Title

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A Linear Control Theory Analysis of Transverse Coherent Bunch Instabilities Feedback Systems (The Control Theory Approach to Hill's Equation)

J. Bengtsson, D. Briggs and G. Portmann

Introduction

There is an on-going effort to build a feedback system for transverse coherent bunch instabilities for ALS [1,2,3]. The beam dynamics issues were already addressed in the conceptual design report [4] and more detailed studies have been carried out [5]. On-going work is the development of a general simulation code including the full 6-dimensional dynamics for coherent bunch instabilities (by using Taylor series maps) as well as related feedback systems [6]. Recently, there has been some confusion about how to choose the gain matrix in the feedback loop. In particular, the current analytical formulas were found (from numerical simulations by D. Briggs using the newly developed simulation code) to only be valid if $\alpha_k = 0$ at the kicker. This motivated us to perform a more careful design study of the transverse feedback system based on linear control theory. This paper presents the general formulas for tuning the system. Also, by a careful analytical study of the performance of the system, based on linear accelerator theory combined with linear control theory for sampled systems, we discovered that the performance of the system can be dramatically improved by slightly changing one of the two coefficients in the gain matrix.

Full State Feedback

The general structure of the transverse feedback system is outlined in Fig. 1.

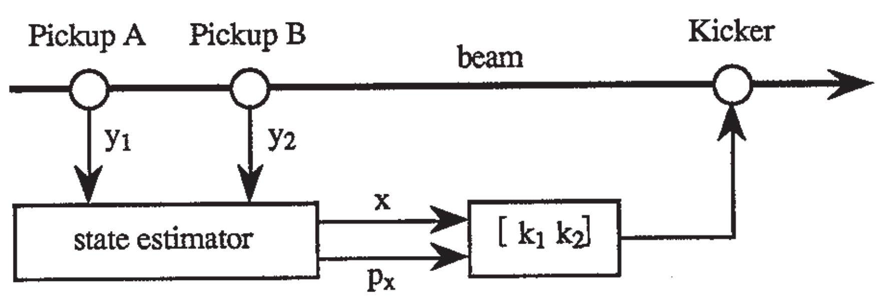

Fig. 1: Transverse Feedback System

The linear betatron motion is described by Hill's equation [7]

$$\boxed{x'' + K(s)\,x = u(s)\,\delta(s - n\,L)}$$

where L is the circumference of the accelerator. By introducing the state variable

$$\overline{x} \equiv \begin{bmatrix} x \\ p_x \end{bmatrix}$$

one obtains the corresponding discrete-time system state equations [8]

$$\boxed{\begin{aligned} \overline{x}_{n+1} &= A\,\overline{x}_n + B\,u_n = A\,\overline{x}_n + A\begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n \\ \overline{y}_n &\equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C\,\overline{x}_n \end{aligned}}$$

where the input, u_n, is the magnitude of the kick, A is the one-turn matrix at the kicker

$$\boxed{A = \begin{pmatrix} \cos(2\pi\nu) + \alpha_k \sin(2\pi\nu) & \beta_k \sin(2\pi\nu) \\ -\dfrac{1 + \alpha_k^2}{\beta_k} \sin(2\pi\nu) & \cos(2\pi\nu) - \alpha_k \sin(2\pi\nu) \end{pmatrix}}$$

and C is the transfer matrix from the kicker to the two BPMs. The Courant and Snyder paper is essentially a straightforward application of linear control theory [9], however, modern control theory is rarely utilized in the field of accelerator design. The following analysis will initiate a control theoretic approach to transverse feedback.

Using full state feedback, $u_n = -K\,\overline{x}_n$, leads to the following state equation

$$\boxed{\overline{x}_{n+1} = \left\{ A - A\begin{bmatrix} 0 \\ 1 \end{bmatrix}[\,k_1\ k_2\,] \right\} \overline{x}_n = \begin{bmatrix} a_{11} - a_{12}\,k_1 & a_{12}(1 - k_2) \\ a_{21} - a_{22}\,k_1 & a_{22}(1 - k_2) \end{bmatrix} \overline{x}_n \equiv A_{cl}\,\overline{x}_n}$$

where A_{cl} is the state matrix for the closed-loop system. The eigenvalues of A_{cl} determines the dynamics of the system. It is easy to show

$$\text{rank } W_c = 2\,, \quad \text{rank } W_o = 2$$

where

$$W_c = [\, B \;\; AB \,], \quad W_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

are the controllability and observability matrix respectively. Hence the system is both observable and controllable.

