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A TECHNIQUE FOR THE CALCULATION OF ACCELERATION WAVES IN NONLINEAR VISCOELASTIC MATERIALS

by

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Summary

A boundary-value problem giving rise to a one-dimensional acceleration wave in a nonlinear viscoelastic material is shown to have a solution, valid in a region behind the wave front, such that stress, strain and velocity are expressed as power series in time measured from the arrival of the wave; the coefficients are functions of position and obtainable by quadrature from first-order differential equations. The results may be used to determine short-time viscoelastic behavior by means of wave-propagation experiments.

1. Introduction

Acceleration waves in nonlinear viscoelastic materials have been the subject of recent studies by Varley [1], Coleman, Gurtin and Herrera [2], Coleman and Gurtin [3-5], and Dunwoody and Dunwoody [6]. In particular, Refs. 1 and 3 contain independent derivations of the equations governing the speed of propagation of a one-dimensional wave and the variation of the strength of the wave as it propagates (including possible growth into a shock wave). For the remainder, the aforementioned studies deal with various types of three-dimensional waves, thermodynamic effects, etc., but not with the solution of boundary-value problems.

The present study is concerned with the explicit solution of one-dimensional boundary-value problems giving rise to an acceleration wave, with the variables (stress, strain, velocity) expressed as power series in time measured from the arrival of the wave front at a station X, the coefficients being functions of X. If uniform convergence of the series is assumed, then the equations of motion and continuity and the constitutive equation may be satisfied term by term, leading to equations governing the coefficients. It turns out that the first two terms yield the wave-speed and wave-strength equations referred to above; the subsequent terms show the variation in the state variables in a short time following the wave-front arrival.

A similar method has been used previously [7] in studying the propagation of shock waves in semilinear material, i.e., materials whose instantaneous response is linear.

2. Definitions and Formulation

We consider a longitudinal deformation $x(X,t)$ of a material half-space $X > 0$; we define strain and velocity as

$$\epsilon(X,t) = \frac{\partial x}{\partial X} - 1, \quad v(X,t) = \frac{\partial x}{\partial t}, \tag{1}$$

inferring the equation of continuity

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial v}{\partial X}. \tag{2}$$

In the absence of body force the equation of motion is

$$\frac{\partial \sigma}{\partial X} = \rho \frac{\partial v}{\partial t}, \tag{3}$$

where $\sigma(X,t)$ is the stress and ρ the rest density, assumed uniform. The constitutive equation is written in the form appropriate to small finite deformations [8]:

$$\sigma = \sum_{n=1}^{N} S_n \tag{4}$$

where

$$S_n = \int_0^t \cdots \int_0^t K_n(\tau_1, \ldots \tau_n)\, \dot{\epsilon}(t-\tau_1) \ldots \dot{\epsilon}(t-\tau_n)\, d\tau_1 \ldots d\tau_n, \tag{5}$$

with $\dot{\epsilon}(t)$ written for $\frac{\partial \epsilon}{\partial t}(X,t)$.

The problem described by Eqs. (2-5) is complete with a boundary condition such as

$$\epsilon(0,t) = \epsilon_0(t). \tag{6}$$

It is assumed that x=X for $t \le 0$, and that $\epsilon_o(t)$ is expressible as a power series in t:

$$\epsilon_o(t) = \sum_{n=1}^{\infty} A_n t^n. \tag{7}$$

The stress, strain, and velocity are now expressed in the form

$$\begin{aligned} \sigma &= \sum_{n=1}^{\infty} c_n \left\langle t - \frac{X}{U} \right\rangle^n \\ \epsilon &= \sum_{n=1}^{\infty} a_n \left\langle t - \frac{X}{U} \right\rangle^n \\ V &= \sum_{n=1}^{\infty} b_n \left\langle t - \frac{X}{U} \right\rangle^n \end{aligned} \tag{8}$$

where U is the material wave speed (to be determined), and the bracket $\langle\ \rangle$ equals its argument if the latter is positive, and vanishes if the argument is negative. The a_n, b_n, and c_n are functions of X.

