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SAN DIEGO STATE UNIVERSITY

# Teachers' Understanding of Algebraic Generalization

A dissertation submitted in partial satisfaction of the  
requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

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2016

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Chair

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2016



## DEDICATION

This dissertation is dedicated to my family, specifically to my wife, who constantly found time to support me even while she took care of our three girls, the house, and every other issue that came up and I neglected, and to my three daughters whose hugs and inspirational notes encouraged me to keep going.

## TABLE OF CONTENTS

|                                                                                          |     |
|------------------------------------------------------------------------------------------|-----|
| SIGNATURE PAGE .....                                                                     | iii |
| DEDICATION .....                                                                         | iv  |
| LIST OF FIGURES .....                                                                    | xi  |
| LIST OF TABLES .....                                                                     | xiv |
| VITA .....                                                                               | xv  |
| ABSTRACT OF THE DISSERTATION .....                                                       | xvi |
| CHAPTER 1: INTRODUCTION .....                                                            | 1   |
| My Experience Teaching Figural Patterns as Generalizing Tasks .....                      | 1   |
| Noyce Mathematics Master-Teaching Fellowship .....                                       | 2   |
| Teachers' Understanding of Algebraic Generalization.....                                 | 4   |
| Algebraic Generalization .....                                                           | 5   |
| Two gaps in the literature. ....                                                         | 6   |
| <i>Gap 1: Extending analyses of generalized patterns to include their symbolization.</i> | 6   |
| <i>Gap 2: Analysis of teachers' understanding of algebraic generalization.</i> .....     | 8   |
| Vignette #1.....                                                                         | 9   |
| <i>Teacher's interpretation of algebraic generalization.</i> .....                       | 11  |
| Research Question 1. ....                                                                | 12  |
| Mathematical knowledge for teaching.....                                                 | 13  |
| <i>Teachers' quantitative understanding of algebraic symbols.</i> .....                  | 14  |
| Research Question 2. ....                                                                | 16  |
| Teachers' Understandings of Students' Thinking of Algebraic Generalization: A            |     |
| Guide to Responsive Teaching .....                                                       | 18  |
| Responsive Teaching .....                                                                | 19  |
| Professional Noticing of Students' Mathematical Thinking.....                            | 20  |
| MKT for Responsive Teaching: A Trajectory of Student Thinking.....                       | 21  |
| Vignette #2.....                                                                         | 23  |
| Research Question 3. ....                                                                | 27  |

|                                                                              |           |
|------------------------------------------------------------------------------|-----------|
| Recapitulation of Research Questions and Overview of Dissertation Study..... | 28        |
| <b>CHAPTER 2: LITERATURE REVIEW .....</b>                                    | <b>30</b> |
| New Notion of Algebra.....                                                   | 30        |
| Mathematical Practices—Ways of Thinking .....                                | 33        |
| Figural-Pattern Generalization.....                                          | 35        |
| Theoretical Perspectives on Generalizing.....                                | 35        |
| Generalizing Strategies and Reasoning .....                                  | 37        |
| Role of Verbalization and Symbolization in Generalizing .....                | 40        |
| Embedding Student Thinking in the Figure.....                                | 42        |
| Importance of Justification in Generalizing.....                             | 44        |
| Representations .....                                                        | 46        |
| Characterization of Current Symbolic Focus.....                              | 47        |
| Role of Multiple Representations .....                                       | 50        |
| Symbolization .....                                                          | 53        |
| Reification.....                                                             | 55        |
| Quantitative Reasoning.....                                                  | 58        |
| Teachers’ Knowledge .....                                                    | 60        |
| Mathematical Knowledge for Teaching.....                                     | 61        |
| Professional Noticing.....                                                   | 64        |
| Classification of Noticing Studies .....                                     | 66        |
| Noticing-Related Findings.....                                               | 69        |
| Components of Professional Noticing .....                                    | 73        |
| Relationship Between Noticing and MKT.....                                   | 76        |
| <b>CHAPTER 3: RESEARCH METHODS .....</b>                                     | <b>78</b> |
| Participants.....                                                            | 79        |
| Overview of Research Design .....                                            | 81        |
| Professional Development. ....                                               | 82        |
| <i>Professional development rationale.</i> ....                              | 82        |
| <i>Professional development structure.</i> .....                             | 84        |
| Research Question 1 .....                                                    | 85        |
| Research Question 2 .....                                                    | 87        |
| Instruments.....                                                             | 88        |
| <i>Analyzing personally generated expressions.</i> .....                     | 88        |

|                                                                                                                              |         |
|------------------------------------------------------------------------------------------------------------------------------|---------|
| <i>Analyzing student-generated expressions.</i> .....                                                                        | 90      |
| Research Question 3 .....                                                                                                    | 91      |
| Research design for Phase II.....                                                                                            | 92      |
| <i>Clinical interview.</i> .....                                                                                             | 94      |
| <i>Daily preinstructional interview.</i> .....                                                                               | 95      |
| <i>Class-instruction observation.</i> .....                                                                                  | 95      |
| <i>Semistructured video-recall interview.</i> .....                                                                          | 97      |
| Data Analysis .....                                                                                                          | 99      |
| Grounded theory and coding.....                                                                                              | 99      |
| Data analysis: Research Question 1.....                                                                                      | 101     |
| Data analysis: Research Question 2.....                                                                                      | 101     |
| Data analysis: Research Question 3.....                                                                                      | 102     |
| <i>Analysis of individual teachers.</i> .....                                                                                | 103     |
| <i>Analysis across the case studies.</i> .....                                                                               | 105     |
| <br>CHAPTER 4: TEACHERS' UNDERSTANDING OF ALGEBRAIC<br>GENERALIZATION.....                                                   | <br>106 |
| Mathematical Goals Associated With Generalizing .....                                                                        | 106     |
| General Goals: Classroom Norms and Mathematical Practices .....                                                              | 107     |
| Detailed Goals: Facts and Procedures Related to a Computational Orientation .....                                            | 110     |
| Teachers' Quantitative Understanding of Algebraic Symbols: Associated<br>Conceptual Challenges and Possible Resolutions..... | 115     |
| Interpreting the Coefficient of $x$ .....                                                                                    | 117     |
| Review of figural-pattern task. ....                                                                                         | 117     |
| Limited quantitative understanding of the symbols.....                                                                       | 118     |
| <i>Denise's initial approach to pattern generalization.</i> .....                                                            | 119     |
| <i>Denise's understanding of the coefficient.</i> .....                                                                      | 120     |
| <i>Denise's understanding of the coefficient imposed on the pattern.</i> .....                                               | 124     |
| Detailed and flexible quantitative interpretation of the symbols.....                                                        | 129     |
| Quantitative, yet fixed interpretation of the symbols.....                                                                   | 133     |
| Interpreting Expressions in Which the Variable Appears More Than Once .....                                                  | 138     |
| Interpreting variables as contextually consistent: A literal translation. ....                                               | 139     |
| <i>Denise's initial approach and interpretation of the variables.</i> .....                                                  | 139     |
| <i>Denise's attempt to coordinate symbols' referents and the resulting conceptual<br/>        shift.</i> .....               | 140     |
| <i>Denise's articulation of her conceptual difficulty.</i> .....                                                             | 144     |
| <i>Denise's oversimplification of the symbols' referents.</i> .....                                                          | 147     |

|                                                                                                                  |     |
|------------------------------------------------------------------------------------------------------------------|-----|
| Associating quantities numerically.....                                                                          | 148 |
| <i>Clara's initial approach and interpretation of the variables.</i> .....                                       | 148 |
| <i>Clara's attempt to coordinate symbol's referents and the resulting conceptual shift.</i> .....                | 149 |
| <i>Clara's conceptual breakthrough.</i> .....                                                                    | 152 |
| Associating variables contextually: A new interpretation. ....                                                   | 153 |
| <i>Jack's initial approach and interpretation of the variables.</i> .....                                        | 154 |
| <i>Jack's conceptual breakthrough.</i> .....                                                                     | 156 |
| <i>Jack's reflection on his conceptual difficulties.</i> .....                                                   | 160 |
| Conclusion .....                                                                                                 | 161 |
| Conceptual Complexity of Interpreting Algebraic Expressions .....                                                | 161 |
| Complexities of Clara's ability to flexibly interchange the meanings of the variable and coefficient.....        | 162 |
| Complexities of Jack's ability to interpret each variable as the number of groups. ....                          | 165 |
| Significance.....                                                                                                | 166 |
| Implications for Teaching.....                                                                                   | 167 |
| <br>CHAPTER 5: TEACHERS' UNDERSTANDING OF THE PROCESS BY WHICH STUDENTS' LEARN TO GENERALIZE ALGEBRAICALLY ..... | 169 |
| Overarching Instructional Goals .....                                                                            | 170 |
| Goal 1: Developing a Meaningful Understanding of Algebraic Notation.....                                         | 171 |
| Goal 2: Understanding Functions as a Relationship Between Two Quantities .....                                   | 173 |
| Goal 3: Addressing Misconceptions Surrounding Variables.....                                                     | 174 |
| Identifying the Differences .....                                                                                | 176 |
| Connections Between the Roles of Describing and Calculating .....                                                | 179 |
| Generalizing through describing and calculating: A conceptual analysis. ....                                     | 180 |
| Clara: Disjointed view. ....                                                                                     | 181 |
| <i>Description of daily outcomes</i> .....                                                                       | 181 |
| <i>Instructional decisions: Ambiguity toward role of calculating.</i> .....                                      | 184 |
| <i>Reflection on challenges.</i> .....                                                                           | 191 |
| Jack: Connected view. ....                                                                                       | 193 |
| Connections Between Expressions and Figure .....                                                                 | 200 |
| Level of details connecting symbols and figure. ....                                                             | 202 |
| <i>Jack's symbol-by-symbol approach.</i> .....                                                                   | 202 |
| <i>Clara's general approach to connecting symbols to the figure.</i> .....                                       | 204 |

|                                                                                  |     |
|----------------------------------------------------------------------------------|-----|
| <i>Clara's overall method.</i>                                                   | 205 |
| <i>Clara's response to student-generated numerical expressions.</i>              | 209 |
| Use of multiple decompositions.                                                  | 217 |
| <i>Jack's individual use of multiple decompositions in succession.</i>           | 218 |
| <i>Clara's simultaneous use of multiple decompositions.</i>                      | 219 |
| Understanding of an expression.                                                  | 222 |
| <i>Jack: Understanding stems from articulating details of single expression.</i> | 223 |
| <i>Clara: Understanding stems from exposure to multiple expressions.</i>         | 225 |
| Instructional Trajectory                                                         | 229 |
| Trajectory model explanation.                                                    | 230 |
| Applying the model.                                                              | 238 |
| <i>Clara's Instructional Trajectory.</i>                                         | 240 |
| <i>Jack's Instructional Trajectory.</i>                                          | 249 |
| <i>Instilling multiple ways of thinking.</i>                                     | 251 |
| <i>Pedagogical content tool.</i>                                                 | 254 |
| Teachers' implementation of their instructional trajectories.                    | 256 |
| <i>Guiding the flow of ideas.</i>                                                | 257 |
| <i>Creating learning opportunities.</i>                                          | 260 |
| <i>Noticing of students' mathematical thinking.</i>                              | 264 |
| <i>Clara's noticing.</i>                                                         | 264 |
| <i>Attending.</i>                                                                | 264 |
| <i>Interpreting.</i>                                                             | 266 |
| <i>Deciding how to respond.</i>                                                  | 269 |
| <i>Jacks' noticing.</i>                                                          | 271 |
| Conclusion                                                                       | 273 |
| Teachers' Understanding of Algebraic Generalization                              | 274 |
| Teacher Knowledge for Responsive Teaching                                        | 277 |
| CHAPTER 6: CONCLUSIONS                                                           | 280 |
| Summary of Findings and Significance                                             | 280 |
| Interpretation of Generalizing Tasks                                             | 280 |
| Quantitative Understanding of Algebraic Notation                                 | 282 |
| MKT for Algebraic Generalization                                                 | 284 |
| Coordination between actions and representations                                 | 285 |
| Decomposition of instructional trajectory.                                       | 287 |
| <i>Theoretical contributions of the trajectory.</i>                              | 288 |
| <i>Pedagogical implications of the trajectory.</i>                               | 290 |
| Anticipation as part of expert professional noticing.                            | 291 |

|                                                                                         |            |
|-----------------------------------------------------------------------------------------|------------|
| One final reflection .....                                                              | 293        |
| Limitations .....                                                                       | 294        |
| Directions for Future Research .....                                                    | 297        |
| <b>APPENDICES .....</b>                                                                 | <b>299</b> |
| Appendix A: Pre-Interview Protocol (part I) .....                                       | 299        |
| Appendix B: Pre-Interview Protocol (part II) .....                                      | 301        |
| Appendix C: Post Professional Development Interview Protocol .....                      | 309        |
| Appendix D: Preinstructional Observation Protocol .....                                 | 311        |
| Appendix E: Postinstructional Observation Stimulated Recall Interview Protocol<br>..... | 312        |
| <b>REFERENCES .....</b>                                                                 | <b>313</b> |

## LIST OF FIGURES

|                                                                                                                                           |     |
|-------------------------------------------------------------------------------------------------------------------------------------------|-----|
| Figure 1.1. Figural pattern used by Lobato et al. (2013). .....                                                                           | 9   |
| Figure 1.2. Domingo’s association of the stage number and the squares per arm.....                                                        | 9   |
| Figure 1.3. John’s quadratic generalization task.....                                                                                     | 24  |
| Figure 3.1. Possible decomposition of Pattern 1.....                                                                                      | 89  |
| Figure 3.2. Alternative decomposition of Pattern 1.....                                                                                   | 89  |
| Figure 3.3. Possible decomposition of Pattern 2.....                                                                                      | 90  |
| Figure 3.4. 3-Step research design.....                                                                                                   | 94  |
| Figure 4.1. Figural patterns from interview protocol.....                                                                                 | 118 |
| Figure 4.2. Denise’s initial decomposition of Pattern 1.....                                                                              | 120 |
| Figure 4.3. Denise’s identification of “the original five” dots.....                                                                      | 121 |
| Figure 4.4. Denise’s decomposition to match $4x + 1$ . .....                                                                              | 122 |
| Figure 4.5. Denise’s derivation of the expression for Pattern 2.....                                                                      | 125 |
| Figure 4.6. Denise’s first interpretation of Pattern 2. ....                                                                              | 126 |
| Figure 4.7. Denise’s second interpretation of Pattern 2.....                                                                              | 127 |
| Figure 4.8. Denise’s explanation of the coefficient of 3.....                                                                             | 127 |
| Figure 4.9. Denise’s depiction of Stage 4 in Pattern 2.....                                                                               | 128 |
| Figure 4.10. Clara’s decomposition of Pattern 1. ....                                                                                     | 130 |
| Figure 4.11. Clara’s decomposition of Pattern 2. ....                                                                                     | 131 |
| Figure 4.12. Clara’s depiction of Stage 10. ....                                                                                          | 132 |
| Figure 4.13. Jack’s numerical approach to Pattern 2.....                                                                                  | 134 |
| Figure 4.14. Jack’s illustration for his interpretation of Pattern 1 showing the figure<br>increasing by a group of 4 at each stage. .... | 135 |



|                                                                                                           |     |
|-----------------------------------------------------------------------------------------------------------|-----|
| Figure 4.15. Jack's reconceptualization of the individual dots as a collective orbit of three tiles. .... | 136 |
| Figure 4.16. Denise's illustration of her interpretation of Pattern 1. ....                               | 140 |
| Figure 4.17. Denise's symbolic expression for Pattern 1. ....                                             | 141 |
| Figure 4.18. Denise's interpretation of $(x - 1)$ . ....                                                  | 142 |
| Figure 4.19. Denise's representation of how "minus 1" creates a group of four dots. ...                   | 143 |
| Figure 4.20. Denise highlighting the group of five dots and group of four dots in Stage 2. ....           | 144 |
| Figure 4.21. Denise's decomposition of Pattern 1 into groups of four dots. ....                           | 146 |
| Figure 4.22. Clara identifying $n$ as the first group of five dots. ....                                  | 150 |
| Figure 4.23. Jack's contextual interpretation of $5s - (s - 1)$ . ....                                    | 155 |
| Figure 4.24. Jack's symbolizing one dot from each of the $s$ groups. ....                                 | 157 |
| Figure 4.25. Jack demonstrating how a group of five dots becomes a group of four dots. ....               | 158 |
| Figure 4.26. Flexible conceptualizations of variable and coefficient. ....                                | 163 |
| Figure 4.27. Jack's and Clara's diverging interpretations of Pattern 2. ....                              | 164 |
| Figure 5.1. Instructional task used by Clara. ....                                                        | 177 |
| Figure 5.2. Instructional task presented as three subtasks by Jack. ....                                  | 178 |
| Figure 5.3. Illustration of hypothetical student's view of the pattern. ....                              | 194 |
| Figure 5.4. Number Sentence with contextual explanations. ....                                            | 195 |
| Figure 5.5. Joe's sketch of Stage 100,002. ....                                                           | 199 |
| Figure 5.6. Jack making connection between symbols and context explicit. ....                             | 204 |
| Figure 5.7. Representation of group's decomposition of the pattern. ....                                  | 206 |
| Figure 5.8. Clara's documentation of the group's understanding. ....                                      | 206 |

|                                                                                                            |     |
|------------------------------------------------------------------------------------------------------------|-----|
| Figure 5.9. Clara’s expression for students’ decomposition.....                                            | 207 |
| Figure 5.10. Model of teachers’ views of the roles of generalizing actions. ....                           | 231 |
| Figure 5.11. Model of teachers’ views of the roles of representations in algebraic<br>generalization. .... | 231 |
| Figure 5.12. Model of merging actions and representations.....                                             | 232 |
| Figure 5.13. Model of increasingly general ways of thinking. ....                                          | 236 |
| Figure 5.14. Clara’s instructional trajectory. ....                                                        | 240 |
| Figure 5.15. Jack’s instructional trajectory.....                                                          | 240 |
| Figure 5.16. Student’s diagonal decomposition of the pattern. ....                                         | 242 |
| Figure 5.17. Chris’s decomposition of the pattern.....                                                     | 244 |

## LIST OF TABLES

|                                                                                                                              |     |
|------------------------------------------------------------------------------------------------------------------------------|-----|
| Table 3.1. Research Design Components and Participants.....                                                                  | 81  |
| Table 4.1. Conceptual Challenges and Resulting Conceptualizations Associated With<br>Contextualizing Algebraic Symbols ..... | 116 |
| Table 5.1. Connecting Students' Understandings of Expressions and Context.....                                               | 201 |
| Table 5.2. Clarification of Ways of Thinking Associated With Generalizing.....                                               | 237 |

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## Publications

- Hawthorne, C., & Rasmussen, C. (2015). A framework for characterizing students' thinking about logical statements and truth tables. *International Journal of Mathematical Education in Science and Technology*, 46(3), 337-353.
- Philipp, R. A. & Hawthorne, C. (2015). Unpacking referent units in fraction division. *Teaching Children Mathematics*, 22(4), 240-247.
- Hawthorne, C., & Rasmussen, C. (2014). Understanding students' conceptualizations of logical tools. In T. Fukawa-Connelly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17<sup>th</sup> annual conference on Research in Undergraduate Mathematics Education* (pp. 105–119), Denver, CO.

# ABSTRACT OF THE DISSERTATION

## Teachers' Understanding of Algebraic Generalization

by

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Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2016

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Generalization has been identified as a cornerstone of algebraic thinking (e.g., Lee, 1996; Sfard, 1995) and is at the center of a rich conceptualization of K–8 algebra (Kaput, 2008; Smith, 2003). Moreover, mathematics teachers are being encouraged to use figural-pattern generalizing tasks as a basis of student-centered instruction, whereby teachers respond to and build upon the ideas that arise from students' explorations of these activities. Although more and more teachers are engaging their students in such generalizing tasks, little is known about teachers' understanding of generalization and their understanding of students' mathematical thinking in this domain.

In this work, I addressed this gap, exploring the understanding of algebraic generalization of 4 exemplary 8th-grade teachers from multiple perspectives. A significant feature of this investigation is an examination of teachers' understanding of the generalization process, including the use of algebraic symbols. The research consisted of two phases. Phase I was an examination of the teachers' understandings of the underlying quantities and quantitative relationships represented by algebraic notation. In Phase II, I observed the instruction of 2 of these teachers. Using the lens of professional noticing of students' mathematical thinking, I explored the teachers' enacted knowledge of algebraic generalization, characterizing how it supported them to effectively respond to the needs and queries of their students.

Results indicated that teachers predominantly see these figural patterns as enrichment activities, disconnected from course content. Furthermore, in my analysis, I identified conceptual difficulties teachers experienced when solving generalization tasks, in particular, connecting multiple symbolic representations with the quantities in the figures. Moreover, while the teachers strived to overcome the challenges of connecting different representations, they invoked both productive and unproductive conceptualizations of the symbols. Finally, by comparing two teachers' understandings of student thinking in the classroom, I developed an instructional trajectory to describe steps along students' generalization processes. This emergent framework serves as an instructional tool for teachers' use in identifying significant connections in supporting students to develop understanding of algebraic symbols as representations that communicate the quantities perceived in the figure.

## **Chapter 1: Introduction**

My study, at least my interest in my study, began years ago when I was a middle school mathematics teacher. At the time, although I had had many years of experience teaching upper level high school mathematics courses, I was teaching algebra for the first time. My middle school students not only were younger than my high school students but also had had limited exposure to abstract symbolic mathematics. In my initial teaching attempt, I engaged my students in a range of procedures involving predominantly the manipulation of symbolic expressions, rituals that have historically been identified with algebra (Kieran, 1992). After one year, I saw clearly that this approach had failed to provide my students with a sufficiently meaningful understanding for the symbols or any overarching sense for the purpose of these algebraic activities.

### **My Experience Teaching Figural Patterns as Generalizing Tasks**

I therefore decided to begin Year 2 of my algebra teaching with a very different approach, one that centered on generalizing tasks. Armed with a book of figural patterns, a gift from an elementary school teacher-colleague of my wife, I optimistically planned to provide my students experiences in which the rationale for all the algebraic symbols and associated rules might come to life. Although I cannot speak to how effective I was in reaching my students, this experience helped me see the importance of these generalizing tasks. I engaged my students in authentic mathematical activities and provided them with a context from which algebra and algebraic thinking grew. Although I was not fully aware of the value at the time, through this transition to focusing on generalization, my students experienced algebra as generalized arithmetic; symbols had meaning, and my students participated in mathematical practices critical for

developing algebraic thinking.

### **Noyce Mathematics Master-Teaching Fellowship**

Relatively early in my doctoral studies, I began working as an assistant for the Noyce Mathematics Master-Teaching Fellowship, a research and professional development project. Through this initiative, 16 secondary mathematics teachers were selected from an applicant pool of 61 teachers to participate annually in 10 full-day professional development sessions throughout the school year and summer. During the first 2 years, we worked to help teachers improve their instructional practices, focusing specifically on the intersection of algebra and student thinking. Teachers were asked to conduct mathematical interviews with students, collect and analyze student work, read literature on student understandings of various algebraic topics, and share their own conceptualizations of the mathematics, all to familiarize them with students' algebraic thinking and to foster this lens in their classroom instruction. In addition, particular attention was paid to algebraic generalization; the Fellows were assigned, on more than one occasion, to lead their classes through figural-pattern tasks.

This work with Noyce and my participation with this cohort of teachers, in conjunction with my own teaching experience, was the stimulus for my dissertation study. These teachers impressed me with their thoughtful comments, nuanced questions, and critical eye in the classroom. In addition, the community that formed projected a collective spirit of genuine investment in attending to student thinking and changing pedagogical approaches so that student ideas were more visible in the classroom and instruction aligned with and built on student thinking.

Furthermore, as part of my role with Noyce, I visited the classrooms of many of



these teachers, watched videos of their students and their instruction, and participated in activities designed to help them unpack their students' engagement with these tasks. Through this work, I witnessed first-hand how challenged students, in particular students from low-income schools who have been historically marginalized, were in transitioning their interpretations of the figures into meaningful understandings of the associated algebraic expressions.

Having personally experienced both the importance and challenges associated with algebraic generalization, as well as having access to a group of teachers with unique experience and understanding, I sensed the fortuitous opportunity these circumstances presented. From this context my dissertation emerged. Leveraging this opportunity, I became interested in exploring this group of teachers' understandings of algebraic generalization as well as their enactment of this knowledge in the classroom when they strove to create instructional environments in which student thinking played a central role.

As such, I sought to investigate the understanding of algebraic generalization of a subset of this group of teachers. Investigating the understandings of these teachers who have substantial experience in algebraic reasoning can provide unique insight into teachers' perspectives and understandings of algebraic generalization. This study affords opportunities to develop not only more nuanced understanding of the challenges associated with algebraic generalization but also productive components of instructional landscapes in which teachers have already achieved success. Drawing on their experiences, these teachers are better equipped than most classroom teachers to articulate with specificity their observations about the effectiveness of particular strategies as well

as areas that continue to elude them. Although a study of a more representative group of teachers might provide more generalizable information, this study is designed not to document the current status but to identify particularly challenging areas as well as productive ways to support teachers moving forward. By analyzing the understanding that these teachers hold, I hope to use their experience and knowledge to develop a more informed awareness of the areas of success and struggle that await all teachers when they engage students in algebraic generalization.

My overall purpose in this dissertation is to explore different aspects of the understanding possessed by this select group of teachers with regard to algebraic generalization. To do so, I will conduct the study in two stages. In the first part, I aim is to investigate aspects of the teachers' understanding of the mathematics content related to algebraic generalization. The second part is dedicated to analyzing the teachers' understanding of student thinking associated with algebraic generalization and how this guides their instruction and in-the-moment decision making in the classroom. To be clear, these two types of knowledge are not mutually exclusive, as much of the associated understandings overlap. The distinction I draw is useful, however, for organizing the study. These two areas of knowledge differ to the extent that I will draw on different theoretical models to analyze them.

### **Teachers' Understanding of Algebraic Generalization**

I begin this section with an overview of the research related to algebraic generalization to highlight the need to study teachers' understanding within this domain. I then provide one vignette, analyzed from two perspectives, to frame the first two of my three research questions.

## **Algebraic Generalization**

The importance of generalization and its role in algebra have been noted by many scholars. For example, Lee (1996) maintained that “algebra, and indeed all of mathematics is about generalizing patterns” (p. 103). Others have established generalization as the basis of algebra, defining *algebraic reasoning* as the ability to generalize and use formal symbols to express generalities (Carpenter & Levi, 2000; Kaput, 1999). Furthermore, over the past two decades, generalization has been established as the center of a new conceptualization of algebra, one that diverges from its traditional image as a subject consisting of primarily symbol manipulation procedures (e.g., Kaput, 2008; Smith 2003). At the forefront of this reform is an effort to incorporate algebraic instruction into the elementary grades by *algebrafying* early computation through generalization (Kaput, 1998).

The significance of generalization is also evident in the research literature; many researchers have explored this topic. Most researchers have focused on students’ understanding of generalization, attempting to develop a better understanding of the challenges that students face and the resources available to them to overcome these difficulties. Details of the generalization process have been the most researched area (e.g., Ellis, 2007; Radford, 2000, 2003, 2006; Rivera, 2010), leading to identification of nuanced actions and conceptual tools that support students in this process. Another area of focus has been to explore and classify various generalizing strategies (e.g., Jurdak & Mouhayar, 2014; Lannin, Barker, & Townsend, 2006; Stacey, 1989). These strategies are ways of thinking, both productive and unproductive, in which students engage when presented generalizing tasks. Finally, a third focus has been on classifying task-related

features according to how they tend to affect students' abilities to generalize (Rivera & Becker, 2008). Through this line of inquiry, researchers have identified characteristics of figural decompositions that make them more accessible as well as those make them more challenging for students.

**Two gaps in the literature.** Although many researchers have investigated generalization, in my review of the literature, I found a limited attention in two areas. The first area is studies in which researchers extend their analyses of generalization to include the symbolization of generalized patterns. The second gap in the generalizing literature is in the analysis of teachers' understanding of algebraic generalization. In the two sections that follow, I elaborate on each of these areas and provide a rationale for further investigating each.

***Gap 1: Extending analyses of generalized patterns to include their symbolization.*** Several researchers have articulated the difference between the act of generalizing and that of formalizing this thinking through algebraic symbolizing (Blanton & Kaput, 2001; Mason, 1996; Zazkis & Liljedahl, 2002), but few researchers have explicitly extended their analysis of the generalizing process to the symbolic stage. In many studies, students might have generated symbols, but the focus of the analysis was on the generalizing process, as opposed to being on meaning students developed of the algebraic symbols through the generalizing process. The one notable exception is the work of Radford (2000; 2003; 2008), who has written extensively on the semiotic systems that are involved in bridging this gap. In addition, MacGregor and Stacey (1997) documented the difficulties students have symbolizing, in particular, highlighting typical errors associated with mastering algebraic syntax.

Although most of those researching generalization have treated the ability to symbolize ones' generalization as separate from the act of generalizing, choosing not to focus their examination on this component, I believe that the inclusion of such understanding as a natural extension of generalization could be beneficial, especially inasmuch as studies in the field continue to show the difficulties students experience with developing a meaningful interpretation of algebraic symbols (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Sfard & Linchevski, 1994). Therefore, throughout this work, I use the term *algebraic generalization* not only to refer to different acts of generalizing but also to include the symbolization of the generalization. In particular, I define *algebraic generalization* to mean the process by which students come to use and understand algebraic notation as communicating a generalized pattern.

Such a definition aligns strongly with work of Radford (2008), who distinguished algebraic generalization from arithmetic generalization and naïve inductions. He used the term *arithmetic generalization* to indicate the observation of patterns without the ability to symbolize them and *naïve inductions* to denote the act of generating expressions through means other than inferring a commonality in the pattern. In addition to these distinguishing features, one further distinction between my view of algebraic generalization and that of Radford is that for me, the process of algebraic generalization is based not on simply observing a pattern but also on an understanding that is linked closely with the figure. As such, algebraic generalization results in the specific interpretation of algebraic expressions as symbols that communicate the quantities and quantitative relationships that have been conceptualized within the figure. Such an understanding is referred to in the second mathematical practice of the CCSS (2010) as

the ability to contextualize notation. Students who possess this understanding can identify the referents for the symbols involved; further, they can compute with the symbols but also attend to the meanings of the quantities they represent.

***Gap 2: Analysis of teachers' understanding of algebraic generalization.*** The second gap in the generalizing literature is in the analysis of teachers' understanding in the area of algebraic generalization. Most of the limited number of studies in this area have been dedicated to investigating preservice teachers' (PSTs') abilities to generalize (Liang Chua, & Hoyles, 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002). In the few examples I found, their results indicated that secondary level PSTs possessed strong and flexible facilities with figural generalizing tasks, whereas elementary school PSTs tended to struggle.

Beyond studies of teachers' abilities to generalize, I found only two additional studies in which other aspects of teachers' understanding of algebraic generalization were explored. Mouhayar and Jurdak (2013), in their quantitative analysis of in-service teachers' understanding of student thinking, did not characterize the teachers' understandings but rather reported on their success rates. They found that just fewer than 50% of the 83 teachers could appropriately explain the meanings of student-generated generalization expressions. Callejo and Zapatera (2016) studied prospective elementary school teachers' professional noticing of students' generalization and found that the majority of the PSTs studied failed to identify significant mathematical elements about students' generalization strategies. Only 5% of these PSTs were able to provide detailed interpretations of the students' strategies, with 50% able to include some specifics.

Given the proliferation of studies on generalization, I was surprised to find a limited number of studies exploring teachers' understanding of algebraic generalization. Although the field has deemed algebraic generalization a cornerstone topic for algebra, and for mathematics in general, we math teacher educators have limited knowledge of teachers' understanding or interpretation of the topic. Such a void in the literature highlights a strong need for us to better understand this topic from the teachers' perspectives. I now provide a vignette to further highlight this need as well as to introduce my first two research questions.

**Vignette #1.** Lobato, Hohensee, and Rhodehamel (2013) explored the relationship between students' collective classroom focus and their postinstructional reasoning. In their report, Lobato et al. chronicled the decisions of a Grade 7 teacher during a class episode in which students were exploring a rule for the  $n$ th case of the pattern shown in Figure 1.1.

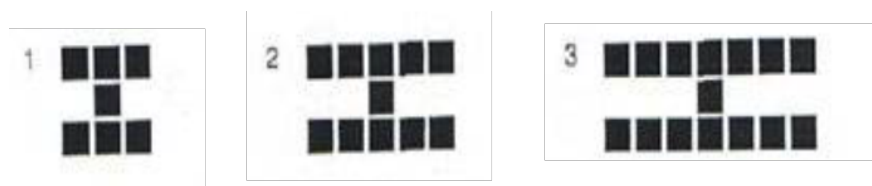


Figure 1.1. Figural pattern used by Lobato et al. (2013).

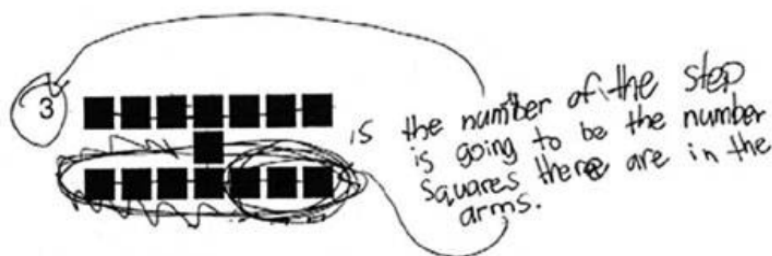


Figure 1.2. Domingo's association of the stage number and the squares per arm.

The following sequence occurred during Day 3 of a unit on linear figural patterns.

In the first two meetings the students appeared to have viewed the previous two patterns recursively, noting only the change in number of squares for figures in consecutive steps without attending explicitly to the step number. On this day, though, a shift seemed to have occurred. Students began to connect the step number with the pattern. This transformation was captured by Domingo, who noticed, "The number of the step is going to be the number [of] squares there are in the arms" (p. 833). He circled the stage number (3) as well as three squares in one arm of the associated figure (see Figure 1.2). He then explained his understanding of the rule to find the number of squares per figure, stating, "You multiple [sic] the number of the step by the number of arms, then add the 3 middles" (p. 833). As Lobato et al. (2013) noted, the identified association of the number of squares per arm and the stage number was promising because it explained the multiplicative relationship between 4 and  $x$ . Using Domingo's description, one can interpret the expression  $4x + 3$  as follows: The 4 is the number of arms, the  $x$  represents the number of squares per arm (also equivalent to the stage number), and the 3 corresponds to the three middle squares that remain constant. As such, the multiplication of 4 and  $x$  is sensible because it results in a new quantity, namely the total number of squares in the arms at a given stage. With this understanding, the 4 can be viewed as the ratio of the number of objects in the arms to the step number.

The teacher, though, chose not to explore and cultivate this perspective and instead projected a different conception of the quantities involved. In the discussion that followed, she replaced the student's use of "number of arms" with "growth." Later, when the class revisited a second pattern made of toothpicks (that can be generalized with the expression  $3n + 1$ ), she repeatedly asked the students to identify by how much the pattern



was growing and to determine the number of toothpicks in Step 0. Eventually, she wrote the expression  $3n + 1$  on the board, labeling the 3 "growth," the  $n$  "Step #," and the 1 "Step 0." While continuing through the unit, she repeatedly used the language of "growth" to describe the coefficient of the variable, language eventually adopted by the class. Such a construal of the coefficient detached from the figure is problematic. For example, the number 4 in the previous pattern, which the teacher referred to as growth, actually relates to the number of arms in the figure, which is, in fact, constant. Thus, the growth interpretation is confusing for a student trying to consolidate his or her understanding of the various representations to develop deeper meaning of the notation. In addition, the teacher's approach proceduralizes the acts of generalizing, providing a specific method to arrive at the function, rather than using the task to develop meaning of either the concept or the mathematical act of generalizing.

***Teacher's interpretation of algebraic generalization.*** Although this vignette shows only a single instance, I present the episode as a somewhat representative case because it highlights questions I have concerning teachers' understanding of these tasks. In reflecting on this vignette, I wondered about the teacher's instructional decision to reframe the quantities involved and provide the students with a precise procedure for writing an algebraic expression. Instead of pursuing and building on the student's understanding of the figure, she directed the class to a straightforward way (although inconsistent with the figure) to identify the coefficient of  $x$  and the constant term. Although this instructional approach gave the students a relatively direct procedure for generating a symbolic expression, the teacher's intervention in effect deterred the students from continuing to engage in productive algebraic generalization. The final

algebraic expression was not a reflection of the students' generalizations. Furthermore, such an approach presented a view of algebraic notation that is free from meaningful quantitative referents, what Harel (2007) referred to as *nonreferential symbolic* thinking.

I can imagine multiple reasons for this teacher's instructional choices and various lenses through which one could explore her approach. Perhaps the teacher's motivation, her primary instructional goal surrounding these figural patterns, was to provide the students a method to produce a symbolic expression that can be used to find the number of dots for a given stage. As noted, she seemed to interpret generalization as synonymous with writing an expression, and she directed students accordingly. She conflated the symbolic representation with the act of generalizing, interpreting generalization as the resulting algebraic expression.

**Research Question 1.** Although many teachers engage their students in activities involving figural patterns, the ultimate goals driving their instruction are unclear. As this vignette indicates, teachers may believe that they are supporting students in algebraic generalization by guiding them to the symbolic expression, but their actions might, in fact, limit students' engagement in the generalizing process if their instructional goals are focused on other areas. In the first part of this study I aim to gain insight into this area by pursuing the following research question.

RQ1. What mathematical goals do teachers who have experience with student thinking, in general, and algebraic generalization, specifically, associate with figural patterns?

My primary intention in attempting to answer this question is to explore what teachers view as the instructional purpose of these tasks and what mathematics they interpret as associated with these figural patterns. Because more and more teachers are being

encouraged to use these tasks with their students, understanding how teachers perceive their purpose is important for those of us in the field.

In addition, a significant component of my reason for posing this question is my desire to investigate the views of teachers who have had significant exposure, prior to this study, working with students attempting to generalize figural patterns. As I stated earlier, my interest stems from working with the Noyce Fellows, teachers who possess a unique set of experiences and expertise. Although extrapolating from such a group of teachers to draw conclusions about the general population is difficult, I believe that studying such a group of teachers can provide unique insight. Drawing from their experiences of having engaged with such tasks and reflected on their students' understanding as a result of using these tasks, they will be better equipped than other teachers to speak reflectively about how they interpret the associated purpose. Consequently, their responses will serve to provide a nuanced analysis of what teachers see in these activities and in what ways their visions correspond with algebraic generalization as I have defined it.

**Mathematical knowledge for teaching.** Although educators have long agreed that content knowledge is critical to effective instruction, what constitutes this knowledge has been ill defined (Ball, Thames, & Phelps, 2008). This debate shifted significantly when Shulman (1986) proposed a new type of subject-matter-specific professional knowledge, which he referred to as *pedagogical content knowledge* (PCK), which he conceptualized as a bridge between specific content knowledge and more general knowledge about the practice of teaching. He argued that effective teaching requires more than a thorough knowledge of conventional mathematics content. Teachers must

know the specific ways of representing and formulating the subject that make it comprehensible to others (Shulman, 1996). In the decades since Shulman made his contribution, a growing number of researchers in the field of mathematics education (e.g., Hill et al., 2008; Silverman & Thompson, 2008) have adapted and refined his original conception into what is now characterized as *mathematical knowledge for teaching* (MKT), the knowledge necessary for successfully teaching mathematics.

Ball and colleagues (Ball et al., 2008) introduced one, now well known, model in which they expanded the previous two areas (subject matter knowledge and pedagogical content knowledge) into six distinct, yet connected, categories within MKT. In developing this model, Ball and colleagues took as their charge the identification of various areas of knowledge that are enacted during *high-quality* mathematical instruction. As such, this model has been useful for recognizing types of teacher knowledge that might not previously have been well recognized and consequently are less developed. One significant contribution of this model was the identification of *specialized content knowledge* (SCK) as a type of knowledge distinct from common content knowledge (CCK). Ball and colleagues described SCK as mathematical knowledge specific to teaching (Ball et al., 2008). Teachers use SCK in their instruction, but do not explicitly teach it. Within this broader definition they highlighted specifically the ability to "interpret and analyze student work, provide a mathematical explanation that's intelligible to young learners, and forge links between mathematical symbols and pictorial representations" (Hill & Ball, 2009, p. 70).

***Teachers' quantitative understanding of algebraic symbols.*** Viewed again, but this time through the lens of specialized content knowledge (SCK), the teacher's actions

in Vignette #1 and the understanding they reflect underscore the role and importance of a particular type of SCK in the area of algebraic generalization. Although the teacher demonstrated the CCK to correctly generate a functional rule, she may not have possessed the SCK necessary to reason quantitatively in relation to this problem and consequently build on students' developing conceptualizations to foster a meaningful understanding of the notation. Although such knowledge might not be significant or necessary for the teacher to write the linear function (and she could have written the function without generalizing the pattern herself), I argue that this knowledge is needed by students who are learning to both generalize algebraically and develop their understanding of variables and algebraic notation.

I argue that to help students broaden their often fragile understanding of algebraic symbols, teachers need a specialized mathematical understanding that enables them to draw meaningful connections between the numerical and algebraic symbols being used in class and the specific quantities in the figure the symbols represent. As explained earlier, the interpretation of the notation promoted by the teacher in Vignette #1 did not accurately correspond to the context. Without connecting the symbolic expression with the figural quantities (and to her students' mathematical ideas), her projected interpretation was detached from any substantive meaning related to the figure, making combining symbols in a contextually meaningful way impossible. Furthermore, without relating the algebraic representation to the quantities involved, as Lobato et al, (2014) noted, the students not only developed a limited view of generalizing and the resulting algebraic notation but also struggled to cultivate an understanding of rate as the multiplicative relationship between two quantities.

**Research Question 2.** An analysis of Vignette #1 through the lens of SCK highlights the need for teachers to possess a specialized understanding of mathematics that enables them to correctly interpret and build on student thinking. In the particular domain of algebraic generalization, teachers need to be able to relate, with precision, the various mathematical representations to the contextual quantities they represent to help students develop deep understandings of the algebraic tools and concepts involved. As I elaborate in chapter 2, scholars have developed theoretical rationales to explain the importance of expressing the meaning of numeric and algebraic figures as a skill foundational to the understanding of mathematical notation, a skill that should be actively fostered with students.

This understanding seems especially important in student-centered, reform-based classrooms. In such an environment, teachers must possess the specialized content knowledge to quickly interpret a myriad of student responses to effectively build from and expand students' conceptualizations. Often, when students engage in inquiry-based activities, their thinking is naturally embedded in the context provided. Therefore, to help students effectively bridge the gap between their developing understandings and the formal mathematical concepts and representations, teachers must possess the SCK to connect the algebraic symbols to the quantities in the context and to do so with precision. Furthermore, to help students navigate the challenges associated with quantitative understandings of algebraic notation, teachers must be familiar with typical misconceptions as well as possible constructive conceptualizations to readily identify conceptual difficulties and eventually support students to overcome them.

Analyzing this specialized content knowledge in teachers is at the heart of my

second research question, which follows:

RQ2: What conceptual hurdles do secondary school teachers encounter when they attend to the quantitative meanings of algebraic symbols that result from algebraic generalization, and what conceptualizations support them to overcome these challenges?

To answer this question, I seek to develop a better image of the SCK teachers possess, and need to possess, to express the meaning of the numeric and algebraic symbols used in pattern generalization in terms of the associated figures. Again, by investigating a collection of teachers with substantial experience in this area, I hope to identify not only particular challenges associated with connecting algebraic symbols to the context but also beneficial conceptualizations teachers have formulated to overcome these conceptual difficulties. To date, because researchers have rarely incorporated the act of symbolization in studies of generalization or focused on teachers' understanding of algebraic generalization, little is known in this domain. Answering this question would address this gap.

Furthermore, to develop a broader view of teachers' understanding, as part of my analysis, I will investigate teachers' understandings not only of individual symbols but also of collections of symbols. For instance, I am curious how teachers interpret not only the 4 and the  $x$  in the linear example  $4x + 3$  from Vignette #1 but also the combination of  $4x$ , including the quantitative relationship between the two symbols. In addition, I hope to capture teachers' understandings of the symbols throughout various stages of algebraic manipulation. In mathematical modeling, the conventional method is to quickly translate the problem into abstract symbols and then tackle the problem syntactically, before transitioning the final answer(s) back to the contextual situation.

Although such an approach might be the preferred method for experts who possess the experience and syntactic understanding to manipulate symbolic forms with ease, I argue that it is not for students who are only developing an understanding of algebraic notation. Therefore, I will ask participants to comment not only on their interpretations of various initial symbolic rules but also on intermediate expressions that arise when they act on the symbols syntactically. I am interested to see to what degree the participants can reorganize their interpretations of the figural shapes to (if necessary) match these manipulations.

### **Teachers' Understandings of Students' Thinking of Algebraic Generalization: A Guide to Responsive Teaching**

In the previous section, I explained my interest in exploring the teachers' understandings of the various aspects of the subject-matter knowledge relevant to algebraic generalization, including the teachers' associated goals and their abilities to interpret the resulting algebraic symbols quantitatively. These two topics are somewhat isolated in nature and, as such, can be conceptualized, as well as analyzed, as independent knowledge bases. I now transition to the second part of my study in which I focus on teachers' understanding of student thinking, in particular the type of understanding that enables teachers to effectively respond and build on students' ideas during classroom instruction.

To situate this part of the study, I first describe the rich, student-centered nature of instruction that frames the type of understanding I want to explore. Second, I clarify the role that professional noticing of students' mathematical thinking plays in this type of instruction. Third, I describe the nature of the detailed and structured knowledge of



student thinking teachers need to effectively engage in responsive teaching and, by association, professional noticing. Finally, I provide a second vignette to illustrate the complexity at the intersection of these three components, highlighting that all three lenses are necessary to understand what guides the in-the-moment decisions of teachers when they aim to support and respond to students grappling with algebraic generalization.

### **Responsive Teaching**

Traditionally, teacher expertise has been measured by the teacher's ability to clearly explain mathematical ideas and demonstrate efficient solutions. Teachers applied their mathematical understanding to develop and present a cohesive sequence of instruction (Resnick & Ford, 1981). Teachers' decision making took place primarily during the planning stage and was based almost exclusively on teachers' understanding of the subject matter, not on their interpretations of the students' conceptualizations. Over the past two decades, with the research community's establishing a strong link between student understanding and teachers' attention to student mathematical thinking (Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Sowder, 2007), a new pedagogical model has emerged. This new model is characterized by adaptive and responsive teaching whereby the instructional focus is equally shared between the content and the students' varying conceptualizations of the material (Ball, 1995; Hammer, Goldberg, & Fargason, 2012; Sherin, Jacobs, & Philipp, 2011). Such a model requires teachers to take on a different instructional perspective, one in which teachers must strive to be attuned to student thinking and flexibly react to the often unexpected student ideas that might arise (National Council of Teachers of Mathematics (NCTM), 2014). Teachers must honor and use students' experiences as resources for instruction, negotiating a balance between

students' ever-changing individual conceptualizations of the mathematical ideas and the conventional demands of the discipline.

### **Professional Noticing of Students' Mathematical Thinking**

Meeting such lofty instructional goals is no doubt challenging, placing new demands on teachers. To make sense of the various developing mathematical ideas in the classroom and quickly decide what is mathematically and pedagogically relevant requires a variety of skills and understanding. One approach researchers have taken in attempting to capture and understand the resources necessary to effectively engage in responsive teaching is through the lens of professional noticing. The ability of teachers to quickly filter, make sense, and ultimately navigate the multitude of demands placed on them in the moment of classroom instruction has come to be known as *professional noticing* (Sherin et al., 2011). Professional noticing of students' mathematical thinking is a specific type of noticing that does not simply incorporate how teachers process their instructional environments in general but, rather, is focused explicitly on how they attend to, interpret, and decide how to respond to children's mathematical thinking, a framework conceptualized by Jacobs, Lamb, and Philipp (2010). In their work they emphasized that the three practices occur concurrently and are conceptually associated. In the midst of instruction, teachers do not simply take note of and reflect on the classroom without acting. The profession inherently involves making decisions about how best to move the class forward. "The work of teaching orients teachers to constantly consider their next moves" (Jacobs, Lamb, Philipp, & Schappelle, 2011, p. 99). Teachers notice in order to act on these observations (Erickson, 2011). They look for particular information, so they can act in particular ways. From this point of view, all three components come together

to constitute teachers' professional noticing and must be examined together to provide a full picture of how teachers engage in this practice.

### **MKT for Responsive Teaching: A Trajectory of Student Thinking**

The ability to quickly interpret and react to the mathematical details, patterns, and nuances in the students' strategies also requires a particular type of knowledge. To date, the details of the type of understanding necessary to effectively respond to and build on student thinking has only begun to be studied. Although some scholars have noted a link between teachers' MKT and their professional noticing, the majority of researchers have analyzed teachers' professional noticing as a practice without exploring any direct connection to the underlying knowledge. One exception is Tyminski et al. (2014), who interpreted their results of professional noticing through a lens of MKT. They highlighted how specific components of teachers' specialized content knowledge, knowledge of content and students and knowledge of content and teaching (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008), must be held for teachers to notice particular aspect of students' thinking. In the end they posited, "Engaging in the professional noticing of children's mathematics, requires that these knowledge bases are accessed and utilized in a coordinated and integrated manner" (Tyminski et al., 2014, p. 6). Similarly, other scholars have noted that effectively responding to novice students' often messy and emerging mathematical ideas requires not only a broader and more nuanced knowledge but also a much more integrated organization of the various knowledge bases (Gess-Newsome, 1999; Silverman & Thompson, 2008). Likewise, Ball and her colleagues asserted that the usefulness of a teacher's content knowledge for teaching is determined not just by the mathematical knowledge a teacher holds but also

by how it is held (Ball et al., 2008; Hill & Ball, 2004). In particular, to negotiate a balance between students' ever changing individual conceptualizations of the mathematical ideas and the conventional demands of the discipline, teachers' understandings of the subject matter must be highly connected with their understandings of students' mathematical thinking (Philipp et al., in press).

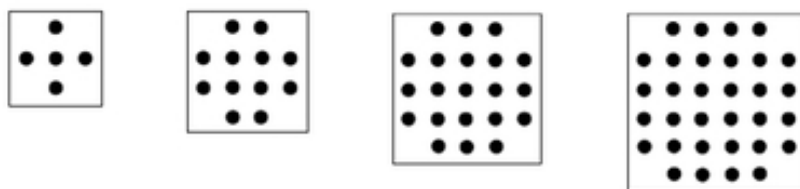
To support teachers in developing the understanding necessary for such a student-centered form of instruction, several professional development projects have been established. Most notable is the Cognitively Guided Instruction project (Carpenter, Fennema, Franke, Levi, & Empson, 2000), a research-based teacher professional development program for elementary school mathematics teachers. Built on extensive research in children's early mathematical thinking, this work has been designed to provide teachers opportunities to explore student thinking, understand how children's ideas about arithmetic develop, and plan ways to build on students' knowledge in their mathematics instruction. A key characteristic of this model of professional development is that teachers are provided a student-thinking framework in which strategies are categorized into a hierarchy of ever more sophisticated ways of thinking (National Research Council, 2007; Sherin, 2014). Such frameworks can be powerful for teachers in explicating nuances of student thinking, thereby framing student learning as a continuum, rather than simply as right or wrong (Furtak, 2009). Furthermore, they can be used by teachers to interpret student understanding, enabling them to better attend to and interpret the details of student thinking in the moment and even make advance predictions (Wilson, Mojica, & Confrey, 2013).

At the secondary level such trajectories of student thinking are uncommon. Teachers attempting to engage in responsive teaching are unable to rely on well-tested frameworks to guide their instruction and explain how they might leverage and build on students' thinking to structure student learning. Consequently, secondary school teachers must learn to navigate the mathematical terrain on their own. To do so effectively, they must develop and organize their own map of potential paths for how students' informal ideas might move them toward more and more sophisticated understanding.

**Vignette #2.** Although detailed and well-structured knowledge for secondary-level topics is undoubtedly rare, the extent of this challenge became even clearer to me near the end of my first year working with the Noyce teachers when I was invited by one of the Fellows, John, to observe his instruction on algebraic generalization. Although I visited John relatively early in the program, the teachers had participated in several professional development sessions focused specifically on algebraic generalization. In particular, they had engaged in multiple activities designed to help them unpack the conceptual phases through which a student might progress in attempting to generalize a pattern and generate a formal symbolic expression. As a follow-up assignment to one of these sessions, John had used one of the tasks from the professional development with his ninth-grade algebra class. He was so pleased with the level of participation and the variety of student ideas generated around the task that he was eager to teach a similar lesson, but this time using a pattern represented by a quadratic relationship.

Before the 90-minute class, I asked John to explain what he saw as the goal(s) for the lesson, which he described as demonstrating equivalence of different expressions by

illustrating how students predict the number of squares in the figure. John showed me the task shown in Figure 1.3 and highlighted two decompositions he anticipated his students would produce and that he wanted to compare. (These decompositions are captured by the algebraic expressions  $n^2 + 4n$  and  $(n + 2)^2 - 4$ .)



- a) Draw **FIGURE 5** and write a sentence describing how you drew **FIGURE 5**.
- b) Write a number sentence describing the number of dots in **FIGURE 10** and **Figure 50**.

*Figure 1.3.* John's quadratic generalization task.

John began the lesson by presenting the problem and then giving the students 20 minutes to work individually on the task. During this time, he observed the students' work and asked questions for clarity. I observed that although the students were fully engaged in exploring the figure, trying to make sense of the pattern on their own, most were struggling. Except for three or four students, the class was analyzing the pattern recursively, trying, with limited success, to articulate the pattern using what they interpreted as an associated number sentence.

John then asked two students to share their approaches for Figure 50, strategies that corresponded to the two he had identified beforehand ( $52^2 - 4$  and  $50^2 + 4 \times 50$ ). While he listened to the students present their thinking about the patterns, John questioned them repeatedly to ensure that their reasoning was clear to the class as a whole. In addition, he probed the two students so that they articulated how their numerical expressions were connected to the figures, attempting to embed their thinking

in the diagram. This exchange took 20–30 minutes.

After these two students had shared their approaches, a third student offered yet another way to describe the number of dots in Figure 50, as  $50 \times 54$ . When asked to explain her solution, she said that it gives the same answer. John then wrote this expression on the board as a third possibility but asked no further clarifying questions. He then moved on, quickly providing the symbolic representation for the three ways  $[(n + 2)^2 - 4, n^2 + 4n, n(n + 4)]$  and using algebraic manipulations to show their equivalence.

Reflecting on this experience, I noted John's ability to elicit student thinking and use their mathematical contributions to structure the lesson. The overall thinking guiding the class was student generated, indicating that John was transitioning to a responsive style of teaching. At the same time, I was surprised to see where John concentrated his instructional energy, because his attention did not seem to align with the predominant students' mathematical ideas that arose during the lesson. Whereas I observed the students grappling with such key mathematical ideas as the difference between recursive and explicit thinking, developing a meaningful understanding of notation, and equivalence, John did not focus on these issues. From my perspective, he had identified and effectively presented the thinking of the few students who successfully generalized the pattern but did not engage with the ideas of the rest of the class. He seemed able to recognize and sequence productive student thinking that he expected but did not attend to the needs and insights of other students who were struggling or had alternative views.

At the end of the lesson, John made a telling comment that seemed to shed light on the difficulties he perceived navigating the instructional landscape within this model

of instruction. Before leaving his classroom, I asked him how he thought the lesson had gone. He initially smiled and then turned a bit reflective, responding, "I am asking questions and I am wondering, 'Why I am asking this question? What I am looking for?'" Upon hearing this reply, I began wondering what understanding of the mathematics had guided John's instruction. In particular, I was curious about what type of conceptual map he possessed of students' algebraic generalization and how this map had informed what he attended to and, ultimately, his instructional decisions. His understanding of the topic seemed adequate to structure the thinking of those already productively moving forward, but he had not sufficiently unpacked the generalization process to effectively attend to and respond to the needs of other students struggling to do so.

When teachers begin to shift their teaching practices from a purely content focus to a more responsive instructional model, they seem to need a different type of understanding of the mathematical landscape, one that includes a highly detailed and well-structured understanding of the mathematics from the students' perspective. To effectively engage in professional noticing of students' mathematical thinking by recognizing seeds of productive thinking, making decisions about what ideas to pursue, and acting on this information, teachers must possess a clear idea of not only their ultimate goals but also intermediate conceptualizations students may develop or need to develop. They must have incorporated in this knowledge understandings of the questions and interventions that might support the desired ways of thinking. In effect, they must possess some type of internal learning trajectory (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). Such in-depth and cohesive knowledge is no doubt difficult to attain, requiring not only experience with students grappling with a particular topic but also a



particular perspective to attend to and understand the details of students' mathematical ideas. Reconsidering my experience with John, I recognized that he was a carefully selected teacher who had already experienced more than 10 days of professional development focused on student thinking, and he had access to a rich task designed to elicit student thinking. If under these circumstances, conditions that seem favorable to promoting responsive teaching, John was still struggling to elevate the mathematics embedded in the activity and guide students in their learning, then how might other teachers who have less training and who do not hold the same orientation to student thinking develop the necessary understanding?

Furthermore, I argue that such knowledge is particularly elusive within the content domain of algebraic generalization. As research indicates (Liang Chua & Hoyles, 2009), most secondary school teachers possess, as did John, the understanding to flexibly generalize patterns and communicate their understanding of the figure symbolically. At the same time, students' struggles to both generalize and develop meaningful understanding of algebraic notation are well documented (Kieran, 1992; Knuth et al., 2005; Sfard & Linchevski, 1994). Such a discrepancy highlights the significant challenge that teachers face when they try to unpack their highly compressed knowledge of algebraic generalization and identify the constituent elements that students struggle to develop (Ball & Bass, 2000; Thompson & Thompson, 1996). It is this challenge that provides the context for my final research question.

**Research Question 3.** My main goal in this part of my study is to characterize the understanding necessary for teachers to effectively support and respond to students who are learning to generalize algebraically. As such, the final research question that will

guide this investigation is as follows:

RQ 3. What subtle differences in knowledge and practice distinguish two teachers who use students' conceptions as the bases of their instruction when they strive to engender in their students meaningful understanding of algebraic generalization?

A component essential to investigating the type of detailed and organized understanding necessary to effectively engage in responsive teaching is an analytic comparison of two well-chosen teachers attempting to support their students in algebraic generalization. My hope is that by juxtaposing two teachers who are similar in their understandings of algebraic generalization and in their orientations to responsive teaching, I will be able to identify the subtle differences in their understandings of student thinking and their professional noticing.

### **Recapitulation of Research Questions and Overview of Dissertation Study**

My overall purpose for this dissertation is to characterize the type of understanding that supports teachers when they strive to effectively respond to students attempting to generalize a figural pattern and develop a quantitative understanding of the associated algebraic notation. To do so, I explore teachers' understandings of algebraic generalization from three perspectives, guided by the following research questions:

1) What mathematical goals do teachers who have experience with student thinking, in general, and algebraic generalization, specifically, associate with figural patterns?