Observing the States

By using two BPMs, the two states at the kicker are directly observable. Since C is invertible for this particular case (by using two BPMs), the states can be computed from the output equation

$$\bar{x}_n = C^{-1}\, \bar{y}_n$$

If one assumes that there are no nonlinear elements between the two BPMs, the matrix formalism for the linear betatron motion gives

$$\bar{y}_2 = M_{1\to 2}\, \bar{y}_1$$

where

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\,[\cos \mu_{12} + \alpha_1 \sin \mu_{12}] & \sqrt{\beta_1 \beta_2}\, \sin \mu_{12} \\ -\frac{(1 + \alpha_1 \alpha_2) \sin \mu_{12} + (\alpha_2 - \alpha_1) \cos \mu_{12}}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}\,(\cos \mu_{12} - \alpha_2 \sin \mu_{12}) \end{pmatrix}$$

is the transport matrix between the two BPMs and $\bar{y}$, the phase space vector $(y,\, p_y)$. It follows

$$y_2 = m_{11}^{1\to 2}\, y_1 + m_{12}^{1\to 2}\, p_{y1}\,,$$

$$p_{y2} = m_{21}^{1\to 2}\, y_1 + m_{22}^{1\to 2}\, p_{y1}$$

Solving for p_{y2}

$$p_{y2} = \left(m_{21}^{1\to2} - \frac{m_{22}^{1\to2}\, m_{11}^{1\to2}}{m_{12}^{1\to2}} \right) y_1 + \frac{m_{22}^{1\to2}}{m_{12}^{1\to2}} y_2$$

The estimated state of the bunch at the kicker is then

$$\overline{x} = M_{2\to k}\, \overline{y}_2$$

where $M_{2\to k}$ is the transport matrix between the second BPM and the kicker.[1] The two states can therefore be reconstructed using

$$\boxed{\overline{x}_n = \begin{pmatrix} -\dfrac{m_{12}^{2\to k}}{m_{12}^{1\to2}} & \dfrac{m_{12}^{1\to2} m_{11}^{2\to k} + m_{22}^{1\to2} m_{12}^{2\to k}}{m_{12}^{1\to2}} \\ -\dfrac{m_{22}^{2\to k}}{m_{12}^{1\to2}} & \dfrac{m_{12}^{1\to2} m_{21}^{2\to k} + m_{22}^{1\to2} m_{22}^{2\to k}}{m_{12}^{1\to2}} \end{pmatrix} \overline{y}_n}$$

Note that the state (phase) space can be fully reconstructed (i.e. if one is willing to use four coefficients in the BPM electronics instead of the minimal two). In other words, if the electronics is implemented as presented, the state space motion can be observed directly, e.g. on an oscilloscope! The tremendous advantage of such an approach is obvious for tuning, fault diagnosis, performance analysis, effects of disturbances, modelling errors, accelerator R&D, etc. Of course, this kind of implementation (observing the states) does not exclude the more old-fashioned approach of breaking the loop and observing the magnitude and the argument of the total transfer function.

A State Estimator

For completeness we present a straightforward implementation of a state estimator, i.e. an estimation of the states based on a single output signal y_n (using only one BPM), which is possible since the system is observable. We find

$$\boxed{\overline{x}_n = A\, W_o^{-1} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \left\{ \begin{bmatrix} B & A^{-1}B \end{bmatrix} - A\, W_o^{-1} \begin{bmatrix} 0 & 0 \\ CB & 0 \end{bmatrix} \right\} \begin{bmatrix} u_{n-1} \\ u_n \end{bmatrix}}$$

[1]Nonlinear elements between the second bpm and the kicker can be accounted for by generalizing to a Taylor series map.

