, $\{vvu\} = 0$. (By symmetry, the only non-trivial assertion to prove is that $\{vvu\} = 0 \Rightarrow D(v, u) = 0$; assuming $\{vvu\} = 0$, by the main identity, $\{vuv\} = \{vu\{vvv\}\} = 2\{vv\{vuv\}\}$, so that $\{vuv\} \in A_1(v)$, and by Peirce calculus, $\{vuv\} \in A_2(v)$, so that $\{vuv\} = 0$. Again by the main identity (written in operator notation),

$$[D(v, v), D(v, u)] = D(v, u) + D(v, \{vvu\}) = D(v, u)$$

and

$$[D(v, u), D(v, v)] = D(\{vuv\}, v) - D(v, \{uvv\}) = 0.$$

Hence $D(v, u) = 0$, as required.) We note here for use in the proof of Lemma 6.3 that the result just proved also holds for arbitrary elements (see [8, Lemma 1], a reference which was pointed out to the authors by the referee).

In the case of a tripotent u in a JBW^*-triple A with predual A_*, there is a corresponding Peirce decomposition of the normal functionals:

$$A_* = A_2(u)_* \oplus A_1(u)_* \oplus A_0(u)_*$$

in which $A_2(u)_*$ is linearly spanned by the normal states of the JBW^*-algebra $A_2(u)$. The norm exposed face $\{f \in A_* : f(u) = 1 = \|f\|\}$ is automatically a subset of $A_2(u)_*$ and coincides with the set of normal states of $A_2(u)$.

The set of tripotents in a JBW^*-triple, with a largest element adjoined, forms a complete lattice under the order $u \leq v$ if $v - u$ is a tripotent orthogonal to u. This lattice is isomorphic to various collections of faces in the JBW^*-triple and its predual ([12]). A maximal element of this lattice other than the artificial largest element is simply called a **maximal tripotent**, and is the same as an extreme point of the unit ball of the JBW^*-triple. Equivalently, a maximal tripotent is one for which the Peirce 0-space vanishes, and it is also referred to as a *complete tripotent*.

Given a JBW^*-triple A and f in the predual A_*, there is a unique tripotent $v_f \in A$, called the **support tripotent** of f, such that $f \circ P_2(v_f) = f$ and the restriction $f|_{A_2(v_f)}$ is a **faithful positive** normal functional on the JBW^*-algebra $A_2(v_f)$. The support tripotent of f is the smallest tripotent on which f assumes its norm. It is known that for any tripotent u, if $f \in A_j(u)_*$ $(j = 0, 1, 2)$, then $v_f \in A_j(u)$. The converse is true for $j = 0$ or 2 but fails in general for $j = 1$ (however, see the proof of Lemma 5.1).

We shall occasionally use the joint Peirce decomposition for two orthogonal tripotents u and v, which states that

$$\begin{aligned} A_2(u+v) &= A_2(u) \oplus A_2(v) \oplus [A_1(u) \cap A_1(v)], \\ A_1(u+v) &= [A_1(u) \cap A_0(v)] \oplus [A_1(v) \cap A_0(u)], \\ A_0(u+v) &= A_0(u) \cap A_0(v). \end{aligned}$$

Let A be a JB^*-triple. For any $a \in A$, there is a triple functional calculus, that is, a triple isomorphism of the closed subtriple $C(a)$ generated by a onto the commutative C*-algebra $C_0(\operatorname{Sp} D(a,a) \cup \{0\})$ of continuous functions vanishing at zero, with the triple product $f\overline{g}h$ (see [26, Cor.1.15]). Any JBW^*-triple has the propertly that it is the norm closure of the linear span of its tripotents. This is a consequence of the spectral theorem in JBW^*-triples, which states that every element has a representation of the form $x = \int \lambda du_\lambda$ analogous to the usual spectral theorem for self-adjoint operators, in which $\{u_\lambda\}$ is a family of tripotents [12, Lemma 3.1].

For any element a in a JBW^*-triple, there is a least tripotent, denoted by $r(a)$ and referred to as the support of a, such that a is a positive element in the JBW^*-algebra $A_2(r(a))$ ([12, Section 3]). It is known that $y \perp u$ is equivalent to $r(y) \perp u$ for a tripotent u. For each element a of norm one in a JBW^*-triple A, denote by $u(a)$ the unique tripotent of A for which

$$\{f \in A_* : f(a) = \|f\| = 1\} = \{f \in A_* : f(u(a)) = \|f\| = 1\}.$$

The tripotent $u(a)$ is the supremum of the set of tripotents u with $\{uau\} = u$ and is the weak*-limit of the sequence $\{a^{2n+1}\}$ ([12, Lemmas 3.2,3.4]).

A closed subspace J of a JBW^*-triple A is an **ideal** if $\{AAJ\} \cup \{AJA\} \subset J$ and a weak*-closed ideal J is complemented in the sense that

$$J^\perp := \{x \in A : D(x, J) = 0\}$$

is also a weak*-closed ideal and $A = J \oplus J^\perp$. A tripotent u is said to be a **central tripotent** if $A_2(u) \oplus A_1(u)$ is a weak*-closed ideal. In this case $A_2(u) \oplus A_1(u)$ is orthogonal to $A_0(u)$. This definition is implicit in [22, 2.7] where instead the notion of **central e-projection** is defined. Our definition of central tripotent differs from the one in [13, p. 262].

A tripotent u is an **abelian tripotent** if $A_2(u)$ is an associative triple, that is, the identity $\{xy\{abc\}\} = \{\{xya\}bc\}$ holds (See [22, Definition 4.8]). The structure theory of JBW^*-triples has been well developed, using these and other concepts in [23] and [24].

The following lemma, [18, Lemma 1.6], will be used repeatedly.

Lemma 2.1: *If u is a tripotent in a JBW^*-triple and x is a norm one element with $P_2(u)x = u$, then $P_1(u)x = 0$. Put another way, $x = u + q$ where $q \perp u$.*

2.2. Some general lemmas.

Lemma 2.2: *Let u_λ be a family of tripotents in a JBW^*-triple B and suppose $\sup_\lambda u_\lambda$ exists.*

(a) *If $u_\lambda \perp y$ for some element $y \in B$, then $\sup_\lambda u_\lambda \perp y$.*
(b) *If $u_\lambda \in B_1(t)$ for some tripotent t, then $\sup_\lambda u_\lambda \in B_1(t)$.*

Proof. (a) If $y \perp u_\lambda$ for all λ, then $r(y) \perp u_\lambda$. If we let $z = \sup u_\lambda$ and $z = z_2 + z_1 + z_0$ be the Peirce decomposition with respect to $r(y)$, then by Peirce calculus, $u_\lambda = \{u_\lambda z u_\lambda\} = \{u_\lambda z_0 u_\lambda\}$ so that by Lemma 2.1, $z_0 = u_\lambda + b_\lambda$ with $b_\lambda \perp u_\lambda$. Therefore $r(z_0) \geq u_\lambda$, which implies $z \leq r(z_0) \in B_0(r(y))$ and so $z \in B_0(r(y))$ and therefore $z \perp y$.

(b) Write $\sup u_\lambda = x_2 + x_1 + x_0$ with respect to t. Since $D(u_\lambda, u_\lambda)(\sup u_\lambda) = u_\lambda$, by Peirce calculus we have $D(u_\lambda, u_\lambda)x_1 = u_\lambda$ and $D(u_\lambda, u_\lambda)x_2 = D(u_\lambda, u_\lambda)x_0 = 0$. By (a), $x_2 \perp \sup u_\lambda$ and $x_0 \perp \sup u_\lambda$ so that $0 = D(x_2, x_2)(x_2 + x_1 + x_0) = \{x_2 x_2 x_2\} + \{x_2 x_2 x_1\}$. By Peirce calculus, $\{x_2 x_2 x_2\} = \{x_2 x_2 x_1\} = 0$, so that $x_2 = 0$.

Similarly, $0 = D(x_0, x_0)(x_2 + x_1 + x_0) = \{x_0 x_0 x_0\} + \{x_0 x_0 x_1\}$, $\{x_0 x_0 x_0\} = \{x_0 x_0 x_1\} = 0$, so that $x_0 = 0$. ∎

Lemma 2.3: *If x and y are orthogonal elements in a JBW^*-triple and if z is any element, then*

$$D(x,x)D(y,y)z = \{x\{xzy\}y\}.$$

In other words, $D(x,x)D(y,y) = Q(x,y)^2$ for orthogonal x, y.

Proof. By the main identity,

$$\{zy\{xxy\}\} = \{\{zyx\}xy\} - \{x\{yzx\}y\} + \{xx\{zyy\}\},$$

and the term on the left and the first term on the right are zero by orthogonality. ∎

Lemma 2.4: *If w is a maximal tripotent, and if u and v are tripotents with $v \in B_1(u) \cap B_2(w)$ and $u \in B_1(w)$, then $B_1(w) \cap B_0(u) \subset B_0(v)$.*

Proof. Let x be a tripotent in $B_1(w) \cap B_0(u)$. By Peirce calculus with respect to w, $D(x,x)v = 2D(x,x)D(u,u)v = 2\{x\{xvu\}u\} = 0$ so that $x \perp v$. The spectral tripotents of an element $x \in B_1(w) \cap B_0(u)$ also lie in $B_1(w) \cap B_0(u)$, and the result follows. ∎

3. Local Jordan multipliers

In this section, we define and establish some properties of Jordan multipliers, and introduce the pullback map, which is a key concept in this paper.

Let $\psi : B_* \to A_*$ be a linear isometry, where A and B are JBW^*-triples. Then ψ^* is a normal contraction of A onto B and, by a standard separation theorem, ψ^* maps the closed unit ball of A onto the closed unit ball of B. Let w be an extreme point of the closed unit ball of B. Since $(\psi^*)^{-1}(w) \cap \text{ball}\, A$ is a non-empty weak*-compact convex set, it has an extreme point v, and in fact v is an extreme point of the closed unit ball of A.

LEMMA 3.1: *With the above notation,* $\psi^*[A_1(v)] \subset B_1(w)$ *and* $P_2(w)\psi^*[A_2(v)] = B_2(w)$.

Proof. If f is a normal state of $B_2(w)$, then $\psi(f)$ has norm one and $\psi(f)(v) = f(\psi^*(v)) = f(w) = 1$ so that $\psi(f)$ is a normal state of $A_2(v)$. Now let $x_1 \in A_1(v)$ and suppose $\psi^*(x_1) = y_2 + y_1$ with $0 \neq y_2 \in B_2(w)$ and $y_1 \in B_1(w)$. There is a normal state of f of $B_2(w)$ such that $f(y_2) \neq 0$. Then $\psi(f)(x_1) = f(\psi^*(x_1)) = f(y_2) \neq 0$, a contradiction since $\psi(f)$, being a state of $A_2(v)$, vanishes on $A_1(v)$.

To prove the second statement, let $z \in B_2(w)$. Then $z = \psi^*(a_2+a_1)$ with $a_j \in A_j(v)$ and, by the first statement, $z = P_2(w)z = P_2(w)\psi^*(a_2) + P_2(w)\psi^*(a_1) = P_2(w)\psi^*(a_2)$. ∎

3.1. A CONSTRUCTION OF KIRCHBERG. The following lemma was proved by Kirchberg [28, Lemma 3.6(ii)] in the case of von Neumann algebras. His proof, which is valid for JBW^*-algebras, is repeated here for the convenience of the reader.

LEMMA 3.2: *Let T be a normal unital contractive linear map of a JBW^*-algebra X onto another JBW^*-algebra Y, which maps the closed unit ball of X onto the closed unit ball of Y. For a projection $q \in Y$, let $a \in X$ be of norm one such that $T(a) = 1_Y - 2q$. If c is the self-adjoint part of a, then*

(i) $T(c^2) = T(c)^2$,
(ii) $T(x \circ c) = T(x) \circ T(c)$ *for every* $x \in X$.

Proof (*Kirchberg* [28, Lemma 3.6(ii)]). With $a \in X$ such that $T(a) = 1_Y - 2q$, let $c = (a + a^*)/2$. Since T is a positive unital map on X, $T(c) = (T(a) + T(a^*))/2 = (T(a) + T(a)^*)/2 = 1_Y - 2q$ and, by Kadison's generalized Schwarz inequality ([36]), $1_Y \geq T(c^2) \geq T(c)^2 = (1_Y - 2q)^2 = 1_Y$, which proves (i).

