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Authors

Fortin, Jean-François
Grinstein, Benjamín
Stergiou, Andreas

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Scale without Conformal Invariance: An Example

Jean-François Fortin, Benjamín Grinstein and Andreas Stergiou

Department of Physics, University of California, San Diego, La Jolla, CA 92093 USA

We give an explicit example of a model in $D = 4 - \epsilon$ space-time dimensions that is scale but not conformally invariant, is unitary, and has finite correlators. The invariance is associated with a limit cycle renormalization group (RG) trajectory. We also prove, to second order in the loop expansion, in $D = 4 - \epsilon$, that scale implies conformal invariance for models of any number of real scalars. For models with one real scalar and any number of Weyl spinors we show that scale implies conformal invariance to all orders in perturbation theory.

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jffortin@physics.ucsd.edu, bgrinstein@ucsd.edu, stergiou@physics.ucsd.edu

Erratum

The original claim of this paper was that the set of couplings given in Eq. (5.7) define a theory that is scale invariant without being conformal in $D = 4 - \epsilon$ spacetime dimensions. This claim is false, as was later realized by the authors [1]. The correct interpretation of Eq. (5.7) is that the theory defined by these couplings is fully conformal, although it lives on a limit cycle of the traditional dim-reg beta function when the anomalous dimension matrix is chosen to be symmetric. This paper remains posted due to this novel feature of the solution (5.7). For more details the reader is referred to [1].

1. Introduction

Can a theory display a symmetry under dilatations but not be invariant under conformal transformations? The answer to the converse, whether a theory can be invariant under conformal transformations but not under scaling, has long been known. The algebra of the conformal group gives the generator of dilatations in terms of conserved generators of conformal transformations and translations. Hence, conformal plus Poincaré invariance implies dilatation invariance. But whether scale implies conformal has remained an elusive open question.

Using an argument of Zamolodchikov, Polchinski has proved, in $D = 2$ space-time dimensions, that a unitary model cannot be scale-invariant without also being conformal [2]. The assumption of unitarity seems to play an essential role. Indeed, Riva and Cardy have exhibited a model with $D = 2$ Euclidean dimensions with scale but not conformal symmetry [3]. But the model is not reflection-positive, the Euclidean version of unitarity. An earlier model by Hull and Townsend [4] seems to contradict Polchinski's result, but this may be attributed to the violation of a technical assumption in Polchinski's argument, namely the existence of finite correlators of the stress-energy tensor. More recently other counterexamples have been given, however every case violates one of the assumptions of the theorem (unitarity, existence and finiteness of correlators) [5].

Polchinski went on to show that, at one-loop order, a scalar field theory in $D = 4 - \epsilon$ is necessarily conformal if it is scale invariant. We will review his argument below and show that it can be extended to one higher order in the loop expansion. A recent one-loop analysis by Dorigoni and Rychkov [6] extended Polchinski's argument to the case of theories with both scalars and fermions. We show that their result can be extended to all orders in

perturbation theory for models with an arbitrary number of spin-$\frac{1}{2}$ fields and no more than one real scalar.

Polchinski also reviewed the literature on the subject to that date. Let us briefly survey salient, mostly recent work on the relation between dilatation and conformal invariance since his review. Antoniadis and Buican studied $\mathcal{N} = 1$ supersymmetric theories with an R-symmetry [7]. They proved that any unitary fixed point is either superconformal or corresponds to a model that has at least two real non-conserved dimension-two scalar singlet operators. They also demonstrated that any IR fixed point reached from a flow from a UV superconformal fixed point is itself superconformal, provided some technical assumptions are met. Nakayama has taken a fresh approach to the question. He considers the AdS/CFT correspondence to claim that the null energy condition in the bulk gravitational theory guarantees the equivalence in the boundary theory between scale and conformal invariance [8].

El-Showk, Nakayama and Rychkov have pointed out that Maxwell theory in $D \neq 4$ is scale but not conformally invariant [9]. Jackiw and Pi have shown that Maxwell's action integral in any D is invariant under conformal transformations if the field strength tensor $F_{\mu\nu}$ is taken as a primary field of scaling dimension $D/2$, but then Maxwell's equations are not covariant under conformal transformations [10]. We note in passing that if the scaling dimension is taken to be $D-2$, then Maxwell's equations are conformally covariant, but the action integral is not invariant, unless $D = 4$. We hasten to indicate that Maxwell theory is a free field theory, and hence of limited interest. Furthermore, in the presence of sources the model is no longer invariant under dilatations in $D \neq 4$.

To summarize the situation until this work, for $D > 2$ there appears to be neither an interacting counterexample nor a proof that scale implies conformal invariance. We have found that, in $D = 4 - \epsilon$, a model of two Weyl spinors and two real scalars is invariant under scaling but not under conformal transformations. More specifically, we will show that there exist points in the parameter space of the model for which a combined transformation by scaling and by an internal rotation among the scalars is a symmetry. We will argue that these points must not be isolated, but lie on RG trajectories that are scale but not conformally invariant and, most remarkably, discover that these RG trajectories form closed loops or display ergodic behavior. To the best of our knowledge neither limit cycles nor ergodic RG trajectories have ever before been reported for a relativistic field theory.