Fig. 2 shows tracking (using Tracy [10]) in the case of ALS for an injected beam displaced by 5 mm in both planes (x) and corresponding estimated values (o). The discrepancy between tracking and estimated values is due to nonlinear contributions from sextupoles (note that the states are estimated after the second turn).

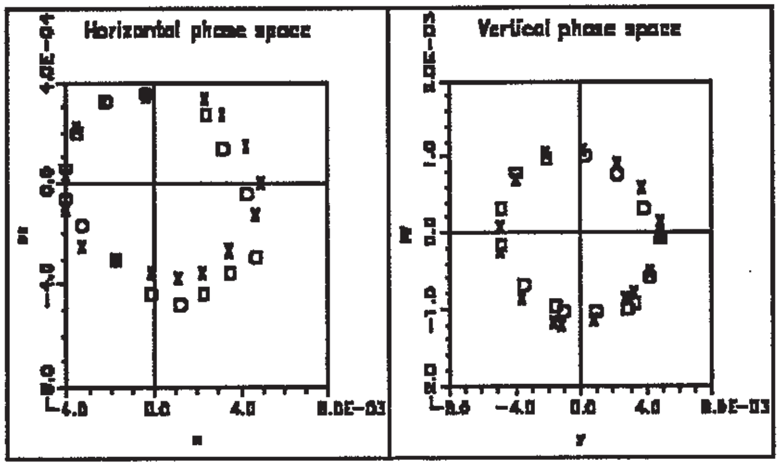

Fig. 2. State Estimator, with sextupoles

When these are turned off agreement is recovered, see Fig. 3

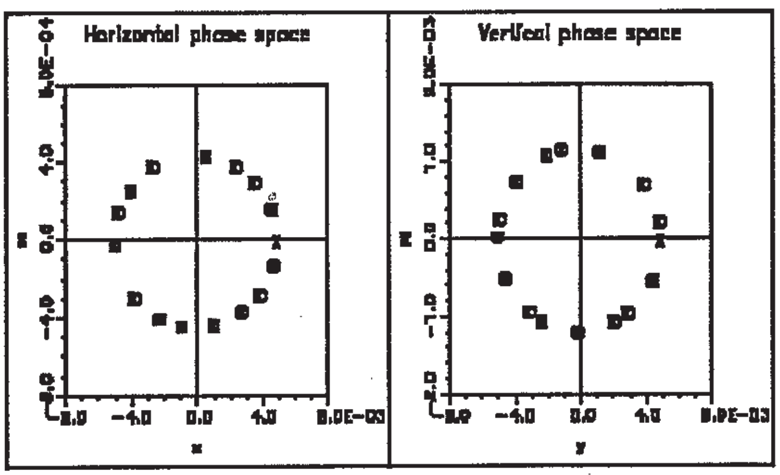

Fig. 3. State Estimator, no sextupoles

The estimator could be improved by generalizing to Taylor series maps. In particular, the corresponding expression in Truncated Power Series Algebra is simply

$$\bar{x}_n = \mathcal{A}\,\mathcal{W}_0^{-1}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \left\{ \begin{bmatrix} \mathcal{B} & \mathcal{A}^{-1}\mathcal{B} \end{bmatrix} - \mathcal{A}\,\mathcal{W}_0^{-1}\begin{bmatrix} 0 & 0 \\ \mathcal{C}\mathcal{B} & 0 \end{bmatrix} \right\} \begin{bmatrix} u_{n-1} \\ u_n \end{bmatrix}$$

where the linear matrices have been replaced by Taylor series maps. Good agreement is obtained when terms are kept to fourth order (using DA-Pascal [11]) as shown in Fig. 4. Independently, a Kalman filter could be used to optimize for sensor noise and plant disturbances.

We would like to make it clear that we are not proposing a state estimator for the current implementation. An analog system with two BPMs seems quite reasonable -- given the nonlinear dynamics and the additional problem to store information about the previous turn. However, it is a possibility when the use of two BPMs are not feasible.

