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Statistical analysis of the SDMP method for parameter estimation of multiple transients

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Abstract. We present a statistical analysis of the subspace decomposition and matrix pencil (SDMP) method for estimating parameters of multiple transient signals arriving at a uniform linear array. This analysis supports several observations obtained via simulation.

Zusammenfassung. Wir präsentieren eine statistische Analyse der Unterraumzerlegung und des Matrix-Pencil-Verfahrens (SDMP) zur Schätzung der Parameter vielfacher transienter Signale, die von einem gleichförmigen linearen Array empfangen werden. Diese Analyse stützt verschiedene Beobachtungen, die mittels Simulationen gemacht wurden.

Résumé. Nous présentons une analyse statistique du sous-espace de décomposition de la méthode SPDM pour l'estimation des paramètres des signaux transitoires multiples arrivant à un tableau linéaire et uniforme. Cette analyse appuie plusieurs observations obtenues par simulation.

Keywords. Multiple transients; parameter estimation; subspace decomposition; matrix pencil; first-order perturbation; statistical analysis.

1. Introduction

In a recent paper by Hua and Sarkar [3], the subspace decomposition and matrix pencil (SDMP) method was developed for estimating temporal and spatial parameters of multiple transient signals arriving at a uniform linear array. Each transient is modeled as the sum of complex exponentials. The temporal parameters are frequencies, damping factors and residues of each transient. The spatial parameters are the angles of arrival of multiple transient waves. The SDMP method is based on an eigen-decomposition of a generalized data matrix which is formed by stacking a sequence of submatrices of array outputs. The eigen-decomposition is then used via matrix pencil to yield the desired parameters. The SDMP method was shown via simulation to have a near-optimum accuracy. In this paper, we present a first-order perturbation analysis of this method. Results of this analysis support several observations reported in [3].

The approach of our analysis follows that shown in [1, 2], where several basic perturbation properties of matrix pencil are provided. We note that there are indeed many approaches available for statistical analysis

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of array processing algorithms, see e.g. [4–7]. But due to the unique structure inherent in the signal model and the SDMP method, there is no other approach as efficient as that in [1, 2].

Section 2 summarizes the SDMP method developed in [3] for easy reference. The notations used are consistent with those in [3]. Section 3 outlines the first-order analysis of the SDMP method and provides expressions for the first-order perturbations in the estimated parameters: damping factors, frequencies and time delays (due to angle of arrival). Section 3 provides a few numerical examples which show some insights into the SDMP method.

2. Summary of the SDMP method

Given the array outputs $y_{k,h}$ for $k = 0, 1, \ldots, K-1$ and $h = 0, 1, \ldots, H-1$, where K is the number of snapshots and H the number of sensors, the SDMP algorithm (for details see [3]) is as follows.

- Choose 2-D moving window sizes M and N. These two integers should satisfy $K - P \geqslant M \geqslant P$ and $H - P \geqslant N \geqslant P$, where P is the sum of number of poles (or exponentials) in each impinging transient wave.
- Form the following generalized data matrix of the size $(H-N)(K-M)\times(N+1)(M+1)$:

$$Y = \begin{bmatrix} Y_0 & Y_1 & \cdots & Y_N \\ Y_1 & Y_2 & \cdots & Y_{N+1} \\ & & \cdots & \\ Y_{H-N-1} & Y_{H-N} & \cdots & Y_{H-1} \end{bmatrix},$$

where

$$Y_h = \begin{bmatrix} y_{0,h} & y_{1,h} & \cdots & y_{M,h} \\ y_{1,h} & y_{2,h} & \cdots & y_{M+1,h} \\ & & \cdots & \\ y_{K-M-1,h} & y_{K-M,h} & \cdots & y_{K-1,h} \end{bmatrix}.$$

