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ABSTRACT

New properties and relations to Clebsch-Gordan coefficients have been found and proved for a certain subset of isospin crossing matrix elements. These properties, coupled with the well-known fact that the crossing matrix is simply related to a real orthogonal matrix, provide a quick method to calculate crossing matrices of rank ≤ 4 . Procedures of construction and tables are given for all crossing matrices involving $I \leq \frac{3}{2}$. For the sake of completeness as well as clarification of the errors made in previous treatments, we include a systematic discussion of crossing relations and a general expression for crossing matrices applicable to arbitrary phase conventions. For the crossing matrix in which both direct and crossed reactions are elastic, some amusing inequalities among elements in the first and last columns are noticed and their physical implication is briefly discussed.

INTRODUCTION

The relation between the isospin crossing matrix (shorthand as C. M. hereafter) for a four-line connected part and the $6 - j$ symbol¹ has been known² for some time. Several previous general treatments³⁻⁵ of this matter are mainly aimed at giving a set of phase rules consistent with isospin invariance for the crossing symmetry relations.⁶ Once this has been done, an expression for the C. M. in terms of $6 - j$ symbols, with convention-dependent phase factors, can readily be obtained. However, it should be pointed out that there are errors made in Refs. 3 and 4. As will be shown later, so long as we adopt the natural phase convention that the imaginary part of elastic amplitudes is positive, each row of the elastic C. M. will sum to unity.⁷ The C. M.'s for NN^* and N^*N^* given in Ref. 3 turn out to have the wrong sign in this respect. The error in Ref. 4 lies in the fact that the rules fail to give distinct crossing phases for crossed particles and anti-particles of half-integral isospin. For the sake of completeness and clarification we feel that a systematic discussion of the crossing relation is necessary.

Our main interest, however, is to investigate whether we can calculate the low rank C. M. by just making use of their general properties. One method of this kind, which has been used for⁸ πN and for⁹ $\pi\pi$ scattering, is to obtain the C.M. by explicitly constructing projection operators of definite isospin in both direct and crossed reactions and then searching

for their relations. Unfortunately, the approach along this line is quite complicated except for a few simple elastic cases. In the following we want to present an algebraic method for calculating the C.M. We have found certain simple phase properties of the outermost layer of C.M. elements and their relations with Clebsch-Gordan coefficients. In addition we have proved some useful sum rules for the trace and for the column and row elements in elastic C.M.'s. These properties, coupled with the well-known fact that the C.M. is simply related to a real orthogonal matrix, provide us enough constraints to calculate all C.M.'s of rank equal to or lower than four. So far the largest isospin for established particles with $B = 0,1$ is $\frac{3}{2}$; hence our method will cover all C.M.'s involving known baryons and mesons.

As a by-product of our investigation, we have noticed that there exist interesting inequalities for the matrix elements of an arbitrary elastic C.M. The physical implication of this fact will be discussed in Section V.

In Section I we shall discuss the phase convention used in this note and derive a general expression of the C.M. which covers all different conventions. The general properties of the C.M., together with simple sum rules among matrix elements, are given in Section II. In Section III we give procedures for constructing the C.M. and also tabulate all C.M.'s involving $I \leq \frac{3}{2}$. Formulas for all elastic C.M.'s up to rank four are also given; these might be of use for spin crossing matrices in a static model.⁸ Finally in Section IV we give the proof of those properties stated in Section II.

I. PHASE CONVENTION

For a general reaction involving two-particle channels, if we let $a + b \rightarrow c + d$ be the s-reaction, as usual we call $a + \bar{c} \rightarrow \bar{b} + d$ and $a + \bar{d} \rightarrow c + \bar{b}$ the t- and u-reactions respectively. The letters a, b, c and d stand not only for particle type but also for the isospin value of each particle; the same letters with and without bar are anti-particle conjugate to each other. Throughout this note the Condon-Shortley convention¹ is used. We shall use a_α to denote a charge state of multiplet a with I_z -component α , and $\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle$ to denote the amplitude for a typical reaction $a_\alpha + b_\beta \rightarrow c_\gamma + d_\delta$. The phases of all amplitudes, or equivalently the phases of all particles, are adjusted in such a way that isospin invariance is expressed by

$$\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle = \sum_{\alpha' \dots \delta'} D_{\alpha' \alpha}(u) D_{\beta' \beta}(u) D_{\gamma' \gamma}^*(u) D_{\delta' \delta}^*(u) \times \langle c_{\gamma'} d_{\delta'} | A | a_{\alpha'} b_{\beta'} \rangle, \quad (1)$$

where u is any group element of $SU(2, C)$ ¹⁰ and $D(u)$ is the usual irreducible representation of u with dimension $(2I + 1)$ for each particle; star means complex conjugation.