Define a continuous Y-valued bilinear form $\tilde{T}$ on $X_{\text{s.a.}}$ by

$$\tilde{T}(x,z) = T(x \circ z) - T(x) \circ T(z).$$

By Kadison's inequality again, $\tilde{T}(x,x) = T(x^2) - T(x)^2 \geq 0$ so that for any state ρ of Y, the Schwarz inequality for positive bilinear functionals yields

$$|\rho \circ \tilde{T}(x,y)| \leq [\rho \circ \tilde{T}(x,x)]^{1/2}[\rho \circ \tilde{T}(y,y)]^{1/2} \leq \|\tilde{T}(x,x)\|^{1/2}\|\tilde{T}(y,y)\|^{1/2}.$$

Then by the Jordan decomposition for normal functionals, for any element $\rho \in Y_*$,

$$|\rho \circ \tilde{T}(x,y)| \leq 4\|\tilde{T}(x,x)\|^{1/2}\|\tilde{T}(y,y)\|^{1/2}.$$

Since $\tilde{T}(c,c) = 0$ we have $\tilde{T}(c,z) = 0$ for all $z \in X_{\text{s.a.}}$, and (ii) follows. ∎

With the notation of Lemma 3.2, define a **Jordan multiplier** (with respect to the data (X, Y, T)) to be any element of the set

$$M(X,Y,T) = \{x \in X : T(x \circ z) = T(x) \circ T(z) \text{ for all } z \in X\}.$$

Corollary 3.3: *Let $\psi : B_* \to A_*$ be a linear isometry, where A and B are JBW^*-triples. Let w be an extreme point of the closed unit ball of B and let v be an extreme point of the closed unit ball of A with $\psi^*(v) = w$. We set $V = P_2(w)\psi^*|A_2(v)$ and note that V is a normal unital contractive (hence positive) map of $A_2(v)$ onto $B_2(w)$. Then*

- (a) *For each projection $q \in B_2(w)$, there is an element $a \in A_2(v)$ of norm one such that $V(a) = w - 2q$.*
- (b) *If c is the self-adjoint part of the element a in (a), then*
  - (i) *$V(c^2) = V(c)^2$,*
  - (ii) *$V(x \circ c) = V(x) \circ V(c)$ for every $x \in A_2(v)$.*

Proof. Part (a) follows from Lemma 3.1 and part (b) follows from Lemma 3.2. ∎

With the notation of Corollary 3.3, a Jordan multiplier (with respect to the pair of extreme points $w \in B, v \in A$ with $\psi^*(v) = w$) is any element of the set

$$\begin{aligned} M &= M(A_2(v), B_2(w), V) \\ &= \{x \in A_2(v) : V(x \circ y) = V(x) \circ V(y) \text{ for all } y \in A_2(v)\}, \end{aligned}$$

where $V = P_2(w)\psi^*|A_2(v)$. We shall let s denote the support of the positive unital normal mapping V, that is, $s = \inf\{p : p \text{ is a projection in } A_2(v), V(p)=w\}$. Note that s is a multiplier by Lemma 3.2.

The following two lemmas could easily have been stated and proved if $A_2(v)$ and $B_2(w)$ were replaced by arbitrary JBW^*-algebras and V was replaced by a normal unital contraction with support s mapping the closed unit ball onto the closed unit ball. This fact will be used explicitly in the proof of Lemma 3.13.

In the rest of Section 3, A and B denote JBW^*-triples, $\psi : B_* \to A_*$ is a linear isometry, and $V = P_2(w)\psi^*$, where w is a maximal tripotent of B.

LEMMA 3.4: *Let $x \in A_2(s)$ be such that $0 \le x \le s$ and $V(x)$ is a projection q in $B_2(w)$. Then $x \in M$.*

Proof. We have $V(2x-s) = 2q-w$ and, by the functional calculus, $\|2x-s\| \le 1$. Then Lemma 3.2 shows that $2x-s \in A_2(s)$ is a multiplier with respect to (w, v), hence $2x - s \in M$ and $x \in M$. ∎

LEMMA 3.5: (a) *M is a unital JBW^*-subalgebra of $A_2(v)$.*
(b) *$V|M$ is a normal unital Jordan *-homomorphism of M onto $B_2(w)$ satisfying $V(\{xyx\}) = \{V(x)V(y)V(x)\}$ for all $x \in M, y \in A_2(v)$.*
(c) *$V|M_2(s)$ is a normal unital Jordan *-isomorphism of $M_2(s)$ onto $B_2(w)$, where $M_2(s) = A_2(s) \cap M$.*

Proof. M is clearly a weak*-closed self-adjoint linear subspace of $A_2(v)$. To prove it is a JBW^*-subalgebra, it suffices to show that if $c = c^* \in M$, then $c^2 \in M$, equivalently that $\tilde{V}(c^2, c^2) = 0$, where $\tilde{V}(x, y) = V(x \circ y) - V(x) \circ V(y)$.

Using the Jordan algebra identity, namely $(b \circ a^2) \circ a = (b \circ a) \circ a^2)$, and the fact that c is a self-adjoint multiplier, we have $V(c^2) \circ V(c^2) = V(c)^2 \circ V(c)^2 = V(c) \circ (V(c) \circ V(c)^2) = V(c) \circ (V(c) \circ V(c^2)) = V(c) \circ (V(c \circ c^2)) = V(c \circ (c \circ c^2)) = V(c^2 \circ c^2)$. Thus $\tilde{V}(c^2, c^2) = V(c^2 \circ c^2) - V(c^2) \circ V(c^2) = 0$, proving (a).

By the definition of multiplier, V is a Jordan *-homomorphism of M into $B_2(w)$. To show that it is onto, let q be a projection in $B_2(w)$. By Corollary 3.3 there is a self-adjoint multiplier c with $V(c) = w - 2q$ and so $q = (w - V(c))/2 = V((v - c)/2)$. By the spectral theorem in $B_2(w)$, $B_2(w)_{\text{s.a.}} \subset V(M)$ proving that $B_2(w) \subset V(M)$ and hence $B_2(w) = V(M)$. The last statement in (b) follows from the relation $\{xyx\} = 2x \circ (x \circ y^*) - y^* \circ x^2$.

To prove (c), note that the kernel of $V|M_2(s)$ is a JBW^*-subalgebra of $M_2(s)$ and is hence generated by its projections. If it contained a non-zero projection p then we would have $V(s-p) = w$, contradicting the fact that s is the support of V. Thus the kernel of $V|M_2(s)$ is zero. Finally, since $V(P_2(s)m) = V(m)$ for any $m \in M$, $V|M_2(s)$ maps onto $B_2(w)$. ∎

3.2. The pullback map.

Remark 3.6: Starting with an extreme point $w \in B$, every choice of extreme point $v \in A$ with $\psi^*(v) = w$ determines the objects V, s, M. This notation will prevail throughout this section. For use in the next three lemmas, we define $\phi : B_2(w) \to M_2(s)$ to be the inverse of the Jordan *-isomorphism $V|M_2(s)$.

Lemma 3.7: *If $u = \sup_\lambda u_\lambda$ in the lattice of tripotents of B, where each u_λ is a tripotent majorized by a fixed maximal tripotent w, then $u \in B_2(w)$ and $\phi(u) = \sup_\lambda \phi(u_\lambda)$ in the lattice of tripotents of A.*

Proof. In $A_2(s)$, $\phi(u_\lambda) \le \sup_\lambda \phi(u_\lambda) \le \phi(u) \le s$ so that $u_\lambda = V(\phi(u_\lambda)) \le V(\sup_\lambda \phi(u_\lambda)) \le u \le w$ and therefore $u = \sup_\lambda u_\lambda \le V(\sup_\lambda \phi(u_\lambda)) \le u$. Thus $u = V(\sup_\lambda \phi(u_\lambda))$ and, since u is a projection in $B_2(w)$ and $\sup_\lambda \phi(u_\lambda) \ge 0$, $\sup_\lambda \phi(u_\lambda)$ is a multiplier by Lemma 3.4. Therefore $\phi(u) = \phi(V(\sup_\lambda \phi(u_\lambda)) = \sup_\lambda \phi(u_\lambda) \le \phi(u)$, proving the lemma. ∎

Lemma 3.8: *Let f be a normal functional on B and let w be a maximal tripotent in B with $v_f \le w$, giving rise to v, M, s in A and $\phi : B_2(w) \to M_2(s)$. Recall that v_f denotes the support tripotent of f. Then $v_{\psi(f)} = \phi(v_f)$.*

Proof. Since $B_2(v_f) \subset B_2(w)$, $f \in B_2(w)_*$. Thus

$$\langle \psi(f), s\rangle = \langle \psi(P_2(w)_* f), s\rangle = \langle f, P_2(w)\psi^*(s)\rangle = f(w) = f(v_f) = \|f\| = \|\psi(f)\|,$$

so that $v_{\psi(f)} \le s$ is a projection in $A_2(s)$.

We also have

$$\langle \phi(v_f), \psi(f)\rangle = \langle P_2(w)\psi^*(\phi(v_f)), f\rangle = \langle v_f, f\rangle = \|f\| = \|\psi(f)\|,$$

and therefore

$$\phi(v_f) \ge v_{\psi(f)}. \tag{1}$$

Let $b = P_2(w)\psi^*(v_{\psi(f)})$ so that $0 \le b \le w$ in $B_2(w)$ and

$$\langle b, f\rangle = \langle \psi^*(v_{\psi(f)}), f\rangle = \langle v_{\psi(f)}, \psi(f)\rangle = \|\psi(f)\| = \|f\|.$$

Thus b belongs to the weak*-closed face in B generated by f (that is, $\{x \in B : \|x\| = 1, \langle x, f\rangle = \|f\|\}$) and therefore by [12, Theorem 4.6], $b = v_f + c$ with $c \perp v_f$.

We then have $v_f + c = b = P_2(w)\psi^*(v_{\psi(f)}) \le P_2(w)\psi^*(\phi(v_f)) = v_f$, so that $c \le 0$. The JB^*-subalgebra generated in $B_2(w)$ by the orthogonal elements v_f and c is associative and is thus representable as continuous functions on a locally compact space. The function representing c cannot take on a negative value, since by orthogonality, so would the function represented by b. Thus

$c = 0$ and $P_2(w)\psi^*(v_{\psi(f)}) = P_2(w)\psi^*(\phi(v_f))$. By Lemma 3.4, $v_{\psi(f)} \in M_2(s)$ and the result follows since $P_2(w)\psi^*$ is one to one on $M_2(s)$. ∎

From the previous two lemmas, we can deduce the following lemma, which in turn will be strengthened in Lemma 3.15 to require only that $u \in B_2(w)$ for some maximal tripotent w. Moreover, Lemma 3.15 holds more generally for arbitrary elements of B.

Lemma 3.9: *With the above notation, if u is any tripotent in B and w is a maximal tripotent with $u \leq w$, then $\phi(u)$ depends only on u and ψ. More precisely, if $w' \geq u$ is another maximal tripotent and if v' is a maximal tripotent in A with $\psi^*(v') = w'$ and if M' and s' are the corresponding objects such that $P_2(w')\psi^*$ is a Jordan *-isomorphism of $M_2'(s')$ onto $B_2(w')$, and ϕ' denotes $(P_2(w')\psi^*|M_2'(s'))^{-1}$, then $\phi(u) = \phi'(u)$.*

Proof. By Zorn's lemma, we may write $u = \sup_\lambda v_{f_\lambda}$ for some family f_λ of normal functionals on B. Writing u_λ for v_{f_λ}, we have

$$\phi(u) = \phi(\sup u_\lambda) = \sup \phi(u_\lambda)$$

and

$$\phi'(u) = \phi'(\sup u_\lambda) = \sup \phi'(u_\lambda).$$

By Lemmas 3.7 and 3.8, $\phi(u_\lambda) = v_{\psi(f_\lambda)}$ and $\phi'(u_\lambda) = v_{\psi(f_\lambda)}$. ∎

Definition 3.10: The **pullback** of a tripotent $u \in B$ is defined to be the element $\phi(u)$ in Lemma 3.9. By this lemma, we may unambiguously denote it by u_ψ.

Thus u_ψ is the unique tripotent of A such that for any maximal tripotent w majorizing u and any maximal tripotent v of A with $\psi^*(v) = w$, giving rise to the space of multipliers M and the support s of $P_2(w)\psi^*|A_2(v)$, we have $u_\psi \in M_2(s)$ and $P_2(w)\psi^*(u_\psi) = u$. Note that in this situation, $s = w_\psi$.