- Compute P dominant right singular vectors (or dominant row subspace) of Y, and call them the (P-column) matrix V. Then define
  $V_a = V$ without its $(M+1)$th, $2(M+1)$th, ..., $(N+1)(M+1)$th rows,
  $V_b = V$ without its 1st $(M+1+1)$th, $2(M+1)+1$th, ..., $N(M+1)+1$th rows,
  $V_c = V$ without its last $M+1$ rows, and
  $V_d = V$ without its first $M+1$ rows.
- Compute the generalized eigenvalues (rank reducing numbers) $\{z_p: p = 1, \ldots, P\}$ and $\{g_p: p = 1, \ldots, P\}$ of the matrix pencils $V_b - zV_a$ and $V_d - gV_c$, respectively.
- Pair z_p and g_p by maximizing the MUSIC spectrum (see [3] for details).
- Compute the damping factors $\alpha_p = \mathrm{Re}\{\log(z_p)\}$, the frequencies $\omega_p = \mathrm{Im}\{\log(z_p)\}$ and the time delays $\tau_p = \mathrm{Re}\{\log(g_p)/\log(z_p)\}$.

- Organize α_p and ω_p into groups, where each group is associated with one transient signal, by identifying the distribution clusters formed by τ_p for $p = 1, 2, \ldots, P$.

3. Statistical analysis

Our focus here is to evaluate the first-order perturbations in the estimated damping factors, frequencies and time delays. It is clear that a first-order perturbation analysis is to yield the following:

$$\delta\theta = \boldsymbol{w}^{\mathrm{T}}\boldsymbol{c}, \tag{1}$$

where $\delta\theta$ is the perturbation in an estimated parameter θ, $\boldsymbol{w}$ is the noise vector (which is real in this paper) and $\boldsymbol{c}$ is the sensitivity coefficient vector. The objective of the following discussion is to find $\boldsymbol{c}$.

It is shown in [2] that 'the SVD truncation' does not change the first-order perturbation. Using this observation, we can show that, to the first-order approximation, computing the generalized eigenvalues of $\boldsymbol{V}_{\mathrm{b}} - z\boldsymbol{V}_{\mathrm{a}}$ and $\boldsymbol{V}_{\mathrm{d}} - g\boldsymbol{V}_{\mathrm{c}}$ as required in the SDMP method is equivalent to computing the generalized eigenvalues of $\boldsymbol{Y}_{\mathrm{b}} - z\boldsymbol{Y}_{\mathrm{a}}$ and $\boldsymbol{Y}_{\mathrm{d}} - g\boldsymbol{Y}_{\mathrm{c}}$, respectively, where $\boldsymbol{Y}_{\mathrm{a}} = \boldsymbol{Y}$ without its $(M+1)$th, $2(M+1)$th, $\ldots, (N+1)(M+1)$th columns, $\boldsymbol{Y}_{\mathrm{b}} = \boldsymbol{Y}$ without its 1st, $(M+1+1)$th, $2(M+1)+1$th, $\ldots$, $N(M+1)+1$th columns, $\boldsymbol{Y}_{\mathrm{c}} = \boldsymbol{Y}$ without its last $M+1$ columns, and $\boldsymbol{Y}_{\mathrm{d}} = \boldsymbol{Y}$ without its first $M+1$ columns.

Following the approach in [1], we can show that the first-order perturbations in the generalized eigenvalues of $\boldsymbol{Y}_{\mathrm{b}} - z\boldsymbol{Y}_{\mathrm{a}}$ and $\boldsymbol{Y}_{\mathrm{d}} - g\boldsymbol{Y}_{\mathrm{c}}$ are, respectively,

$$\delta z_p = \frac{\boldsymbol{p}_p^{\mathrm{H}}(\delta\boldsymbol{Y}_{\mathrm{b}} - z_p\delta\boldsymbol{Y}_{\mathrm{a}})\boldsymbol{q}_p}{\boldsymbol{p}_p^{\mathrm{H}}\boldsymbol{X}_{\mathrm{a}}\boldsymbol{q}_p}, \tag{2}$$