2) What conceptual hurdles do secondary school teachers encounter when they attend to the quantitative meanings of algebraic symbols that result from algebraic generalization, and what conceptualizations support them to overcome these challenges?

3) What subtle differences in knowledge and practice distinguish two teachers who use students' conceptions as the bases of their instruction when they strive to engender in their students meaningful understanding of algebraic generalization?

In chapter 2 of this study, I include an overview of the ways in which mathematics education researchers have conceptualized and explored various topics relevant to these questions. In particular, I highlight the importance of generalization in mathematics and its role in algebra. I then review the research on students' understanding of algebraic generalization, followed with a summary of theories put forward to explain the role representations play in algebra. Finally, to conclude chapter 2, I present conceptualizations of professional noticing along with a summary of relevant findings. In chapter 3, I describe the research design of the two-part study reported here.

Chapters 4 and 5 are reports of the results of the study. Chapter 4 has two parts. In the first section, I present a characterization of the goals teachers associate with algebraic generalization, illustrating that they possess narrow interpretations of the mathematics involved in generalizing tasks, interpretations that do not incorporate developing a meaningful understanding of the notation. In the second section, I describe the challenges teachers face as well as various conceptualizations teachers have when connecting algebraic notation to quantities in the figure. Finally, in chapter 5, I offer an empirically grounded framework to conceptualize a learning trajectory of the ways students engage in algebraic generalization, including how they use and think about associated representations. I also include an analysis of ways teachers implement this progression on the bases of details of their professional noticing of students' mathematical thinking.

## **Chapter 2: Literature Review**

This chapter is organized into five main sections. In the first section, I present a reform-based, expanded notion of algebra and what it entails. In the second section, I provide an overview of literature characterizing the variety of ways students approach figural-pattern generalization as well as the instructional methods that have been shown to affect associated student thinking. In the third section I detail theories for the role representation plays in algebraic understanding, describing cognitive developments that must occur for one to develop a rich understanding of algebraic symbols. In the fourth section, I review perspectives toward the content knowledge and the pedagogical knowledge a teacher needs to effectively teach mathematics. Finally, in the last section, I outline the related noticing studies conducted to date, emphasizing researchers' conceptualizations of this practice.

### **New Notion of Algebra**

To review literature on a new model of algebra and its instruction, I first outline its key components, highlighting, in particular, the fundamental role that generalization plays. This summarized description not only illustrates the central role that generalizing tasks play in contemporary views of algebra curriculum and instruction but also contextualizes my study because I examine teachers' understanding through the lens of this characterization of algebra. Analyzing the knowledge and goals of teachers who hold a more traditional conceptualization of algebra would be very different. By developing a clear image of the current characteristics of effective algebra instruction, I can better understand areas of student thinking I should explore with the teachers, given the choice.

Historically, algebra has been characterized by the use and manipulation of symbols (Smith & Thompson, 2007). Unfortunately, this approach has failed to meet the needs of many, if not most, students while they struggle to cope with the abrupt introduction of abstract notation (Kieran, 1992). Without sufficient transitional activities or alternative resources, students often fail to see connections between their previous experience with mathematics and algebra, leading to what Herscovics and Linchevski (1994) characterized as a *cognitive gap* between arithmetic and algebra. For the past two decades, scholars have searched for alternative ways to effectively support student learning of algebra. In attempting to develop new, student-centered approaches, researchers considered what, other than simply notation, distinguishes and embodies the discipline. An expansion of the notion of school algebra that includes a focus on algebraic thinking, the type of reasoning that enables one to identify, communicate, and apply mathematical relationships, has resulted from this examination. Consequently, in the new view of algebra instruction that has emerged, developing the reasoning behind rules, rather than the procedures themselves, is emphasized. The rationale is that with a broader definition of algebra that includes student conceptualizations, teachers will be better able to recognize and develop algebraic reasoning as well as teach algebra in contexts beyond simply the instruction of syntactic rules.

At the center of this reconceptualization of algebra is the significant role generalization plays. First, in examining the difference between arithmetic and algebra, researchers have identified *generality* as a key distinguishing features of algebraic reasoning. Sfard (1995), in her study of the history of algebra, noted that historically algebraic thinking was identified as any attempt "made to treat computational processes

in a somehow general way" (p. 18). She continued, "Generality is one of these salient characteristics that make algebra different from arithmetic" (p. 18). Schliemann, Carraher, and Brizuela (2001) echoed this sentiment, describing the transition from arithmetic to algebra as a "move from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures; from computing numerical answers to describing relations among variables" (p. 145). Finally, Carpenter and Franke (2001) noted that through the act of generalizing, students are able to think of numbers as representing a class of mathematical objects and thus focus on the forms of expressions rather than their values when computed.

The identification of generalization as a core component of algebraic thinking has empowered various researchers to imagine ways to foster such reasoning at an early age. A leader in this effort, Kaput (2008), asserted that by supporting students to make their natural observations about structural patterns of computation more explicit and systematic, teachers can harness children's innate powers of generalization to *algebrafy* topics already in the elementary curriculum. Students can develop algebraic thinking at an early age when they learn to express the generalizations formal algebraic symbols convey through more familiar representational systems, such as natural language and drawings. Thus, algebraic thinking need not be wedded to formal mathematical notation, at least initially. In fact, as Smith (2003) advocated, through generalizing activities, the use of variables can and should occur naturally over a period of years. Over time, students can slowly transform their understanding into more formal, abstract systems of symbols. As I elaborate later in this chapter, scholars now believe that through this lengthy process of attempting to symbolize their generalizations through

increasingly abstract representations, students develop meanings for symbolic forms. By slowly transforming their ideas into formal, abstract systems of symbols over time, students develop deeper understandings of representations because they are linked to their own personal repeated actions and informal descriptions that occur through acts of generalization.

### **Mathematical Practices—Ways of Thinking**

Many researchers who have researched the role generalization plays in helping students develop robust understanding of algebraic concepts, and in particular algebraic notation, have posited that introducing students to algebra through generalization help students develop some of the ways mathematicians approach problems (Kaput, 1998; Kieran, 2004; Lee, 1996), conceptualized by Cuoco (1996) and Driscoll (1999) as *habits of mind* and in Common Core State Standards (2010) as *mathematical practices*. By engaging in the authentic mathematical activities of detecting and articulating relationships, students begin to adopt the mathematical norms and practices through this active participation. As highlighted by the National Research Council's report *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) *mathematical proficiency* requires more than simply a mastery of particular content; it requires familiarity with the approaches and thinking associated with mathematicians. As such, the goal of instruction should not be to simply to help students master specific mathematical topics but also to foster the norms that align with the mathematical community, what Cobb and Yackel (1996) referred to as *socio-mathematical norms*. By engendering socio-mathematical norms, teachers enable students to act increasingly more autonomously, empowering them to apply their thinking, even generate their own understanding of new and unfamiliar mathematical

topics (Cobb & Yackel, 1996).

The distinction, but also interconnected relationship, between content and practices parallels in part Harel and Sowder's (2005) conception of ways of understanding and ways of thinking. *Ways of understanding* convey the reasoning one applies in a local, particular mathematical situation. *Ways of thinking* are more global views about what constitutes valid mathematical activity and results, including beliefs about mathematics, problem-solving approaches, and interpretation of mathematical justification that can be applied to almost any subject matter. These two ideas are connected according to the Duality Principle: When students learn new ways of understanding, they can modify their ways of thinking, and, in turn, new ways of thinking affect how students come to understand concepts. Thus each idea affects the other in a cyclical process.

In terms of figural patterns, the difference between ways of understanding and ways of thinking seems to manifest itself in the ability to arrive at a generalized expression for a particular class of problems versus developing an appreciation for how the mathematics community understands the practice of generalizing. Because the teacher described in Vignette 1 focused on providing her students a method for writing a symbolic expression without deep understanding, she seemed to foster in them a way of thinking about algebraic generalization that is quite removed from mathematicians' understanding of this process. In teaching algebraic thinking through generalization, teachers must help students approach and ultimately develop ways of understanding about figural patterns that engender a global way of thinking about generalization, in alignment with the thinking of the mathematical community.



### **Figural-Pattern Generalization**

My purpose for this study is to gain insight into teachers' knowledge of algebraic generalization from multiple perspectives. In particular, I focused Research Question 3 on how teachers' understandings of student thinking in this domain shape their practices, specifically their support and responses to students who are engaging in the generalization process. As such, any analysis of teachers' understanding of students' thinking in the area of algebraic generalization requires, at a minimum, a familiarity with the understanding that students bring to the subject. I take the stance that novices' thinking, in this case students' thinking, about mathematics differs from the thinking of experts. Without the knowledge and experience that come with years of practicing mathematics, students engage in thinking that, while possessing some commonalities with that of experts, differs and, as such, cannot be measured by the field's conventional methods of analysis. This view affects my study because I am interested in the mathematics that the teachers understand, not as depicted and organized by the discipline but rather from the perspective of the students with their formative views of the material. I outline in the following section research identifying and classifying ways students have approached figural-pattern generalization as well as instructional foci that have positively affected their engagement. These findings help me better appreciate the topic from the students' perspectives and, in turn, support me in analyzing the teachers' understanding of student thinking in this domain.

### **Theoretical Perspectives on Generalizing**

The act of generalizing patterns requires attending to and identifying commonalities across cases (Kaput, 2000). Such a skill requires focusing on certain

properties and relationships while disregarding other characteristics. Attending to particular details and ignoring others is not a transparent activity but a subjective and learned action (Goodwin, 1994). When two people look at a sequence of figures or numbers, although looking at the same objects, they may see different patterns. Rivera and Becker (2008) referred to this distinction as *visual perception* versus *cognitive perception*. The former is simply a sensory activity involving the observation of objects without the perception of any specific features. The latter is characterized by the act of recognizing particular relevant facets of the object(s) and identifying properties within the object. Several studies have shown that although students easily derive generalizations, they struggle to identify relevant mathematical phenomena and generalize in mathematically productive ways (Lee, 1996; Stacey & MacGregor, 2001). Whereas generalizing is an inherent human tendency, generalizing in mathematical ways is not, and therefore must be developed.

Although most educators and researchers view generalizing as an individual, cognitive act, others have begun to take a more situated perspective (Ellis, 2011; Jurow, 2004; Lee, 1996; Smith, 2003). As Smith (2003) said, "Learning mathematics is gaining the ability to construct patterns that are compatible with, or can be communicated to, others" (p. 137). According to this perspective, mathematics, and more specifically generalizing, is a culturally mediated activity.

To see useful patterns and learn appropriate ways to express them, students must be enculturated into the mathematical community. Through their participation, students gradually become oriented to recognize what is relevant in and across situations.

Although students ultimately generalize on their own, this activity occurs in collaboration

with others in a particular social context. The participants involved, their histories, their interactions, the artifacts used, and the sociomathematical norms under which they operate affect which features they see, how they see them, and how they generalize about them (Lobato, Ellis, & Munoz, 2003). From this perspective, learning to generalize patterns in mathematically useful ways and communicate these using appropriate language and notation is ultimately an act of enculturation into the mathematical community. As such, the teacher's role is to lead students in this activity, orienting them to relevant details and guiding them to interpret and communicate the phenomena in mathematical ways.

### **Generalizing Strategies and Reasoning**

Over the past decade, while the idea of generalization as a beneficial path to algebraic thinking has been flourishing, significant research has been devoted to understanding the various ways in which students generalize. Two fundamental ways in which all patterns can be described have been recognized. One can attend either to the change between two or more successive stages, referred to as *recursive* or *covariational reasoning*, or to the structure in a given, single state, denoted as a *correspondence* or *explicit relationship* (Smith, 2003). *Covariational reasoning* requires students to identify a unit of change. Often students fail to coordinate a change observed between terms with the change in the term number, viewing the observed change as an independent quantity. Alternatively, *correspondence* entails recognizing a connection between the characteristics in a single stage and the corresponding stage number of that term, a functional relationship between the dependent and independent variables. Overwhelmingly, students tend to focus more on the recursive relationships and not look

for functional relationships (Orton & Orton, 1999; Stacey & Macgregor, 2001). Part of this tendency can be attributed to inherent qualities of these generalizing tasks because figures are naturally presented as discrete successive stages. As Harel (2008) noted in his elaboration of the *necessity principle*, without an intellectual need to reason beyond recursive thinking, students struggle to construct new knowledge. Therefore, transitioning to functional relationships is challenging for students and requires careful and well-timed instructional facilitation.

Jurdak and Mouhayar (2014) refined these areas by consolidating the research on student pattern generalization over the past 15 years, identifying five dominant strategies employed by students. In the first method, *counting from a drawing*, a student counts the elements of a particular figural term. This strategy can be used on *near generalization* tasks (Stacey, 1989), in which the student is asked only to solve for the number of elements of an early stage. Second, students can use a *recursive* approach, in which, as mentioned above, they identify the common difference between pairs of consecutive terms and use this difference to repetitively add from term to term to continue the pattern. Again, this method is more suitably applied to near generalization tasks. A third technique is called *chunking*: To extend the recursive method to far generalization tasks, a student might combine the multiple common differences between pairs of consecutive terms. By identifying the number of intermittent steps, multiplying the difference by this number, and then adding the result to the initial figural term, students can generalize beyond a few initial cases. Although not highlighted by Jurdak and Mouhayar, this technique is applicable only to linear progressions. In a fourth method, *whole-object*, a student solves for the value of a term by using a multiple of a previous term or by adding

two previous terms, a technique characterized by incorrectly applying proportional reasoning. Finally, the fifth strategy is *functional*, which, as described above, involves relating parts of the pattern directly to the figure step number. Jurdak and Mouhayar (2014) clarified that these strategies are not exhaustive, nor do they represent a sequential developmental progression. Students might never engage in a particular technique and might employ different strategies depending on the problem.

Jurdak and Mouhayar (2014) distinguished these forms of generalizing from the reasoning in which students engage when they employ these approaches, describing four levels of reasoning (unistructural, multistructural, relational, and extended abstract) that occur within these strategies. The authors distinguished levels of reasoning by the number of characteristics the student observes and implements in his or her strategy. They noted that although the particular strategies students apply do not dictate their reasoning, they definitely influence the reasoning. In consolidating their results, they found that students' responses based on the counting from a drawing, recursive, or whole-object strategies tended strongly to remain within the lower levels of generalizing, whereas chunking and functional approaches enabled students to reason at higher levels. Their comprehensive K–12 study also showed that students' use of higher levels of reasoning tended to increase with age. They attributed this development of reasoning less to students' natural maturation and more to their experiences with generalizing inasmuch as the level of reasoning demonstrated varied significantly within each grade level. Finally, the authors found that the nature of the task, specifically whether the pattern was linear or quadratic, significantly affected students' abilities to generalize, whereas the visual complexity of the figures had no statistical effect.

### **Role of Verbalization and Symbolization in Generalizing**

Many researchers have tried to identify specific practices that facilitate students' transitioning to higher forms of generalization. One area that seems to influence the type of generalizing in which students engage is the degree to which students endeavor to verbalize their observations. Stacey and MacGregor (2001) cited various examples from their research to illustrate the prominent role of verbal communication in students' pattern generalization. They first noted that those students who clearly articulated their patterns verbally in their study were significantly more likely than other students to write a functional rule. In addition, they highlighted several cases of students who were able to calculate missing terms when presented with various numerical patterns in tabular form but unable to correctly articulate the patterns verbally or in terms of algebraic notation. From this observation, they inferred that these students had begun to generalize—to make use of particular properties associated with the numerical patterns to engage in the process of computation—but had not yet developed strong structural understandings of the patterns. These students seemed unable to fully recognize the functional relationship between the numbers without having expressed the pattern verbally. Mason (1996) expounded upon this point, explaining that verbal articulation actually shifts the way we see a pattern. He asserted that although physically manipulating objects associated with the pattern helps one get a sense of the generalities, vocalizing a perceived relationship causes previously abstract notions to become increasingly concrete and manipulable. From this perspective, expressing patterns verbally is an important, if not necessary, component of generalizing at a functional level.

Although verbal communication seems significant in pattern generalization, a rush

to formal mathematical notation seems often to have the opposite effect (Rivera & Becker, 2008). MacGregor and Stacey (1997) found that students new to algebraic notation constructed nonstandard algebraic forms to describe the relationships they saw and misinterpreted algebraic letters presented to them. Instead of supporting their ideas, the early introduction of formal mathematical notation distracted from their understanding. In fact, many would argue as Smith (2003) did that "a symbolic language is more than just a tool that expresses generalizations; it is a tool that requires the learner to reconceptualize those generalizations" (p. 138). Sfard and Linchevski (1994) described this transition as an epistemological hurdle that took mathematicians centuries to overcome. As such, insisting that students who are beginning to develop algebraic thinking use formal notation limits their abilities to communicate mathematical ideas and, consequently, to think about mathematical ideas. Instead of pushing students into formal mediums, teachers need to first spend time encouraging students to articulate their ideas in other, more familiar, forms of communication. Zazkis and Lijendak (2002) supported this idea, asserting that teachers need to allow students to be more flexible in the ways they choose to communicate. They found that whereas their students actively sought ways to communicate their generalizations, their attempts to use algebraic notation were not helpful. Instead their algebraic thinking emerged through alternative forms of communicating their findings. They observed a gap between students' abilities to express generality verbally and to employ algebraic notation comfortably. Therefore, Zazkis and Lijendak (2002) maintained,

Rather than insisting on any particular symbolic notation, this gap should be accepted and used as a venue for students to practice their algebraic thinking. They should have the opportunity to engage in situations that

promote such thinking without the constraints of formal symbolism. (p. 400)

Students initially allowed the flexibility to communicate their generalizations in familiar forms of communication will engage in the act of generalizing more comfortably than others. Instead of feeling inadequate by being unable to express their ideas with formal algebraic notation, students will feel confident to express and manipulate their ideas.

Although I agree that pushing students to symbolize their understandings using a notational system can be problematic, I believe that students' struggles in using and understanding formal algebraic notation indicate an area in which teachers need to focus instructional support. Algebraic notation is a powerful tool that once mastered can create a shift in one's understanding and ability to engage in algebraic thinking (Sfard & Linchevski, 1994). As such, students' grappling with this challenge should be viewed as a both cautionary and critical area of algebraic instruction, one that teachers should not force, but at the same time should explicitly address.

### **Embedding Student Thinking in the Figure**

Another area researchers have shown to affect students' generalizing is the use of numerical versus figural approaches to generalizing. Although most students tend to approach these pattern tasks with numerical strategies, (Becker & Rivera, 2005; Krebs, 2003; Stacey & Macgregor, 2001), various researchers have seen the need to position students' thinking within the diagrams. Many studies have shown that students attempting to numerically generalize often use random and disconnected trial-and-error strategies to devise rules and tend to resist exploring other methods (Becker & Rivera, 2005; Healy & Hoyle, 1999; Lannin, 2005; Lannin, Barker, & Townsend, 2006; Rivera &



Becker, 2008; Stacey, 1989). Although students might arrive at a correct formula, their strategies are often void of understanding, and the students are consequently unable to make suitable adaptations when contextual information changes (Lannin, Barker, & Townsend, 2006). Students might, therefore, produce symbolic expressions without having generalized the situation (emblematic of Vignette 1). The focus of the exercise becomes the answer, not the process. Further complicating the use of numerical strategies, many students lack sufficient understanding of the meanings of the underlying mathematical operations to make the necessary connections. For example, unable to interpret repeated addition as multiplication, students do not see how a constant difference translates into multiplication in the general rule. In contrast, students who use figural approaches tend to demonstrate more success generalizing (Becker & Rivera, 2005; Bishop, 2000; Healy & Hoyle, 1999; Krebs, 2003; Lannin, 2005; Lannin, Barker, & Townsend, 2006). These students usually combine their thinking with numerical approaches, develop the flexibility to try other approaches, and draw possible connections between different forms of representation. Krebs (2003) noted that although students already familiar with certain patterns might effectively use numerical tables to simply recognize these known relationships, focusing on and analyzing shapes visually seemed important for students trying to develop patterns that were not well known.

The exhibited benefit of students' connecting their thinking to the figure highlights the significance of the transition from near generalizing to far generalizing. Asking students to extend their generalization into figures with larger stage numbers encourages the switch from covariational to correspondence reasoning (Zazkis, 2008) because this shift must be undertaken without the aid of diagrams, but it also encourages students to

abandon sense making and resort to inappropriate methods. As a consequence, Lannin, Barker, and Townsend (2006) argued that educators should not dismiss recursive thinking and rush students into explicit forms, but rather should strive to help students make connections between recursive and explicit reasoning, using understanding of the figure to facilitate this link. They found that many students were able to connect their recursive reasoning and explicit forms by adding a calculated number of differences to a known term, or *chunking*. Furthermore, they suggested that teachers pursue students' thinking regardless of the form and discuss its efficiency instead of pushing students prematurely into functional thinking.

### **Importance of Justification in Generalizing**

The significance that connecting numerical patterns to geometric patterns has demonstrated on students' ability to develop functional relationships points to the importance of identifying why the process occurs, rather than simply finding a pattern through empirical means. This difference is characterized in Harel's (2008) distinction of *result pattern generation* (RPG) and *process pattern generation* (PPG). In *result pattern generation*, the student simply recognizes that the pattern is correct for a select number of cases. It is a guess-and-check or procedurally oriented approach. On the one hand, in figural pattern generalization, students might try multiple formulas until one is found that is correct with the numbers they have, or they simply apply the known rule (e.g., that a repeated difference of 3 means the coefficient of  $x$  is 3). On the other hand, PPG is characterized by the ability to reason in terms of underlying structures of numerical and geometric patterns. In PPG the student is able to provide a rationale for why the pattern works. Such thinking can manifest itself within pattern generalization

through both numerical and figural approaches. A student able to express Stage 4 as  $2 + 3 + 3 + 3 + 3$  and understand that this can be written as  $2 + 4 \times 3$  because repeated addition is multiplication would demonstrate PPG in a numerical sense. Or, if a student decomposed the figure into a certain number of identified shapes and coordinated the number of these shapes with the term number, the student has used PPG in a figural sense.

In general, many studies have highlighted the critical link between justification and generalizing (Ellis, 2007; Rivera & Becker, 2008; Lannin, Barker, & Townsend, 2006). Most notably, Ellis (2007) found that when students were asked to defend their generalizations, their thinking became more sophisticated. They revisited and built upon their generalizing actions, leading to increasingly more powerful approaches. Likewise, their more sophisticated forms of generalizing led to higher forms of justification, indicating that the two processes support each other and mutually develop.

Unfortunately, students working on pattern generalizing tasks rarely take it upon themselves to justify their work naturally (Lannin, 2005). Consequently, many students tend to overgeneralize or simply speculate about relationships without engaging in the necessary critical thinking and reflecting on the appropriateness of their conjectures (Stacey, 1989). This absence of a natural inclination to justify leads to the important role that the teacher plays in compelling students to justify their generalizations. Ellis (2007) suggested rather than leaving justification to late in the generalization process, as is more of the norm, teachers should begin to incorporate justifications early in assignments to help students move ahead and avoid difficulties before they have taken root. In addition, Lannin, Barker, & Townsend (2006) observed that asking students to justify their

generalizations pushed students to remain connected with the figural representation and avoid the desire to apply a guess-and-check strategy. As such, students tended to approach generalizing tasks less from a purely numerical means and focus more on the associated quantities and their relationships. Rivera and Becker (2008) also found that students who used numerical approaches employed less sophisticated, often empirical approaches to justify their thinking, again highlighting the interconnectedness between these two practices. In Rivera and Becker's (2008) teaching experiment, the classroom norm of connecting all observed patterns to the diagram(s) and discussing them figurally became established. As a consequence, students engaged in constructing and validating their formulas at the same time.

### **Representations**

The importance of representations in mathematics has been noted by various scholars. Duval (2006) emphasized that mathematical objects are never directly accessible, and only through representations are we able to engage with mathematical ideas. As such, teachers' understanding of representation seems paramount. As I highlighted in Chapter 1, Ball et al. (2008) acknowledged the importance of such understanding, stating specifically that "recognizing what is involved in using a particular representation" (p. 400) and "linking representations to underlying ideas and to other representations" (p. 400) is part of the specialized content knowledge that teachers need to help students make sense of mathematics. Although several researchers have investigated students' understanding of representations in algebra (e.g., Knuth, 2000; Nathan & Kim, 2007, Stylianou & Silver, 2004; Swafford & Langrall, 2000), less attention has been given to examining teachers' understandings of representations and,

specifically, to their abilities to see relationships between them. Stylianou (2010) studied middle school teachers' beliefs, but not their knowledge, about the instructional use of multiple representations. Harel, Fuller, and Rabin (2008) documented ways in which teachers failed to support students to develop meaningful interpretations of symbols, but without exploring the potential causes for the failure. Although we lack empirical studies addressing teachers' understandings of mathematical representations, considerable thought has been devoted to establishing the importance and role of representations theoretically. In the following section, I summarize the main arguments for studying both teachers' abilities to draw explicit connections between representations as well as their instructional orientations to elevate these relationships in class. I begin by describing the limited role representations have played historically in algebra instruction and the more recent effort to remedy this situation. I then outline broader theoretical arguments scholars have made for how representations facilitate mathematical understanding, in particular, algebraic thinking.

### **Characterization of Current Symbolic Focus**

Traditionally, the instructional approach to algebra in the United States has been to emphasize symbols and their manipulation over other methods that make use of other representations such as tables and graphs (Kieran, 2007; Yerushalmy & Chazan, 2002). When these other representations are incorporated, they usually play a less significant role than algebraic symbols. Teachers' perspectives of mathematical representations undoubtedly largely determine the emphasis on symbolic manipulation. Stylianou (2010) found that teachers regarded symbolic representations not simply as one of the ways a concept is embodied but as the concept itself. Rather than seeing the inclusion of

multiple representations as providing students more tools to make sense of the mathematical concepts, they saw nonsymbolic representations as supplementary formats that simply add to the list of content they must teach. Furthermore, many teachers contended that nonsymbolic representations should be introduced only to the more advanced students, and some even indicated that multiple representations would be detrimental as their inclusion confuses students (Stylianou, 2010). Moreover, teachers reported a strong belief that symbol manipulation must be taught before contextual problems are introduced, what Nathan and Koedinger (2000a) referred to as the *symbol-precedence view*. This partiality to abstract symbols is built on the belief that formalisms provide the structure to guide students' conceptual understanding of algebraic concepts (Nathan, 2012). Nathan and Petrosino (2003) found that the symbol-precedence view is stronger in teachers possessing degrees in mathematics or science, ostensibly those with more subject-matter knowledge. Such a preference is mirrored and reinforced in our textbooks, in which contextual problems appear predominantly toward the end of exercises and chapters (Nathan, Long, & Alibali, 2002). This sequencing contradicts findings that show students at different grade levels solved story and word-equation problems more readily than problems presented in formal symbolic terms (Koedinger & Nathan, 2004; Nathan & Koedinger, 2000b) and demonstrated a preference for more informal strategies when solving word problems (Koedinger & Nathan, 2004; Stylianou & Silver, 2004).

The U.S. approach to instruction not only is focused heavily on symbol manipulation but also lacks systematic exposure across multiple representations. In comparing U.S. and Japanese curricular documents, Mayer, Sims, and Tajika (1995)

found that seventh-grade textbooks in the United States emphasized the coordination of different representations to a much smaller degree than Japanese texts, with the instruction of different representations often organized and addressed in separate units. Consequently, Manly and Ginsburg (2010) posited that U.S. students learn to see the various representations as disconnected, reflecting different procedures and content.

Over the past two decades, mathematics educators have urged teachers to expand the use of multiple representations in an effort to move away from the emphasis on formalisms and the symbol-precedence view of algebra. Researchers have advocated for the introduction of algebra through inquiry-based activities grounded in concrete representations such as tables, situations, and words, introducing and translating to formalisms more gradually as the need arises (Koedinger & Nathan, 2004; Nathan, 2012). Likewise, the introduction and use of various representations has been viewed as an integral part of mathematics education as noted in the past two standards documents (National Council of Teachers of Mathematics, 2000; Common Core State Standards, 2010). *The Principles and Standards of School Mathematics* (NCTM, 2000) included representations as one of five process standards, establishing it as a necessary goal of effective mathematics instruction. According to this standard, multiple representations should be incorporated at all ages, and students should be encouraged to capture mathematical concepts or relationships in forms that make sense to them, even if their first representations are not conventional ones. The emphasis is not only on flexibility but also on seeing representations as both a process and a product. The standard promotes the use of different types of representations and drawing connections among them. The CCSS (2010), in its process standards, reiterates drawing connections

between representations, stating specifically that students need to contextualize symbols by pausing "as needed during the manipulation process to probe into the referents for the symbols involved" (p. 6). Here the focus is on connecting formal symbols to context so that symbol-manipulation becomes more meaningful and students relate their abstract and quantitative reasoning, what Harel, Fuller, and Rabin (2008) referred to as *referential symbolic reasoning*, "in which symbols are manipulated according to a coherent system of referents" (p. 117).

### **Role of Multiple Representations**

The emphasis on multiple representations in standard documents follows from the findings and theoretical underpinnings put forward by multiple scholars over the past two decades. Lesh, Landau, and Hamilton (as cited by Lesh, Post, & Behr, 1987) found that students working through mathematics problems seldom came to the solutions successfully using a single representational mode. Such a result highlights that each representation has specific strengths and disadvantages, depending on the situation, and therefore provides different insights into the problem at hand (Cuoco, 2001; Lesh et al., 1987). For example, in terms of figural-pattern generalization, tables can, over time, illuminate relationships that are not directly accessible purely through visual perception (Jurow, 2004). At the same time, although numerical patterns, more easily expressed in tables than in figures, can impart the idea of a variable as a generalized number, this format conceals the various relationships between quantities within the context (English & Warren, 1998).

Note that representations differ not only in the way information is expressed but also in terms of the information itself, a position emphasized by Dreyfus and Eisenberg



(1996), who stated, "Any representation will express some, but not all of the information, stress some aspects and hide others" (pp. 267–268). Subsequently, mathematical ideas are not embodied by a single representation, but rather lie, in some sense, at the intersection of these representations. As such, the understanding of a mathematical concept involves the ability to recognize and engage with it in a variety of representational systems to capitalize on their various strengths (Lesh et al., 1987). When students begin to see and work with the concepts through different mediums, they become more familiar with the concepts' various characteristics. Thus, exposure to multiple representations may promote learning when different representations come to express different aspects more clearly (Ainsworth, Bibby, & Wood, 1998).

Moreover, the ability to translate between representations has been shown to be a characteristic of more robust and flexible knowledge. In a study comparing novices and experts, Tabachneck, Leonardo, and Simon (1997) reported that novices in economics did not attempt to associate information between different representations and had difficulty doing so when asked, whereas experts closely connected the use of multiple representations in their explanations instinctively without being prompted. Lesh et al. (1987) offered the following explanation for the importance of making connections between representations. They asserted that requiring students to establish a relationship or mapping from one representational system to another encourages them to coordinate meaning and recognize what structural characteristics are preserved, an exercise similar to translating ideas between languages. Because the process of coordination makes the structural components involved more explicit, especially in representations in which they are backgrounded and less apparent, alternating between representations develops a

stronger understanding of the various relationships and properties within the situation. In addition, it is an effective way to explore and check the reasonableness of a particular technique, insight, or answer (Lesh et al., 1987). Unfortunately, investigating this ability has shown that the vast majority of students struggle considerably with the translation process between different forms and, as a consequence, often become dependent on a single representation (Dreyfus & Eisenberg, 1996; Lesh et al., 1987, Lobato et al., 2003). Therefore, as Gagatsis, Elia, and Kyriakides (2003) pointed out, teachers should not assume that simply the exposure to multiple representations will in itself help students develop mathematical understanding. They must work to support students in making connections.

But what exactly are the structural components in terms of figural-pattern generalizing? I argue that the mathematical structure is derived from the figure. Although patterns have both recursive and explicit structure, these qualities stem from the features within the figural pattern. Therefore, recognizing the structure entails identifying the characteristics in the figure that remain constant and those that vary. Such recognition is at the core of generalizing. Only through connecting the symbolic expression to the figure can one see the mathematical relationships between the quantities in the abstract algebraic notation. At the same time, only by relating the diagram to a numerical representation does the commonality of quantities across cases become explicitly enumerated. This idea was highlighted by Radford (1996), who stated, "Symbolic vehicles give precision to algebraic ideas" (p. 110). It is through this symbiotic association that the generalization becomes clearly articulated. Without making these representational links, any functional expression is only a tool for

calculating, rather than a symbolic representation highlighting the structural relationships between objects.

### **Symbolization**

Many scholars have noted that representations are not transparent; they do not speak for themselves, but their meanings depend on the contexts. Although this is a strength of notation because one does not need to attend to the changes in meaning and associated referent during manipulation, it is also a source for confusion. This confusion is not based simply on difficulty interpreting what the symbols represent but also in understanding the epistemic role they play. When students look at an algebraic expression, do they see "a string of symbols, a description of a computational process, a number, a function, or a member of a family of functions" (Huntley, Marcus, Kahan, & Miller, 2007, p. 116)? What one actually sees depends not only on the context of a problem but also on what one is able to perceive and is prepared to notice, particularly with respect to algebraic symbols because they are compact and abstract in nature (Stylianou, 2010).

For one to develop rich understanding and tap into the power that algebraic notation provides, various cognitive developments must take place. Kaput, Blanton, and Moreno-Armella (2008) described a process they refer to as *symbolization*, in which one's understanding and experiences in working with mathematical ideas become infused in the mathematical objects used to represent the phenomenon. Cognitively, instead of the symbols' representing the referent as a separate entity, the two become interpreted as one. Actions applied to the symbols are construed as actions on the referent itself. At this stage, a student does not look *at* symbols, but *through* them, seeing the mathematical

phenomenon and the notation as one.

Kaput et al. (2008) noted that the process of symbolization occurs over time, involving multiple iterations of reflection upon more and more densely compressed forms of symbolization. Initially, students use such representations as oral, written, and drawn descriptions to express their experiences. They then use these symbols to reflect on this same experience. This process leads to a newly mediated conceptualization of the mathematical phenomenon and possibly to new representations. Each interaction with the mathematical phenomenon, whether individual and or socially mediated, results in a new conceptualization. Eventually these conceptualizations converge into a conventional and compact symbolic form, establishing an increasingly rich, densely packed interpretation of the mathematical phenomenon. More experienced students often begin with representations that are the result of previous symbolizations, such as verbal, tabular, and graphical notation. An example of this situation is the transition from arithmetic to algebra, during which the symbolization results in a transfer from numerical to variable notation. Kaput et al. (2008) highlighted that without a student's individually constructing a connection between notation and referents through the symbolization process, actions in the notational system must be guided strictly by the rules of that notational system, without support from the previously learned structure of the reference field. As such, knowledge is more fragile with students tending to overgeneralize symbolic rules such as  $(a + b)^2 = a^2 + b^2$ . The process of symbolization seems to be at the core of the CCSS standard emphasizing the connections between abstract and quantitative reasoning.

Looking at the example from Vignette 1 in the introductory chapter, I see that the

lens of symbolization provides insight into both the strategy and the conceptualization of the expression  $4x + 3$  the teacher promotes. First, limited time was provided for the class to build a robust understanding of the notation used. The instructional focus seemed to be producing a symbolic expression of the pattern, and, as such, students were not encouraged to relate and develop their own understandings through alternative representations. Instead, the teacher quickly moved to the symbolic form, supporting an interpretation of  $x$  as the step number and the 4 as growth. With this conceptualization, the teacher had not infused the meaning of the quantities from the figure into the notation. Without such a connection, understanding why the 4 and the variable have a multiplicative relationship and how the fixed number of arms contributes to a constant increase are problematic. Furthermore, as noted by Lobato et al. (2013), without the coordination of the notation to the quantities in the diagram, students did not come to understand the coefficient of  $x$  as the multiplicative relationship between quantities, eventually demonstrating instead a univariate and additive view of rate of change. Thus, symbolization is valuable not only in developing meaning of the notation but also in fostering robust understanding of associated concepts. Only by connecting the stage number with the number of squares per arm,  $x$ , and understanding that the 4 corresponds to the number of arms, can one make sense of why the 4 and  $x$  should be multiplied. Having made this connection, one can see that the resulting  $4x$  represents the total number of squares in the four arms at Stage  $x$  and that the increase at each stage comes from adding one square to each of the four arms. The  $+ 3$  symbolizes the constant central column of three squares.

### **Reification**

Another example of a conceptual transition associated with a robust understanding of algebraic symbols is what Sfard (1991) referred to as reification. *Reification* is the process of transitioning one's view of algebraic notation from a purely operational conception to a dual process-product interpretation. Sfard and Linchevski (1994) highlighted that all notation can be viewed in two linked, yet distinct, ways. When first working with algebraic notation, students see an expression as some sort of calculation shorthand, a set of instructions dictating particular processes to be carried out. At this stage the conception of the symbols is operational. For example, the algebraic expression  $3(x + 5) - 1$  would be interpreted as the operations "add 5," "multiply by 3," and then "subtract 1." When students continue to carry out these procedures, familiarizing themselves with the notation, they begin to see the algebraic expressions embodying a structural conception, in which the notation is perceived as an object, specifically the result of these operations. With this interpretation, the process has been reified, and the sequence of various operations is condensed into a single, complete, now manageable unit. Once reified, the notation can be seen by the student both as a set of processes and as a completed object. In this conception, the expression can be thought of as representing a class of possible numbers and used as an example of a particular category to explore relationships and properties.

Sfard (1991) connected reification to what Skemp (as cited by Sfard, 1991) referred to as *relational understanding*, which students come to when they can provide sound reasoning for the rule being used. Such a conception contrasts with an *operational view*, knowing what to do but without understanding the rationale behind the process. In the context of pattern generalization, the distinction between operational and

structural view seems to parallel the difference in a student's producing a generalization and the student's having generalized the pattern. A student with an operational conception can devise an expression to correctly find the number of blocks per pattern without having generalized the properties that explain why the formula generates the correct answer.

Although a structural understanding of notation is advantageous, Sfard and Linchevski (1994) highlighted that such a conception is not complete without an operational view as well. They warn against what they refer to as a *pseud structural conceptualization*, in which students are introduced to powerful symbolic notation along with various procedures to apply to them but fail to develop an underlying grasp of the processes the notation embodies. Sfard (1995) noted that notational expressions become viewed as “meaningless symbols governed by arbitrary established transformations” (p. 30). The manipulation itself becomes the focus of the activity and the symbolic results are seen as producing the answer. This distinction highlights that teachers must attend to not only the meaning students attribute to the symbols, but also the conceptions that develop regarding them.

One can use the lens of reification to analyze students' methods for pattern generalization to explain students' conceptions and struggles. I believe that a student who arrives at the correct value for a particular stage by using the technique referred to as *chunking* (Jurdak & Mouhayar, 2014), which involves identifying the number of intermittent steps between a new and established stage, multiplying the common difference by this number and then adding the result to the known figural term, has yet to develop a structural understanding of the notation. Similarly, if students use indicated

arithmetic to decompose the resultant value per stage into an organized structure, they can identify a pattern and use it to find the number of dots per stage, but such a technique reflects a calculational perspective. Without connecting the structure of the figures to the structure of notation, the variable takes on the role of simply counting the number of times the constant change has been added. Therefore, the function is perceived as an iterated process, supporting an operational view.

I argue that by elevating the structural components that the numbers and variables represent, the symbolic representation becomes viewed less as a tool for calculating and more as a way of communicating the structural relationships between objects. This practice helps students make the psychological transition from holding an operational view of the notation to developing a structural understanding. The work of Schliemann, Brizuela, and Earnest (2001) supports this view. In examining children's understanding of the difference between differences, Schliemann et al. attributed students' reification of additive differences to their exposure and use of multiple representations. The authors asserted that because mathematical concepts "express their properties diversely in different representational systems" (p. 164), only through engaging with a concept in various contexts can students appreciate the multiple conceptualizations. Consequently, they advocated a broadened notion of *reification* to mean an understanding across multiple contexts.

### **Quantitative Reasoning**

Finally, many scholars have argued the importance of bridging the gap between symbols and their referents because they see quantitative reasoning as a valuable, if not the primary, vehicle for developing algebraic reasoning (Ellis, 2011; Kaput, 1994; Steffe



& Iszak, 2002; Smith & Thompson, 2008). According to Thompson (1995), "Quantitative reasoning is not reasoning about numbers, but reasoning about objects and their measurements (i.e., quantities) and relationships among quantities" (p. 8).

Quantities are attributes of objects or phenomena that are measurable. They can be quantities in their own right or the relationship between quantities. What makes something a quantity is its capacity to be measured, whether the measuring is carried out or not. Therefore, reasoning about quantities does not require knowing their numerical values (Thompson, 1994). Smith and Thompson (2008) argued that if algebra is based on the type of reasoning that enables one to identify, communicate, and apply mathematical relationships, then algebraic thinking must be grounded in quantitative referents. They contended that too often algebra becomes an ungrounded method of matching the correct operations with the correct numbers. Most educators, in moving quickly from problem situations to algebraic expressions and manipulations, fail to develop the necessary bond between concrete, intuitive, situation-specific patterns of reasoning and formal, abstract reasoning. Without sufficient experiences to ground their thinking, students are unable to make sense of the abstract notation. Alternatively, by focusing on quantitative reasoning, students come to view symbols as tools that express quantitative and structural relationships rather than as specific computations.

As reported in the previous section on figural algebraic patterns, students who embedded their thinking in diagrams engaged in more sophisticated ways of reasoning than did other students. They looked for and developed rationales for their patterns, revisiting their thinking and more naturally infusing the practice of justification. In contrast, students who were disconnected from the quantities simply grabbed at random

relationships and focused on the result rather than the process. As such, focusing on quantitative reasoning as a core component of algebraic thinking appears to help students not only develop meaning in abstract algebraic notation but also develop generalizing as a mathematical way of thinking, not simply as a method to arrive at an answer.

### **Teachers' Knowledge**

The issue of what types of knowledge are essential for teaching mathematics has been the subject of debate and exploration in the mathematics education community for more than 50 years (Mewborn, 2001). Although mathematical content knowledge has always been deemed essential, scholars have yet to come to consensus on the exact details of what constitutes this knowledge (Baumert, 2010). For many years, researchers sought to establish a link between content knowledge and student achievement, taking a seemingly naïve and simplistic view of content knowledge (Mewborn, 2001). They used such gross measures as level of education, number of postsecondary mathematics courses, college major (mathematics or not), and grade-point average to explore this relationship in large-scale quantitative studies. The results of these examinations showed minimal to no correlation between teachers' subject matter knowledge and students' academic achievement (Begle, 1979; Eisenberg, 1977; Gess-Newsome, 1999; Goldhaber & Brewer, 1997, 2000; Hill et al., 2007; Mewborn, 2001), indicating that simply exposure to more mathematics did not lead to more successful teaching of mathematics.

In an effort to describe the strengths and weaknesses of teacher knowledge in particular content areas, researchers began conducting more qualitative studies. For example, Borko and colleagues (1992) followed several preservice teachers into the classroom during their student teaching. They reported on one teacher, Ms. Daniels, who

was particularly noteworthy because she had the most extensive mathematical background of the participants, having taken many postsecondary pure mathematics courses before deciding to pursue elementary school education. Nonetheless, she struggled to support her students mathematically. In the lesson of interest, after Ms. Daniels demonstrated to the class a rule for dividing fractions, one student asked if she could explain why, to divide fractions, one inverts and multiplies. Ms. Daniels attempted to answer the student by presenting a problem with a related diagram. Her example, however, involved fraction multiplication rather than division. Moreover, in a postinterview, she was unable to explain her error. Although Ms. Daniels could correctly solve the mathematical task, she did not possess the knowledge to explain the rule she used or produce a contextual example that corresponded to the operation. Such studies served to uncover the complexities of the knowledge necessary for effective mathematics instruction, highlighting that the relationship between teachers' knowledge and their practices is anything but straightforward.

### **Mathematical Knowledge for Teaching**

Because more and more studies have shown that “knowing mathematics is not the same as knowing how to teach mathematics” (Mewborn, 2001, p. 31), researchers have begun to explore the types of knowledge necessary for effective mathematics instruction in an effort to develop a more comprehensive image of teachers' mathematical knowledge. A significant breakthrough in this pursuit was Shulman's (1986) introduction of *pedagogical content knowledge* (PCK). Shulman conceptualized PCK as a domain-specific instructional knowledge that blends subject-matter knowledge (SMK) and pedagogical knowledge. He argued that knowing mathematical facts and possessing the

ability to solve mathematical problems was not sufficient. At the same time, an understanding of general pedagogical strategies such as classroom management and lesson planning was not specific enough to support students in developing deep mathematical understanding. Teachers needed a specialized professional knowledge that enabled them to facilitate the learning of that subject matter and adapt instruction to meet the needs of diverse learners. Shulman included knowledge of applicable representations, examples and counterexamples, as well as knowledge of what makes learning a particular concept easy or difficult.

Many scholars have since elaborated on Shulman's notion of PCK, in an effort to depict a mathematical knowledge for teaching (MKT) that characterizes the types of understanding a teacher needs to effectively support students in developing rich mathematical proficiency (Ball & Bass, 2000; Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008). Probably the most well-known and comprehensive model was one developed by Ball and colleagues in which they attempted to identify various specific types of knowledge that teachers need to enact "high-quality" mathematical instruction (Ball et al., 2005). To do so, they examined the practice of teaching itself. To examine broadly the expertise involved in effective instruction, they analyzed videos of teachers, lesson plans, interviews, and student work (Ball & Bass, 2003). They identified six categories of professional knowledge, dividing both subject-matter knowledge (SMK) and pedagogical content knowledge (PCK) into three categories (Hill et al., 2008). They decomposed SMK into the domains (a) *common content knowledge* (CCK), (b) *specialized content knowledge* (SCK), and (c) *knowledge at the mathematical horizon* (KMH). They separated PCK into the domains (a) *knowledge of content and students*

(KCS), (b) *knowledge of content and teaching* (KCT), and (c) *knowledge of curriculum* (KC).

This Hill et al. (2008) model had several notable features. First, as highlighted in chapter 1, Ball and colleagues conceptualized *specialized content knowledge* (SCK) as distinct from the *common content knowledge* (CCK) that Shulman originally interpreted as encapsulating all subject-matter knowledge. SCK is mathematical knowledge particular to teachers of mathematics, knowledge that goes beyond the knowledge of topics as they are used in other mathematically rich disciplines. It includes the ability to evaluate whether students' methods are generalizable or to what degree they are applicable. Particularly relevant to this study, SCK also encompasses a familiarity with multiple interpretations and symbolic representations of mathematical ideas.

Another significant contribution of the model is to show the importance of student thinking. Ball and colleagues identified two distinct, yet related, categories of knowledge dedicated to the understanding students' mathematical thinking. The first category is *knowledge of content and students* (KCS). Beyond knowing the multiple ways a mathematical topic can be conceptualized, teachers need to understand how students think about, interact with, and learn particular topics. They need to have awareness of what makes learning a particular topic difficult for children, common preconceptions and misconceptions about the topic, and typical methods or approaches employed by students. A teacher who possesses this type of knowledge can better identify student thinking by the students' questions and responses and ground his or her planning of instruction in probable student expectations. A second category dedicated to student thinking is *knowledge of content and teaching* (KCT). Such knowledge is not

simply an awareness of possible student difficulties but also knowledge of how to overcome them. KCT includes the teacher's ability to choose problems or plan a sequence of activities that will stimulate discussion and guide students to make productive connections. A teacher with this form of knowledge possesses an awareness of how a particular question, example, or representation might build on student thinking, linking students' thinking to a more sophisticated approach.

In addition to contributing these theoretical underpinnings of MKT, the researchers have developed an assessment of teachers' MKT that is based on refinement of the model of teacher knowledge. Ball and colleagues (Ball et al., 2005) found that teachers' scores on this test strongly correlated with student gains in mathematics in early elementary grades, whereas other factors, such as teachers' experience, did not. These results confirm that teachers' content knowledge consists of more than simply strong understandings of mathematics held by well-educated adults. Furthermore, they highlight that the various categories of this model map onto distinct areas of knowledge necessary for effective teaching, indicating that we as teacher educators must support teachers in developing these areas of knowledge.

### **Professional Noticing**

In broadening one's interpretation of algebra beyond simply a list of content standards and incorporating mathematical practices and algebraic thinking as pedagogical goals, an instructor cannot concentrate solely on the answers students generate. If teachers focus only on correct and incorrect solutions, this limited awareness will not enable them to help students develop robust algebraic thinking. Additionally, teachers must strive to familiarize themselves with the various conceptions that individual

students possess (Ball, 2011; Schifter, 2001). For students to meaningfully understand both the mathematical content and the ways of thinking associated with the discipline, they must be engaged, first hand, in authentic mathematical tasks, and their reasoning must guide the class. Consequently, for teachers to know and be able to present the mathematics is insufficient. They must see and interpret the mathematics through the eyes of students to grasp the meanings students are developing. By perceiving the content through the lenses of both the expert and the student, teachers can better connect students' ideas to powerful mathematical ideas. As pointed out in Chapter 1, this view is foundation to a new model of responsive instruction (Ball, 1995; Sherin, Jacobs, & Philipp, 2011). According to this model, teachers enter the classroom with specific goals in mind and a general plan for how the lesson will proceed, but upon beginning to implement this plan in a classroom, they must react to the unexpected student ideas that arise, make decisions about how to proceed, and adapt the lesson in the midst of instruction on the basis of the situation they encounter. As Sherin and Star (2011) noted, "Bombarded with a blooming, buzzing confusion of sensory data" (p. 69), teachers must inevitably make choices about what is pedagogically relevant and direct their attention accordingly.

Navigating this situation requires specialized understanding. The heightened abilities of experts in a particular field to see and make sense of phenomenon in ways that are beneficial for the specific needs of their profession have come to be known as *professional noticing*. Expert teachers, like all professionals, filter the complexity of the classroom by choosing specific, pertinent information to which they attend. Recently, when researchers have begun to see the role of a teacher as an active guide, responding

quickly and effectively to the dynamic situation in the classroom, they have found teachers' noticing a productive area of study for understanding teachers' expertise in terms of how they recognize and react to the ever-changing needs of their students (Russ, Sherin, & Sherin, 2011). Although the construct has only a short history in mathematics education research, already widely varying studies have emerged in which professional noticing is used as both a lens to understand teachers' perceptions of the classroom learning environment and the basis for the decisions they make. In addition, although this area of research has grown and expanded, various conceptions of what professional noticing entails have emerged, differing in the foundational elements attributed to this skill as well as the methodologies perceived to best examine this practice. In the next section I first examine and classify related, but different, noticing studies to situate my study within the larger literature base. I then summarize findings related to professional noticing as well as ways researchers have conceptualized *professional noticing*, explaining their applicability to my study.

### **Classification of Noticing Studies**

In one area of research initiated by van Es and Sherin (2002), researchers use the construct of *professional noticing* to identify and categorize the features of the instructional environment that teachers find salient and document how the features change over time (Russ & Luna, 2013; McDuffie et al., 2014; Star, Lynch, Perova, 2011; Star & Strickland, 2008; van Es & Sherin, 2009). Although they typically focus on student thinking and ways to help teachers make noticing a more prominent feature in their instruction, these studies have typically not been embedded in any particular mathematical domain, only attempting to analyze overall shifts in teachers' attention and



interpretation of classroom events.

A second branch of research, first conducted by Jacobs et al. (2010), is concentrated on teachers' attention to how students conceptualize a specific area of mathematics. In what they referred to as *professional noticing of children's mathematical thinking*, Jacobs et al. compared teachers' in-the-moment understanding of student thinking with research findings on children's methods and understandings of whole number operations. They used these results to highlight differences between groups of teachers with varying amounts of professional development experience. Similar work has been repeated in other domains. Schack et al. (2013) and Tyminski et al. (2014) conducted studies in the same domain of children's understanding of whole number operations. Their work differed, though, in that they employed pre and post interviews to measure the growth of preservice teachers' noticing after a semester of instruction. Fernández, Llinares, and Valls (2011, 2013) carried out parallel pre and post studies of preservice teachers' noticing in the area of proportional reasoning at the secondary and primary levels. Finally, Callejo, and Zapatera (2016) explored preservice primary-grades teachers' noticing of students' understanding of pattern generalization. Researchers in all these studies measured teachers' professional noticing using instantiations of practice in which the teachers themselves were not participants, providing teachers either short videoclips or written student work to investigate what the teachers noticed. Furthermore, except in the study by Jacobs et al. (2010), the participants were preservice teachers whose exposure to students' thinking was limited to their teacher-preparation programs.

In another group of studies, researchers attempted to capture teachers' professional

noticing of students' thinking during the act of teaching (Choppin, 2011; Zapatera & Callejo, 2013). In one study, Choppin (2011) explored how secondary school teachers' attention to student thinking shaped how teachers adapted tasks while they strove to meet the changing intellectual needs of their students. As part of his study, he videotaped 10 class sessions and then conducted semistructured video-stimulated interviews related to short clips he selected from this compilation. Although such a methodology incorporated teachers' analyzing their own instruction, the delay between class session and interviews presents a question of the degree to which the teachers' comments reflected their in-the-moment thinking versus their later analysis of these instructional episodes. Also, although these various studies were grounded in students' mathematical thinking, the teachers' understandings were analyzed at a more global level, and their rationales were described in general terms. Consequently, results of these studies provided a broader characterization of teachers' attention to students' mathematical thinking.

One other group of researchers with research in this category has used a new methodological approach, which I discuss further in Chapter 3 (Colestock, 2009; Sherin, Russ, Colestock, 2011; Walkoe, 2010). These researchers attempted to capture teachers' in-the-moment views of the classroom by attaching cameras to the teachers themselves and asking them to identify interesting moments. Although Sherin et al.'s study had no particular mathematical focus, Walkoe embedded her study specifically in the context of algebra, exploring what teachers identified as moments of students' algebraic thinking. One advantage of this method is that the teachers select the classroom instances to discuss and describe their rationales for selecting each moment. The teachers'

explanations of the significance of the clip serve to frame their understandings within the context of the larger instructional unit. Furthermore, this process avoids, as much as possible, the teachers' analyzing the moment retrospectively when they report on their motivations for choosing the clip during the classroom instruction.

### **Noticing-Related Findings**

Although some researchers have explored patterns of teachers' professional noticing from a broad perspective (e.g., Star & Strickland, 2008), the majority of studies have focused their analysis in the context of students' thinking. By asking teachers to reflect on classroom videos consisting of examples of students engaged in mathematical problem solving, researchers have begun to gain insight into both what teachers notice and how they notice. One significant finding is that simply attending to student thinking is challenging for teachers (Empson & Jacobs, 2008; Jacobs et al., 2010, Sherin & Han, 2004; van Es & Sherin, 2009). U.S. teachers' comments focused predominantly on the teachers—their personalities, classroom-management styles, or instructional moves, with limited details on student thinking (Jacobs et al., 2010; Miller & Zhou, 2007; van Es & Sherin, 2009). When asked to respond to students' work, teachers primarily offered suggestions that appear almost predetermined, without any evidence of relating to the students' thinking (Choppin, 2011; Jacobs et al., 2010). Moreover, Jacobs et al. (2010) found no significant difference in abilities to decide how to respond to students' mathematical thinking between experienced teachers who had yet to start professional development focused on children's mathematical thinking and preservice teachers. This finding indicates that the responding component of professional noticing of students' mathematical thinking is a specialized skill that one does not develop simply through

teaching experience. That said, significant shifts in teachers' expertise in professional noticing have been noted through prolonged professional development concentrated on students' mathematical thinking (Jacobs et al., 2010; Shack, 2013; Sherin & Han, 2004; Tyminski et al., 2014; van Es & Sherin, 2009; Walkoe, 2014). In particular, van Es and Sherin (2009) found that after a year of participation in a video club, teachers began commenting less on the pedagogical decisions of the teacher and reflected more about the students' mathematical understandings. In addition, the teachers were able to support their observations with more specific details of evidence. This finding indicates a trajectory in substance of teachers' noticing that begins with attention on the teacher's instruction and transfers to a more detailed focus on students' mathematical ideas after sustained professional development.

Researchers have observed not only a shift in the area of focus in teachers' noticing as a result of long-term, directed professional development but also a change in teachers' stances toward noticing. This finding has been documented and characterized in many ways. First, before teachers have much experience explicitly analyzing student thinking, they are quite evaluative in their comments, commenting on what they believe the teacher should have done differently or classifying instances as good or bad (Empson & Jacobs, 2008; Mason, 2011; Nemirovsky, DiMattia, Ribeiro, & Lara-Meloy, 2005; van Es & Sherin, 2009). When asked about the students, teachers identify and characterize perceived difficulties without providing evidence to support their claims (Choppin, 2011). Instead of attending to details of the video, they simply reflect on and draw from their own experiences (Nemirovsky et al., 2005).

Second, with more exposure to examining student thinking, teachers become less

evaluative and begin to recount events from the video in detail without imposing their own biases. Nemirovsky et al. (2005) noted that when teachers reach this phase, their discourse is characterized by a grounded narrative in which they provide evidence from the video to support their comments and distinguish between what they have seen and what they have inferred. Mason (2011) highlighted this distinction in his use of the terms *accounting for* and *accounting of*. The former involves interpretation and the latter is simply a descriptive account. Mason argued that teachers routinely engage in the former without identifying and reflecting on the actual details of the incident. Instead, they theorize about a situation by creating a single, simple narrative to explain the incident on the basis of assumptions and unsubstantiated interpretations of past events.

Van Es and Sherin (2009) identified a third category of teacher response, which they denoted as interpretive. In this phase, teachers draw connections between the student understanding and teacher actions. They move from merely reporting events in classroom and begin to synthesize their observations, seeing patterns and themes. In contrast with *accounting for* though, the teacher's interpretations of the situations are not based on preconceived notions but are supported by evidence from the clip. In fact, the teachers can envision more than one possible explanation for the student's thinking and speak about the differences among them.

Similarly, Empson and Jacobs (2008) have identified three classifications of teachers' listening: directive, observational, and responsive, which seem to mirror the three previously outlined phases of response but which characterized not teachers' reflections after watching a video but their in-the-moment exchanges with students.

They note that directive listening is by far the most pervasive in the current teaching culture. It involves teachers guiding students to a particular destination and attending only to whether students' answers match their preconceived, intended responses. Instead of providing opportunities for students to express their idea, teachers quickly cut off students or take over student exchanges with excessive support. They often overgeneralize students' comments, imposing their own thinking on the ideas the students provide. With training, teachers move to observational listening; their interactions with students become characterized by excessive wait time and generic follow-up questions with little or inconsistent probing even when solutions are unclear. Finally, teachers reach the stage at which they are able to listen responsively. They begin to attempt to draw out students' mathematical ideas and make them more explicit. Not only do teachers at this stage listen carefully, they are also able to support and extend the students' understanding. They ask the students to think deeply about their own, personal conceptions. Empson and Jacobs (2008), in their description of this trajectory, emphasized the difficulty and length of this transition, especially to the last stage.

