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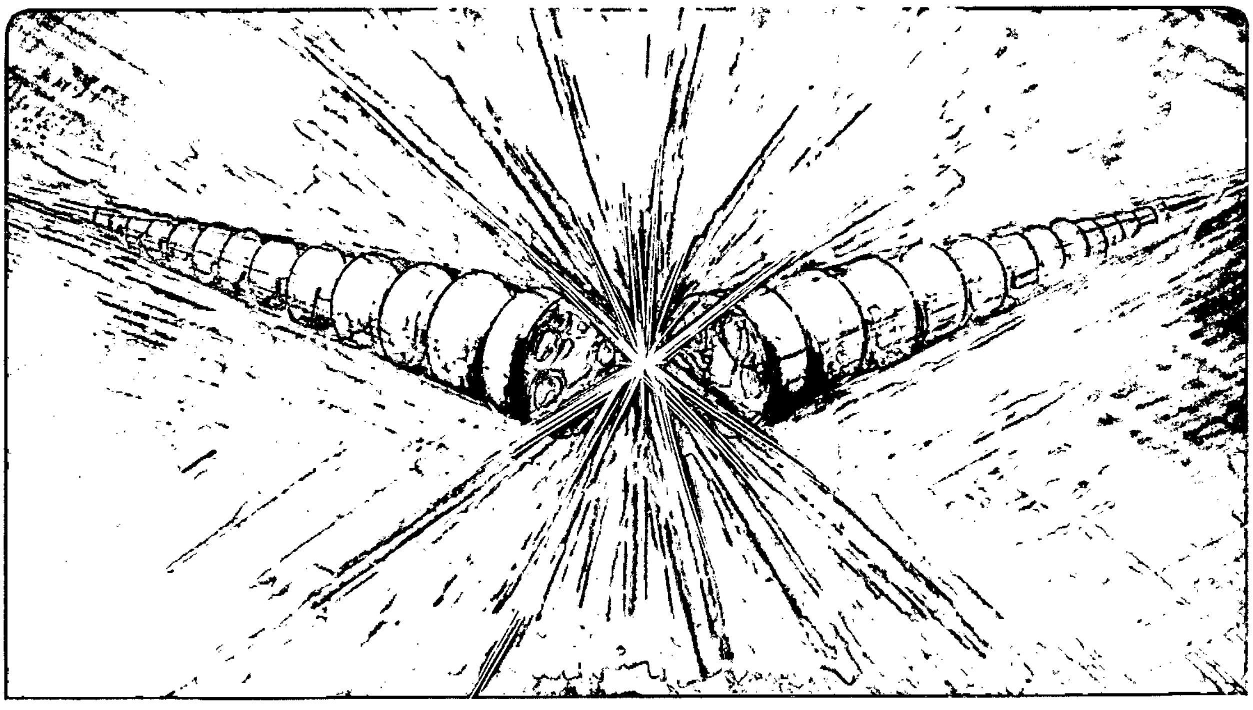

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Hamiltonian Structure of Two-Fluid Plasma Dynamics*

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Abstract

We present the Hamiltonian structure of two-fluid electrodynamics, with the Hamiltonian functional equaling the energy. The Poisson bracket on functionals of the fluid variables and the electric and magnetic fields is bilinear, antisymmetric, and satisfies the Jacobi identity.

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Of the three standard non-dissipative models for plasma dynamics, the Hamiltonian structure has previously been presented for two. Morrison and Greene[1] have treated ideal MHD, while Marsden and Weinstein[2] have derived the Poisson bracket appropriate for the Maxwell-Vlasov system. For the third model, two-fluid dynamics, we have followed the approach of Marsden and Weinstein to determine the Poisson structure for functionals on the phase space consisting of the fluid variables and the Maxwell field variables. The bracket $\{\,,\,\}$ so constructed automatically satisfies the requisite properties of a Poisson structure; letting E, F, and G be functionals on phase space and letting α be a scalar, these properties are

i) antisymmetry: $\{E, F\} = -\{F, E\}$

ii) bilinearity: $\{\alpha E + F, G\} = \alpha\{E,G\} + \{F, G\}$

iii) the Jacobi identity:

$$\{\{E, F\}, G\} + \{\{F, G\}, E\} + \{\{G, E\}, F\} = 0 \quad .$$

We now define the physical system of charged fluids under consideration. Label fluid species with the subscript s; each is composed of structureless particles of mass m_s and charge q_s. Let $a_s = q_s/m_s$; $q_s = 0$ is allowed. Our treatment holds for an arbitrary number of species, but two (oppositely charged) species is the situation most commonly discussed. Then, in terms of the fluid velocities $\underline{u}_s$, mass densities ρ_s, specific entropies σ_s, electric field $\underline{E}$, and magnetic field $\underline{B}$, the equations of ideal multi-fluid dynamics, in rationalized units, are

$$\underline{\nabla} \cdot \underline{E} = \sum_s a_s \rho_s \tag{1a}$$

$$\underline{\nabla} \times \underline{B} = \sum_s a_s \rho_s \underline{u}_s + \dot{\underline{E}} \tag{1b}$$

$$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}} \tag{1c}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{1d}$$

$$\dot{\rho}_s + \underline{\nabla} \cdot (\rho_s \underline{u}_s) = 0 \quad (2a)$$

$$\dot{\sigma}_s + \underline{u}_s \cdot \underline{\nabla}\sigma_s = 0 \quad (2b)$$

$$\rho_s \dot{\underline{u}}_s + \rho_s (\underline{u}_s \cdot \underline{\nabla})\underline{u}_s = a_s\rho_s (\underline{E} + \underline{u}_s \times \underline{B}) - \underline{\nabla} P_s \quad (2c)$$

where the specific internal energy $U_s(\rho_s, \sigma_s)$, expressed as an equation of state, yields the (partial) pressure P_s according to

$$P_s = \rho_s^2 \, \partial U_s / \partial \rho_s \quad . \quad (3)$$

Eqs. (1) are the Maxwell equations, and eqs. (2) and (3) are the laws of compressible ideal fluid dynamics. We neglect heat flow, and therefore express entropy convection by the adiabatic equation (2b).

It is natural in our construction to replace the velocity field variable $\underline{u}_s$ with the momentum density $\underline{M}_s \equiv \rho_s \underline{u}_s$. Then phase space consists of the set of quintuples of dynamical variables ($\underline{M}_s$, ρ_s, σ_s, $\underline{E}$, $\underline{B}$), while the energy of the system is

$$H (\underline{M}_s, \rho_s, \sigma_s, \underline{E}, \underline{B}) = \sum_s \int (\tfrac{1}{2} \rho_s^{-1} |\underline{M}_s|^2 + \rho_s U_s (\rho_s, \sigma_s)) d^3x + \int (\tfrac{1}{2} |\underline{E}|^2 + \tfrac{1}{2} |\underline{B}|^2) \, d^3x, \quad (4)$$

where the integrals are over the region in space occupied by the fluids. (We will sometimes use the notation of writing, e.g., $\underline{M}_s \equiv (\underline{M}_1, \ldots, \underline{M}_k)$ for k species. Whether this is the case or whether $\underline{M}_s$ refers to the single species s will always be clear from the context.)

The purpose of this paper is to express eqs. (1b), (1c), and (2) in the form of Hamiltonian evolution equations

$$\dot{Z} = \{ Z, H \} \quad (5)$$

where Z represents one of the dynamical variables, and the Hamiltonian H is given by the energy (4). Because we are abandoning canonical coordinates in favor of these physically appealing variables, the bracket in (5) is not expected to have the form of a standard Poisson bracket. We do require, however, that it satisfies the essential properties of the usual Poisson bracket mentioned above.

We regard the system defined by (1), (2), (3), and (4) as the coupling of the vacuum Maxwell equations to ordinary fluid dynamics. Therefore, we will briefly review the Hamiltonian structures of these theories.

The equations of motion for a single fluid species composed of uncharged particles are eqs. (2) and (3), with s=1 and a_s=0. The Hamiltonian is the first integral in (4). The Poisson bracket for this has been given by Morrison and Greene[1], and rederived by Marsden and Weinstein (private communication) in such a way that the Jacobi identity is automatically satisfied. Suppressing the species label and using a dynamical variable as a subscript to denote the functional derivative with respect to that dynamical variable, Morrison and Greene's result is

$$\{F,G\}(\underline{M}, \rho, \sigma) = -\int \underline{M} \cdot \left[(F_{\underline{M}} \cdot \underline{\nabla}) G_{\underline{M}} - (G_{\underline{M}} \cdot \underline{\nabla}) F_{\underline{M}} \right] d^3x \qquad (6)$$

$$-\int \rho \left[F_{\underline{M}} \cdot \underline{\nabla} G_\rho - G_{\underline{M}} \cdot \underline{\nabla} F_\rho \right] d^3x$$

$$-\int \sigma \left[F_{\underline{M}} \cdot \underline{\nabla} G_\sigma - G_{\underline{M}} \cdot \underline{\nabla} F_\sigma \right] d^3x \quad .$$

The equations of motion now follow from (5).