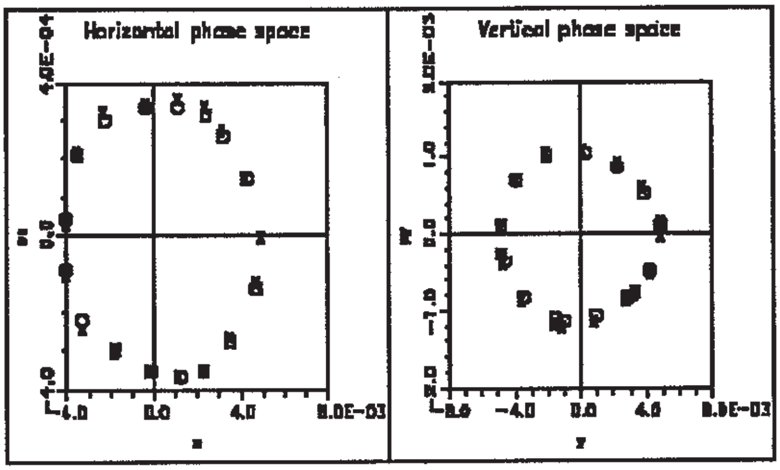

Fig. 4. Nonlinear State Estimator, with sextupoles

Naive Selection of the Feedback Gain Matrix

Naively, one would expect the magnitude of the kick to match the estimated momentum of each bunch at the kicker. However, this is only correct if $\alpha_k = 0$ (at the kicker) which is clear from Fig. 5.

Floquet space (i.e. a linear transformation from ellipse to circle) is defined by the following canonical transformation

$$\tilde{\bar{x}} = S\,\bar{x}$$

where

$$S = \begin{pmatrix} \frac{1}{\sqrt{\beta_k}} & 0 \\ \frac{\alpha_k}{\sqrt{\beta_k}} & \sqrt{\beta_k} \end{pmatrix}$$

A simple minded choice of gain coefficients is obtained by transforming to normalized phase space and selecting

$$\tilde{p}_x - \Delta \tilde{p}_x = 0$$

which leads to

$$\Delta p_x = \frac{\alpha_k}{\beta_k} x_k + p_x$$

and finally

$$\boxed{k_1 = \frac{\alpha_k}{\beta_k} = \frac{a_{11} - a_{22}}{2\, a_{12}}, \quad k_2 = 1}$$

The corresponding state equation is

$$\boxed{\overline{x}_{n+1} = \begin{bmatrix} \frac{a_{11} + a_{22}}{2} & 0 \\ a_{21} + \frac{a_{22}(a_{22} - a_{11})}{2\, a_{12}} & 0 \end{bmatrix} \overline{x}_n = \begin{bmatrix} \cos(2\pi\nu) & 0 \\ -\frac{\alpha_k \cos(2\pi\nu) + \sin(2\pi\nu)}{\beta_k} & 0 \end{bmatrix} \overline{x}_n}$$

It follows that the system will damp the motion according to

$$\boxed{\overline{x}_n = \begin{bmatrix} \cos^n(2\pi\nu) & 0 \\ -\cos^{n-1}(2\pi\nu)\frac{\alpha_k \cos(2\pi\nu) + \sin(2\pi\nu)}{\beta_k} & 0 \end{bmatrix} \overline{x}_0}$$

Feedback Design (Placement of the Eigenvalues)

A more systematic approach for selecting the gain coefficients is to study the eigenvalues of the state matrix which can be moved freely, since the system is controllable. A common approach is to make the system matrix nilpotent (i.e. deadbeat control)

$$A_{cl}^k = [0], \quad k \le n'$$

where n is the dimension of A. This is justified by

$$\overline{x}_n = A_{cl}^n \, \overline{x}_0$$

i.e. it will only take k turns to damp any initial condition. This is done by placing all the system eigenvalues at the origin in the complex plane. The eigenvalues are given by

$$\det\left(A_{cl} - \lambda I\right) = \begin{vmatrix} a_{11} - a_{12} k_1 - \lambda & a_{12}(1 - k_2) \\ a_{21} - a_{22} k_1 & a_{22}(1 - k_2) - \lambda \end{vmatrix} = 0$$