As will be shown in the Appendix, the crossing relation compatible with (1) takes the general form

$$\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle = \eta(b,\beta) \eta^*(d,\delta) \langle c_\gamma \bar{b}_{-\beta} | A | a_\alpha \bar{d}_{-\delta} \rangle, \quad (2)$$

where the η 's are phase factors which can be written as $\eta(j,m) = \eta_j (-1)^m$ with η_j arbitrary and independent of m . Thus we can take all crossing phases of the type $\eta(j,m)$ to be real for convenience, and rewrite (2) as

$$\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle = \eta(b,\beta) \eta(d,\delta) \langle c_\gamma \bar{b}_{-\beta} | A | a_\alpha \bar{d}_{-\delta} \rangle. \quad (3)$$

In real phase convention, which we shall use hereafter, there is two-fold freedom in selecting η_a for a given iso-multiplet a , i.e.,

$$\begin{aligned} \eta_a &= -\eta_{\bar{a}} = \pm i && \text{if } 2a \text{ is odd,} \\ \eta_a &= \eta_{\bar{a}} = \pm 1 && \text{if } 2a \text{ is even.} \end{aligned}$$

We are now in a position to define the C.M. unambiguously:

Let A_s , A_t , and A_u denote the amplitudes of definite isospin in s , t , and u channels respectively; they are defined through¹¹

$$\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle = \sum_s \langle a, \alpha; b, \beta | s \rangle \langle c, \gamma; d, \delta | s \rangle A_s \quad (4)$$

$$\langle \bar{b}_{-\beta} d_\delta | A | a_\alpha \bar{c}_{-\gamma} \rangle = \sum_t \langle a, \alpha; c, -\gamma | t \rangle \langle b, -\beta; d, \delta | t \rangle A_t \quad (5)$$

$$\langle c_\gamma \bar{b}_{-\beta} | A | a_\alpha \bar{d}_{-\delta} \rangle = \sum_u \langle a, \alpha; d, -\delta | u \rangle \langle c, \gamma; b, -\beta | u \rangle A_u. \quad (6)$$

The C.M.'s C_{st} and C_{su} are such that

$$A_s = \sum_t C_{st} A_t, \quad (7)$$

$$A_s = \sum_u C_{su} A_u. \quad (8)$$

Notice that the phases of the A 's depend on the particle order of writing the Clebsch-Gordan coefficients, since¹

$$\langle j_1, m_1; j_2, m_2 | j \rangle = (-1)^{j_1+j_2-j} \langle j_2, m_2; j_1, m_1 | j \rangle.$$

In our convention, (4) through (6), we simply let the particle written in the "first" ("second") position in each channel also be the one located at the first (second) position in each C - G coefficient. Thus defined, it can be easily shown that

$$C_{st}(a + b \rightarrow c + d) = (-1)^{2d+b+c-s-u} C_{su}(a + b \rightarrow d + c), \quad (9)$$

where the reaction in the bracket specifies the s-reaction for each C.M. Because of (9) we shall confine our attention to C_{su} from now on.

With the crossing relation (3) and the definition of the $6 - j$ symbol, together with (4) and (6) we finally get

$$C_{su} = \eta_b \eta_d (-1)^{b+d+2s} (2u+1) \begin{Bmatrix} a & b & s \\ c & d & u \end{Bmatrix}. \quad (10)$$

The phase factor in (10) exhibits a symmetrical dependence on the two crossed particles, and in the general phase convention $\eta_d (-1)^d$ is simply replaced by $[\eta_d (-1)^d]^*$. There are only two cases in which the phase of (10) will be independent of convention: one case is when b and d belong to the same particle multiplet, then $\eta_b (-1)^b [\eta_d (-1)^d]^* = 1$; the other case is when b and d belong to two multiplets which are particle-antiparticle conjugate to each other, by (A-9) we have

$$\eta_b (-1)^b [\eta_d (-1)^d]^* = \eta_b \eta_{\bar{b}}^* = (-1)^{2b}.$$

In the following sections it will be convenient to consider a matrix \mathcal{C}_{su} whose phase differs from that of C_{su} by the rule

$$\begin{aligned} C_{su} &= \eta_b \eta_d (-1)^{d+b+2s} (-1)^{a+b+c+d} \mathcal{C}_{su} \\ &= \eta_b \eta_d (-1)^{a+2b+3c+4d} \mathcal{C}_{su}. \end{aligned} \quad (11)$$

The last form is written just for mnemonic reasons.

II. GENERAL PROPERTIES OF THE CROSSING MATRIX

From the last section, C is given by

$$C_{su} = (-1)^{a+b+c+d} (2u+1) \begin{Bmatrix} a & b & s \\ c & d & u \end{Bmatrix}. \quad (12)$$

With this relation and the properties of 6-j symbols we then infer the following:

$$(A) \quad C_{su} = \left(\frac{2u+1}{2s+1} \right)^{1/2} O_{su}, \quad (13)$$

where O is a real orthogonal matrix; thus

$$C_{su} = \left(\frac{2u+1}{2s+1} \right) C_{us}^{-1}. \quad (14)$$

(B) The last row and last column of C are all positive whereas the first row and first column are alternating in sign.

(C) The last row can be expressed in terms of Clebsch-Gordan coefficients as follows:

If $c+d \geq a+b$,

$$C_{a+b,u} = \frac{\langle a, a; d, c-a-b|u \rangle \langle c, c; b, -b|u \rangle}{\langle c, c; d, a+b-c|a+b, a+b \rangle}. \quad (15)$$

If $a+b \geq c+d$,

$$C_{c+d, u} = \frac{\langle c, c; b, a - c - d | u \rangle \langle a, a; d, -d | u \rangle}{\langle a, a; b, c + d - a | c + d, c + d \rangle} \quad (16)$$

It is clear (15) and (16) are related by the exchange $a \leftrightarrow c$ and $b \leftrightarrow d$.