If the expressions (8) are inserted into Eqs. (2) and (3) and the series equated term by term, we obtain

$$b'_{n-1} - \frac{n}{U} b_n = n\, a_n, \tag{9}$$

$$c'_{n-1} - \frac{n}{U} c_n = \rho\, n\, b_n, \tag{10}$$

for n=1,2...., it being understood that $a_o=b_o=c_o=0$. By ( )' we denote $\frac{d}{dX}(\ )$.

On eliminating b_1 from Eqs. (9-10) we obtain

$$\frac{c''_{n-2}}{n(n-1)} - \frac{2\,c'_{n-1}}{n\,U} + \frac{c_n}{U^2} = \rho\, a_n, \tag{11}$$

n=1,2..., with $c_{-1} = 0$.

We must now find a way of converting Eqs. (5-6) into a relation between the c_m's and the a_m's.

3. Short-Time Viscoelastic Behavior

To describe short-time viscoelastic behavior, we expand the kernels $K_n(\ldots)$ as series:

$$K_n\,(\tau_1,\ldots\tau_n) = \sum_{k_1} \ldots \sum_{k_n} K^{(n)}_{k_1\ldots k_n}\, \tau_1^{k_1} \ldots \tau_n^{k_n}, \tag{12}$$

where the $K^{(n)}_{k_1\ldots k_n}$ are constants, symmetric with respect to interchange of any two subscripts; then

$$S_n = \sum_{k_1} \ldots \sum_{k_n} K^{(n)}_{k_1\ldots k_n}\, \epsilon_{k_1} \ldots \epsilon_{k_n}, \tag{13}$$

where

$$\epsilon_k = \int_o^t (t-\tau)^k\, \dot{\epsilon}(\tau)\, d\tau \tag{14}$$

Introducing the second of expressions (8) and the definition of the beta function, we find

$$\epsilon_k = \sum_{n=1}^{\infty} \frac{k!\,n!}{(k+n)!}\, a_n \left\langle t - \frac{X}{U} \right\rangle^{k+n}. \tag{15}$$

Consequently, if we write z for $\left\langle t - \frac{X}{U} \right\rangle$, we have

$$S_1 = K_o^{(1)} a_1 z + (K_o^{(1)} a_2 + \frac{1}{2} K_1^{(1)} a_1) z^2$$

$$+ (K_o^{(1)} a_3 + \frac{1}{3} K_1^{(1)} a_2 + \frac{1}{3} K_2^{(1)} a_1) z^3$$

$$+ 0 (z^4),$$

$$S_2 = K_{00}^{(2)} a_1^2 z^2 + (2 K_{00}^{(2)} a_1 a_2 + K_{01}^{(2)} a_1^2) z^3$$

$$+ 0 (z^4),$$

$$S_3 = K_{000}^{(3)} a_1^3 z^3 + 0 (z^4),$$

$$S_4 = 0 (z^4), \quad \text{etc.}$$

Hence

$$c_1 = K_0^{(1)} a_1, \tag{16.1}$$

$$c_2 = K_0^{(1)} a_2 + \frac{1}{2} K_1^{(1)} a_1 + K_{00}^{(2)} a_1^2, \tag{16.2}$$

while, for n ≥ 3,

$$c_n = K_0^{(1)} a_n + (\frac{1}{n} K_1^{(1)} + 2 K_{00}^{(2)} a_1) a_{n-1} \tag{16.n}$$

$$+ \text{terms in } a_1, \ldots a_{n-2};$$

in particular,

$$c_3 = K_0^{(1)} a_3 + (\frac{1}{3} K_1^{(1)} + 2 K_{00}^{(2)} a_1) a_2$$
$$+ \frac{1}{3} K_2^{(1)} a_1 + K_{01}^{(2)} a_1^2 + K_{000}^{(3)} a_1^3. \tag{16.3}$$

4. Solutions for the a_n

For each specific value of n, Eq. (11) will be denoted by (11.n). Thus, Eqs. (11.1) and (16.1) are a set of two homogeneous linear equations in a_1 and c_1, having a non-trivial solution only if

$$\rho U^2 = K_0^{(1)} \quad . \tag{17}$$

Equation (17) agrees with the results of Refs. 1 and 2.