In the case of $k_1 = k_2 = 0$ (i.e. the open loop system) one has

$$\lambda_{1,2} = \frac{1}{2}(a_{11} + a_{22}) \pm \sqrt{\frac{1}{4}(a_{11} + a_{22})^2 - 1} = e^{\pm i 2\pi v}$$

as expected. In general,

$$\boxed{\lambda_{1,2} = \frac{1}{2}[a_{11} - a_{12} k_1 - a_{22}(1 - k_2)] \pm \sqrt{\frac{1}{4}[a_{11} - a_{12} k_1 + a_{22}(1 - k_2)]^2 + 1 - k_2}}$$

It follows that both eigenvalues are zero when

$$\boxed{k_1 = \frac{a_{11}}{a_{12}} = \frac{\cos(2\pi v) + \alpha_k \sin(2\pi v)}{\beta_k \sin(2\pi v)} \;, \quad k_2 = 1}$$

Note that $k_1 = 0$ for

$$\alpha_k = -\frac{1}{\tan(2\pi v)}$$

The state equation is now

$$\boxed{\overline{x}_{n+1} = \begin{bmatrix} 0 & 0 \\ a_{21} - \frac{a_{22} a_{11}}{a_{12}} & 0 \end{bmatrix} \overline{x}_n = \begin{bmatrix} 0 & 0 \\ -\frac{\sin(2\pi v)}{\beta_k}\left\{1 + \frac{1}{\tan^2(2\pi v)}\right\} & 0 \end{bmatrix} \overline{x}_n}$$

It follows that

$$\boxed{\bar{x}_{n+2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \bar{x}_n}$$

i.e. the oscillation is completely damped in two turns!

For comparison, the eigenvalues in the naive case are

$$\boxed{\lambda_1 = 0\,, \quad \lambda_2 = \frac{a_{11} + a_{22}}{2} = \cos(2\pi\nu)}$$

which is clearly far from optimal damping. Fig. 5 models one bunch in the PEP-II LER, with no cavity HOMs, and no resistive wall effects. The bunch is injected with an initial horizontal deviation of 1 mm. After 100 turns of open-loop operation, the loop is closed. After closing the feedback loop, the bunch damps.

The fact that the beam can be damped in two turns using only one kicker might seem a bit surprising but it is simply a matter of choosing the first kick so that the displacement is zero after the following turn as can be seen in Fig. 6.

In the case of the B-factory, using the working point

$$\nu_x = 36.570\,, \quad \nu_y = 34.640$$

one finds

$$\cos(2\pi\nu_x) = -0.90483\,, \quad \cos(2\pi\nu_y) = -0.63742$$

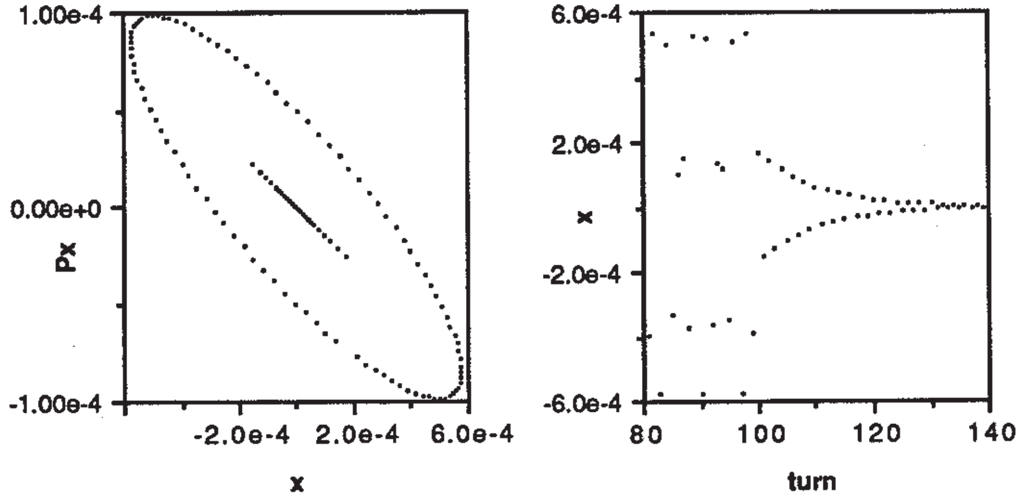

Fig. 5. Damping using the naive choice of gain matrix.