The structure of the vacuum Maxwell equations as a Hamiltonian system is also known[2]. The Hamiltonian is the second integral in (4), and the Poisson bracket, for functionals of the electric and magnetic fields, is

$$\{F, G\} (\underline{E}, \underline{B}) = \int \left[F_{\underline{E}} \cdot (\underline{\nabla} \times G_{\underline{B}}) - G_{\underline{E}} \cdot (\underline{\nabla} \times F_{\underline{B}}) \right] d^3x \ . \qquad (7)$$

One then obtains the vacuum Maxwell equations for $\dot{\underline{E}}$ and $\dot{\underline{B}}$ in the form of (5).

We now state our results for the Hamiltonian structure of the combined system of charged fluids plus the Maxwell equations. With the Hamiltonian given by (4), we require eqs. (1b), (1c), and (2) to be of the form (5). This is accomplished with the Poisson bracket

$$\begin{aligned} \{F,G\}(\underline{M}_s, \rho_s, \sigma_s, \underline{E}, \underline{B}) = \sum_s \{F,G\}(\underline{M}_s, \rho_s, \sigma_s) + \{F,G\}(\underline{E}, \underline{B}) \qquad (8) \\ + \sum_s \int a_s \rho_s \, (F_{\underline{M}} \cdot G_{\underline{E}} - G_{\underline{M}} \cdot F_{\underline{E}}) \, d^3x \\ + \sum_s \int a_s \rho_s \, \underline{B} \cdot (F_{\underline{M}} \times G_{\underline{M}}) \, d^3x \end{aligned}$$

where the first and second terms are defined in (6) and (7), and species subscripts have been suppressed in the functional derivatives.

We observe that the first term of (8) involves only the fluid variables and that the second is purely electromagnetic, while the third and fourth provide the coupling of the fluids to the electric and magnetic fields, respectively. Bilinearity, skew symmetry, and the Jacobi identity all follow for (8) by the methods used in its derivation. In addition it is readily verified that the correct evolution equations for the phase space variables, in the form (5), follow from (8) and (4). Additional body forces, such as gravity, can easily be incorporated into eq. (2c) by the inclusion of an appropriate term in the Hamiltonian. Finally, eqs. (1a) and (1d), rather than being postulated separately as initial conditions, follow from the gauge invariance of electromagnetism.

The restriction of multi-species electrodynamics to the Coulomb case, in which $\underline{B} = 0$, can also be treated. The scalar potential ϕ is expressed in terms of the mass densities ρ_s by

$$\phi(\underline{x}) = \frac{1}{4\pi} \int \frac{\sum_s a_s \rho_s(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d^3x'$$

and $\underline{E} = - \underline{\nabla} \phi$. Eqs. (1) are then replaced by the Poisson equation $\nabla^2 \phi = - \sum_s a_s \rho_s$, and the Lorentz force term in (2c) is replaced by $a_s \rho_s \underline{E}$. The Hamiltonian structure is obtained by taking the Hamiltonian on the phase space of sets of triples $(\underline{M}_s, \rho_s, \sigma_s)$ to be the total energy

$$H(\underline{M}_s, \rho_s, \sigma_s) = \sum_s \int (\tfrac{1}{2} \rho_s^{-1} |\underline{M}_s|^2 + \rho_s U_s(\rho_s, \sigma_s)) \, d^3x$$

$$+ \frac{1}{8\pi} \iint \frac{1}{|\underline{x} - \underline{x}'|} (\sum_s a_s \rho_s(\underline{x}))(\sum_{s'} a_{s'} \rho_{s'}(\underline{x}')) \, d^3x \, d^3x'$$

and letting the Poisson bracket on phase functionals be given by the first term of (8). The correct equations of motion for the dynamical variables ($\underline{M}_s$, ρ_s, σ_s) now follow in the form (5).

The mathematical details of the derivation of (8) will appear elsewhere.[3]

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