These results from noticing studies inform my study in two ways. First, the finding that noticing is a challenging, yet learnable practice guided me in my research design. Analyzing teachers with limited understandings of student thinking in the area of algebraic generalization would have most likely resulted in a deficit study in which I would have primarily documented the various ways in which the teachers struggled. To avoid this type of study, I incorporated a 5-day, professional development in which all the teachers were able to engage in professional noticing of algebraic generalization and discuss their observations and understandings. Through this experience teachers develop

richer understandings of the generalizing process from the students' perspective. Second, knowing how pervasive directive listening is among teachers who are just beginning to engage in professional noticing, I chose participants with a significant amount of experience reflecting on student thinking in general. The goal of this study was to characterize the understanding of teachers who were endeavoring to interpret and respond to student thinking. As such, I wanted teachers with a particular stance toward noticing to ensure that the learning environment was student centered and that the teachers were genuinely trying to support and build on student ideas.

### **Components of Professional Noticing**

In addition to differing with respect to the methods they use to capture and analyze teachers' professional noticing, researchers also differ regarding the aspects they conceptualize as making up the practice. Some researchers investigate only to what teachers attend, whereas others see professional noticing as comprising interpreting as well as deciding how to responding. Star, Lynch, and Perova (2011) characterized noticing as including only attending. Although the authors acknowledged that all three components are important, they emphasized that attending is the most foundational of the three practices (Star & Strickland, 2008), explaining that a teacher cannot interpret or act on that to which they do not attend. This description implies a conceptualization of noticing in which the three components build on one another in a sequential manner. A teacher first gathers information, then reasons about it, and finally acts on it. In their study, Star et al. (2011) sought to identify and categorize the observations preservice teachers make when watching videos of classroom interaction and to examine how these observations shifted during a semester methods course. With such a focus, they

backgrounded the role the participants' knowledge played in noticing. They suggested that to interpret and respond to events is beyond preservice teachers' abilities while they struggle to connect events in the classroom to broader principles of teaching (Star & Strickland, 2008). Furthermore, the authors portrayed noticing as a general skill, one they believed would be "activated" (Star & Strickland, 2008, p. 120) throughout the semester, so they chose not to focus specifically on noticing of students' mathematical thinking. For these researchers who view noticing as a step-by-step process and who deemphasize the effect of teachers' understanding, attending becomes a stand-alone skill, meaningful in its own right. As such, investigating solely what aspects of the classroom interaction teachers found salient provided some insight into their instruction.

Sherin and Star (2011) argued that attending and interpreting are not mutually exclusive, but rather interact with each other simultaneously in a dynamic manner, constantly reshaping one another. Accordingly, they maintained that noticing consists of both deciding what to selectively attend to as well as reasoning about it. Although Sherin and Star (2011) acknowledged that, methodologically, including interpreting may make the study's scope too broad, they noted that how a teacher interprets a situation influences to what he or she attends and, therefore, cannot be considered separately. Often teachers first establish interpretations and then search for the same interpretations in instances they observe. Sherin and Star highlighted that what we see is driven by our knowledge and expectations—quoting Gibson (1958) who stated, "Perception is active, not passive. It is exploratory, not merely receptive" (p. 43, as cited by Sherin & Star, 2011) From this perspective, teachers do not simply survey the classroom for information, they interact with the class to produce desired outcomes. Teachers' ongoing



analysis guides their observations. Supporting this view, Sherin and Han (2004) found that when teachers reasoned about student ideas, their analysis influenced what they perceived to be important and subsequently to what they attended. From this point of view, the context of the situation as well as a teacher's knowledge and beliefs about how students learn serve as a filter for what he or she sees. Attending and interpreting influence each other, necessitating that the two be examined jointly.

Finally, Jacobs, Lamb, and Philipp (2010) conceptualized the construct of professional noticing of children's mathematical thinking as including all three features: attending, interpreting, and deciding how to respond. They emphasized that the three practices occur concurrently and are conceptually associated. Similarly, Erickson (2011) argued that teachers notice in order to act on these observations. They look for particular information so they can act in particular ways. Jacobs et al. (2010) cited Sherin (2001) and Schoenfeld (1998), who seemed to support this stance, stating, "The work of teaching orients teachers to constantly consider their next moves" (p. 99). In the midst of instruction, teachers do not simply take note of and reflect on the classroom without acting. The profession inherently involves making decisions of how best to move the class forward. From this point of view, all three components come together to constitute teachers' professional noticing and must be examined together to provide a full picture of how teachers engage in this practice. Studying how one responds or intends to respond informs researchers regarding to which details the teacher attended and how he or she interpreted them.

As I commented in Chapter 1, because my data are from teachers actively engaged with students and in the process of assisting and understanding their students'

thinking, I incorporated all three elements of noticing, including teachers' decisions about how to respond, as part of my analysis. I believe that teachers' motivations guide their attending and interpretation of the situation. Teachers, in the act of instruction, naturally try to interpret, anticipate, and even possibly shape students' thinking. This type of reflection cannot be removed from their thinking processes, and, as such, should not be omitted from the analysis. Furthermore, a critical piece of this study was capturing not merely what the teachers interpreted about students' thinking but also how the teachers' understandings of the generalizing process enabled them to frame the students' responses within the context of a larger instructional unit and the overall learning of the class. As such, the teachers' follow-up questions or proposed actions provided greater details about exactly what they understood about the generalization process in which students were engaging.

### **Relationship Between Noticing and MKT**

To date, the relationship between teachers' professional noticing of students' mathematical thinking and their underlying mathematical knowledge for teaching has not been well defined. Most scholars describe and investigate noticing as a practice without attaching any direct connection to knowledge. In contrast, Schoenfeld (2011) asserted, "What teachers notice, and how they act on it, is a function of the teachers' knowledge and resources, goals, and orientations" (p. 233). Similarly, Tyminski et al. (2014), drawing links between PSTs' noticing and categories of the Ball model, argued that various aspects of teachers' knowledge influence their professional noticing. Although the connection has yet to be clearly established, the ability to interpret and productively respond to the mathematics embedded in students' ideas seems to require a specific type

of knowledge. Moreover, effective professional noticing of student thinking necessitates an integration of various types of knowledge; in particular, a combination of teachers' specialized content knowledge, knowledge of content and students, and knowledge of content and teaching are needed for a teacher to effectively engage in professional noticing of children's mathematical thinking (Tyminski, 2014).

In this study, I used professional noticing as a theoretical lens to see and make sense of the classroom through the eyes of the teacher. Analyzing how teachers attended to, interpreted, and decided how to respond to students' thinking enabled me to investigate and characterize the type of MKT necessary for teachers to support students' productive engagement in the generalization process within the context of a student-centered learning environment.

### **CHAPTER 3: Research Methods**

In the first two chapters, I presented an argument for the relevance of my three research questions. In particular, I highlighted the importance of generalization in mathematics and the significant role it plays in the new conception of algebra. I also showed that although a large research base is dedicated to students' understanding of algebraic generalization, little work has been devoted to exploring teachers' understanding, with even less dedicated specifically to exploring their understanding of students' thinking in this domain. Furthermore, I argued for expanding the analysis of algebraic generalization to include symbolization, providing a theoretical basis for the benefit of helping students develop a strong understanding of the connection between algebraic notation and the associated figure.

Finally, I described the rich and integrated knowledge teachers need to effectively engage in responsive teaching and provided a rationale for studying expert teachers during their instruction as a productive context to provide insight into this knowledge. At the intersection of these various arguments, lie my three research questions, aimed at investigating teachers' understanding of algebraic generalization from various different perspectives.

In this chapter, I present the research methods used to answer these three questions. I begin with a description of the four participants because I think that their selection was critical for the study. I then provide a brief overview of the research design, including a description of the professional development project that supported this study, followed by a more detailed review of the specific methods and data sources used to address each question. Finally, I elaborate on the data analysis, giving an

overview of grounded theory before describing in detail how this approach was used to analyze data related to all three questions.

### **Participants**

As I explained in chapter 1, a vital feature of this study is the group of teachers selected to participate. My overall guiding objective for this study is not to document the current state of teachers' knowledge but rather to characterize the integrated MKT necessary to support students to productively engage in algebraic generalization as well as develop insight into associated challenges. This type of knowledge can be examined only in working with teachers with a specific background and skill set.

With that restriction in mind, I chose the participants for this study because of their overall experience as educators, in general, and more specifically because of their participation in the Noyce fellowship. In particular, the Noyce fellowship was focused on teachers' developing an orientation to student thinking and a strong understanding of various algebraic concepts. As part of this work, a specific emphasis was placed on algebraic generalization. As such, the participants had completed several assignments, including video analysis, one-on-one interviews, as well as various teaching experiments specifically targeting algebraic generalization to enable the teachers to explore and reflect on student thinking within this content area. Typically, after each assignment the teachers would debrief in small groups, sharing observations and surprises about students' strategies and misconceptions. Such experiences provided a unique foundation from which the teachers could draw.

Within this group of Noyce teachers, the four teachers I studied were purposefully selected to ensure that all participants taught comparable student bodies yet were

otherwise quite varied. At the time of the study, all four volunteers taught 8th-grade mathematics at public schools in the San Diego area, with one of these being a charter institution. The middle schools in which these teachers worked and wherein the Phase II data collection took place were all high-need schools, with the vast majority of students being of lower socioeconomic status. Although the participants shared the common experience of Noyce and taught a common grade level, they varied in terms of gender, ethnicity, and years of experience (gender: 1 male, 3 females; ethnicity: 2 Caucasian, 1 Asian, 1 African American; years of teaching experience: 7, 9, 12, 27). Therefore, I believe that the structural similarities in their instructional environments, together with their diverse backgrounds, provide a beneficial context from which to draw rich comparisons and distinctions in their understanding.

Furthermore, as I learned during this study, in addition to the rich set of experiences these teachers had engaged in as part of their participation with Noyce, all four teachers brought significant experience of algebraic generalization from their own practices; each included relevant assignments as part of their normal instruction. Two teachers (Jack and Palila) commented that they engaged students in generalizing tasks throughout the year, with Jack specifying that he liked to use them at the beginning of the year and on days following long breaks. The other two teachers (Denise and Clara) both used curricula that included a heavy emphasis on generalization. Denise's textbook (*CPM-College Preparatory Mathematics*) included an entire unit on the topic, with figural patterns provided as a way to help students see characteristics associated with linear equations (slope and  $y$ -intercept) in varied representations (table, graph, rule, pattern). Clara's materials interspersed figural patterns in a more spiral format so that

students revisited the topic, adding more complexity and targeting different outcomes throughout the school year.

### Overview of Research Design

The research design used in this dissertation study comprised of two main phases, an interview phase and a classroom phase. In addition, between these two phases, each of the 4 teachers also participated in a weeklong, student-centered professional development organized around algebraic generalization. To provide an understanding of the overall project as well as explain the purpose and structure of the intermediate professional development, I present a short synopsis of the various components. Later, when I clarify the exact data resources for each research question (RQ), I provide a more detailed description of the two phases.

Table 3.1. Research Design Components and Participants

| Teacher | Phase I<br>Interview |      | Professional<br>Development | Phase II<br>Classroom |
|---------|----------------------|------|-----------------------------|-----------------------|
|         | RQ 1                 | RQ 2 |                             | RQ 3                  |
| Palila  | X                    | X    |                             |                       |
| Denise  | X                    | X    |                             | X                     |
| Jack    | X                    | X    |                             | X                     |
| Clara   | X                    | X    |                             | X                     |

The interview phase (Phase I), a 2-hour clinical interview (Ginsburg, 1997) of each of the 4 teachers, was used to address the first two research questions. The interview consisted of three components. The teachers first reflected on their goals associated with these figural tasks (RQ1). They then solved various generalizing tasks and explained their interpretations of the numeric and algebraic representations they produced (RQ2).

Finally, they offered their analysis of student work on the same tasks (RQ2). Although all 4 teachers participated in Phase I, as I will explain, only 3 were included in the analysis for RQ2.

Phase II, which was designed to answer my third research question, involved 3 of the 4 teachers and was conducted on-site in the classroom of each participating teacher. (Although, as I will explain, data for only 2 teachers were analyzed.) Phase II data consisted primarily of my observations of the instruction of 2 or 3 lessons by each teacher. These lessons were planned and carried out by the teachers as part of a unit on algebraic generalization. Before each class I conducted a short interview with the teacher to understand the desired learning outcomes and proposed lesson structure. Then after each lesson, the teacher participated in a stimulated video-recall interview (Busse & Borromeo Ferri, 2003), reflecting on understandings of particular moments during the lesson. In addition, following the professional development and preceding my classroom visits, I conducted a 2-hour clinical interview with each teacher to reexamine the teacher's overarching goals and motivation for the generalization unit.

**Professional Development.** As I stated above, between Phases I and II, all 4 teachers participated in 5 half-days of professional development. A mathematics teacher educator with 25+ years of experience and I led these sessions.

***Professional development rationale.*** The week of professional development served two interrelated purposes. The first objective was to develop a collective understanding of algebraic generalization among the participating teachers. As a research team, we wanted the teachers to see how the generalizing process, grounded in each figure, could lead to meaningful, quantitative understanding of the algebraic



notation in which the resulting symbolic expressions could be used to communicate a generalized understanding of the figure. As I will show in my response to the first research question, the participants came into the study with their own varying interpretations of algebraic generalization as well as differing types of associated understandings they hoped students would develop. The professional development afforded the opportunity to present a particular image of algebraic generalization. As a consequence, this common experience served to align, at least partially, the participants' views of the content with one another's and with mine. I argue that for me to effectively make nuanced comparisons and identify subtle differences among the teachers in terms of their knowledge and practices, they must have comparable views of the final instructional goals. Although the differences in their understandings make for rich data, drawing comparisons among teachers with very different pedagogical aims would be unproductive and challenging.

The second rationale for the weeklong professional development was to provide the teachers with an experience that would foster an integrated and comprehensive understanding of algebraic generalization. Specifically, it offered them a rich context in which to develop a better understanding of algebraic generalization from the students' perspectives. By being placed in a live learning environment in which students were grappling to generalize, the teachers could engage in professional noticing of students' mathematical thinking without the added complexities of leading instruction. As such, they could see first-hand how students engage in the generalization process by observing and internalizing nuances of the different ways students approach algebraic generalization, the various conceptual benchmarks through which students progress, and

the effects of certain pedagogical interventions in facilitating these understandings. Furthermore, by reflecting on the students' struggles, teachers would have opportunities to develop detailed understandings of possible alternative student conceptions as well as potential support structures. In general, I hoped that the the intimate and interactive environment of the week-long professional development would provide the teachers a rich learning experience through which they would broaden and consolidate their understandings of the process by which students learn to generalize algebraically, understandings that would ultimately guide their own instruction and make the eventual analysis of their MKT in the classroom fruitful.

***Professional development structure.*** The professional development consisted of 5 half days devoted to algebraic generalization. Each day (except the first) was divided into two parts. The first half (1.5–2 hours) consisted of live instruction, led by the first mathematics teacher educator, involving a class of 6 eighth-grade students from the school of one of the participating teachers. These students were chosen explicitly to provide a learning environment that aligned as closely as possible with the teachers' own classrooms. The structure was somewhat similar to a classroom setting and served as a possible model of responsive teaching for a unit on algebraic generalization. During this instructional time, the 4 participating teachers observed but were also free to engage with the students to explore their thinking in more detail through students' responses to questions the teachers posed. On occasion, they also had the flexibility to approach the mathematics educators during instruction and suggest ideas for moving forward. In addition, each of the 4 teachers conducted two one-on-one interviews with one of the participating students. The first took place at the beginning of the week and served to

develop a baseline of the students' understanding of algebraic generalization. The second took place at the end of the week and was designed specifically to investigate the students' understanding of variables.

The second half of each day was devoted to unpacking the morning lesson and planning for the next day. The 4 teachers first reflected individually, writing observations about moments during the lesson they deemed critical. A whole-group discussion then followed; the teachers and professional development leaders shared their thinking about the lesson, the student learning, and possible paths forward. Although the teachers did not design the unit instruction, they did have input into many of the planning decisions. Moreover, special care was taken to explain the leadership's rationale for various instructional choices made during the week, giving the teachers access to their decision-making process while they attempted to navigate the learning environment.

Although the purpose of the professional development was primarily intellectual in nature, as described above, it also provided data. The teachers' written responses and the group discussions during the daily reflections were collected and recorded, respectively. Their comments were used to contextualize the teachers' understanding of algebraic generalization that played a role in answering Questions 1 and 3.

### **Research Question 1**

Research Question 1 follows: What mathematical goals do teachers who have experience with student thinking, in general, and algebraic generalization, specifically, associate with figural patterns? To answer this question, I conducted individual semistructured clinical interviews with each of the four teachers (Ginsburg, 1997). These interviews took place at the beginning of Phase I, before the teachers had engaged

in any other activities as part of the study. During the interviews, participants were asked to elaborate on their goals in general and to articulate the mathematical content they associated with these figural tasks. In addition, to further probe their understanding, they were given hypothetical situations such as why they would or would not include a unit on generalization if they were designing their own curricula (see Appendix A for full interview protocol).

Clinical interviews are well suited for exploring the details of teachers' instructional objectives, because they are designed for the researcher to observe and make inferences concerning the perceptions and capabilities of the interviewee (Ginsburg, 1997). Although such a format has limitations, most notably that two teachers might use similar words to describe their goals and mean different things, a methodological advantage of clinical interviews is their flexibility and adaptability. To develop a clearer and more complete understanding of the participants' thinking, the researcher is encouraged to engage in *hypothesis testing* (Clement 2000; Ginsburg, 1997); that is, although a consistent protocol guides the interview, the researcher is encouraged to make inferences about the the participants' understanding and formulate, in the moment, questions or scenarios to test these interpretations. He or she is given freedom to introduce unscripted questions as deemed appropriate to investigate the exact meaning or extent of a comment. For example, many times during the interview I provided the teachers with different contextual features to explore the situated nature of their understanding. Therefore, while the same major questions were posed to each teacher, many of the follow-up questions diverged slightly to further probe each teacher's thinking (Ginsburg, 1997). The rationale behind such a technique is that only through

follow-up questions, relevant and specific to the participant's individual comments, can a detailed understanding of the full extent of their thinking be achieved.

## **Research Question 2**

Research question 2 follows: What conceptual hurdles do secondary school teachers encounter when they attend to the quantitative meanings of algebraic symbols that result from algebraic generalization, and what conceptualizations support them to overcome these challenges? To answer this question, I again engaged the teachers in a clinical interview as part of Phase I. Similar to the approach for answering Question 1, such an approach is beneficial because it enabled me to flexibly probe and explore the nuances and boundaries of the teachers' thinking beyond initial generalities they offered. Furthermore, a key goal I had in pursuing this research question was to move beyond the teachers' current understandings to identify conceptual resources these teachers might draw upon while they endeavored to overcome challenges they encountered. Responses from each of the four participants showed that I was asking questions they had not thought about in detail prior to the interview. While they grappled with the concepts, struggling at times to articulate their thinking, flexibly producing follow-up questions enabled me to introduce clarifying questions, ask about apparent contradictions in their explanations, and explore the consistency of their interpretations. Also, to obtain a rich record of the data related to teachers' understanding, I used two cameras for this part of the interview, one focused on their written work and one on them. Thus, I ensured that video captured teachers' verbal reports, gestures, and written inscriptions. In addition, all written work was collected.

**Instruments.** My investigation of the teachers' quantitative understanding of algebraic notation within the interview consisted of two approaches. Teachers were asked to analyze the expressions that they themselves generated as well as reflect upon student work.

*Analyzing personally generated expressions.* The teachers were each given two patterns to analyze (see appendix B). They were then asked to solve for the number of dots for various stages and write expressions for the  $n$ th stage. For each expression, numerical or algebraic, I asked the teacher to explain the meaning of each symbol in the expression as well as many combinations of these symbols formed by grouping different pieces. I also asked the participants to explain their interpretations of symbols in intermediate expressions that arose in subsequent stages when they acted on the symbols syntactically. Although my initial questions about their interpretations of the symbols were general in nature, if their responses did not incorporate exact features of the figure, I followed up by asking if they could connect them to the figure, even asking them to clarify their interpretations physically on the figure when appropriate.

Although the full interview protocol is provided in Appendix B, I highlight a few characteristics of the tasks chosen. Although both patterns chosen are linear, the nature of their decomposition and the associated quantities differ in meaning. Among the many different conceptualizations possible, the first pattern can be imagined as  $n$  groups of four

dots (with  $n$  corresponding to the stage number) with a constant dot at the very beginning (or at the very end depending on your perspective) (see Figure 3.1).

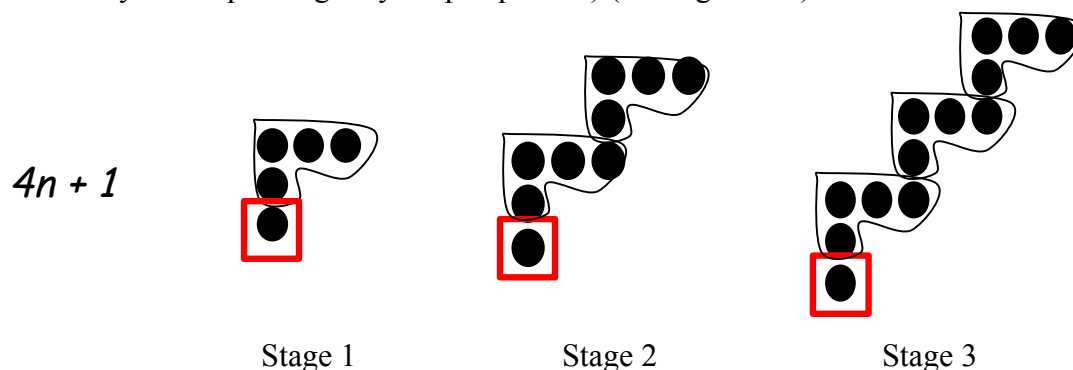


Figure 3.1. Possible decomposition of Pattern 1.

Alternatively, this figure can be interpreted as  $n$  groups of 5, which requires the removal of  $n - 1$  overlapping dots (see Figure 3.2). I asked teachers to reflect upon a noteworthy aspect of this second expression: It consists of two instances of the variable. In addition, the teachers were asked to explore the meaning of the simplified form of this expression. To provide such an analysis, the teachers had to compare the two decompositions, which share the same final simplified form but originate from different conceptualizations.

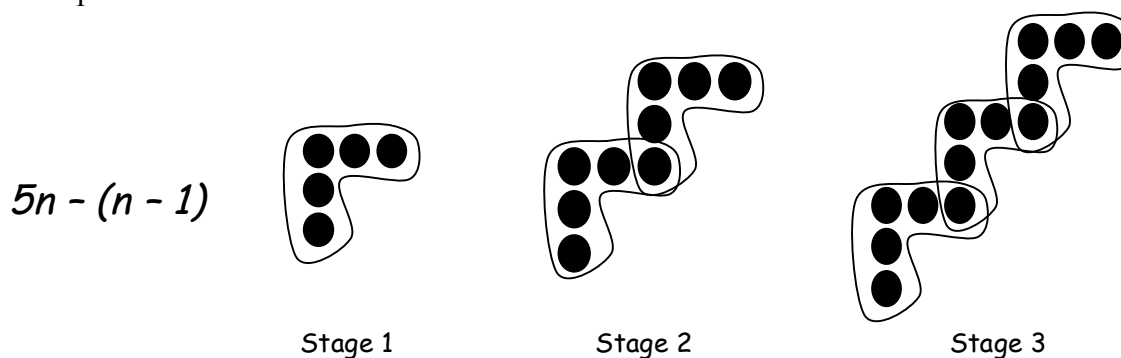


Figure 3.2. Alternative decomposition of Pattern 1.

The teachers also analyzed a second linear pattern (see Figure 3.3). Although symbolically it corresponds closely to the first interpretation of Pattern 1, the meanings of

the symbols differ considerably. As shown, in this situation the number of arms is constant, but their sizes grow with each stage. Therefore, from this interpretation, it consists of three groups of  $n$  dots with one constant dot (as opposed to  $n$  groups of four dots in Pattern 1). In effect, the roles of the symbols have reversed in the pattern. Another noteworthy detail is that the constant dot does not align with the vertical tower as one might assume. My goal was to investigate the teachers' grappling with such details.

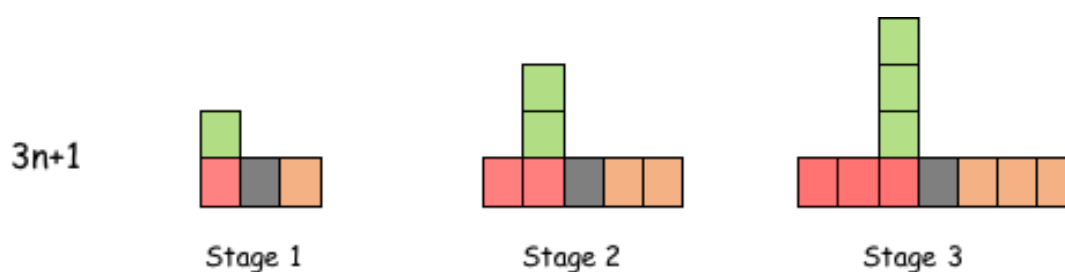


Figure 3.3. Possible decomposition of Pattern 2.

**Analyzing student-generated expressions.** In addition to asking the teachers to explain the meaning of expressions they generated, I presented them with students' responses for the first task and asked them to offer possible interpretations based on the students' work. I began this portion of the interview by asking what the teachers' noticed in general and then asked what meaning they thought the students were making of the algebraic notation. At times I showed the teachers only one particular part of the students' solutions, and at other times I showed an entire page of work.

I solicited student responses from ninth-grade students attending the high school many of the participating teachers' students eventually attend. Of the more than 30 student artifacts I received, I presented the teachers 3 responses that I had selected to



provide a diversity in terms of interpretations of the visual pattern, accuracy in their responses, and ways of reasoning about the number of dots. The overall diversity provided data for how the teachers analyzed incorrect expressions as well as expressions that were the result of both numerical and visual approaches to the pattern. I also purposefully included responses that aligned with each of the two decompositions outlined above in Figures 3.1 and 3.2. This choice served two purposes. First, it ensured that each teacher analyzed both views of the pattern, providing more consistent data across the teachers. Second, some teachers dissected the pattern in only one way on their own; thus, I asked these teachers to analyze an expression that did not correspond to a way they had interpreted the pattern. This request created a situation in which the teachers had to search for a meaningful interpretation of the figure that corresponded to the expression, rather than vice versa.

### **Research Question 3**

Research Question 3 follows: What subtle differences in knowledge and practice distinguish two teachers who both use students' conceptions as the basis of their instruction when they strive to engender in their students a meaningful understanding of algebraic generalization? To answer this question, I examined data from two teachers as case studies, a feature inherent in the question. To be clear, my goal for investigating this question was to develop theory, not test it, that is, to understand these teachers' knowledge and practices, not measure them. By comparing these two teachers, I aimed to characterize the MKT one needs to effectively support students to generalize algebraically in an instructional environment that follows a responsive-teaching model. As I reported earlier, inquiry into teachers' knowledge about algebraic generalization, in

general, and the type of knowledge required to successfully navigate this domain in a student-centered classroom, in particular, has been limited. To generate data for the type of detailed account that I hoped to produce, a case study approach was well suited because it allows for rich empirical exploration, description, and explanation (Yin, 2003). Moreover, a case study approach is extremely useful for comparing the interpretation and actions between teachers with similar perspectives and across similar settings, so that a phenomenon can be studied in-depth from different perspectives (Yin, 2003). In Yin's view, studying multiple cases is analogous to performing multiple experiments. As such, these relatively comparative contexts provided a useful perspective for me to better interpret the significance of various details involved. The similarities helped me to triangulate my observations, enabling me to test assertions about what characteristics of the teachers' MKT affected the learning environment, ultimately enabling me to better understand each teachers' knowledge.

**Research design for Phase II.** To develop a comprehensive understanding of the MKT these teachers possessed, I examined both the knowledge that informed their instructional decisions outside of class when they organized their lessons as well as the knowledge they drew from during instruction to assess the learning environment and make in-the-moment decisions. To gain insight into such areas of knowledge, I investigated the teachers' thinking while they planned in preparation for the lessons as well as their thinking during live instruction, namely their noticing of students' mathematical thinking. To do so I adopted a modified version of the 3-Step Design presented by Busse and Borromeo Ferri (2003). As the name implies, the design consisted of three components, preinterview (Step 1), classroom observations (Step 2),

and semistructured stimulated-recall interview (Step 3) (see Figure 3.4). These three elements together inform the researcher's interpretation of the phenomenon and triangulate inferences he or she makes. Each phase of data collection informs the researcher's subsequent data collection and analysis and vice versa. Busse and Borromeo Ferri (2003) argued that because the three phases of data collection are conceptually intertwined, temporal sequencing is not a concern. For example, a teacher's comment during class as part of Step 2 can be used to motivate inquiry during the stimulated-recall interview in Step 3 and to analyze any data obtained during the preinterview in Step 1. Such a conceptualization was especially useful because my data collection for each teacher consisted of two cycles, so that all six components could be considered and used as a single data source to draw and confirm conclusions about the teachers.

In addition, before I collected data via the aforementioned 3-Step Design, I conducted a clinical interview with each participating teacher to establish the teachers' overarching goals and motivation for the generalization unit following the professional development (see Appendix C for full interview protocol). Figure 3.4 illustrates the modified 3-Step Design including the preinterview. The details of the preinterview and each of the three steps in the research design follow.

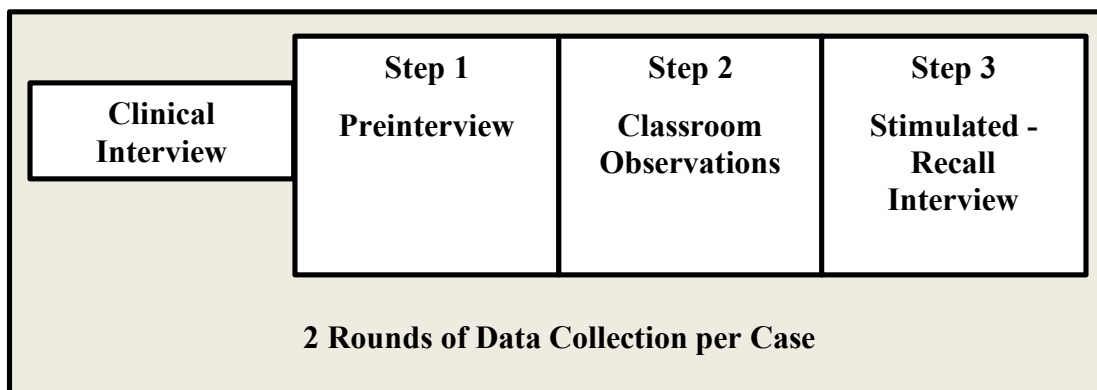


Figure 3.4. 3-Step research design.

**Clinical interview.** The main purpose for this interview was to identify globally how the teachers interpreted algebraic generalization after the week-long professional development. Because one goal for the week was to instill in the participating teachers a richer MKT associated with algebraic generalization and to reshape their associated goals, their developing a quantitative understanding of the notation was a critical piece. To examine the two teachers' views of algebraic generalization in preparation for the teaching component of the study, I again asked them to describe the goals they associated with these figural tasks and what value they saw in teaching a unit on algebraic generalization. Whereas I initially left the question quite open, I eventually pressed the teachers in an attempt to see what types of specific mathematical goals they held. In addition, as in the first clinical interview, I gave them hypothetical situations to contextualize their responses. I also asked them to rate and weight various goals that emerged during the week as a way to compare those among the teachers and with my view as well. Again, through these questions, I aimed to develop a comprehensive understanding of the two teachers' views of algebraic generalization, in preparation for the classroom-observation component of the study. Once again I used a clinical-

interview format to probe the teachers' understanding so that I could follow-up with specific questions tailored for each individual teacher as a way to test my interpretation of their responses during the interview.

***Daily preinstructional interview.*** Before each classroom observation, I spent approximately 15 minutes with the teachers asking them about their upcoming lesson. Specially, I asked them to explain their goals for the day, to describe details of their instructional plan, and to articulate their expectations for students' responses (See Appendix D). These day-to-day descriptions provided me detailed information about how the teachers endeavored to sequence the students' learning, providing evidence for their envisioned instructional trajectory for how students would come to generalize. The nuances with which they were able to anticipate student thinking and learning played a critical part in my analysis because one distinguishing feature in their MKT relates to anticipatory knowledge. In addition, this prelesson commentary helped me better understand the events of the class from the teachers' perspectives, giving me insight into what they saw and understood and assisting me in identifying moments to choose to analyze in the postinterview.

***Class-instruction observation.*** Each lesson was filmed from a single, stationary camera placed in the back of the room. I positioned the camera at all times to be focused on the teachers to ensure that I captured all the questions and decisions they made. In addition to recording data about the teachers' questions, comments, and reactions, having the camera trained on the teachers gave me access to the same classroom information that was available to the teachers, most notably, all student contributions. Such access proved invaluable in helping me develop a detailed sense of the flow of student

understanding from the teachers' perspectives, giving me insight into what the teachers were understanding and their decision-making processes. To maintain quality audio during the filming, each teacher wore a microphone with the speaker positioned next to the camera. In addition, any time the teacher engaged with a small group, I placed a small, handheld recording device on the table to capture details of these conversations. Finally, during the lessons I took observational notes that helped me begin to formulate both hypotheses about the teachers' understanding and subsequent questions to introduce during the subsequent stimulated-recall interviews.

Although such methods provided me observational data, to gain detailed access to teachers' in-the-moment thinking and understanding of the instructional environment, each teacher wore a small, portable head camera. This camera, designed by Dejaview (Reich, Goldberg, & Hudek, 2004), continually records and erases footage in a loop mode until the record button is pushed. When the user pushes this button (located on a small battery box worn on the user's belt), the video of the 30 seconds previously recorded is saved. The small hard drive has space for up to sixteen 30-second clips to be saved; they can then be downloaded, stored, and viewed on a computer.

Before each lesson the teachers were asked to identify significant moments during the lesson. Specifically, they were given the following directions to guide their selection:

You described to me the goals of the lesson. In general, I am interested in what you identify as significant moments, or aspects of the lesson as they pertain to the goals of the lesson, and in particular what incidents of students' thinking seemed salient.

The wording of this question was purposely designed to convey a certain type of instance which is rich with student thinking. The type of knowledge that I was trying to characterize is focused on students' mathematical thinking. As such, I was not interested in the teachers' understanding of other classroom features. At the same time, I attempted to craft the question to allow the teachers flexibility in deciding and interpreting significant moments.

These clips were later used as part of the stimulated-recall interview. During the interview, the teachers were asked to review the clips and explain why they had selected each piece and what they were thinking at the time. The purpose of this methodology was two-fold. First, the method had the advantage of breaking the continuous classroom instruction into discrete pieces, elevating particular moments. Second, the teachers instead of the researcher selected the moments, providing insight into what the teachers were interpreting about the instructional environment in general and students' understanding in particular (Sherin, Russ, & Colestock, 2011). Although knowing whether the teachers' comments represented their thinking during classroom instruction or their analysis of the situation afterward is impossible, I believe that this design is superior to other methods for capturing teachers' immediate thoughts about the moment, as opposed to their reflections after the fact.

***Semistructured video-recall interview.*** On the day of each classroom observations or the subsequent day, I conducted a semistructured video-recall interview approximately 90 minutes in length. Because Jack's class was in the morning, I could prepare for the interview and return at the end of the school day, 6 hours later for his interviews. Clara's class occurred later in the school day but did not meet every day, so I returned on

the off days to conduct the video-recall interviews 27 hours after class, before the next day of instruction. Between the time of the class and the interview, I reviewed the various clips selected by the teacher and selected three or four other moments that I found particularly rich in terms of student thinking or that seemed critical from my perspective for the overall sequencing of the lesson. In addition, I selected two or three moments from the classroom instruction that had not been identified by the teacher but that I interpreted as important or that I wanted to press for clarity. For these I simply marked beginning and ending times on the stationary-camera video as well as the times on the sound recording for the moments that occurred during small-group interactions.

I began each interview by asking the teacher to select the clip that was most significant. I then showed the clip and asked the teacher to explain his or her thoughts as they had occurred during the moment in the video (See Appendix E). I encouraged the teachers to stop the video when appropriate to discuss specific moments in the clips that they deemed particularly noteworthy. In addition, I usually followed up by asking the teachers to explain what they thought the students were understanding in these moments. Also, when pertinent, I asked what instructional decisions the teachers were considering at the time and why they ultimately made the decisions they did. Eventually, after the teachers had reflected on the three or four videos they had selected, I repeated the process with the two or three clips that I had chosen. During the interview, I also probed teachers' thinking by asking more specific clarifying question. For example, on several occasions I asked whether a particular response or action was indicative of their overall general instruction. As I stated earlier, a distinguishing characteristic of semistructured interviews, is that the researcher is to begin to develop and test hypotheses. The



structure of this methodology inherently provided time to do this. During the break between the classroom observation and the follow-up interview, I reflected on the different classroom moments and formulated additional specific questions about the teachers' understanding and motivations.

### **Data Analysis**

I began data analysis by creating transcripts of all interviews in Phases I and II and of each teacher's classroom instruction in Phase II using Inqscribe (Inquirium, 2013). Although details of the analysis for each question differed considerably, the overall approach for all three questions was quite similar, involving grounded theory (Strauss & Corbin, 1994; 1998), a methodology appropriate for constructing theory from qualitative data. I therefore begin by outlining the various components of grounded theory in general and then explain the details of applying this method for each research question.

**Grounded theory and coding.** *Grounded theory* is a general methodology designed to produce viable and cohesive justifications for the patterns identified among the concepts perceived in the data (Strauss & Corbin, 1994; 1998). In a grounded theory approach, the investigator strives to understand the themes that unify the concepts and explain their relationships, to generate a conceptual *theory* or explanation for the phenomenon recorded. In the end, the researcher develops a cohesive theory to not simply describe the events but to provide a more abstract model to predict what events will occur, given certain circumstances, or what outcomes follow given inputs.

For data to be reliably converted into categories and eventually to a coherent theory, data must pass through three phases of coding (Corbin & Strauss, 1990; Strauss, 1987). During the initial process of *open coding*, data are separated into discrete

segments, which, in turn, are examined and grouped by themes. One reason for labeling of similar segments is to reduce the data from a fractured set containing many individual segments to a set containing related segments and, thus, fewer categories. Pre-existing codes are often used in cases for which previous, related analysis has been done.

Because no other researchers have explored teachers' MKT in the domain of algebraic generalization, an a priori coding scheme was not available. Consequently, I used an inductive process of open coding in which categories of codes emerge from the data (Miles & Huberman, 1994).

After data have been open coded, they are subject to the process of *axial coding*. During this phase, the data become more comprehensively structured while the researcher looks for unifying themes. Previous organization is often tested against further data, resulting in more refined categories and subcategories. Eventually, the various emerging groups of data are woven into threads of relationships surrounding a category of interest, leading to the third and final stage of *selective coding*, in which the researcher identifies a comprehensive, unifying theory to connect the various conceptual categories and brings overall clarity to the phenomenon (Corbin & Strauss, 1990; Strauss, 1987).

Grounded theory's reliability results from researchers' using the process of *constant comparisons* (Corbin & Strauss, 1990; Strauss, 1987; Strauss & Corbin, 1994). Throughout the stages of the coding process, after phenomena have been described and recorded, one examines the codes by repeatedly comparing their fit with other similar data, either of other instances from the same individual or from other comparable participants. One does not analyze anew each additional occurrence, but compares it to determine whether previous codes adequately account for details present in the new data.

This process of revisiting segments and groups of data, looking for similarities and differences, helps the researcher refine and revise the possible meanings of the data (Corbin & Strauss, 1990; Merriam, 2009). Eventually, the systematic identification, review, and refinement of categories converges to a consistent model.

**Data analysis: Research Question 1.** The majority of data used to answer Question 1 came from the teachers' responses to a short series of questions during the Phase I clinical interview. Having only one source from which to make inferences and the clarity of the teachers' comments made the analysis relatively straightforward. Because the teachers clearly articulated their interpretations either as a topic for setting classroom norms or for specific procedural outcomes, two categories quickly emerged. After determining these two categories, I reviewed the teachers' written and verbal reflections in which they were asked to explain how their views of algebraic generalization changed during the week of professional development. These data, too, were extremely beneficial, because the teachers provided clear details about their previous goals when they juxtaposed their prior understandings of algebraic generalization with the type of learning outcomes they were experiencing in the professional development. Overlaying these comments with my previous analysis helped me confirm my characterization of the mathematical goals these teachers associated with figural patterns.

**Data analysis: Research Question 2.** I began my analysis for Question 2 by locating all instances in the initial interview of teachers' struggles or hesitations in connecting the algebraic notation to the quantities in the figure. Through this process, I isolated exact data to focus my analysis. I then compared teacher difficulties across the

4 teachers and discovered two similar challenges in 3 of the 4 teachers; thus, I reduced my final analysis to these 3 teachers (Denise, Clara, & Jack). Comparing the answers and apparent conceptualizations among these 3 teachers helped me understand the details in their thinking and ultimately characterize these challenges. I then returned to the data to explore what actions, comments, or comparisons made by the teachers seemed to either help them overcome these challenges or, in contrast, appeared to complicate their understanding. In addition, I analyzed how the teachers conceptualized similar items on different problems to get a clearer notion of the nature of their conceptualizations.

Again, the nature of the teachers helped tremendously in my analysis, because all 3 teachers were open and communicative in their thinking. Even when they realized that they had not thought about issues I was probing, they became increasingly expressive about the ways of thinking they attempted when they tried to make sense of the challenges they faced.

**Data analysis: Research Question 3.** For Question 3, because of the case-study approach I used as part of my research design, I conducted analysis on two levels: (a) within individual cases and (b) across the three cases (Stake, 2006). Analyzing the individual cases helped me identify characteristics of the teachers' understanding unique to that individual, whereas examining cross cases helped me better understand the significance of particular details, enabling me to make general theoretical assertions about the overall nature of teachers' MKT in the domain of algebraic generalization (Stake, 2006).

After collecting the data on 3 teachers (Denise, Clara, and Jack), I decided to reduce my analysis to 2 teachers (Clara and Jack) because of the stark contrast in the 2

teachers' students' abilities to generalize algebraically. Although both teachers articulated rich learning goals for students and genuinely endeavored to make instructional decisions on the basis of their students' thinking, their instruction led to very different learning outcomes.

***Analysis of individual teachers.*** I began my individual analysis of the 2 teachers by first reviewing each teacher's comments from the daily pre-instructional interviews and the stimulated-recall interviews collectively. Attempting to understand the instructional environment through the teachers' lenses, I looked in their responses for themes about their construal of specific events in the classroom and, in particular, about their motivations for particular decisions. To guide my analysis, I used the Professional Noticing of Students' Mathematical Thinking framework by Jacobs et al. (2010). In reviewing the teachers' comments about each of the identified clips, I first looked for elements of how the teachers attended to student thinking. To do so, I identified details of students' thinking the teachers focused on in the clips. I also highlighted aspects of student thinking that I saw as significant but that the teachers did not mention. I then examined the teachers' interpretations of the students' mathematical conceptualizations; in particular, I tried to capture what type of understanding of algebraic generalization informed their descriptions of how the students were thinking. Finally, I analyzed the explanations they provided for their decisions, or in some cases, for their hypothetical decisions, for how they responded to students' ideas. If their reasoning was not clearly stated, I inferred their motivation by looking at their actions. I then looked across the various cases for evidence to suggest trends in the type of understanding that might have guided these decisions.

For each of these three components, I continually referred to the teachers' comments about their goals for the day to help further explain what understanding of algebraic generalization seemed to be guiding their responses during the stimulated-recall interviews. Revisiting the data from the pre and post interviews provided me a balance of specific details and a global perspective of the teachers' understanding of how their students were engaging in the process of algebraic generalization. Eventually, in considering consistencies between these various areas, I identified 4–7 themes for each teacher.

I then turned to the classroom videos. During this part of the analysis, instead of trying to understand classroom events through the teachers' lenses, I interpreted them from my own perspective. I continued to use the professional noticing framework to inform my analysis of the students' thinking but now attempted to make inferences about the teachers on the bases of their actions in class. For this analysis, I first made narrative accounts of each class and then identified moments of teacher interaction that I saw as critical for students' learning, both productive and unproductive. Finally, I attempted to formulate my own rationale for these events and outcomes, at times imagining what I as a teacher might have done. This analysis helped me to develop an understanding for the class from my own perspective. I then compared my interpretation of classroom events with the themes that emerged in the teachers' reflections.

Purposefully attempting to separate my analysis of the teachers' responses and my interpretation of the classroom instruction served several purposes. First, it helped me to better understand the classroom from the teachers' perspective. By separating the two analyses, I identified more clearly what I saw and what the teacher was seeing and, in

turn became more descriptive and less evaluative in my analysis. In addition, through this approach, I was able to produce two sets of data points: (a) the teachers' comments about the generalization process and (b) the classroom events as described through my interpretation.

*Analysis across the case studies.* After I had fully characterized both teachers independently, I compared and contrasted data for the two teachers, looking for connections among the identified themes. Although some themes were common to the 2 teachers, more were not. To consider themes that were not shared, I returned to the data and attempted to analyze each teacher through the lens of the other. By this, I mean that I looked for evidence in Jack's comments and instruction that corresponded to themes identified for Clara and vice versa. Through this activity, I identified both the significance of particular overarching characteristics and attributes in each teacher that originally had not been salient. Furthermore, much of this process of comparing and contrasting the two teachers mirrored the constant comparative method when I tested my initial interpretations of the teachers against the data of another teacher. This process helped to ensure the validity and reliability of my final conclusions.

## **Chapter 4: Teachers' Understanding of Algebraic Generalization**

This chapter is a report of the perception and mathematical understandings of algebraic generalization the four participating teachers held at the beginning of the study. In the first of the chapter's two sections, I provide an analysis of the mathematical goals these teachers held for a unit on algebraic generalization. In the second section, I describe the teachers' personal approaches to solving these generalizing tasks and present the results of my examination of their abilities to explain the meanings of their resulting numerical and algebraic expressions in terms of the quantities they represent in the figure.

### **Mathematical Goals Associated With Generalizing**

As stated, in the first part of this chapter, I characterize the various learning objectives this group of experienced teachers had for engaging their students in figural generalizing tasks. In particular, it serves to answer my first research question:

What mathematical goals do teachers who have experience with student thinking, in general, and algebraic generalization, specifically, associate with figural patterns?

Two categories of goals emerged from analysis of the various descriptions provided by the teachers. The first category included goals that were general in nature. These teachers did not see these generalizing tasks associated with any particular mathematical content, but rather used them in class to instill norms and universal ways of thinking. In contrast, the second category consisted of a detailed list of topics, characterized mostly as facts and procedures that were seen as related to these tasks. Although specific in nature, the outcomes were so prescribed and couched within a particular model of direct instruction that the goals arguably did not encompass algebraic generalization. Therefore, in both cases, the goals were considerably lacking in terms



of mathematics and seemed to be associated with a style of instruction as much as any specific content area. A closer look sheds light into how these teachers viewed these tasks.

### **General Goals: Classroom Norms and Mathematical Practices**

Three of the 4 teachers, Clara, Palila, and Jack, provided goals that were quite general in nature. Their comments not only lacked specificity, but the goals they highlighted could be goals for a wide range of mathematical activities. In spite of their depth of exposure to this topic, these teachers provided limited depth and clarity in their related mathematical content goals. Although their descriptions revealed an enthusiasm for the topic, their motivation for having students engage in such tasks was vague and often stemmed from a desire to instill classroom or sociomathematical norms. To illustrate the generality of these comments, first consider Clara, who offered the most general descriptions. She stated that she included these tasks because she wanted students “to develop thinking skills, logic and thinking abstractly, seeing patterns, pattern recognition.” She went on to say, “It is helping you see connections. You are training—you’re exercising your brain to look at patterns.” She used the existence of patterns in various everyday encounters such as the stock market to emphasize the importance of these skills. She emphasized the generality of these learning goals by making an analogy to literature and how reading specific books broadens the mind and makes one more intelligent. She added that through these tasks students will “develop a skill that [they] are going to apply to jobs that we might not know about now.” Connecting these activities to a broad array of topics, seemed to underscore the general nature of her goals. Similarly, Jack and Palila provided rationales for this unit with little distinction

mathematically from other investigative or student-centered tasks. Palila stated, “I would like them to recognize patterns. I would like for them to make relationships between different models of representations. I want them to problem solve.” Jack highlighted that he thought these tasks were good for students because engaging with them exposed students to fun and different types of problems. He added that he found them beneficial to “erase [the idea] that 'there is one way to do it' and 'this is the right answer thing'.” Although these are all laudable goals, such outcomes can be attributed to any open, exploratory task, not specifically to algebraic generalization.

All three teachers did highlight specific CCSS mathematical practices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and included other descriptions that aligned with elements of these standards. Again, though, their comments indicated that, at least to some degree, their open-ended, accessible nature, not necessarily the content, made these tasks worthwhile. First and foremost, the three teachers in this category clearly called out SMP #1, make sense of problems and persevere in solving them, as the relevant standard. They emphasized that they liked engaging students with these problems because they instilled the idea that students can, and need to, make sense of problems and persist in their solutions. Palila explained that these tasks require students to apply skills and strategies in contexts they have not seen before. These three teachers also referred to SMP #7, seeing and making use of structure, as related to the tasks. Clara alluded to this practice, stating that to solve these tasks, students must “take things apart.” Palila and Jack said that they valued these tasks because students will see the patterns differently, explicitly highlighting this as an example of learning structure.

In their framing of these goals though, the three teachers seemed to see these outcomes as a result of a style of instruction associated with the tasks as much as of the mathematics. Palila and Jack explicitly distinguished their teaching of these tasks from other content-driven lessons that were designed to introduce students to new mathematical topics. Palila said she liked this type of tasks because unlike her other lessons, they did not require “precoaching.” Jack stated he was trying to figure out how to give his students similar “tasks in a new content they haven’t seen before,” contrasting such tasks with these “doable” tasks, in which students can engage without instruction. According to these characterizations, the teachers clearly saw connections to various mathematical practices but did not relate the unit to any particular area of mathematics. In fact, as Jack and Palila alluded, they saw this detachment from mathematical content as an advantage, believing that such an attribute made the topic more accessible for their students.

Furthermore, when I followed up and pressed each teacher to identify specific mathematical content goals, no teacher provided any. Jack openly admitted that he could not think of one content goal. He reiterated that he used these tasks specifically to build the culture of his classroom, emphasizing the classroom norms of different paths to the answer and the need for students to engage in discussions around their thinking. Palila and Clara also responded without clearly stating mathematical goals. Clara said, “Kids are going to have to utilize arithmetic ... and then to generalize that, they are applying the algebra.” Palila added that she wanted her students “to be able to see the patterns..., to come up with a way to predict the next step, and, again, hopefully they can use that to make a table or a graph.” In these two cases, the teachers included global aspects that

seem inherent in arriving at answers to these tasks, but without articulating any associated mathematical concepts. Clara underscored that arithmetic and algebra would be involved, and Palila reiterated that students would see patterns and make use of different representations. Neither of these descriptions provides insight into the mathematical understandings students will gain from these tasks. Even when I pressed for specific content goals, these three teachers were unable to add any more details to their answers. Although these teachers valued these activities and had a significant amount of experience working with these tasks, they appreciated this topic as a more general experience for students. As Zazkis and Liljedahl (2002) found in their study, these teachers saw these generalizing tasks as “recreational enrichment rather than curriculum core.”

Although the effect these vague and disconnected goals would have on teachers’ instruction is unclear, I predict that without concrete mathematical content objectives, teachers might overlook areas in which students struggle. Teachers may not recognize the concepts with which their students are grappling and consequently may fail to offer the necessary support or may miss opportunities to push students in new directions. In addition, teachers may tend to focus simply on helping students arrive at the answer and assume that their students are learning, without identifying exactly what mathematics the students have or have not understood.

### **Detailed Goals: Facts and Procedures Related to a Computational Orientation**

In contrast to these three teachers who struggled to articulate clear mathematical content goals, the fourth teacher, Denise, provided a comprehensive list of topics she associated with these tasks. She began by explaining that the main goal was for students

to “learn how to see the growth, the slope, in various representations [and] how to find the y-intercept in various representations.” She specified that she would want her students to connect these two elements of linear relationships across the four representations: a table, a graph, a rule, and the pattern. She then included a list of subtopics, adding that she would want students to learn how to graph a line without plotting points from a table, be able to identify proportional relationships graphically, and, finally, determine which relations are and are not functions. Furthermore, for each of these areas, she described the precise responses she hoped students would produce to indicate mastery. Overall, Denise seemed to have a very detailed sense of her objectives for this unit.

That being said, although Denise specified a list of topics, and articulated the exact facts and procedures she saw associated with these topics, I argue that her goals were as indistinct mathematically as those put forward by the other teachers in the previous category. Moreover, her outcomes seemed removed from generalization, both in substance and in nature. Providing a list of related topics does not serve to explicate the mathematics embedded in these generalizing tasks. Although particular patterns may be modeled by a linear equation or represent a proportional relationship, these are not foundational to learning how to generalize or communicate generalizations using algebraic notation. In addition, as her comments throughout the week of the professional development illustrated, Denise’s view of these various content areas was based in what has been referred to as a *calculational orientation* (Thompson, Philipp, Thompson, & Boyd, 1994). Her description of the objectives communicated an image of instruction in which the focus was not on developing particular ways for students to reason about the

mathematical content but rather on familiarizing them with specific facts and calculational sequences to ensure they would arrive at the desired answer.

During the week of professional development, when Denise was exposed to alternative possible goals and began juxtaposing her previous understanding with this new perspective, she began to recognize a difference. In the postinterview, she acknowledged that at the beginning of the professional development she was unsure how “we were going to generalize without finding figure 0.” According to her understanding, the purpose of these tasks was to provide a way to show students how the two components of the slope-intercept form manifested themselves in different representations. As the week progressed though, Denise became cognizant that she had been focusing on ensuring that students develop particular solution methods rather than ways of thinking. By the end of the professional development, she commented that she had come to see a significant “difference between getting students to generalize ... and getting students to see connections between various representations.” In the postinterview, she reflected on this difference, stating, “Before we were just interested in that rule, the growth and the starting point— not really the generalization, not really trying to understand that you can see it differently than this person, and we can describe that algebraically.” Reflecting on her students engaging in this activity, she realized that she had not been developing their thinking, but “imposing something on them.” Her students could write a rule because they identified the two things she had told them to look for: the growth and starting point; but, as she recognized, her students were not generalizing. In the end her students would “all have the same rule, the same starting

point. And it's either right or wrong because it's growing at the same rate and its  $y$ -intercept is the same.”

When I asked Denise what she perceived might be some drawbacks to her previous approach, she commented that she had already observed that her students were often confused trying to remember where the “growth” went in the formula. She clarified their misunderstanding, explaining,

It doesn't make sense that the growth is multiplying by  $x$  because it is adding. It seems like ... the growth goes with that plus [referring to the constant term in the expression] because we are adding. We're not multiplying anything. It is difficult to explain why the growth goes there other than telling them.

Providing students with a sequence of steps so that they can write a rule is not supporting them to generalize. Clearly, as Denise reflected, she had already witnessed effects of her calculational approach. Her students, having been shown a procedure they had not developed, were unable to attach meaning to the various symbols and consequently struggled to correctly write the rule.

Finally, Denise's approach to algebraic generalization was to decontextualize the problems. She instructed her students to immediately extract the numbers and focus purely on the numerical relationships. Reflecting on her instruction, she explained that she had not been asking students to analyze and make sense of the visual patterns themselves, but only to look at the numerical values to find the change. Clara echoed this approach, stating that she previously had used the patterns only as a basis for the numerical data. She had thought that “you just use this visual pattern to get to a table and then from the table you get the constant difference, and you develop the rule.” She added that she felt that the associated context made the task more interesting, instead of

simply presenting a dry table of data. Clearly both teachers had encouraged their students to disregard the context, believing that a numerical approach would enable their students to more easily arrive at the expression. Their comments also indicated that they saw the figure as purely a source of the data, not as a resource for students to use to reason about and make sense of the problem.

Denise detailed the contrasts in her old and new goals:

Before, I don't think they were making meaning, but they were able to count and say how it is changing. It feels very differently than just looking for a way to do a problem rather than making meaning of the math that they see. If they are decomposing the pattern, if they are breaking it down and seeing what's happening, then they are able to write an expression that states what they see is happening.

In this reflection, Denise explained how her approach of focusing primarily on providing a procedure to identify and convert specific information into a particular form had not provided her students with the necessary foundation to make sense of the algebraic expression. Guided by the principle goal that students would generate a certain expression, she provided instructions that did not direct her students towards developing the understanding of various necessary related concepts, and, consequently, they were struggling. Although many of her students developed the ability to write a linear equation, they were not developing the skills to generalize in other contexts. Furthermore, this approach fostered in the students a limited interpretation of the notation they were using. They had been told the meanings of the symbols in isolation, but such a definition did not provide for flexibility in their use or support students in making sense of the symbols collectively or in new contexts.



If we assume that these data are representative of the field and extrapolate from these findings, teachers seem either to see as unit associated with the figural tasks as providing students with a rich mathematical experience but without connecting specific content or they see these tasks connected to a wide range of topics, but in a superficial way, without reflecting on the associated understanding developed.

### **Teachers' Quantitative Understanding of Algebraic Symbols: Associated Conceptual Challenges and Possible Resolutions**

Attending to contextual quantities often supports students in writing and manipulating algebraic expressions because their understanding of quantitative relationships informs them about the viability and appropriateness of operations between symbolic elements. However, attempting to define the quantities with precision and to coordinate their meanings across the expression can elevate particular difficulties of which teachers, who often have a more abstract approach and understanding, are not fully aware. My goal in the remaining part of this chapter is to answer my second research question:

What conceptual hurdles do secondary teachers encounter when they attend to the quantitative meanings of algebraic symbols that result from algebraic generalization, and what conceptualizations support them to overcome these challenges?

By comparing the struggles among the teachers when they attempted to connect and make meaning of notational descriptions of the generalized patterns, I identified two conceptual challenges with this process. For each challenge, I describe multiple conceptualizations, both unproductive and productive, that the teachers developed in attempting to overcome these challenges (see Table 4.1). The explanations and

examples of the various conceptual approaches in which the teachers engaged will clarify the nature and significance of the conceptual hurdles. To demonstrate each difficulty and conceptualization that emerged, I compare and contrast 3 teachers, Denise, Clara, and Jack, whose engagement with the task best illustrated the associated struggles and possible resolutions. To be clear, these conceptual challenges, though having pedagogical implications, relate to teachers' understanding of algebraic generalizations, not explicitly to teaching.