In preparation for our analysis, we review the arguments of [2, 6] in Sec. 2. We will argue in Sec. 3 that, in theories with scale but not conformal invariance, RG flows must have either limit cycles or ergodic behavior. We also make some general comments about

the possibility of uncovering fixed points with enhanced internal symmetries. We show (at least to second order in the loop expansion) that scale implies conformal invariance in two classes of models in Sec. 4. This is interesting in its own right, but also sets the stage for discovering models with scale but not conformal invariance. An example of the latter we present in Sec. 5. For conciseness we present explicitly the analysis and results for the simplest model only, consisting of one Weyl spinor and two real scalars which also displays a limit cycle with scaling symmetry. However, on this cycle this model's scalar potential is unbounded from below. We close with a short summary and a brief discussion of some interesting new open questions.

2. Preliminaries

In order to establish notation and for completeness, we begin by reviewing the conditions for scale and conformal invariance [2, 11, 12]. The dilatation current is of the form

$$\mathcal{D}^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)\,. \tag{2.1}$$

Here $T^{\mu\nu}(x)$ is any symmetric stress-energy tensor and $V^\mu(x)$ is any current that does not depend explicitly on x^μ. The freedom to choose among different symmetric stress tensors is compensated through changes in the current V^μ. The improved stress-energy tensor can be particularly useful since it does not get renormalized [11]. Given a choice of stress-energy tensor, scaling will be a symmetry if it is possible to find a current V^μ such that

$$T_\mu{}^\mu = \partial_\mu V^\mu\,. \tag{2.2}$$

Conformal invariance is equivalent to the existence of a traceless stress-energy tensor. However, the stress-energy tensor in Eq. (2.2) need not be traceless. For $D > 2$ it is sufficient that

$$T_\mu{}^\mu = \partial_\mu \partial_\nu L^{\mu\nu} \tag{2.3}$$

for some local tensor operator $L^{\mu\nu}$, for then one can explicitly construct a traceless stress-energy tensor out of $T^{\mu\nu}$ and $L^{\mu\nu}$. It follows that the condition that a model has scale but not conformal invariance is to satisfy Eq. (2.2), with the additional condition that the current V^μ cannot be written as a conserved current J^μ plus the divergence of a two index symmetric tensor $L^{\nu\mu}$,

$$T_\mu{}^\mu = \partial_\mu V^\mu, \text{ where } V^\mu \neq J^\mu + \partial_\nu L^{\nu\mu} \text{ with } \partial_\mu J^\mu = 0. \tag{2.4}$$

Finding candidates for a current that one can use in the test Eq. (2.4) is not difficult. In D space-time dimensions the scaling dimension of the current must be $D-1$. In a perturbative setting with a collection of real scalars,[1] ϕ_a, and Weyl spinors, ψ_i, the most general candidate is [6]

$$V_\mu = Q_{ab}\phi_a\partial_\mu\phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}_\mu\psi_j. \tag{2.5}$$

Without loss of generality one may take Q_{ab} to be antisymmetric, $Q_{ab} = -Q_{ba}$. Furthermore, in order for V^μ to be Hermitian, P_{ij} has to be anti-Hermitian, $P^*_{ij} = -P_{ji}$. The unknown coefficients Q_{ab} and P_{ij} are to be determined by satisfying Eq. (2.4). One may well expect that they depend on the coupling constants of the model: the dimension of the operators $\phi_a\partial_\mu\phi_b$ and $\bar{\psi}_i\bar{\sigma}_\mu\psi_j$ that go into V_μ is not generally $D-1$ in an interacting model, so the coefficients may have to make up for the difference. But in a perturbative model this difference is small. Hence, operators with naive dimensions that differ from $D-1$ are not included in the candidate current.

To proceed further we need to specify the model some more. In order to have both UV and IR fixed points, so we may study the relation between scale-invariant and conformal theories, we consider only $D = 4-\epsilon$ at small ϵ. The action integral defines the coupling constants as follows:

$$S = \int d^Dx \left(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i - \frac{\lambda_{abcd}}{4!}\phi_a\phi_b\phi_c\phi_d - \frac{y_{a|ij}}{2}\phi_a\psi_i\psi_j - \frac{y^*_{a|ij}}{2}\phi_a\bar{\psi}_i\bar{\psi}_j\right). \tag{2.6}$$