We next improve the last assertion in Lemma 3.5 by replacing $V|M_2(s)$ by $\psi^*|M_2(s)$.

Lemma 3.11: *ψ^* agrees with V on $M_2(s)$. In particular, $\psi^*(u_\psi) = u$ for every tripotent u of B and $\psi^*|M_2(s)$ is a normal unital Jordan *-isomorphism of $M_2(s)$ onto $B_2(w)$.*

Proof. We use the notation of Lemma 3.5. Since $V(s) = w$, we have $\psi^*(s) = w + x_1$ where $x_1 = P_1(w)\psi^*(s)$. Then by Lemma 2.1, $x_1 = 0$, so that $\psi^*(s) = w$.

It suffices to show that ψ^* maps projections of $M_2(s)$ into $B_2(w)$. So let p be any projection in $B_2(w)$. Since $V(p_\psi) = p$, we have $\psi^*(p_\psi) = p + y_1$ where $y_1 = P_1(w)\psi^*(p_\psi)$. Since $p \le w$ and $y_1 \in B_1(w)$, $P_2(p)y_1 = \{p\{py_1p\}p\} = 0$ by Peirce calculus with respect to w, so that by Lemma 2.1, $y_1 \perp p$. Similarly, $\psi^*(s - p_\psi) = w - p - y_1$ and, by Lemma 2.1, $y_1 \perp w - p$. Hence $y_1 \in B_0(w) = \{0\}$. ∎

The following lemma will be improved in Lemma 5.4 to include the case of the Peirce 2-space. As it stands, it extends the first statement of Lemma 3.1.

LEMMA 3.12: *Let v be a tripotent in B. Then*

(a) $\psi^*(A_1(v_\psi)) \subset B_1(v) + B_0(v)$,
(b) $\psi^*(A_0(v_\psi)) \subset B_0(v)$.

Proof. Let f be a normal state of $B_2(v)$. Then $\langle \psi(f), v_\psi\rangle = f(v) = 1 = \|f\| = \|\psi(f)\|$ so that $\psi(f)$ is a normal state of $A_2(v_\psi)$ and hence $\psi[B_2(v)_*] \subset A_2(v_\psi)_*$.

Now if $x \in A_1(v_\psi)$ and $f \in B_2(v)_*$ is arbitrary, $\langle f, \psi^*(x)\rangle = \langle \psi(f), x\rangle = 0$ and therefore $\psi^*(x) \in B_1(v) + B_0(v)$. This proves (a).

Now let $x \in A_0(v_\psi)$ and suppose $\|x\| = 1$. Then $\|v_\psi \pm x\| = 1$ and therefore by Lemma 3.11,

$$\begin{aligned}\|v \pm P_2(v)\psi^*(x)\| &= \|P_2(v)\psi^*(v_\psi) \pm P_2(v)\psi^*(x)\|\\ &\le \|\psi^*(v_\psi) \pm \psi^*(x)\| = \|\psi^*(v_\psi \pm x)\| \le 1,\end{aligned}$$

and since v is an extreme point of the unit ball of $B_2(v)$, we have $P_2(v)\psi^*(x) = 0$. We now have $\|v + P_1(v)\psi^*(x) + P_0(v)\psi^*(x)\| = \|v + \psi^*(x)\| = \|\psi^*(v_\psi + x)\| \le 1$ and, by Lemma 2.1, $P_1(v)\psi^*(x) = 0$. ∎

LEMMA 3.13: *Suppose $\psi^*(x) = v$ for a tripotent $v \in B$ and an element $x \in A$ with $\|x\| = 1$. Then $x = v_\psi + q$ for some $q \perp v_\psi$.*

Proof. Let w be a maximal tripotent of B majorizing v and let v' be a maximal tripotent of A with $\psi^*(v') = w$.

If $z \in A_2(v_\psi)$, then $z = \{v_\psi\{v_\psi z v_\psi\}v_\psi\}$. Since v_ψ is a multiplier with respect to $A_2(v')$, for all $c \in A_2(v')$ we have $P_2(w)\psi^*(v_\psi \circ c) = v \circ P_2(w)\psi^*(c)$. Using this and the general formula $\{zyz\} = 2z \circ (z \circ y^*) - y^* \circ z^2$ we obtain $P_2(w)\psi^*\{v_\psi z v_\psi\} = \{v, P_2(w)\psi^*(z), v\}$. For the same reason, $P_2(w)\psi^*(z) = \{v, P_2(w)\psi^*\{v_\psi z v_\psi\}, v\} = \{v\{v, P_2(w)\psi^*(z), v\}v\} \in B_2(v)$, proving that

$P_2(w)\psi^*[A_2(v_\psi)] \subset B_2(v)$. In fact, $P_2(w)\psi^*[A_2(v_\psi)] = B_2(v)$, since if p is any projection in $B_2(v)$, then $p_\psi \le v_\psi$, so that $p_\psi \in A_2(v_\psi)$ and $P_2(w)\psi^*(p_\psi) = p$.

Decomposing $x = x_2 + x_1 + x_0$ with respect to v_ψ, we notice that by Lemma 3.12, $P_2(v)\psi^*(x_2) = v$, and since $P_2(v)\psi^*$ is a contractive unital, hence positive, hence self-adjoint map of $A_2(v_\psi)$ onto $B_2(v)$, $P_2(v)\psi^*(x_2') = v$ where x_2' is the self-adjoint part of x_2 in $A_2(v_\psi)$.

Now x_2' is a norm one self-adjoint element of the JBW^*-algebra $A_2(v_\psi)$ which $P_2(v)\psi^*$ maps to the identity v of $B_2(v)$. Thus by Lemma 3.2, we see that x_2' is a multiplier with respect to $A_2(v_\psi)$.

We show next that v_ψ is the support of the map $P_2(v)\psi^*$. Let $p \le v_\psi$ be a projection with $P_2(v)\psi^*(p) = v$. Then $P_2(w)\psi^*(p) = v$, so that by Lemma 3.4, $p \in M_2(s)$, and since $P_2(w)\psi^*$ is one-to-one there, $p = v_\psi$.

Now, since v_ψ is the support of the map $P_2(v)\psi^*$, it is a multiplier with respect to $A_2(v_\psi)$, and we have $x_2' = v_\psi$ by Lemma 3.5 (replacing $B_2(w)$ there by $B_2(v)$ and $A_2(v)$ by $A_2(v_\psi)$).

Thus $x_2 = x_2' + ix_2'' = v_\psi + ix_2''$ with x_2'' self-adjoint and, by a familiar argument, if $x_2'' \neq 0$, then $\|x_2\| = \|v_\psi + ix_2''\| > 1$, a contradiction. We now have $x_2 = v_\psi$ and the proof is completed by applying Lemma 2.1 to show that $x_1 = 0$. ∎

Definition 3.14: Suppose x lies in B and let w be a maximal tripotent majorizing $r(x)$. The Jordan*-isomorphism $(\psi^*|M_2(s))^{-1}$ of $B_2(w)$ onto $M_2(s)$ carries $B_2(r(x))$ onto $M_2((r(x)_\psi)$. We let x_ψ denote the image of x under this map so that $\psi^*(x_\psi) = x$. This is an extension of the pullback of a tripotent defined in Definition 3.10.

The following lemma shows that x_ψ may be computed using any maximal tripotent w for which $x \in B_2(w)$, that is, $r(x)$ need not be majorized by w. This fact will be critical in the proofs of Theorem 2 and elsewhere in this paper (for example, Lemmas 5.7 and 6.2).

Lemma 3.15: *Suppose x is an element in $B_2(w)$, where w is a maximal tripotent not necessarily majorizing $r(x)$. Let M be the space of multipliers corresponding to a choice of maximal tripotent v such that $\psi^*(v) = w$. Then $x_\psi = (\psi^*|M_2(w_\psi))^{-1}(x)$.*

Proof. We shall consider first the case that $x = u$ is a tripotent. Let w' be a maximal tripotent majorizing u, so that by Lemma 3.12, $\psi^*|M_2'(s')$ is a

Jordan*-isomorphism onto $B_2(w')$, $u_\psi = (\psi^*|M_2'(s'))^{-1}(u)$ and let m denote $(\psi^*|M_2(s))^{-1}(u)$. Here, of course, $s = w_\psi$ and $s' = w'_\psi$.

Since $\psi^*(m) = u$, by Lemma 3.13, $m = u_\psi + q$ with $q \perp u_\psi$. Furthermore, $\psi^*(q) = 0$.

Note that since m and u_ψ are tripotents, cubing the relation $m = u_\psi + q$ shows that q is also a tripotent. We claim that u_ψ and q belong to $A_2(s)$. First of all, since $m \in A_2(s)$, we have $A_2(m) \subset A_2(s)$, and since $u_\psi \leq m$ and $q \leq m$, $u_\psi, q \in A_2(m) \subset A_2(s)$, proving the claim.

It remains to show that $q = 0$. To this end, note first that in $A_2(s)$, $\{qqs\} = q \circ q^*$ and $\{mqs\} = m \circ q^*$. Using this and the fact that m is a multiplier, with $V = P_2(w)\psi^*$, we have

$$V(q\circ q^*) = V\{qqs\} = V\{mqs\} = V(m\circ q^*) = V(m)\circ V(q^*) = V(m)\circ V(q)^* = 0.$$

Now we have $V(s - q \circ q^*) = w$ so that, by Lemma 3.4, $s - q \circ q^* \in M_2(s)$. Thus $q \circ q^* \in M_2(s)$ and, since V is bijective on $M_2(s)$, $q \circ q^* = 0$ and $q = 0$.

Having proved the lemma for tripotents, we now let $x = \int \lambda du_\lambda$ be the spectral decomposition of x and let w' be a maximal tripotent majorizing $r(x)$. Then for any spectral tripotent u_S, we have $u_S \in B_2(w)$ and $u_S \leq w'$ so that, by the special case just proved, $(u_S)_\psi = \phi(u_S)$ where $\phi = (\psi^*|M_2(s))^{-1}$. Approximating x by $y = \sum \lambda_i u_{S_i}$, we have

$$\begin{aligned} y_\psi &= (\psi^*|M_2'(s'))^{-1}\Big(\sum \lambda_i u_{S_i}\Big) = \sum \lambda_i (\psi^*|M_2'(s'))^{-1}(u_{S_i}) \\ &= \sum \lambda_i \phi(u_{S_i}) \\ &= \phi(y), \end{aligned}$$

which completes the proof, as the maps in question are continuous. ∎

Remark 3.16: We will henceforth refer to elements x_ψ as multipliers without specifying the Peirce 2-space containing x. By embedding two orthogonal elements x and y of B into $B_2(w)$ for some maximal tripotent w, it follows that $x_\psi \perp y_\psi$. This fact will be used explicitly in the rest of this paper.

4. Analysis of tripotents and pullback of the Peirce 1-space

Our next goal is to prove, in the case where B has no summand isometric to $L^\infty(\Omega, H)$, that the pullback map respects Peirce 1-spaces, that is, if u is any tripotent in $B_1(w)$ for some maximal tripotent w, then $u_\psi \in A_1(w_\psi)$. This will be achieved in this section (see Theorem 2 below) after some analysis of tripotents in a JBW^*-triple.

4.1. Rigid collinearity.

Proposition 4.1: *If u is a tripotent in $B_1(w)$ and w is a maximal tripotent, then the element $2\{uuw\}$, which we shall denote by w_u, is a tripotent in $B_2(w)$ which is collinear to u and $\leq w$. Moreover, u and w_u are rigidly collinear.*

The proof will be contained in Lemmas 4.2 to 4.6 in which the standing assumption is that w is a tripotent in B and u is a tripotent in $B_1(w)$. This proposition was proved in [23, Lemma 2.5] for w not necessarily maximal but under the additional assumption that $B_2(u) \subset B_1(w)$, which follows from the maximality of w. On the other hand, Lemmas 4.3 and 4.4 are stated here with an assumption weaker than maximality and will be used in that form later on. For this reason, we include the proof of Proposition 4.1 here.