Table 4.1. Conceptual Challenges and Resulting Conceptualizations Associated With Contextualizing Algebraic Symbols

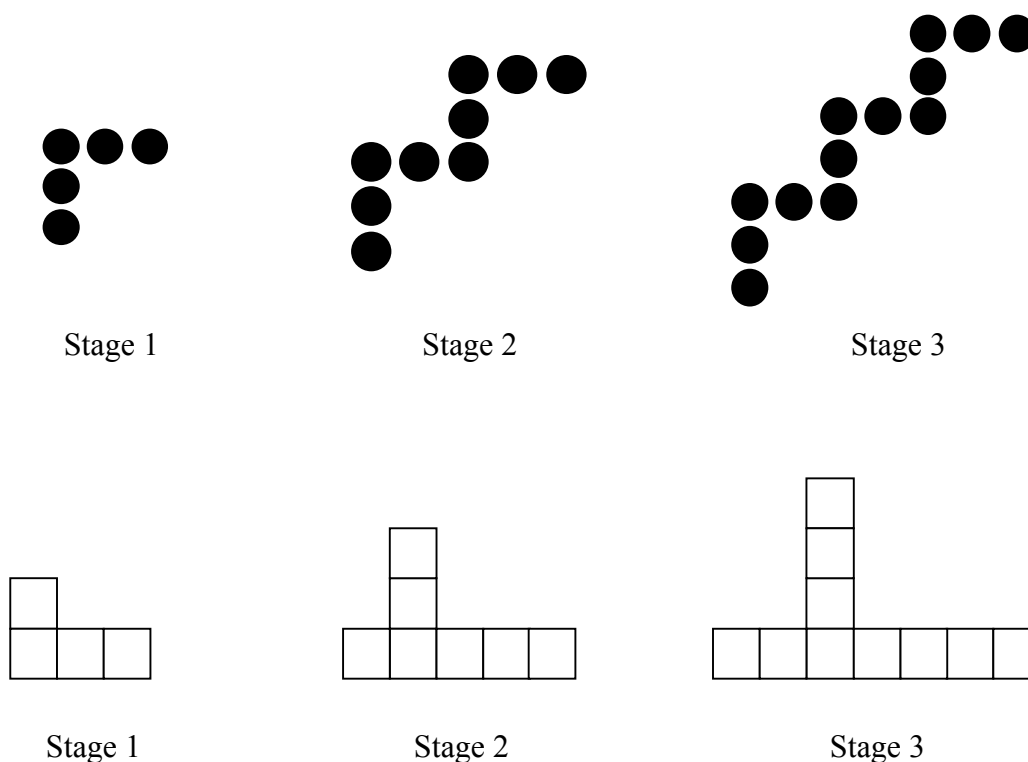
| <b>Challenge #1. Interpreting the coefficient of <math>x</math></b><br>Developing an understanding of the quantitative meanings of symbols so that the operations that bind them make sense. In particular, a variable and its coefficient must be conceptualized so that multiplication is a sensible operation linking the two symbols.                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                                          |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Unproductive conceptualization                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | Productive conceptualizations                                                                                                                                                                                                                                                                                                                            |
| Detaching meaning of the symbols from details of context. In particular, interpreting the coefficient as the constant difference between stages.                                                                                                                                                                                                                                                                                                                                                             | a) Interpreting the variable as the number of groups and coefficient as a ratio of dots or tiles per group, with the flexibility to exchange this mapping to match the details of the context<br>b) Reinterpreting the contextual situation to match single quantitative understanding of variable as number of groups and coefficient as size of groups |
| <b>Challenge #2. Interpreting expressions in which the variable appears more than once</b><br>Coordinating the meanings of referents so that the simplification of more complex expressions is contextually viable. Algebraic notation can be simplified by decontextualizing symbols and combining them according to algebraic rules. Attending to their referents often creates new challenges because the same symbols might be interpreted as different quantities, making their combination unfeasible. |                                                                                                                                                                                                                                                                                                                                                          |
| Unproductive conceptualization                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | Productive conceptualizations                                                                                                                                                                                                                                                                                                                            |
| Attempting to impose the interpretation of one variable onto another                                                                                                                                                                                                                                                                                                                                                                                                                                         | a) Retaining the different quantitative meanings but coordinating their numerical values so that their combination is viable<br>b) Reinterpreting both symbols so that their quantitative meanings align                                                                                                                                                 |

### **Interpreting the Coefficient of $x$**

The first conceptual difficulty that emerged among the four teachers centered on their understandings of the coefficient of  $x$ . Although all the teachers successfully wrote expressions to correctly model the number of dots or tiles per stage, their articulations of the meaning of the coefficient of  $x$  and its relationship to the variable were not always coherent. To highlight the associated challenges, I first explain Denise's attempt to provide a detailed description of the various symbols in her generalized expression of the pattern and the obstacles associated with her conceptualization. I then describe Clara's and Jack's interpretations, both of which were productive, but distinct, conceptualizations to clarify the relationship between the coefficient and the variable. To help the reader fully understand these three teachers' conceptualizations, I begin my analysis by outlining each teacher's method for deriving the expression and then explain each teacher's understanding of the symbols used. Finally, in addition to explaining the teachers' conceptualizations, I depict their struggles to develop this understanding and the associated ramifications.

**Review of figural-pattern task.** Before analyzing the teachers' responses and interpretations of the notation, I first review the two tasks posed in the interview. Two sequences of patterns were presented to the teachers (see Figure 4.1). For each pattern, the teachers were first asked to first draw Stage 4 and determine the number of dots in that stage. They were then asked to solve for the number of dots in Stage 10 and describe what it looks like. If the teachers did not provide a number sentence as part of their explanations, I asked them to produce one for Stage 10 and explain the meanings of

the symbols. Finally, I asked the teachers to provide a general rule for the number of dots in any stage and to once again communicate their interpretation of each symbol. The details of the protocol used can be found in Appendix C, and an analysis of the possible decompositions is included in Chapter 3.



*Figure 4.1.* Figural patterns from interview protocol.

**Limited quantitative understanding of the symbols.** Denise's responses highlighted the challenges associated with developing an understanding of the quantities represented by a variable and its coefficient necessary to explain their multiplicative relationship. As I will show, although she approached the problem quantitatively, decomposing the pattern and communicating her understanding of the various components symbolically, Denise was unable to articulate precisely the contextual quantities the coefficient and variable represented, expressing her understanding more

generally. Ultimately, she failed to explain not only the multiplicative relationship between the coefficient and variable but also their connections to the context. When she attempted to apply her understanding to other problems, the disconnect of her view became more visible and the associated obstacles of this conceptualization became more salient.

***Denise's initial approach to pattern generalization.*** I describe Denise's overall approach to Pattern 1 as quantitative. That is, she worked to first understand the pattern contextually, identifying the quantities in the figure, and then used algebraic symbols to express her understanding of the figure's structure. Her work and explanations demonstrated this approach. When asked to analyze the pattern and solve for Stage 10, she first decomposed the figure into recurring overlapping groups of five dots, circling each subsequent group (see Figure 4.2) and repeating, "So there would be a group of five here, but I would subtract 1." She then provided a numerical expression for Stage 10 that corresponded to her decomposition of the figure, writing  $10(5) - 9$  and explaining, "There are going to be 10 of these L's, but then there's going to be that one overlapping dot." Finally, she arrived at the general expression  $5x - (x - 1)$ , explaining, "So the stage number times 5 minus the stage number minus 1." Although Denise did not mention the exact quantities in the figure in her description of this generalized form, her overall analysis and her multiple explanations indicated that she had derived the algebraic expression from her decomposition of the figure and that the various expressions she had created matched her understanding of the pattern, at least at a general level. I include this analysis to document that Denise did not use a procedure to derive the expression but, rather, understood the symbols sufficiently to generate an expression that was

connected to her view of the pattern. Denise's understanding of the symbols' referents was, however, quite general and not detailed enough to explain how exactly the expression modeled the constituent parts of the figure. Such a view proved problematic when she tried to impose it on other contexts.

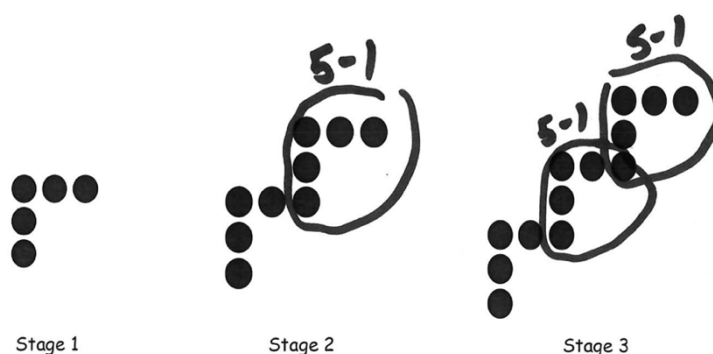


Figure 4.2. Denise's initial decomposition of Pattern 1.

***Denise's understanding of the coefficient.*** After Denise derived the algebraic expression  $5x - (x - 1)$ , I asked her to explain the meaning of the first  $x$  that appeared in her expression in terms of the figure. Initially she seemed confused by my request to indicate its meaning in terms of the figure. In her response, she began by repeating with a questioning, almost exploratory, tone, "the number of groups of five; the number of groups of five, minus one; the number of groups of five." Eventually, she looked up and with a reticent voice stated, "It [ $x$ ] could represent the number of groups of five." From her reply, although she had initially demonstrated a general understanding of the connection between the notation and the context in developing the expression, it appeared she lacked a clear understanding of the meaning of the individual symbols.

I then asked her to explain what the coefficient of 5 represented. She answered by identifying the 5 as "basically this [drawing a rectangle around the first group of five dots in Stage 1; see Figure 4.3], that group of five." Upon hearing Denise refer to both

symbols as groups of five, I pointed out the apparent contradiction and asked for clarification, “So the  $x$  represents the group of five and the 5 represents the group of five?” She then replied,

No, no, no .... Wait, wait, wait. This is the ... how many groups of five that I have (pointing to the first  $x$  in  $5x - (x - 1)$ ). I have one group of five; two groups of five. (Each time she pointed to the  $x$  and then the 5.) That's just, that's just five dots. (referring to the coefficient of 5). That's just like the original five here (drawing a rectangle around the first five dots in Stage 1 and the same five dots in Stage 2; see Figure 4.3). Well, ... yeah, ... umm. Yeah, it is actually the number of groups (pointing to  $x$ ). That is just 5 (pointing to the 5).

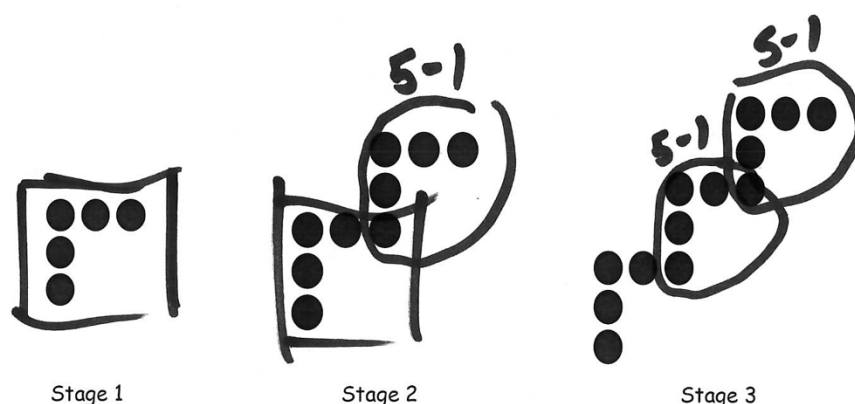


Figure 4.3. Denise's identification of "the original five" dots.

In this one quote, Denise interpreted the 5 first as the original "group of five," then "five dots," then again as "the original five" dots, and finally as "5," depicting it as a decontextualized number. Although her language indicated that she understood the combined expression  $5x$  to represent an increasing numbers of groups of 5, the overall exchange shows that she was not sure exactly what quantity the 5 represented. Furthermore, none of the characterizations she offered explained the relationship between the 5 and the  $x$  or provided a contextual rationale for why they should be multiplied together.

I then asked Denise if she could say any more about the 5. Instead of providing more details about the 5, she began reviewing her decomposition of the pattern. Eventually she changed her focus to other contextual issues without responding to my query. However, later in her examination of the pattern, she did return to talk about the meaning of the coefficient. After simplifying the previous expression to  $4x + 1$ , she began to analyze this new form. First, she returned to the figure, decomposing the pattern relative to this new expression (see Figure 4.4).

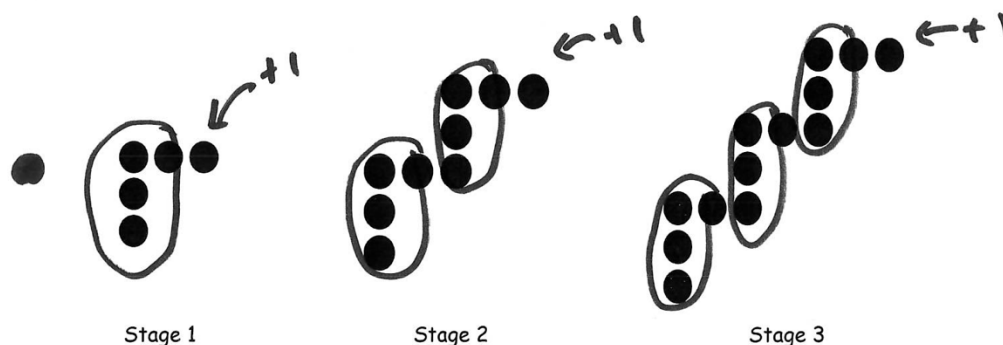


Figure 4.4. Denise's decomposition to match  $4x + 1$ .

Then, unsolicited, in an apparent attempt to make sense of the meanings of the symbols, Denise started to verbalize her interpretation of the entire expression. She repeated several times, "The number of groups of 4, plus 1; the number of groups of 4, plus 1;... the number of groups of 4, plus 1." I interpreted this description to indicate that Denise saw the  $4x$  collectively representing the number of groups of four dots, similar to her view of  $5x$ . Again, wanting a more detailed understanding of her interpretation of the individual symbols, I asked her about the meanings of each symbol in this new expression.

Interviewer: Okay, and the  $x$  here represents?  
 Denise: The number of groups of four.



- Interviewer: And the 4 represents
- Denise: Four .... The 4 represents ... four.
- Interviewer: And what do you mean by ... four.
- Denise: Oh, it represents four dots, yeah, four dots. The number of groups of four dots
- Interviewer: Are there specifically four dots it represents or ... what four dots?
- Denise: The four that are being added. You add on .... You are adding on a group of four each time. The number of dots that you add each time you go up; you increase in the stage. The number of dots that is added each time, that it's growing by.

Similar to her explanation of the coefficient 5, Denise again vacillated among interpretations. In this one exchange, she described the coefficient first as a decontextualized “4.” It then became “4 dots.” Finally, she concluded that it was the “the number of dots that is added each time.” These varying explanations show that Denise was confused about how the symbols and the figure correspond, almost as if she were searching for and trying out possible solutions. Although her development of the expression and explanation of the overall expression indicated that she possessed a general understanding of how the notation connected to the figure, she struggled to identify the referents of the symbols separately. She knew that the  $5x$  and  $4x$  collectively represented groups of dots, but was unable to disentangle the meanings of the two symbols (coefficient and variable) and explain the quantities that each represented independently.

Furthermore, Denise’s last interpretation of the coefficient as “the number of dots that is added each time” was particularly interesting because it aligned strongly with her instructional method. As I explained regarding the teachers’ goals, before the

professional development sessions, Denise's orientation to this topic was calculational (Thompson et al., 1994). She had come to view these exercises not as opportunities to understand and communicate quantitative relationships symbolically but as tasks in which students should learn to identify the change and the starting point in various representations to efficiently identify and write linear equations. In Denise's class, students were to interpret the coefficient as the constant change and use this fact to derive the expressions for other linear-generalization tasks. When Denise struggled to make sense of the individual symbols, she adopted this constant-change interpretation for the coefficient. Such a conceptualization, along with its complications, became clear later in the interview when she attempted to impose this understanding on a second pattern, demonstrating how such an interpretation is quite removed from the contextual situation.

*Denise's understanding of the coefficient imposed on the pattern.* Denise's approach to deriving an algebraic expression for the second pattern was somewhat different from her approach for the first. Instead of first developing a contextual understanding of the pattern and then using symbols to capture this interpretation, she derived part of the expression by noticing a numerical pattern and creating an expression that produced this numerical sequence. Specifically, she arrived at the solution by connecting the number of tiles in the vertical tower to the stage number and then observing that the base of the figure consisted of an increasing odd number of tiles. Denise then determined that multiplying the stage number by 2 and adding 1 produced

the correct number for the base. Finally, she combined the  $2s + 1$  and the  $s$  to get  $3s + 1$  (see Figure 4.5).

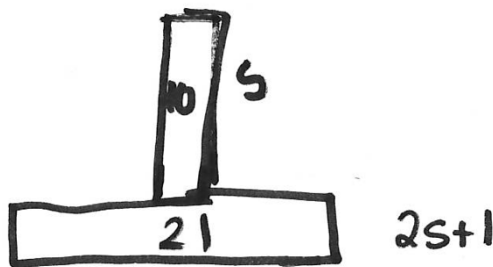


Figure 4.5. Denise's derivation of the expression for Pattern 2.

When Denise arrived at this algebraic expression, I again asked her to explain her interpretation of the individual symbols. She first explained the constant 1 as the one tile from stage 0 that is always there, drawing a single tile to represent stage 0 (see Figure 4.6). I then moved onto her interpretation of the coefficient, as demonstrated in the following exchange.

Interviewer: And what does the 3 represent?

Denise: The number of groups of three. Three tiles.

Interviewer: Could you maybe highlight that on one of these stages?

Denise: So there's three, there's three, there's three (circling each group of three tiles; see Figure 4.6.)

Interviewer: So the 3 represents what?

Denise: Three tiles

Interviewer: And when you say three tiles, again, are you saying specifically three tiles, or?

Denise: The three tiles it grows by (with an explanatory voice inflection). It grows by 3. So the three tiles that are being added.

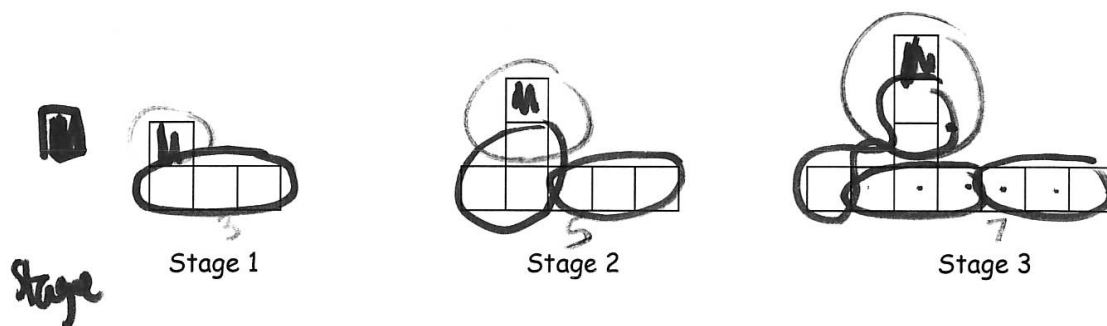


Figure 4.6. Denise's first interpretation of Pattern 2.

Again Denise offered multiple interpretations of the coefficient 3 (number of groups of three, three tiles, three tiles it grows by), before ultimately concluding that it represented the three tiles that were being added at each stage. Moreover, when asked to identify the 3 in the pattern, she appeared to randomly select the tiles. Her arbitrary choice of location indicated that her understanding of the 3 was not connected to the figure. It seemed that Denise was not initially seeing groups in the figure, but imposing them on the figure to make sense of the notation she used in the expression. Unable to identify the meaning of the coefficient, she adopted the view of 3 as constant change and then reverted to the idea of groups from the previous problem, imposing this inferred understanding on the figure.

I then asked Denise about the meaning of the 1. She explained, "The 1 is constant .... I could put the 1 anywhere." I asked Denise if it matters whether where these tiles she had identified were in the figure. Denise then took out another copy of the pattern and began exploring an alternative decomposition of the figures in Pattern 2. She first shaded the top tile of each tower for the constant of 1 and explained, "I don't really think of it as specific or not. I thought it was the top because it seems consistent, that they'll be one tile there at the top." She then transitioned to talk about the 3:

I don't know where the three are coming into; two there, one there, two there, two there [shading the one far left tile and the two far right tiles in Stages 2 and 3; see Figure 4.7]. That seems to be more of a pattern.

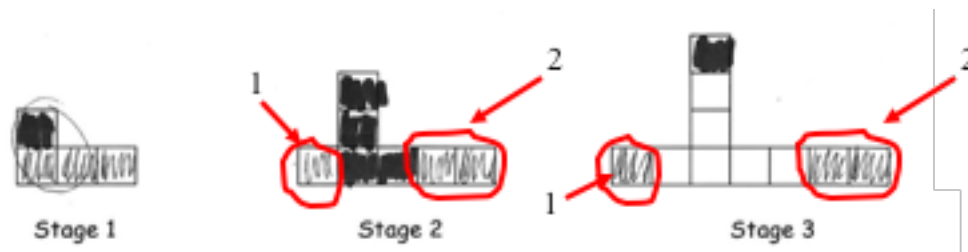


Figure 4.7. Denise's second interpretation of Pattern 2.

Wanting to better understand Denise's interpretation of the "three added on," I asked her to clarify her interpretation by asking her to highlight the three on Stage 3. As an indication of her interpretation of the coefficient of constant difference, she responded by questioning, the three "that were added?" Denise then highlighted once again the first tile on the left on Stage 3 and the last two tiles on the right by drawing arrows to them and labeling them as "adding 3" (see Figure 4.8).

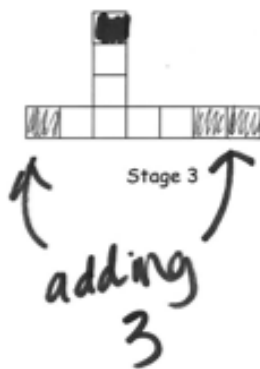


Figure 4.8. Denise's explanation of the coefficient of 3.

Once more, I asked Denise to clarify by identifying the three on Stage 4. Denise correctly drew Stage 4 and again shaded the same tiles as the constant 1 in  $3s + 1$  and the other three being added (see Figure 4.9).

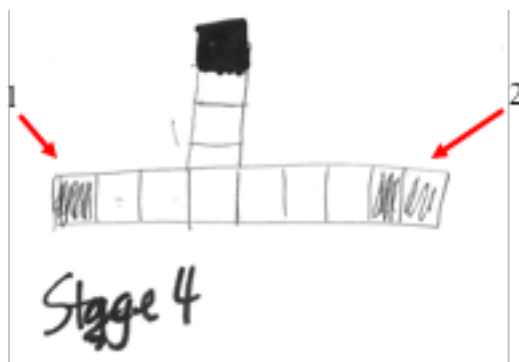


Figure 4.9. Denise's depiction of Stage 4 in Pattern 2.

Finally, I revoiced her explanation, pointing to the corresponding tiles.

After she confirmed that was her interpretation, I followed up by asking, “What is the  $s$ ?” After a long pause, Denise responded:

Oh, I was using the stage number as  $s$ , but you're saying within the diagram, yeah? The number of groups of three tiles. Here is one group of three tiles, two groups of three tiles, three groups of three tiles (pointing to each successive stage, but not to anything specific in the figure.) So you got all of these groups of three.

Again, Denise implied that she understood the 3s as representing increasing numbers of groups of three, but was unable to explain what each symbol (i.e., 3 and  $s$ ) represented separately. The notion of a group was sometimes assigned to the variable, sometimes to the coefficient, and sometimes to the combination of the two symbols. Furthermore, although in Pattern 1 such an interpretation, while vague, seemed to align with her decomposition of the pattern, in Pattern 2 such a construal is problematic because her notion of groups of three does not correspond with her view of the picture. Moreover, when asked to explain the coefficient 3 in isolation, she appeared to divorce its meaning from the idea of a group and attempted to impose her depiction of “three tiles it grows by,” onto the diagram. She selected tiles that were no longer part of a group and seemed

to encapsulate the idea of growth. Finally, her choice of tiles contradicts how the pattern changes between stages. The two tiles on the base would overlap with the addition of each successive stage and she did not account for the increase of tiles in the vertical tower. It seemed that when she was asked to identify the 3, Denise changed her conceptualization from adding groups to adding three separate tiles for each consecutive stage; her circling did not indicate a conception of groups of three tiles. Regardless, both interpretations seem to illustrate how disconnected her understanding of the symbols was from the figure.

Denise's vague and even problematic explanations of the various symbols' referents are evidence that she was not using symbols to communicate her interpretation of the figure. Instead, she had derived the expression using a partially numerical approach, and when asked to explain the expression in terms of the figure, she reinterpreted the coefficient as representing a decontextualized growth factor and attempted to improvise a quantitative interpretation on the figure. These examples from Denise's interview demonstrate that although interpreting the coefficient as the difference between successive stages might be an effective strategy to produce a correct symbolic expression, it does not convey a meaningful interpretation of the symbols. Moreover, such a conception of additional dots seems to lead one to characterize the expression as consisting of various symbols that correctly calculate the number of dots rather than as representing quantitative relationships.

**Detailed and flexible quantitative interpretation of the symbols.** Clara's and Jack's descriptions demonstrate an understanding different from that of Denise. Clara demonstrated not only a strong contextual understanding of the symbols but also ability

to flexibly adapt her understanding of the symbols to correspond to the details of different patterns. To analyze Pattern 2, Clara, similar to Denise, dissected the figure into overlapping groups of five dots (see Figure 4.10) and generated the expression  $5n - (n - 1)$ . Although their methods were quite similar, Clara and Denise demonstrated different understandings of the notation.

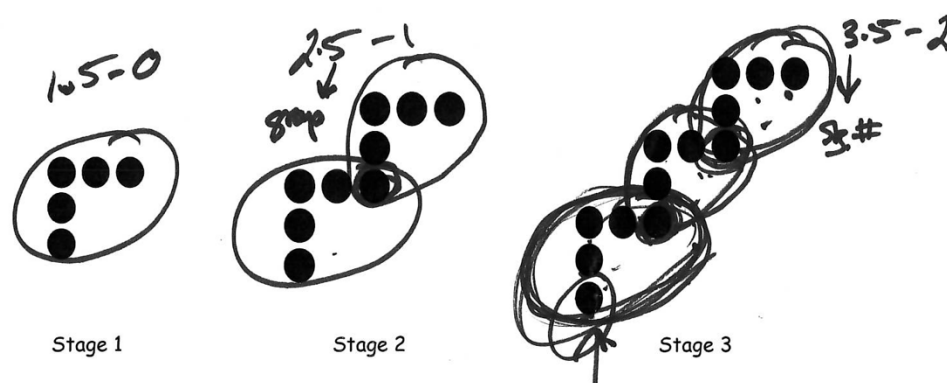


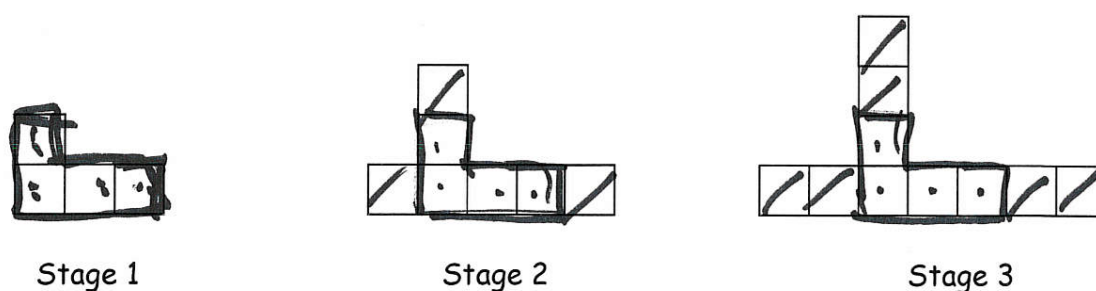
Figure 4.10. Clara's decomposition of Pattern 1.

When asked to explain the symbols in her expression, Clara offered a succinct explanation of the two symbols  $n$  and 5 separately as well as their combined meaning, stating "There's  $n$  number of groups, and there are five dots in each group. So  $5n$  is the total number of dots." This interpretation differs from Denise's in a variety of ways. First, Clara articulated the role of each symbol independently without any overlap in their meanings. Her description fully separated the meaning of  $n$ , which she interpreted as the number of groups, and the meaning of the coefficient, which she interpreted as the size of groups. In contrast, Denise included the size of the groups in her description of the variable, seeing  $n$  as the number of groups of five dots, not simply as the number of groups. This construal of the variable subsumes the role of 5, in effect leaving nothing for the 5 to mean, possibly contributing to Denise's confusion.



Second, Clara appeared to conceive of the coefficient, not as a single quantity (i.e., a number of dots), but rather as a ratio of dots per group. Although Denise's interpretation of "dots being added" indicates a ratio of dots per stage, her description was never explicit. Without having clearly articulated that the coefficient is a ratio involving two quantities, she seemed unable to see the relationship between the coefficient and the variable, and ultimately to separate the two meanings of these two symbols.

When presented with Pattern 2, Clara again engaged in reasoning consistent with a conception of the variable and coefficient representing the number of groups and dots per group. She was then able to modify her interpretation to accommodate the characteristics of the new figure. As before, Clara first decomposed the pattern and then looked for relationships between the resulting quantities and stage numbers. Specifically, she outlined the center L and then marked the new pieces added around the shape at each stage (see Figure 4.11).



*Figure 4.11.* Clara's decomposition of Pattern 2.

When asked about Stage 10, she drew a representation (see Figure 4.12) indicating her interpretation that each of the three ends of the L was growing and wrote the expression  $3(10 - 1) + 4$  to match her understanding of the pattern.

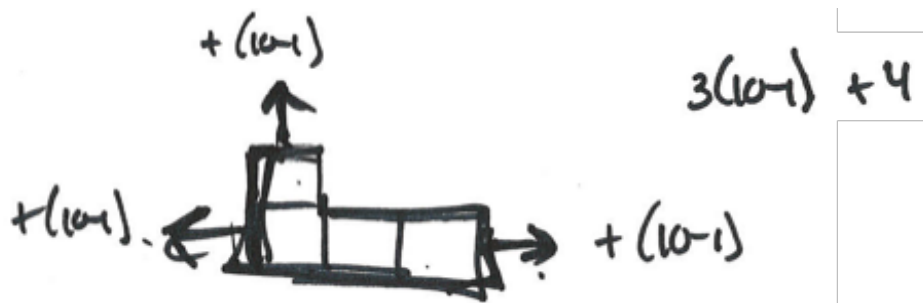


Figure 4.12. Clara's depiction of Stage 10.

Finally, Clara produced the expression  $3(n - 1) + 4$  as the general case. At this point, I asked her to explain, symbol by symbol, what each represented.

- Interviewer: What does the 4 represent?  
 Clara: The beginning stage, the base, the base of the figure.  
 Interviewer: And what does  $n - 1$  represent?  
 Clara:  $n - 1$  are the number of blocks that are going to be added onto each end of the base figure.  
 Interviewer: And then the 3?  
 Clara: There's three places where they are added onto.

In reflecting on Clara's interpretation of 3 in  $3(n - 1)$  as "places where the [blocks] are added onto," her depiction of "places" suggests the idea of groups, similar to that of her previous conceptualization for the variable. Likewise, her explanation of  $(n - 1)$  as the "the number of blocks added onto each end," appeared to denote the number of blocks in each of these groups. In the picture she drew to explain Stage 10 (see Figure 4.12), this idea of groups and tiles per group seems even more evident. She marked the three ends with arrows implying groups and notated the number of tiles per end as  $(10 - 1)$ . Clara's descriptions of  $(n - 1)$  and the coefficients in Pattern 2 seem comparable to her explanation of Pattern 1, but the roles of the symbols have been reversed. In the former, Clara translated the variable as the number of groups and the coefficient as the size of each group. For Pattern 2, although I did not ask her specifically about her

understanding of  $n$ , her conceptualization of the coefficient maps onto the idea of number of groups and the expression containing the variable,  $(n - 1)$ , refers to the size of these groups.

Clara not only demonstrated a strong quantitative understanding of the symbols in both patterns but also adapted her interpretation of the notation to correspond to her understanding of the figure. In both cases she denoted one symbol as the number of groups and the other as the ratio of dots per group, but was able to flexibly match their assignments with the details of the context.

**Quantitative, yet fixed interpretation of the symbols.** Whereas Clara flexibly adapted her interpretations of the symbols to accommodate her quantitative understandings of the figures, Jack was able to constructively modify his view of the figure to create a quantitatively grounded expression that leveraged his interpretation of the coefficient and variable. For Pattern 2, Jack reconceptualized the quantities in the figure so that he could apply his previous understanding from Pattern 1 and ultimately write an expression that was connected to the figure.

Initially, Jack approached Pattern 2, at least in part, numerically (see Figure 4.13). He identified the numerical values of the first stages and attempted to write an expression that correctly produced these numbers as well as corresponded to what he noticed in the figure. He explained his thinking at the time:

I saw that it kept going by one block out each leg, but I totaled them up, so I got seven. So I knew something plus 4 had to equal 7. So this number (pointing to the box) had to be 3. So how could I use ..., you know, this, this, and these (pointing to each of the blocks added in Stage 2) together?

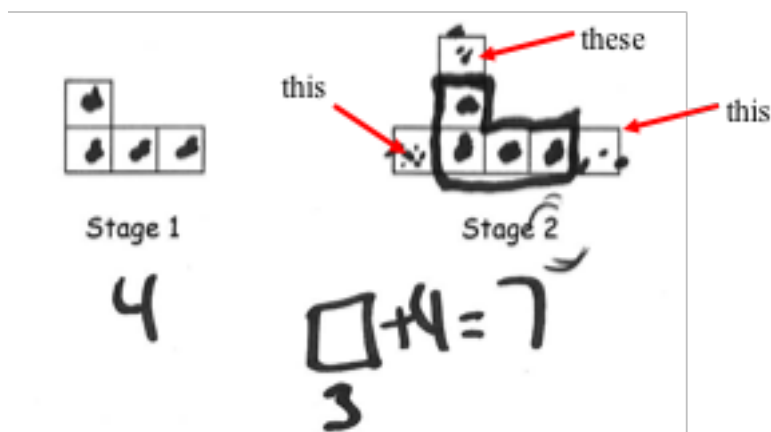


Figure 4.13. Jack's numerical approach to Pattern 2.

At this point in the interview, Jack was confused by what he saw as an inconsistency. He had noticed that each “leg” increased by one tile, which was different from the numerical value of 3 needed to correctly complete the number sentence  $3 + 4 = 7$ . To resolve this situation, he reflected on his solution to Pattern 1 to see how he could apply his previous understanding to make sense of this new pattern. Eventually Jack found a solution to this problem by reconceptualizing Pattern 2 to correspond to his decomposition of Pattern 1. This shift in his interpretation of the figure enabled him to apply his prior conception of the notation. I explain his articulation of his initial confusion and ultimate resolution of what he interpreted as a contradiction in the following quote.

First Jack reiterated his interpretation of Pattern 2 as consisting of three individual legs, each increasing independently by one tile. He then compared the total resultant numerical value of 3 to the stage number, which is 2.

So I saw there was my base and, ... I saw these as individuals (pointing to the three new blocks in Stage 2 of Figure 4.13), or each 1, 2, 3 [new blocks in Stage 2], which was one more than [the stage] 2.

Jack then retrieved his work from Pattern 1 (see Figure 4.14) and described how, in contrast, he saw this previous figure increasing by one group. Again he compared this value of 1 [group] to the stage number of 2.

Where here (pointing to his solution to Pattern 1), I saw Stage 2 and the difference, well I saw Stage 2 as one group. So this 1 (referring to the number of groups added) was less than this 2 (referring to the stage number).

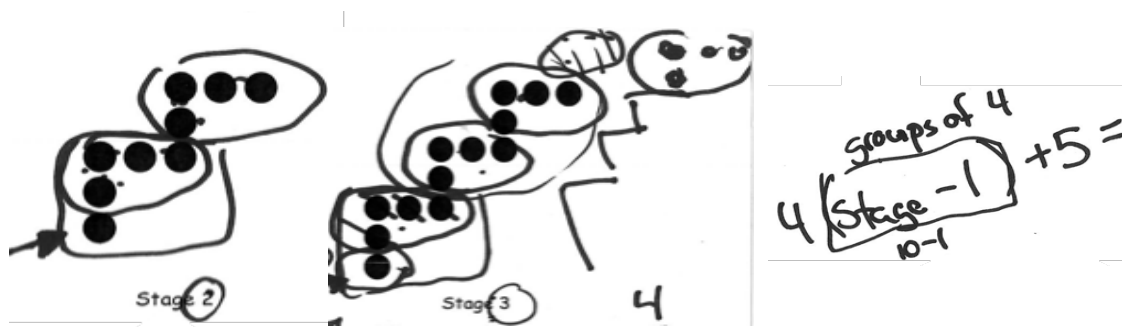


Figure 4.14. Jack's illustration for his interpretation of Pattern 1 showing the figure increasing by a group of 4 at each stage.

Jack then clarified the contrast between his interpretation of Pattern 2, which he conceptualized as a growth of 3 independent pieces, and his understanding of Pattern 1, which he saw increasing as a collective group of four dots.

Here (pointing to Pattern 2) I counted them individually and here (pointing to Pattern 1) I grouped them all together.

Finally, Jack explained how he was able to use his understanding of Pattern 1 to reconceptualize the individual blocks in Pattern 2 (which increased by three tiles per stage) as a whole group (which increased by 1 group).

So my initial thought, because I still had them [tiles added] more than the stage number; 1, 2, 3 [tiles added at each stage] instead of 1 [group], so I was thinking I had to add something to this number (referring to the 1) or do something to make it work. But when I went to the box and figured out that this [value in the box] had to be 3. How could I do that? The three

went out the window, and I realized in circling them (see Figure 4.15), that I counted them individually, and those I clumped together, and that's why it made it [value added each stage] less than a stage [number] instead of more.

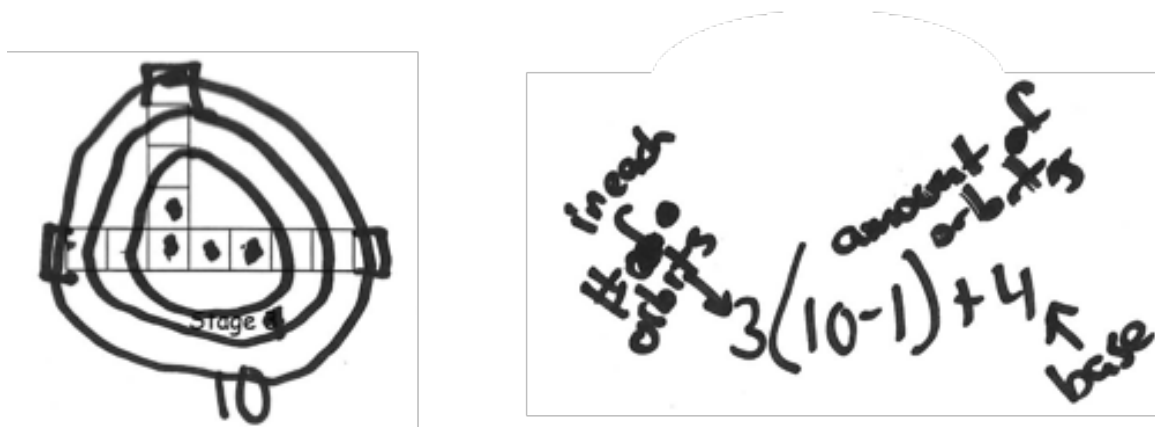


Figure 4.15. Jack's reconceptualization of the individual dots as a collective orbit of three tiles.

As you can see in his picture, and as he explained later, Jack interpreted the groups of three blocks added each stage as orbits encircling the original base of four tiles (see Figure 4.15). Shifting his interpretation of the pattern from legs with individual blocks, to collective orbits consisting of a collection of blocks, enabled him to leverage his previous understanding and generate the expression  $3(x - 1) + 4$ .

Jack's understanding of the figure and how he saw the notation representing this view becomes clear in the following exchange when Jack explained his interpretation of the various symbols he used in the expression  $3(10 - 1) + 4$  for Stage 10.

- Interviewer: So what does this represent then (pointing to the  $10 - 1$ )?  
 Jack: This is going to be the amount of orbits that I have circulating.  
 Interviewer: And this is (pointing to the 3)?  
 Jack: The number in each orbit.  
 Interviewer: And the 4 is what?

Jack: The 4 is my sun. So, yeah, in Stage 10, there would be 27 tiles or 9 orbits, each with three tiles in it. So there would be 27 planets with my sun in the middle.

Jack was explicit that the term with the variable (in this case  $(10 - 1)$  or  $(x - 1)$  in the general form) represented the number of groups or orbits and the coefficient 3 was the number of tiles per orbit. Furthermore, his articulation of the quantities clarified why the two symbols were multiplied and his overall metaphor indicated that he understood how the various quantities combined to make the total number of tiles. Although he struggled to convert his initial interpretation of the figure into an algebraic expression, he continued to analyze and grapple with the pattern quantitatively, eventually reconceptualizing the pattern as growing by a single orbital group at each stage. With such a conception, he was able to model the pattern symbolically, once again interpreting the term with the variable as representing the number of groups surrounding his sun and the coefficient denoting the size of each group.

Although Jack did not possess the notational flexibility exhibited by Clara, he did have a clear quantitative approach to symbolizing the pattern and was able to adapt his interpretation of the pattern to align with this understanding. In a sense, for the flexibility he lacked in his symbolic understanding, he was able to compensate with flexibility in his quantitative understanding. Like Denise, he accommodated his view of the figure to align with his understanding of the symbols. He differed from Denise, though, in that his understanding of the symbols was always quantitatively grounded. After he reconceptualized the pattern, he was able to use his previous conception to develop an expression that grew explicitly out of his understanding of the figure, providing him a detailed interpretation of each individual symbol as well as their combinations.

### **Interpreting Expressions in Which the Variable Appears More Than Once**

The second conceptual challenge that emerged for the teachers was negotiating the meanings of symbols (in particular, variables) that appeared more than once in a single expression or between expressions after algebraic manipulation. Such a situation exists in Pattern 1, first with the expression  $5x - (x - 1)$  and then in the subsequent simplified expression  $4x + 1$ . What made this particularly challenging for the teachers was that Denise, Clara, and Jack all initially interpreted the two  $x$ s in  $5x - (x - 1)$ , as well as the  $x$  in  $4x + 1$  as representing different quantities in the figure. Specifically, I provide evidence to support the claim that in the expression  $5x - (x - 1)$ , they saw the  $x$  in  $5x$  as referring to the number of groups of five dots and the  $x$  in the expression  $(x - 1)$  as the number of corner dots that are counted twice in the overlapping groups of five dots. Then, in the expression  $4x + 1$ , they interpreted the  $x$  as the number of groups of four dots. Overcoming this challenge proved difficult. I begin my analysis with Denise's explanations; she was the most articulate of the teachers in communicating the cause of her confusion and offered the best account of the complexity of this topic. Although she identified the meanings of the  $x$ s as isolated symbols and clearly articulated her difficulty in coordinating them, she was unable to conceptualize the variables to resolve this apparent inconsistency. I then illustrate how Clara and Jack grappled with the same conceptual hurdle, providing details of their struggle and describing their various efforts to arrive at a solution. In addition to explaining Clara's and Jack's final understandings of the symbols, I describe the specific actions that contributed to their conceptual breakthroughs and their eventual productive conceptualizations.



**Interpreting variables as contextually consistent: A literal translation.** In the following section, I describe Denise's overall approach for deriving the expression  $5x - (x - 1)$ , her various endeavors to coordinate the divergent meanings of the variables within the expression  $5x - (x - 1)$  and between the expressions  $5x - (x - 1)$  and  $4x + 1$ , and finally her reflection on this struggle. As noted, although Denise conceived of meaningful interpretations for each of the  $x$ s in isolation and clearly articulated her conceptual difficulties, she could not make sense of the multiple quantities when making comparisons among them. Attempting to find a consistent interpretation for the variables among the various expressions, she appeared to substitute her understood meaning of one variable for another, causing her to misconstrue some associated quantities. Unable to satisfactorily resolve these issues, she eventually formulated an interpretation of  $x$  that was quite removed from the quantitative reality of the figure. This construal of the variable afforded her a coherent view among the various symbols, but she failed to incorporate and account for many contextual details.

***Denise's initial approach and interpretation of the variables.*** As described previously, Denise arrived at the expression  $5x - (x - 1)$  by decomposing the figure into overlapping groups of five dots (see Figure 4.2) and creating an algebraic expression that matched her understanding of the pattern. Although she was later unable to clearly articulate the meaning of the symbols she had used, her initial method was grounded in the figure. To highlight details of her original interpretation, I present her description of Stage 10, which she had written numerically as  $10 \times 5 - 9$ .

It's going to have groups of five [dots], so stage number. Well, there's going to be 10 of these  $L$ s (sketching out a sequence of upside down  $L$ s;

see Figure 4.16), but then there's going to be that one overlapping dot (highlights these on her picture by shading the corners).



*Figure 4.16.* Denise's illustration of her interpretation of Pattern 1.

Although Denise did not state precisely how she interpreted each symbol in the expression  $10 \times 5 - 9$ , I inferred from her description that she was coordinating the number sentence and the quantities in the figure at least at a general level. From her explanation, I deduced that the first part of the expression corresponded to the groups of five dots (or the *Ls*) and the second part of the expression corresponded to the overlapping dots.

***Denise's attempt to coordinate symbols' referents and the resulting conceptual shift.*** After describing and generating a numerical expression for Stage 10, Denise wrote a general rule for the pattern. Although her expression matched her previous description of the pattern, she no longer incorporated concrete elements of the figure in her explanation, referring to and writing out the variable  $x$  as the stage number (see Figure 4.17).

$$\begin{array}{l} (\text{stage \#})5 - (\text{Stage \#} - 1) \\ 5x - (x - 1) \quad \boxed{x = \text{stage \#}} \end{array}$$

Figure 4.17. Denise's symbolic expression for Pattern 1.

To explore her contextual understanding of this expression in more detail, I asked Denise if the  $x$  could be interpreted as a specific component in the figure. At this point she hesitated, asking both herself and me clarifying questions while she tried to find a possible interpretation other than the stage number. After 1 minute and 45 seconds of exploration, she answered, "The number of groups of five" (pointing to the first  $x$ ). She then paused and continued, "The number of groups of five minus 1" (pointing to the second  $x$ ). Finally, she looked up and tentatively concluded, "It could represent the number of groups of 5." Her tone clearly showed that she was still exploring the viability of this interpretation. She seemed to have decided that the first  $x$  represented the number of groups of five dots and then tried to substitute this for the second expression as a direct translation of  $x$ . As will be shown, this effort was based, at least in part, on an assumption that identical symbols should have equivalent meanings.

I then asked Denise to explain what the  $(x - 1)$  represented. As her response and later comments indicated, her substitution method seemed to have triggered a modification in her interpretation. She replied,

If I had a group of five (pointing to the  $x$  in  $x - 1$ ) and I take away one (pointing to the 1 in  $x - 1$ ), it's like that overlapping (pointing to the one overlapping dot in stage 2, see Figure 4.18). I would say, "Here is a group of five (circling the second group of five dots in Stage 2, see Figure 4.18)

that I take away the one dot that is overlapping (pointing again to the one overlapping dot, see Figure 4.18), 'cause they share that dot."

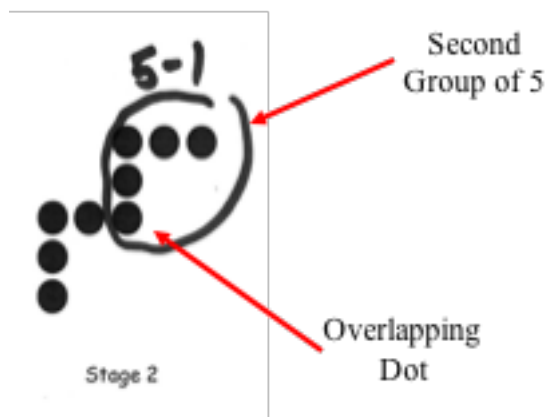


Figure 4.18. Denise's interpretation of  $(x - 1)$ .

This comment shows seeds of a new interpretation of the symbols. Denise's repeated use of simply "group of 5" as well as her highlighting in the diagram a single group indicated that in searching for a "number of groups of 5, minus 1," she had changed her interpretation of  $x$  from the *number of groups* of five dots, to simply *a group* of five dots. Furthermore, Denise seemed to construe the minus 1 as representing the one dot in the group that overlapped. It appeared that Denise, in interpreting 1 as the overlapping dot, began seeing the minus 1 as accounting for the difference in sizes of the groups of five and the groups of four (i.e., the difference in the number of dots) rather than the difference between the total number of groups of five and the number of overlapping dots.

While Denise continued to explore the meaning of the symbols, this construal of  $x$  as a group of five dots and  $(x - 1)$  as a group of four dots became clearer. After reiterating that the first  $x$  represented the number of groups of five (dots), she attempted

to clarify how the expression  $5x - (x - 1)$  matched her understanding of the pattern by sketching a picture and coordinating her understanding of the symbols:

Okay, so here is a group of five (drawing an  $L$ ; see Figure 4.19). And if it is not overlapped, here's also a group of 5 (drawing in a second  $L$  on top; see Figure 4.19). But then I take—take away the one (first pointing to the 1 in the expression and then marking on the diagram a dash to represent the deletion of the overlapping dot), and that becomes a group of four.

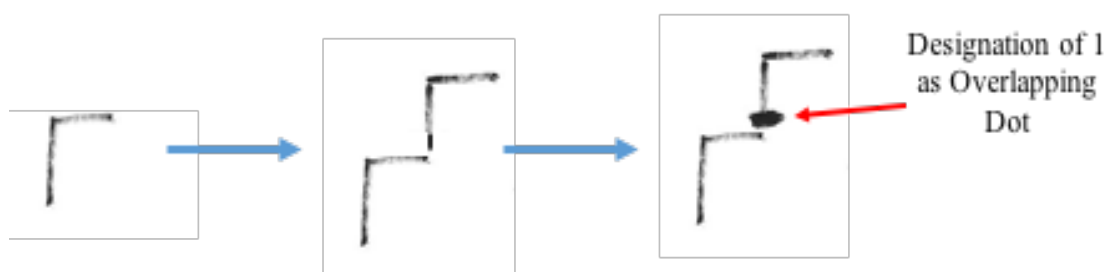


Figure 4.19. Denise's representation of how "minus 1" creates a group of four dots.

Although Denise did not explain clearly what part of the notation corresponded to the group of five dots, she explicitly identified the 1 in the diagram as the removal of the dot that creates a group of four dots, further denoting her shift in thinking. From this explanation, I inferred that she was interpreting the second  $x$  in the expression  $5x - (x - 1)$  as a group of five dots and the subtraction of 1 as decreasing the number of dots to four. With such a construal,  $(x - 1)$  would then represent "a group of four."

Finally, still seeming in search of clarity about the quantitative relationships among the various notational pieces, Denise tried to make sense of the quantities in the expression  $5x - (x - 1)$  by evaluating  $x$  with different numbers and endeavoring to coordinate these values with the figure. In her attempt she focused on Stage 2, stating,

For Stage 2, if I said the number of groups of five, then it [groups of five] would be one here (pointing to  $5x$ ) and again 0 here (pointing to  $(x - 1)$ ). Because this would have to be a group of five and a group of four (circling

the first group of five in Stage 2 and then the group of four on top; see Figure 4.20).

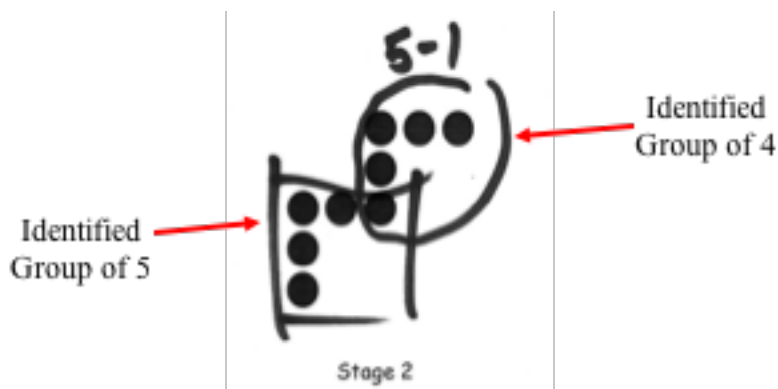


Figure 4.20. Denise highlighting the group of five dots and group of four dots in Stage 2.

Although how  $x$  and  $x - 1$  became 1 and 0 for Stage 2 is unclear, her language and gestures indicated that she associated the  $5x$  with the group of five dots and the  $x - 1$  with the group of four dots. According to Denise's new construal of the notation,  $x - 1$  did not represent the "number of groups minus 1," as in her initial substitution method, but became an actual group with one dot fewer than other groups (i.e., a group of four dots). In the end, she seemed to adopt a different interpretation of  $x$  in the second part of the expression in an attempt to match the quantity "number of groups of 5, minus 1" with her visual understanding of the pattern.

***Denise's articulation of her conceptual difficulty.*** During the interview, Denise continued to grapple with the inconsistencies in the quantities she saw connected with the variable  $x$ . She returned to this issue without being prompted, searching for an explanation to appropriately coordinate their meanings. Throughout her exploration for a feasible solution, she also offered multiple reflections indicating an acute awareness of precisely what she understood about the quantities involved, the exact elements that were

confusing her, and the conceptual resources from which she was drawing. In particular, two of her responses included the most detail.

The first instance occurred immediately after the previous exchange, when she again stopped to discuss the nature of her challenge in understanding the expression  $5x - (x - 1)$ .

I am trying to make sense of how this  $x$  (pointing to  $5x$ ) represents something else in the diagram. If it represented one group of five, then it would have to be 1 every time. It can't represent 1 here (pointing to  $5x$ ) and 1 here (pointing to  $(x - 1)$ ), I don't think.

In this reflection Denise stated clearly that what was troubling her was her belief that a symbol cannot represent different quantities in the same context. She seemed to have identified the difference in her interpretations of  $x$  and was confused by this apparent contradiction. At the same time, she was questioning her assumption that the variable should have the same meaning in each instance. While Denise struggled to find a viable conceptualization, she was grappling with both the mathematical ideas and her understanding of the associated mathematical conventions. This statement also seems to provide further evidence about her strategy for connecting the expression to the context. Thinking that every  $x$  must represent the same quantity throughout the expression and after syntactic manipulations, Denise attempted to translate the symbols into words verbatim, substituting the phrase she understood for the first  $x$  into the  $(x - 1)$  expression.

A second example occurred later when Denise began comparing the  $x$  in the expression  $4x + 1$  with her understanding of  $x$  in  $5x - (x - 1)$ . Trying to make sense of the meaning of the notation, she evaluated the expression  $4x + 1$  using various numbers. She verbalized her interpretation in context: "One group of four plus one. Two groups of

four plus one.” In each case she circled the corresponding number of groups of four dots in the diagram and concluded, “The number of groups of four, plus one. My number of groups will work for that one” (see Figure 4.21). Then, to clarify her construal of the symbols in the expression, I asked her what the  $x$  and the 4 represented in  $4x + 1$ . Denise explained that the  $x$  is “the number of groups of four,” and the 4 represents “four dots.”

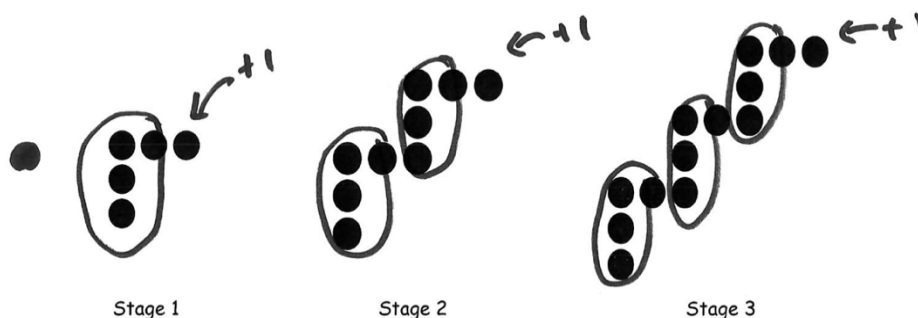


Figure 4.21. Denise's decomposition of Pattern 1 into groups of four dots.

Denise then tried to build on her understanding of this  $x$  by returning to the meaning of  $x$  in the previous expression,  $5x - (x - 1)$ , and searching for a relationship between the two quantities. Struggling to do so, she again articulated the exact nature of her challenge:

I am trying to see how these two (pointing to the  $5x - (x - 1)$  and the  $4x + 1$ ) have a relationship, ... but I can't really see. ... If this (pointing to  $4x + 1$ ) is the number of groups of four dots, this (pointing to  $5x$ ) has to be the number of groups of five dots.”

In this reflection, Denise was clear about what she understood and the difficulty she was experiencing. She indicated that the  $x$  in  $5x$  represented the number of groups of five dots, and the  $x$  in the  $4x$  represented the number of groups of four dots, but she was unable to clarify the relationship. She had observed the discrepancy in meanings in these two expressions but was unable to explain how the symbols' referents changed. Although



part of her confusion was caused by her inability to disentangle the meanings of the  $x$  and its coefficient, continually vacillated between whether  $x$  or  $x$  combined with its coefficient represented these groups of distinct size, she had clearly identified the cause of her uncertainty.

Juxtaposing these two expressions and articulating their corresponding meanings seemed to focus Denise's attention and narrow the conceptual challenge. She then attempted to bridge this gap by considering the intermediate expression  $5x - x + 1$ . Underlining the  $5x - x$ , she stated, "This can also represent the number of four dots." She then pointed to just the  $5x$ , reiterating, "but this has to be the number of groups of five dots." Although her organized approach seemed to move her closer to a conceptual breakthrough, the connection between the symbols continued to allude her. She was able to articulate her exact challenge and her understanding of the various symbols but not to develop a conceptualization that would fully connect their meanings.

***Denise's oversimplification of the symbols' referents.*** Finally, after almost 12 minutes of attempting to find a viable interpretation of the various  $x$ s, Denise eliminated the inconsistencies in the referents, but she did so by assigning an interpretation that was only loosely connected to the figure, stating, "Well, it could just represent the number of dots;  $x$  could represent the number of dots. ... That would make more sense with these two when comparing them." She then went on to complete several calculations using this interpretation to illustrate its viability.

If this  $[x]$  is a dot and you multiply it by 4 (pointing to the expression  $4x + 1$ ), then there are four dots, plus the one there is five dots. Then if [in Stage 2] you have two dots times 4, you have eight dots, plus the one, that's nine dots. That is the same here (pointing to the expression  $5x - (x - 1)$ ). You have, that  $[x]$  is a dot, times 5, is five dots, minus one dot;

that's four dots, plus that one, you get back to five. So I think  $x$  is going to represent just a dot.