$$\delta g_p = \frac{\boldsymbol{r}_p^{\mathrm{H}}(\delta\boldsymbol{Y}_{\mathrm{d}} - g_p\delta\boldsymbol{Y}_{\mathrm{c}})\boldsymbol{s}_p}{\boldsymbol{r}_p^{\mathrm{H}}\boldsymbol{X}_{\mathrm{c}}\boldsymbol{s}_p}, \tag{3}$$

where $\delta\boldsymbol{Y}_{\mathrm{a}}$, $\delta\boldsymbol{Y}_{\mathrm{b}}$, $\delta\boldsymbol{Y}_{\mathrm{c}}$ and $\delta\boldsymbol{Y}_{\mathrm{d}}$ are the additive noise matrices in $\boldsymbol{Y}_{\mathrm{a}}$, $\boldsymbol{Y}_{\mathrm{b}}$, $\boldsymbol{Y}_{\mathrm{c}}$ and $\boldsymbol{Y}_{\mathrm{d}}$, respectively. $\boldsymbol{p}_p$ and $\boldsymbol{q}_p$ are, respectively, the left and right generalized eigenvectors of the noiseless matrix pencil $\boldsymbol{X}_{\mathrm{b}} - z\boldsymbol{X}_{\mathrm{a}}$, $\boldsymbol{r}_p$ and $\boldsymbol{s}_p$ are, respectively, the left and right generalized eigenvectors of the noiseless matrix pencil $\boldsymbol{X}_{\mathrm{d}} - g\boldsymbol{X}_{\mathrm{c}}$. z_p and g_p are the noiseless generalized eigenvalues. Note that the $\boldsymbol{X}$'s are the noiseless version of the $\boldsymbol{Y}$'s.

We know (see [3] for details) that

$$\boldsymbol{X}_{\mathrm{a}} = \boldsymbol{Z}_{K-M,H-N}\,\boldsymbol{B}\,\boldsymbol{Z}_{M,N+1}^{\mathrm{T}},$$

$$\boldsymbol{X}_{\mathrm{b}} = \boldsymbol{Z}_{K-M,H-N}\,\boldsymbol{B}\,\boldsymbol{Z}_{\mathrm{d}}\boldsymbol{Z}_{M,N+1}^{\mathrm{T}},$$

$$\boldsymbol{X}_{\mathrm{c}} = \boldsymbol{Z}_{K-M,H-N}\,\boldsymbol{B}\,\boldsymbol{Z}_{M+1,N}^{\mathrm{T}},$$

$$\boldsymbol{X}_{\mathrm{d}} = \boldsymbol{Z}_{K-M,H-N}\,\boldsymbol{B}\,\boldsymbol{G}_{\mathrm{d}}\boldsymbol{Z}_{M+1,N}^{\mathrm{T}},$$

where $\boldsymbol{B}$ is the diagonal matrix of the signal amplitudes b_p, $\boldsymbol{Z}_{\mathrm{d}}$ the diagonal matrix of the poles z_p, $\boldsymbol{G}_{\mathrm{d}}$ the diagonal matrix of the poles g_p, and the other $\boldsymbol{Z}$ matrices are defined by z_p and g_p. Then we can show (following the approach in [1]) that

$\boldsymbol{p}_p^{\mathrm{H}} = \boldsymbol{r}_p^{\mathrm{H}} =$ pth row of pseudoinverse of $\boldsymbol{Z}_{K-M,H-N}$,
$\boldsymbol{q}_p =$ pth column of pseudoinverse of $\boldsymbol{Z}_{M,N+1}^{\mathrm{T}}$,
$\boldsymbol{s}_p =$ pth column of pseudoinverse of $\boldsymbol{Z}_{M+1,N}^{\mathrm{T}}$,

and then

$$\boldsymbol{p}_p^{\mathrm{H}} \boldsymbol{X}_{\mathrm{a}} \boldsymbol{q}_p = b_p, \tag{4}$$

$$\boldsymbol{r}_p^{\mathrm{H}} \boldsymbol{X}_{\mathrm{c}} \boldsymbol{s}_p = b_p, \tag{5}$$