Lemma 4.2: *If w is maximal, then $B_2(u) \subset B_1(w)$.*

Proof. If $x \in B_2(u)$, then $x = P_2(u)x = \{u\{uxu\}u\} \in B_1(w)$ by Peirce calculus with respect to w and the maximality of w. ∎

Lemma 4.3: *If $\{uwu\} = 0$ (in particular, if w is maximal), then $w_u \in B_1(u)$.*

Proof. By the main identity,

$$\{wuu\} = \{wu\{uuu\}\} = \{\{wuu\}uu\} - \{u\{uwu\}u\} + \{uu\{wuu\}\}$$

and the middle term is zero by assumption. Hence

$$w_u/2 = \{w_u uu\}/2 + \{uuw_u\}/2 = \{uuw_u\}. \quad ∎$$

Lemma 4.4: *If $\{uwu\} = 0$ and $u \neq 0$ (in particular, if w is maximal), then w_u is a nonzero tripotent and $w_u \leq w$.*

Proof. Clearly w_u is non-zero since $u \neq 0$ does not lie in $B_0(w)$. By the main identity,

$$\{uu\{www\}\} = \{\{uuw\}ww\} - \{w\{uuw\}w\} + \{ww\{uuw\}\}$$

so that

$$\{w\{uuw\}w\} = 2\{\{uuw\}ww\} - \{uuw\} = 2\{uuw\} - \{uuw\} = \{uuw\},$$

proving that w_u is a self-adjoint element of $B_2(w)$.

It remains to show that w_u is an idempotent in $B_2(w)$. To this end use the main identity to obtain

$$\begin{aligned} \{w_u w w_u\} &= 2\{w_u w\{uuw\}\} \\ &= 2\left[\{\{w_u wu\}uw\} - \{u\{ww_u u\}w\} + \{uu\{w_u ww\}\}\right]. \end{aligned} \tag{2}$$

Since $w_u \in B_2(w)$, the third term in the bracket on the right is equal to $\{uuw_u\} = w_u/2$ by Lemma 4.3. It remains to show that the first two terms on the right side of (2) cancel out. In the first place, by the main identity

$$\begin{aligned} u/2 &= \{uu\{wwu\}\} \\ &= \{\{uuw\}wu\} - \{w\{uuw\}u\} + \{ww\{uuu\}\} \\ &= \{\{uuw\}wu\} - \{w\{uuw\}u\} + u/2, \end{aligned}$$

so that $\{\{uuw\}wu\} = \{w\{uuw\}u\}$, that is, $\{ww_u u\} = \{w_u wu\}$.

On the other hand, by the main identity,

$$\begin{aligned} \{uww_u\} &= 2\{uw\{wuu\}\} \\ &= 2[\{\{uww\}uu\} - \{w\{wuu\}u\} + \{wu\{uwu\}\}] \\ &= 2[u/2 - \{ww_u u\}/2 + 0] = u - \{ww_u u\}, \end{aligned}$$

and it now follows that $\{uww_u\} = \{ww_u u\} = u/2$, proving that the first two terms in (2) do cancel out. ∎

Lemma 4.5: *If w is maximal, then $B_2(u) \subset B_1(w_u)$.*

Proof. By the joint Peirce decomposition and Lemma 4.2,

$$B_2(u) \subset B_1(w) = B_1(w_u) \cap B_0(w - w_u) + B_1(w - w_u) \cap B_0(w_u).$$

Now

$$2D(u,u)(w - w_u) = w_u - 2D(u,u)w_u = w_u - w_u = 0,$$

so that $u \perp (w - w_u)$ and therefore $B_2(u) \perp (w - w_u)$. This shows that $B_2(u) \subset B_1(w_u) \cap B_0(w - w_u) \subset B_1(w_u)$. ∎

Lemma 4.6: *If w is maximal, then $B_2(w_u) \subset B_1(u)$ (this completes the proof of the rigid collinearity of w_u and u).*

Proof. Let $x \in B_2(w_u)$. By Lemma 4.3 and Peirce calculus with respect to u, $\{w_u, P_0(u)x, w_u\} \in B_2(u)$ and, by Lemma 4.5, $B_2(u) \subset B_1(w_u)$. By compatibility of u and w_u, $P_0(u)x \in B_2(w_u)$ and, by Peirce calculus with respect to

w_u, $P_0(u)x = \{w_u\{w_u, P_0(u)x, w_u\}w_u\} = 0$, since the middle term belongs to $B_1(w_u)$, as just shown. On the other hand, by Lemma 4.5, $P_2(u)x \in B_1(w_u)$ so that $P_2(u)x = 0$ also. ∎

The next two lemmas give important properties of w_u. Note that by definition, $w = w_u$ if and only if $w \in B_1(u)$.

Lemma 4.7: *If $u \in B_1(w)$ and w is maximal, then $B_1(w) \cap B_0(u) \subset B_0(w_u)$. In particular, if $w_u = w$ (hence $u \top w$), then u is maximal.*

Proof. The first statement holds by Lemma 2.4.

Suppose now that $w = w_u$ so that $u \top w$. We shall show that $B_0(u) \subset B_0(w)$, which implies the second assertion. By Lemma 4.6, $B_2(w) = B_2(w_u) \subset B_1(u)$. If $x \in B_0(u) = [B_0(u) \cap B_2(w)] + [B_0(u) \cap B_1(w)]$, say $x = x_2 + x_1$ with respect to w, then by the first statement, $x_1 \in B_0(w_u) = B_0(w) = 0$. On the other hand, $x_2 \in B_2(w) \cap B_0(u) \subset B_1(u) \cap B_0(u)$, so $x_2 = 0$. ∎

Lemma 4.8: *Suppose that tripotents $u_1, u_2 \in B_1(w)$ with w a maximal tripotent in B. If $u_1 \leq u_2$ then $w_{u_1} \leq w_{u_2}$ and $w_{u_2 - u_1} = w_{u_2} - w_{u_1}$.*

Proof. If $u_1 \leq u_2$, then $u_2 - u_1 \perp u_1$, $\{wu_1u_2\} = \{wu_1u_1\}$ and

$$\begin{aligned} w_{u_2-u_1} &= 2\{w, u_2 - u_1, u_2 - u_1\} \\ &= 2\{w, u_2 - u_1, u_2\} - 2\{w, u_2 - u_1, u_1\} \\ &= 2\{wu_2u_2\} - 2\{wu_1u_1\} - 0 = w_{u_2} - w_{u_1}. \end{aligned}$$

On the other hand, if $v_1, v_2 \in B_1(w)$ and $v_1 \perp v_2$, then by Lemma 4.7, $v_2 \perp w_{v_1}$ and, since $w_{v_1} \perp w - w_{v_1}$,

$$\begin{aligned} \{w_{v_1}w_{v_1}w_{v_2}\} &= 2\{w_{v_1}w_{v_1}\{wv_2v_2\}\} \\ &= 2\{\{w_{v_1}w_{v_1}w\}v_2v_2\} - 2\{w\{w_{v_1}w_{v_1}v_2\}v_2\} + 2\{wv_2\{w_{v_1}w_{v_1}v_2\}\} \\ &= 2\{\{w_{v_1}w_{v_1}w_{v_1}\}v_2v_2\} - 0 + 0 = 2\{w_{v_1}v_2v_2\} = 0. \end{aligned}$$

Combining the results of the previous two paragraphs, if $u_1 \leq u_2$, then $u_1 \perp u_2 - u_1$, $w_{u_2-u_1} \perp w_{u_1}$, $(w_{u_2} - w_{u_1}) \perp w_{u_1}$ so that $w_{u_1} \leq w_{u_2}$. ∎

4.2. Central tripotents.

Lemma 4.9: *Let w be a maximal tripotent of B and suppose that v is a tripotent $\leq w$, u is a tripotent in $B_1(w)$ and $u \top v$. Then either $B_1(w) \cap B_1(u) \cap B_0(v) \neq 0$ or u is a central tripotent in B.*

Proof. If $v = w$ then we are in the situtation of the second sentence in Lemma 4.7, so u is maximal, hence central. So we assume $v \neq w$. Suppose that $B_1(w) \cap B_1(u) \cap B_0(v) = 0$ and let $e \in B_1(v) \cap B_1(w - v) \subseteq B_2(w)$ be a tripotent. We proceed to show, using Peirce calculus, that $e = 0$ and then that u is central.

We first note that, by the joint Peirce decomposition,

$$B_1(w) = B_1(v) \cap B_0(w - v) + B_1(w - v) \cap B_0(v),$$

and therefore $u \in B_1(w) \cap B_1(v) \subset B_0(w - v)$ so $w - v \in B_0(u)$. Then $\{u, e, w - v\} \in B_1(w) \cap B_1(u) \cap B_0(v) = 0$ and $\{uev\} \in B_2(v) \cap B_1(w) \subset B_2(w) \cap B_1(w) = 0$, so that $\{uew\} = \{u, e, w - v\} + \{uev\} = 0$. Clearly $\{euw\} = 0$ as well.

We next show that $u \perp e$. By the main identity, $\{uee\} = \{ue\{eww\}\} = \{\{uee\}ww\} - \{e\{euw\}w\} + \{ew\{uew\}\}$. The last two terms are zero and, since $\{uee\} \in B_1(w)$, the first term is equal to $\{uee\}/2$. Hence $\{uee\} = 0$ and $u \perp e$.

Finally, we show that $e = 0$. Note that $\{uve\} \in B_1(w) \cap B_1(u) \cap B_0(v)$ so $\{uve\} = 0$ and, by Peirce calculus with respect to w, $\{vue\} = 0$. Hence, by the main identity, $0 = \{vu\{uve\}\} = \{\{vuu\}ve\} - \{u\{uvv\}e\} + \{uv\{vue\}\} = \{vve\}/2 - \{uue\}/2 + 0 = e/4$.

From the fact just proved, namely, that $B_1(v) \cap B_1(w - v) = 0$, it follows from the joint Peirce decomposition that $B_2(w) = B_2(v) \oplus B_2(w - v)$, which by [22, Theorem 4.2(2)] implies that $B = C \oplus D$ where C and D are orthogonal weak*-closed ideals generated by $B_2(v)$ and $B_2(w - v)$, respectively. Again from [22, Theorem 4.2 (3)], $C = B_2(v) \oplus B_1(v)$ so that v is a maximal tripotent in C. Since $u \top v$, Lemma 4.7 assures that u is a maximal tripotent of C, so that $C = B_2(u) \oplus B_1(u)$, showing that u is a central tripotent. ∎

The proof of the following remark is identical to the proofs of Lemmas 4.3 and 4.4. Recall that, as noted above, those two lemmas are valid without assuming the maximality of w there and u here.

Remark 4.10: Let w be a maximal tripotent and let $u \in B_1(w)$ be a tripotent. Assume that u is not a central tripotent of B and that $w_u \neq w$. Let a be a non-zero tripotent of $B_1(u) \cap B_0(w_u) \cap B_1(w)$ (which is non-zero by Lemma 4.9). Then u_a $(:= 2\{aau\})$ is a tripotent $\leq u$ by Lemma 4.4, noting that $\{aua\} = 0$ by Peirce calculus with respect to w_u. Also, u_a lies in $B_1(a)$ by Peirce calculus since $P_2(a)u = \{a\{aua\}a\} = 0$.

Lemma 4.11: *With the notation of Remark 4.10,* $w_{u_a} \top u_a$.

Proof. By assumption, $a \in B_1(w)$. Therefore $u_a := 2\{uaa\} \in B_1(w)$ and the result follows from Proposition 4.1. ∎

Proposition 4.12: *Let B be a JBW^*-triple with no direct summand of the form $L^\infty(\Omega, H)$, where H is a Hilbert space of any positive dimension. Then every tripotent of B is the supremum of the non-central tripotents that it majorizes.*

Proof. Given a tripotent u in B, let v denote the supremum of all non-central tripotents majorized by u, or zero, if there are none. Let us suppose that $u \neq v$. By the definition of v, $u - v$ is a central tripotent and any tripotent majorized by $u - v$ is also a central tripotent. Hence $u - v$ is an abelian tripotent, that is, $B_2(u - v)$ is associative and hence a commutative C^*-algebra.