(D) Coupling (13) and (C), we can get the last column of C in terms of last row of C^{-1}

$$C_{s, u_M} = \left(\frac{2u_M + 1}{2s + 1} \right) C_{u_M, s}^{-1}, \quad (17)$$

where u_M is the maximum of u .

(E) If $I_b = I_d$, i.e., the two crossed particles have the same isospin, then $C^2 = 1$ and C is symmetrical.

(F) For the case $I_a = I_c$ and $I_b = I_d$, we have sum rules for the matrix elements:

$$\sum_u C_{su} = 1 \quad (18)$$

$$\sum_s (2s + 1) C_{su} = (2u + 1) \quad (19)$$

$$\text{Tr } C = \sum_{\alpha} C_{\alpha\alpha} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \quad (20)$$

where n is the rank of C .

(G) Again for the elastic case only, the elements in the last row are simply

$$C_{a+b,u} = \langle a, a; b, -b | u, a - b \rangle^2 \quad (21)$$

while those in the first column are given by (without loss of generality we take $a \geq b$)

$$C_{s,a-b} = (-1)^l \langle a, a - l; b, -b + l | a - b, a - b \rangle^2, \quad (22)$$

where $l \equiv (a + b) - s$, taking values from 0, 1 ... to $2b$.

(H) Another special case worth mentioning is that when $I_a = I_d$ and $I_b = I_c$, then

$$C_{s0} = (-1)^{a+b-s} [(2a + 1)(2b + 1)]^{-1/2}. \quad (23)$$

The properties stated above are useful for constructing the C.M. Following are properties with interesting dynamical implications.

(I) For $C_{su}(a + b \rightarrow a + b)$ and $a \geq b$,

(i) $C_{a-b,a+b}$ is greater than any other element.

(ii) The elements in the last column decrease with increasing s .

(iii) The elements in the first column increase their magnitude with increasing s for $a \neq b$, and become all equal in magnitude when $a = b$.

(iv) The elements in the first row increase in magnitude with increasing u .

(J) For $C(a + b \rightarrow a + b)$ with $\frac{a}{b} \gg 1$, the anti-diagonal elements are close to unity to the order of $\frac{b}{a}$, while others are all vanishingly small in the order

$$C_{su} \approx 0 \left(\left(\frac{b}{a} \right)^{|s+u-2a|} \right);$$

and the signs of C_{su} are given by

$$\text{sgn } C_{su} = (-1)^{s+u-2a} \quad \text{for } s + u < 2a,$$

$$\text{sgn } C_{su} = 1 \quad \text{for } s + u \geq 2a.$$

(K) For $C_{su}(a + a \rightarrow b + b)$, the element with largest magnitude in each column is the one with $s = 0$.

Properties (A), (E), and (H) are well known,¹ (K) has been proved by Masuda,¹² and the remaining ones will be proved in Section IV.

III. METHOD OF CONSTRUCTION AND TABLES
FOR CROSSING MATRICES

Since by (11) we can obtain the most general crossing matrix for different conventions and particle sets, it is sufficient for us to consider the C matrix for various set of isospins. From the relation (12) and symmetry properties of the $6 - j$ symbol, it is clear that $C(a + b \rightarrow c + d)$ and $C(b + a \rightarrow d + c)$ are identical, while $C(a + b \rightarrow c + d)$ and $C(a + d \rightarrow c + b)$ are inverse to each other. Hence for a given set of isospin a, b, c and d there are at most two independent C matrices to be considered, namely, $C(a + b \rightarrow c + d)$ and $C(a + b \rightarrow d + c)$.

The procedures for constructing C matrices are as follows:

Case of rank one

By (13) we trivially obtain

$$C_{su} = \left(\frac{2u + 1}{2s + 1} \right)^{1/2} .$$

Case of rank two

By (13) and (B) we may put

$$C_{su} = \left(\frac{2u + 1}{2s + 1} \right)^{1/2} O_{su} \quad \text{with} \quad O = \begin{vmatrix} -\alpha & (1 - \alpha^2)^{1/2} \\ (1 - \alpha^2)^{1/2} & \alpha \end{vmatrix}$$

where α can be determined from any element in the last row by using (c).

Case of rank three or four

Although our method can cover all C. M. up to rank four, we shall consider only those in which either (or both) the s and u reaction is elastic (in the sense of isospin), since others necessarily involve particles with $I > \frac{3}{2}$, in which we are not interested. First consider $C(a + b \rightarrow a + b)$; the last row and first column are immediately obtained by using (G). [It is amusing to notice that the numbers given by (21) and (22) also appear in a column and row in the table of Clebsch-Gordan coefficients.] With (14) relating symmetrical elements about diagonal for the case $C = C^{-1}$, we easily get the first row and last column. The remaining elements are then determined by sum rules in (F).

As for the case $a + b \rightarrow b + a$, we use (C) and (D) to get the last row and column, and (H) to get the first row; then the rest can be determined by orthogonality constraints imposed on O_{su} .