We can now test these models for scale and conformal invariance. The trace of the energy momentum tensor is

$$T_\mu{}^\mu = -\frac{\beta_{abcd}}{4!}\phi_a\phi_b\phi_c\phi_d - \frac{\beta_{a|ij}}{2}\phi_a\psi_i\psi_j - \frac{\beta^*_{a|ij}}{2}\phi_a\bar{\psi}_i\bar{\psi}_j,$$

up to terms that vanish by the equations of motion.[2] The divergence of the candidate current, after using the equations of motion, is

$$\partial_\mu V^\mu = -\tfrac{1}{4!}\mathcal{Q}_{abcd}\phi_a\phi_b\phi_c\phi_d - \tfrac{1}{2}(\mathcal{P}_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}),$$

where

$$\mathcal{Q}_{abcd} = Q_{ae}\lambda_{ebcd} + 3 \text{ permutations}, \tag{2.7a}$$

$$\mathcal{P}_{a|ij} = Q_{ab}y_{b|ij} + (P_{ki}y_{a|jk} + i \leftrightarrow j). \tag{2.7b}$$

[1] Indices from the beginning of the roman alphabet are in scalar-flavor space, while indices from the middle are in fermion-flavor space. In this work we don't consider gauge fields [13].

[2] Equations of motion can be used and are not renormalized, see Ref. [14].

With these results, condition (2.4) becomes

$$\beta_{abcd} - \mathcal{Q}_{abcd} = 0, \tag{2.8a}$$

$$\beta_{a|ij} - \mathcal{P}_{a|ij} = 0. \tag{2.8b}$$

The problem has been reduced to solving these algebraic equations. Here, by solving the equations we mean finding Q_{ab} and P_{ij} that satisfy these equations for some values of the coupling constants. To reiterate, the equations need not be satisfied identically, that is, for every value of the couplings. Still the equations are not trivial because there are more equations than free variables in Q_{ab} and P_{ij}, so generally we expect no solutions or, if the theory has fixed points, a trivial solution in which both Q_{ab} and P_{ij} vanish. If these are the only solutions, then we have that scale implies conformal invariance.

3. Trajectories and enhanced-symmetry fixed points

If a solution with $\beta \neq 0$ is found, then there exist a point in parameter space with scale but not conformal invariance. Perhaps it is because such a point has never been found that it is universally referred to in the literature as a fixed point. But clearly the point cannot be fixed because the beta functions do not all vanish there. The point must lie on an RG trajectory. Physical properties are common to the complete trajectory. It follows that the whole trajectory displays scale but not conformal invariance.

RG trajectories are defined by

$$\frac{d\bar{\lambda}_{abcd}(t)}{dt} = \beta_{abcd}(\bar{\lambda}_{abcd}(t), \bar{y}_{a|ij}(t)) \qquad \text{and} \qquad \frac{d\bar{y}_{a|ij}(t)}{dt} = \beta_{a|ij}(\bar{\lambda}_{abcd}(t), \bar{y}_{a|ij}(t))\,, \tag{3.1}$$

but now Eqs. (2.8) hold along the whole trajectory. So, it must also be true that

$$\frac{d\bar{\lambda}_{abcd}(t)}{dt} = \mathcal{Q}_{abcd}(\bar{\lambda}_{abcd}(t), \bar{y}_{a|ij}(t)) \qquad \text{and} \qquad \frac{d\bar{y}_{a|ij}(t)}{dt} = \mathcal{P}_{a|ij}(\bar{\lambda}_{abcd}(t), \bar{y}_{a|ij}(t))\,. \tag{3.2}$$

Therefore, the running couplings solve both sets of equations (3.1) and (3.2) simultaneously. This is a remarkable condition.

If the dependence of Q_{ab} and P_{ij} on coupling constants is simple, one may integrate the equations readily. In fact, in the explicit examples we give below, Q_{ab} are constants (independent of coupling constants) and P_{ij} vanish. Hence, $\mathcal{Q}$ and $\mathcal{P}$ are linear in the couplings. Moreover, since Q_{ab} is real and antisymmetric, it has purely imaginary eigenvalues, and thus the trajectories must be periodic or quasi-periodic. These correspond to the existence of limit cycles and ergodicity, respectively, in the classification of possible behaviors

of RG trajectories [15]. There do not seem to be any reported examples in relativistic field theory displaying either of these. There appears to be a tight connection between these behaviors and scale but not conformal symmetry, and no examples of such models were known before this work [13].

There is a simple reason to expect periodic or quasi-periodic RG trajectories. The conserved dilatation current, Eq. (2.1), is a combination of a scaling and a rotation in field space. The latter is a transformation in a compact group. A curve in a compact space must be periodic or quasi-periodic. Now, a scale transformation gives a translation along the RG trajectory.[3] So as the RG trajectory is traversed, the field rotation eventually goes back to the identity, or arbitrarily close to the identity. Hence, the scale transformation itself must go back to, or arbitrarily close to, the identity and therefore, by continuity in parameter space, the RG trajectory must return to, or arbitrarily close to, the starting point.