We thus now know that $B_2(u - v) \oplus B_1(u - v)$ is a weak*-closed ideal which is an ℓ^∞ summand of B containing a complete (=maximal) abelian tripotent, namely $u - v$. By [23, Theorem 2.8] (see also [23, p. 277] for the definition of type I_1 and [23, Proposition 2.3] for the other terminology used in this theorem), $B_2(u-v) \oplus B_1(u-v)$ is a direct sum of spaces of the form $L^\infty(\Omega_m, H_m)$ where H_m is a Hilbert space of dimension m for a family of cardinal numbers m. This contradicts our assumption, proving that $u = v$ ∎

4.3. Pullback of the Peirce 1-space. We are now ready to prove the main result of this section.

Theorem 2: *Assume that the JBW^*-triple B has no direct summand of the form $L^\infty(\Omega, H)$, where H is a Hilbert space of dimension at least two. Suppose $w \in B$ is a maximal tripotent and u is a tripotent in $B_1(w)$. If A is a JBW^*-triple and $\psi : B_* \to A_*$ is an isometry into, then $u_\psi \in A_1(w_\psi)$.*

Proof. Since commutative JBW^*-triples have no Peirce 1-spaces, it follows easily using a joint Peirce decomposition of w that we may assume B also has no summands $L^\infty(\Omega)$, so that the hypothesis of Proposition 4.12 holds. Thus we can write $u = \sup_{\lambda \in \Lambda} u_\lambda$ where each u_λ is a non-central tripotent belonging to $B_1(w)$, by Lemma 4.2. Then by Lemma 4.9 and Definition 4.10, for each $\lambda \in \Lambda$, $v_\lambda := \sup_a (u_\lambda)_a$ exists, where the supremum is over all non-zero tripotents a in $B_1(u_\lambda) \cap B_0(w_{u_\lambda}) \cap B_1(w)$.

We claim that $u = \sup_{\lambda\in\Lambda} v_\lambda$. Indeed, setting $v = \sup_\lambda v_\lambda$, if $v \neq u$ we would have that $u - v$ is the supremum of non-central tripotents majorized by $u - v$ and hence by u. Let u_{λ_0} be one of these non-central tripotents. Then $v_{\lambda_0} \leq u_{\lambda_0} \leq u - v$, which contradicts $v = \sup_\lambda v_\lambda$. This proves the claim.

Explicitly, we have proved

$$u = \sup_\lambda \sup_{a_\lambda} \{(u_\lambda)_{a_\lambda} : a_\lambda \in B_1(w) \cap B_1(u_\lambda) \cap B_0(w_{u_\lambda})\}$$

and this is the same as

$$u = \sup\{(u_\lambda)_{a_\lambda} : \lambda \in \Lambda, a_\lambda \in B_1(u_\lambda) \cap B_0(w_{u_\lambda}) \cap B_1(w)\}.$$

In the rest of this proof, we shall use the fact, just established, that u is the supremum of a family of tripotents v_a for certain $v \leq u$ and certain tripotents $a \in B_1(v) \cap B_0(w_v) \cap B_1(w)$ where, by the argument at the end of Remark 4.10, v_a lies in $B_1(a)$. Note that Lemma 3.15 will be used several times, as indicated below.

We note first that $w_{v_a}, v_a \in B_2(w_v + a)$ and $v_a \in B_1(w_v)$. Indeed, from $v_a \leq v$ we have from Lemma 4.8 that $w_{v_a} \leq w_v$ so $w_{v_a} \in B_2(w_v) \subset B_2(w_v + a)$. On the other hand, by Lemma 4.5, $v_a \in B_1(a) \cap B_2(v) \subset B_1(a) \cap B_1(w_v) \subset B_2(w_v + a)$.

We claim next that $(v_a)_\psi \in A_1((w_u)_\psi)$. Indeed, since by Lemma 4.8, $w_v \perp w_u - w_v$, we have, by Remark 3.16 and the joint Peirce decomposition,

$$A_1((w_v)_\psi) \cap A_0((w_u - w_v)_\psi) \subset A_1((w_u)_\psi). \tag{3}$$

Since $w_v, v_a \in B_2(w_v + a)$ and $\{w_v w_v v_a\} = v_a/2$, it follows (using Lemma 3.15) that $\{(w_v)_\psi, (w_v)_\psi, (v_a)_\psi\} = (v_a)_\psi/2$ so $(v_a)_\psi$ lies in $A_1((w_v)_\psi)$. Also, $v \perp w - w_v$ since

$$\begin{aligned}\{w - w_v, w - w_v, v\} &= \{wwv\} - \{w_v wv\} - \{ww_v u\} + \{w_v w_v v\} \\ &= \{wwv\} - \{w_v w_v v\} - \{w_v w_v v\} + \{w_v w_v v\} \\ &= \{wwv\} - \{w_v w_v v\} = v/2 - v/2 = 0.\end{aligned}$$

Hence $v_a \leq v$ lies in $A_0(w - w_v) \subseteq A_0(w_u - w_v)$. Embedding v_a and $w_u - w_v$ in $B_2(v_a + w_u - w_v)$, we see that $(v_a)_\psi$ lies in $A_0((w_u - w_v)_\psi)$ and the claim follows from (3).

We now have from Lemma 3.7 and Lemma 2.2 that $u_\psi \in A_1((w_u)_\psi)$. As before, $u \perp (w - w_u)$, so application of Lemma 3.15 and Remark 3.16 yields $u_\psi \in A_0((w - w_u)_\psi)$. Finally, $u_\psi \in A_1((w_u)_\psi) \cap A_0((w - w_u)_\psi) \subset A_1(w_\psi)$. ∎

5. The space of local multipliers

In this section, we establish some deeper properties of Jordan multipliers.

We retain the notation of the previous two sections, that is, $\psi : B_* \to A_*$ is a linear isometry, where A and B are JBW^*-triples and w is an extreme point of B giving rise to the objects v, M, s in A. We also assume that B satisfies the condition in Theorem 2, that is, it has no direct summand of the form $L^\infty(\Omega, H)$ where H is a Hilbert space of dimension at least two.

Lemma 5.1: $\psi[B_1(w)_*] \subset A_1(s)_*$.

Proof. If $f \in B_1(w)_*$, then $v_f \in B_1(w)$ and, by Lemma 3.8 and Theorem 2, $v_{\psi(f)} = (v_f)_\psi \in A_1(s)$.

To show that $\psi(f) \in A_1(s)_*$, let $g = \psi(f)$ and Peirce decompose it with respect to s: $g = g_2+g_1+g_0$. Since $\langle g_0, A_0(s)\rangle = \langle g, A_0(s)\rangle = \langle f, \psi^*[A_0(s)]\rangle = 0$ we have $g_0 = 0$. It remains to show $g_2 = 0$. We may assume that $\|f\| = 1$.

Since $g = g_2 + g_1$ and $v_g \in A_1(s)$, $g_1(v_g) = g(v_g) = 1 = \|g\| \geq \|g_1\|$ so that $\|g_1\| = 1$ and $g_1 \in A_2(v_g)_*$. Since obviously $g \in A_2(v_g)_*$, we have $g_2 \in A_2(v_g)_*$. By [18, Lemma 1.1], we have $\|\lambda g_2 + g_1\| = \|g_2 + g_1\| = 1$ for every complex λ of modulus 1. The local argument given in [1, Theorem 3.1] can be easily extended to apply to JBW^*-algebras to show that g_1 is a complex extreme point of the unit ball of the predual of the JBW^*-algebra $A_2(v_g)$, and thus we must have $g_2 = 0$. ∎

Corollary 5.2: $\psi^*(A_2(s)) \subset B_2(w)$.

Proof. If $x \in A_2(s)$, let $\psi^*(x) = y_2 + y_1$ be the Peirce decomposition of $\psi^*(x)$ with respect to w. If $f \in B_1(w)_*$, then $\langle f, y_1\rangle = \langle f, \psi^*(x) - y_2\rangle = \langle f, \psi^*(x)\rangle = \langle \psi(f), x\rangle = 0$ since $\psi(f) \in A_1(s)_*$ and $x \in A_2(s)$. Thus $y_1 = 0$. ∎

In view of this Corollary, we may improve the statement of Lemma 3.4 by replacing V by ψ^* We restate this improved lemma here.

Lemma 5.3: *Let $x \in A_2(s)$ be such that $0 \leq x \leq s$ and $\psi^*(x)$ is a projection in $B_2(w)$. Then $x \in M_2(s)$.*

The following is the announced improvement of Lemma 3.12.

Lemma 5.4: *Let u be a tripotent in B. Then*

(a) $\psi^*(A_1(u_\psi)) \subset B_1(u) + B_0(u)$,

(b) $\psi^*(A_j(u_\psi)) \subset B_j(u)$ *for* $j = 0, 2$.

Proof. Part (a) and the case $j = 0$ of part (b) have been proved in Lemma 3.12.

To prove the case $j = 2$ of (b), note first that by Lemma 5.3 $u_\psi \in M_2(s)$. (Recall that $u_\psi \le s \le v$ where v is a maximal tripotent of A with $\psi^*(v) = w$ and w is a maximal tripotent majorizing u.)

If $x \in A_2(u_\psi)$, then $x = \{u_\psi\{u_\psi x u_\psi\}u_\psi\}$ and, by definition of multiplier and using Corollary 5.2, $\psi^*(u_\psi \circ c) = u \circ \psi^*(c)$ for all $c \in A_2(v)$. Using this and the general formula $\{xyx\} = 2x \circ (x \circ y^*) - y^* \circ x^2$ we obtain $\psi^*\{u_\psi x u_\psi\} = \{u, \psi^*(x), u\}$. For the same reason, $\psi^*(x) = \{u, \psi^*\{u_\psi x u_\psi\}, u\} = \{u\{u, \psi^*(x), u\}u\} \in B_2(u)$, proving the case $j = 2$ of (b). ∎

Lemma 5.5: *Suppose* $x \in A$. *If* $\psi^*(x^{2n+1}) = (\psi^*(x))^{2n+1}$ *for all positive integers* n, *then* $x = (\psi^*(x))_\psi + q$, *where* $q \perp (\psi^*(x))_\psi$.

Proof. We may assume $\|x\| = 1$. Let $W(x)$ be the JBW^*-triple generated by x. By assumption and weak*-continuity, ψ^* restricts to an isomorphism of $W(x)$ onto $W(\psi^*(x))$. For each closed subset S of $(0, 1]$, if we let $u_S \in W(x)$ be the corresponding spectral tripotent for x, then $\psi^*(u_S)$ is the spectral tripotent v_S of $\psi^*(x)$ (or zero, if S has no intersection with the spectrum of $\psi^*(x)$).

Choose a maximal tripotent $w \ge r(\psi^*(x))$. If $\psi^*(u_S)$ is not zero, then by Lemma 3.13, $u_S = (v_S)_\psi + q_S$ where q_S is a tripotent which is perpendicular to $(v_S)_\psi$.

Now suppose $S \cap T = \emptyset$ and u_S and u_T are non-zero. Then $u_T \perp u_S$ and hence $(u_T)_\psi$ and q_T are each orthogonal to $(v_S)_\psi$ and q_S (subtripotents of orthogonal tripotents are orthogonal).

We now use approximation to show that $x = (\psi^*(x))_\psi + q$, where $q \perp (\psi^*(x))_\psi$. Indeed, approximate x as a norm limit of finite sums $y = \sum \lambda_i u_{S_i}$ with the S_i disjoint, and $\sum u_{S_i} = r(x) = r(y)$. Then $y = \sum \lambda_i u_{S_i} = \sum \lambda_i[(v_{S_i})_\psi + q_{S_i}] = (\sum \lambda_i v_{S_i})_\psi + \sum \lambda_i q_{S_i} = (\psi^*(y))_\psi + q$ where, since $q_{S_i} \perp (v_{S_j})_\psi$ for all i, j, the element $q = \sum \lambda_i q_{S_i}$ is orthogonal to $\sum \lambda_i (v_{S_i})_\psi = (\psi^*(y))_\psi$. The result follows from continuity. ∎

Note that by the spectral theorem, Theorem 2 is valid for arbitrary elements $x \in B_1(w)$. We now extend Theorem 2 to not necessarily maximal tripotents.

Lemma 5.6: *If* u *is any tripotent of* B *and if* $x \in B_1(u)$, *then* $x_\psi \in A_1(u_\psi)$.

Proof. Consider first a tripotent $v \in B_1(u)$. Write

$$v_\psi = P_2(u_\psi)v_\psi + P_1(u_\psi)v_\psi + P_0(u_\psi)v_\psi := (v_\psi)_2 + (v_\psi)_1 + (v_\psi)_0.$$

Then for any $f \in B_1(u)_*$ with $f(v) = 1 = \|f\|$, by Lemma 5.4

$$\begin{aligned} 1 = f(v) = \psi(f)(v_\psi) &= \psi(f)((v_\psi)_2 + (v_\psi)_1 + (v_\psi)_0) \\ &= f(\psi^*[(v_\psi)_2] + \psi^*[(v_\psi)_1] + \psi^*[(v_\psi)_0]) \\ &= f[\psi^*[(v_\psi)_1]] = \psi(f)[(v_\psi)_1]. \end{aligned}$$

Moreover, by [12, Theorem 4.6] and Lemma 3.8, $(v_\psi)_1 = v_{\psi(f)} + b_f = (v_f)_\psi + b_f$ where $b_f \perp v_{\psi(f)}$.