On combining Eqs. (11.2) and (16.2) we find that a_2 and c_2 drop out, and, on eliminating c_1, we obtain an equation on a_1:

$$a_1' + \beta a_1 - \gamma a_1^2 = 0, \tag{18}$$

where

$$\beta = -\frac{K_1^{(1)}}{2K_0^{(1)} U} , \quad \gamma = \frac{K_{00}^{(2)}}{K_0^{(1)} U} . \tag{19}$$

Equation (18) describes the variation of the strength of the acceleration wave (in this case measured by the strain-rate discontinuity a_1) as it propagates on the characteristic X = Ut. The solution, taking into account the initial condition (7), is

$$a_1 = \frac{\beta A_1}{\gamma A_1 - (\gamma A_1 - \beta) e^{\beta X}} \tag{20}$$

The strength of the wave grows or decays if γA_1 is algebraically greater or less than β, respectively (β must, on thermodynamic grounds, be non-negative). In the former case, the acceleration wave will grow into a shock wave at $X = \frac{1}{\beta} \ell n \left(\frac{m}{m-1} \right)$, where $m = \gamma A_1/\beta$, unless it is previously overtaken by another wave.

Equations analogous to (18) have been derived by other methods and discussed by Varley [1], Coleman and Gurtin [2], and Dunwoody and Dunwoody [6].

If we go on to substitute expressions from Eqs. (16) into (11.n) for $n \geq 3$, we find that the terms in a_n drop out, leaving an equation of the form

$$a'_{n-1} + (\beta - n\gamma a_1)\, a_{n-1} = f_{n-1} \tag{21}$$

where f_{n-1} is a function of $a_1, \ldots\, a_{n-2}$. The solution of Eq. (21) is

$$a_n = (a_1/A_1)^{n+1} e^{n\beta X} \left[A_n + \int_0^X e^{-n\beta y} \left[\frac{A_1}{a_1(\bar{y})} \right]^{n+1} f_n(y)\, dy \right]. \tag{22}$$

In particular,

$$f_2 = k_1 a_1 + k_2 a_1^2 + k_3 a_1^3, \tag{23}$$

where

$$\begin{aligned} k_1 &= \frac{K_2^{(1)}}{2K_0^{(1)} U} - \frac{3}{4} \beta^2 U, \\ k_2 &= \frac{3K_{01}^{(2)}}{2K_0^{(1)} U} + \frac{9}{4} \beta \gamma U, \\ k_3 &= \frac{3K_{000}^{(3)}}{2K_0^{(1)} U} - \frac{3}{2} \gamma^2 U. \end{aligned} \tag{24}$$

Hence, on substituting a_1 from (20) and performing the integrations, we obtain

$$a_2 = [m + (1-m)\, e^{\beta X}]^{-3}\, e^{2\beta X}\, [A_2 + (1-m)^2\, k_1\, A_1\, X$$

$$+ \frac{1-m}{\beta}\, (2\, m\, k_1\, A_1 + k_2 A_1^{\,2})\, (1-e^{-\beta X}) \tag{25}$$

$$+ \frac{1}{2\beta}\, (m^2 k_1 A_1 + m k_2 A_1^{\,2} + k_3 A_1^{\,3})\, (1-e^{-2\beta X})\ ],$$

with $m = \gamma\, A_1/\beta$ as before.

For $m > 1$, a_2 blows up as X goes to $\frac{1}{\beta}\,\ell n\,(\frac{m}{m-1})$, as does a_1. In particular, for $m = \infty$ (corresponding to $\beta = 0$) we have

$$a_1 = \frac{A_1}{1-\gamma\, A_1 X}\ , \tag{26.1}$$

$$a_2 = (1-\gamma\, A_1 X)^{-3} \left\{ A_2 + \frac{k_1}{3\gamma}\ [1-(1-\gamma\, A_1 X)^3\ ] \right.$$

$$\left. + \frac{k_2\, A_1}{2\gamma}\ [1-(1-\gamma\, A_1 X)^2\ ] + \frac{k_3\, A_1^{\,2}}{\gamma}\ (\gamma\, A_1 X) \right\} \tag{26.2}$$