Denise was finally able to provide a uniform conception of  $x$ , a condition she saw as necessary, but she did so by oversimplifying the meaning of the variable, interpreting it as simply a “dot.” Such a conceptualization essentially removes any quantitative meaning of the symbols. As her example using dots in context demonstrated, interpreting  $x$  as dots, the same quantity that is the input of the function is the output. Under such an interpretation,  $x$  functions like a label and the coefficients are simply decontextualized numbers. Furthermore, this interpretation fails to explain any of the quantitative relationships between the symbols.

**Associating quantities numerically.** Although Denise was able to conceive of meaningful interpretations for  $x$  considered in isolation, she was unable to reconcile the diverging meanings of the same symbols within the expression  $5x - (x - 1)$  and between the expressions  $5x - (x - 1)$  and  $4x + 1$ . Clara and Jack, experienced similar difficulties coordinating these different meanings, but they were finally able to articulate viable, yet different solutions. Consider first Clara, who continued to hold different interpretations for the multiple  $x$ s, but was able to coordinate their equivalence numerically.

***Clara’s initial approach and interpretation of the variables.*** Throughout the interview, Clara demonstrated a strong coordination between the algebraic symbols she used and the quantities they represented within the figure. Her overall approach was to generalize the figural pattern visually and use the notation to communicate her understanding. She initially interpreted the first  $n$  in  $5n - (n - 1)$  as having two meanings, “the number of groups of five, which is also the stage number.” She

explained the entire  $(n - 1)$  as “how many you are counting doubly (circling the corners). You are double counting each of the intersecting dots except the first one.” These quotes, along with her general approach, indicate that she possessed a strong quantitative understanding of the symbols she was using to communicate the generalization algebraically.

***Clara’s attempt to coordinate symbol’s referents and the resulting conceptual shift.*** Although Clara was confident that  $(n - 1)$  represented the dots that were counted twice, she struggled to identify the meaning of the single  $n$  in the expression  $5n - n + 1$  obtained by removing the parentheses in  $5n - (n - 1)$ . In attempting to make sense of  $n$  in this new form, Clara first reviewed what it had represented previously, commenting, “Well, it starts in here (pointing at  $(n - 1)$ ), and it is the number of groups. So as long as it is inside the parenthesis, it’s the total number of groups that I have. Now, what does it mean?” Having reminded herself that she interpreted the first  $n$  in  $5n - (n - 1)$  as the number of groups, Clara then attempted to answer her own question about what the single  $n$  in the expanded expression  $5n - n + 1$  represented:

That’s going to mean ... all of the groups (moving her hand to indicate each of the the groups of five dots in Stage 3) except for the first one. So it’s (pointing first to the first group of five dots in Stage 3 [see Figure 4.22], and then to the  $- n$  in the expression  $5n - n + 1$ ) going to represent the first group.

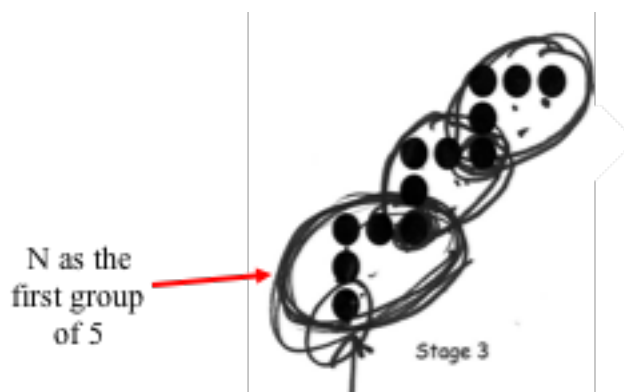


Figure 4.22. Clara identifying  $n$  as the first group of five dots.

Although this statement aligns somewhat with Clara's original interpretation of  $n - 1$ , in that the number of dots overlapping occurs in all groups except the first, the meanings of the individual symbols and their roles in the context have shifted. Trying to make sense of  $n$  in this new context, Clara seemed to reinterpret the various elements of the expression. In particular, she no longer saw  $n$  as a variable describing the changing number of groups, but now interpreted  $n$  as an actual group, specifically the first group. This construal was clear inasmuch as she explicitly coordinated the first group in the figure with the symbol  $-n$  in the expression with a combination of her fingers and language.

As the interview proceeded, Clara confirmed this interpretation as she continued to refer to  $-n$  as the removal of the first group. For example, moments later, she stated,

So if I have...So five for each group (pointing to the  $5n$  in the expression  $5n - n + 1$  and then highlighting the three groups of 5 in the figure), but let's take a group away (pointing to the  $-n$ ). So I am taking this group away (pointing to the first group in stage 3, see Figure 4.22). I am taking the first group away...from the total.

Again, Clara clearly indicated with her gesturing and verbal commentary that she had reinterpreted the new expression and that the  $5n$  denoted the collection of all the groups of five dots and the  $-n$  represented the removal of the first group. With this new construal of the expression, not only did Clara switch her interpretation of  $n$  from the number of groups to a specific group, she also changed the symbol to which she attributed the act of removing the first group. In her previous interpretation, the removal of the first group (or specifically the first dot) had been represented by the  $-1$ , and in her new construal the removal of the first group was denoted by  $-n$ . It seemed that the idea of removing first the dots and then the group followed the subtraction sign when Clara transferred this conception (removing the first group) from the  $-1$  to the  $-n$ .

Over the next few minutes Clara's comments and gestures continued to indicate an interpretation of  $n$  as the first group when she repeatedly coordinated removing the first group in the diagram with the  $-n$  in the expression. She clarified that she had not been referring to the 1 during her exploration when she paused and then purposefully transitioned to the  $+1$ , pointing to the  $+1$  in the expression and stating, "And then the plus 1. What am I adding?" This shift clearly showed that the  $+1$  was not part of her initial interpretation of the first group. Much like Denise, Clara tried to make sense of the expression by using a literal translation of the words. Borrowing from her previous interpretation of  $n$ , she transformed the quantity "number of groups" into simply "a group." In an attempt to coordinate the symbols and her understanding of the figure, she reinterpreted the removal of the first dot, represented by the subtraction of 1, as the removal of the first group, represented by the subtraction of  $n$ . However, in contrast to Denise, Clara's struggle led to a different view of  $n$ . She interpreted the subtraction of  $n$

as the removal of the first group, whereas Denise misconstrued  $(x - 1)$  as a group of four dots. In both cases though, substituting words for symbols caused the teachers to incorrectly create a new quantity by trying to match their view of the figure with the symbols and operations in the notation.

***Clara's conceptual breakthrough.*** Clara struggled to conceptualize the various interpretations of  $n$ . She spent almost 4 minutes pursuing different interpretations, mostly referring to  $n$  as the first group in different ways. The length of time as well as the depth of her misconstrual highlight the conceptual challenge involved in coordinating the various meanings of the symbols involved. Unlike Denise though, Clara was eventually able to find a way to resolve her varying understandings of the variable.

Clara's conceptual breakthrough occurred after she continued to grapple to coordinate the two interpretations of  $n$ . At one point, I asked what she meant by "taking away the first stage." Referring to the expression, she again tried to clarify how the various notational pieces accounted for the dots. She explained, "The  $5n$  represents (pointing to  $5n$  in the expression  $5n - n + 1$ ) .... It's telling me the total number of dots;  $5n$  represents dots. It's too many ..., so I have to take away." Explaining how subtracting  $n$  compensated for the "too many" dots in the  $5n$  seemed to provided her clarity. In her answer, she stated, "The number of *dots* I have to take away is equal to the number of *groups* there are." She then paused. The significance of this correlation between the number of dots and the number of groups seemed to resonate with her while she slowly and methodically repeated this phrase again. By making the connection between the values of these quantities explicit, she was able to clarify her confusion and

find a meaningful interpretation of  $n$ . Ultimately, to explain each element of  $5n - n + 1$ , she confidently stated,

So the  $n$  .... So if there were three groups, there were three dots. And this is saying there were three dots that were doubly counted. So I have to take that many away. But, oh, this one wasn't doubly counted (pointing to the first dot). So I have to put that back again.

In this final explanation, Clara not only articulated that the number of groups and the number of dots were the same but also alluded to how the addition of 1 in the expression  $5n - n + 1$  represented the first dot that was not doubly counted. The minus  $n$  removed all the overlapping dots as well as the first one, which must be reinserted to compensate for their deletion.

Although Clara's final interpretation mirrored her initial description, her understanding changed in that she realized that whereas the  $n$ s represented different quantities, the numerical relationship between the quantities was invariant. The number of groups of five dots, which  $n$  initially represented, is always equal to the number of overlapping dots to be removed. Clara was very clear about this connection, going through several numerical examples to highlight these two quantities and the numerical relationship between them. Although the same variable takes on different referents, namely number of groups and number of overlapping dots, the numerical value of each of these quantities is always the same and is always the stage number.

**Associating variables contextually: A new interpretation.** Now consider Jack, who also struggled coordinating the different meanings of the symbols, but in contrast to Clara, did not overcome this conceptual challenge by connecting the referents numerically. Instead, he conceived of a single, consistent interpretation of all the

variables throughout. The data show that Jack was eventually able to detach the size of the groups from his interpretation of the variable and conceive of each  $x$  as solely representing the number of groups with the coefficient indicating the size. For example, in the expression  $5x$ , the  $x$  represented the number of groups (as opposed to the number of groups of five d), and the 5 designated that each group had five dots. This change was most significant when Jack analyzed the single  $x$ . Instead of thinking of  $x$  as the number of overlapping dots, Jack reconceptualized the number of dots as the number of groups comprised of one dot. In the following section I elaborate on his approach and his struggle and highlight the key elements that led him to make this shift in thinking.

***Jack's initial approach and interpretation of the variables.*** Unlike Denise and Clara, Jack did not create the expression  $5s - (s - 1)$  on his own; he was exposed to and made sense of this formulation in the interview when asked to analyze the work of a student who had produced it (see Appendix B). Initially, like the other two teachers, Jack identified the first  $s$  as “the number of groups of 5” and  $(s - 1)$  as “the overlapping dots that need to be taken away.” Not only did he indicate this interpretation verbally, he further demonstrated his understanding of the symbols through his actions. To emphasize that  $s$  represented a group of five dots, he drew boxes around a sequence of groups of five dots on a diagram the student had drawn for Stage 10 (see Figure 4.23). In addition, he layered pieces of paper atop one another, referring to each one as a group of five. He also highlighted his understanding of “overlapping dots” on the diagram, by including small arrows in each corner (see Figure 4.23).





Figure 4.23. Jack's contextual interpretation of  $5s - (s - 1)$ .

After Jack had indicated a firm understanding of  $s$  and  $(s - 1)$  in the initial expression as representing the number of groups of five dots and the number of overlapping dots, respectively, I asked him to interpret variables in expressions that followed after symbolic manipulation. I expanded the original expression, writing both  $5s - s + 1$  and  $4s + 1$ . Then, pointing to the isolated  $s$  in the first resultant expression  $5s - s + 1$ , I asked Jack what the “ $s$  or the minus  $s$  represented in the diagram.” After pausing to reflect for a few moments, Jack initially responded, “Seems like you are taking away one group. You are taking away one of the ... (gesturing a circle over a group of five dots in Stage 3).” Although Jack quickly changed this view to seeing the  $s$  aligned with his previous notion of overlapping dots, I find noteworthy that he, just like Denise and Clara, momentarily reinterpreted  $s$  as a group, instead of as the number of groups. Such a consistent misconstrual of the variable seems to highlight the importance of clearly communicating the meaning of the variable as the *number of groups* as opposed to simply *a group*.

***Jack's conceptual breakthrough.*** Although Jack quickly moved away from his initial interpretation of  $s$  as a group, he continued to struggle to make sense of  $s$  in the new expanded expression, searching for possible interpretations for more than 4 minutes. During his search, he clearly articulated his various conceptions of  $s$  as well as the exact conceptual hurdle he was trying to overcome. At one point, he compared his understanding of the variables in the two expressions  $5s - s + 1$  and  $4s + 1$ :

I'm trying to think of what ... this amount (pointing to the  $s$  in the  $5s$ ) of groups of five (pointing to the 5 in the  $5s$ ), and you're taking away this amount (pointing to the singular  $s$ ), and I get this many groups (pointing to the  $s$  in the  $4s$ ) of four (pointing to the 4 in the  $4s$ ).

Jack indicated that he understood the referent for  $s$  in both  $5s$  and  $4s$  as the number of groups, with the coefficients representing their respective sizes. Using similar language, he referred to the  $s$  as "this amount," suggesting that he saw  $s$  as a number of groups as well, but without indicating any size. He explained further, "So I was trying to think what it (referring to the  $s$ ) would be in the picture." In this comment, Jack seemed to allude to the cause of his confusion. Although Jack interpreted  $s$  as the number of groups, he seemed unable to see in the diagram these "groups" for the isolated  $s$ .

Verbalizing his understanding of the various variables in the two expressions led Jack to then describe  $s$ : "He (referring to the student whose work he was analyzing) is taking away one of the dots in each group." At this point, while Jack identified  $s$  as the dots in the figure, he still seemed to be interpreting  $s$  as the number of groups in coordination with his view of  $s$  in  $5s$  and  $4s$ . His description of the dots *in each group* indicated that he had not conceptualized the dots as groups themselves, but rather as

members of the groups he interpreted as  $s$ . This conceptualization becomes clearer in a statement Jack made moments later.

Jack then rephrased this transformation one more time, adding more detail while he described his understanding of the various symbols in the expression  $5s - s + 1$ .

So you are taking away one dot (inserting a coefficient of 1 for the second  $s$ ; see Figure 4.24) from each group. So you have this (pointing to the first  $s$ ) many groups of five (pointing to the 5), and you are taking away 1 (pointing to the new coefficient 1) from the same amount of groups (pointing to the second  $s$ ). So you are taking away 1. This is the same amount of groups. So one dot for each of these groups.

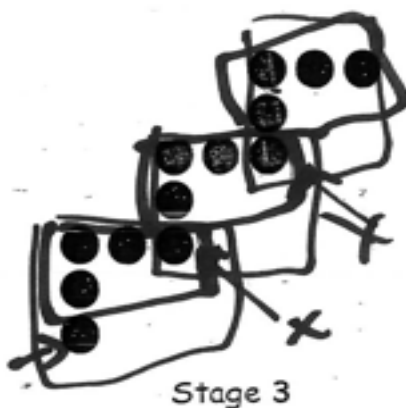


*Figure 4.24.* Jack's symbolizing one dot from each of the  $s$  groups.

This description, while similar to Clara's final interpretation of  $s$  as dots being removed, is noticeably different. One difference is that Jack did not simply recognize that the number of dots and the number of groups coincided but connected these two quantities physically, highlighting that the overlapping dots were members of the groups of five, a detail that seemed key to his final interpretation. Probably even more significant were his actions surrounding this description. Although Jack stated, "You're taking away one dot from each group," he simultaneously wrote a coefficient of 1 for the second  $s$ . Although Jack's motivation for writing the 1 is unclear, this act served to create a notational bridge between the two  $s$ 's, reinforcing the connection that appeared in Jack's statement that the two symbols are associated with the same groups. Inserting this numeral 1 seemed to standardize and consequently emphasize the idea that the size of each group is denoted by the coefficient. Without a coefficient of 1, this connection is

particularly challenging to see because the number of overlapping dots and the number of groups is the same.

Having added this coefficient, Jack returned to the figure and seemed to attempt to connect his interpretations of the three variables ( $5s$ ,  $s$ , and  $4s$ ) within these two expressions to the figure. He first circled the groups of 5. He then went group by group, highlighting the dot (or group of 1) that he removed, and circled the remaining four dots (see Figure 4.25). Eventually he summarized his actions and understanding: “You still have the same amount of groups, but now they are of four instead of five.” Again, while Jack was not explicit, he seemed to imply, through his actions and description, that the same groups change in size, an indication that  $s$  represented groups of varying size throughout the various expressions.



*Figure 4.25.* Jack demonstrating how a group of five dots becomes a group of four dots.

Sensing that Jack was unsure about his interpretation, I asked if he could be clearer about what each  $s$  represented in  $5s - 1s + 1$ . His response seemed to confirm his confusion. He answered, “This is the amount of groups you have,” while at the same time laughing. His verbal response indicated a quantitative connection between the

symbols, but his laughter communicated that he had not come to fully understand the meaning of this description. He then went on to clarify his thinking:

It makes sense [that] they're all the same; they're all *ses*, so they're all the same number. I can place them in here and see they're all the same number. But I am not sure how to answer your question other than relating it back to the stage number.

With this comment Jack again established the numerical relationship between the symbols that Clara had used to associate the various symbols, but he also implied that he was searching for more of a connection, a conceptualization of *s* that would not simply relate the quantities numerically but would provide a way of thinking about the variables so that they were actually the same. I asked Jack one last time to explain his thinking about the meaning of *s* in  $5s - 1s + 1$ .

Well, for this one (pointing to the first *s*), the *s* represents the amount of groups that I have. So there's this many groups of five, and I can take one away from however many groups there are. And then I am going to have the same amount of groups, but they now each have four in it .... But I guess the answer would be how many groups ... how many groups there are, ... but groups of what? But I guess it changes. That is where I get confused.

Although Jack's description seemed logical and coherent, he continued to comment on his displeasure with his interpretation of the various *ses*. He was able to clearly state that *s* represents the number of groups (without attaching a size), recognized that the dots are members of these groups, indicated that their removal resulted in the group size changing from five dots to four dots in each of these groups, and had previously clearly described the roles of the 5 and 4 as determining the sizes of the groups. Still, he was unable to connect these understandings and interpret the various *ses* as representing the same groups with the coefficients indicating the different group sizes. I argue that the missing

piece conceptually for Jack was his inability to fully conceive of the dots as *groups of 1*. Although he coordinated the dots and groups, thus, seeing the dots as being the difference between the groups of four and five, and even wrote the isolated  $s$  in a consistent form with a coefficient of 1, he failed to fully appreciate the 1 as indicating the size of  $s$  groups. Consequently, unable to reinterpret the overlapping dots as groups of one, he was unable to consolidate his understandings and arrive at a clear interpretation. Although he had produced an interpretation of  $s$  consistent throughout the various expressions and manipulations, he was unable to fully appreciate the significance of his conceptualization.

***Jack's reflection on his conceptual difficulties.*** At the end of the interview, I asked Jack if he could articulate his confusion with coordinating the symbols and add any details about what troubled him. His answered,

My initial confusion was, I guess, you're not taking away .... Well, once I put the 1 here [before  $s$  in  $5s - 1s + 1$ ], it seemed to make more sense to me. I think that however many groups you have, you're taking away one [dot] from each one [group]. And that's how I made sense that now you have a group of four instead of a group of five. And I think [previously] I was going, "Well you take away one group, but that doesn't give you groups of four anymore. It will give you one less group of five." But in the picture, I need to find groups of four. So this was a way I could do that.

Jack's response revealed a high level of metacognition. First, he identified the exact part of his struggle that had led him to a conceptual breakthrough. He clearly articulated that writing the 1 helped him to coordinate the different meanings of the symbols and to see all the  $s$ es as representing the number of groups. In addition, he explained how his initial interpretation, which he and Clara had both explored, was inconsistent. By reflecting on

the quantities, he recognized that when  $s$  is (erroneously) thought of as a group, subtracting one group changes not the size of the groups but, instead, the number of groups. Furthermore, although he had named the  $5s$  and  $4s$  as groups of five dots and groups of four dots, respectively, he still referred to the single  $s$  as 1, implying one dot. It seemed he still had not conceptualized the  $s$  as a group comprised of one dot.

### **Conclusion**

As this analysis reveals, contextualizing algebraic notation is challenging, even for experienced teachers. There are many nuances that experts overlook when they use algebraic expressions to communicate their generalizations and then manipulate the symbols to solve problems. Helping students develop a rich understanding of algebraic notation requires focusing on quantitative relationships rather than on simply numerical values. Focusing on the quantitative relationships is fraught with many challenges and inconsistencies. In this study, I identified two challenges, along with various conceptualizations, both productive and unproductive, that teachers formulated to overcome them.

In this last section I draw from these findings and discuss various related conclusions that stem from these results for both the teaching and research communities. In particular, I reflect on complexities associated with the conceptualizations that emerged among the teachers. In addition, I highlight ways this work contributes to the field and discuss instructional implications of this study.

### **Conceptual Complexity of Interpreting Algebraic Expressions**

I first analyze the conceptual complexities embedded in the type of understanding achieved by the participating teachers by articulating the details of the cognitive

challenges involved to illustrate that the conceptualizations these teachers ultimately formulated were far from straightforward. Moreover, this analysis serves to emphasize that the quantities the teachers came to see in the notation were not characteristics inherent in the figure but, rather, mental constructs the teachers created.

**Complexities of Clara's ability to flexibly interchange the meanings of the variable and coefficient.** One challenge that emerged in this study was articulating, with precision, the meaning of each variable and its coefficient in the notation as well as the relationship between the two. To overcome this difficulty, Clara not only interpreted the  $x$  as the number of groups and the coefficient as the size of each group, but also reversed this mapping and conceptualized the coefficient as the number of groups and the variable as representing group size. To illustrate the complexity of such an understanding and to highlight the flexibility necessary to see and move between the two interpretations, I use an alternative figure for which the transition between these two views requires only a subtle shift in defining the group.

The pattern shown in Figure 4.26 can be modeled with the expression of  $3n$ . Depending on your perspective, though, the 3 and the  $n$  take on different meanings. I have included arrows to help the reader see the two interpretations.



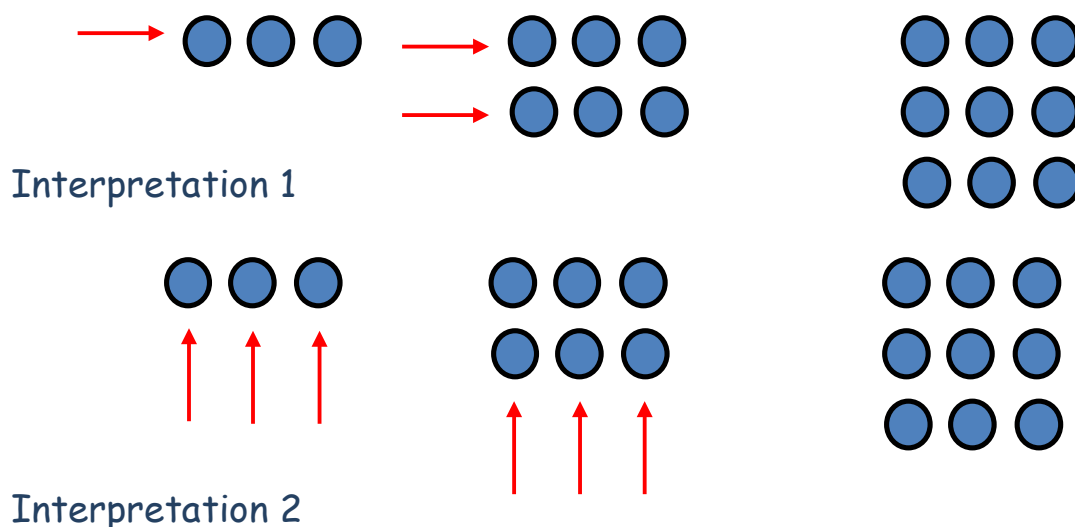


Figure 4.26. Flexible conceptualizations of variable and coefficient.

For Interpretation 1, I have indicated the rows. According to this view, the figure consists of an increasing number of rows, each containing three dots. If each row is a group, the  $n$  would represent the number of groups and the 3 would indicate the size of each group, or the number of dots per group. For Interpretation 2, I have marked the columns.  $3n$  still models the pattern, but I have switched the perspective. If the columns are the groups, the number of groups is fixed at 3, but the group size increases at each new stage. With this new view, the 3 represents the number of groups (or columns) and the  $n$  is the varying number of dots per group.

Although another figure might require a more abstract interpretation of the symbols to be seen in both ways, such a mental shift between these two views can be made with any pattern. Jack's and Clara's diverging interpretations of Pattern 2 provide another example (see Figure 4.27) of the interchange of coefficient and variable. Clara thought of the three arms of tiles as fixed groups, symbolized by the coefficient, and the variable representing the size of these groups, always 1 less than the stage number. In

contrast, Jack interpreted the number of three-dot orbits he pictured as the variable, inferring their constant size as the coefficient.

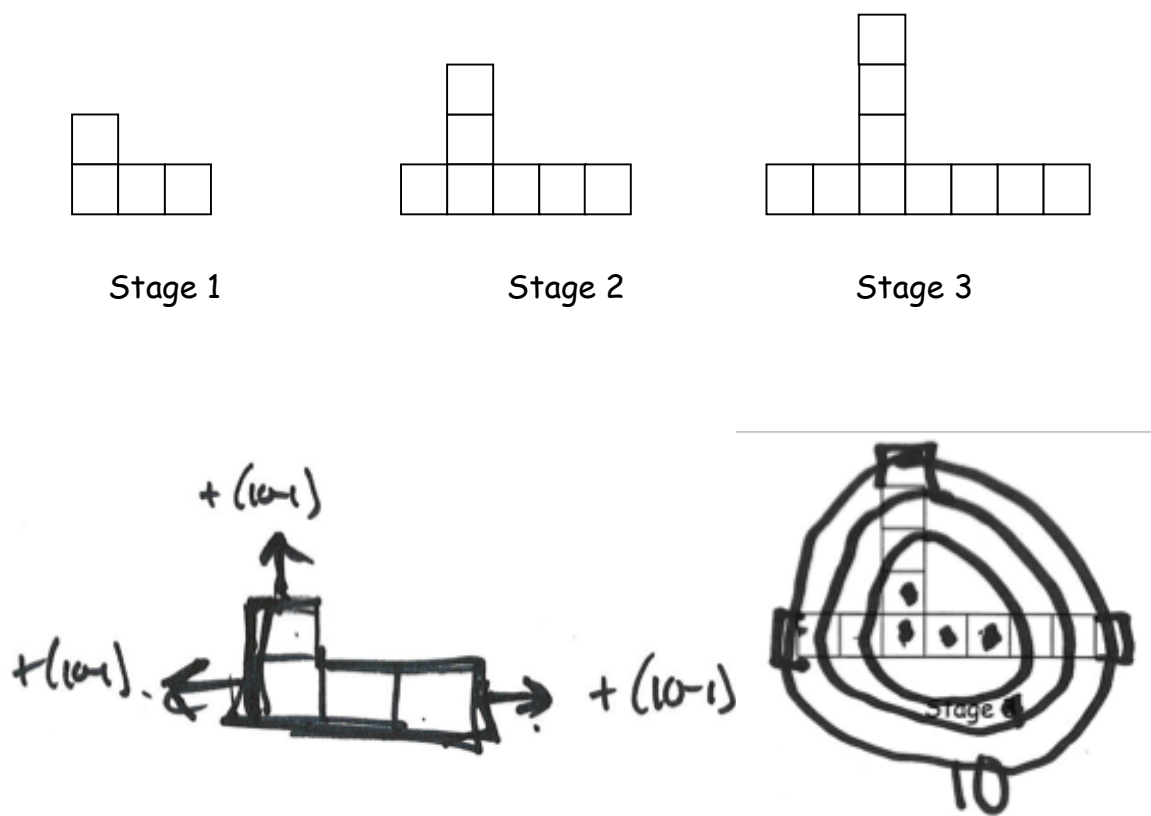


Figure 4.27. Jack's and Clara's diverging interpretations of Pattern 2.

I argue that to conceptualize the symbols in these two ways, one needs a strong understanding of the algebraic notation, skill to flexibly decompose and imagine quantities in the figure in multiple ways, and the ability to coordinate these skills. Clara's capacity to interpret the symbols using both perspectives demonstrated that she possessed all these components. As these two examples show, these understandings are not intrinsic properties in the various representations but must be developed by the person working to understand them.

**Complexities of Jack's ability to interpret each variable as the number of groups.** Another challenge that emerged among the teachers in this study was the ability to develop an interpretation for the multiple variables, either within a single expression or between two algebraically manipulated expressions, that explained their relationship quantitatively. For Pattern 1, Jack was able to generate such an interpretation by conceptualizing all the  $x$ es in the various expressions as the number of groups, with the varying coefficients representing the numbers of dots within their respective groups. Such a conceptualization proved quite powerful because it not only enabled him to explain why the  $x$ es in each variable expression ( $5x$ ,  $x - 1$ ,  $x$  and  $4x$ ) had the same value but also provided meaning to the operations that served to combine the variables. If one interprets all the  $x$ es as the same number of groups and the coefficients as their sizes, the subtraction of variables corresponds to a reduction in the sizes of the groups according to the relative number of dots, which symbolically results in a decrease in their coefficients.

At the same time, such a conceptualization involved many underlying conceptual complexities. First, as noted in the analysis, Jack had to reconceptualize the overlapping dots as groups of size one dot. To do so Jack had to not only reinterpret the quantities in the figure but also restructure and reconceptualize the symbolic expression. To reinterpret the symbolic form, he had to first conceive of a coefficient of 1 and then conceptualize it as representing the size of a group. As Jack's response indicated, such a construal is quite challenging, inasmuch as he was never able to fully interpret the dots as groups of one dot.

In addition, such a construal introduces another conceptual hurdle. By interpreting  $x$  as only the number of groups without reference to the group size, the

variable takes on a more abstract meaning, one that cannot be concretely highlighted in the figure. One cannot circle a group without size. For any group circled, the meaning of  $x$  takes as part of its interpretation the size, which is incorporated in the definition. Therefore, one must be able to understand the variable as the number of groups, independent of the picture. Denise, who struggled to disentangle the meanings of the coefficient and variable, showed that interpreting a symbol as a quantity that does not have an exact physical representation can be challenging.

### **Significance**

This study makes several contributions to the field; I highlight two here. Although the fact that algebra classrooms are dominated by a symbolic focus without attention to meaning has been documented, little is known about teachers' understanding of algebraic notation. As I pointed out in Chapter 2, the few studies that have focused on teachers' understanding have shown that teachers often avoid embedding their instruction in contextual situations because they see abstract symbols as much easier for students to understand (Nathan & Petrosino, 2003), and they interpret other representations as supplementary (Stylianou, 2010). In addition, Harel, Fuller, and Ravin (2008) reported on teachers enacted understanding of symbols, detailing ways in which they failed to support students to develop mathematical meaning.

This study adds to these findings, but differs in two distinct ways. First, these previous investigations focused predominantly on teachers' orientations to algebraic symbols, identifying ways in which teachers' syntactic approaches to algebraic instruction were linked to their beliefs as mathematics instructors. I explicitly investigated teachers' understandings of the notation, and the results show that the

challenge to transform the current symbolic focus in algebra classrooms is not simply an issue of beliefs. By documenting teachers' struggles to articulate a clear connection between the symbols and the quantities they represent as well as identifying many conceptual hurdles, I demonstrate that, at least in part, the difficulties teachers experience in shifting their instruction is connected to their knowledge bases.

Second, earlier work was reported primarily on the difficulties teachers experienced engaging students in meaningfully understanding algebraic notation. The results of this study, while characterizing some of the challenges, also include various constructive conceptualizations that teachers formulate to overcome these challenges. As such, these findings help document not only the hurdles, but some of the conceptual resources that will help the field begin to support teachers in expanding their specialized content knowledge and eventually promote and support students to develop a deeper, quantitative understanding of the notation. In particular, Clara's and Jack's conceptualizations provide specific details about two potential characteristics of rich understanding of the notation: (a) flexibility in interpreting the variable and coefficient as number of groups and size of groups, respectively, and vice versa and (b) ability to see both the numerical and quantitative relationships between variables.

### **Implications for Teaching**

Finally, I highlight two interrelated pedagogical implications that stem from the results of this study. First, as the struggles of these three teachers indicate, developing the type of detailed understanding that enables one to clearly articulate the quantities represented by the algebraic notation is not easy. Writing an expression and understanding the meanings of the symbols are not the same. Each of these teachers

decomposed the figures and generated expressions that aligned with their views of the patterns. Still, when asked to explain the meanings of the symbols, each experienced significant difficulty. One conclusion that can be drawn from the teachers' struggles is that supporting students to develop a quantitative understanding of algebraic symbols will require a significant shift in our algebraic instruction. Clearly the current focus on generating symbolic expressions without emphasizing the underlying conceptual understandings will not support students in interpreting algebraic notation as a representation that communicates quantitative relationships. Only through a concerted and dedicated effort will students begin to learn to make rich connections between representations, which is central to developing a meaningful understanding of algebra.

With this challenge in mind, I see a second related implication in terms of teacher preparation. For teachers to support students in developing understanding of algebraic notation that goes beyond a calculational orientation, we mathematics teacher educators must help them acquire more detailed understanding of symbolic representations. Moreover, for teachers to shift their practices to align with a more responsive model of instruction will require them to develop a whole new knowledge base of specialized content knowledge. A specific focus of the mathematical preparation of teachers must be placed on ensuring that they have opportunities to grapple with the meanings of the representations they use. In particular, this study, by highlighting the difficulties teachers experienced making connections between representations and documenting some of the details of the complexity involved, contributes to this endeavor in identifying for MTEs specific areas on which to focus their instructional attention to support teachers in developing components of this knowledge.

## **Chapter 5: Teachers' Understanding of the Process by Which Students' Learn to Generalize Algebraically**

In this chapter I report on how differences in the teachers' understandings of algebraic generalization translate into the instructional environment, affecting how they interpret and react to student thinking. In particular, I answer Research Question 3: What subtle differences in knowledge and practice distinguish two teachers who use students' conceptions as the basis of their instruction when they strive to engender in their students a meaningful understanding of algebraic notation? I analyzed instruction of only two of the four participants, Clara and Jack. I selected these two teachers because although both articulated rich learning goals for students and genuinely endeavored to make instructional decisions on the basis of their students' thinking, their instruction led to very different outcomes. By exploring this significant contrast, I sought insight into the nuances of algebraic generalization. Revealing the details that distinguished these two similar teachers, yet contributed to such different results, helps identify key components necessary to support students while they learn to generalize and communicate symbolically their understanding of these figural patterns.

To situate this study, I characterize the goals the other leader of the professional development and I (hereafter referred to as *the researchers*) held at the time of the study. I then provide an overview of the goals communicated by the two teachers in the postinterview conducted after the one-week professional development. These data indicate that the teachers' views of the mathematics associated with these generalizing tasks had begun to align closely with the goals the researchers had envisioned. Furthermore, at the level of detail at which the researchers held and articulated these

goals at the time of the professional development, the teachers' descriptions were similar. However, in a deeper study of the understandings of these two teachers, in particular, their enactments of their goals in the classroom and their reflections about their instructional decisions, details emerged that illustrated significant distinctions between the two. My primary focus in this chapter is to explain these differences. I identified several factors that distinguished the teachers' understandings of the process by which students come to generalize symbolically. In addition, I present a detailed analysis of the mathematical terrain, including a detailed description of the conceptual benchmarks and instructional strategies identified as part of the teachers' instructional trajectories and used to analyze the teachers' instruction.

### **Overarching Instructional Goals**

A unit on algebraic generalization can have multiple goals. As explained in the previous chapter, teachers' goals for these tasks include instilling sociomathematical norms, developing the mathematical practices, or illustrating characteristics associated with linear equations. Teachers' orientations, their mathematical understandings, the needs of their students, the curriculum, as well as the time of the year can all factor into what they see and pursue in a given unit or activity. The main focus of the instruction in the professional development, from the perspective of the researchers, was to provide an experience for the students that engendered in them a richer understanding of algebraic notation. Too often in algebra students experience prescribed procedures and rules of algebraic manipulation. Such a decontextualized approach often leads to "number grabbing," in which students select numbers and operations and then blindly compute with symbols (NRC, 2001). In contrast, our goal was for students to dissect the figural



patterns by identifying visual quantities within them and then use algebraic notation as a tool to communicate their understood decompositions. We hoped to spend time ensuring first that students had strong understandings of the visual pattern and then slowly transition this understanding to a symbolic representation. With such an approach, our objective was for students to see algebraic expressions not simply as a tool to calculate the figural number but also as a representation that can be used to describe the pattern. By developing the algebraic expressions with a strong connection to the context, students would see how one expression captures a particular structure of the figure perceived by the student whereas another expression depicts an alternative view. The researchers' ultimate goal was for students to learn to coordinate the meanings between the symbols in the algebraic expression and the quantities in the pattern. Such an understanding of algebraic notation is described in SMP #2 (CCSS, 2010): Students who develop this practice learn to contextualize notation, and when suitable, they probe the meaning of the symbols and see the quantities and quantitative relationships they express. The process by which this understanding is developed is articulated in Kaput's depiction of symbolization (Kaput et al, 2008). Over time, students do not simply see symbols as objects themselves but develop the ability to see in the symbols the quantities they represent. Eventually, they interpret the mathematical phenomenon and the notation as the same.

### **Goal 1: Developing a Meaningful Understanding of Algebraic Notation**

After the week-long professional development, both Clara and Jack clearly articulated that they saw the ability to contextualize the symbols as central to the unit. In the interview that followed, Clara detailed the importance of connecting the symbols to

the figure, emphasizing the contrast between this view and her previous understanding associated with the purpose of these tasks. She stated that in the past, she thought that working in a context in which the numbers had a source would be more stimulating for the students, but she did not have them actually analyze the figure. Instead, students immediately worked with the generated numbers in a table and looked for numerical patterns. After seeing the rich understanding of the notation the students developed by engaging with the figure, she changed her view, saying, “I see that the pattern itself is a very valuable tool. It is a valuable tool for being useful for explanation for *variable*, for explanation of *expression*, for explanation of *function*.” She went on to clarify:

That's the context for where they are going to develop their expression .... Now they're going to see that this expression can have a meaning. There is a reason that, you know, this expression is written this way and it looks different from this one.... The numbers mean something. The numbers really do represent that.

Clara's description mirrors the goals of the researchers at the time of the professional development. Her comment reflected a changed view, and she now considered the figure an integral part of her learning goals. Clara had come to see that by working directly with the patterns, students could develop meanings for the symbols and see how different interpretations of the pattern could map onto different expressions.

Jack also alluded to this goal in his interview. He stated that instead of just seeing the symbols, he wanted students to understand what the numbers in the visual pattern mean. He differentiated this understanding from the view of expressions to which he thought students are normally exposed. Although this was Jack's only specific reference to contextualizing symbols in the interview, he later reflected on his instructional decisions in the classroom; he held a deep understanding of this connection,

and, like Clara, saw instilling a meaningful understanding of symbols as a significant goal for the unit.

## **Goal 2: Understanding Functions as a Relationship Between Two Quantities**

As a consequence of the professional development, two additional related topics emerged as significant goals. The first was supporting students' developing understandings of function beyond simply a rule that produces outputs. Clearly, according to their comments, teachers saw that through generalization students could begin to understand how a rule is developed and see the function as a relationship that connects the input and output quantities. When Jack planned to teach the unit, he read both the CCSS content standards and his textbook chapter on functions to determine how the topic corresponded to the requirements. He emphasized that he believed “seeing the relationship between the step number that’s changing and the pattern that’s changing” was a significant component of algebraic generalization. Clara also noted that this context was particularly conducive for showing students that functional notation represents a relationship, highlighting that the visual context of the pattern supports students in understanding “that a change in the input affects the output.” Jack echoed this sentiment, stating that “your increasing something by 1 is going to effect the other side. This is (a) nice way to see it happening. You can see the pattern is changing as the stage number increases.” He continued to reflect on the use of pattern generalization as an approach to help students understand functions, contrasting it with the approach outlined in his book. He noted that the book presented functions as abstract equations and symbols, without meaning. In contrast, in the context of algebraic generalization, students develop the symbolic representation of a functional relationship that is depicted

in the figure rather than simply being presented the symbolic notation “as a magic thing that happens as you plug in a number.” He thought that because students would establish the functional relationship and generate the associated notation, they would develop better understandings and “have a better chance of remembering it and using it in other situations.” Clara offered a similar analysis:

Before it was about creating an expression that produced the correct number and then focusing on the linear characteristics. Now it is about seeing two quantities and understanding and communicating how they are related ... [and] making sense of that connection; not simply using an expression without knowing why it produces these value and what these represent.

Comments from both teachers conveyed that they hoped that by developing the formula from context and connecting it to the figure, students would derive understanding for the source of the functional notation and the relationship it communicates.

### **Goal 3: Addressing Misconceptions Surrounding Variables**

The final goal that emerged for the teachers during the professional development was the role of *variable* in algebraic generalization. During the week, the researchers and teachers recognized that the student participants held superficial understandings of variables' meanings and roles in algebra. The extent of their understanding of variable was as a symbol that represented and could be interchanged for a number. The students did not see the symbols as connected to quantities, and they struggled significantly to symbolize quantitative relationships that began to emerge in the patterns. For example, they depicted *a quantity two more than n* as  $p$ ,  $n2$ , and  $n^2$ . Furthermore, they were unfamiliar with the convention that symbols used in a single algebraic expression must have the same value, believing that one  $n$  could be larger than  $n + 1$  “depending on the

numbers.” They also demonstrated difficulty simplifying expressions. Many rewrote  $2n + n + 1$  as  $n$ , thinking that  $n$  could represent anything, whereas others believed the expression  $2n + n + 1$  could not be combined because  $n$  and  $n + 1$  represented different quantities. By the end of the week, helping students develop a more contextual and nuanced understanding of a variable became an important goal of the group.

The significance of understanding variables was confirmed by Clara and Jack, who both commented on the struggles students displayed with variables. Clara clearly stated that the participating students demonstrated multiple misconceptions about variable. She also noted that she felt that these tasks “help reveal what a variable is.” Jack echoed this sentiment, stating, “I think we opened up a lot of cans of worms through this unit, with the variables and the misconceptions.” Jack then offered a more detailed overview of his particular concerns, highlighting specific notational issues he wanted to address with his own students. These comments on students’ struggles to properly use symbols showed that a focus on variables had become a central goal for both teachers.

In summary, after participating in the week-long professional development, three instructional goals emerged as significant for Jack and Clara while they prepared to teach a unit organized around these figural patterns. First, both teachers included instilling a meaningful understanding of symbols as an important goal. In particular, they indicated that they wanted students to see algebraic notation as communicating a particular decomposition of the figure. Second, by supporting students to develop strong understandings of the connection between the symbolic expression and the figure, they wanted students to see functional notation as a relationship between input and output quantities. Both commented that they had previously had students work with functions

in symbolic form but without focusing on students' understanding why the expression produced particular numbers. As a second goal, they wanted students to understand functions, particularly the symbolic representations, as a relationship between two quantities. The last goal communicated by the teachers was to address students' misconceptions about variables and familiarize them with certain conventions associated with algebraic notation. Both teachers believed that equivalent expressions provided a productive context to help students with this issue. When these teachers prepared to teach the unit I observed, these were the three goals they held.

### **Identifying the Differences**

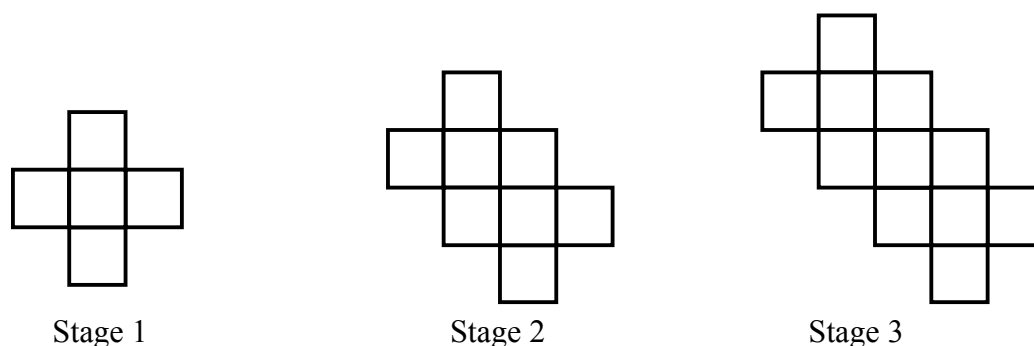
As stated before, the aim of this chapter is to answer the final research question:

What subtle differences in knowledge and practice distinguish two teachers who use students' conceptions as the basis of their instruction when they strive to engender in their students a meaningful understanding of algebraic notation?

As evidenced in the first section, both Jack and Clara articulated comparable goals at a general level. Their comments indicated that they wanted students to develop strong connections between the algebraic symbols and the associated figures and to ultimately understand that notation serves as a tool to communicate the quantitative relationships within the figure. By observing them into the classroom and comparing details of their day-to-day instructional goals, their pedagogical decisions, and their reflections on classroom interactions, I identified three connected, yet distinct, differences in their understandings of the processes by which students come to generalize using algebraic notation. These factors are (a) their understandings of the connections between the roles of describing and calculating, (b) their understandings of the connections between

expressions and figures, and (c) their understandings of a possible trajectory of cognitive milestones in the process of symbolic generalization. Because these categories are not mutually exclusive, some data that illustrate distinctions in one category also reflect differences in other categories.

Before presenting my analysis of the differences between the two teachers, I describe the tasks and lesson structure each teacher used. Additional details of their teaching and instructional plans follow this description. Clara focused her instruction around a pattern (See Figure 5.1) used during the week of professional development, chosen, she explained later, because of the many ways to decompose the pattern. The pattern could be seen as increasing by an upside down *L*-Shape in the top left or bottom corner, by adding a row or a column, as overlapping crosses, or as three diagonals. Her class used Ipads, and students were to draw, using color, the pattern as they understood it.



*Figure 5.1.* Instructional task used by Clara.

Jack used his own pattern and presented the overall task as three related, but distinct, subtasks (See Figure 5.2). For each subtask, he circled pieces of the figure to illustrate a particular decomposition of the pattern. He characterized the drawings as depicting a student's understanding and asked students to analyze the pattern each time according to the way of thinking represented by the drawing.

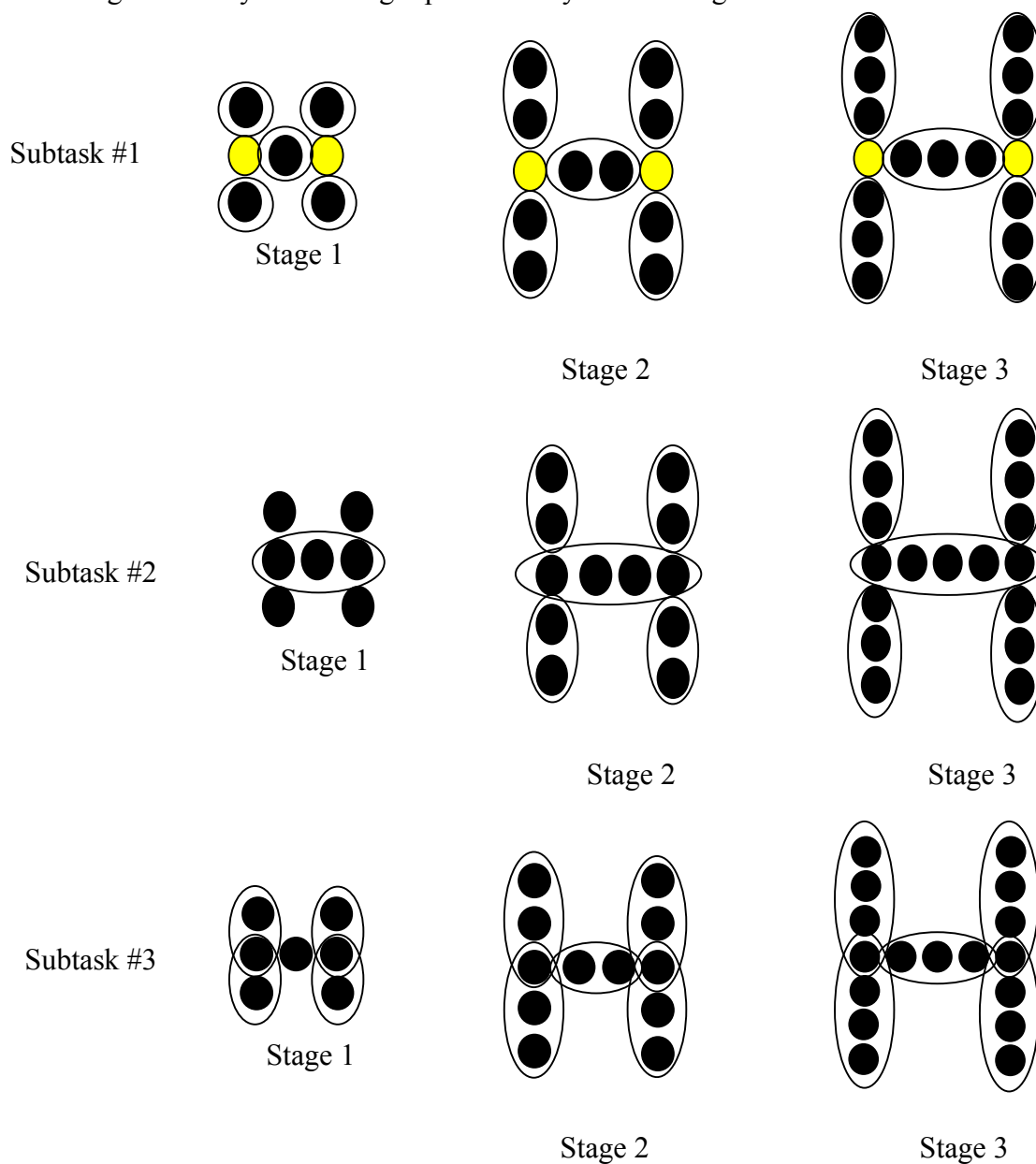


Figure 5.2. Instructional task presented as three subtasks by Jack.



Another distinction between the two teachers was the timing of my observations relative to the overall unit and class period lengths. In Clara's class I visited the first three classes devoted to the unit; they were 120 minutes, 60 minutes, and 120 minutes, respectively. My visits to Jack's class corresponded to Days 5 and 6 of the unit, and each class period was 60 minutes in length. Previously, Jack had worked with his students using two figures without decompositions indicated.

### **Connections Between the Roles of Describing and Calculating**

In the next section, I demonstrate how Clara's comments and actions indicated that she conceptualized describing patterns and calculating elements as distinct features of generalizing, whereas Jack saw these activities as interconnected. Clara, in an attempt to develop meaning, discouraged the use of abstract symbols, which she associated with less meaningful calculations. Having developed an awareness that students often can calculate without making sense of the quantities represented or developing deep understanding, Clara was hesitant to engage students in calculating, seeming to avoid any type of calculational or symbolic approach. Instead, she focused only on students' describing the pattern, ultimately conveying that she believed that verbal descriptions could be seamlessly transformed into algebraic expressions. Seeing a descriptive approach as the best, possibly only, way for students to connect algebraic notation to the figure, she appeared to separate the acts of describing and calculating. Consequently, Clara, on the one hand, struggled to help her students see the two acts as related. Jack, on the other hand, supported students in making connections across the two components of generalizing. He encouraged them to engage in describing and calculating simultaneously, even when a particular representation foregrounded a particular way of

thinking. Before illustrating this difference, I explain the affordances describing and calculating provide and how they interact to support students' developing robust understandings of generalization.

**Generalizing through describing and calculating: A conceptual analysis.** The process of identifying and communicating generalizations incorporates both the acts of describing observed quantities in the figure and calculating exact values for those quantities. Both elements are necessary, requiring different ways of thinking and relating to different properties. Verbalizing serves to elevate contextual characteristics, making both quantities and relationships between quantities explicit. Students attempting to describe patterns tend to focus on relationships (Stacey & MacGregor, 2001) because only through descriptions can students explain and justify connections they perceive. Being asked to identify quantities and the ways they change from stage to stage encourages students to decompose patterns, ultimately leading to a stronger structural understanding of the pattern. Finally, verbal descriptions tend to be less formal in nature than algebraic notation, providing a familiar modality in which students can comfortably express their thinking (Zazkis & Lijendak, 2002).

Alternatively, calculating helps students see precise numerical relationships that might be obfuscated in the diagram. Without numbers, students struggle to precisely quantify relationships. For example, using a table supports students in simultaneously analyzing a collection of stages and identifying commonalities across cases. In addition, calculating serves as a specific activity that anchors student thinking, providing a familiar and explicit pursuit for engaging students in making sense of a pattern. At the same time, without description, students can produce computations without reflecting on the

sources of the results. Too often students abandon sense making and begin searching for possible numerical relationships (Becker & Rivera, 2005; Healy & Hoyle, 1999; Lannin, 2005; Lannin, Barker, & Townsend, 2006; Rivera & Becker, 2008; Stacey, 1989).

Numerical and algebraic symbols, like all mathematical representations, are abstract in nature, without inherent meaning. The symbols can represent multiple quantities and can change referents through manipulation. Vocalizing a perceived relationship causes previously abstract notions to become increasingly concrete (Mason, 1996). Therefore, both actions are needed to support students' developing understanding of algebraic generalization. Ideally, over time the two components become connected, interacting and supporting each other.

**Clara: Disjointed view.** I begin analyzing the two teachers' views of describing and calculating, with Clara's view. She possessed a disjointed view of the role of these activities in generalizing. To illustrate this perspective, I examine (a) her depiction of students' expected daily outcomes, (b) her instructional decisions and the corresponding responses by the students, and (c) her reflection on challenges experienced by the class. Throughout these three aspects, her comments and actions, as well as the understanding exhibited by the students, seemed to reflect a separation of these two elements.

***Description of daily outcomes.*** Throughout the three classes I observed, Clara's depiction of what students would do and learn indicated that she desired for students to engage with the task, almost exclusively, in a verbal, descriptive manner and avoid numerical approaches. On Day 1 she announced to the class that the daily learning target was to "analyze a visual pattern and describe in words how it develops." She reiterated that the work on this day would be done "in words." She emphasized that this

activity differed from their normal work with patterns, stating, “We will eventually look at these patterns and handle them the way the curriculum is presenting them, but we felt there was a step that was missing.” After class I asked her to explain what she meant by this comment and how she saw her goals for this unit different from those in her curriculum. She explained,

They [the curriculum authors] want the kids to be able to predict the number of tiles. Then it's all about the numbers. It's all about the number, number, number of tiles. It is all about manipulating the numbers, and there is no connection to what they mean.... What's happening, what we are asking the kids to do here is stay in the figure and give me that description; give me that expression based on how, in one particular way, the pattern is growing.

In this quote, Clara first expounded upon how her previous emphasis on calculating did not support students' developing meaningful understandings. In contrast, she wanted to instill in her students that algebraic symbols should convey meaning. Second, she articulated her belief that the way to ensure this understanding was to first focus on describing. If an expression is to be viewed as communicating particular quantitative connections, then it must be built from a descriptive approach. Later in the interview she confirmed this view when she explained why staying in the figure was important for students. She stated, “When it's just numbers, ... there's not a real understanding of, ...'Hey, this means something; ... these represent something.' The bigger idea in math is you should be able to describe.” Again, she emphasized her concern for a numerical approach and reiterated that her focus was on instilling in her students a view that mathematics serves to describe. She seemed not only to value a descriptive approach but also to avoid a calculational one.

On the next two days, when explaining what she hoped students would do and learn, she again expressed her focus on describing. She began class on Day 2 by telling the students that they would “look for relationships between the figure and the stage number, between the number of tiles,” and then “describe any relationships.” After class, I again asked her to clarify what it would mean if students achieved her goal. In her response, she seemed to intentionally deemphasize any calculational approach by highlighting the use of words as the primary representation. She explained that students would be “generalizing in words,” “express[ing] in words ... the relationship between what is changing in a stage number, and then [they] can for any stage number describe how you would build it.” On Day 3, Clara's goal was for students to create algebraic expressions, but again she highlighted that they were to arrive at this symbolic form through verbal descriptions. Her instructions to the class were to first write “an expression in words” of how the pattern relates to the stage number. She then clarified, “I want this written very tight, very succinct, very precise, as short as you can.” Throughout the 2-hour class, she reiterated these instructions to students while the whole class struggled to write any relationship. After class I asked her to clarify her plan for how she thought students would arrive at the expressions. She replied, “I thought that when they would write these descriptions and that we could just replace—'Okay, instead of this word, we're going to replace it with some algebra.'” This final explanation explicated her instructional plan for the week: for students to describe in words their patterns and then convert their descriptions into algebraic expressions without calculating or using any symbolic tool. Clara, in her attempt to develop meaningful expression, seemed intentionally to avoid use of any abstract symbols. At no point in the three days

of instruction did she encourage her students to calculate the value of a particular stage, continually emphasizing that students were to describe even numerical relationships in words. In fact, when I asked Clara before Day 3, “Are there any other representations that you hope will come out or that you will push to come out?” she construed representation to mean different pattern decompositions. So far from her instructional plan were such computational approaches that she was unable to interpret representation as a table or numerical expression as I had intended. Furthermore, as I show in the next section, her actions throughout the lessons downplayed the role of calculating. Not only did she discourage students from exploring the patterns numerically, even when students did pursue a numerical approach, she did not engage with it, leading to a confusion for the students as to the role of calculating in these tasks.

***Instructional decisions: Ambiguity toward role of calculating.*** Although Clara continued to emphasize describing the pattern, many students began pursuing calculational approaches. In each instance, instead of looking for ways to relate the student's calculational and descriptive thinking, Clara offered limited support for the student's calculational method, engaging it only minimally. Furthermore, to elevate a descriptive approach, she consistently downplayed the use of numbers to the point that students were unclear about the role of calculating and to what degree numbers should be used. This reaction was evident on several occasions throughout the three days of instruction. In one case, on Day 1, a group asked if they should include the number of tiles in their description. Clara neither answered their question nor clarified the role of numbers in the task: she asked, “Do you think this [numbers] should be part of analyzing the pattern?” Instead of supporting or pressing the students’ thinking, she neither

clarified the students' thinking nor communicated what relationship, if any, was desirable between describing and calculating.

Later, toward the end of the same day, Clara offered a similar response to the whole class. She had asked the class to explore Stage 10, providing the specific instructions, "Describe what Stage 10 would look like without building it, drawing it, or referring to Stage 5, as if you were describing it on the phone." Although her instructions encouraged a descriptive approach, more than half the class began calculating and most of the student groups did so inaccurately. With the class gathered for discussion following their small-group work, Clara asked one group to read their answer. The group representative shared their incorrect calculation: "Thirty-five, because 10 times 3 plus 5." Clara responded by recognizing the shift to a numerical approach, but again did not pursue the students' thinking or try to connect it to their descriptive view. Instead she replied vaguely, announcing to the class, "One of the things that I'm starting to hear is that you're talking about the number of tiles. Is that an important idea? Is it important to know the number? That's where we're going to continue." When I asked Clara her thinking in that moment, she said, "I remember thinking at the time, 'I do want to bring out the fact that the number's important. ... We spent—we carefully stayed away from there, but we do need to bring it in.'" In this reflection, Clara explicitly stated that she worked to avoid numbers. She then commented that calculating is important but offered another unclear response about its role. Clara seemed to want students to focus on describing as a way to ensure a meaningful understanding of the pattern. Consequently, to elevate the act of describing, she tried to reduce the focus on calculating. At the same time, she seemed to understand, although

not necessarily explicitly, that numerical details are part of generalizing, creating a tension for her students and even for herself. Students began calculating the values of stages, sometimes instead of describing the quantities as Clara had requested. Instead of drawing connections between the two acts, Clara emphasized describing and deemphasized calculating, creating ambiguity about their joint roles and reinforcing their separation.