These two equations simplify the denominators of (2) and (3). The numerators of (2) and (3) can be treated as follows. Based on the structures of $\delta \boldsymbol{Y}_{\mathrm{a}}$, $\delta \boldsymbol{Y}_{\mathrm{b}}$, $\delta \boldsymbol{Y}_{\mathrm{c}}$ and $\delta \boldsymbol{Y}_{\mathrm{d}}$, we can show that

$$\delta \boldsymbol{Y}_{\mathrm{b}} - z_p \delta \boldsymbol{Y}_{\mathrm{a}} = \delta \boldsymbol{Y}(\boldsymbol{P}_{\mathrm{b}} - z_p \boldsymbol{P}_{\mathrm{a}}), \tag{6}$$

$$\delta \boldsymbol{Y}_{\mathrm{d}} - g_p \delta \boldsymbol{Y}_{\mathrm{c}} = \delta \boldsymbol{Y}(\boldsymbol{P}_{\mathrm{d}} - g_p \boldsymbol{P}_{\mathrm{c}}), \tag{7}$$

where $\delta \boldsymbol{Y}$ is the additive noise matrix, i.e. $\{n_{k,h}$: row index $k = 0, 1, \ldots, K-1$; column index $h = 0, 1, \ldots, H-1\}$, in the generalized matrix $\boldsymbol{Y}$ formed from the array outputs, $\boldsymbol{P}_{\mathrm{a}}$ is the identity matrix $\boldsymbol{I}_{(M+1)(N+1)\times(M+1)(N+1)}$ without its $(M+1)$th, $2(M+1)$th, $\ldots, (N+1)(M+1)$th columns, $\boldsymbol{P}_{\mathrm{b}}$ is the identity matrix $\boldsymbol{I}_{(M+1)(N+1)\times(M+1)(N+1)}$ without its $(M+1+1)$th, $(2(M+1)+1)$th, $\ldots, (N(M+1)+1)$th columns, $\boldsymbol{P}_{\mathrm{c}}$ is the identity matrix $\boldsymbol{I}_{(M+1)(N+1)\times(M+1)(N+1)}$ without its last $M+1$ columns, and $\boldsymbol{P}_{\mathrm{d}}$ is the identity matrix $\boldsymbol{I}_{(M+1)(N+1)\times(M+1)(N+1)}$ without its first $M+1$ columns.

Furthermore, we can show that

$$\boldsymbol{p}_p^{\mathrm{H}} \delta \boldsymbol{Y} = \boldsymbol{r}_p^{\mathrm{H}} \delta \boldsymbol{Y} = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{P}_p, \tag{8}$$

where the noise vector $\boldsymbol{w}$ is formed by cascading the columns of the noise matrix $\delta \boldsymbol{Y}$, and $\boldsymbol{P}_p$ is a $KH \times (M+1)(N+1)$ matrix defined by the $H \times (N+1)$-block matrix:

$$\boldsymbol{P}_p = \begin{bmatrix} \boldsymbol{P}_{p,0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{P}_{p,1} & \boldsymbol{P}_{p,0} & \cdots & \\ & \boldsymbol{P}_{p,1} & \cdots & \\ \boldsymbol{P}_{p,(H-N-1)} & & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_{p,(H-N-1)} & \cdots & \boldsymbol{P}_{p,0} \\ & \boldsymbol{0} & \cdots & \boldsymbol{P}_{p,1} \\ & & \cdots & \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{P}_{p,(H-N-1)} \end{bmatrix},$$

where the $K \times (M+1)$ matrices $\boldsymbol{P}_{p,t}$ for $t = 0, 1, \ldots, (H-N-1)$ are defined as