For m=1, i.e., $\gamma A_1=\beta$, we have

$$a_1 = A_1, \tag{27.1}$$

$$a_2 = e^{2\beta X}\ [A_2 + \frac{1}{2\beta}\, (k_1\, A_1 + k_2\, A_1^{\,2} + k_3\, A_1^{\,3})\, (1-e^{-2\beta X})\ ]\ . \tag{27.2}$$

For $m < 1$, we consider three special cases of interest: (i) $\gamma = 0$, (ii) $A_1 = 0$, and (iii) $\beta = 0$ with $\gamma A_1 < 0$. Case (i) reads

$$a_1 = A_1 e^{-\beta X} \tag{28.1}$$

$$a_2 = e^{-\beta X} [A_2 + k_1 A_1 X + \frac{k_2}{\beta} A_1^2 (1-e^{-\beta X}) + \frac{k_3}{2\beta} A_1^3 (1-e^{-2\beta X})]. \tag{28.2}$$

Case (ii) (which is not actually an acceleration wave) is obtained simply by setting $A_1 = 0$ in Eqs. (28), to wit

$$a_1 = 0 \tag{29.1}$$

$$a_2 = A_2 e^{-\beta X} \tag{29.2}$$

Case (iii) is given by Eqs. (26), with γA_1 negative. It is to be noted that in this case

$$\lim_{X\to\infty} a_1 = 0$$

$$\lim_{X\to\infty} a_2 = - k_1/3\gamma = - K_2^{(1)}/6 K_{00}^{(2)}$$

Equations (26) represent the behavior of an <u>elastic</u> material if k_1 and k_2 vanish.

Equations (28) represent the behavior of a linear material if k_2 and k_3 vanish. (This reduces to a result due to Bland [9]).

5. Discussion of the Results

The validity of the solution established here is clearly limited to the domain of convergence of the series. In addition, any interruption in the analyticity of the boundary conditions as given by Eq. (7) gives rise to a wave which likewise nullifies the validity of the series solution, as does, obviously, the blowing up of the coefficients in the case $m > 1$. Furthermore, a shock wave may be produced behind the acceleration wave if the envelope of the characteristics has a cusp. There is a possibility, however, that the trajectory of such a wave coincides with the limit of convergence of the series; this question is yet to be investigated.

The results of this paper give the possibility of an experimental evaluation of the short-time behavior of the one-dimensional response functions of a general nonlinear viscoelastic material describable by Eqs. (4-5), in the form of the coefficients $K^{(1)}_{m_1 \ldots m_2}$ of the series expansion (12) of the kernel function $K_n(\tau_1 \ldots \tau_n)$. Let us consider an acceleration wave produced by longitudinal impact on a long, straight bar having strain gages affixed at a number of sufficiently closely spaced stations. A record of the first arrival time of the wave gives the speed U, and hence $K^{(1)}_0$. From strain-time records we may calculate $a_1 = \dot{\epsilon}|_{r=X/U}$, $a_2 = \ddot{\epsilon}|_{t=X/U}$, etc. We plot these quantities as functions of X. To the data for a_1 we try to fit a curve given by Eq. (20), having at our disposal the parameters β and γ; from the values of the parameters giving the best fit (e.g., by least squares) we obtain $K^{(1)}_1$ and $K^{(2)}_{00}$ by Eq. (19). Similarly, the values of k_1, k_2 and k_3 giving best fit to the data for a_2 by Eq. (25) yield $K_2^{(1)}$, $K_{01}^{(2)}$, and $K_{00}^{(3)}$.

I remark further that "short-time behavior" is used in the present context differently from the quasi-static treatment of Huang and Lee [10]. The significance of short-time approximations in viscoelasticity has been discussed elsewhere [11].

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Résumé

On montre qu'un problème d'un corps viscoélastique non linéaire où les conditions à la limite provoquent une onde d'accélération undimensionnelle possède une solution, valable dans une certaine région derrière l'onde, telle que la contrainte, la déformation et la vitesse s'expriment sous forme de séries en puissances du temps relatif a l'arrivée de l'onde; les coefficients sont des fonctions de la position qui s'obtiennent par intégration d'équations différentielles de premier ordre. Ces résultats permettent de déterminer les constantes qui régissent le comportement viscoélastique à courte durée au moyen d'essais de propagation des ondes.