Clara's interaction with another group struggling to make sense of Stage 10 further highlighted the divide Clara interpreted between the two actions and the confusion that was created by her attempt to distance them. Similar to the previous instance, this group had incorrectly calculated the number of tiles. In contrast though, they erroneously concluded that the value of Stage 10 was double the value of Stage 5. Clara's response was to encourage the students to count the number of tiles as a way to highlight their wrong answer. She then had them count the numbers of tiles in Stages 2 and 4 to confirm that the latter were not double the former. I later asked what she was thinking at that moment. She explained that she was attempting to illustrate numerically that their answer was incorrect so "they can let go of that." In this exchange, although Clara did engage in the students' numerical thinking, it was only to demonstrate their error. Calculating served to demonstrate wrong methods but not to support generalizing. Although this group was struggling specifically because they had ceased to try to describe the pattern and were attempting to generalize by simply calculating, Clara did not try to refocus the students' attention and encourage them to combine their calculations with their earlier descriptions of the pattern. Instead of engaging the students in both describing and calculating for their mutual support, Clara pushed the



group to continue to pursue their numerical approach. In fact, by having the students repeatedly count the number of dots, she tried to highlight that their calculational method was incorrect by asking them to calculate.

Finally, on Day 2, Clara reinforced her disconnected view of the two actions when she reflected on the students' thinking from the previous day and described her instructional goals for the upcoming class. She acknowledged that the students the day before had pursued calculational approaches, stating, "We're noticing that the number of tiles is important. That the kids, in the description, they say, 'Yeah, you know, that's an important part of it too. How many total?'" I use the word *acknowledged* because Clara's tone inferred that she was reflecting on the role of calculation, seeming to realize anew that calculating is an important part of generalizing. She then segued into the goals for the day, identifying both describing and calculating, but listing them as separate. She stated that she wanted students in class to think about two things: (a) "the relationship between the figure and the stage number," and (b) "the number of tiles." She then qualified the second one as "good, at least ... I mean that's a relationship they need to consider, but I want them to still stay in the figure." Again, Clara articulated that she saw calculating as part of generalizing, but portrayed it as distinct from her goals. Furthermore, she explicitly diminished its role, seemingly as a way to elevate describing.

Clara's lack of emphasis on calculating as well as her lack of clarity seemed to shape the students' thinking. First, on several occasions, when Clara asked students to describe connections between the stage number and the figure, their responses indicated a lack of numerical precision. As the following exchange illustrates, on the one hand, Clara's questions appeared to indicate the belief that students could forego a numerical

step and generalize this relationship verbally. The students' responses, on the other hand, suggest this was not the case, that they needed more experience with calculational details to support their thinking.

- Clara: Okay, so you are starting with two [tiles]. So if you would write those words, what you just said to me, what would you write? If you had somebody writing down exactly what you said, what would that person write?
- Student A: You start with two things.
- Student B: Okay, tiles, whatever. So you are starting with two [tiles].
- Student A: Then you add an upside down  $L$  to the left.
- Student B: And you keep on adding a three [tiles].
- Clara: Okay, and how many times will you do that?
- Student C: How many times? As many stages as there are.
- Clara: So tell me. So I have so far, "Start with two tiles, and then you add three tiles at a time," but how many times do you add that?
- Student C: For each stage, you add one. So one time.
- Student A: Oh, one time.
- Clara: For each stage?
- Student A: You add three.

In this excerpt, the students repeatedly provided vague, mostly recursive responses each time Clara asked how many times the  $L$  or 3 needed to be added. Clearly, what Clara anticipated did not correspond with the students' responses. Clara's questioning also provides insight into how she seemed to distance describing from calculating. At no point did she introduce numbers during the exchange, even while students continued to provide unclear responses.

Later, similar exchanges occurred when Clara wanted the students to provide a general description of the explicit pattern, but they repeatedly responded with vague or recursive answers. Clara once asked students if they could "describe it [quantities in the

figure] for any pattern?” Although Clara wanted the students to provide a description of the quantities within the pattern relative to the stage number, the students misinterpreted the request and provided a recursive view, stating, “It starts off with four red squares on the sides and one yellow square in the middle. Each stage, it grows by adding one yellow square in the middle and two red squares at the sides.” In another case, Clara asked how many groups of three needed to be added, in general. Again, the students responded in a vague way, stating, “It depends.” Either because of the students’ lack of experience calculating specific stages or Clara’s unclear goal, the students repeatedly demonstrated their confusion. Without calculating specific stages, they were unable to identify the exact relationship between the stage number and quantities in the figure. Of note, in this second example, Clara eventually shifted focus to specific stages, asking for the number of groups of three in Stage 4. Although the students did respond with numerical details, they were initially unable to provide the correct answer. Possibly because this was Clara’s first request for numerical details in their description, students were still struggling to see and quantify the connection with precision.

On the basis of evidence, I conclude that the chasm Clara projected between describing the visual pattern and calculating the numbers had a second consequence: At least one group of students’ understood the pattern one way for describing and a different way for calculating. They provided distinct and disconnected answers depending on whether they interpreted the question as about their descriptive view versus their calculational view.

To illustrate this divide, I provide background information. On Day 2, this particular group of students produced a method for calculating the number of tiles for a

given stage purely by identifying numerical patterns. When Clara asked them to explain their thinking, they replied, “Instead of adding 3, we multiplied by 3 and added 2.”

When Clara asked whether they had viewed the pattern in a specific way, related to that calculation, they were confused by the question said, “No”. When asked to reflect on this moment, Clara acknowledged that their thinking was purely numerical, explaining, “I think she’s seeing the numbers. She’s not relating it to the shape. She’s looked at the table, and she’s looked at the number, and she made that [expression]. She’s come up with a rule.”

On Day 3, Clara returned to this group, asking the students to share their description of the pattern. Clara emphasized that she was looking for a verbal description, which had been modeled by various groups beforehand. The spokesperson for the group said, “You have to draw a cross. Then at the bottom you’re going to add three [tiles].” Clara then showed a picture with the figure shaded according to this description, and the group agreed that it reflected their thinking. Clara then asked, “Where is the stage-number relationship coming in here? Did your group talk about how the stage number relates here?” Seeming to interpret this question as focusing on calculational aspects of the figure, the group then reverted to their numerical rule from the day before, “Yeah, you have to multiply the stage number by 3 and add 2.” Noticing the disconnect, Clara then asked if their “method builds off the cross,” to which they replied with another numerical example: “On Stage 4, we multiply by 3, which equals 12, and then add 2.” Finally, after the students provided another numerical example of their calculational method, Clara funneled them through the pattern according to their descriptive view, eventually writing a detailed description for this decomposition. When

asked to reflect on this episode, Clara explained, “They [the students] weren’t specifically seeing that they’re double counting. If they do that, they’re double counting the cross shape.” Clara’s explanation of the situation indicated that she believed that the students’ two answers corresponded to the same view of the pattern, but that they simply had not attended to the misalignment of the formula and their view. I argue that the students held the two as completely separate answers for what they interpreted as two separate requests and were consequently not attempting to connect them. They had created a method to calculate the number of tiles for a given stage, and they could describe one of the given decompositions, but they viewed these as different activities.

***Reflection on challenges.*** The last piece of evidence that shows how Clara saw the role of describing and calculating as distinct took place at the end of the three days when she reflected on the experience in general and commented specifically on what she might change if she taught this unit again. She first acknowledged that “none of the students came up with the expression on their own” and that she imagined that they probably felt discouraged because she had “changed the game on them without telling them.” Explaining this change and expressing a bit of her frustration at the lack of success, she said,

How do I get them to stay in the figure and come up with that algebraic expression, when in the end it's, you know, when you talk about three  $n$  plus two, it's not really about the shape? That really doesn't describe the shape at all. I mean, it's really going to come down to the number. I mean, it's how you see these, how you see you're counting them. So it really is the number. When it comes time to go back and to generalize, it's, it's really not going to be about the shape. At least it's not for the three  $n$  plus 2.

She then explained further,

I asked the kids to describe what the figure looks like.... But when I'm writing [the expression] ... how many... I removed the upside down *L*-shape, which is exactly what the figure looks like. So I removed that idea.... I thought they're probably...thinking, "Well wait a minute, you asked me to describe the figure for you and now your taking away the very thing that shows how the figure is described," because the essence in the end will be "It's about the number."

In these statements, not only did Clara treat calculating and describing as distinct, but from her characterization, I inferred that she was struggling to conceive of them as coexisting. According to her interpretation, the students were unsuccessful because she had focused on describing the pattern, which is disconnected from the calculational role of the expression. In an attempt to engender an understanding of the symbols as a form of communication, she had her students repeatedly provide verbal descriptions of the way they were seeing the pattern. In the end, though, such an approach was unsuccessful because in her mind the final expression served to simply calculate. Clara was unable to see the two components as connected. Furthermore, not only did she see describing and calculating as separate, it seemed she was only able to associate to a representation a way of thinking that was more explicit in the representation. She failed to recognize (even though she demonstrated this understanding herself in Part I) that although such descriptions are not inherent in the notation, mentally we can see and connect these quantities to the notation. In fact, a major goal for this unit was to enable students to see through the notation and envision the constituent part of the pattern each notational element represents. Her repeatedly characterizing the acts of describing and calculating as separate showed that Clara had lost sight of this goal and no longer saw describing and calculating as interconnected.

**Jack: Connected view.** Similar to Clara, Jack viewed descriptive and calculational approaches to generalization as distinct (or at least not equivalent), but he seemed to conceptualize them in a more connected way and to strive to blend them for his students. Significantly, he worked to infuse descriptive thinking into calculational representations and calculational thinking into descriptive representations. To illustrate this finding, I first highlight Jack's use and thinking of numerical expressions as a descriptive representation. Second, I review an episode during which he worked to inject numerical details into a student's, and eventually the whole class's, verbal description of the pattern.

On the first day I observed Jack's instruction, he introduced the  $H$  pattern shown in Figure 5.2; his students had explored other patterns the previous 4 days. Instead of asking the students for their ways of viewing this pattern, as he had done with previous patterns, he focused their attention on a particular decomposition to align the class's thinking and to ensure that everyone approached the figure similarly. After presenting the image shown in Figure 5.3 as the work of one student, Jack asked his students to share what they noticed about the pattern. Several students gave their interpretations and offered various descriptions of the pattern. Jack then directed the class: "Draw Stages 4 and 5. Circle what you think this student would have circled—how he would have thought about it." After some time, Jack asked a student to draw Stage 4 and explain his interpretation of the student's thinking.

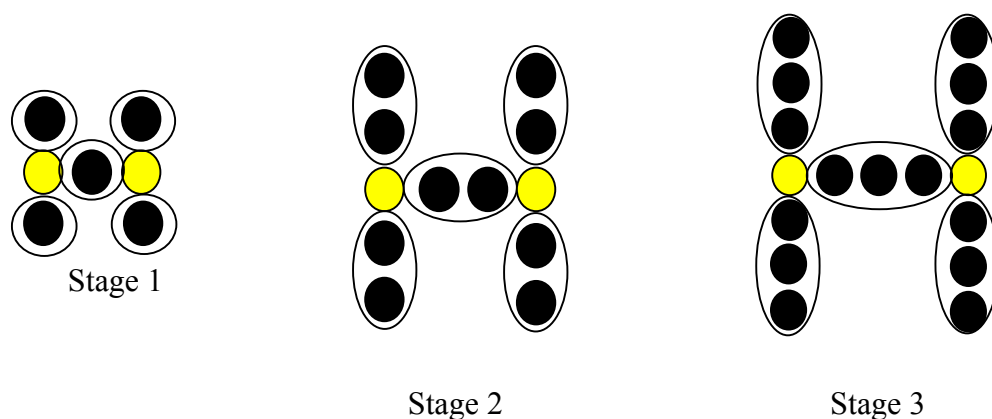


Figure 5.3. Illustration of hypothetical student's view of the pattern.

With Stage 4 drawn on the board and several students having made various observations, Jack shifted the class from a descriptive view to a calculational view of the pattern by asking if anyone could create number sentences to represent Stage 4 and then Stage 5. After a student volunteered her method, Jack asked her to explain what was represented, symbol by symbol, writing her description beside each individual symbol (see Figure 5.4). Jack also sought details, ensuring that the student gave contextual quantities and did not simply refer to the symbols as the stage number. He also revoiced the students' explanation, drawing a clear distinction between the *number of groups* and the *number of dots per group*. Throughout the 2 days, Jack repeatedly used this approach, asking students to write a numerical expression for a particular stage and to connect the symbols to the figure. Eventually, accustomed to this question, students began automatically providing verbal descriptions of the symbols when Jack asked for a number sentence.



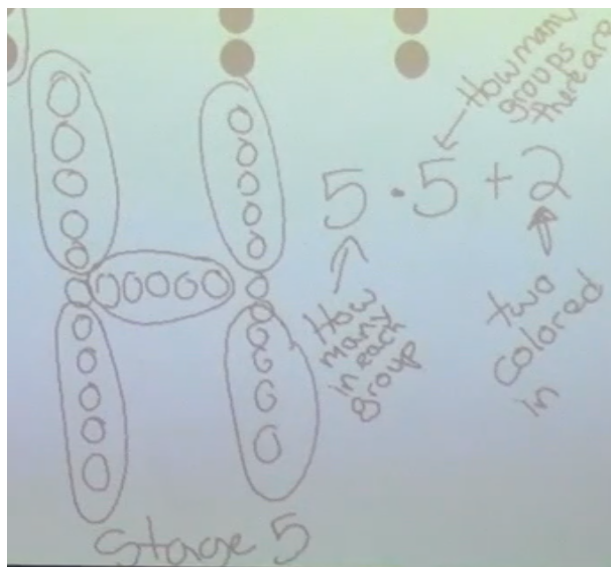


Figure 5.4. Number Sentence with contextual explanations.

Jack consistently asked students to first calculate, to provide a numerical representation, then to add a descriptive understanding of the pattern to the symbols. Jack not only alternated between descriptive and calculational approaches, he also combined the two in writing on the same representation, helping students to visually connect the two ways of thinking. Furthermore, in his description of his plan prior to class, his view of these two components seemed more interconnected than Clara's. After Jack outlined his proposed instructional sequence for the day, I asked what he meant by "a number sentence." He replied, "Five times 4, plus 2, like ... to describe stage, Stage 5. And then we'll come up with 'Describe Stage 10.'" In his response, he used the word *describe* to denote the act associated with the number sentence, implying that for him a number sentence is a form of description. With this particular representation, Jack seemed to conflate describing and calculating, considering describing and writing a number sentence as quite similar, if not synonymous.

Soon after the classroom episode described above, Jack moved to Stage 100,002. For this stage, instead of first requesting a number sentence and then combining descriptions, Jack first asked students to describe the pattern and then supported them to include numerical attributes. The sequence began with Jack asking, “Does anyone think they could describe what that picture would look like for Stage 100,002? What is that  $H$  going to look like? Could anybody describe what that  $H$  would look like?” While the class explored this task, Jack probed individual students with questions and assessed their understanding. Eventually, he engaged with Ale, who Jack described later as a student who often struggles in mathematics class.

- Jack: How big is this  $H$  going to be, do you think?  
 Ale: Pretty big  
 Jack: Pretty big. Do we know how many groups this stage would have?  
 Ale: Umm, what?  
 Jack: How many groups would this  $H$  have?  
 Ale: Five groups.  
 Jack: Five groups. And how many dots are going to be each in those, in each of those groups?  
 Ale: 100,002  
 Jack: 100,002. And then are there any other dots around?  
 Ale: Two yellow.  
 Jack: The two yellow. So we have, we still have these five groups, but now they have 100,002 [dots] and the two yellow.

In this exchange, initially Ale could describe the stage only in general terms, depicting the  $H$  as “big” and “pretty big.” Jack then asked Ale explicitly for numerical details, infusing calculational details into Ale’s initial descriptive approach. In addition, Jack’s leading question supported Ale to move away from a general description and quantify, with numerical details, what “big” meant, illustrating how the two approaches can support each other.

Later, after viewing this clip, Jack commented on what Ale understood:

I think for specific cases he might be able to realize that there's still five groups and that it's just getting bigger.... But I think he got the idea that this thing is—the stages grow. And it is the number of dots in the group that's growing.

Although Jack's response is not detailed, I deduced that Jack believed that Ale understood exactly what made the  $H$  bigger. By being asked the values of particular quantities, Ale came to realize that the number of dots in a group increases the size of the  $H$ .

Jack next explained what he meant by specific case:

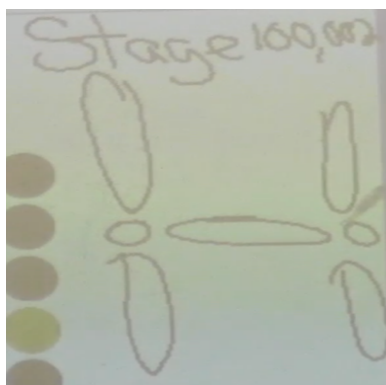
Ale does not know what the general pattern is, but he can describe what a particular stage looks like... I think if you ask him to write a number sentence, ... I think he could write something and explain what he wanted to say, but I'm not sure it would be represented—what he had written down would be a representation of what he verbalized.

I find Jack's characterization of a verbal description and a calculational representation as different very telling. On other occasions he explained how his students could describe properties of patterns without being able to express them numerically. Like Clara, Jack sees the two approaches and their associated representations as distinct. In contrast though, Jack seems to identify this distinction as an inherent challenge, but one that can be overcome by engaging students in both and, more important, actively drawing connections between the two.

After working with Ale individually, Jack led the whole class in a discussion about the details of Stage 100,002. As the following exchange shows, Jack continued to push students to include numerical details in their descriptions.

Jack: Marco, Stage 100,002. Is it going to be a big  $H$  or a small  $H$ ?

- Marco: It depends on how many dots.
- Jack: Is that something we can figure out?
- Class: Yes (collectively).
- Jack: Brian, can you help Marco? How big is the  $H$  going to be?
- Brian: (Pause, shrugs his shoulders.)
- Jack: How many groups would it have?
- Sarah: Five [groups].
- Jack: It would have five groups. Marco, could you tell me where those five groups would be? (Jack pauses.) Do you see five groups here (placing his hand over Stage 5 drawn on the board; see Figure 5.4)? For stage 100,002, can you picture where five groups would be for that one? Do you think they would be in the same place? Maybe a different place?
- Marco: Same.
- Jack: Do you think there would still be five groups, or do you think there would be a different number of groups?
- Marco: Different.
- Jack: So you think it would be different. Good, so we are starting to think what this  $H$  would look like. Ale (the student with whom he had the previous exchange), do you think there would be five groups?
- Ale: Yeah.
- Jack: Do you think they would be in the same place?
- Ale: Yeah.
- Jack: What's different about these groups? The size of the dots, how I draw it is bigger? Or the amount of dots in them?
- Ale: The amount of dots in them.
- Jack: The amount of dots in them. How many dots are going to be in Stage 100,002? How many are going to be in one of the groups (circling one arm in Stage 5 on the board)?
- Ale: 100,002.
- Jack: 100,002. We still have five groups, but now each group has 100,002 dots. Emma, you said that there are still some left over. There's still something going on. What are we missing?
- Emma: The two yellow dots.
- Jack: The two yellow dots. Marco, we still have an  $H$  that looks like this (drawing a skeleton of the  $H$ ; see Figure 5.5). But instead of five dots, it's going to have 100,002, 100,002, 100,002 (each time pointing on the picture where these values would be).



*Figure 5.5.* Joe's sketch of Stage 100,002.

Similar to Ale's individual response earlier, other students' initial responses in the group discussion were vague descriptions of Stage 100,002. To ensure that the students added precision to their descriptions, Joe repeatedly asked for numerical details to be added. This request also supported the class by making the connection between the stage number and quantities in the figure explicit. Moreover, Joe did not simply ask for numerical values; he endeavored to relate students' calculational responses to other descriptive features of the figure. Most notably, he continually requested and included precise descriptions of the quantities to which the numerical values referred and then linked the numbers students provided to the quantities in a visual description of Stage 100,002.

Compared to Clara's, Joe's descriptions and actions linked the descriptive and calculational approaches integral to students' effectively engaging in the generalization process. Clara repeatedly depicted the two actions as separate and struggled to see how they related and interacted. To elevate descriptive features, she did not request students to incorporate calculational details or support students in connecting their numerical approaches to their descriptions when they pursued them. In contrast, Jack interpreted numerical and algebraic expressions as descriptive representations, seeing a strong link

between describing and calculating. He not only actively worked to ensure that students engaged in both approaches but also used specific instructional techniques to support students in drawing connections between the two.

### **Connections Between Expressions and Figure**

The second characteristic that distinguished Jack and Clara was their differing perceptions of how to develop a rich quantitative understanding of the algebraic expressions used in generalizing figural patterns. Both teachers, according to their post interviews following the professional development and their instructional plans for the unit had as a goal to engender in their students understanding of how algebraic notation relates to particular views of the pattern. They differed in the level of detail they attributed to understanding this connection. Jack's actions and comments that he felt students needed to grapple with the meaning of each symbol individually to develop rich understanding of how the overall expression communicated particular quantities in the pattern. He consistently implemented strategies to support students in developing strong contextual understandings of the symbols by making this connection explicit. Although Clara held similar goals in general, her approach differed significantly. Her instruction seemed designed to expose students to multiple decompositions and their corresponding algebraic notation to cultivate the understanding that different expressions communicate different views of the figure. She strove to instill meaning, not by developing a detailed understanding of the quantities represented by each constituent part, but by familiarizing students with multiple views of the pattern and comparing those expressions. Although she hoped that students would understand how notation was connected to the figure, she struggled to support those who had developed a disconnected view. In the end, Clara

focused on comparing expressions, without appreciating the nuances necessary to foster a strong contextual understanding of the symbols.

To illustrate these two differing understandings, I contrast the two teachers in three linked, yet distinct, areas (see Table 5.1). First, I analyze details of their pedagogical decisions and overall instructional methods for connecting students' understanding of numerical and algebraic expressions to the figure. Second, I examine how the two teachers used multiple decompositions in differing ways and explain how their actions aligned with their own perceptions about the role multiple decompositions play in helping students understand the connection between algebraic expressions and the figure. Finally, I compare the teachers' views on the importance of connecting algebraic notation to context. To better communicate the similarities and differences between these two teachers, I proceed with my analysis category by category, describing Jack and Clara relative to each other within each of the three areas. I believe by juxtaposing the two teachers, their nuanced differences within each category become much more salient.

Table 5.1. Connecting Students' Understandings of Expressions and Context

| Detailed connection (Jack)                                                                                                                                                             | General connection (Clara)                                                                                                                                                                                                        |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ul style="list-style-type: none"> <li>• Related notation to context, symbol by symbol</li> <li>• Implemented strategy to explicate connection between notation and context</li> </ul> | <ul style="list-style-type: none"> <li>• Related notation to context at level of entire expression</li> <li>• Struggled to support students with disconnected view of symbols and context</li> </ul>                              |
| <ul style="list-style-type: none"> <li>• Focused on details of one decomposition in isolation</li> </ul>                                                                               | <ul style="list-style-type: none"> <li>• Worked with multiple decompositions simultaneously</li> </ul>                                                                                                                            |
| <ul style="list-style-type: none"> <li>• Believed understanding of an expression stems from a detailed understanding of constituent parts</li> </ul>                                   | <ul style="list-style-type: none"> <li>• Believed understanding of an expression stems from exposure to multiple expressions</li> <li>• Believed meaning comes from knowledge that mathematics is connected to context</li> </ul> |

### **Level of details connecting symbols and figure.**

*Jack's symbol-by-symbol approach.* In terms of the teachers' instruction, the most visible distinction between the two was the level of detail with which they asked their students to relate symbolic expressions to the figure. To illustrate the difference, I begin with Jack, who consistently asked students to provide precise details of this connection. The level of his specificity becomes apparent in an analysis of his general approach. Throughout the two days of instruction, with each new pattern or view of a pattern, Jack began by presenting images of the first three stages. He did not simply present the three figures but circled various pieces within the drawings to highlight a particular decomposition, ultimately identifying various quantities within the pattern. He told the class that the markings reflected a student's perception of the pattern and asked the students to consider the pattern according to the thinking they believed was illustrated. (The rationale and affordance of this strategy is discussed in my analysis in the final section.) Jack then gave the students time to think about the patterns before asking them to share general observations about what they noticed and how they might draw subsequent stages.

After the students had familiarized themselves with the decomposition of the pattern depicted, Jack asked them to provide a number sentence according to "how this student thought about the pattern." Jack always started with Stages 4 and 5 and then moved to higher stages, for which the figure was more challenging to draw. After a student offered a number sentence, regardless of its form or accuracy, Jack asked the student to explain, symbol by symbol what the notation represented. Jack insisted that



students' explanations were embedded in the picture. For example, every time a student referred to a numeral's referent as the stage number, Jack directed him or her back to the figure (i.e., "Is it also in the picture somewhere?")

Jack also pressed students to include details. In one instance he followed a vague response about the symbol's representing "the groups, the 5 that are in each group," accompanied by a circling motion in the air unclearly indicating the referent, by asking, "Are you talking about the five [dots] that are inside each group or you talking about the group?" (In this case both values were 5.) In addition, to ensure that the whole class was engaged and grappling with these connections, Jack often had other students repeat their classmates' explanations for the meanings of the symbols. Furthermore, on occasion Jack, himself, revoiced and summarized students' observations to help students contextualize the connection and not simply regurgitate their responses. For example, after one student explained that the 4 and 5 in  $4 \times 5 + 2$  represented 4 groups and 5 dots in each group, respectively, Jack asked another student, "He's saying that there are four groups and each group has how many dots in it?" Instead of merely accepting the student's response of 5, Jack then asked the student to connect his answer to the picture: "Can you see four groups of five dots?" This line of questioning assisted students in seeing the contextual relationship between the two symbols (in this case the four groups and the five dots). Although Jack never explicitly explained or addressed the connection between the two symbols, his phrasing alluded to their relationship and the reason the two quantities were multiplied. Finally, in a few cases, Jack asked the class to predict the meanings of symbols after an expression had been introduced but before the student explained his or her thinking. Jack used this technique when a student introduced an

algebraic expression that included a letter, asking the class what they imagined the student thought the variable  $n$  represented.

Jack also attempted to reinforce the connection between the figure and the symbols visually. First, he presented students' numerical expressions on the whiteboard next to the corresponding diagram drawn to represent the inferred decomposition. He emphasized that he wanted students to generate a numerical expression for early stages so that they "could relate it back to the picture that was on the board ... and have a concrete figure to refer back to and see what each symbol represented." He then asked students to explain, symbol by symbol, the contextual meaning of each element of notation and wrote their description beside the corresponding symbol (see Figure 5.6). Jack's meticulous questioning reveals his belief that by explicitly identifying the referent of each symbol, students would develop contextual understandings of the overall expressions.

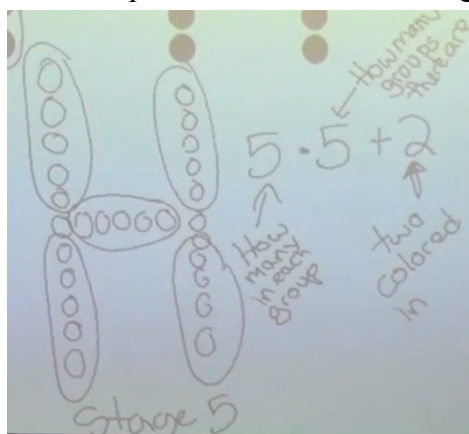


Figure 5.6. Jack making connection between symbols and context explicit.

***Clara's general approach to connecting symbols to the figure.*** Jack's careful instruction differed from Clara's; Clara also supported her students in connecting algebraic expressions with the figure, but at a general level. To illustrate the difference in the degree of nuance between Clara's and Jack's approaches, I present my analysis in two

parts. As I stated in the previous section, Clara did not ask students to generate numerical expressions. Instead, she attempted to transition students' verbal descriptions to more and more concise phrasing and eventually to algebraic expressions. Although she discouraged the use of calculating symbols, some students explored the pattern numerically. To show clearly the level of detail Clara promoted between symbols and context, I analyze examples of two cases: first, by describing the level of connection associated with her overall method, and, second, by showing the type of support Clara provided when students did generate numerical expressions. Examining instances of both cases illustrates the level of generality in Clara's approach. In both cases, although Clara seemed to want students to connect their understanding of the pattern with the algebraic expression, her supporting actions left the connection vague, requiring students to infer much of the meaning on their own.

*Clara's overall method.* To demonstrate the level of detail associated with Clara's overall method, I analyze an exchange that occurred on Day 3. I chose this episode to best illustrate Clara's general approach to connecting symbols and the figure. During the class Clara worked with various groups, helping them convert their understanding of the pattern into a more succinct form, similar to an algebraic expression. Each interaction took place in front of the class with a different group, each group sharing a different decomposition of the pattern. Clara specifically chose these groups and organized her class around different views of the pattern, a strategy I discuss in the next section. The following example, the first of four or five similar exchanges, is indicative of Clara's level of detail during this process.

Before this moment, the students in the group had recognized that the pattern consisted of two tiles plus multiple upside down  $L$ s (This interpretation of the pattern is demonstrated in Figure 5.7). Although they had successfully decomposed the pattern, they were still seeing the pattern recursively as adding a new upside down  $L$  shape composed of three tiles at the bottom right of each stage.

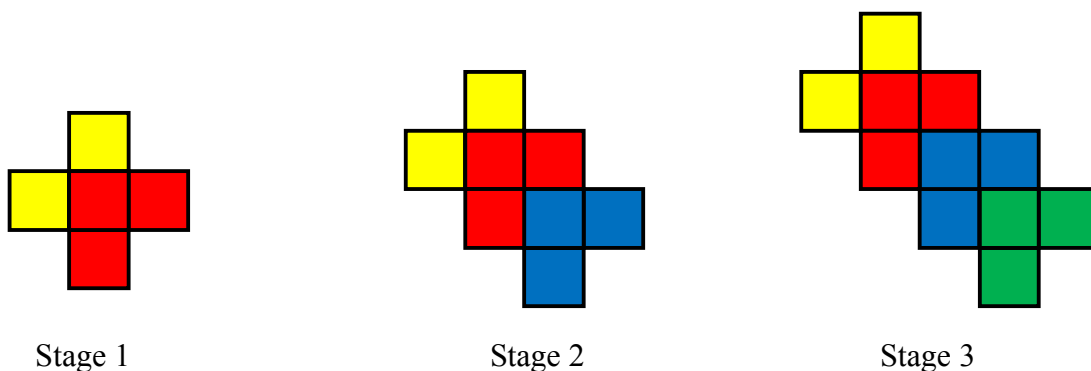


Figure 5.7. Representation of group's decomposition of the pattern.

Clara started the conversation by documenting the group's understanding in writing on the overhead (see Figure 5.8). Clara then questioned the group in an attempt to elevate particular pattern features needed to transform their thinking into an explicit expression but that were still eluding the students. In each case, Clara led with a question, but ultimately provided each detail herself, because students were unable to articulate her desired answer.

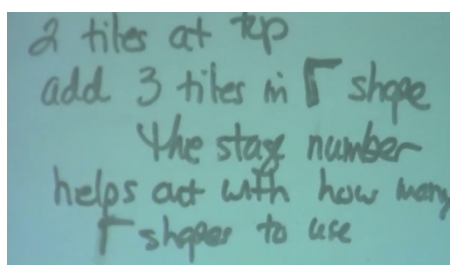


Figure 5.8. Clara's documentation of the group's understanding.

First, Clara asked about the relationship between the stage number and the number of upside down  $L$ -shapes, repeatedly asking, “How many  $L$ s do I know to add?” After each inquiry, the students in the group answered recursively, stating either “three tiles” or “one upside down  $L$ .” As the students indicated in their description in Figure 5.8, they had observed that the number of upside down  $L$ -shapes was related to the stage number but had not explicitly identified the relationship between the two quantities, continuing to think of the pattern mostly recursively. When the students continued to struggle in seeing this connection, Clara shifted her questioning, asking the class to state the number of  $L$ s for specific stages. After students provided the correct numerical responses for a few cases, Clara generalized this relationship for the students, stating the link explicitly: “The stage number is how many upside down  $L$ -shapes there are. The stage number is the  $L$ -shapes.” Then, to lead the students to the operation connecting the three and the stage number, Clara asked several times, both the group and the class, “What should I do with the stage number?” After the class shouted out various responses, Clara answered her own question: “Okay, multiply by 3.” She then wrote  $3(\text{stage \#})$  in place of the bottom half of the students’ description. Finally, Clara refocused the group’s attention on the initial two dots, informing them of the role the dots played in the expression, and stating, “Is that it then? Do you agree you have to add 2 then? Two tiles at the top.” She then added  $+2$  to the previous expression, erasing all the students’ earlier description and leaving only her algebraic expression (see Figure 5.9).

A photograph of a piece of paper with the handwritten algebraic expression  $3(\text{stage \#}) + 2$  in dark ink. The handwriting is casual and slightly slanted. The expression is written on a light-colored background.

Figure 5.9. Clara’s expression for students’ decomposition.

This example illustrates that although Clara did attempt to connect individual symbols to the figure to develop meaning of the expressions, she did not articulate the exact relationships in as much detail or clarity as Jack did. For example, an interpretation of 3 as the number of dots in an  $L$  could be deduced from the sequence of comments in this exchange, but such an understanding was left to the class to infer because it was never made explicit. To clarify this connection required students to make several conceptual leaps. At the beginning of this episode, the students explained that each upside down  $L$ -shape had three dots, a critical detail that was not elaborated or mentioned again. Instead, Clara quickly moved the discussion to the relationship between the stage number and the number of  $L$ s, a connection Clara stated but had not developed. Finally, Clara provided the operation between the 3 and the stage number, without explaining why or how these symbols were related. Combining these three discussion points, a student could have interpreted the 3 as dots per  $L$  added, but this understanding would require reasoning through and connecting each of these newly established relationships, not to mention internalizing many of these new quantities.

Such an interpretation of 3 was made even more challenging because in the expression provided, the 3 was written as the multiplication by the *stage number*, which is not a contextual quantity within the figure. Consequently, to deduce the meaning of the numeral 3 within the figure, the students had to first transform the statement by substituting the previously mentioned fact that the stage number is equal to the number of upside down  $L$ -shapes and then discern how the 3 relates to this somewhat hidden quantity. Furthermore, Clara provided many of the contextual details on her own, without allowing time for the students to make sense of them and connect their meanings

to the two representations. Additionally, I argue that the expression was developed in such a way that it promoted a characterization of the notation less as a description of the pattern and more like a shorthand for how to calculate the number of tiles according to this way of thinking. The stage number was defined as the number of  $L$ s being added and the 3 was introduced in the expression as an operation, implying a more calculational role. Subsequently, the 3 was portrayed less like a representation of a quantity and more like a number that correctly completes a process.

Overall, although Clara did allude to many contextual details in her explanation, she did so rather quickly and with an approach that required students to make many connections and inferences on their own from earlier statements and without the support of an explicit connection to the diagram. Clara seems to assume that if students understand the figural pattern in context and are able to write an expression, they will understand the connection between the two without making it explicitly.

*Clara's response to student-generated numerical expressions.* As I stated above, although Clara did not ask students to generate numerical expressions, on a few occasions students did approach the problem calculationally, and their responses could be characterized as verbalized numerical expressions. In each case, Clara directed the students to the figure, but in ways that did not help them develop a deeper understanding of the connection between the symbols they were using to calculate and the quantities they represented. In general, Clara's responses indicated that although she engaged students in activities and discussions in which they had opportunities to connect the symbols and figure, she did not support students, in particular those students who struggled to see this connection on their own, in doing so. To illustrate this point, I

highlight Clara's responses in three episodes, two in which the numerical expressions were incorrect and one in which the expression accurately aligned with the pattern but was based on a numerical pattern.

The first two examples occurred on Day 1 when students began to explore Stage 10, the first stage they were to analyze without the aid of a figure. As mentioned, although Clara asked the students to describe the figure, several groups answered with calculations. Two responses in particular were of interest: the first because Clara identified it as significant and both because Clara reflected on her understanding and actions during the follow-up interview. Although both cases represented incorrect calculations, they differed in that one was based on an incorrect assumption without any apparent connection to the figure, whereas the other appeared to be an attempt to capture the students' decomposition of the figure, but without fully attending to the details involved. Clara's attempts to support students struggling to connect the two representations and her analysis of these exchanges show her understanding of the role of the figure in algebraic generalization and its connection to developing meaning of symbolic expressions.

In the first case, the group had arrived at the value of Stage 10 by incorrectly doubling the Stage 5 value. Upon noticing this, Clara asked to see the work that led to this answer. Whereas one student explained that they had doubled Stage 5, another student pointed out that they had also identified that number of dots in each stage increased by 3. Clara then tried to leverage this observation, asking the students to use the idea of adding by 3 to find Stage 10. When this approach provided a numerical contradiction, both students were quite confused. Still believing the doubling method



was correct, the two assumed that they had miscounted, stating so several times. To reinforce the idea that the value of stages did not double with the stage numbers, Clara had the group compare Stages 2 and 4, choosing these specifically because they had figures already drawn. Again, the students assumed that Stage 4 would have double the number of dots in Stage 2 and were surprised to see that it did not. At this point, after Clara had left the group, one student continued to assume that they had miscounted, while the other had accepted that their doubling method was incorrect without understanding why.

When I asked Clara what she was thinking during this exchange, she explained that she was using the pictures to help the students understand that the numbers of dots did not double with the stage numbers, stating, “To me, if you can present it to them visually about why it doesn’t work just to double it, then they can let go of that idea of, okay, I doubled 5 to get 10.” Although Clara directed the students to the figure, she did not elevate any quantities in the drawings or encourage the students to use symbols as a way to communicate a particular decomposition. Instead, she asked the students to use the figure to count the number of squares, hoping that the calculational contradiction would create disequilibrium and convince them that their method was incorrect. Such a strategy seemed to be based on a belief that students must be convinced of the inconsistencies in their existing strategy before they can move on and pursue other productive ways of thinking.

I then asked Clara to explain what the students understood at the end of this exchange. Concerning the first student, Clara replied, “He knows that the way they were thinking was not going to work. I think he knows that. I don’t think he has a way that

will work yet.” She then described both students, saying “They don’t have a method for understanding the pattern, and they probably had no incentive or desire to investigate.”

In this reflection Clara acknowledged that her engagement with the students had not provided them a constructive path forward. She thought that her questioning and use of the figure might have led the students to see that their method was incorrect but did not support them in making sense of the pattern or provide a strategy to move forward. I asked Clara what other options she might have pursued if she had had more time to work with the students. She initially provided a strategy similar to the one she implemented in class, but then hedged on this suggestion, deciding that it too would fail to support students in understanding the pattern.

I think I would explicitly show what's happening when you put these two together, the stages together. But that's kind of not, still not getting them to understand this pattern (referring to the figure) over here. We're trying to move forward. That's probably the same reason I wouldn't. I don't think that it's going to get us where we want to go.

Clara recognized that her hypothetical approach would show students that their present method was flawed without providing a way of thinking that would enable them to generalize the pattern (and develop a quantitative understanding of the notation). Although Clara wanted students to connect their understandings of the pattern and the symbols, she did not possess a method to help students develop this relationship.

Although responding in this case might be construed as particularly challenging because the students’ numerical expression was detached from the figure, Clara responded similarly to a separate group of students whose numerical expression appeared to align with their contextual understanding of the pattern. Again, Clara first showed the students their errors without supporting them to better understand the connection between

their symbolic calculations and the pattern. In this case, just as before, the group in question was struggling to describe Stage 10. The group spokesperson explained, “The number of tiles for Stage 10 is 35 because 10 times 3 equals 30, plus 5 equals 35.” This calculation seems somewhat connected to the pattern for which a viable expression could be  $10 \times 3 + 2$  or  $9 \times 3 + 5$ . As reported in the previous section, because class was just about to end, Clara did not engage with the group about their thinking or attend to the details of their response. However, in the postinterview Clara had time to make sense of the student’s response and immediately said that she needed to go back to the student “because she has the wrong answer.... She needs that misunderstanding—you know, we need to deal with that.” Just as before, Clara’s answer indicates that she felt compelled to engage with the students, not to build on their thinking but to correct their misconceptions. Clara’s follow-up response confirms this sentiment. Hypothesizing what she would now do, she said:

I would have brought her back to Stage 5, because it’s there. It’s drawn out. And, I would have said, “Okay, so based on what you said, how many would there be in Stage 5? Using how you get to [Stage] 10, using that, how many would there be in [Stage] 5?” Okay, so I would have expected her to say, “There’s five groups of 3, plus 5. Twenty.” And then we would count and see that it wasn’t 20 [dots]. {I’d say,} “So, what happened to those [dots]? Okay, let’s build Stage 5. Let’s take the five 3s away and let’s see what we have left.” And we’ll see that we only have 2 tiles left.

Two issues raised in Clara’s response seem significant. First, in her hypothetical response for Stage 5, Clara anticipated the student would possess a rather strong connection between the symbols and the context. Although the students’ initial response was purely calculational, “Ten times 3 plus 5,” Clara transformed and contextualized the

student's understanding of Stage 5 to be "Five groups of 3." Although Clara probably did not intend to impute a difference in understanding, her inadvertent interchange between these two distinct interpretations shows that she was not attuned to the details of these differences. Although Jack focused his instruction on elevating and connecting these two interpretations, Clara seemed to attend more to accuracy of the students' answers rather than the details of their conceptualization.

In the second case, Clara's reaction was again to use the pattern to demonstrate that the student's method was calculationally incompatible with the pattern. Rather than asking the students to explain or connect their understanding of the symbols to the figure, Clara directed the students' thinking to the figure primarily to cause disequilibrium, not to instill meaning. Here, a comparison between Clara and Jack can be made almost directly because a student in Jack's class offered a similar numerical expression, in that it was connected to the figure but incorrect. In contrast to Clara's approach, Jack's approach was to continue as if the student were correct, asking him to explain his interpretation of every symbol. Initially, the student offered an ambiguous explanation of the incorrect symbols, which Jack documented without correcting him. Jack then asked another student to provide his expression and explain his interpretation. Later in the period, the first student asked Jack if he could change his expression, having identified his initial mistake. Although the results in the classrooms is not the focus of this paper, the difference in approaches is important. Whereas Clara sought to correct the students' error, Jack chose to reinforce the idea that symbols are connected to quantities, an action that ultimately also helped the student see his mistake. In general, although Clara's inclination was to direct students to the pattern, in analyzing her actions and her

motivation, I find that her main intent was to illustrate to the students that their answers were incorrect. Without an orientation to having the students connect the symbols and the figure and an explicit way to support students to do so, Clara's pedagogical response was to highlight incorrect answers.

Although the previous two examples show that Clara's inclination was to direct students with incorrect expressions to the figure as a means to highlight their miscalculations, the following example illustrates her response to a group of students who formulated a correct expression, but one based on a numerical generalization with no connection to the figure. Her actions and reflection highlight that she also struggled to support these students to develop understanding of the link between their calculations and quantities in the figure. This exchange took place on Day 2 with a group who had determined that the number of tiles could be calculated by multiplying the stage number by 3 and then adding 2. Clara first engaged with the students by asking them to explain their method. One student responded, "Instead of adding, you can multiply. It's the same thing." Recognizing that their procedure was numerically generated (which Clara noted in the post interview), Clara probed to determine the students' contextual understanding of the expression, asking, "So is your method—did it depend on a specific way of seeing the pattern?" When the students replied, "No," Clara again pushed for a connection to the figure, asking, "So would your method work for the people who see the plus sign or the cross first and then add 3 every time? Would your method still work?" Confused, the students responded by reiterating their explanation that multiplication is repeated addition.

In the subsequent interview, I asked Clara about possible follow-up actions she might have pursued to help these students if she had had more time. Clara replied, “I would definitely say, 'Okay, show me where this [the expression] is. Show me. Can you relate it back to the figure?' Just their  $3m + 2$ .” I find Clara’s hypothetical response indicative of her perspective of helping students connect their thinking of the figure with the symbols. Clara wanted students to understand the connection between the notation and the context and asked questions to elicit students’ understanding of this relationship, but she did not have a method to help students who did not see the connection or an understanding of the mathematical details that eluded them. By comparison, Jack elevated the specific meanings of each symbol, whereas Clara’s strategy was to ask about the whole expression. Much like her queries in class, this question might elicit the understanding from a student who possessed it, but it would not build understanding for those who lacked understanding. Clara recognized that the notation/pattern association was important, but had not developed an understanding of what exactly the association entails and was consequently unable to support students to develop this understanding.

In the post interview I also asked Clara about the students’ understanding during this exchange. Clara stated clearly that the students had arrived at their procedures numerically without any connection to the figure, contrary to her goal for the unit. She then explained her understanding of the role making this connection plays in this unit:

She makes a beautiful table..., a numerical pattern, and that's great. But, but she can do more. She can show there's a relationship in the figure. So, if she can do it and it's something that she can demonstrate, then let's have her do it. Cause the whole idea is that there is meaning there.

In this response, Clara seemed to characterize the connection between notation and the figure as a supplementary understanding for only a few capable students. Such a portrayal contrasts significantly with Jack's view; his actions showed that he saw developing this connection as the essential part of the lesson. Furthermore, Clara portrayed understanding this relationship more as a challenge to be overcome. As shown later, Jack not only believed that this connection was at the core of the unit but also saw having students grapple with this connection, even if difficult, was essential to developing their understanding of algebraic generalization.

With respect to the instructional methods and responses of these two teachers, Jack possessed a clear, detailed way to help students connect the symbols and the context, whereas Clara did not. Although Clara valued this connection and recognized that her responses were not sufficiently supportive for most of her students, she did not know how to instill this understanding in struggling students. When students had a disconnected perception of the symbols, she directed them to the figure, not in a way that promoted their understanding but, instead, to highlight their lack of understanding. Her overall method was based on establishing a quantitative understanding of the symbols, but she did not support students to explicitly connect details of their numerical and algebraic expressions to the figure. Without a nuanced understanding of the components necessary for students to effectively engage in algebraic generalization, she left understanding the details of making this connection to the students.

**Use of multiple decompositions.** A second feature distinguishing the two teachers' understandings of the figure and notation was their instructional planning, specifically how they chose to familiarize their students with different views of the

pattern. Both teachers intentionally selected figures amenable to multiple decompositions and generalized patterns. Each teacher explained that such patterns allowed for focusing their discussions on ways students saw the pattern and that juxtaposing different expressions could be leveraged to raise issues of equivalency. That being said, how they decided to expose students to these varying decompositions differed.

***Jack's individual use of multiple decompositions in succession.*** Jack was clear in outlining his plan for having the entire class analyze the same pattern decomposition: He provided a pattern with circles around particular pieces of the figure to communicate one student's interpretation of the first three stages (see Figure 5.2) and then asked the class to draw and analyze the pattern using the same way of thinking. As explained in the previous section, after students engaged with this view of the pattern, he asked students to write number sentences for stage after stage and eventually to write an algebraic expression using a variable. When asked about this approach, Jack explained that students had previously been successful in seeing and decomposing the pattern in different ways. Now he wanted to engage the class in discussing the meaning of the variable as well as focus on specific notational issues. He highlighted his rationale behind the various decompositions he chose (which I explain later), indicating the exact notational concerns he would target. He added that when the students looked at the figure in different ways, having them focus on the same issues was difficult. Furthermore, although Jack did not specifically say so, I inferred from the details he provided about his motivation for deciding to focus the class on particular views of the pattern that he wanted to shift the emphasis from seeing the pattern to working on communicating the



generalization symbolically. By scaffolding students' dissection of the pattern with the circled figures, he in effect removed the act of decomposing a particular generalization. Separating this activity from writing expressions allowed him to focus almost exclusively on developing understanding of the symbols. Dividing up these two skills and having the entire class engage with a single decomposition enabled Jack to focus the class's attention on particular details about how algebraic expressions are connected to the figure, details he thought supported students' developing understanding of this connection. Tackling different decompositions, but in succession, enabled him to both elevate and compare specific features.

***Clara's simultaneous use of multiple decompositions.*** In contrast to Jack's approach, Clara chose a pattern that included six decompositions and consistently encourage her students to look at the pattern in multiple ways. For example, she decided to begin on Day 2 and again on Day 3 to present all six possible views, many of which her students had not identified previously on their own. On both days, she presented the class with drawings of the first three stages, colored to represent different decompositions. She discussed each decomposition quite quickly, giving the students roughly a minute to make sense of each pattern. After showing these to the students, she seemed to emphasize their collective importance, stating, "We do want to work with all of those methods." On other occasions, Clara emphasized that students should "think of all the other ways of seeing the pattern." When I asked her about this decision, she explained that some of the views might be more accessible than others, implying that exposing students to various decompositions would facilitate their generalizing of the pattern.

From my perspective, the introduction of these various views of the pattern served not only to diffuse the class's focus during the subsequent discussions, it also did not align with the pedagogical needs of the class. In the case of Day 2, the previous lesson had ended with each of the group's providing their understanding of Stage 10. Of the 9 groups in the class, only 1 group had provided a coherent description of the pattern. The other groups had displayed significant complications in describing the pattern, calculating the value of Stage 10, or connecting their views of the pattern with their calculations. On Day 3, Clara organized the class around the various views of the pattern. She moved from decomposition to decomposition, calling on the different groups that had analyzed each pattern. Although none of the groups was able to write a correct expression (ultimately requiring Clara to provide it), in each case Clara quickly moved onto the next decomposition. From my perspective, if the groups that had focused their attention on a particular view of the pattern were all struggling, then the other students in the class, for whom this was a completely new way of thinking about the pattern, would need time to grapple with and internalize the meaning of this new expression.

In general, Clara seemed quite focused on breadth rather than the depth during the three days of instruction. Such an emphasis seemed to communicate that for Clara, student understanding of the connection between the notation and figure would come from being exposed to multiple expressions; that is, Clara thought that by seeing how different decompositions could be represented by different expressions, students would begin to understand how to use algebraic notation to communicate generalizations. Consequently, Clara took every opportunity to expose students to different views of the

pattern, without elevating the details of how or why a particular decomposition and expression did or did not match.

Reflecting on the students' learning during the week of the professional development, in the Phase II clinical interview, Clara commented on the role different views of the pattern had played:

What we did see was that kids have misconceptions about variable, and I think that through work with these patterns, especially through the ones that you can see in different ways, you know, obviously you can ....., And that's what we started to look at—What does the  $n$  mean here? What does the  $n$  mean here?—so kids can start to understand variable.

Clara seemed to attribute the students' improved understanding of variable, at least partially, to working with different decompositions. During the interview she reiterated this belief, explaining, "I think it is a good thing they [the students] look for different ways to see these. It helps reveal what a variable is." Again, Clara credited the students' exposure to multiple decompositions as supporting their understanding of variable. Finally, later in the interview, Clara debated with herself about why generalizing patterns might be a better approach to developing a deeper understanding of variables than the "fruit salad" treatment (a method used to facilitate adding like terms in which variables are characterized as different fruits with the coefficient representing the quantity of each fruit):

I wonder if that [understanding of a variable] would really come out in the fruit-salad-variable situation. Probably wouldn't come out. Because those situations aren't as dynamic as the pictures; those don't have those different ways of looking at them as the pictures. Maybe that is what it is about the pictures. They have to have different ways of seeing it.

Clara again posited that thinking about the figures in multiple ways contributes to students' rich understandings of a variable. In each case, she associated exposure to different decompositions and different expressions with a deeper understanding, but she did so without including details about how students made sense of the symbolizing or connected their understanding to the quantities in the figure. As such, her comments seemed to imply that she had developed an oversimplified view of the connection between multiple interpretations of the pattern and understanding of variable, that is that simply by exposing students to these different views, they will learn to correctly interpret the different expressions, leading to a better understanding of the role and meaning of the variable.

Although I agree that the multiple decompositions used during the professional development elevated various student misconceptions and afforded rich discussions about the meaning and conventions associated with variables, the students' eventual understanding stemmed from explicit conversations about the meanings of the symbols. Clara seemed to see a direct relationship between experience with multiple expressions, regardless of the exact interaction, and understanding. During the 5 days of professional development, Jack seemed to internalize nuances of the discussions surrounding these multiple expressions that supported students' understanding, whereas Clara internalized the interactions at a more general level, without fully digesting the details associated in scaffolding this understanding.

**Understanding of an expression.** The final distinction between the two teachers was in the values they associated with connecting the symbols and the context and how these values corresponded to their views of what is entailed in understanding an

expression and how this understanding is developed. During the stimulated-recall interviews, both Clara and Jack spoke about wanting to foster in their students an understanding of the connection between the notation and the figure. To better understand the teachers' views of this connection, I asked them why they considered this relationship important and how they believed this connection supports students in algebraic generalization. With this question, I sought the teachers' views of the pedagogical significance of developing this connection. I wanted to draw a distinction between the role that this relationship plays in the learning process and the resulting understanding.

***Jack: Understanding stems from articulating details of single expression.*** Jack provided a purposeful and specific rationale for supporting students to develop a detailed understanding of the connection between expressions and the figure. He explained his reason for repeatedly asking students to articulate the referent for each symbol:

Because I wanted them to see some consistency in some things. Like all these numbers aren't just always randomly changing. That there's, there's some consistency. There's always this 2 and it always—this 2 represents, in each of these things, [it] represents those extra 2 [dots]. And it's—to me, by going back and forth, its creating kind of a building block of "Okay, I can see this 2. And it is also here in this stage. It is the same 2, and it is the same place.... And this 5, ... it's the number of groups. And it's the same every time. I can see it." So the part that is changing, they can focus their attention on "What does it mean that its changing? What's happening? Is it growing dots?" I want them to see the structure of it. That there [are] some things that don't change, and, you know, there are some things that do change and build that.

Two features of this response are particularly noteworthy. First is the level of detail Jack provided. In explaining the importance of this connection, he went through each and

every symbol in the expression, specifying their exact contextual meaning. The level of detail in his answer seems to reflect his belief that understanding an expression stems from recognizing the contextual meaning of its constituent parts. Second, Jack articulated that he sees contextualizing each symbol as supporting students' understanding of the pattern. By making this connection explicit, students see not simply a string of numbers, some changing and some constant, they see why each number changes or remains constant. Consequently, they anticipate and make sense of these patterns. Connecting each element of the notation to a quantity in the figure enables students to see consistencies in the symbols, even those that are changing. For example, although the number of dots per group is increasing, the referent for these varying symbols is constant. Jack referred to this feature of the notation as *structure*, alluding to reflective abstraction, whereby students see a characteristic not as an isolated case or relative to a few examples but become aware of the wider applicability of their observations (von Glasersfeld, 1995). Such an awareness is the basis for generalizing.

Jack then summarized his view of the value of contextualizing these symbols:

I guess I am trying to paint a *visual picture* for them when they get to these other different numbers or bigger numbers [later stages], that it can make sense. That these make sense. That it's not just 100,000 times 5 plus 2 just because.... That 2 here is *that* 2 (highlighting the 2 constant points with his hands). You can almost *touch it*. And so I'm—I think I'm trying to build a strong *visual background* so they can *see*. So it's not just some, an abstract, weird-looking thing on a table, that ... it's connected.

Most significant for me in this response is the imagery Jack provided. Jack did not simply state that students *will* understand the connection to the pattern. He was very clear about the type of understanding he wanted students to develop: By identifying and

drawing out the details, what each symbol represents, students will develop a mental picture and envision the exact quantities through the notation. In this short quote, although he was speaking about the symbols, Jack used metaphors about seeing and touching the figure four times. For Jack, understanding the algebraic expression seemed to entail connecting each symbol to the figure, developing an embodied relationship between notation and figure. With such a strong connection, the symbols on the page cease to be merely abstract numbers and can be seen as the contextual properties they represent. At this point, the notation and the figural quantities become integrated and interchangeable, similar to Kaput's notion of *symbolization* (Kaput et al., 2008). In the two days I observed, Jack began to achieve this level of understanding with his students. On the second day, Jack asked students to write a number sentence for a particular stage number. After one student provided the numerical expression, Jack quickly replied, “Can you take me through your number sentence?” Without hesitating or needing clarification, the student identified the contextual meanings of each symbol. Such an instinctive response is evidence that the students were starting to envision the quantities in the notation.

***Clara: Understanding stems from exposure to multiple expressions.*** In contrast, Clara provided less detail when explaining the importance of the notation/figure connection. Although she clearly wanted students to develop an understanding strongly embedded in the figure, stating repeatedly that her primary focus for these exercises was for students to “stay in the figure,” her reasoning for developing this relationship was vague. On several instances, Clara elaborated her view of the significance of connecting

the notation to the figure in these exercises. On one occasion, she talked about what students miss when they analyze the patterns purely numerically:

Once they [students] move to just the numbers, there's no meaning ..., you know, we're just manipulating numbers. And that whole connection to "Hey, these represent something." The bigger idea that math—the bigger idea in math [that] you should be able to describe. Because that to me is, is the, the, what math is. Math describes the world around us.

Clara described the connection as providing meaning, but without detailing how meaning might be achieved or what connecting might entail. Instead of highlighting specific quantities that the notation represented as Jack did, Clara provided ambiguous terms such as “something” and “the world.” She seemed to depict meaning as more of an abstract and broad notion, the idea that math, in general, is connected to the world, but did not specify this connection.

Looking for more insight into her thinking, I asked Clara to explain how this connection helps students in their learning. She again alluded to a vague idea that this connection provides meaning, stating, “Because it’s going to have a context. That's the context for where they are going to develop their expression. So that gives them a context... This is where the math is coming from.” Clara explained that with mere numbers, there is no understanding of where the formula comes from or why it produces the correct answer, making an analogy to a black box function machine whose formula magically yields an output. Clara appreciated that connecting students’ thinking to the figure would support their understanding, but she was unclear as to how this occurs or what it entails.

Later, during the follow-up interview for Day 2, Clara again reflected on the significance of connecting the notation and the figure. Having just watched a clip of the



group that approached and reasoned about the problem purely numerically, Clara commented that it was unfortunate that the students were not relating their numerical expression to the shape, inasmuch as the "whole goal" was for students to "show there's a relationship in the figure." She then explained, without prompting, "Cause the whole idea is that there is meaning there .... The numbers really do represent that." Again, she highlighted the importance of connecting the students' thinking to the figure but provided no details for what this entails. She stated simply that the figure provides meaning, referring to the referents as general ideas, using terms such as "context" and "that." Clara explained the importance of this connection only by saying, "Math is profound." She hoped to develop in her students appreciation for the power of mathematics to explain natural phenomena, using as an example the way the Fibonacci sequence describes the natural world. She concluded,

There really are these connections to be made. And if we don't help teach them that they're there, then they're [students] not going to know that they should look for them. They won't know—know that's a thing people do. People can make those connections.

Once again, Clara indicated that symbols should have meaning, but she explained only in terms of a vague connection to the real world. Although I probed several times for more information, Clara repeatedly described this relationship at a broad, general level. She did not contextualize *meaning* in this topic or explain how she would develop it in her students. Again, Clara's view of the process by which students come to use and understand algebraic notation seems oversimplified: By discussing the patterns and seeing different decompositions, students will develop the formula from context and understand the meaning of the symbols.