There may also be solutions to (2.8) that have $\beta = 0$ but for which $Q \neq 0$ or $P \neq 0$. The trace of the stress-energy tensor vanishes and from Eq. (2.2) we see that V^μ is a conserved current. This is the case of a fixed point with an enhanced internal symmetry. On the trajectories that flow towards the fixed point there is no symmetry associated with the current V^μ, since away from the fixed point $T_\mu{}^\mu \neq 0$.

4. Scale implies conformal: two classes of models in $4 - \epsilon$

We now prove, to second order in perturbation theory, that scale implies conformal invariance for models of an arbitrary number of scalars in $D = 4 - \epsilon$. We also prove the same result *to all orders in perturbation theory* for models of an arbitrary number of Weyl spinors and exactly one real scalar.

4.1. Models of scalars

Polchinski showed that, in a model of any number of scalar fields with arbitrary quartic couplings, scale implies conformal invariance at one loop. We review his argument and then extend it to second order in perturbation theory.

The model is given by the Lagrangian in Eq. (2.6) with all of the fermion fields set to zero and $D = 4 - \epsilon$. Recall that we are trying to find a solution to Eq. (2.8a) at a point in parameter space for which the beta function does not vanish. Polchinski's argument is the

[3] Except at a fixed point. We are considering the behavior of a model with scale but not conformal symmetry.

following. Using the explicit form of the β function to one-loop order,

$$\beta^{(1\text{-loop})}_{abcd} = -\epsilon\lambda_{abcd} + \frac{1}{16\pi^2}\left(\lambda_{abef}\lambda_{efcd} + 2 \text{ permutations}\right)$$

and the explicit form of $\mathcal{Q}$ given in Eq. (2.7a), one can verify by explicit computation that

$$\mathcal{Q}_{abcd}\beta^{(1\text{-loop})}_{abcd} = 0\,.$$

From Eq. (2.8a) it follows that $\mathcal{Q}_{abcd}\mathcal{Q}_{abcd} = 0$ which implies that $Q_{ab} = 0$, and thus leaves us only with fixed-point solutions to Eq. (2.8a).

We extend this argument to two loops by brute force. We have used the two-loop expression for the beta function in dimensional regularization [16] to verify that $\mathcal{Q}_{abcd}\beta^{(2\text{-loop})}_{abcd} = 0$. We do not give the details of the computation, there is little to be learned from the explicit and lengthy expressions. One can verify, however, that with all possible contractions of three λ's with four free indices, the only way to get $\mathcal{Q}_{abcd}(\lambda\lambda\lambda)_{abcd} \neq 0$ is to contract two indices in the same λ. But the diagrams that correspond to such a contraction are zero in dimensional regularization, and so we find $\mathcal{Q}_{abcd}\beta^{(2\text{-loop})}_{abcd} = 0$. It follows that scale implies conformal invariance at two loops too.

4.2. Models of Weyl spinors and one real scalar

Dorigoni and Rychkov have extended Polchinski's analysis to the case of models with spinors and scalars described by the Lagrangian density in Eq. (2.6), with $D = 4 - \epsilon$ [6]. They showed that, at one loop, for the conditions in Eqs. (2.8) to be satisfied one must set both $\mathcal{P}$ and $\mathcal{Q}$ to zero. Hence, at any scale-invariant point one must have vanishing beta functions. They argue as follows. First contract Eq. (2.8b) with $\mathcal{P}^*_{a|ij}$ to obtain

$$\mathcal{P}^*_{a|ij}\beta_{a|ij} = \mathcal{P}^*_{a|ij}\mathcal{P}_{a|ij}. \tag{4.1}$$

The right-hand side of Eq. (4.1) is a real number. However, using the explicit form of the beta function at one-loop order one finds that the real part of the left-hand side of Eq. (4.1) is identically zero, and so $\mathcal{P}_{a|ij}$ vanishes. Furthermore, contracting Eq. (2.8a) with $\mathcal{Q}_{abcd}$ and using $\mathcal{P}_{a|ij} = 0$ one finds that $\mathcal{Q}_{abcd}$ vanishes as well. Consequently, Eqs. (2.8) are satisfied only at conformal fixed points.

If we now attempt to extend this result by using the explicit form of the beta function to two-loop order [16], we find that $\mathrm{Re}(\mathcal{P}^*\beta)$ does not vanish in the general case and, therefore, condition (4.1) does not generally require the vanishing of $\mathcal{P}$. In the absence of a general argument we inspect specific cases.