Let us now write, as in Lemma 3.9, $v = \sup_\lambda v_{g_\lambda}$ where $\{g_\lambda\}$ is an orthogonal family of normal functionals on $B_1(u)$. Note that $g_\lambda(v) = g_\lambda(v_{g_\lambda}) = 1 = \|g_\lambda\|$ so that, for each λ, $(v_\psi)_1 = v_{\psi(g_\lambda)} + b_\lambda$, where $b_\lambda \perp v_{\psi(g_\lambda)}$. This implies that the associated tripotent $u((v_\psi)_1)$ defined as in Subsection 2.1 verifies $u((v_\psi)_1) \geq v_{\psi(g_\lambda)}$ and therefore $u((v_\psi)_1) \geq \sup_\lambda v_{\psi(g_\lambda)}$. Indeed, by orthogonality, $((v_\psi)_1)^{2n+1} = v_{\psi(g_\lambda)} + b_\lambda^{2n+1}$ so that in the limit, $u((v_\psi)_1) = v_{\psi(g_\lambda)} + u(b_\lambda)$.

For notation's sake, in this paragraph, let $x := (v_\psi)_1$ and $w := \sup_\lambda v_{\psi(g_\lambda)}$. From the property $\{u(x), x, u(x)\} = u(x)$, we have $P_2(u(x))x = u(x)$, so that by Lemma 2.1, $x = u(x) + c$ with $c \perp u(x)$. Since $u(x) \geq w$, say $u(x) = w + w'$ where w' is a tripotent orthogonal to w, we now have $x = w + w' + c$, with both w' and c orthogonal to w. Thus

$$(v_\psi)_1 = \sup_\lambda v_{\psi(g_\lambda)} + b \tag{4}$$

for some element $b \perp \sup_\lambda v_{\psi(g_\lambda)}$.

By Lemma 3.7, we now have

$$\sup_\lambda v_{\psi(g_\lambda)} = \sup_\lambda (v_{g_\lambda})_\psi = (\sup_\lambda v_{g_\lambda})_\psi = v_\psi = (v_\psi)_2 + (v_\psi)_1 + (v_\psi)_0. \tag{5}$$

For notation's sake, in this paragraph, let $y = v_\psi$. It follows from (4) and (5) that $-b = (y_2 + y_0) \perp y$, or $D(y_2+y_0, y_2+y_0)(y_2+y_1+y_0) = 0$. This yields, upon expansion and comparison of Peirce components, that $\{y_2y_2y_2\} = 0 = \{y_0y_0y_0\}$ so that $y_2 = y_0 = 0$. Thus, v_ψ lies in $A_1(u_\psi)$.

The lemma follows easily for an arbitrary $x \in B_1(u)$ by considering the spectral decomposition of x. ∎

LEMMA 5.7: *Let u and v be compatible tripotents in B (in particular, if u is a tripotent in $B_1(v)$) and let x be an element in $B_2(v)$. Then*

$$P_j(u_\psi)x_\psi = (P_j(u)x)_\psi \quad \text{for } j = 0, 1, 2.$$

In particular, $P_j(u_\psi)x_\psi$ is a multiplier for $j = 0, 1, 2$.

Proof. Since u and v are compatible, $P_j(u)x = P_2(v)P_j(u)x \in B_2(v)$ so that, by Lemma 3.15,

(6) $x_\psi = (P_2(u)x + P_1(u)x + P_0(u)x)_\psi = (P_2(u)x)_\psi + (P_1(u)x)_\psi + (P_0(u)x)_\psi$.

From Lemma 5.6, $(P_1(u)x)_\psi \in A_1(u_\psi)$ and, by Remark 3.16, $(P_0(u)x)_\psi \in A_0(u_\psi)$. Again by Lemma 3.15,

$$\begin{aligned}(P_2(u)x)_\psi &= (\{u\{uxu\}u\})_\psi = (\{u\{u, P_2(u)x, u\}u\})_\psi \\ &= (\{u_\psi\{u_\psi, (P_2(u)x)_\psi, u_\psi\}u_\psi\}),\end{aligned}$$

so that $(P_2(u)x)_\psi \in A_2(u_\psi)$.

By the uniqueness of Peirce decompositions and (6), $P_j(u_\psi)x_\psi = (P_j(u)x)_\psi$. ■

6. Proof of the main results (Theorems 3 and 1)

We again assume in this section that the JBW^*-triple B satisifes the condition in Theorem 2, that is, it has no direct summands of the form $L^\infty(\Omega, H)$, where H is a Hilbert space of dimension at least two.

LEMMA 6.1: *Suppose v is a tripotent in B. Further suppose that x is a tripotent in $B_1(v)$ with $\{x, v, x\} = 0$ and $\{x_\psi, v_\psi, x_\psi\} = 0$. Then $\psi^*\{x_\psi, x_\psi, v_\psi\} = \{x, x, v\}$. Furthermore, $\{x_\psi, x_\psi, v_\psi\} = y_\psi$ for some $y \in B$.*

Proof. We note first that, as shown in Lemma 4.4, $p := 2D(x, x)v$ is a self-adjoint projection in $B_2(v)$. By Peirce arithmetic, using the assumption $\{xvx\} = 0$, p lies in $B_1(x)$ and, by Lemma 5.6, p_ψ lies in $A_1(x_\psi)$. By this fact, the compatibility of p_ψ and x_ψ, and the fact that $p_\psi \le v_\psi$, we have

$$2D(p_\psi, p_\psi)D(x_\psi, x_\psi)v_\psi = 2D(x_\psi, x_\psi)D(p_\psi, p_\psi)v_\psi = 2D(x_\psi, x_\psi)p_\psi = p_\psi.$$

Similarly to the calculation above,

$$q := 2\{x_\psi, x_\psi, v_\psi\}$$

is a self-adjoint projection in $A_2(v_\psi)$ and, since $q \circ p_\psi = 2\{\{x_\psi x_\psi v_\psi\}v_\psi p_\psi\} = 2D(p_\psi, p_\psi)D(x_\psi, x_\psi)v_\psi = p_\psi$, $q \ge p_\psi$ and it follows that $\psi^*(q) \ge p$.

Now $D(x,x)(v-p) = \{xxv\}-\{xxp\} = p/2-p/2 = 0$. Hence, x_ψ is orthogonal to $v_\psi - p_\psi$. By this orthogonality and compatibility, and since $p_\psi \le v_\psi \le w_\psi$ (w is a maximal tripotent majorizing v) so that $\{p_\psi p_\psi v_\psi\} = p_\psi$,

$$\begin{aligned} D(v_\psi - p_\psi, v_\psi - p_\psi)D(x_\psi, x_\psi)v_\psi &= D(x_\psi, x_\psi)D(v_\psi - p_\psi, v_\psi - p_\psi)v_\psi \\ &= D(x_\psi, x_\psi)(v_\psi - p_\psi) = 0, \end{aligned}$$

showing $v_\psi - p_\psi$ is orthogonal to q. We then have

$$\|v - p \pm \psi^*(q)\| \le \|v_\psi - p_\psi \pm q\| = 1$$

so that $v-p$ is orthogonal to $\psi^*(q)$. Since, as shown above, $\psi^*(q) \ge p$, it follows (using Lemma 5.4 to ensure that $\psi^*(q) \in B_2(v)$) that $\psi^*(q) = p$. This proves the first statement. The second follows immediately from Lemma 5.3, since v_ψ is majorized by w_ψ for a maximal tripotent $w \in B$ and ψ^* takes the positive element $2\{x_\psi, x_\psi, v_\psi\} \in A_2(w_\psi)$ to a projection in $B_2(w)$. ∎

Lemma 6.2: *Suppose that y and z lie in $B_2(w)$ for a maximal tripotent w and that x lies in $B_1(w)$. Then*

$$\{x_\psi, y_\psi, z_\psi\}$$

is a multiplier belonging to $A_1(w_\psi) \cap A_2([r(x) + r(z_0)]_\psi)$ (where $z_0 = P_0(r(x))z$), and $\psi^*\{x_\psi, y_\psi, z_\psi\} = \{x, y, z\}$.

Proof. Suppose first that x is a tripotent. Let y_j denote $P_j(x)y$ and $(y_\psi)_j = P_j(x_\psi)y_\psi$ for $j = 0, 1, 2$. Similarly for z. By Lemma 5.7, replacing u, v, x there by x, w, y, respectively, we have $(y_j)_\psi = (y_\psi)_j$ and similarly $(z_j)_\psi = (z_\psi)_j$, for $j = 0, 1, 2$.

Note that in the expansion

$$\{x_\psi, y_\psi, z_\psi\} = \left\{x_\psi, \sum_i (y_\psi)_i, \sum_j (z_\psi)_j\right\} = \sum_{i,j} \{x_\psi, (y_\psi)_i, (z_\psi)_j\},$$

seven of the nine terms are zero, five of them since $y_2 = P_2(x)y = \{x\{x, y, x\}x\} = 0$ by the maximality of w (so also $z_2 = 0$), and two others since x_ψ is orthogonal to $(y_\psi)_0$ and $(z_\psi)_0$. Hence

$$\{x_\psi, y_\psi, z_\psi\} = \{x_\psi, (y_1)_\psi, (z_1)_\psi\} + \{x_\psi, (y_1)_\psi, (z_0)_\psi\}. \tag{7}$$

Let u_S be a spectral tripotent of y_1. By Peirce calculus with respect to w

and w_ψ, $\{u_S, x, u_S\} = 0$ and $\{(u_S)_\psi, x_\psi, (u_S)_\psi\} = 0$. Therefore, by Lemma 6.1, $\{x_\psi, (u_S)_\psi, (u_S)_\psi\}$ is a multiplier in $A_2(x_\psi)$ and $\psi^*\{x_\psi, (u_S)_\psi, (u_S)_\psi\} = \{xu_Su_S\}$. Passing to the limit using the spectral theorem shows that $\{x_\psi, (y_1)_\psi, (y_1)_\psi\}$ is a multiplier in $A_2(x_\psi)$ and $\psi^*\{x_\psi, (y_1)_\psi, (y_1)_\psi\} = \{xy_1y_1\}$. Of course, the same holds for z: $\{x_\psi, (z_1)_\psi, (z_1)_\psi\}$ is a multiplier in $A_2(x_\psi)$ and $\psi^*\{x_\psi, (z_1)_\psi, (z_1)_\psi\} = \{xz_1z_1\}$.

By Lemma 3.15, $(y_1)_\psi + (z_1)_\psi = (y_1 + z_1)_\psi$. Hence the same statement holds for $\{x_\psi, (y_1)_\psi + (z_1)_\psi, (y_1)_\psi + (z_1)_\psi\}$. Thus the statement holds for $\{x_\psi, (y_1)_\psi, (z_1)_\psi\} + \{x_\psi, (z_1)_\psi, (y_1)_\psi\}$. Explicitly,

$$\{x_\psi, (y_1)_\psi, (z_1)_\psi\} + \{x_\psi, (z_1)_\psi, (y_1)_\psi\}$$

is a multiplier in $A_2(x_\psi)$ and

$$\psi^*(\{x_\psi, (y_1)_\psi, (z_1)_\psi\} + \{x_\psi, (z_1)_\psi, (y_1)_\psi\}) = \{xy_1z_1\} + \{xz_1y_1\}.$$

Replacing z by iz shows that the statement holds for $\{x_\psi, (y_1)_\psi, (z_1)_\psi\}$ and $\{x_\psi, (z_1)_\psi, (y_1)_\psi\}$ individually. This proves, in the case that x is a tripotent, that the first term in the right side of (7) is a multiplier in $A_2(x_\psi) \cap A_1(w_\psi)$ and ψ^* is multiplicative on this term.