$$\boldsymbol{P}_{p,t} = \begin{bmatrix} p_{(K-M)t+1,p} & 0 & \cdots & 0 \\ p_{(K-M)t+2,p} & p_{(K-M)t+1,p} & \cdots & 0 \\ & p_{(K-M)t+2,p} & \cdots & \\ p_{(K-M)t+(K-M),p} & & \cdots & \\ 0 & p_{(K-M)t+(K-M),p} & \cdots & p_{(K-M)t+1,p} \\ 0 & 0 & \cdots & p_{(K-M)t+2,p} \\ & & \cdots & \\ 0 & 0 & \cdots & p_{(K-M)t+(K-M),p} \end{bmatrix},$$

where $p_{j,p}$ is the jth element of $\boldsymbol{p}_p$.

Combining (2)–(8), we can easily find $c_{z,p}$ and $c_{g,p}$ so that

$$\delta z_p = w^T c_{z,p}, \tag{9}$$

$$\delta g_p = w^T c_{g,p}. \tag{10}$$

Then it follows that the first-order perturbations in the damping factors, frequencies and the time delays are, respectively,

$$\delta\alpha_p = \mathrm{Re}\left\{\frac{1}{z_p}\delta z_p\right\} = \mathrm{Re}\left\{w^T c_{z,p}\frac{1}{z_p}\right\}, \tag{11}$$

$$\delta\omega_p = \mathrm{Im}\left\{\frac{1}{z_p}\delta z_p\right\} = \mathrm{Im}\left\{w^T c_{z,p}\frac{1}{z_p}\right\}, \tag{12}$$

$$\delta\tau_p = \mathrm{Re}\left\{\frac{\tau_p}{g_p\log(g_p)}\delta g_p - \frac{\tau_p}{z_p\log(z_p)}\delta z_p\right\}$$

$$= \mathrm{Re}\left\{w^T c_{g,p}\frac{\tau_p}{g_p\log(g_p)} - w^T c_{z,p}\frac{\tau_p}{z_p\log(z_p)}\right\}. \tag{13}$$

Provided that the noise vector w is real-valued, from (11)–(13), we can easily find $c_{\alpha,p}$, $c_{\omega,p}$ and $c_{\tau,p}$ so that

$$\delta\alpha_p = w^T c_{\alpha,p}, \tag{14}$$

$$\delta\omega_p = w^T c_{\omega,p}, \tag{15}$$

$$\delta\tau_p = w^T c_{\tau,p}. \tag{16}$$

With (14)–(16), the variance of the first-order perturbations can be obtained for any noise covariance matrix. But for a simple discussion, we assume in the next section that the noise is white, i.e. the noise covariance matrix is a diagonal matrix with the diagonal elements equal to σ^2.

4. Numerical results

To show a few numerical examples and verify the theoretical result obtained in the previous section, we assume a uniform linear array of 10 sensors ($H = 10$) on which there are two arriving transient waves ($I = 2$). For a time interval of 25 snapshots ($K = 25$), the sampled array outputs can be expressed as

$$y_{k,h} = s_1(k + \tau_1 h) + s_2(k + \tau_2 h) + \sigma n_{k,n},$$

with

$$s_1(k) = a_1 \exp(-\alpha_1 k)\cos(\omega_1 k + \phi_1),$$

$$s_2(k) = a_2 \exp(-\alpha_2 k)\cos(\omega_2 k + \phi_2),$$

where $k = 0, 1, \ldots, K-1$, $h = 0, 1, \ldots, H-1$, $a_1 = a_2 = 1$, $\alpha_1 = -0.05$, $\alpha_2 = -0.1$, $\omega_1 = 0.5$, $\omega_2 = 1.0$, $\phi_1 = 0$, $\phi_2 = \pi/2$, $\tau_1 = 0.1$, $\tau_2 = 0.2$ and $n_{k,h}$ is white noise. These are the same data as considered in [3].