Consider a model with an arbitrary number of spinors and only one real scalar field. In this case the Lagrangian in Eq. (2.6) has only a single scalar self-coupling λ, and the Yukawa couplings form a single matrix y_{ij}. The 1×1 antisymmetric matrix Q_{ab} vanishes and so do $\mathcal{Q}$ and, by Eq. (2.8a), β_λ. Hence the condition for scale invariance is solved only if λ is at a fixed point. The beta function β_{ij} for the Yukawa couplings is also a single matrix and is given by the matrix product of y times a real polynomial in $y^\dagger y$, with coefficients that are real functions of λ. Therefore, to all orders in perturbation theory, β is of the form Hy where H is a Hermitian matrix, $H^\dagger = H$, which satisfies $Hy = yH$. It follows that $\mathcal{P}^*_{ij}\beta_{ij} = \mathrm{Tr}(\mathcal{P}^\dagger\beta) = \mathrm{Tr}[(P^\dagger y^\dagger - y^\dagger P^T)Hy] = -\mathrm{Tr}[P(y^\dagger Hy + (yHy^\dagger)^T)]$, that is, the product of the anti-Hermitian matrix P with the Hermitian matrix $y^\dagger Hy + (yHy^\dagger)^T$. Therefore, the trace is purely imaginary and we can now complete the argument: the right-hand side of Eq. (4.1), being a real number, must vanish. Hence β_y must vanish if the condition for scale invariance, Eq. (2.8b), is satisfied. We have shown that, to all orders, if there are scale-invariant points, they are also fixed points.

5. The simplest example

If we attempt to continue our analysis of explicit cases we find a snag: we cannot complete the argument that $\mathcal{Q}$ and $\mathcal{P}$ vanish. In a model with two (or more) real scalars and one (or more) Weyl spinors the matrix Q_{ab} does not automatically vanish, as was the case in models with a single real scalar. Then, using the explicit form of the two-loop beta function [16] for the model (2.6) with at least two real scalars and at least one Weyl spinor, one finds that $\mathrm{Re}(\mathcal{P}^*_{a|ij}\beta^{(\text{2-loop})}_{a|ij}) \neq 0$. This does not mean that non-trivial solutions to Eq. (2.8) must exist. But it indicates a direction to investigate.

The simplest theory of this type has one Weyl spinor ψ and two real scalars ϕ_1 and ϕ_2. The Lagrangian is

$$\mathscr{L} = \text{kin. terms} - \frac{\lambda_1}{24}\phi_1^4 - \frac{\lambda_2}{24}\phi_2^2 - \frac{\lambda_3}{4}\phi_1^2\phi_2^2 - \frac{\lambda_4}{6}\phi_1^3\phi_2 - \frac{\lambda_5}{6}\phi_1\phi_2^3 - \left(\frac{y_1}{2}\phi_1\psi^2 + \frac{y_2}{2}\phi_2\psi^2 + \text{h.c.}\right), \tag{5.1}$$

and the candidate for V^μ can be written as

$$V_\mu = q(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1) - p\bar\psi\bar\sigma_\mu\psi, \tag{5.2}$$

where q and p are real numbers. The connection with the notation for the general model of Eq. (2.6) is that $\lambda_{1111} \equiv \lambda_1, \lambda_{1112} \equiv \lambda_4, \lambda_{1122} \equiv \lambda_3, \lambda_{1222} \equiv \lambda_5,\ \lambda_{2222} \equiv \lambda_2$ and $y_{a|11} \equiv y_a$.

Using the beta functions in Ref. [16] we find

$$\mathrm{Re}(\mathcal{P}^*_{a|ij}\beta^{(\text{2-loop})}_{a|ij}) = \frac{1}{(16\pi^2)^2}[y_a y_b y^*_c y^*_d (Q_{ae}\lambda_{bcde} + Q_{ce}\lambda_{abde}) - \tfrac{1}{24} y_a y^*_b (Q_{ac}\lambda_{bdef} + Q_{bc}\lambda_{adef})\lambda_{cdef}], \tag{5.3}$$

which does not necessarily vanish. We proceed to search for non-trivial solutions to (2.8).

In order to explain the strategy that we follow in solving (2.8) it is useful to sketch the form of these equations. Retaining up to two-loop contributions to the beta functions, we have, schematically,

$$\begin{aligned}
\beta_{abcd} - \mathcal{Q}_{abcd} &\sim -\epsilon\lambda + \frac{\lambda^2 + \lambda(y^*y) + (y^*y)^2}{16\pi^2} + \frac{\lambda^3 + \lambda^2(y^*y) + \lambda(y^*y)^2 + (y^*y)^3}{(16\pi^2)^2} - Q\lambda = 0, \\
\beta_{a|ij} - \mathcal{P}_{a|ij} &\sim -\frac{\epsilon}{2} y + \frac{yy^*y}{16\pi^2} + \frac{y(y^*y)^2 + yy^*y\lambda + y\lambda^2}{(16\pi^2)^2} - Qy - Py = 0.
\end{aligned} \tag{5.4}$$