The main scheme of the above procedures is quite clear: first we calculate the outermost layer of elements; then we use the orthogonality constraints together with sum rules to get the remaining elements. Since the number of parameters needed to describe an arbitrary real orthogonal matrix of rank 2, 3, and 4 is 1, 3, and 6, respectively, we see that the available information is more than necessary to obtain a unique result.

In fact it is interesting to note that for $C(a + \frac{1}{2} \rightarrow a + \frac{1}{2})$, the sum rules from (F) are alone sufficient to fix all elements. By (18) and (20) C can be parameterized as

$$C = \begin{vmatrix} -\alpha & 1 + \alpha \\ 1 - \alpha & \alpha \end{vmatrix} .$$

Now using (19) we have

$$2a(-\alpha) + (2a + 2)(1 - \alpha) = 2a ,$$

so

$$\alpha = \frac{1}{2a + 1} .$$

The following tables for C.M.'s are listed in the order of increasing rank. The reaction written in each case is the s-reaction for C and the u-reaction for C^{-1} .

$$\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$$

$$C = C^{-1} = \begin{vmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$1 + \frac{1}{2} \rightarrow 1 + \frac{1}{2}$$

$$C = C^{-1} = \begin{vmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$\frac{3}{2} + \frac{1}{2} \rightarrow \frac{3}{2} + \frac{1}{2}$$

$$C = C^{-1} = \begin{vmatrix} -\frac{1}{4} & \frac{5}{4} \\ \frac{3}{4} & \frac{1}{4} \end{vmatrix}$$

$$a + \frac{1}{2} \rightarrow a + \frac{1}{2}$$

$$C = C^{-1} = \begin{vmatrix} -\frac{1}{2a+1} & \frac{2(a+1)}{2a+1} \\ \frac{2a}{2a+1} & \frac{1}{2a+1} \end{vmatrix}$$

$$1 + \frac{1}{2} \rightarrow \frac{1}{2} + 1$$

$$C = \begin{vmatrix} \frac{-1}{\sqrt{6}} & 1 \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{vmatrix} \quad C^{-1} = \begin{vmatrix} \frac{\sqrt{6}}{3} & \frac{2\sqrt{6}}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{vmatrix}$$

$$\frac{3}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{3}{2}$$

$$C = \begin{vmatrix} -\frac{\sqrt{2}}{4} & \frac{\sqrt{10}}{4} \\ \frac{\sqrt{2}}{4} & \frac{3}{2\sqrt{10}} \end{vmatrix} \quad C^{-1} = \begin{vmatrix} -\frac{3\sqrt{2}}{4} & \frac{5\sqrt{2}}{4} \\ \frac{\sqrt{10}}{4} & \frac{\sqrt{10}}{4} \end{vmatrix}$$

$$a + \frac{1}{2} \rightarrow \frac{1}{2} + a$$

$$C = \frac{1}{[2(2a+1)]^{\frac{1}{2}}} \begin{vmatrix} -1 & \left[\frac{3(a+1)}{a}\right]^{\frac{1}{2}} \\ 1 & \left[\frac{3a}{(a+1)}\right]^{\frac{1}{2}} \end{vmatrix}$$

$$C^{-1} = \frac{1}{[2(2a+1)]^{\frac{1}{2}}} \begin{vmatrix} -2a & 2(a+1) \\ 2\left[\frac{a(a+1)}{3}\right]^{\frac{1}{2}} & 2\left[\frac{a(a+1)}{3}\right]^{\frac{1}{2}} \end{vmatrix}$$

$$\frac{1}{2} + 1 \rightarrow 1 + \frac{3}{3}$$

$$C = \begin{vmatrix} -\frac{1}{2} & \frac{5\sqrt{3}}{6} \\ \frac{\sqrt{5}}{2\sqrt{2}} & \frac{\sqrt{5}}{2\sqrt{6}} \end{vmatrix}$$

$$C^{-1} = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{10}}{3} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{15}} \end{vmatrix}$$

$$\frac{1}{2} + 1 \rightarrow \frac{3}{2} + 1$$

$$C = C^{-1} = \begin{vmatrix} -\frac{2}{3} & \frac{\sqrt{10}}{3} \\ \frac{\sqrt{10}}{6} & \frac{2}{3} \end{vmatrix}$$

$$\frac{1}{2} + \frac{3}{2} \rightarrow \frac{3}{2} + \frac{3}{2}$$

$$C = C^{-1} = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{5}}{2} \\ \frac{3}{\sqrt{20}} & \frac{1}{2} \end{vmatrix}$$

$$1 + 1 \rightarrow 1 + 1$$

$$C = C^{-1} = \begin{vmatrix} \frac{1}{3} & -1 & \frac{5}{3} \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{vmatrix}$$

$$\frac{3}{2} + 1 \rightarrow \frac{3}{2} + 1$$

$$C = C^{-1} = \begin{vmatrix} \frac{1}{6} & -\frac{2}{3} & \frac{3}{2} \\ -\frac{1}{3} & \frac{11}{15} & \frac{3}{5} \\ \frac{1}{2} & \frac{2}{5} & \frac{1}{10} \end{vmatrix}$$