We now consider the second term in the right side of (7), still in the case that x is a tripotent. Since $x \perp z_0$ (recall that $z_0 = P_0(x)z$), we can choose a maximal tripotent w' such that $B_2(x + r(z_0)) \subset B_2(w')$, so that x_ψ and $(z_0)_\psi$ are multipliers in $A_2(x_\psi + r(z_0)_\psi) = A_2([x + r(z_0)]_\psi) \subset A_2(w'_\psi)$. We next note that for every $a \in A$,

$$\psi^*\{x_\psi, a, (z_0)_\psi\} = \{x, \psi^*(a), z_0\}. \tag{8}$$

Indeed, by Peirce calculus $\{x_\psi, a, (z_0)_\psi\} = \{x_\psi, P_2(w'_\psi)a, (z_0)_\psi\}$ and, by properties of multipliers and the Jordan algebra relation,

$$\{abc\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^* \tag{9}$$

(cf. Lemma 3.5), and Lemma 5.4,

$$\begin{aligned}\psi^*\{x_\psi, a, (z_0)_\psi\} &= \psi^*\{x_\psi, P_2(w'_\psi)a, (z_0)_\psi\} = \{x, \psi^*(P_2(w'_\psi)a), z_0\} \\ &= \{x, P_2(w')\psi^*(a), z_0\} \\ &= \{x, \psi^*(a), z_0\},\end{aligned}$$

proving (8). In particular, $\psi^*\{x_\psi, (y_1)_\psi, (z_0)_\psi\} = \{x, y_1, z_0\}$ so that

$$\psi^*\{x_\psi, y_\psi, z_\psi\} = \{x, y_1, z_1\} + \{x, y_1, z_0\} = \{x, y, z\}.$$

We still must show that $\{x_\psi, (y_1)_\psi, (z_0)_\psi\}$ is a multiplier. By the joint Peirce decomposition and the relation $D(u,u) = P_2(u) + P_1(u)/2$,

$$\begin{aligned} P_2(x_\psi + r(z_0)_\psi)(y_1)_\psi &= [P_2(x_\psi) + P_2(r(z_0)_\psi) + P_1(x_\psi)P_1(r(z_0)_\psi)](y_1)_\psi \\ &= P_1(r(z_0)_\psi)(y_1)_\psi \\ &= [2D(r(z_0)_\psi, r(z_0)_\psi) - 2P_2(r(z_0)_\psi)](y_1)_\psi \\ &= 2D(r(z_0)_\psi, r(z_0)_\psi)(y_1)_\psi. \end{aligned}$$

The right side of the preceding equation is a triple product of multipliers in $A_2(w_\psi)$, and is hence a multiplier in $A_2(w_\psi)$ by (9) and the fact that the multipliers form a Jordan algebra. Hence $P_2(x_\psi + r(z_0)_\psi)(y_1)_\psi$ is a multiplier in $A_2(w_\psi)$. Since $\{x_\psi(y_1)_\psi(z_0)_\psi\} = \{x_\psi, P_2(x_\psi + r(z_0)_\psi)(y_1)_\psi, (z_0)_\psi\}$, using Lemma 3.15, $\{x_\psi(y_1)_\psi(z_0)_\psi\}$ is a multiplier in $A_2([x + r(z_0)]_\psi)$. This proves the lemma in the case that x is a tripotent.

Now let x be an arbitrary element of $B_1(w)$. Approximate it by sums $\tilde{x} = \sum \lambda_i u_i$, where the elements $u_i \in B_1(w)$ are orthogonal spectral tripotents with $\sum u_i = r(x)$. Decomposing y and z with respect to $r(x) = r(\tilde{x})$, it follows as in (7) that

$$\{\tilde{x}_\psi y_\psi z_\psi\} = \{\tilde{x}_\psi, (y_1)_\psi, (z_1)_\psi\} + \{\tilde{x}_\psi, (y_1)_\psi, (z_0)_\psi\}. \tag{10}$$

By the previous discussion surrounding (7), with y, z there replaced by $y_1, z_1 \in B_2(w)$ and since $u_i \in B_1(w)$, $\{(u_i)_\psi, (y_1)_\psi, (z_1)_\psi\}$, which lies in $A_2(r(x)_\psi)$ by Peirce calculus, is a sum of a multiplier in $A_2((u_i)_\psi) \subseteq A_2(r(x)_\psi)$ and a multiplier belonging to $A_1(w_\psi)$ which must also lie in $A_2((r(x))_\psi)$. Also, ψ^* is multiplicative on these products. Hence the first term in the right side of (10) is a multiplier in $A_2(r(x)_\psi) \subseteq A_2([r(x) + r(z_0)]_\psi)$ and ψ^* is multiplicative on it.

The second term equals $\sum \lambda_i \{(u_i)_\psi, (y_1)_\psi, (z_0)_\psi\}$. Since $z_0 \perp u_i$ (recall that $z_0 = P_0(r(x))z$), the same argument used above shows that $\{(u_i)_\psi, (y_1)_\psi, (z_0)_\psi\}$ is a multiplier in $A_2([u_i + r(z_0)]_\psi) \subseteq A_2([r(x) + r(z_0)]_\psi)$ and that ψ^* is multiplicative on these products. Hence the second term in (10) is a multiplier in $A_2(r(x)_\psi) \subseteq A_2([r(x) + r(z_0)]_\psi)$ and ψ^* is multiplicative on it. The lemma is proved. ∎

Lemma 6.3: *If q lies in $A_0(v_\psi)$ for some maximal tripotent $v \in B$, then $\psi^*\{q, x, y\} = 0$ for all $x, y \in A$; in particular, $\psi^*\{q, q, x\} = 0$ for all $x \in A$. Also, $q \perp x_\psi$ for all $x \in B$, that is, $A_0(v_\psi) \perp \{x_\psi : x \in B\}$.*

Proof. Let z be a maximal tripotent in $A_0(v_\psi)$ such that $q/\|q\|$ is a self-adjoint element with respect to z (see [22, Lemma 3.12(1)]). Clearly $v_\psi + z$ is maximal. Because ψ^* preserves orthogonality with v_ψ and v is maximal, $\psi^*(q) = \psi^*(z) = 0$ and therefore ψ^* maps the self-adjoint element $v_\psi + q/\|q\|$ to the unit v of $B_2(v)$ and maps $v_\psi + z$ to v. By Corollary 3.3, $v_\psi + q/\|q\|$ is a multiplier in $A_2(v_\psi + z)$. Since v_ψ is a multiplier there, so is q. On the other hand, if we let $x = x_2 + x_1 + x_0$ be its Peirce decomposition with respect to v_ψ, then $\{qqx\} = \{q, q, x_1 + x_0\}$ so that $\psi^*\{qqx\} = \psi^*\{qqx_1\}$ since $\{qqx_0\} \in A_0(v_\psi)$. If we now expand x_1 in its Peirce decomposition with respect to z, say $x_1 = (x_1)_2 + (x_1)_1 + (x_1)_0$, then $\{qqx_1\} = \{q, q, (x_1)_2 + (x_1)_1\}$ and, since v_ψ and z are compatible, $(x_1)_2 + (x_1)_1 \in A_2(z) + A_1(z) \cap A_1(v_\psi) \subset A_2(v_\psi + z)$. Since q is a multiplier in $A_2(v_\psi + z)$, we now have $\psi^*\{qqx_1\} = \{\psi^*(q), \psi^*(q), \psi^*((x_1)_2 + (x_1)_1)\} = 0$, proving that $\psi^*\{qqx\} = 0$.

Letting $x, y \in A$ and Peirce decomposing them with respect to v_ψ, we have

$$\psi^*\{qxy\} = \psi^*\{q, x_1 + x_0, y_2 + y_1 + y_0\} = \psi^*\{q, x_0, y_1 + y_0\} + \psi^*\{qx_1y_2\}. \tag{11}$$

Since $\{qx_1y_2\} \in A_1(z)$ (by Peirce calculus), we have

$$\{qx_1y_2\} = 2\{z, z, \{qx_1y_2\}\} = 2\{z, v_\psi + z, \{qx_1y_2\}\}$$

and therefore, since z is a multiplier in $A_2(v_\psi + z)$, $\psi^*\{qx_1y_2\} = \psi^*(z) \circ \psi^*\{qx_1y_2\} = 0$. Thus the second term on the right side of (11) is zero.

For the first term on the right side of (11), we have

$$\psi^*\{q, x_0, y_1 + y_0\} = \psi^*\{q, x_0, y_1\} + \psi^*\{q, x_0, y_0\} \tag{12}$$

and the second term in (12) is zero since $\{q, x_0, y_0\} \in A_0(v_\psi)$. Peirce decomposing x_0 and y_1 with respect to z and expanding the first term in (12) leads to

$$\begin{aligned}\psi^*\{q, x_0, y_1\} = {}&\psi^*\{q, (x_0)_2, (y_1)_2\} + \psi^*\{q, (x_0)_2, (y_1)_1\} \\ &+ \psi^*\{q, (x_0)_1, (y_1)_1\} + \psi^*\{q, (x_0)_1, (y_1)_0\}.\end{aligned}$$

The first and third terms here are zero since $(y_1)_2$ and $\{q, (x_0)_1, (y_1)_1\}$ belong to $A_1(v_\psi) \cap A_2(z)$, which is zero since $v_\psi \perp z$. The second term is zero since $\{q, (x_0)_2, (y_1)_1\}$ lies in $A_1(v_\psi) \cap A_1(z) \subseteq A_2(v_\psi + z)$ and $\{q, (x_0)_2, (y_1)_1\} = 2\{z, z, \{q, (x_0)_2, (y_1)_1\}\} = 2\{z, v_\psi + z, \{q, (x_0)_2, (y_1)_1\}\}$ so that

$$\psi^*\{q, (x_0)_2, (y_1)_1\} = \psi^*(z) \circ \psi^*\{q, (x_0)_2, (y_1)_1\} = 0.$$

The proof that the fourth term is zero is similar. This proves that $\psi^*\{qxy\} = 0$.

To prove the last statement, it may be assumed that both q and x are tripotents. Decompose x_ψ with respect to q: $x_\psi = (x_\psi)_2 + (x_\psi)_1 + (x_\psi)_0$ and note that by the first two parts of this lemma, $\psi^*((x_\psi)_2 + (x_\psi)_1) = 0$, so that $\psi^*((x_\psi)_0) = x$. By Lemma 3.13, $(x_\psi)_0 = x_\psi + \tilde{q}$ where $\tilde{q} \perp x_\psi$. Thus $\tilde{q} = -(x_\psi)_2 - (x_\psi)_1$ is orthogonal to $(x_\psi)_2 + (x_\psi)_1 + (x_\psi)_0$. Considering the components of

$$0 = D((x_\psi)_2 + (x_\psi)_1, (x_\psi)_2 + (x_\psi)_1 + (x_\psi)_0)(x_\psi)_2$$

we immediately see that $\{(x_\psi)_2(x_\psi)_2(x_\psi)_1\} = 0$, so that $(x_\psi)_2 \perp (x_\psi)_1$ and therefore $\{(x_\psi)_1(x_\psi)_1(x_\psi)_2\} = 0$ and $((x_\psi)_2)^3 = 0$, $(x_\psi)_2 = 0$. Considering $0 = D((x_\psi)_1, (x_\psi)_1 + (x_\psi)_0)(x_\psi)_1$ we see that $(x_\psi)_1 = 0$. The lemma follows. ∎

Corollary 6.4: *If $x \in B_2(w)$ for a maximal tripotent w and $y, z \in B_1(w)$, then $\{y_\psi, x_\psi, z_\psi\} = 0$.*

Proof. Let $\alpha := \{y_\psi, x_\psi, z_\psi\}$. By Peirce calculus with respect to w_ψ, $\alpha \in A_0(w_\psi)$ so, by Lemma 6.3, $y_\psi, z_\psi, x_\psi \perp \alpha$. By the main identity,

$$\{\alpha\alpha\alpha\} = \{\alpha\alpha\{y_\psi x_\psi z_\psi\}\} = \{\{\alpha\alpha y_\psi\}x_\psi z_\psi\} - \{y_\psi\{\alpha\alpha x_\psi\}z_\psi\} + \{y_\psi x_\psi\{\alpha\alpha z_\psi\}\}$$

and each term is zero, hence $\alpha = 0$. ∎

Lemma 6.5: *Suppose x_ψ is a multiplier belonging to $A_1(w_\psi)$ for a maximal tripotent $w \in B$ and that y_ψ is a multiplier belonging to $A_2(w_\psi)$. Then $\{x_\psi, x_\psi, y_\psi\}$ is a multiplier and ψ^* is multiplicative on this product.*

Proof. Suppose first that x is a tripotent. By Corollary 6.4, $\{x_\psi y_\psi x_\psi\} = 0$ and hence $P_2(x_\psi)y_\psi = 0$. Then by Lemma 5.7,

$$\begin{aligned}\{x_\psi x_\psi y_\psi\} &= D(x_\psi, x_\psi)y_\psi = (P_2(x_\psi) + P_1(x_\psi)/2)y_\psi \\ &= P_1(x_\psi)y_\psi/2 = (P_1(x)y)_\psi/2,\end{aligned}$$

proving that $\{x_\psi, x_\psi, y_\psi\}$ is a multiplier. Moreover, $\psi^*\{x_\psi x_\psi y_\psi\} = P_1(x)y/2 = (2D(x,x) - 2P_2(x))y/2 = \{xxy\}$, since by Peirce calculus with respect to the maximal tripotent w, $\{xyx\} = 0$.