The form of these equations suggests that we search for solutions as an expansion in ϵ. To lowest order the solutions should correspond to the fixed points obtained from balancing the "classical" term, $\sim \epsilon\lambda$, against the first quantum corrections, i.e. the one-loop terms. The Polchinski–Dorigoni–Rychkov argument tells us that we should ignore Q and P at this order. So we take

$$\lambda_{abcd} = \sum_{n=1}^{\infty} \lambda^{(n)}_{abcd}\epsilon^n, \qquad y_{a|ij} = \sum_{n=1}^{\infty} y^{(n)}_{a|ij}\epsilon^{n-\frac{1}{2}}, \tag{5.5}$$

$$Q_{ab} = \sum_{n=2}^{\infty} Q^{(n)}_{ab}\epsilon^n, \qquad P_{ij} = \sum_{n=2}^{\infty} P^{(n)}_{ij}\epsilon^n. \tag{5.6}$$

Notice that the nature of the expansions for the coupling constants is dictated by the two lowest-order terms, that is, by the location of a would-be fixed point. Meanwhile, the expansions for Q and P start at order ϵ^2 or higher, since they must vanish if only up to one-loop terms are retained in the beta functions.

For this particular model there are nine equations to solve, corresponding to the beta functions for two complex y's and five real λ's in Eqs. (5.4). Adding p and q to the list of coupling constants, there are eleven variables. Thus, the system is under-constrained. This is as it should be if we are to have solutions along trajectories, but it is not computationally convenient. Instead it is best to fix some variables. We can set $\mathrm{Im}(y_2) = 0$ by redefining the fields by a phase rotation of the Weyl spinor.

There are many fixed-point solutions, and one must be careful to check that a solution to (2.8) is not also a fixed point, even for nonzero q and/or p. One must also check that

the solution does not give a scalar potential that is unbounded from below. Unfortunately, in the simplest example with one Weyl spinor and two real scalars, we have not found a scale-invariant point with a bounded-from-below scalar potential that is not conformally invariant. We have verified that such scale-invariant points exist in more general models, e.g., with two Weyl spinors and two real scalars [13], but, to keep the presentation simple, we only display here the scale-invariant point (and the trajectory it lies on) in the model with one Weyl spinor and two real scalars. In an ϵ-expansion we find the scale-invariant solution

$$
\begin{aligned}
\lambda_1 &= \tfrac{821326-5427\sqrt{419802}}{607836}\pi^2\epsilon + \tfrac{518735723529516-118790842537\sqrt{419802}}{195971150186496}\pi^2\epsilon^2 + \ldots, \\
\lambda_2 &= \tfrac{7(373922-141\sqrt{419802})}{607836}\pi^2\epsilon - \tfrac{23(6387330973\sqrt{419802}-5101825968812)}{65323716728832}\pi^2\epsilon^2 + \ldots, \\
\lambda_3 &= \tfrac{469(222+\sqrt{419802})}{607836}\pi^2\epsilon + \tfrac{74835485902788+225616637735\sqrt{419802}}{195971150186496}\pi^2\epsilon^2 + \ldots, \\
\lambda_4 &= \tfrac{7\sqrt{\frac{469}{74}(3601+6\sqrt{419802})}}{8214}\pi^2\epsilon \\
&\qquad + \tfrac{\sqrt{\frac{7(259576132595538832854006422950 7+40506730535260862254181781 12\sqrt{419802})}{543700157374}}}{17272267776}\pi^2\epsilon^2 + \ldots, \\
\lambda_5 &= \tfrac{67\sqrt{\frac{469}{74}(3601+6\sqrt{419802})}}{8214}\pi^2\epsilon \\
&\qquad + \tfrac{\sqrt{\frac{67(6689894769565669970572147430 17+10514452509065147909765526 40\sqrt{419802})}{170413482162}}}{17272267776}\pi^2\epsilon^2 + \ldots, \\
y_1 &= \sqrt{2}\pi\sqrt{\epsilon} + \tfrac{1737927\sqrt{2}-3350\sqrt{209901}}{12616704}\pi\epsilon^{3/2} \\
&\qquad + \tfrac{\sqrt{\frac{258756594352544587227002131-322169380386272743890676\sqrt{419802}}{3782}}}{1434065043456}\pi\epsilon^{5/2} + \ldots, \\
y_2 &= 0, \\
q &= \tfrac{511\sqrt{\frac{469}{74}(3601+6\sqrt{419802})}}{33644544}\epsilon^3 + \ldots, \\
p &= 0.
\end{aligned}
\tag{5.7}
$$

The reader will notice that the expansion for q begins at third order in epsilon—this corresponds to the third order in the loop expansion of the beta function in Eqs. (2.8), which we have not included. It would seem that the solution is inconsistent and may disappear when the next order in beta is included. However, we have verified that this does not happen. That q here begins at third order is due to a numerical accident of the two-loop Yukawa beta-functions. All these issues will be discussed elsewhere.