$$a + 1 \rightarrow a + 1$$

$$C = C^{-1} = \begin{vmatrix} \frac{1}{a(2a+1)} & -\frac{1}{a} & \frac{2a+3}{2a+1} \\ \frac{-(2a-1)}{a(2a+1)} & \frac{a^2+a-1}{a(a+1)} & \frac{2a+3}{(a+1)(2a+1)} \\ \frac{2a-1}{2a+1} & \frac{1}{a+1} & \frac{1}{(a+1)(2a+1)} \end{vmatrix}$$

$$\frac{3}{2} + 1 \rightarrow 1 + \frac{3}{2}$$

$$C = \begin{vmatrix} \frac{\sqrt{3}}{6} & -\frac{\sqrt{10}}{4} & \frac{5}{2\sqrt{6}} \\ -\frac{\sqrt{3}}{6} & \frac{1}{\sqrt{10}} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{2\sqrt{10}} & \frac{\sqrt{6}}{12} \end{vmatrix}$$

$$C = \begin{vmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & \sqrt{3} \\ -\frac{\sqrt{10}}{6} & \frac{2\sqrt{10}}{15} & \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{6}} & \frac{4\sqrt{6}}{15} & \frac{\sqrt{6}}{10} \end{vmatrix}$$

$$a + 1 \rightarrow 1 + a$$

$$C = \begin{vmatrix} 1 & -3 \left[\frac{(a+1)}{2a} \right]^{\frac{1}{2}} & \left[\frac{5(a+1)(2a+3)}{2a(2a-1)} \right]^{\frac{1}{2}} \\ \frac{1+a}{[3(2a+1)]^{\frac{1}{2}}} & -1 \frac{3}{[2a(a+1)]^{\frac{1}{2}}} & \left[\frac{5(2a-1)(2a+3)}{2a(a+1)} \right]^{\frac{1}{2}} \\ 1 & 3 \left[\frac{a}{2a(a+1)} \right]^{\frac{1}{2}} & \left[\frac{5a(2a-1)}{2(a+1)(2a+3)} \right]^{\frac{1}{2}} \end{vmatrix}$$

$$C^{-1} = \frac{1}{[3(2a+1)]^{\frac{1}{2}}} \begin{vmatrix} (2a-1) & -(2a+1) & (2a+3) \\ -(2a-1)\left[\frac{(a+1)}{a}\right]^{\frac{1}{2}} & \frac{(2a+1)}{[2a(a+1)]^{\frac{1}{2}}} & (2a+3)\left[\frac{a}{2a(a+1)}\right]^{\frac{1}{2}} \\ \left[\frac{(2a-1)(a+1)(a+3)}{10a}\right]^{\frac{1}{2}} & (2a+1)\left[\frac{(2a-1)(2a+3)}{10a(a+1)}\right]^{\frac{1}{2}} & \left[\frac{(2a-1)a(2a+3)}{10(a+1)}\right]^{\frac{1}{2}} \end{vmatrix}$$

$$\frac{3}{2} + \frac{3}{2} \rightarrow \frac{3}{2} + \frac{3}{2}$$

$$C = C^{-1} = \begin{vmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{5}{4} & \frac{7}{4} \\ \frac{1}{4} & -\frac{11}{20} & \frac{1}{4} & \frac{21}{20} \\ -\frac{1}{4} & \frac{3}{20} & \frac{3}{4} & \frac{7}{20} \\ \frac{1}{4} & \frac{9}{20} & \frac{1}{4} & \frac{1}{20} \end{vmatrix}$$

$$a + \frac{3}{2} \rightarrow a + \frac{3}{2}$$

$$C = C^{-1} = \frac{1}{(2a+1)} \begin{vmatrix} -\frac{3}{a(2a-1)} & \frac{1}{2a-1} & -\frac{3(a+1)}{a} & 2(a+2) \\ \frac{6(a-1)}{a(2a-1)} & -\frac{8a^2+4a-13}{(a+1)(2a-1)} & \frac{2(a^2+a-3)}{a} & \frac{3(a+2)}{(a+1)} \\ -\frac{3(a-1)}{a} & \frac{2(a^2+a-3)}{(a+1)} & \frac{8a^2+8a-9}{a(2a+3)} & \frac{6(a+2)}{(a+1)(2a+3)} \\ 2(a-1) & \frac{3a}{a+1} & \frac{6}{2a+3} & \frac{3}{(a+1)(2a+3)} \end{vmatrix}$$

IV. PROOFS OF THE GENERAL PROPERTIES

To prove (B) we use a formula¹

$$\left\{ \begin{matrix} j_1 & j_2 & l_1 + l_2 \\ l_1 & l_2 & l_3 \end{matrix} \right\} = (-1)^{j_1 + j_2 + l_1 + l_2} \left[\frac{(2l_1)! (2l_2)! (j_1 + j_2 + l_1 + l_2 + 1)!}{(2l_1 + 2l_2 + 1)! (j_1 + j_2 - l_1 - l_2)!} \right. \\ \left. \times \frac{(j_1 + l_1 + l_2 - j_2)! (j_2 + l_1 + l_2 - j_1)! (j_1 + l_3 - l_2)! (j_2 + l_3 - l_1)!}{(l_1 + j_2 + l_3 + 1)! (j_1 + l_2 - l_3)! (l_2 + l_3 - j_1)! (j_1 + l_2 + l_3 + 1)! (l_1 + j_2 - l_3)! (l_1 + l_3 - j_2)!} \right]^{\frac{1}{2}} \quad (24)$$