For example, in order to damp to 1% of the initial amplitude it would take

$$n_x \approx 42\,, \quad n_y \approx 9$$

number of turns on average (the actual number depends on the initial conditions), i.e. more than an order of magnitude more than for the nilpotent system! Note that issues related to possible saturation effects have so far not been studied for either case. However, straightforward application of control theory can accommodate these effects.

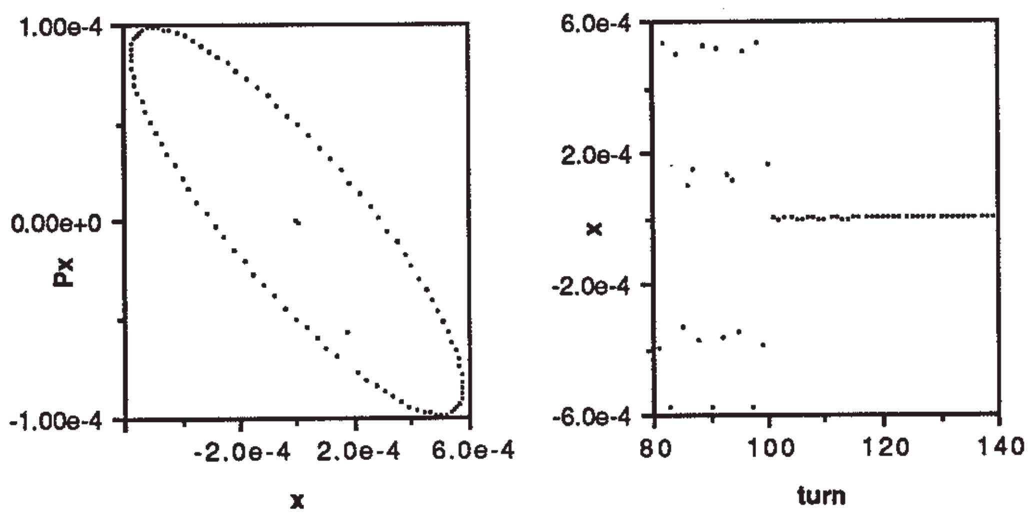

Fig. 6. Damping in the case of deadbeat feedback.

Conclusions

We have shown that the combination of linear control theory with linear accelerator theory is an extremely valuable tool for analysis and design of feedback systems. In particular, it has been shown to be a valuable design tool for the transverse feedback systems being designed for the ALS and PEP-II [4, 12]. The theory selects optimal gain coefficients based on performance criteria and furthermore predicts the corresponding system performance. This is done by analyzing the eigenvalues of the state matrix. Further studies should address possible saturation effects, sensitivity to sensor noise, disturbances and unmodeled dynamics. One may also consider more sophisticated gain coefficients selection methods, e.g. PID control, adaptive control, Kalman filters, etc. We have also indicated how the combined theory can be generalized to the nonlinear case by using Taylor series maps.

Note that the simplest implementation of the feedback system only requires one BPM if $\alpha_k = 0$ and the phase advance to the kicker is $90^\circ \pm n\ 180^\circ$, $n = 0, 1, 2, \cdots$. If one chooses to use two BPMs, then one might as well take full advantage of them by observing the states. Hence we recommend that the implementation of the electronics support full observation of the states. This may require as little as a couple of signal splitters and a few more voltage controlled attenuators.

Application of control theory has strongly benefitted other fields such as aerospace engineering and process control. We would again like to point out that the classical paper by Courant and Snyder is essentially a straightforward application of linear control theory to study the stability of betatron motion. However, advances in modern control theory are rarely utilized in the field of accelerator design. We strongly recommend the combination of accelerator physics and control theory to be utilized as an extremely valuable tool in the design of accelerators and related feedback system.

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