Now that we have discovered a solution that is not a fixed point we can uncover the

scale-invariant trajectory from Eqs. (3.2). For the scalar couplings, for example, organized as a five-dimensional vector, we have a system of coupled linear differential equations,

$$\frac{d\vec{\lambda}}{dt} = \mathsf{Q}\vec{\lambda},$$

where the matrix Q is

$$\mathsf{Q} = q \begin{pmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -2 & 2 \\ -1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{pmatrix}.$$

Once the matrix Q is diagonalized the system decouples and is easy to solve. The solution, including the Yukawa couplings, is

$$\begin{aligned}
\bar{\lambda}_1(t) &= \lambda_1 \cos^4 qt + \lambda_2 \sin^4 qt + \tfrac{3}{2}\lambda_3 \sin^2 2qt + 4\lambda_4 \sin qt \cos^3 qt + 4\lambda_5 \sin^3 qt \cos qt, \\
\bar{\lambda}_2(t) &= \lambda_1 \sin^4 qt + \lambda_2 \cos^4 qt + \tfrac{3}{2}\lambda_3 \sin^2 2qt - 4\lambda_4 \sin^3 qt \cos qt - 4\lambda_5 \sin qt \cos^3 qt, \\
\bar{\lambda}_3(t) &= \tfrac{1}{4}\lambda_1 \sin^2 2qt + \tfrac{1}{4}\lambda_2 \sin^2 2qt + \tfrac{1}{4}\lambda_3(3\cos 4qt + 1) - \tfrac{1}{2}\lambda_4 \sin 4qt + \tfrac{1}{2}\lambda_5 \sin 4qt, \\
\bar{\lambda}_4(t) &= -\lambda_1 \sin qt \cos^3 qt + \lambda_2 \sin^3 qt \cos qt + \tfrac{3}{4}\lambda_3 \sin 4qt + \tfrac{1}{2}\lambda_4(\cos 2qt + \cos 4qt) \\
&\qquad + \lambda_5 \sin^2 qt\,(2\cos 2qt + 1), \\
\bar{\lambda}_5(t) &= -\lambda_1 \sin^3 qt \cos qt + \lambda_2 \sin qt \cos^3 qt - \tfrac{3}{4}\lambda_3 \sin 4qt + \lambda_4 \sin^2 qt\,(2\cos 2qt + 1) \\
&\qquad + \tfrac{1}{2}\lambda_5(\cos 2qt + \cos 4qt), \\
\bar{y}_1(t) &= y_1 \cos qt + y_2 \sin qt, \\
\bar{y}_2(t) &= -y_1 \sin qt + y_2 \cos qt.
\end{aligned} \tag{5.8}$$

Here the initial values for the scalar and Yukawa couplings on the right-hand side, as well as the frequency q are the solutions given in Eqs. (5.7) above. As discussed above, the scalar potential is unbounded from below for these values. Note that the imaginary parts of the Yukawas vanish, hence the theory does not violate CP. Note, furthermore, that these statements remain true throughout the cycle, as indeed they should. The couplings are plotted in Fig. 1. How do these solutions depend on ϵ? From Eqs. (5.7) it follows that the scale-invariant trajectory disappears in $D = 4$. A sketch of the shrinking trajectories in parameter space is shown in Fig. 2.

The t-dependent solutions (5.8) can be obtained by replacing

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos qt & \sin qt \\ -\sin qt & \cos qt \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