For the general case it suffices to assume that x is a finite sum $\sum \lambda_i x_i$ of pairwise orthogonal tripotents x_i in $B_1(w)$. By the special case just proved, $\{(x_i)_\psi(x_i)_\psi y_\psi\}$ is a multiplier and ψ^* is multiplicative on it. Therefore,

$$\{x_\psi, x_\psi, y_\psi\} = \sum \lambda_i^2 \{(x_i)_\psi(x_i)_\psi y_\psi\}$$

is also a multiplier and ψ^* is multiplicative on it. ∎

Lemma 6.6: *Suppose that z is a tripotent in B and that w is maximal tripotent in B. Then, letting $z_2 = P_2(w)z$ and $z_1 = P_1(w)z$, we have $z_\psi = (z_2)_\psi + (z_1)_\psi$.*

Proof. It follows from Corollary 6.4 and Lemmas 6.2 and 6.5 that

$$\psi^*[((z_2)_\psi + (z_1)_\psi)^3] = z.$$

Indeed,

$$((z_2)_\psi + (z_1)_\psi)^3 = \sum_{i,j,k=1}^{2} \{(z_i)_\psi, (z_j)_\psi, (z_k)_\psi\},$$

and ψ^* is multiplicative on each term on the right side as follows. For the terms corresponding to $(i,j,k) = (2,2,2)$ and $(1,1,1)$, this is because ψ^* is a Jordan homomorphism on the set of local multipliers. For the terms corresponding to $(i,j,k) = (2,2,1)$ and $(1,2,2)$ (which are the same), this is because of Lemma 6.2. For the terms corresponding to $(i,j,k) = (2,1,1)$ and $(1,1,2)$ (which are the same), this is because of Lemma 6.5. For the term corresponding to $(1,2,1)$, this is because of Corollary 6.4 and the maximality of w. For the term corresponding to $(2,1,2)$, this is because of Peirce calculus. Thus

$$\psi^*[((z_2)_\psi + (z_1)_\psi)^3] = \sum_{i,j,k=1}^{2} \{z_i z_j z_k\} = (z_2 + z_1)^3 = z^3 = z, \tag{13}$$

as required.

Now if we Peirce decompose $((z_2)_\psi + (z_1)_\psi)^3$ with respect to w_ψ we obtain

$$P_2(w_\psi)[((z_2)_\psi + (z_1)_\psi)^3] = ((z_2)_\psi)^3 + 2\{(z_2)_\psi, (z_1)_\psi, (z_1)_\psi\}, \tag{14}$$

$$P_1(w_\psi)[((z_2)_\psi + (z_1)_\psi)^3] = ((z_1)_\psi)^3 + 2\{(z_2)_\psi, (z_2)_\psi, (z_1)_\psi\}, \tag{15}$$

and

$$P_0(w_\psi)[((z_2)_\psi + (z_1)_\psi)^3] = 0.$$

By Lemma 6.5, the right side of (14) is a sum of three multipliers, and hence a multiplier itself in $A_2(w_\psi)$.

On the other hand, the first term on the right side of (15) is obviously a multiplier in $A_2(r(z_1)_\psi) \subseteq A_2([r(z_1) + r(P_0(r(z_1))z_2)]_\psi)$. By Lemma 6.2, the second term is also a multiplier in $A_2([r(z_1) + r(P_0(r(z_1))z_2)]_\psi)$. Hence the sum is a multiplier. It follows that $((z_2)_\psi + (z_1)_\psi)^3$ is again a sum of two multipliers $(z_2')_\psi + (z_1')_\psi$, where the indices indicate Peirce components of z' with respect to

w. Since $\psi^*((z_2')_\psi + (z_1')_\psi) = z_2' + z_1'$, (13) tells us that $z = z_2' + z_1'$, and therefore $z_2 = z_2', z_1 = z_1'$ and $(z_2)_\psi + (z_1)_\psi$ is a tripotent. We may use Lemma 5.5 (with x there equal to the tripotent $(z_2)_\psi + (z_1)_\psi$) to see that $(z_2)_\psi + (z_1)_\psi = z_\psi + q$, where $q \perp z_\psi$ and $\psi^*(q) = 0$.

To show that $q = 0$, suppose first that z is maximal. It follows from Lemma 6.3 that $q \perp [(z_2)_\psi + (z_1)_\psi]$, from which it follows that $q = 0$. Now suppose z is a general tripotent less than a maximal tripotent v. Let $u = v - z$. Then $(z_2)_\psi + (z_1)_\psi + (u_2)_\psi + (u_1)_\psi = z_\psi + q + u_\psi + p = v_\psi + p + q = (v_2)_\psi + (v_1)_\psi + p + q$.

Note that $(z_2)_\psi + (u_2)_\psi = (z_2 + u_2)_\psi = (v_2)_\psi$ and therefore

$$(v_2)_\psi + (z_1)_\psi + (u_1)_\psi = (v_2)_\psi + (v_1)_\psi + p + q,$$

which tells us that $p + q \in A_1(w_\psi)$. Repeating this argument with $-u$ instead of u shows that $p - q \in A_1(w_\psi)$ so that both p and q belong to $A_1(w_\psi)$.

From $(z_2)_\psi + (z_1)_\psi = z_\psi + q$ with $q \in A_0(z_\psi) \cap A_1(w_\psi)$ and $z_\psi = (z_\psi)_2 + (z_\psi)_1 + (z_\psi)_0$ (Peirce decomposition with respect to w_ψ) we have $q \perp (z_\psi)_1$; indeed, $0 = \{z_\psi q q\} = \{(z_\psi)_2 q q\} + \{(z_\psi)_1 q q\} + \{(z_\psi)_0 q q\}$ and all three terms are zero since they lie in different Peirce spaces.

Thus $(z_1)_\psi = (z_\psi)_1 + q$ with $q \perp (z_\psi)_1$ and therefore

$$r(z_1)_\psi = r((z_\psi)_1) + q \text{ with } q \perp r((z_\psi)_1). \tag{16}$$

By (16), $\psi^*(r((z_\psi)_1)) = r(z_1)$ showing, by Lemmas 3.4 and 3.5(c), that $r(z_1)_\psi = r((z_\psi)_1)$, that is $q = 0$. ∎

THEOREM 3: *Let ψ denote an isometry of B_* into A_* where A and B are JBW^*-triples. Assume that B has no $L^\infty(\Omega, H)$ summand, where H is a Hilbert space of dimension at least two. Let C be the weak*-closure of the linear span of all multipliers: $C := \overline{\text{sp}}^{w*}\{x_\psi | x \in B\}$. Then C is a JBW^*-subtriple of A, and ψ^* restricted to C is a weak* bi-continuous isomorphism onto B with inverse $x \mapsto x_\psi$ for $x \in B$.*

Proof. We first consider three tripotents u, v and w in B and show that $\{u_\psi, v_\psi, w_\psi\}$ is a sum of multipliers and that ψ^* is multiplicative on this product. Choose a maximal tripotent $z \geq v$ and decompose with respect to it: $u = u_2 + u_1$ and $w = w_2 + w_1$. It follows from Lemma 6.6 and Corollary 6.4 that the above product equals

$$\{(u_2)_\psi, v_\psi, (w_2)_\psi\} + \{(u_1)_\psi, v_\psi, (w_2)_\psi\} + \{(u_2)_\psi, v_\psi, (w_1)_\psi\}.$$

The first product satisfies the desired conditions by the work in Section 3. The second and third products also satisfy these conditions by Lemma 6.2. It follows from Section 3 and separate weak*-continuity of the triple product that C is a weak*-closed subtriple of A and that ψ^* restricted to C is a weak*-continuous homomorphism onto B. Let $C = I \oplus K$ where K denotes the kernel, which is a weak*-closed ideal and I is the complementary ideal $K^\perp$. Suppose u is a tripotent in B. Let P and $P^\perp$ be the projections of C onto I and K. $P(u_\psi)$ and $P^\perp(u_\psi)$ are orthogonal tripotents that sum to u_ψ and $\psi^*(P(u_\psi)) = u$. By Lemma 3.13, $P(u_\psi) = u_\psi + q$ where $q \perp u_\psi$. Hence $q = -P^\perp(u_\psi)$, which forces $q^3 = 0$ because $q \perp Pu_\psi - u_\psi = q$. Thus $K = 0$ and ψ^* is a weak*-continuous isomorphism from C onto B. ∎

An immediate consequence of the proof is the following corollary.

Corollary 6.7: *Retain the notation of the theorem. Then* $C = \{x_\psi | x \in B\}$.

The next two corollaries constitute a proof of Theorem 1.

Corollary 6.8: *Suppose that A, B, C and ψ are as in Theorem 3. Let ϕ denote the inverse of $\psi^*|C$ and let $P : A_* \to A_*$ be the linear map with $P^* = \phi \circ \psi^*$ (which exists by the automatic weak* continuity of JBW^*-triple isomorphisms). Then P is a contractive projection of A_* onto $\psi(B_*)$*

Proof. For $f \in B_*$ and $a \in A$, $\langle P(\psi(f)), a\rangle = \langle f, \psi^*(\phi(\psi^*(a)))\rangle = \langle f, \psi^*(a)\rangle = \langle \psi(f), a\rangle$. The statement follows. ∎

In the next corollary we use the following fact from the structure theory of JBW^*-triples: every JBW^*-triple U can be decomposed into an ℓ^∞-direct sum of orthogonal weak*-closed ideals U_1 and U_2, where U_1 is a direct sum of spaces of the form $L^\infty(\Omega, C)$, with C a Cartan factor, and U_2 has no abelian tripotents (see [24, (1.16)] and [23, (1.7)]). In particular, since Hilbert spaces are Cartan factors, we can write $B = B_1 \oplus B_2$ where $(B_1)_*$ is an ℓ^1 direct sum of spaces isomorphic to $L^1(\Omega_\lambda, H_\lambda)$, where H_λ is a Hilbert space of dimension at least two, and $(B_2)_*$ has no non-trivial ℓ^1-summand of the from $L^1(\Omega, H)$, with H a Hilbert space of dimension at least two.

Corollary 6.9: *Suppose that A and B are JBW^*-triples and ψ is an isometry from B_* into A_*, and let $B = B_1 \oplus B_2$ be the decomposition described* *above. Then there is a contractive projection P from A_* onto $\psi((B_2)_*)$ which annihilates $\psi((B_1)_*)$*

Proof. Denote by ψ_i the restriction of ψ to $(B_i)_*$. It is immediate from the previous corollary that there exists a contractive projection P from A_* onto $\psi_2((B_2)_*)$ with $P^* = \phi_2 \circ \psi_2^*$. Suppose $f \in \psi_1((B_1)_*)$. Pick a tripotent $u \in B_2$. Using Lemmas 3.7 and 3.8,

$$u_{\psi_2} = \phi_2(u) = \phi_2(\sup_\lambda v_{g_\lambda}) = \sup_\lambda \phi_2(v_{g_\lambda}) = \sup_\lambda v_{\psi_2(g_\lambda)}$$

for a family of pairwise orthogonal normal functionals $g_\lambda \in (B_2)_*$ (see the proof of Lemma 3.9). Since $f \perp \psi_2(g_\lambda)$, $f(v_{\psi_2(g_\lambda)}) = 0$ and so, by [22, (3.23)], $f(u_{\psi_2}) = 0$. Hence $f(\phi_2(u)) = 0$. It follows that $f(\phi_2((\psi_2)^*(A))) = 0$ and $P(f) = 0$. ∎

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