Another example further illustrates the generality of Clara's understanding of the process through which students develop a rich contextual understanding of algebraic notation. In reflecting at the end of Day 3, after acknowledging that her approach had not been successful in supporting students to write an algebraic expression, Clara began to explore other options for helping students connect the contextual meaning with symbols. She was particularly frustrated because she had worked diligently for three days to focus the discussion, and consequently the class's attention, on contextual details, but these ideas seemed completely lost when they were converted into an algebraic expression. Searching for a solution, she said,

So instead of erasing—so instead of getting rid of the rows and keeping stage [the] number, maybe if .... There has to be some realization.... They know stage number equals rows. That's not evidenced in the algebraic expression. Unless, I mean you need to set up, "Okay, stage number equals row or stage number equals columns." And maybe then it comes with using  $r$  for rows or  $c$  for columns.

Her solution was to use letters that correspond to the first letters of the quantities they represent. Although this suggestion reiterates her desire to support students, it also highlights the generality of her appreciation for the process of developing a strong contextual meaning of notation. Whereas Jack demonstrated his understanding of the complexity of this process through the variety, depth, and modality of the strategies he employed, Clara appeared to underestimate the challenges of the understanding students need to make this connection.

Furthermore, as I mentioned earlier, Clara's perception that the contextual information is lost in an algebraic expression shows that she was focusing only on features readily visible in the representational form, not on the understandings that must

be imposed and developed through working with multiple representations. Such quantitative understanding is not implicit in abstract mathematical symbols. Clara had not made explicit the idea that seeing the quantities in the expressions is a cognitive state, a mental construct that must be developed and mentally imparted to the notation, not simply presented. Using a particular letter does not provide this understanding. On the one hand, although Clara possessed a strong contextual understanding of the symbols, she was unable to assist her students in developing the same. With an oversimplified view of this relationship, she transitioned from assuming that the students would relatively easily connect the two to seeing the two as completely separate.

Jack, on the other hand, described a type of understanding in which the quantities of the figure were mentally coupled with the notation. He possessed a method that foregrounded concepts that the representations backgrounded and supported students in visualizing the contextual ideas in the abstract symbols. Thus, although both teachers had similar goals and worked conscientiously to achieve them, their actions and comments revealed very different levels of detailed understanding. Clara believed that such understanding could be instilled by demonstrating multiple ways patterns could be communicated using algebraic expressions, whereas Jack believed that understanding was derived by developing explicit connections, symbol by symbol.

### **Instructional Trajectory**

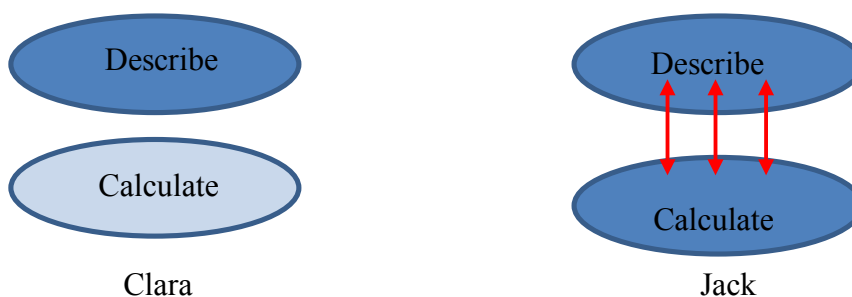
The third and final feature that distinguished Jack's and Clara's instruction was the trajectory of thinking in which each envisioned students would engage while generalizing the pattern with increasing detail, ultimately developing a quantitative understanding of the algebraic notation used to capture the pattern generalization. In analyzing the

teachers' comments and actions, two different models of student thinking emerged that seemed to guide their instructional attention and decisions. Although both teachers, by means of their questioning and the student responses they chose to elevate, fostered certain ways of thinking about the pattern and particular interpretations of the associated representations, Jack was more intentional in working to ensure that certain ideas were introduced, discussed, and ultimately established. Furthermore, he had identified and was able to engage students in more specific areas of focus along the trajectory than Clara. In addition, the descriptions he provided of his instructional plan as well as his reflections on the effectiveness of various pedagogical interventions indicated that he possessed a more nuanced understanding of the various components involved as well as a more interconnected model overall.

**Trajectory model explanation.** My interpretation of the teachers' instructional trajectories builds on my analysis from the preceding two sections. It is the teachers' differing perspectives on describing versus calculating as well as the connection between context and symbols that provide the basis for the framework I used to understand Jack's and Clara's models of student thinking. Therefore, I first review these results and then explain how I transitioned these findings into a tool to examine and communicate the teachers' understanding of the process in which students engage while they learn to generalize symbolically.

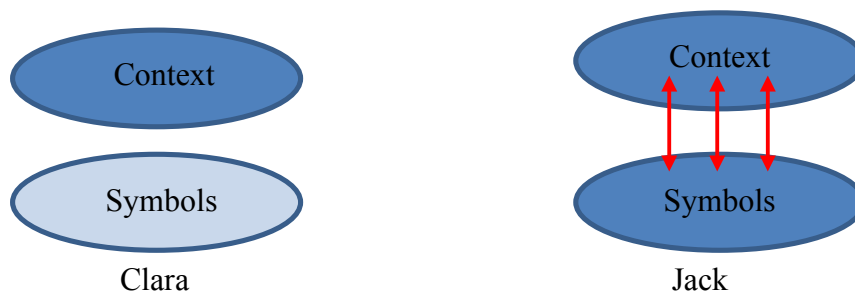
As explained above, Clara, on the one hand, attempted to guide students to a meaningful interpretation of the notation by consistently encouraging them to describe their observations about the pattern, while deemphasizing the act of calculating. When students did pursue a calculational approach, she engaged minimally in their thinking and

did not support them in combining their calculational and descriptive views of the pattern. Jack, on the other hand, encouraged students to employ both forms of thinking and worked to draw connections between the two. The representation in Figure 5.10 shows the differing types of engagement and understanding the teachers attempted to foster among their students relative to these two acts.



*Figure 5.10.* Model of teachers' views of the roles of generalizing actions.

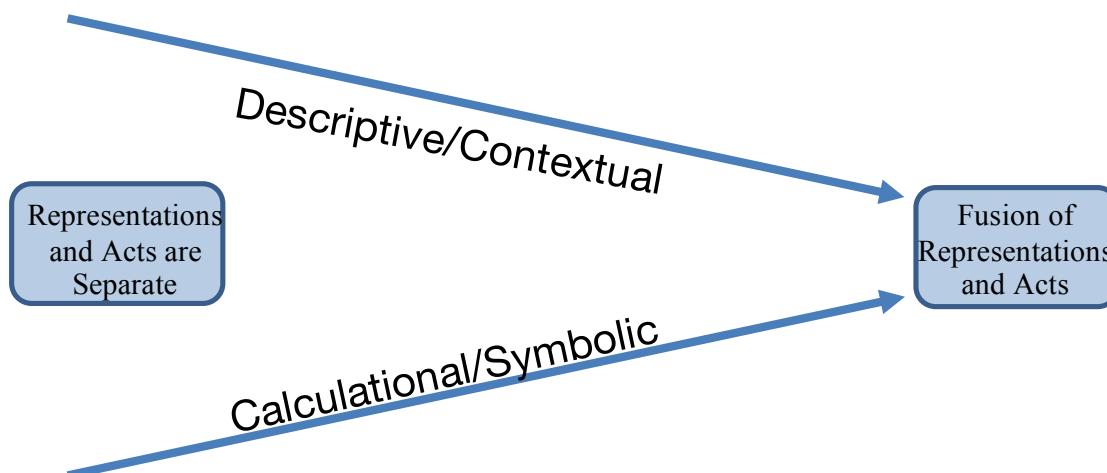
Similarly, as the representation in Figure 5.11 illustrates, Clara endeavored to avoid the use of abstract symbols, both numerical and algebraic, and was unsuccessful in assisting students to relate their numerical expressions to the figure when they did produce numbers. In contrast, Jack actively worked to support students in connecting symbols and the context. He not only made this relationship explicit but also elevated mathematical ideas that were challenging to see in certain representations.



*Figure 5.11.* Model of teachers' views of the roles of representations in algebraic generalization.

Furthermore, Jack's ability to support his students in engaging in descriptive and calculational approaches concurrently, as well as in linking algebraic symbols and the figure, helped them connect the patterns' qualities and forms. Eventually, when the connections between the two acts (describing and calculating) and two representations (symbols and patterns) were reinforced, the boundaries between these components blurred. Jack's students began to see numerical symbols as descriptive representations of the quantitative relationships within the figure. Likewise, their descriptions became more precise, including more calculational details.

To capture the process by which the understanding associated with these acts and representations merged, I created the model shown in Figure 5.12.



*Figure 5.12.* Model of merging actions and representations.

Instead of characterizing describing/calculating and context/symbols as distinct and fixed entities, this model depicts them as two continua to emphasize a development toward a sophisticated understanding. Initially, as captured on the left side of the figure, when students begin to analyze a figural pattern, these two acts and associated representations

are quite detached; students engage with each in isolation. Students' descriptions tend to focus on iconic qualities without numerical details, and the students produce numerical values that lack contextual elements. With increased familiarity with the problem at hand, as well as generalizing overall, the two acts, describing and calculating, become more connected, interacting and supporting each other. Similarly, the two representations become more closely linked, almost fused. This coordination is represented on the right side of the figure, where the two continua draw closer together.

In addition, a notable feature of the model is the combination of describing and context on one continuum and calculating and symbols on another. To be clear, the decision to merge these was an attempt to facilitate the communication of the teachers' view of the generalization process, not to convey that only a particular representation aligns with, or supports, a certain act. In fact, one affordance of this study is to decouple the acts and the representations and analyze them separately. However, the acts and representations are not completely unrelated. To some degree, certain representations inherently encourage particular actions, and certain actions are most easily performed using particular representations. For example, a table, by its very nature, requires calculation, and relationships between quantities are expressed most clearly using words. At the same time, though, a student who completes a table using number sentences is not only calculating, but in choosing which values to include and omit in the numerical expressions, he or she is also identifying specific quantities, albeit implicitly in many cases. Inherent in the act of recognizing quantities, even those expressed with numerical symbols, are many descriptive qualities. Similarly, verbal descriptions of qualities often incorporate numerical values. For example, in both classes some students' descriptions

consisted of merely the calculational process expressed in words. Therefore, although the choice of representation does not dictate the type of thinking that is associated and vice versa, representations do influence thinking. I argue that for novices beginning to generalize, a particular act corresponds strongly with a particular representation. With experience, however, one's interpretation and ability to use a given representation expand and the ways of thinking become less fixed by the representation. Qualities inherent in one representation become visible in other forms, and vice versa. Both teachers held developing such awareness as the overarching goal for this unit. However, this level of nuance is difficult to communicate in a model. Over time, the acts and representations merge, and their alignment becomes disentangled. Therefore, to simplify the model, I forgo these details and combine the actions with their more typically associated representations.

Having conceptualized these two elements of the generalizing process as continua, I then analyzed the teachers' actions and reflections as well as the type of thinking in which their students engaged to identify various steps along each continuum. The schematic in Figure 5.13 highlights these steps on the two continua, and Table 5.1 provides more detailed features of the steps. To clarify, these categories reflect students' comments and actions that are associated with a particular type of thinking in which students engaged using a particular representation. They are not attempts to capture the learning processes that enable a student to develop these types of understanding. In addition, these categories are neither discrete nor mutually exclusive. Just like there are many examples of comments that combine descriptive and calculational approaches and expressions that cross between different representations, there are multiple ways of



reasoning that blend components from different classifications. For example, students' comments that "the pattern is adding dots below" and "the pattern increases by a cross each time" have recursive attributes but lack detail or include inaccuracies and thus reflect qualities associated with iconic descriptions. Furthermore, although I contend that on both continua, the type of thinking highlighted from left to right increases in generality, I do not suggest that this trajectory is a fixed linear progression in which students must master one type of thinking before engaging in the next. For instance, one student in Clara's class described the quantities in a particular stage without first having articulated the pattern recursively. Jack's entire class wrote out numerical expressions and then later used these to create a table values. In addition, for organizational purposes, I have attempted to match elements of the descriptive/contextual and calculational/symbolic approaches that loosely correspond in terms of the level of thinking involved. I have placed these opposite one another in the table (see Table 5.1). I reiterate that although I have characterized these steps as increasing levels of thinking, they are based on my own interpretation of the mathematics and what I have observed to be more and less accessible for students in general. Their order serves to organize the continua but does not communicate an absolute sequencing. Finally, just as students need not pass through these steps sequentially, they need not engage in all the ways of thinking to develop rich, quantitative understandings of the algebraic notation. That being said, developing understanding in one category, or at least engaging in the type of thinking associated with the category, often supports the reasoning involved in the subsequent way of thinking. In Clara's class, most students were unable to arrive at a general description of the pattern because they were not supported to describe a particular stage. Moreover, it

seems likely that the more ways of reasoning in which students engage, especially those students with limited experience generalizing, the stronger understanding they will eventually develop.

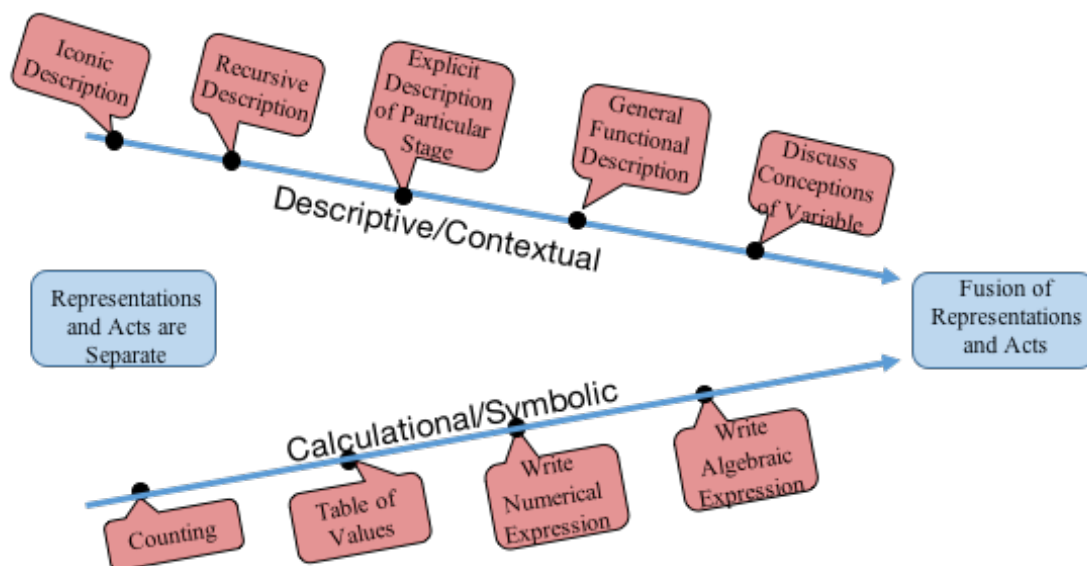


Figure 5.13. Model of increasingly general ways of thinking.

Table 5.2. Clarification of Ways of Thinking Associated With Generalizing

| <b>Describing<br/>contextual representations</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                               | <b>Calculating<br/>symbolic representations</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       |      |   |    |   |    |   |    |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|------|---|----|---|----|---|----|
| <p><b>Iconic description.</b> The figure is described in general without attending to specifics, or specific parts are described without exact numerical values “It is an <i>H</i>. At every stage the sides and middle get bigger.”</p>                                                                                                                                                                                                                                                                       | <p><b>Counting.</b> A student is able to calculate only individual elements, the shapes provided, without leveraging any structural properties.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                   |       |      |   |    |   |    |   |    |
| <p><b>Recursive description.</b> The pattern is described in terms of how it changes from stage to stage. “Each new stage adds an upside down <i>L</i>-shape.” “The two sides grow by two dots at each stage, and the middle grows by just one.” This description may simply explain how the overall shape increases in number of elements, what changes in the figure, or how constituent pieces grow.</p>                                                                                                    | <p><b>Table of values.</b> A student calculates the number of dots in various stages. A student could draw the stage and count or add recursively from the previous stage but give no indication of understanding how the number relates to the figure. Still, table creation requires the minimal ability to generalize the pattern and create further stages whether numerically or pictorially.</p> <table border="1" data-bbox="886 657 1076 816"> <tr> <th>Stage</th><th>Dots</th></tr> <tr> <td>3</td><td>17</td></tr> <tr> <td>4</td><td>22</td></tr> <tr> <td>5</td><td>27</td></tr> </table> | Stage | Dots | 3 | 17 | 4 | 22 | 5 | 27 |
| Stage                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Dots                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |       |      |   |    |   |    |   |    |
| 3                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | 17                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |      |   |    |   |    |   |    |
| 4                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | 22                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |      |   |    |   |    |   |    |
| 5                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | 27                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |      |   |    |   |    |   |    |
| <p><b>Explicit description of particular stage.</b> Stage 10 would have the 2 dots that are always there, and then each piece would have 10 dots. *Note. A description could be a hybrid of recursive and explicit (or what I might call <i>pseudo explicit</i>). “Stage 10 will add 5, six more times.” Or “Stage 10 will add 5, 10 times to the initial 2.” In this case, the student does not describe the stage, but explains how to calculate the number of dots by using a recursive generalization.</p> | <p><b>Open numerical expression.</b> A student writes the number of dots for a particular stage using a number sentence according to how he or she sees the pattern. “Stage 10 is <math>4 \times 10 + 10 + 2</math>.” Although calculating the number of dots, the student includes explicit understanding of the pattern, revealing his or her structural interpretation.</p>                                                                                                                                                                                                                        |       |      |   |    |   |    |   |    |
| <p><b>General functional description.</b> Student decomposes the stage and connects the description of the pieces to the stage number. “In each stage the four legs equal the stage number and the middle is two more.”</p>                                                                                                                                                                                                                                                                                    | <p><b>Algebraic expression.</b> <math>5n + 2</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |       |      |   |    |   |    |   |    |
| <p><b>Discussion of conception of variables.</b> Student reflects on meaning and conventions surrounding the use of variables</p>                                                                                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       |      |   |    |   |    |   |    |

**Applying the model.** Whereas the model introduced above represents the conglomeration of actions and comments in both classes as structured through my understanding, the schematics shown in Figures 5.14 and 5.15 illustrate its application in terms of Clara and Jack, capturing their personal instructional trajectories. Each figure has been modified to highlight variations in their practices and understandings. First, the distinct shadings of the categories denote differing emphases and levels of support provided by the teachers during the unit. Although most types of thinking occurred in both class (except for the final two for Clara's class, which have been marked through), the two classes varied significantly in the frequencies and the roles each type of thinking played in the overall attention and understanding of the students. The lightly shaded categories reflect approaches in which only a few students engaged and which were not elevated or supported by the teacher. The darker categories indicate ways of thinking that were more prevalent and more central to overall instruction. In addition, the positions that have been circled (red for Jack and green for Clara) represent the areas of the most significant instructional focus for the teacher. These are ways of thinking that not only were demonstrated consistently but also that the teacher actively worked to foster. The ways the teachers guided students to engage in these particular ways of thinking are discussed later in the chapter. Finally, the arrows on the schematic representing Jack's instructional trajectory serve to highlight the strong relationship he developed between ways of thinking and the corresponding representations. As examples in the previous two sections show, Jack did not simply support students to engage in the various types of thinking in isolation; he linked the underscored approaches explicitly.

Note that the lessons analyzed for each teacher took place at slightly different points in the overall unit. My study of Clara's instruction occurred at the beginning of the unit, hours 1–5 of the overall unit. I began observing Jack after he had begun the unit, so my observation was of Hours 5 and 6 of his overall unit. Although I observed Hour 5 in both classes, the timing difference accounts, at least to some degree, for the foci of Jack's lessons being further along the trajectory. He had already concentrated on the types of thinking that typically occur in the early stages of generalizing, but, with each new figure, students did continue to engage in these types of thinking expected early in the trajectory. Jack encouraged this thinking as a way for students to familiarize themselves with the pattern. He then collectively transitioned the class to areas along the trajectory that constituted his primary instructional attention.

In the following section I illuminate each teacher's instructional trajectory in more detail, explaining the two teachers' differing instructional foci. I illustrate how the two teachers worked to establish particular ideas, providing examples of how they elevated certain ways of thinking and deemphasized others. In addition, I contrast the teachers' methods for implementing their plans, what they did to create circumstances to promote and develop productive thinking and how they reacted to students' conceptions. Finally, I discuss how the teachers differed in implementing their instructional trajectories in terms of their professional noticing of students' mathematical thinking.

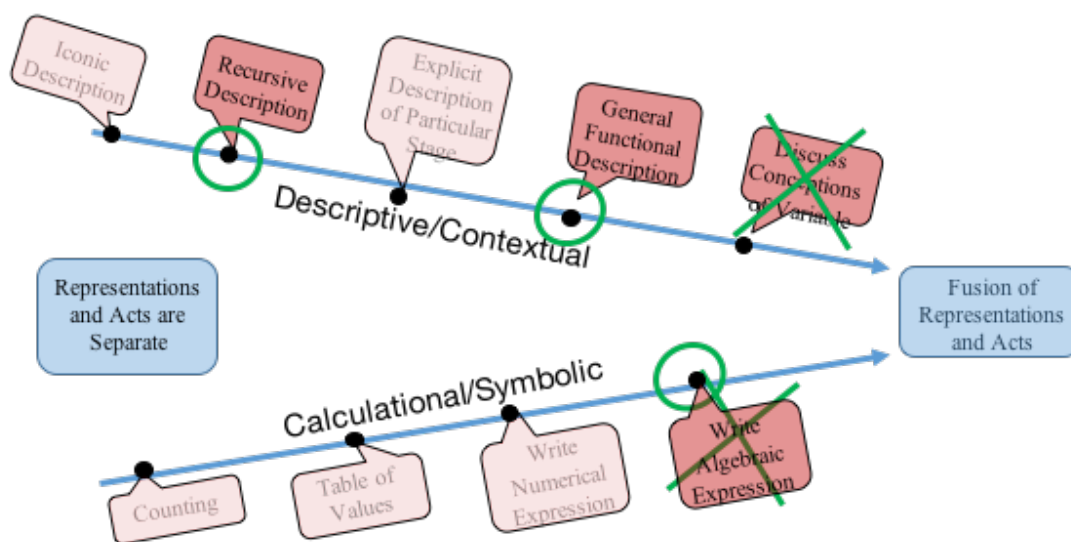


Figure 5.14. Clara's instructional trajectory.

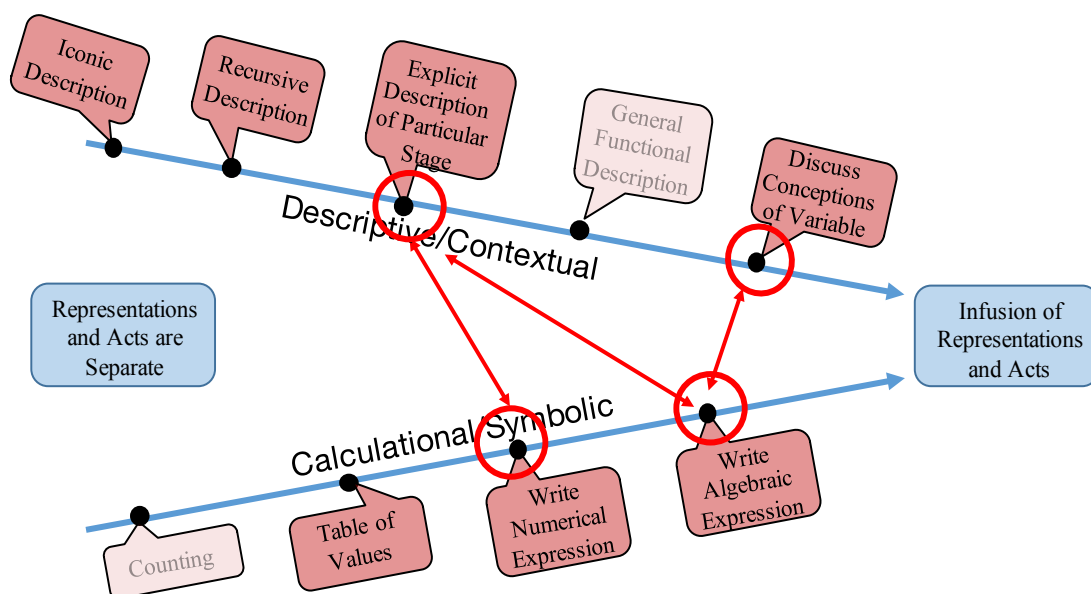


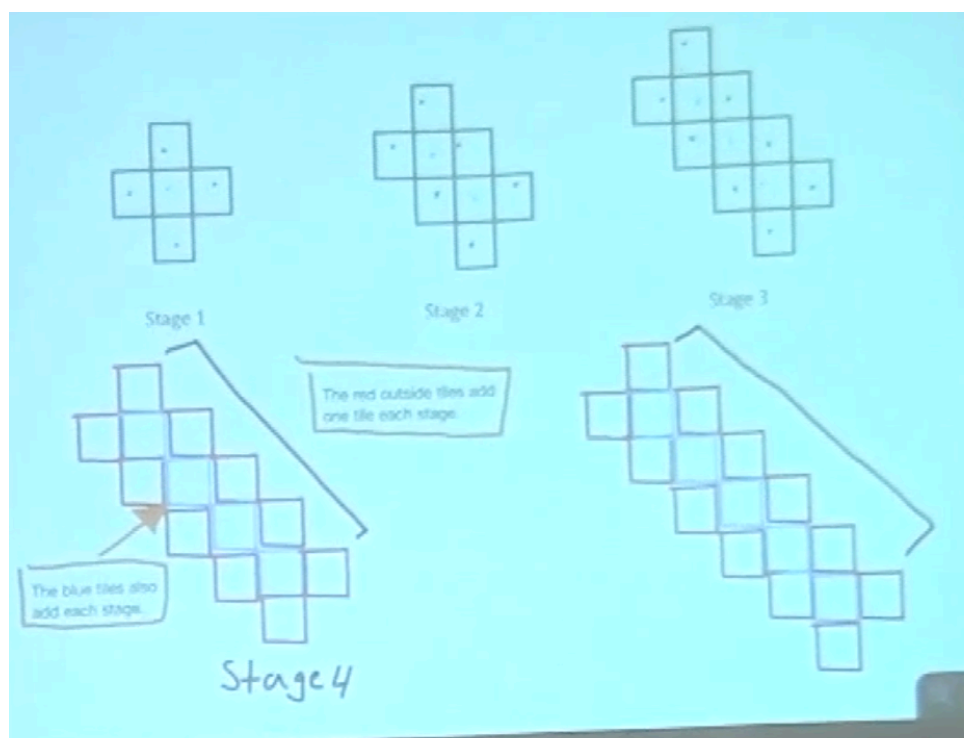
Figure 5.15. Jack's instructional trajectory.

**Clara's Instructional Trajectory.** As shown in the model of Clara's instruction (see Figure 5.14), she focused on two major areas. She devoted the first 2.5 hours to

encouraging students to describe the pattern recursively and the majority of the remaining 2.5 hours asking students to describe the pattern in an overall, general way relative to the stage number. Although she did direct the students through writing algebraic expressions toward the end of the third day, this attempt was unsuccessful; the students were visibly confused and unable to follow her lead. Without a doubt Clara's primary method for elevating these types of thinking was her questioning. As I demonstrate, the focus of her questioning was so pronounced and so consistent that it served to detract from students' engaging in other approaches. To illustrate, I provide examples from each of the two instructional foci that seemed to comprise the unit.

Clara's emphasis on recursive thinking was apparent from the outset when she communicated her learning goal for Day 1. She wrote on the board that students would "analyze a visual pattern and describe in words how it develops." She then clarified her expectations by telling the class that she wanted them to "discuss what is changing from stage to stage, and see if you can find more than one way to describe the change." In both her written and verbal articulation of her goals, Clara directed the students' attention to the change in the pattern, encouraging a recursive approach. She then presented the problem and visited different groups while they explored the pattern, pursuing their thinking with her questions. During this time, every follow-up question, without exception, refocused the students' thinking on the recursive pattern. I counted 13 requests for students to describe the change between stages (e.g., "What did we notice about how the pattern changed from stage to stage? In what way did it grow and did you find more than one way? Can each of you show me which three [dots] you are adding? Can you describe where you are adding 3?"). Although Clara asked many of these questions to

support students struggling to make sense of the pattern, in several instances, she redirected other, quite productive, thinking. Most notably she shifted student thinking midway through the first period when she asked multiple students to present their thinking to the entire class. In spite of Clara's emphasis on recursive thinking, one student had developed an emerging explicit understanding of the pattern, which he shared with the class. Providing the drawing shown in Figure 5.16, he stated, "For this Stage 3, it ended with four red tiles, and this Stage 4, it ended with 5. Stage 3 it had three blue tiles, and in Stage 4 it had four blue tiles."



*Figure 5.16.* Student's diagonal decomposition of the pattern.

Instead of attempting to leverage the students' explicit thinking, Clara asked, "And what are you saying is happening each time? When you say build the next stage, what do you need to think about?" This question redirected the student's thinking, and

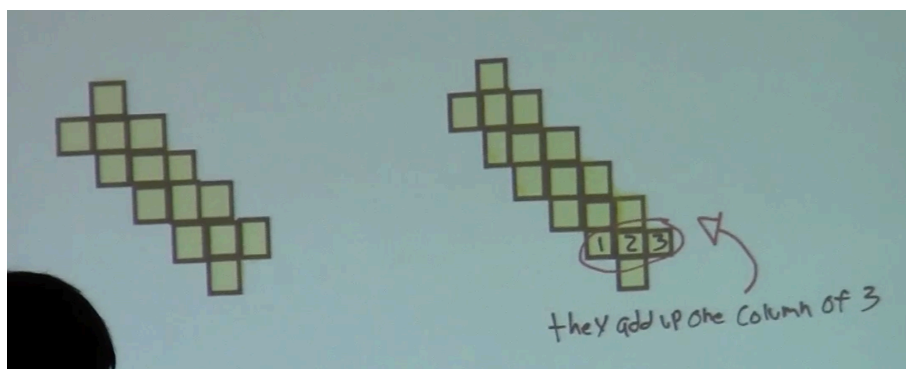


he responded with a recursive interpretation of the pattern. Clara then asked another student to describe the first student's view, who again answered with a recursive view, stating, "So basically you add the 2 and the 1." Finally, another student explained the overall interpretation of the pattern by summarizing, "It's just adding by 3." Clara endorsed these recursive views by clarifying, "Yes, it [the addition of one dot in the middle and two dots on the edges] is just how we are seeing those three (pointing to the three new dots in Stage 4)." Apparently wanting to draw attention that there are various ways to see the pattern can grow, Clara attended not to the thinking in which students were engaging but only to the various decompositions.

In the postinterview, I asked Clara about the significance of this student's thinking, which was part of a group discussion she had identified as a noteworthy episode prior to his sharing his approach to the whole class. She had chosen it because this student and another student in his group had argued over their differing decompositions of the pattern. In her reflection, she did not mention the difference in their types of generalization (recursive or explicit) or any other mathematical details about the students' approaches. Clara seemed so focused on drawing attention to their various ways to see the pattern, she had not attended to the type of thinking in which students were engaging. Instead she elevated how the pattern was changing as a way to differentiate the differing decompositions.

Later in the class, when the students were asked to explore later stages, another student saw the pattern explicitly. First, though, another student (Chris) shared his recursive interpretation of the pattern of adding a new row to each stage (which he referred to as a *column*). The student showed the picture in Figure 5.17 and presented

his recursive understanding, stating, “They add up like 3, and like each column (incorrectly referring to the rows as columns) has three [squares].”



*Figure 5.17.* Chris’s decomposition of the pattern.

Afterwards, Clara asked the class if anyone could “explain how Chris [was] seeing the pattern grow.” Although she again directed the class to reflect on the pattern recursively, one student offered his explicit view, what he characterized as an “interesting fact”: “There is an interesting fact that on Stage 1 there is one column, on Stage 2 there are two columns, and on Stage 3 there are three columns, yada, yada, yada.” Although Clara then paused a bit theatrically to acknowledge this different type of thinking and emphasized it, stating, “Just an interesting fact,” she did not clarify what was interesting about the pattern and did not have anyone revoice the idea. She clearly recognized that this thinking differed from that previously expressed, as did the student who made the contribution, but again she did not support the class in engaging with it.

At the end of the 2-hour lesson, Clara asked each group to share their view of stage 10. The students’ responses showed that Clara’s focus on a recursive approach had affected the student thinking; eight of the nine groups either described the pattern in terms of its change or offered a numerical answer. The one exception was from Chris,

the student mentioned above, who had previously seen the pattern recursively as rows. He had since shifted his thinking, still thinking about rows, but now with an explicit view of the pattern. The following excerpt illustrates his group's confusion but also shows the productive elements in their thinking that were neither supported nor promoted.

- Clara: So what would Stage 10 look like. How would you tell someone to build Stage 10?
- Student: There's 10 [dots] in the ... rows.
- Clara: Columns. Not 10 in them.
- Student: There's three [dots].
- Clara: What's 3? You can help her.
- Another student in the group: There's three of these thingies and 10.
- Clara: Three what?
- Student: Rows, columns.
- Clara: What are there three of?
- Student: Columns.
- Clara: There are 3 columns?
- Student: Rows, columns ....

At this point Clara stopped the conversation telling the group to “get their story together.” Once more, instead of building on the students’ thinking or elevating the significance of this type of thinking, Clara quickly moved on to the next group. Clara did not comment on this moment in the postinterview, so I did not hear her rationale for not focusing on the group’s explicit understanding of the pattern.

On Day 2, Clara’s repeated and focused questioning with the class continued, although the instructional aim of her queries changed dramatically midway through the period. She began the class once again directing the students to think recursively. Reviewing the possible decompositions, she showed a different representation of each view of the pattern and repeatedly asked the students to think, “How does the pattern

grow?” “What are they adding?” “How does it change?” For this first 30 minutes of review, Clara continued to direct the class to analyze the multiple patterns recursively. Clara then stopped to introduce the next facet of the activity, and her language clearly indicated that her focus changed. She started by explaining the new instructional goals, writing on the board the learning target: “Analyze a visual pattern, and describe how it relates to the stage number.” She elaborated verbally, confirming the shift and clarifying the new aim: “We have looked at how the pattern is growing, so now I want you to think about how the figure and how it changes might relate to the stage number.” She continued, saying, “In general, I want you to describe any relationship you find to the stage number, any way you can relate your pattern, and how you’re thinking of it, to the stage number.” She then gave the students time in groups to tackle this assignment while she engaged with the many small groups and monitored their progress. With each group, she repeated a version of her previously explained goal, asking for example, “Can you share a relationship to the stage number that you recognized, or does someone want to explain a relationship they saw?”

Through this line of questioning, although Clara asked students only to identify a relationship, she seemed to hope that students would describe the various constituent parts of the figure and explain the relationship to the stage number. Most students struggled significantly with this question, with six of the nine groups making limited progress. Without a model, they seemed unsure of the exact objective. To support those students who were confused, Clara responded in one of two ways. In three cases, she engaged with the students by providing a specific stage number and asking them to explain a relationship for that stage. To engage in this exchange though, students had to

have already identified some of the quantities in the pattern relative to the decomposition they were pursuing. For example, in one group a student recognized that the pattern consisted of increasing numbers of columns. Clara then asked the student to describe the number of columns in Stage 13, to which the student replied 13. Such straightforward questioning did not seem to deepen the students' understanding of the pattern, but rather simply removed the cognitive demand, enabling students to provide answers by noticing and regurgitating simple numerical patterns. At no point did Clara ask students to give an overall description of a specific stage, always asking for the numerical value of a particular quantity in a particular stage.

The other three groups who were struggling to make progress on the task had yet to identify quantities in the figure. Clara asked the students to think about how the pattern was changing from one stage to another as a way to support their understanding. For example, she asked one group,

You guys already wrote directions for how to build Stage 10. Right? So if you wrote directions for the figure in Stage 11, could you use those same directions? Would you have to change something? What would you change? And think about what you would change. Does that help you see where the stage number fits in the figure?

Although Clara never explained the pedagogical purpose behind this approach, I see this question as having the potential to help students identify quantities within the figure. Seeing and articulating how the pattern is changing makes the structure of the pattern more evident. This type of question also seemed to be linked to Clara's previous focus of describing the pattern recursively. As mentioned earlier, Clara appeared to ask about how the pattern was changing as a way to elevate the particular quantitative composition of the pattern.

Finally, although three groups were able to connect quantities in the figure to the stage number, only one group (in fact one student in that group) provided a detailed description of the pattern. In contrast, the other two groups described the relationship of individual quantities in terms of the stage number but did not describe the entire figure. In these two groups, the students recognized that the number of dots in the diagonal was equal to the stage number. When asked about the remaining dots, they stated that the number on the sides was one more than the stage number. At no point did Clara ask students in these groups to describe a full stage and connect the quantities within the figure. Clara's method of asking students to connect features in the figure to the stage number seemed to encourage the students to analyze one quantity and its relationship to the stage number in isolation. Moreover, her questioning further reinforced the students' independent treatment of the quantities because she consistently asked students about one quantity at a time.

In the end, Clara had these three groups share their thinking, but they did so rather quickly in the last 7 minutes of the class. Furthermore, she did not have the other students repeat or engage in their thinking as a way to elevate or distribute these ideas. She simply had the students present their understandings.

In reflecting on the students' difficulties, a couple of sources seem possible. First, the task, as presented by Clara, was not specific. The vague instructions did not provide enough guidance for the students to know exactly what the final outcome entailed or how to pursue it. In addition, the students seemed to have only begun to identify the quantities within the pattern. Without more exposure to the various components of the figure, they lacked the familiarity with the quantities needed to

describe their general properties. Finally, students seemed to need some tool, vehicle, or representation to organize and present the structure of the pattern so that the qualities of the embedded quantities, and the relationships among quantities, were more visible.

Clara, in her instructional trajectory, avoided two areas that specifically support students in developing these types of thinking. In the acts of describing a specific stage and writing out a numerical expression, one must identify the quantities, at least implicitly, and with support can do so explicitly. Engaging in these types of thinking also requires elevating numerical details about the quantities to make connections between the quantities more obvious. Comparison of the instructional support afforded by Jack's more detailed trajectory with Clara's trajectory indicates that such an omission might have caused students difficulty. Both numerical expressions and descriptions of specific stages help students develop this understanding and provide an intermediary step in the generalizing process. The absence of these transitional stages from Clara's instructional trajectory suggest that she underestimated the complexity of algebraic generalization. She seemed to expect students to skip from one type of thinking to another supported by her questioning alone.

***Jack's Instructional Trajectory.*** Clara's and Jack's instructional trajectories differ dramatically. Most notably, as the shaded and circled areas indicate, Jack was able to facilitate students' engaging in more types of thinking than was Clara. In addition, whereas Clara focused students' attention on particular types of thinking, in effect isolating these areas, Jack drew explicit connections among pattern components, creating a cohesive flow for the students. Comparing the two, Jack's trajectory was both denser,

consisting of more nodes, and more connected, with direct relationships made between different approaches.

Jack's students engaged in more types of thinking than Clara's. A wide range of Jack's students pursued thinking in most, if not all of the highlighted areas. In addition to the diverse thinking exhibited in class, the majority of Jack's students grappled productively with the later areas along the trajectory, including the last two ways of thinking, which none of Clara's students reached.

Charting the types of thinking throughout the class showed that Jack's instruction separated students' engagement with the pattern into two halves of the trajectory; whereas Clara's students were directed to pursue the problem in two specific ways on the trajectory, Jack's students were encouraged to employ multiple ways of thinking, first using strategies that aligned with ways of thinking from the left side of the trajectory and then the right side. With the introduction of each new pattern decomposition, students initially explored the generalization, thinking about the pattern iconically, recursively, or using a table of values (ways of thinking on the left half). After 15–20 minutes, Jack transitioned the class to create numerical and algebraic expressions as well as connect their thinking to the figure through explicit descriptions of specific stages (ways of thinking on the right half). Such a partition appeared to be a result of Jack's instructional focus. Initially, to ensure that students familiarized themselves with the pattern, Jack supported students' using any way of thinking was most accessible. Later, when he began to pursue his specific instructional goals for the lesson, his approach changed. In an effort to ensure that students engaged in more challenging forms of generalizing, he



became more visibly deliberate in his instructional approach. (I include the modifier *visibly* because Jack's initial, less-directed questioning also appeared quite purposeful.)

*Instilling multiple ways of thinking.* A significant factor in students' engaging in multiple forms of thinking was Jack's ability to communicate throughout that students were to approach the pattern using the types of thinking that made sense to them as opposed to trying to reproduce a particular form endorsed by the teacher. Students were encouraged to develop ownership of their own understanding; Jack instilled in his students the value of thinking in multiple ways. Clara's approach, in contrast, was to repeatedly reinforce to her class that they were to think about the problem in a particular way. Although part of Jack's students' flexible engagement with the tasks surely stems from a semester of instilled classroom norms, Jack's presentation of the problem and his interaction with his students illustrate his ability to foster this open exploration in this unit.

First, Jack's initial presentation of the problems was much more open than Clara's, seeming to provide students space to explore patterns in various ways. For example, he introduced the second decomposition stating the following:

This might not be the way you thought about it, but we are trying to think, "What is this student thinking?" So take a minute just to look at it, think about it;... try to figure out how this student was thinking about this  $H$ .

In this statement, Jack did not attempt to fix the students' attention on any specific attributes. He guided them to a particular decomposition, but then they were encouraged to make sense of it as they chose.

In addition, Jack engaged with his students without any specific directive, probing their thinking and supporting them in their understanding, irrespective of the approach

they might be pursuing. His comments were in no way evaluative, but rather he conveyed a genuine interest in the students' thinking. That being said, although Jack did not try to *funnel* (Herbal-Eisenmann & Breyfogle, 2005) students into thinking in a particular way, he did gently nudge them to include more details. He often asked students to explain their thinking or highlight particular pieces in their thinking, actions that served to deepen their understanding and at times move them to other areas along the trajectory. Notably, he consistently attempted to connect students' calculational and descriptive approaches. The following two exchanges illustrate Jack's supportive, yet slightly subtle pressing, interaction with students. Both instances took place on Day 2 while the students explored the second decomposition.

- Student: I don't know. Like I'm just adding more dots.
- Jack: Okay. And so when you said you were adding more dots, and you pointed to Stage 2, Stage 3, and Stage 4, can you tell me a bit more about what you were adding?
- Student: I'm just adding one dot. Cause this, like the pattern, its going up; you just keep adding one. This is 1, 2, 3, 4. 5 [dots].
- Jack: Where is the five? Going across or going up and down?
- Student: Across.
- Jack: Okay, so you are looking right here (pointing to the middle group).

In his initial question ("Can you tell me a bit more about what you were adding?"), Jack probed for more numerical details, encouraging the student to articulate what he was adding. After the student provided specific values, Jack asked him to connect these numbers to the figure ("Where Is the 5? Going across or going up and down?"). Through his questioning, he guided the student from an iconic description, with limited to no detail, to a numerical recursive view—what would align with elements of a table of values—and then finally eliciting contextual details to encourage the student to describe

the pattern with more figural qualities. Although the students' understanding did shift, Jack's questioning was centered on the student's current thinking. At no point did he try to reconfigure the student's attention or request him to pursue a different way of thinking.

Similarly, in the following exchange, Jack engaged with a student who could describe the legs in the second  $H$ , because they were the same configuration as in the first  $H$ , but had not made sense of the middle group.

- Student: In Stage 4, there would be four in each group (referring to the legs) and in Stage 5 there's five in each group, except for the middle.
- Jack: Cool, and if I had to draw Stage 6, how many do you think there would be in each group?
- Student: Six
- Jack: Did you make any sense of this middle part? How did you know how many were in there?
- Student: Cause it's two more.
- Jack: Two more? Tell me more.
- Student: So it's two more here than here.
- Jack: So it's two more in the middle than on the sides.

Jack began by questioning the student about the figure's connection to the stage number in other stages. He then asked the student to explain her understanding about other quantities in the figure, namely the middle section. He also asked the student to provide a rationale for her observation, which encouraged her to provide more detail, presumably deepening her understanding. Finally, Jack asked for more specific articulation of what the student meant by "two more," revoicing the student's thinking using quantities in place of simply pronouns. Again, through his questioning, Jack nudged the students' description from an explicit description of a specific stage to a more general view of the pattern, by having her articulate the quantitative relationships. As these two examples

illustrate, Jack instilled a flexible approach in his students while providing, through his questioning, support to deepen their understandings.

*Pedagogical content tool.* As I mentioned, after the students had a chance to make sense of the presented decomposition and share their understandings of the pattern verbally and pictorially, Jack transitioned by asking students to provide a number sentence for Stage 4 (as well as subsequent stages) that would communicate a way to find the total number of dots according to the indicated decomposition. Afterward, regardless of the form the students generated, Jack asked them to identify the quantities in the figure that the various symbols represented. By asking students to create a numerical expression and supporting them in relating it to the figure, Jack used what Rasmussen and Marrongelle (2006) referred to as a *pedagogical content tool*. This is a “device that a teacher intentionally uses to connect to student thinking while moving the mathematical agenda forward” (p. 389). Unlike Clara, who attempted to shift students’ thinking by repeatedly delivering a somewhat vague question with no associated action, Jack provided students a specific task and a reasonably familiar representation in which a transition in thinking would result from, or be a potential consequence of, engaging in the activity. Rather than asking students to produce a new way of thinking on their own, this pedagogical content tool served to gently push the students to flexibly build on their own thinking and approach the problem differently, creating a more seamless and supportive transition.

More specifically, the task and representation served multiple purposes. First, the act of creating the number sentence initiated a shift for many students from recursive thinking to an explicit view of the pattern. When students determined the number of

dots for each constituent part, they began to see the relationship between the figural pieces and the stage number, no longer relying on the change between stages. Second, using a numerical expression forced the students to symbolize their understanding, transforming their views of the pattern to be concise, abstract representations. Verbalizing a particular view of the pattern and using mathematical symbols to communicate this view require different skills and a shift in thinking. Third, this act highlighted the decomposition's structure when students organized the various pieces into a single representation. The creation of successive numerical expressions illustrated which pieces remained constant and which were changing, enabling in the generalization process. Finally, Jack's strategy of asking students to verbalize the figural quantities that corresponded to the respective numerals elevated the contextual quantities, helping students see more than simply abstract numerals. This act shifted the number sentence from a tool for calculating the number of dots to an object that represented the quantitative structure of the figure. I argue that the numerical expressions as well as the discourse generated around them served as a pedagogical content tool for Jack, and these outcomes were an almost inherent consequence. As such, Jack's students shifted their understandings and moved along the trajectory.

Comparing Clara's and Jack's instructional trajectories, and in particular the number of areas along the trajectory in which students engaged, showed that many of the differences reflected their general instructional approaches. Clara encouraged students to engage with different types of views of the pattern, but not with a variety of types of thinking. She limited their understanding of the generalization process by fixing their attention on specific attributes and requesting a particular way of thinking without

supporting them to do so. In contrast, Jack encouraged students to think about the pattern using approaches that were most accessible to them. His instructional tools enabled students to build on their own understandings of the pattern and to engage in more challenging ways of thinking. As a consequence, Jack's students engaged in more areas along the trajectory than Clara's, with stronger connections between them.

**Teachers' implementation of their instructional trajectories.** In addition to the difference in the overall composition of each teacher's instructional trajectory, the teachers also contrasted in how purposeful and with what level of detail they attempted to implement their trajectories. Jack's approach was more deliberate and meticulous than Clara's. He devised a nuanced plan of the exact type of thinking in which he wanted students to engage throughout the class. Such an organization guided his instructional decisions, enabling him to proactively anticipate and sequence students' ideas as well as envision instructional interventions that might encourage and support such thinking. Clara, on the other hand, although she had identified specific conceptualizations she wanted students to develop, had not anticipated either the type of thinking that might lead to such conceptions or instructional strategies that would promote them. She viewed these types of thinking as discrete points, not as part of a continuous progression. She possessed no clear plan of the details necessary to foster particular ways of thinking, resulting in reactive and less-structured engagement with student ideas and, eventually, a direct and demonstrative style of instruction. In the next section, I contrast the two teachers' implementations of their instructional trajectories, highlighting how they differed in terms of their management of the flow of ideas, their instructional planning, and their noticing of student thinking.

***Guiding the flow of ideas.*** Jack possessed a very clear plan of instruction for both lessons I observed. In the preinterview for each class, he articulated in detail not only the exact activities in which students would engage, but also the flow of ideas he hoped to shape. He portrayed his instructional plan as leading the students on a particular conceptual path. On multiple occasions he described the experience he had planned for his students as a “journey,” signifying that he planned to follow a specific route. Probably the biggest indication of his detailed, sequential plan was his disappointment on Day 1 when a student offered an idea too early in the lesson, according to his plan. Although the concept the student correctly explained aligned strongly with Jack’s eventual goal for the lesson, Jack was quite unhappy. He explained that he “was hoping to build that foundation through the journey.” He wanted students to grapple with this particular idea at the end of the period, after they first had experience with other activities. Jack reiterated his frustration several times in the debrief, stating how he had “let the cat out of the bag” and describing his lesson as being “jumbled back and forth” and not “fluid.” He explained, “So it kind of threw me for a loop, and so I was kind of kicking myself for not stopping.... It was kind of a different pathway. And so it felt awkward.” So precise was his plan that he considered the lesson unsuccessful because it had not followed the conceptual path he had designed.

A key for Jack’s successfully implementing his plan was his effort to guide the flow of ideas by calling on specific students to share their thinking. Jack spoke about how he normally has specific reasons for calling on particular people. In reflecting on how Day 1 had not gone as planned, he explained, “Usually I have a better idea of who I am calling on and why.” He then cited two examples of students he had called on

without preknowledge of their exact contributions, ideas that in his mind had diverted the flow of ideas. “I assumed a lot this period. And that's something that I don't usually do.” Later in the interview, Jack explained how he normally makes decisions about selecting students to share:

Usually, I have a thought of the order I want to do things. So like there ... you know different people have different things. But things like drawing the  $H$ , which more people can do, then I'll call on people that haven't,—that are reluctant to get up or that are nervous, you know, about getting up. Umm ... then I'll do ... then I was thinking that, you know, I called on four or five boys in a row or something, and then—so ... it's usually a mix of everything.

Jack's remark about the ordering of the class shows that a mathematical plan guides his instruction and his selection of student work. In addition, Jack indicated that he attends to issues of participation and equality, but focuses on these at times when he believes the mathematics is less significant. Although his comment did not provide details about how he navigates the ‘different things’ students offer and how he uses them to direct his mathematical agenda, later, when I describe Jack's noticing, the mathematical nuances that guide his instructional decisions, in particular how he incorporates students' conceptualizations, will be clearer.

Clara identified a singular focus for each class, but had not anticipated other conceptions that might lead up to this type of thinking or possible instructional strategies to promote them. Specific features of her instructional plan centered on the decompositions, rather than about how to progress to and through these ideas. Consequently, without a detailed plan for creating particular conceptualizations, she simply reiterated the same question over and over. As I explained earlier, her instruction



over the three days was divided into two major segments. With each she had a specific instructional goal, but it was held in an isolated fashion, without a plan for navigating various student ideas that might arise. In contrast to Jack's metaphor of a journey as an instructional plan, Clara's metaphor seemed to incorporate specific stops, but not a path to connect them.

Additionally, unlike Jack, Clara did not use students' conceptions productively to organize and shape the collective understanding of the class or to support the understanding of others. In general, all student ideas—correct, incorrect, sophisticated, less sophisticated—were given equal credence. As I have illustrated, Clara did not act to elevate particular ways of thinking by discussing the beneficial attributes of given contributions or by having other students revoice them. When I asked Clara how she makes decisions about which students to have share, she emphasized, “The main thing is the math.” She then gave a description for longer than a minute of her system of calling on students to instill the importance of students' roles in their groups. In analyzing Clara's actions, I found that issues surrounding participation guided her instruction more than the mathematics. For example, several times she stopped students from sharing their productive ideas when their groups had not reached consensus. In each case, she directed them to consult with their group, emphasizing that contributions by the spokesperson must represent every member in the group. Moreover, she instituted routines that seemed to reinforce the significance of participation. On Days 1 and 3, instead of targeting specific ideas, she had every group share. On other occasions she chose groups randomly, using a card-drawing method. In fact, on Day 3, she required a

new student, during her first period in attendance at the school, to share even though she had missed the previous two days of instruction.

From my perspective, Clara's fine-grained pedagogical decisions seemed based less on students' understanding and more on participation norms. Even when she did focus on mathematically related issues, they tended to involve less significant qualities. The one occasion I explicitly asked Clara to explain why she had chosen a particular group to share first, she said it was because their way of seeing the pattern was the most common. Such a lack of focus on mathematical nuance seemed particularly noteworthy when Clara herself acknowledged that she was unable to help students move forward in their thinking if they did not arrive at the particular way of thinking on their own. In the end, unable to support students in making connections her only resource was to provide ideas and ways of thinking herself.

***Creating learning opportunities.*** In addition to (or possibly as a consequence of) the differing levels of detail in the teachers' daily instructional plans, Clara and Jack also contrasted in their perspectives on planning. Jack was proactive. He identified particular conceptualizations toward which he wanted to work. He then organized his instruction so that students would have opportunities to grapple with the key issues involved, anticipating as well several misconceptions or difficulties that might be associated. Clara was more reactive. She identified her mathematical goals without including how students might engage in them. She did not speak about expecting particular difficulties or what she might do to tackle them. In her planning she seemed to assume that students would productively engage in the task without any challenges.

To illustrate this difference, I contrast the two teachers' preparation in tackling the goal of helping students symbolize their generalizations. This was a primary instructional focus identified by both teachers, Jack for both days and Clara for Day 3. Significantly, Jack deliberately designed his lesson so that particular challenges would arise. He spoke specifically about trying to set up conceptual difficulties so that he could address them later by leveraging possible associated struggles for discussion. He chose three specific decompositions for the class to analyze, offering a detailed rationale for selecting each. The first  $(5n + 2)$  he characterized as more "straightforward, easy to see." This involved five groups, all with dots equal to the stage number, which he circled, and two constant dots, which he shaded. Jack highlighted that the previous day a student in class had expressed his interpretation of the variable " $s$ " as always representing the 19<sup>th</sup> stage because it was the 19<sup>th</sup> letter of the alphabet. He explained that he chose this simpler view as a way of confronting and revisiting this misconception. The second pattern involved four groups like the previous one, but the fifth group was combined with the previous two constant dots, resulting in an expression  $4n + (n+2)$ . Jack explained that he selected this decomposition because he knew that students would struggle symbolizing something two more than the stage number. He referenced students' confusion from the week of professional development:

I knew that the  $n + 2$ , from our week, was going to bring up some issues.... Verbally they'll be able to say it's two more than the stage number, ... but writing it out, I see it's going to be a problem; we're going to run into similar issues.

Jack then listed multiple alternative symbolizations he anticipated: “ $n$  to the second power,  $p$ , or  $n^2$ .” The last decomposition involved overlapping groups for which students would have to subtract dots as compensation. Jack stated that he chose this view because of the challenge that these overlapping groups posed.

Later in class, Jack leveraged the first decomposition to lead a conversation about the use and meaning of the variable  $s$ . As he had expected, students readily made sense of the decomposition, providing an ideal situation to lead a discussion about the role of the symbol  $s$ . Specifically targeting the previous misconception, Jack engaged the class in questions such as “Is  $s$  always 19?” and “When, if ever, would it make sense for  $s$  to be 19?” For the second decomposition, although students did struggle to make sense of the pattern, the class as a whole was able to write an expression without much difficulty. Jack was surprised, almost disappointed, saying, “I think I was almost trying to set it up. Like I wanted a  $n^2$  or  $p$ .” He emphasized how important the creation of these conceptual hurdles is for his instructional plan by speaking at length about his regret that students had not experienced the problems he had anticipated. As I explain in the next section, he had spent a fair amount of energy and attention on finding such cases so that he could address what he perceived as notational challenges.

In analyzing Jack’s instructional preparation, I concluded that his stance on learning seemed to influence his desire to elevate misconceptions with which students must grapple. For example, when talking about revisiting the idea that  $s$  does not always represent 19, he described his goal as having “a little conversation there to again bring up what it is and, you know, I don’t plan on solving everyone’s issue there.” Several similar comments indicated that Jack sees learning not as a binary construct in which students

either understand or do not, but as a process such that over time they slowly develop deeper understanding. Holding such a belief prompts Jack to provide experiences in which students face conceptual challenges. Deep mathematical understanding will not be achieved in one day or by one act, but only through revisiting and discussing, especially areas that are conceptually challenging.

Clara's preparation for student thinking, as communicated in her preinterview comments, was quite different from Jack's. On none of the 3 days I observed, did she mention any conceptual difficulties she anticipated or would target. When asked to explain her goals and expectations, she simply read from them her daily learning target, exactly as she would present them to the class. Even when I pressed for specifics, Clara offered no more depth to her instructional plan. For example, on Day 3, after she said that she wanted students "to create an algebraic expression that generalizes how it develops," I asked for more details. She highlighted that she wanted students "to understand what the variable means in the pattern." She then added that she wanted students to write an expression for each of the six views of the pattern. When I asked if she would focus on anything else in particular with any of these decompositions, she replied, "They're going to need to look at the  $n - 1$  idea and an  $n + 1$  idea," identifying this symbolization as a challenge, but without any more reflection. Finally, I probed one last time, asking, "Any idea how they'll—how you think the students will think about them?" Clara replied, "I think they're going to be pretty .... I think there's a few of them that are ready, very ready, "So, okay, I can do this." She then proceeded to explain that she would focus on describing the pattern first, without providing any details about how students might struggle with writing expressions. Similar to Jack, she brought up these

notational issues, remembering that they were challenging. In contrast, though, she did not elaborate about how students might think about these challenges or how she hoped to facilitate a discussion around these concerns. Unlike Jack, who centered his planning around nuances of ways of thinking, both anticipating them and creating situations in which they were more likely to occur, Clara focused her instructional energy on the mathematics, not on how students might engage in it.

***Noticing of students' mathematical thinking.*** The final area in which the two teachers' implementation of their instructional trajectories differed was in the practice of professional noticing students' mathematical thinking. Not only did Clara and Jack vary in terms of how they anticipated students' mathematical thinking, they also contrasted in how they attended to, interpreted, and acted on mathematical ideas raised by students in class. Jack focused on the mathematical details of students' thinking to make in-the-moment instructional decisions, whereas Clara decision's, seemed less informed by such details. Consequently, she struggled to leverage student contributions to orchestrate richer mathematical discussions and ultimately promote a deeper understanding of the mathematical concepts.

*Clara's noticing.* In comparing the two teachers' noticing, I begin with Clara's. In the following section, I outline the primary emphasis of her reflections during the 3 days, describe the detail and nature of her interpretations of these comments, and finally highlight how she used student thinking in her classroom to make pedagogical decisions.