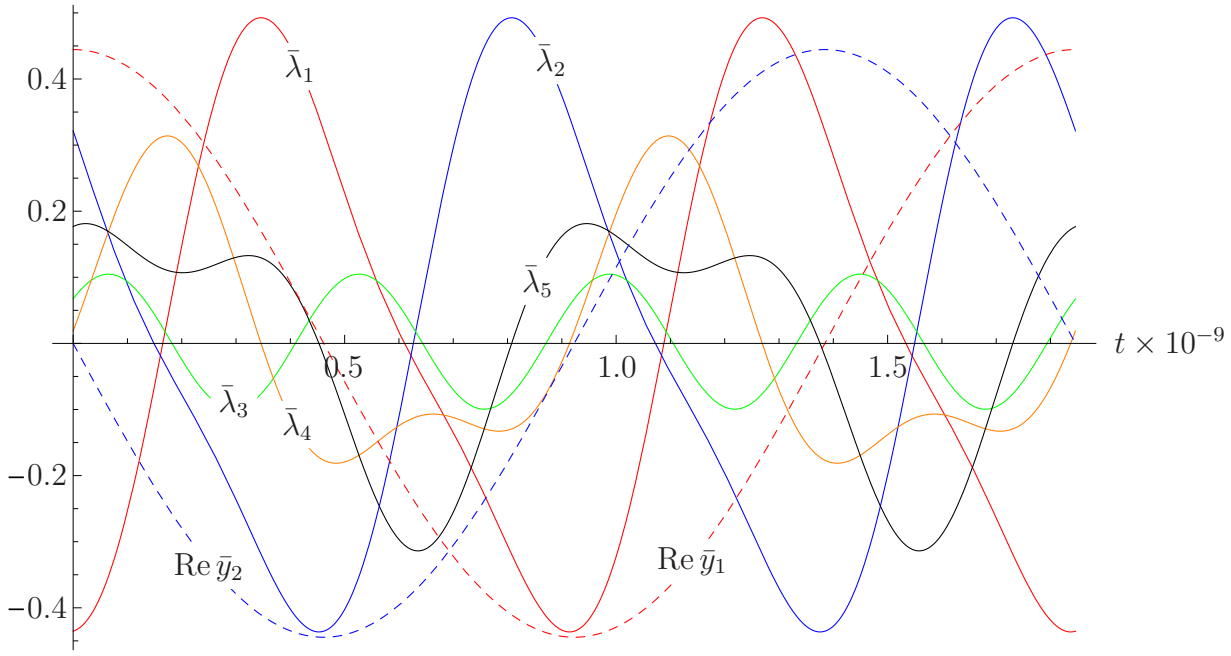

Fig. 1: The couplings of the model with one Weyl spinor and two real scalars on an RG cycle, as a function of RG time. Here $\epsilon = 0.01$ and the starting conditions are the solutions to (2.8) given in the text, Eqs. (5.7).

in the Lagrangian (5.1) and then reading off the new coupling constants as coefficients of the separate monomials of the potential in $\mathscr{L}$. This is not surprising since the transformation corresponds to the action of the charge $\int d^3x\, V^0$, and the current in (5.2) generates rotations among the two real scalars.

We have also studied two additional cases. For a model of one Weyl spinor and three real scalars we find closed trajectories, but this model also has an unbounded scalar potential. For a model of two Weyl spinors and two real scalars we find scale-invariant trajectories with non-vanishing Q and undetermined P. The potential is bounded from below and the model displays a limit cycle. The same limit cycle is found for a model consisting of two real scalars and one Dirac spinor (which dispels any concerns that may arise from the use of Weyl spinors in non-integral dimensions). We will give the details of this model in a forthcoming publication [13].

6. Discussion and conclusion

The dearth of examples of scale but not conformally invariant theories has led to the general belief that scale implies conformal invariance. As a result, a vast amount of knowledge has

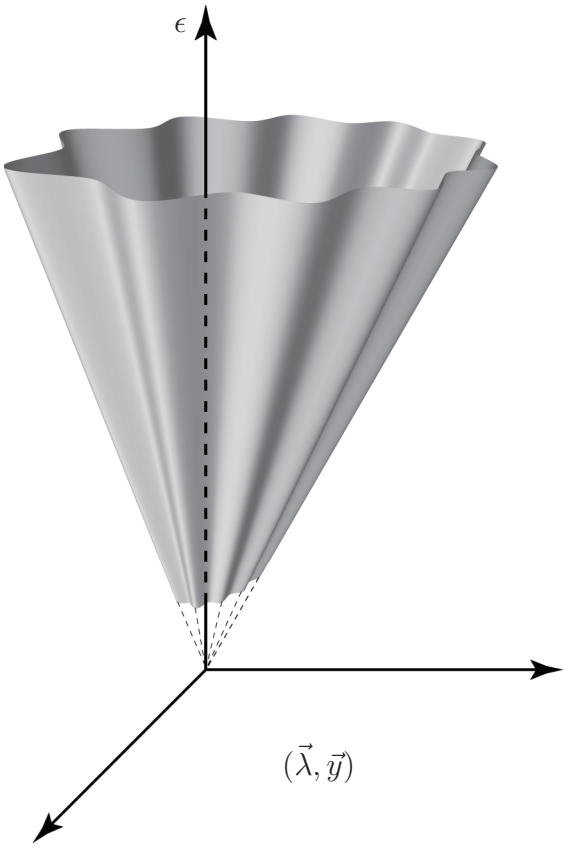

Fig. 2: Artistic rendition of scale-invariant trajectories as a function of ϵ.

been amassed on the behavior of both conformal theories and scale non-invariant ones. We know virtually nothing about scale but not conformally invariant theories. The examples we have found open the door for the exploration of this radically new class of relativistic quantum field theories.

Are the trajectories we have found attractors, much like infrared fixed points? What is the scalars' effective potential? Are there models in physical space-time dimensions, i.e., integral D, which exhibit limit cycles? What about supersymmetric models? Can one systematize the search for models in $D = 4 - \epsilon$? What restrictions are imposed by scale invariance on Green's functions? Are there phenomenological, model-building applications? We plan to address some of these questions in a forthcoming publication [13].

The question of models in integral dimensions is particularly exciting. At the moment the best hopes are to either convincingly extend our results here to $D = 3$, much like in the theory of critical phenomena, or to study $D = 4$ models with a Yang–Mills fixed-point coupling playing the role of ϵ in the $4 - \epsilon$ examples. In light of our present results, we are happy to abandon the gloomy pessimism of the past and plan to search for one such

example in earnest.

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