Figures 1–3 show the 3-D plots of the deviations (normalized by the Cramer–Rao lower bound or CRB) of the estimated α_1, ω_1 and τ_1, respectively. The plots for α_2, ω_2 and τ_2 are omitted because of their similarity. The simulation results shown in these plots are based on 200 independent runs with the (Gaussian white)

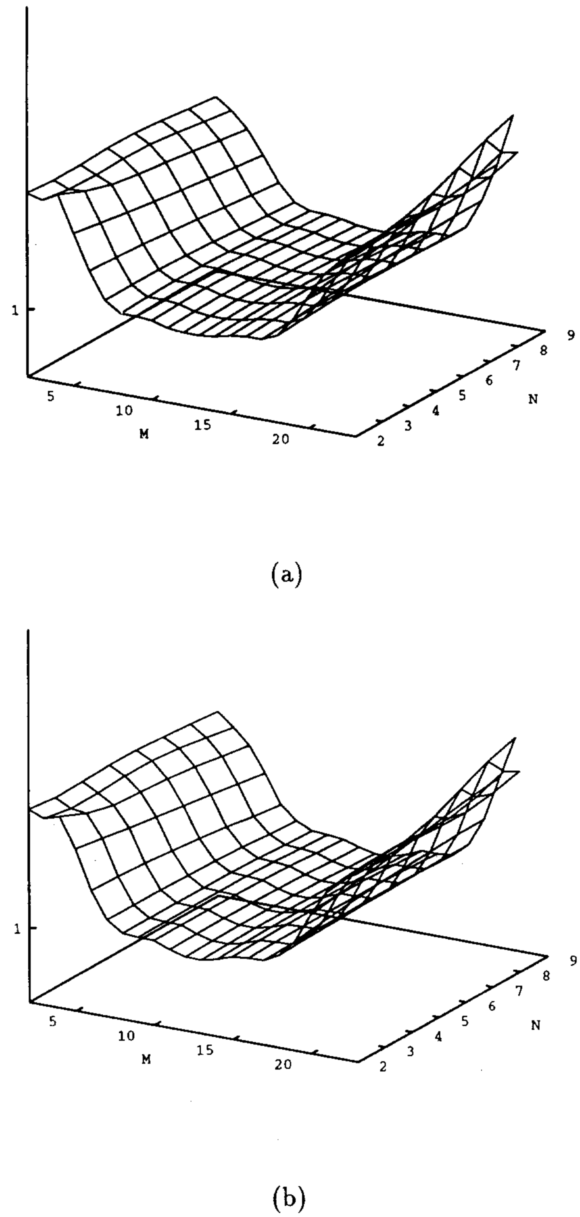

Fig. 1. Theoretical (a) and simulated (b) $\text{dev}(\delta\alpha_1)/\text{CRB}^{1/2}$ versus M and N (normalized CRB = 1).