*Attending.* In all, Clara spoke about 13 classroom episodes. Although her focus in every instance was about various aspects of specific students, mathematical thinking was not always her emphasis. In roughly a third of the total moments, Clara's comments

centered on nonmathematical topics, issues such as student affect or social norms. For example, she explained that one moment was significant because the student talking was usually a silent student who rarely participated. In another, she mentioned only that the group had failed to work together. Although her other comments were mathematical in nature, they tended to be vague, without details. For example, she tended to focus on how students dissected the pattern. She often began her reflections by reiterating the specific decomposition used and, in two cases, identified moments as significant purely on the basis of the students' views of the pattern. In one instance, she focused on a student who inferred a hole in the middle of each successive stage, and in another she talked only about how the group tried to reconcile two different dissections. Only once during the three days did she comment on the type of generalization (recursive or explicit). In general, she shared limited mathematical details of the students' thinking, even after I asked what she thought the students understood.

Of note, Clara's attention to mathematical issues did increase noticeably throughout the 3-day unit. On Day 1, half of Clara's comments emphasized mathematics, whereas by Day 3, every comment was mathematical in nature. Although I cannot say why this change occurred, I speculate it was because of a transition in Clara's instructional goals. Initially, while students familiarized themselves with the pattern on Day 1, her learning outcomes were not clearly defined. By Day 3, though, this changed and she had a specific target in mind. She wanted students to write an algebraic expression and was determined to guide them to this outcome. This specific outcome-driven instructional goal might have prompted her to increase her focus on mathematical issues. Regardless of the reason behind this change, by the last day, although Clara

continued to include reflections about social dynamics and participation norms, her comments for each of the moments analyzed focused primarily on the mathematical understanding of the students. That being said, her reflections still lacked details and, at times, were not fully appropriate interpretations. Furthermore, even when Clara offered a thorough and accurate depiction, I saw no evidence that this understanding shaped her instructional decisions. To illustrate the nature of Clara's noticing, in particular how she interpreted and decided how to respond to student thinking, I analyze in more detail a few Day 3 instances in which Clara reflected in depth, representative of her noticing of students' mathematical thinking.

*Interpreting.* The first example occurred on Day 3 and was Clara's most extensive reflection throughout. I share this example to highlight that even when Clara was quite focused on the students' understanding, her comments often lacked a detailed appreciation of the key mathematical issues with which the students were grappling. In this episode Clara was working with a group of students who were all still seeing the pattern recursively. She repeatedly asked the group questions that would require an explicit view of the pattern to productively answer. Instead they consistently provide responses that indicate their recursive thinking.

- Clara: Okay, so you are starting with two [tiles]. So if you would write those words, what you just said to me, what would you write? If you had somebody writing down exactly what you said, what would that person write?
- Student 1: You start with two things.
- Student 2: Okay tiles, whatever. So you are starting with 2.
- Student 1: Then you add an upside down  $L$  to the left.
- Student 2: And you keep on adding a 3.
- Clara: Okay, and how many times will you do that?
- Student 3: How many times? As many stages as there are.



- Clara: So tell me. So I have so far. Start with two tiles and then you add 3 tiles at a time, but how many times do you add that?
- Student 3: For each stage, you add 1 [tiles]. So one time.
- Student 1: Oh, 1 time
- Clara: For each stage?
- Student 1: You add three
- Clara: How are you doing there, Savannah? Savannah, do you have something written there about how the pattern grows?

Clara to described what she thought was significant about this moment:

They had words to be able to describe, but they ... didn't think that would be sufficient just to write it. Kids are often—are more able to talk about it than describe it. I thought their thinking was good already. They just needed—they just needed some help in figuring out where do they go with that. So, that was about How do I get them to write it down?

Clara's response provides a vague description of the students' understanding. She seemed to think that the students were close to connecting the stage number and the number of *L*-shapes, but were simply struggling to write it down. Although I agree that verbalizing and symbolizing require different understandings, from my perspective, these students' repetitive recursive responses indicate that their main difficulty was not an inability to communicate their understanding in writing but rather that they had not shifted to an explicit understanding. Then, at my request, Clara described the students' thinking:

These folks were thinking about the two tabs and the *L*-shapes. They had an idea that the number of *L*-shapes is about the stage number. They could describe it. But then just get it to words, to get it on paper.

She initially focused on the students' view of the pattern (as opposed to their method of generalizing) and then reiterated a vague description about their abilities to describe the pattern, but not write their description on paper. Although I pressed, Clara offered

limited details of the students' understanding. In class she appeared frustrated that the students were not providing the responses she desired, but from her ambiguous response, it seemed that she had not pinpointed what was constructive about the students' thinking and what was still lacking. During the entire exchange, she did not speak about the students' recursive thinking. She did elevate the difficulty in connecting verbal and written representations, but not the transition between describing a pattern to formulating the idea in a mathematical expression, something that is much more conceptually challenging.

During the interview, I showed Clara a clip from a few minutes later, when she had reengaged with this group as part of a whole-class, teacher-led exchange. She again asked the group several times to explain how many times the pattern changes. Each time the students reiterated their recursive view, answering either three tiles per stage or one  $L$  per stage. When I asked her to reflect on the group's understanding again after this exchange, she replied, "I think that they can, they see the pattern. They can see how it is changing. They can describe how it's changing." She highlighted that "Alexis (the student who spoke for the group) knows how to describe the shape. He sees it, but he's not getting why it is important to talk about it."

Again, she did not provide a clear explanation of what the students understood or what the mathematical concerns were. In her reflection, she couched the students' thinking in more general terms: "They see the pattern." She highlighted that the students "see how it is changing," alluding to a recursive view, but failed to elevate this idea explicitly or mention that the students were unable to transition to an explicit view. Although her questioning indicated that she was observing this implicitly, neither her

explanation afterward nor her actions in class indicated that she had identified this as a primary issue.

*Deciding how to respond.* To highlight how Clara used details of student thinking to inform her instruction, I examine two instances on Day 3. I group my analysis of these two examples together because they are similar in nature. I chose these two instances because in both cases Clara provided a comprehensive description of the students' thinking but still showed no indication that this awareness guided her pedagogical decisions. In particular, she did not work to ensure that the students' insights, which were initially communicated at the group level, were then shared and used to inform the collective thinking at the class level. Such a disconnect seems indicative of Clara's overall instruction, inasmuch as throughout the unit, Clara's interaction with students in full classroom discussions did not seem to be informed by her understanding of individual students.

Both examples stem from Clara's work with two groups in which some of the students had developed a more detailed, explicit understanding of the pattern, whereas the rest of the group still focused on a recursive view. Such a disparity between students' understanding provided interesting context for analysis. Clara's comments on this contrast provided evidence of her detailed understanding of the students' thinking. In addition, the juxtaposition of ideas seemed to highlight the difficulties many students were experiencing, emphasizing the significance of the student thinking. In both cases, not only did Clara explain the difference between the students' thinking, her actions with the small group of students also illustrated that she was attempting to leverage this distinction, working with one half of the group to help the other half. Furthermore, in

her reflection on these moments, she emphasized the significance of the more developed student thinking, referring to it in one case as “Spot on, ... very significant” and in another describing the student’s understanding as “knocking it out of the park.”

Indicative of Clara’s overall instructional practices, however, although she identified the students’ thinking as constructive, she did not attempt to leverage it to shape the overall understanding of the class and support the understanding of other students.

In the classroom discussion that directly followed these two moments, instead of focusing on the two students’ productive understanding, she first called on multiple other groups who were all struggling considerably with different decompositions. Although both groups did eventually share their understanding of the generalization, their contributions were recorded in the same way as all other comments. Clara did not attempt to elevate the significance of their thinking or distinguish their ideas in any way. Furthermore, both groups shared at a time when the rest of the class was visibly confused, in effect minimizing their effect. From my perspective, the difficulties exhibited at the group level were critical issues that were the source of confusion for the entire class. Their relevance was made even more pronounced by the juxtaposition of understanding within the group. Nonetheless, although Clara had attempted to resolve the differing understandings among group members, she did not act on this with the rest of the class. Unlike Jack, who consistently used students’ understanding to shape the overall flow of ideas and understanding of the class, Clara seemed to view her engagement with students on an individual basis, never attempting to coordinate the individual and collective understandings.

*Jacks' noticing.* Jack's noticing of students' mathematical thinking was noticeably different from Clara's. The ability to monitor and eventually shape the flow of ideas in the classroom in the way he did required a very different type of practice. First, although Jack interwove other pedagogical issues into his comments, the primary focus of his reflections on all the classroom moments was the students' mathematical thinking. When Jack moved from group to group, mathematical understanding appeared to be the prominent feature of his attention. In addition, he consistently included nuances of students' understandings in his comments. For example, he distinguished between a student who was able to see the pattern visually but unable to capture it symbolically. In another instance, he characterized a student's understanding as including the ability to generalize for specific cases but not in more general cases. Probably the most significant distinction between Jack's and Clara's noticing was that Jack not only highlighted details of the students' thinking but also regularly incorporated pedagogical relevance, namely how one student's understanding related to the overall flow of ideas he was attempting to develop. As such, the three components of noticing were highly intertwined for Jack. To illustrate their interconnectedness, I provide an example of a method that Jack employed on various occasions where he would identify a particular student's conceptualization and then hold on to it to use later in class to clarify a related misconception that might arise. Such a practice required not only a detailed noticing of student thinking but also a deep understanding of the mathematics from the student's perspective to anticipate possible difficulties.

One example involved the symbolization of  $n + 2$ . At the end of Day 1, Jack presented the class with the second decomposition, which consisted of one group with

two more dots than the stage number. As I explained earlier, Jack had specifically chosen this decomposition because it targeted students' symbolizing quantities that were more than, in this case, two more than the variable. Walking around the classroom, he worked with a group in which one student, Rafael, saw the pattern as "4 times 4, plus 6," interpreting the middle group as a whole entity, whereas another student, Emma, saw it as "4 times 4, plus 4, plus 2," alluding to a view of two more than the stage number. Jack immediately identified this difference as significant, engaging the two students in a discussion at the group's table to compare and make this distinction explicit. In his reflection at the end of the day, Jack further compared the details of the two students' thinking as well as talked about ways to leverage this observation to support other students in the class. He spoke extensively about how to sequence these two students' thinking the next day so that their contributions would be most effective pedagogically. While he reflected, he drew connections to specific student thinking in other classes (a practice he demonstrated on multiple occasions), explaining that other students who saw it Rafael's way (plus 6) were able to write number sentences for Stages 4 and 5, but then struggled to write a sentence for Stage 10. Jack ultimately decided to present Rafael's "4 times 4, plus 6" view because it aligned with what most students had derived. More specifically, he explained that he hoped that students would have difficulty symbolizing so he could then call on Emma to offer her view as an explanation. Later, in his debrief of Day 2, after students, in the end, did not struggle with higher stage numbers, he talked in more detail about his strategy, actually lamenting that he did not need to execute his plan:

I thought, "Oh, when we have problems with this, there's my explanation from her...." I was counting on using Emma to have an explanation that would make sense of  $n + 2$  type of way without it coming from me. Like I almost wanted him to mess up, so I could bring Emma to save the day ... that's why I was saving that card for so long. Because I *knew* it was going to come up, and then it never came up.

After another case when Jack talked about employing the same strategy of recognizing a productive way of thinking and holding onto it for later, I asked him about it. He acknowledged this was, in fact, something he did. When questioned about his motivation, he replied that he wanted answers to not always come from him. He wanted to communicate to his students that "they have some answers within themselves; not everything just comes from me." As this reflection reveals, Jack not only leveraged nuances of student thinking to move the mathematical agenda forward but also to instill sociomathematical norms. By attending to details of students' thinking, he ensured that students saw mathematics as something they produced, he and encouraged students to look to their peers as knowledgeable resources. Relative to Clara, Jack was not only more consistent but also more proactive in his noticing, anticipating what students might think. Furthermore, he supported his detailed observations with a comprehensive plan to contextualize and productively leverage students' mathematical insights.

### **Conclusion**

The question driving the inquiry reported in this chapter of my dissertation study is at the intersection of algebraic generalization and the knowledge teachers need to respond in the moment to students' mathematical thinking. As such, the findings of this chapter provide insight into two areas: (a) the understandings that teachers need to support students in algebraic generalization and (b) the organization of knowledge that

enables teachers to effectively engage in responsive teaching. In the following section, I review the results of this chapter pertaining to these areas and examine associated conclusions that stem from these findings.

### **Teachers' Understanding of Algebraic Generalization**

Examination of the varying understandings and subsequent instructional actions of the two teachers in this study revealed two components significant for supporting students in the process of algebraic generalization. The first is the importance of students' analyzing figural patterns through the actions of both *describing* and *calculating*. Research has shown that students tend to engage primarily in calculational approaches when generalizing (Becker & Rivera, 2005; Krebs, 2003; Stacey & Macgregor, 2001) and that such an approach, involving only the search for numerical patterns, often leads to unproductive engagement with the problem (Becker & Rivera, 2005; Healy & Hoyle, 1999; Lannin, 2005; Lannin, Barker, & Townsend, 2006; Rivera & Becker, 2008; Stacey, 1989). Although many studies have shown the importance of students' verbalizing in pattern generalization (e.g., Stacey & MacGregor, 2001), the results of this study show the benefits of students' engaging in both approaches as well as teachers' helping students connect these two actions.

Clara, in an attempt to help students develop a meaningful understanding, guided students through the generalization process using a predominantly descriptive approach, encouraging her students to repeatedly describe features in the figure without calculating the values of the quantities. Without numerical actions on which to reflect, her students struggled to either develop a precise understanding of the quantities within the pattern or connect these quantities to the stage number. In contrast, Jack supported his students to



simultaneously describe and calculate the quantities within the figure, in effect combining the two actions. As a consequence, his students began interpreting the numerical expressions they were using to calculate as descriptive representations.

The contrast between these teachers' views of describing and calculating in the generalization process as well as between their students' resulting understandings highlight the prominent role teachers have in guiding the way in which students explore these figural patterns. Moreover, these results indicate the importance of teachers' supporting their students not only to engage in both descriptive and calculational approaches but also to infuse the two actions. In particular, teachers must ask students to explain what quantities in the figure their numerical calculations represent as well as ensure that they include numerical details in their descriptions. By helping students analyze patterns from both perspectives and draw connections between these two ways of thinking, teachers can support their students in developing stronger understandings of algebraic generalization.

A second component that emerged as significant was the teachers' perspectives about connecting mathematical representations, specifically their view about connecting algebraic symbols with their referents in the figure. Historically teachers have focused on what students can do with algebraic symbols (Kieran, 2007; Yerushalmy & Chazan, 2002), failing to elevate other representations or focus on the mathematical thinking that students develop in relationship with them. Over the past two decades, educators have made a significant effort to shift the focus to algebraic thinking and work with students to develop mathematical practices and associated habits of mind. The findings of this study show the importance of teachers' seeing these two entities as connected. Teachers

cannot focus on students' use of representations and their algebraic thinking separately but must combine them, with a clear understanding of the type of thinking students will develop through their use of representations.

Results in the chapter show that Clara articulated relatively specific understandings she wanted students to develop but did not include in her planning how representations would afford this type of thinking. Because she did not consider how these tools would support students' thinking and how students would ultimately think when using these tools, she did not work to develop the use of representations effectively in her instruction. Instead, her comments and actions indicated that she believed that if students were engaged in rich thinking, this thinking would become embedded in the associated representations without substantial support. As a consequence, Clara focused on multiple decompositions of the figure, thinking that through the exposure of multiple views of the pattern, students would fuse their understanding of the figure with the notation being used.

In contrast, Jack was explicit about the type of thinking he wanted students to develop with respect to the representations and how these tools supported students to think in particular ways. He not only used numerical expressions as a tool to structure how students analyzed the patterns, but also ensured that the students' understanding of the pattern was connected to the symbolic representations. Having identified the exact type of thinking he wanted students to develop, he actively fostered this understanding through using the algebraic notation by supporting students to connect each symbol to the exact quantity in the figure.

Drawing on Jack's expertise, I conclude that the role of representations must be made explicit in teachers' understandings of mathematical domains. In addition to identifying the conceptualizations they want students to develop, teachers must have detailed knowledge of how representations will support the desired way of thinking and how that way of thinking will ultimately be connected and conceptualized in different representations. In the particular domain of algebraic generalization, such awareness is critical.

### **Teacher Knowledge for Responsive Teaching**

The final conclusion based on results from this chapter relates to the organization of the knowledge one needs to effectively engage in responsive teaching. First, results from the final section indicate that the term *responsive teaching* is a bit of a misnomer. Although teachers responded to students' comments in the classroom, teachers, at least experts in responsive teaching, do not fully react in the moment. As Jack's data show, much of such experts' decision making is premeditated. In their planning, these teachers anticipate the type of thinking in which students will engage, both productive and unproductive, and imagine how to sequence the ideas to effectively support the collective learning of all the students. Each day before class, Jack identified specific ways of thinking he wanted to identify and elevate. He described particular scenarios that might arise and how he would use different anticipated student ideas in these situations. After class, Jack explained how he had effectively managed the flow of ideas according to his plan. Therefore, much of his noticing of students' mathematical thinking was his implementation of this plan according to the details of student thinking that emerged, which were ideas he had preidentified.

To engage in this type of instruction requires not only specific knowledge of students' thinking but also a particular organization of this knowledge—a structure around a connected and continuous trajectory. The two teachers in this study planned, to differing degrees, for how student ideas would develop: Clara had planned around two specific ways of thinking, indicating that her understanding of the generalization process was organized as discrete elements. She worked to develop these two ways of thinking independently and at distinct times during the unit, indicating that she saw them as separate understandings. Interpreting students' thinking through an isolated lens, she failed to situate their mathematical ideas within a larger mathematical landscape. Consequently, she struggled to identify salient mathematical contributions and support students to productively move forward. Instead, she responded by funneling ideas to one of the two specific ways of thinking she had inferred.

In contrast, Jack's image of instruction and of students' thinking was as a path. He had an ultimate goal for the unit, but he envisioned students' thinking in multiple ways along the way. Moreover, he indicated that he saw learning as a journey in which one progresses along this path. Consequently, he interpreted students' ideas along a spectrum, appreciating details of their mathematical contributions relative to a larger instructional framework. With this understanding, Jack engaged with students not to funnel their thinking but rather to build on and develop their understanding in the desired direction. Although having many pedagogical tools to support students distinguished Jack from Clara, the key to his success seemed to stem from his ability to contextualize the students' thinking within a larger trajectory of ideas.

One significant pedagogical implication that follows from this result involves how we as mathematics teacher educators support teachers. Currently, most curricular materials are organized around separate, daily topics. As a teacher for many years, I was instructed to plan my lessons much as Clara did, around specific, discrete goals. Such a focused instructional perspective seems to structure teachers' knowledge around discrete mathematical understandings. Although such an organization might be suitable for a direct instructional model, as the results in this chapter highlight, it is not a productive model for responsive teaching because it promotes a more isolated vision. In an instructional environment in which mathematical ideas stem from student-centered activities, teachers must possess the ability to interpret student thinking along a spectrum to identify and respond to the nuances of the emerging mathematical ideas. Consequently, I argue that we need to move away from thinking of mathematical understanding as consisting of isolated topics and support teachers to develop MKT organized around trajectories. Such a change will require a significant shift for most educators, but it is a necessary one if they are to successfully engage in responsive teaching.

## **Chapter 6: Conclusions**

The results of this dissertation study contribute to the field by broadening our understanding of teachers' Mathematical Knowledge for Teaching (MKT) in the area of algebraic generalization. In the following section, I present a summary of my findings, organized around the three research questions I sought to answer. With the review of results of each question, I present what I see as the significance of each finding, in terms of both the theoretical and pedagogical implications. I then discuss limitations to the study, followed by questions and future research directions that stem from my work.

### **Summary of Findings and Significance**

#### **Interpretation of Generalizing Tasks**

My first area of investigation in this study was an exploration of the goals teachers associate with figural generalizing tasks. In analyzing the teachers' responses, I identified two goal categories. First, 3 of the 4 teachers saw these activities as predominantly enrichment. They specifically distinguished the activities from their normal content-driven instruction and used them to inspire thinking, develop norms, and encourage participation, but did not associate them with their core curriculum. In contrast, the fourth teacher associated these tasks with a unit on linear functions, but approached them with a calculation orientation. She used these figures to highlight specific features in the various associated representations and to introduce a method to convert these components into a symbolic expression. Although the two interpretations seem quite distinct, one unifying characteristic is that the act of generalization was not connected to mathematical content. In the former, the teachers saw the act of generalization as distinct from the mathematics in their curriculum, and in the latter, the

teacher had effectively removed generalization from the task.

Reflecting on this result, I find significant that teachers, even those with considerable experience and professional development in the area of algebraic generalization, did not possess a clear understanding of the mathematical content associated with these figural patterns. This discovery seems to have direct pedagogical implications. I know that many mathematics teacher educators (MTE), myself included, encourage teachers to incorporate these generalizing tasks in their instruction. They provide these as examples of tasks with a “high cognitive demand” (Henningesen & Stein, 1997) or with a “low floor-high ceiling” (Gadanidis, Hughes, Scucuglia, & Tolley, 2009) and promote them in an effort to give students enriched mathematical learning experiences. In light of these results, I believe that MTEs must be explicit with teachers in describing the mathematical outcomes associated with these generalizing tasks, helping them see strong connections to the content they are charged to teach. In the absence of clear mathematical goals, teachers will create their own, but they may do so without elevating key conceptual ideas. Specifically, if generalization is at the core of algebra, I argue that teachers must be supported to see developing a meaningful quantitative understanding of the notation as a central goal of algebraic generalization.

Finally, from a theoretical perspective, I believe that this finding highlights a form of knowledge not well explored in the field. While exploring the question of what teachers understand about a task, I attempted to situate my query within the knowledge literature but struggled to find a model of MKT that addressed the specific type of knowledge related to teachers’ mathematical goals. My interpretation is that those in the field have created models that begin with a particular mathematical topic and detail the

types of knowledge necessary to support students to develop a strong understanding of this topic. In contrast, I see key professional knowledge as the ability to work backward and identify the mathematics that students can learn from a particular activity. While more and more teachers are being encouraged to use rich, open-ended tasks (and while they explore various sources to find them), we educators need to study the professional knowledge necessary to interpret the relevant mathematics within a task.

### **Quantitative Understanding of Algebraic Notation**

The second area of MKT I explored in this study is a type of Specialized Content Knowledge (SCK) relatively specific to algebraic generalization, namely a quantitative understanding of algebraic notation. Findings indicate that whereas teachers were able to provide a general interpretation of algebraic notation to describe the contextual meanings of symbols in isolation, they struggled to create detailed conceptualizations to explain quantitatively the operations, specifically multiplication and addition, on the quantities represented by the symbols. For multiplication, teachers were challenged when describing the meanings of a variable and its coefficient. Issues with respect to addition surfaced for combining variables that appeared twice in a single expression.

Teachers, in their efforts to overcome these challenges, generated both productive and unproductive conceptualizations. In general, teachers generating productive conceptualizations were more flexible in their approaches and strove to create more nuanced and consistent interpretations of the symbols than other teachers. When explaining the relationship between the variable and coefficient, those generating productive conceptualization were able to define each symbol in a way that fully separated their meanings, identifying the variable as the number of groups and the



coefficient as the size of the groups (or vice versa). Likewise, when adding  $5x$  and  $x$ , they were able to transform their conceptualization of the isolated  $x$  from the number of dots to the number of groups of size 1. They did so by including a coefficient of 1 to produce a consistent form among all variables and then interpreting the 1 as the size of the group in correspondence with the other coefficients.

On the contrary, unproductive conceptualizations tended to be characterized by an oversimplification of the quantities involved. When trying to make sense of the coefficient and variable, one teacher detached the meaning of the symbols from the details of the context, interpreting both as simply dots. Later, when attempting to coordinate the addition of the variables, the teacher substituted one interpretation for another as a direct translation. In both cases, her treatment of the variables produced a more abstract interpretation of the symbols that seemed to enable her to rely more on algebraic rules than contextual meanings. Such conceptualizations did not promote a deeper understanding of the situation, and, thus, would likely not be as supportive in work with students.

I believe that one implication of this study is the importance of exposing teachers to problems with quantitative complexities and pressing them to clearly articulate the relationships between the algebraic notation and the figure. In this study, when the participants were asked to clarify details of this connection, the process of overcoming these hurdles seemed to develop in them a more nuanced understanding of the algebraic notation involved. Even when the teachers initially did not understand the relationship, the act of looking for consistencies and verbalizing their thinking helped them to make connections. I conjecture, but do not claim as a study result, that clearly defining this

relationship not only elevates the quantities involved but also helps to foster a deeper understanding of the related quantities themselves. To develop a quantitative explanation of how symbols interact requires grappling with the meanings of the quantities themselves.

Another contribution of this segment of the study is the identification of exact areas on which MTEs can focus their attention to help teachers develop this SCK. Results indicate that teachers should be asked to grapple with the meanings of the variable and coefficient. To promote this grappling, I suggest that teachers be exposed to a variety of contexts in which the number of groups and the size of the groups are represented by the variable and coefficient, respectively, and also to contexts in which these roles are reversed. In addition, the findings indicate that teachers should be exposed to problems in which they must combine variables and be asked to explain the meaning of this operation in contextual terms, as opposed to simply relying on syntactic rules. By engaging in discussions about their interpretations of these specific situations, teachers will develop insight into difficulties their students face in learning to use and make sense of algebraic notation as well as gain knowledge to support students to overcome these challenges.

### **MKT for Algebraic Generalization**

In the last part of this study, I aimed to identify features that distinguish two knowledgeable teachers engaged in responsive teaching in the domain of algebraic generalization. By studying data from these two teachers, I was able to highlight key features of the type of expert knowledge necessary to support students in developing rich understanding of algebraic generalization. By comparing the teachers' anticipated

learning outcomes, instructional decisions, and noticing of student thinking, I identified three significant characteristics that distinguished the teachers' MKT in this domain: (a) coordination between describing/calculating and symbols/context, (b) decomposition of instructional trajectory, and (c) anticipation as part of professional noticing. In addition, on the bases of these differences in the details and organization of the two teachers' collective understanding, I proposed a newly conceptualized framework for MKT for algebraic generalization.

**Coordination between actions and representations.** The first feature that differentiated the two teachers was their conceptualization of the role of and relationship between the two actions, describing and calculating, and the role of and relationship between two types of representations, symbolic and contextual. Although both teachers described the two entities within each of the two sets as different, one teacher (Jack) worked to emphasize and develop both entities within each pair and possessed instructional strategies to bridge the gap between them. In particular, he ensured that students included calculational details with their descriptions of the figure and that they provided clear descriptions of what their calculational symbols represented. Likewise, he had students draw clear connections between elements of the symbolic representation and the quantities represented in the figure. In the end, this more integrated instructional approach correlated with the students' successfully engaging in the generalization process as well as ultimately developing a rich quantitative understanding of the algebraic notation.

One direct implication of these results is the importance of teachers' supporting students not only to develop but also to explicitly infuse these two actions and

representations. MTEs who use these generalizing tasks with teachers need to present a comprehensive perspective to algebraic generalization such that these multiple actions and representations are integral to the task at hand. Most significantly, they need to support teachers in finding ways to draw connections between these different areas. Results indicate that an in-depth understanding and successful engagement with the process of algebraic generalization lie at the intersections of these actions and representations. With that in mind, I propose that the most salient contribution of this section is the specific method by which Jack facilitated this connection. Analysis indicates that the key to Jack's success in bridging the gap between these actions and representations was a particular content-specific pedagogical move he implemented throughout his instruction. This method was centered on the use of number sentences to express the perceived structure of the figure. When asking students to analyze a new stage, Jack consistently asked them to use a number sentence to express their understanding of the pattern and then followed up by asking them to explain the meaning of each symbol in terms of the figure.

The use of this pedagogical move supported students' understanding in numerous ways and as such has significant pedagogical implications. First, it serves to shift students' thinking from recursive to explicit. To create a number sentence, students must begin to see and articulate the relationship between the various figural pieces and the stage number. Second, it infuses the students' descriptive understanding of the figure into the mathematical symbols, transforming the numerical and algebraic expressions into representations that communicate the quantities in the figure. When students state how they see the symbolic expressions representing quantities in the figure, they embed their

descriptive understanding of the pattern in the symbols. Third, this pedagogical move provides a method for linking the mathematical symbols to the context at a detailed level. As such, these often implied connections are not left to students to make (or fail to make) on their own but are explicitly communicated through this instructional exchange.

A theoretical contribution of this portion of the study was to reconceptualize Jack's use of number sentences, with the aforementioned pedagogical affordances and the associated shift in student thinking, as a pedagogical content tool. Viewing these number sentences as a pedagogical content tool provides a powerful lens for teachers and researchers to see the importance of these mathematical representations. Rather than considering them as representations that organize concluded thoughts and understanding, they can be conceptualized as a tool that can be leveraged to support students to grapple with more sophisticated mathematical concepts. I believe that the notion of pedagogical content tool can be expanded to other representations associated with algebraic generalization, such as a table, elevating their potential as instructional instruments that can support students to engage with challenging mathematical concepts.

**Decomposition of instructional trajectory.** A second feature that distinguished the two teachers was the trajectory of the targeted ways they each envisioned and in which they supported students to engage as part of the process of algebraic generalization. Comparing the two, Jack not only identified more specific areas of focus than did Clara, he also possessed a more detailed and interconnected understanding of these various area of focus. As an apparent consequence, the students' learning experience in Jack's class was more seamless; most of Jack's students successfully generalized the various patterns presented and connected their algebraic

expressions to the figure. In contrast, the vast majority of Clara's class struggled to move beyond a recursive view of the pattern, and, in the end, no student in her class was able to generate a meaningful algebraic expression.

By comparing these two teachers' understanding of the process through which students progress while learning to generalize symbolically, I devised a new instructional-trajectory framework for algebraic generalization. Although the final model mirrors Jack's model, the significant details emerged only through a relative comparison with Clara's model. The resultant framework consists of two continua. One denotes the various symbolic representations used to express mostly calculational aspects of the pattern. The second continuum presents different ways of describing the quantities perceived within the figural pattern as well as the generalized relationships between them. In addition, along each continuum, various instructional benchmarks associated with these representations have been identified. While students progress through these benchmarks, their understanding of the associated representations, be they numerical symbols or verbal descriptions, becomes more abstract and more compact in nature. Of note, although the two continua are presented as discrete pieces, a significant characteristic of this framework is the connections teachers support students to make between the two. When these connections grow closer and more integrated, eventually the different acts and representations merge, with the previously calculational tools becoming reconceptualized as descriptive representations of quantities in the figure.

***Theoretical contributions of the trajectory.*** The resulting instructional trajectory encapsulates three main theoretical contributions. First, the framework is an addition to the growing number of learning trajectories (LT) being developed in the field, and one of

the few at the secondary level. Its most distinguishing characteristic is that the included descriptions of students' thinking are characterized relative to various associated representations. Consequently, it incorporates a new conceptualization for LTs, one built around students' strategies and their understanding of relevant mathematical representations. To situate the LT theoretically, I note that the descriptions of the various instructional benchmarks are not characterizations of students' schemes (von Glasersfeld, 1995) but are instead accounts of students' observable behaviors. As such, I believe that the unit of analysis is at a level useful to guide the planning and decision making of a teacher (as I explain later) instead of a level at which a researcher attempts to understand the details of the fine-grained learning process involved.

A second theoretical contribution of this study is the method used for generating the LT. Unlike the derivation of past studies, its derivation stems neither directly from the study of students nor from testing the effectiveness of a researcher-designed teaching experiment. This framework results from a study of the teachers' understanding of student thinking during the act of instruction. Hence, it is a model of student thinking that is based on teachers' interpretations but organized through the researcher's perspective and understanding. Such a method offers researchers many affordances. First, studying expert teachers offers unique insight into the domain by leveraging their wealth of experience and knowledge. Second, observing live classrooms provides an authentic context with results that are grounded in the realities of that particular environment. Third, comparing and contrasting multiple teachers serves to elevate potential contributing factors.

A third theoretical contribution of the resulting framework is as a variation in the

conceptualization of algebraic generalization, one in which the fundamental role that representations play is integrated. By highlighting the different ways representations are used in the generalizing process as well as key connections between them, I describe a trajectory that serves to ground the repeated reasoning necessary for students to generalize in the associated representations. As such, algebraic generalization becomes conceptualized as the iterative process of connecting numerical (and eventually algebraic) symbols to the quantities in the figure through the acts of calculating and describing. By repeatedly coordinating these actions and representations, students are supported not only to generalize the pattern but also to develop particular ways of thinking in terms of the symbols. Ultimately, this process of algebraic generalization leads students to understand the resulting notation as both a tool to calculate the value of the pattern and a form of communication to describe the quantitative structure of a particular decomposition of the pattern.

***Pedagogical implications of the trajectory.*** In practical terms, this framework provides teachers attempting to support students in algebraic generalization specific intermediary areas on which to focus their instructional attention. The identification of these transitional ways students can engage in algebraic generalization supports teachers in a variety of ways. First, such a decomposition helps teachers expand their instructional vision beyond right and wrong answers. As Wilson, Mojica, and Confrey (2013) have shown, such frameworks can support teachers in anticipating student thinking when the teachers begin to view students' informal ideas as more meaningful. As such, the framework elevates in teachers' minds the relevance of particular areas of the generalizing process of which they may have been less aware, helping them to more



effectively notice and interpret the value in students' mathematical thinking.

Furthermore, the progressive nature of the trajectory supports teachers in navigating the variety of student contributions, helping them to decide which ways of thinking to pursue as well as the possible sequencing of student ideas.

Second, the unique composition of the trajectory into various actions and representations affords a particular lens and pedagogical understanding of algebraic generalization. It helps teachers not only attend to how students are thinking in general but also specifically the way they are using and interpreting the associated representations as well as the connections they are making among them. Moreover, the structure of the trajectory provides specific ways in which various mathematical representations and connections between representations support algebraic generalization. As such, the trajectory provides not simply a map to help teachers recognize and organize ideas generated by students but also ways teachers can leverage representations to create situations that might motivate particular ways of thinking.

Finally, I argue that engaging with the framework will develop teachers' MKT associated with algebraic generalization. Helping teachers attend to and interpret particular facets of students' thinking will help them, in turn, foster a deeper understanding of the mathematics. As such this framework has a reciprocal nature, providing a lens to better navigate the mathematical and pedagogical landscape while facilitating a more refined knowledge of algebraic generalization from the students' perspective.

**Anticipation as part of expert professional noticing.** In addition to the structure of the two teachers' MKT, the last feature that distinguished the two teachers

was their implementation of this knowledge through their professional noticing of students' mathematical thinking. Although the teachers' demonstrated similar degrees of detail when describing student thinking, Jack seemed more intentional in his practice of noticing, searching out particular ways of thinking and consistently using details of student thinking to guide his instructional decisions. Furthermore, Jack incorporated details of student thinking in his planning. Imagining how students might engage with the mathematics at a detailed level enabled him to not only quickly identify and build on student thinking but also to create instructional situations in which particular ways of thinking might arise in the classroom. In contrast, Clara did not actively anticipate the development of student thinking at a detailed level when planning her lessons. Consequently, Clara seemed to be reactive when interpreting and responding to student thinking, missing the same opportunities Jack used to build on and extend her students' emergent ideas.

Drawing from this comparison, I suggest that the ability to effectively anticipate student thinking is an integral component of expert professional noticing. Noticing student thinking at an expert level involves not simply noticing particular ideas when they emerge but also proactively searching out specific ways of thinking, even creating circumstances in which they might occur. Such premeditated noticing in the lesson-planning process enables teachers to quickly respond to and leverage relevant student ideas.

This observation has theoretical and methodological implications. For one, the well-established framework for professional noticing of students' mathematical thinking proposed by Jacobs et al. (2011) comprising of attending to, interpreting, and deciding

how to respond might be expanded to incorporate *anticipating* as well, at least in the study of expert professional noticing. Furthermore, to explore how anticipating is related to the other three subpractices, I expect that studying an authentic teaching environment would be a more productive than conducting clinical interviews because anticipating student thinking seems situated in nature. In fact, this finding indicates that professional noticing itself might be a rather situated pedagogical knowledge. For example, noticing the student thinking of the eighth-grade students from high-needs schools in this study requires a different understanding than is needed in the noticing of a different set of students with distinct mathematics backgrounds or within other learning environments. Although content-specific professional noticing has been investigated, the role of context has not.

**One final reflection.** Reflecting on this chapter in totality and my overall experience working with these teachers, I see one last significant implication, or more precisely, one question. These were highly experienced teachers. They not only were members of a researcher-led, long-term professional learning community but also had participated in a week of rich, content-specific professional development. And yet, even under these almost ideal situations, although one teacher was successful, the others struggled to engage their students in a meaningful understanding of algebraic generalization. No doubt, the instructional environments and student populations were major contributing factors. These were large classrooms of students who, for the most part, had yet to experience success in mathematics. But such a situation is not unique. From my experience in and out of local schools, these are the typical environments of our low-income, inner-city schools.

Although this study was of only a small number of teachers, and extrapolating from these results to the general state of mathematics education is impossible, I think that the results raise a serious question for our field and for mathematics teacher educators who are charged to support teachers working with such high-needs populations. Specifically, how do we prepare teachers, both preservice and in-service, to develop the deep mathematical and pedagogical understanding necessary to effectively support students in these environments. Observing first-hand the difficulties of these teachers attempting to effectively respond to the needs of all of their students, develop a rich mathematical understanding that grows from their collective experience seems like a daunting challenge. At the same time, if we genuinely want to effect change and reach out to those students who have been historically marginalized, it is a question we must take seriously.

### **Limitations**

My aim in conducting this study was to investigate the MKT of various expert teachers in the area of algebraic generalization. Exploring this unique set of teachers provided detailed insight into the type of knowledge necessary to support students in developing a strong understanding of algebraic generalization as well as many of the associated challenges. Nonetheless, although the research design proved effective, I encountered several associated constraints. First, having additional data points would have been useful for answering the first two research questions about the goals and quantitative understanding of these teachers. Because of the small number and high knowledge levels of the participants, I am unable to generalize about the larger population of teachers. In particular, although two classifications of goals and two

specific difficulties associated with the teachers' quantitative understanding of the notation emerged, I cannot speak to the consistency of these findings as related to the general population or suggest what other categories might be found in studying less experienced teachers. Essentially, what I gained in depth of analysis in Part II of my study, I lost in breadth in Part I.

Regarding my analysis in Part II of the teachers' enacted knowledge in the classroom, three main limitations in the research design arise. First, whereas focusing the camera exclusively on the teacher throughout the study provided me awareness of the classroom from the teacher's perspective, it limited my understanding of the class's thinking as a whole. As a result, I had access only to the comments, questions, and contributions that were directed to the teacher. What other students said or did that was not verbalized to the teacher was not available for analysis. Consequently, I cannot speak to the students' overall understanding or method of engagement. Analyzing teachers' knowledge, and, in particular, their noticing, relates to understanding not only with which ideas teachers choose to engage but also information on which they choose not to focus. Therefore, I encourage researchers using a similar method in the future to place a small recording device on each student table and to collect student work from all students to provide a comprehensive view of the classroom.

In addition, I encourage future researchers to assess the students' understanding in some way. Part of my analysis of the teacher's MKT was based on the students' abilities to engage in algebraic generalization and their progressive understanding. Pre and post assessment of students' understanding would provide detailed information about the effectiveness of the teachers' instructional path and support. Furthermore, with these

data, one could compare effects of the teachers' understanding and classroom instruction on the students' understanding of algebraic generalization.

A second limitation concerns the number of classroom visits in the research design. Expanding the analysis to three or four visits would have provided a wider perspective of the teachers' understanding of algebraic generalization. In particular, my analysis of Jack's teaching included only a portion of his instructional trajectory after he had completed an initial exploration of the pattern. Therefore, I was unable to speak in detail about his understanding of the beginning areas of the trajectory. A related limitation is that each stimulated-recall interview related to only five or six discrete moments; inevitably other moments would have provided a more comprehensive understanding the teachers' MKT. Moreover, with the exception of one or two instances with each teacher, the clips analyzed did not involve whole-class discussions. Consequently, the method led to considerably more data points from small-group interactions than from large-group conversations. Without teachers' comments from all classroom moments, I inferred some understanding from the teachers' actions rather than from their own reflections.

My final concern is technical in nature. I used a camera based on technology more than 10 years old. The clips it produced were rather short and poor in quality. Consequently, when reviewing clips, the teachers often preferred watching video from the stationary classroom camera. This approach provided information beyond the 30 seconds recorded by the headcam but did not give the same perspective. Moreover, because time between filming and the stimulated-recall interview was insufficient for me to properly edit the video and splice the separately recorded sound, teachers had to watch

the video and then listen to the sound separately. Newer cameras can store footage the length of the entire class as well as tag certain clips, allowing for analysis of longer units exclusively from the teacher's perspective.

### **Directions for Future Research**

Several areas of future work particularly interesting to me grow out of the findings of this dissertation. Moving forward, I would like to test the consistency and replicability of this framework by conducting a teaching experiment with eighth-grade students similar to those in this study—a study in which I lead instruction using this framework as a guide and attempt to purposefully move students through the instructional areas identified. I would like to analyze the framework from both the teachers' and students' perspectives. From the teacher's perspective, I would report on how useful this framework was for me, the teacher, while I led the instruction. In particular, I would examine how it informed my noticing (attending, interpreting, and deciding how to respond). I could use a research design similar to the design of this study, but with a colleague interviewing me before and after each class, or I could simply reflect on my own experience and comment on the ways in which the framework was or was not useful.

To study the effectiveness of this framework from the students' perspectives, I would try to document to what degree students flexibly move through the framework when instruction, questioning, and pedagogical decisions are designed to encourage their progression. I would try to answer some of the following questions: (a) What struggles and challenges do students experience? (b) What other conceptual benchmarks need to be added to the framework? (c) Do other instructional interventions come to mind to support students when they attempt to build a quantitative understanding of the notation?

I would also incorporate spaced student interviews in the study to capture the progression of student understanding throughout the process instead of simply measuring it in whole-class discussions. In particular, I would present a new pattern in a final interview to learn which benchmarks students proceed through without instructional intervention and what type of understanding of the notation, both numeric and algebraic, they possess.

In a second area of future study, I would introduce this framework to middle school teachers and explore how this trajectory supports their teaching. In particular, I would place special emphasis on the use of numerical expressions as pedagogical content tools. My goal would be to report how the trajectory supported these teachers in their practices and how it affected their noticing of students' mathematical thinking.



## **Appendices**

### **Appendix A: Pre-Interview Protocol (part I)**

This first part of the interview is about your professional background.

1. Let's talk about your path to becoming a teacher.
  - a. Is teaching your first profession?
  - b. When did you know that you wanted to be a teacher?
  - c. A mathematics teacher?
2. What did you study in college?
3. How long have you been teaching, at what schools and at what grade levels or subjects?
4. Would you recommend teaching to a good friend or to one of your children? Why or why not?
5. Are there other aspects of your teaching life that are significant to your professional identity? For example, your serving as an administrator, a counselor, a coach, as a department chair, in a union, as a guide teacher, as a professional developer, etc.?
6. I know that you are currently serving as a Noyce Teaching Fellow, which you started about 1.5 years ago. Would you recommend this project to a colleague, and if so, why?
7. Have you been involved in other professional development experiences?
8. I know that you have a master's degree. Can you say a little about that experience and about what you learned? ... Would you say that your graduate work supported your professional development, or was that not really a focus of your studies?
9. Is there anything else you would like to add related to your background or your professional life?

This next part of the interview will focus more specifically on algebraic tasks, looking at generalizing tasks. Some of these questions may be pretty specific, and I don't expect you to have thought about them all to the same extent. If you do not have much to say about some of these, feel free to tell me that and we'll move on.

1. What are your goals in a unit focused on generalizing figural patterns?
2. Imagine that you were free to determine the curriculum. Would you include generalizing figural pattern tasks? Why or why not?
3. Imagine a colleague is thinking about incorporating some of these tasks and asked you what mathematics students would learn from these tasks. What would you say?

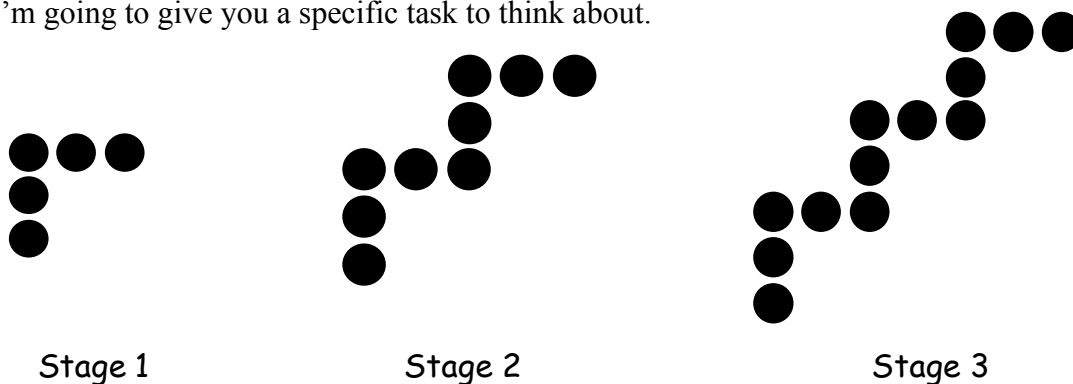
*Follow up:* Can you be specific about the content?

*Follow up:* How do you see these tasks developing an understanding of \_\_\_\_\_?

4. Do you see your instruction in these tasks different from how you normally teach? How so? Why do you think you teach this differently?

## Appendix B: Pre-Interview Protocol (part II)

I'm going to give you a specific task to think about.



1. How many dots does stage 4 have?  
*Follow up: Can you explain how you came up with your answer?*
2. How many dots does stage 10 have?  
*Follow up: Can you explain how you came up with your answer?*  
*Can you describe what stage 10 looks like?*  
*If they provide a number sentence, ask them what various numbers represent?*
3. Can you find a general rule for the number of dots in a particular stage?  
*Follow up: Again, I am curious how you came up with your formula.*  
*Follow up: Imagine that you wanted to explain your formula to someone who had thought about this pattern differently than you had. What would you say?*
4. Ask them to explain specific pieces of their notation. For example,  
 “Can you explain what the 4 represents? The  $n$  represents? The  $4n$  represents?”

If their explanation of the notation does not make use of the diagram.

Is there any way of thinking of the 4 in terms of the diagram? What about  $n$ ?

5. Is there another way you can think about the pattern?
6. Can you use this approach to figure out how many dots are in stage 10? In the  $n^{\text{th}}$  stage?

7. Again, explore how they are thinking about specific pieces in their expression.  
“Can you tell me what the \_\_\_\_\_ represents?”

- A. I have some student work to show you of the same pattern. These were 9<sup>th</sup> graders in an Algebra class. I want you to look through student A's work first. *Give teacher only page one (parts a-c) of student A's work.*
1. Can you tell me what stands out for you or what you notice?
  2. Can you tell me what you learned about this student's understanding?  
*Follow up:* If the participant does not include an explanation of the student's method, ask: Can you explain how you think the student solved part c)?
- B. Show part d) of student A's work and ask questions 1-2 again.
- C. Show part e) of student A's work and ask questions 1-2 again (without the follow-up)
- D. Show student B's work a)-c) and again ask the questions 1-2.
- E. Show student C's work a)-c) and ask questions 1-2.

Student C wrote the formula  $5 \times s - (s - 1)$ . What do you think the  $s$  represents for this student? What do you think the 5 represents for this student? What do you think the  $s - 1$  represents for this student?

If the participant does not make any comparisons to the previous student, ask: "How do you believe student C's work compares with student B's approach? Are there any similarities or differences?"

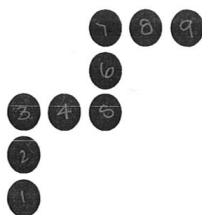
- F. Write out  $5s - (s - 1) = 5s - s + 1 = 4s + 1$
1. If we have not previously talked about the formula  $4n + 1$ , ask: In this form, what do you think the 1 represents?
  2. If we have already discussed  $4n + 1$ , ask about the  $s$  in " $-s$ " in the expression  $5s - s + 1$ : "In this intermediate stage of simplification, can you think of anything the  $s$  represents?"  
*Follow-up:* "Could it represent anything in the figure?"
  3. Move onto the 1 in the expression  $5s - s + 1$  and ask, "In this intermediate stage of simplification, can you explain what you think the 1 represents?"

## Student A

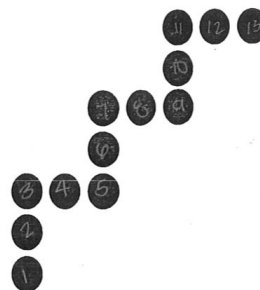
## The Snake Pattern



Stage 1



Stage 2



Stage 3

- a) Determine how many dots are in stage 4 and explain how you know.  
You may use diagrams and/or words to help clarify your idea.



I think there would be  
17 dots in total. The  
Pattern repeats itself  
self by adding 4  
But it starts with 5.

| stage 1 | stage 3 |
|---------|---------|
| 5 dots  | 13      |
| stage 2 | stage 4 |
| 9       | 17      |

- b) Describe in words what stage 10 would like.

Stage 10 would be  
41 dots, because stage  
10 you would keep  
adding 4

- c) Determine how many dots are in stage 10 and explain how you know.  
You may use diagrams and/or words to help clarify your idea.



It would 41 dots

- d) How many dots would stage 37 have? Explain how you know.

$$\begin{array}{r} 27 \\ \times 4 \\ \hline 108 \end{array}$$

$$41 + 108 = 149 \text{ dots}$$

I multiplied 27 representing the dots times 4 being amount of dots.

- e) Can you come up with a rule for the number of dots in any stage, say stage "n"? You can state the rule in words or write it as an algebraic expression. Again, please explain how you came up with your rule.

$$n = \text{stage}$$

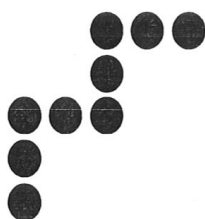
$$(n) 4 + 5$$

↑ ↑ ↑  
stage dots dots you initially start

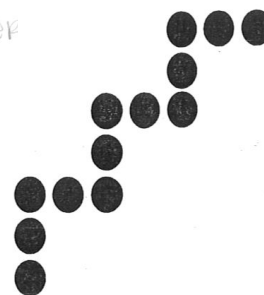
When you multiply the stage by 4 that gives you a amount of dots. But then you'll have to add 5 because of the 5 initial dots

**Student B****The Snake Pattern**

Stage 1



Stage 2



Stage 3

- a) Determine how many dots are in stage 4 and explain how you know.

You may use diagrams and/or words to help clarify your idea.

There are 17 dots in stage 4 because since there are 5 dots in stage 1, you multiply  $5 \times 4$  which is 20. Then you subtract 3 because there is going to be 3 corners, which are connecting the dots.

- b) Describe in words what stage 10 would look like.

stage 10 would look like 10 dots of 3 going up. And 10 dots of 3 going across each column.

- c) Determine how many dots are in stage 10 and explain how you know.

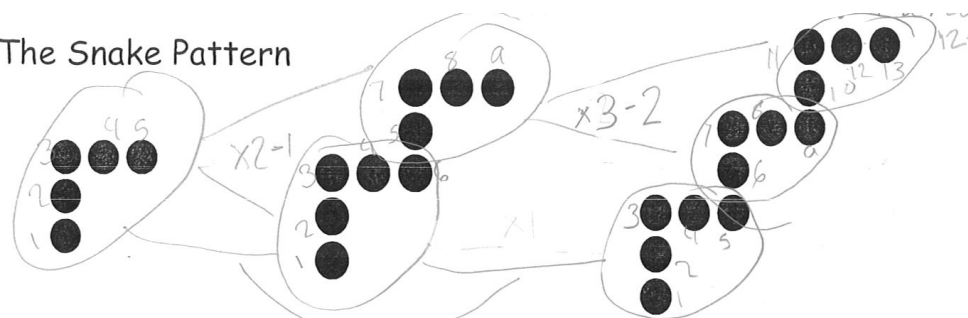
You may use diagrams and/or words to help clarify your idea.

There are 41 dots in stage 10 because you multiply  $5 \times 10$ . You multiply it by 5 because it is the original number of dots in stage 1. Then you get 50. You then subtract 9 because you have 10 subtract the number you get after subtracting 1 from the stage number. So the answer is 41 dots.



## Student C

## The Snake Pattern



Stage 1

Stage 2

Stage 3

- a) Determine how many dots are in stage 4 and explain how you know.  
You may use diagrams and/or words to help clarify your idea.

There are 5 dots in pattern 1 and in stage 2 its doubled but -1. In pattern 3 stage was tripled but -2. So then I presume stage 4 will be stage 1 but  $\times 2 - 3$ . so stage 4 would have 17 dots because  $5 \times 4 - 3 = 20 - 3 = 17$  dots.

- b) Describe in words what stage 10 would like.  
 $5 \times 10 - 9$  it is -9 because I noticed that the number you subtract is one less the number of the stage so  $5 \times 10 - 9 \rightarrow 50 - 9 \rightarrow 41$  dots  
stage ten would have stage 1 as a base and then it would look like stage 3 but longer because its stage 10.

- c) Determine how many dots are in stage 10 and explain how you know.  
You may use diagrams and/or words to help clarify your idea.

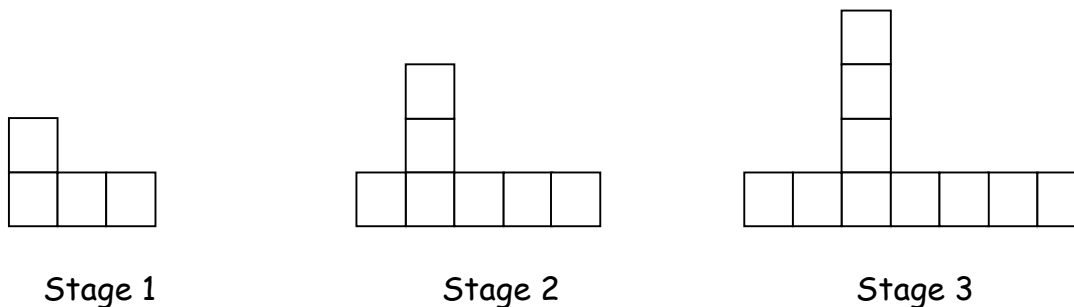
There are 41 dots in stage 10 because of the equation I thought of

$$5 \times S - (S - 1)$$

↑  
number of dots in base

↑  
stage number

↑  
1 less than S



A. How many dots would stage 10 have?

*Follow up:* Can you explain how you came up with your answer?

*Can you describe what stage 10 looks like?*

*If they provide a number sentence, ask them what various numbers represent?*

B. Can you find a general rule for the number of dots in a particular stage?

*Follow up:* Again, I am curious how you came up with your formula.

*Follow up:* If you were working with someone else who got a different answer, how would explain your function so that they would be convinced yours is the correct answer?

C. Ask them what the various pieces of their notation represent.

1. “Can you tell me what the 3 represents?”

2. “Can you tell me what the  $x$  represents?”

3. “What about the  $3x$ ?”

*Follow-up if their understanding of any of these is not connected to the diagram:*

“Do you think that the 3 could represent something in the figure?”

“Can you show me exactly where you see the 3 in the diagram?”

3. Ask specifically about the constant term. “What about the 1? What does that represent?”

*Follow-up:* If they allude to the diagram, but are not specific, ask them, “Can you show me exactly where you see the 1 in the diagram?”

*Follow-up:* If they don’t connect the constant term to the diagram, ask: “Can you show me what that might be in the figure?”

### Appendix C: Post Professional Development Interview Protocol

- I.
  - a) I have a series of questions to ask you, but before we get started, I was just curious, what has been going through your mind as you have been thinking about this unit since last week?
  - b) Also, are there things that you are hoping that I will help you think about today?
- II.
  - a) Imagine you were trying to convince your department (administration) to include a unit on pattern generalization in the 8th grade curriculum. What would you say?

*Follow-up:* How has your answer changed over the course of this PD?

*Follow-up:* Can you state explicitly what mathematics you believe students will learn from these tasks?

*Follow-up:* A colleague comments that these problems are not on the Smarter Balance test. How would you respond?

*Specific Follow-up:* I asked a similar question in the pre-interview and on Thursday. Your answer was: (Read a synopsis of their previous answer) Could you please explain what you mean by this?

- b) A few minutes ago I asked you to imagine that you were trying to convince your colleagues to include a unit on pattern generalization, and you talked about what you would say and why. My question now is a little different. What value do you see in such a unit? And has that value changed for you recently?
- c) Can you bring a particular task you will be using so we can talk about your selection of that task.

In what way does this task help meet your goals for the task and the unit?

As you thought about selecting this task, did you consider other tasks, or did you consider using this task at a different point?

Have you thought about in detail how these will play out?

*Use this at the end:* I am curious how you were thinking about the tasks we used versus the tasks you are planning on using (those that you used previously). Can you tell me a bit about how you see these two groups of tasks? I guess I am trying to understand your selection of particular tasks and how last week's work relates.

*If they plan on using their previous tasks:* Are you mapping something that happened this past week onto a particular task or group of tasks and if so what are you mapping.

III. Imagine you had 60 points to allocate. For example, if you thought they were equaling that would be 10, 10....

The purpose of these generalizing tasks is:

- a) To introduce linear equations, including slope and y-intercept
- b) To engage students in authentic mathematics, so they participate first hand in the mathematical practices.
- c) To develop community norms, for example discourse practices.
- d) To help students understand the purpose and role of algebraic notation
- e) To see and quantify relationships between different quantities
- f) To familiarize students with different representations, highlighting particular attributes of equations, for example change.

IV. a) Going into teaching this unit on your own, what are you most excited to try out or learn? What are you most apprehensive about?

b) Is there something as you go about preparing for this unit that you believe would be of interest to me?

## Appendix D: Preinstructional Observation Protocol

1. Please describe the goals for today's lesson.
2. Describe how you believe the students will respond to your task. Please provide an explanation for your thinking.

## Appendix E: Postinstructional Observation Stimulated Recall Interview

### Protocol

Now we are going to watch *x-number* clips from the second group. I am interested in what you were thinking at the time of your interactions with the students. I can see how you were interacting with the students, but I don't know what you were thinking at that moment. What I'd like you to do is tell me what was going through your mind at the time you were watching or interacting with the students.

I will start the video and I will let you pause the video at any time that you want. So if you want to tell me anything you just push pause. In addition, if I have a question about what you were thinking, then I will push pause and ask you to talk about that segment of the video. Adopted from, Gass & Mackey, 2000)

#### Follow-up questions

- a) Describe the mathematics, the concept or practice, that the student was grappling with.
- b) Please describe in details what you think the student(s) were doing in this clip?
- c) What does this tell you about how the student was thinking about this topic?
- d) What were you interpreting in terms of student thinking at this moment?
- e) Please explain what you learned about the student(s)' understanding about this topic.
- f) i. What follow up question did you ask and why? or  
ii. What follow up question would you have asked and why? Or

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