Let s_M and u_M represent the maximum of the s and u values that appeared in the C.M., while s_m and u_m represent the corresponding minimum values. Since s_M is equal to the smaller value of $(a + b)$ and $(c + d)$, while u_M equals the smaller of $(a + d)$ and $(b + c)$, the $6 - j$ symbols

$$\left\{ \begin{matrix} a & b & s_M \\ c & d & u \end{matrix} \right\} \text{ and } \left\{ \begin{matrix} a & b & s \\ c & d & u_M \end{matrix} \right\}$$

in all cases can be put into the form (24) by use of the following symmetries:

$$\left\{ \begin{matrix} a & b & s \\ c & d & u \end{matrix} \right\} = \left\{ \begin{matrix} c & d & s \\ a & b & u \end{matrix} \right\} = \left\{ \begin{matrix} a & d & u \\ c & b & s \end{matrix} \right\} = \left\{ \begin{matrix} c & b & u \\ a & d & s \end{matrix} \right\} .$$

By (24) and (12) we immediately get

$$\operatorname{sgn} C_{s_M u} = \operatorname{sgn} C_{s u_M} = (-1)^{2a+2b+2c+2d} = 1,$$

where $\operatorname{sgn} x \equiv \frac{x}{|x|}$.

On the other hand, s_m is the larger value of $|a - b|$ and $|c - d|$, u_m the larger of $|a - d|$ and $|b - c|$. For example, let us assume for definiteness that $s_m = a - b$; this implies that

$$a - b \geq |c - d|,$$

so

$$a + d \geq c + b = u_M.$$

Now

$$\begin{aligned} \begin{Bmatrix} a & b & s_m \\ c & d & u \end{Bmatrix} &= \begin{Bmatrix} a & b & a-b \\ c & d & u \end{Bmatrix} \\ &= \begin{Bmatrix} a-b & b & a \\ u & d & c \end{Bmatrix} = \begin{Bmatrix} u & d & a \\ a-b & b & c \end{Bmatrix}. \end{aligned}$$

Applying (24) to the last form, we get the phase $(-1)^{a+d+u}$, hence

$$\operatorname{sgn} C_{(a-b), u} = (-1)^{2a+b+c+2d+u} = (-1)^{u-b-c} = (-1)^{u-u_M}.$$

Similar considerations for all other cases lead to the same result, namely,

$$\operatorname{sgn} C_{s_m, u} = (-1)^{u-u_M}$$

and

$$\text{sgn } C_{s, u_m} = (-1)^{s-s_M}.$$

This completes the proof of assertion (B).

One can verify (C) straightforwardly by direct substitution of the explicit expression for each entry in (15) and (16), with the use of (24) and¹

$$\langle j_1, j_1; j_2, m-j_1 | j, m \rangle = \left[\frac{(2j+1) (2j_1)! (-j_1+j_2+j)! (j_1+j_2-m)! (j+m)!}{(j_1+j_2-j)! (j_1-j_2+j)! (j_1+j_2+j+1)! (-j_1+j_2+m)! (j-m)!} \right]^{\frac{1}{2}}. \quad (25)$$

But we prefer the following proof in view of its physical transparency.

Take the case $c + d \geq a + b$; we consider the following set of charge states,

$$|a, a\rangle + |b, b\rangle \rightarrow |c, c\rangle + |d, a + b - c\rangle. \quad (26)$$

Now the LHS is a pure $I_s = a + b$ state while the RHS has only a fraction

$$\langle c, c; d, a + b - c | a + b, a + b \rangle \text{ of an } I_s = a + b$$

state, so the whole expression represents

$$\langle c, c; d, a + b - c | a + b, a + b \rangle A_{s=a+b}.$$

On the other hand, the crossed reaction

$$|a, a\rangle + |\bar{d}, -a - b + c\rangle \rightarrow |c, c\rangle + |\bar{b}, -b\rangle \quad (27)$$

represents a linear combination of different C_u amplitudes, namely,

$$\sum_u \langle a, a; d, c - b - a | u \rangle \langle c, c; b, -b | u \rangle A_u .$$

Equating this to the previous expression we arrive at (15). Notice that if we restore the crossing phase $\eta(d, a + b - c) \eta(b, b)$ to the amplitude represented by (27) according to (3), then instead of (15) we will automatically have an expression for $C_{a+b, u}$.

The sum rule (18) can be proved most easily as follows: Consider $a = c$ and $b = d$, then $C_{su} = C_{su}$ by (11); now let A be an identity operator, from (4) and (6) we have $A_s = A_u = 1$; since C_{su} is independent of A , (8) is just reduced to $\sum_u C_{su} = 1$.