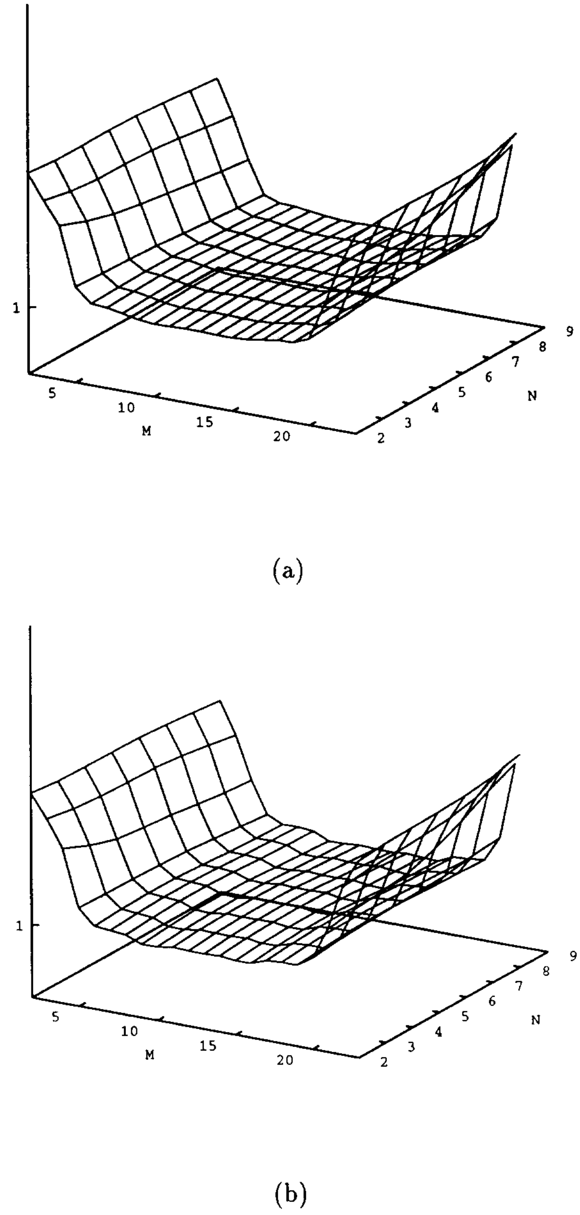

Fig. 2. Theoretical (a) and simulated (b) $\text{dev}(\delta\omega_1)/\text{CRB}^{1/2}$ versus M and N (normalized CRB = 1).

noise deviation $\sigma = 0.001$. There are three major observations that can be obtained from these plots. First, the theoretical results are consistent with the simulation. Second, the variances of all estimated parameters are close to the CRB when good choices of the window sizes M and N are used. The estimation variance is almost invariant to a large number of good choices of M and N around $M = \frac{1}{2}K$ and $N = \frac{1}{2}H$, respectively. Third, the estimation variances of the parameters α_1, ω_1 and τ_1 (associated with the first dimension) are more

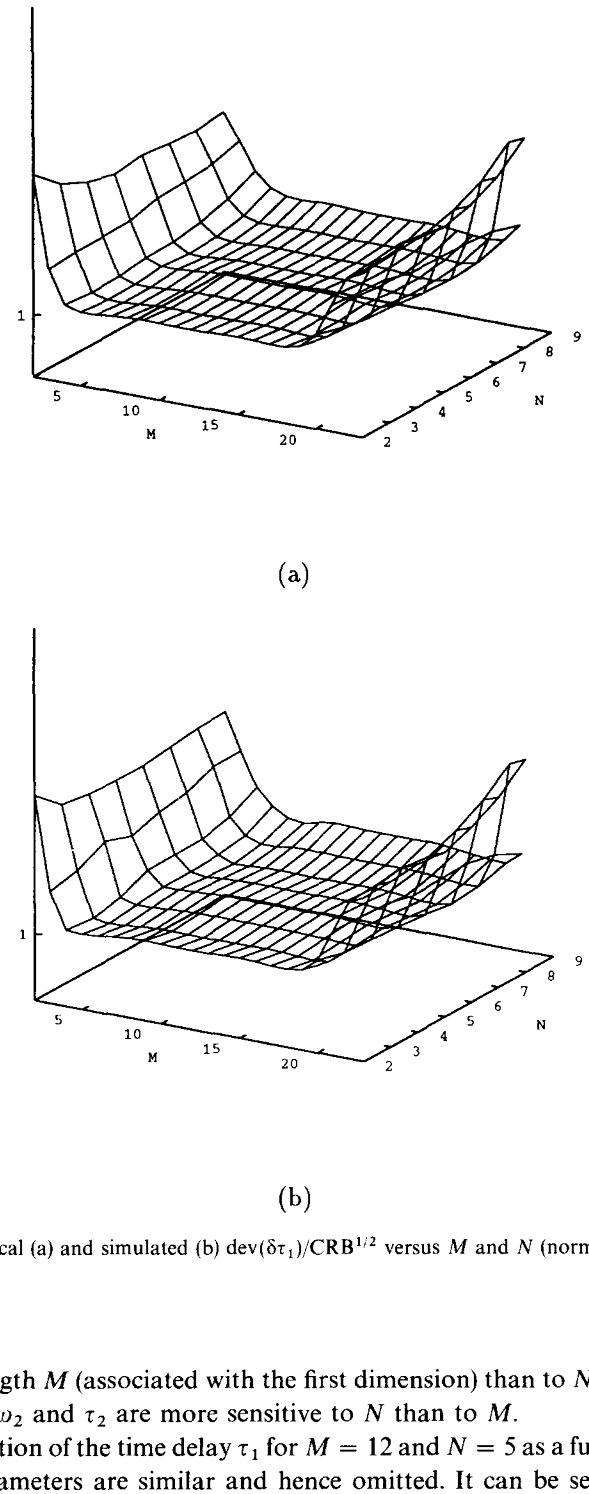

Fig. 3. Theoretical (a) and simulated (b) dev($\delta\tau_1$)/CRB$^{1/2}$ versus M and N (normalized CRB = 1).

sensitive to the window length M (associated with the first dimension) than to N (associated with the second dimension). Similarly, α_2, ω_2 and τ_2 are more sensitive to N than to M.

Figure 4 shows the deviation of the time delay τ_1 for $M = 12$ and $N = 5$ as a function of the noise deviation σ. The plots for other parameters are similar and hence omitted. It can be seen from this figure that the consistency between the theoretical result and the simulation holds up to $\sigma = 0.1$, which is equivalent to

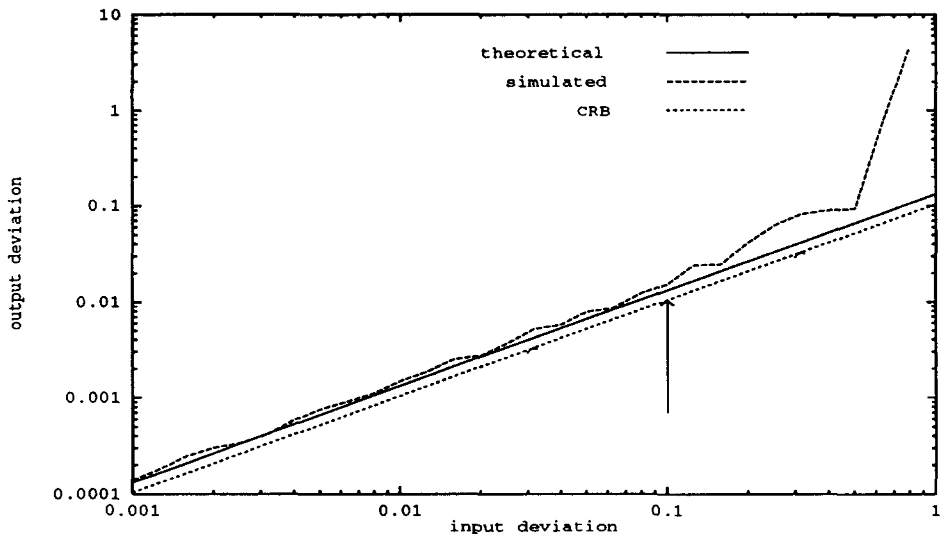

Fig. 4. Input versus output deviation of τ_1 for $M = 12$, $N = 5$; theoretical and simulated results compared to CRB.

SNR $\approx$ 3 dB, where

$$\mathrm{SNR} = 10\log_{10}\left(\frac{\sum_{k,h} x_{k,h^2}}{KH\sigma^2}\right).$$

5. Conclusions

We have presented a sensitivity analysis of the SDMP method for multiple transient waves. This analysis has led to the closed forms of the first-order perturbations in the estimated parameters: damping factors, frequencies and time delays. The theoretical results have also been verified via simulation. The near-optimum performance of, as well as some insights into, the SDMP method has now been supported by both analysis and simulation.

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