To prove the sum rule (19) we use (13) and (18),

$$\begin{aligned} \sum_s (2s + 1) C_{su} &= \sum_s [(2s + 1) (2u + 1)]^{\frac{1}{2}} O_{su} \\ &= \sum_s [(2u + 1) (2s + 1)]^{\frac{1}{2}} O_{us} \\ &= \sum_s (2u + 1) C_{us} = (2u + 1) . \end{aligned}$$

The proof of the sum rules (20) is more elaborate. First we note that $\text{Tr } C = \text{Tr } O$ from (13), and $O^2 = I$ from (E). Since

now the eigenvalues of O can only be ± 1 and its trace is invariant under diagonalization, we conclude that $\text{Tr } C = \text{integer}$. Now using (J), which we shall prove later, we can put

$$C_{su} = \delta_{s+u, 2a} + B_{su}, \quad (28)$$

where $B_{su} \rightarrow 0$ as $a/b \rightarrow \infty$. So for sufficiently large a we see that (20) indeed follows. To extend the proof to all values of a , we fix b and consider $C_{su}(a)$ as a function of the real variable a ; it is actually a rational function of a without poles for $a \geq b$, as can be seen from the analytic expression¹ for the $6-j$ symbol. So, what we have proved about the trace for large a can be analytically continued down to all values of a as low as b ; thus we have (20).

To prove the statements in (I), we use (25) in (21) and (22) to obtain the analytic expression for $C_{a+b, u}$ and $C_{s, a-b}$, and then use (14) to get the expressions for $C_{s, a+b}$ and $C_{a-b, u}$.

Taking ratios we have

$$\frac{C_{s, a+b}}{C_{s+1, a+b}} = \frac{(a+b+2+s)}{(a+b-s)} > 1, \quad (29)$$

$$\frac{|C_{s, a-b}|}{|C_{s+1, a-b}|} = \frac{(s+1+b-a)}{(s+1+a-b)} < 1 \quad \text{for } a > b$$

$$= 1 \quad \text{for } a = b \quad (30)$$

$$\frac{|C_{a-b, u}|}{|C_{a-b, u+1}|} = \left(\frac{2u+1}{2u+3} \right) \left(\frac{u+1+b-a}{u+1+a-b} \right) < 1. \quad (31)$$

These relations confirm (i) through (iii). To show that $C_{a-b, a+b}$ is the largest element, we employ

$$\begin{aligned} \left| (2a+1) \begin{Bmatrix} a & b & s \\ a & b & u \end{Bmatrix} \right| &= \left| C_{aa}(s+b \rightarrow u+b) \right| \\ &= \left| O_{aa}(s+b \rightarrow u+b) \right| < 1, \end{aligned}$$

so

$$\begin{aligned} \frac{|C_{su}|}{C_{a-b, a+b}} &= (2u+1) \left| \begin{Bmatrix} a & b & s \\ a & b & u \end{Bmatrix} \right| \left(\frac{2a+2b+1}{2a+1} \right)^{-1} \\ &< \left(\frac{2u+1}{2a+2b+1} \right) \leq 1. \end{aligned} \quad (32)$$

Finally, for property (J), we have assumed $a \gg b$, so $\frac{a-b}{a+b} \approx 1$ and $s \approx u \approx a$. (We use \approx to mean approximation to the order of b/a .) Using the explicit formula for 6-j symbol, we can write

$$\begin{aligned} C_{su} &= (-1)^{2a+2b} (2u+1) P(abs) P(abu) \sum_z \frac{(-1)^z (z+1)!}{[(z-a-b-u)!]^2} \\ &\times \frac{1}{[(z-a-b-s)!]^2 (2a+2b-z)! (2b+s+u-z)! (2a+s+u-z)!}, \end{aligned}$$

where

$$P(a \ b \ c) \equiv \frac{(-a+b+c)! (a-b+c)! (a+b-c)!}{(a+b+c+1)!}$$

and z takes all integer values such that the arguments in the factorials are non-negative. It is easy to see that the term with highest z in the sum is the one with the leading power in a ; thus (let $s \equiv a + m$, $u \equiv a + n$)

$$\begin{aligned}
 C_{su} &\approx (-1)^{2a-s-u} P(a \ b \ s) P(a \ b \ u) \\
 &\times \frac{(2b + s + u + 1)!}{[(s+b-a)!]^2 [(u+b-a)!]^2 (2a-s-u)! (2a-2b-1)!} \\
 &\approx (-1)^{2a-s-u} \frac{(b-m)! (b-n)!}{(b+m)! (b+n)! (-m-n)!} (2a)^{s+u-2a}
 \end{aligned}$$

for $s + u \leq 2a$;

$$\begin{aligned}
 C_{su} &\approx P(a \ b \ s) P(a \ b \ u) \\
 &\times \frac{(2a + 2b + 1)!}{[(a+b-s)!]^2 [(a+b-u)!]^2 (s+u-2a)! (s+u-2b)!} \\
 &\approx \frac{(b+m)! (b+n)!}{(b-m)! (b-n)! (m+n)!} (2a)^{2a-s-u}
 \end{aligned}$$

for $s + u \geq 2a$. The proof is completed.

Equation (21) is just a special case of (15), while Eq. (22) can be verified by direct substitution:

$$\begin{aligned}
 C_{s, a-b} &= (-1)^{2a+2b} (2a - 2b + 1) \begin{Bmatrix} a & b & s \\ a & b & a-b \end{Bmatrix} \\
 &= (-1)^{2a+2b} (2a - 2b + 1) \begin{Bmatrix} s & b & a \\ a-b & b & a \end{Bmatrix} \\
 &= (-1)^{s-a-b} \frac{(2b)! (s + a - b)!}{(2a + 1)! (s + b - a)!} .
 \end{aligned}$$

In the last step we make use of (24). On the other hand,

$$\begin{aligned}
 &\langle a, a - l ; b, -b + l | a - b, a - b \rangle^2 \\
 &= \left(\frac{2a - 2b + 1}{2a + 1} \right) \langle a - b, a - b ; b, b - l | a, a - l \rangle^2 \\
 &= (2a - 2b + 1) \left[\frac{(2b)! (2a - l)!}{(2a + 1)! (2b - l)!} \right] ,
 \end{aligned}$$

so this is just $(-1)^l C_{s, a+b}$ if we put $l = (a + b) - s$.

V. DISCUSSION IN CONNECTION WITH THE RECIPROCAL BOOTSTRAP

Several years ago Chew proposed a very attractive mechanism,¹³ the so-called reciprocal bootstrap, for the self-consistent calculation of the N, Δ system. One of the main features in the theory is that the crossing matrix plays an important role in determining the forces. Roughly speaking, the force due to the exchange of N will be strongest in the $(3,3)$ -channel, while the force from the exchange of Δ will be strongest in the $(1,1)$ -channel. This results from the fact that the crossing matrix in question is such that the bottom (top) element in the first (last) column is positive and largest in magnitude. From (29) and (30) we see in fact all elastic matrices with $a \neq b$ will exhibit this general character. Furthermore, whenever property (J) is applicable [actually the condition $a \gg b$ for its validity can be relaxed to $a/b \gg b$ for small b , as can be verified by the general formulas for $C(a + b \rightarrow a + b)$ given in Section III.], we expect that the strongest force in each column (or row) will be given by the anti-diagonal elements, and the reciprocal bootstrap mechanism is likely to operate. The sign and rough magnitude of the forces produced by other elements can also be estimated without the explicit C.M.

However, if in a system the spin C.M. or the isospin C.M. or both are of the type with $a = b$, the situation will be quite different due to property (K). In this case, the exchange of a particle with zero spin or zero isospin or both would be important in all the direct channels, and we don't expect this to be a reciprocal bootstrap system.

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APPENDIX

In the following we shall work with the spinless case for simplicity; however, our result will not be affected if we have spin. The most general crossing relation without constraint from (1) can be expressed as

$$\langle K_1 | A | K_2 ; a_\alpha, p \rangle = \eta \langle K_1 ; \bar{a}_{-\alpha} - p | A | K_2 \rangle, \quad (A-1)$$

and

$$\langle K'_1 | A | K'_2 ; \bar{a}_{-\alpha}, -p \rangle = \eta \langle K'_1 ; a_\alpha, p | A | K'_2 \rangle, \quad (A-2)$$

where K 's represent particle sets which are not crossed, and $p(-p)$ is the 4-momentum of $a_\alpha(\bar{a}_{-\alpha})$. That the crossing phase η is independent of p but depends only on the crossed particle can be understood from Lorentz invariance. We shall not attempt to give the proof but refer to Ref. 6 to prove that the same η appears in both (A-1) and (A-2). We take the convention of writing η as $\eta(a, \alpha)$ when a particle a_α in the ket vector (initial state) is crossed; then $\eta^*(a, \alpha)$ is the crossing phase when a_α is crossed from a bra vector (final state). From (A-2) we have

$$\eta(\bar{a}, -\alpha) = \eta(a, \alpha). \quad (A-3)$$

In particular for a four-line connected part the crossing relation is

$$\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle = \eta(b, \beta) \eta^*(d, \delta) \langle c_\gamma \bar{b}_{-\beta} | A | a_\alpha \bar{d}_{-\delta} \rangle. \quad (A-4)$$

Define the matrix Λ^j corresponding to each set of $\eta(j,m)$ so that

$$\Lambda_{mm'}^j = \eta(j,m') \delta_{m,-m'}; \quad (A-5)$$

clearly

$$\Lambda^{j+} = \Lambda^{j-1}$$

Now (A-4) can be rewritten as

$$\langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle = \sum_{\beta'\delta'} \langle c_\gamma \bar{b}_{\beta'} | A | a_\alpha \bar{d}_{\delta'} \rangle \Lambda_{\beta'\beta}^b \Lambda_{\delta'\delta}^{d*}. \quad (A-6)$$

Apply the transformation (1) on both sides of (A-6)

$$\begin{aligned} & \sum \langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle D_{\alpha'\alpha} D_{\beta'\beta} D_{\gamma'\gamma}^* D_{\delta'\delta}^* \\ &= \sum \langle c_\gamma \bar{b}_{\beta''} | A | a_{\alpha'} \bar{d}_{\delta''} \rangle D_{\alpha'\alpha} D_{\beta''\beta'}^* D_{\gamma'\gamma}^* D_{\delta''\delta'} \Lambda_{\beta'\beta}^b \Lambda_{\delta'\delta}^{d*} \\ &= \sum \langle c_\gamma d_\delta | A | a_\alpha b_\beta \rangle D_{\alpha'\alpha} (\Lambda^{b-1} D^* \Lambda^b)_{\beta'\beta} D_{\gamma'\gamma}^* (\Lambda^{d-1} D^* \Lambda^d)_{\delta'\delta}^*, \end{aligned}$$

where primed indices are to be summed over. Comparing both ends of the last equality we conclude that

$$\Lambda^{b-1} D^{b*}(u) \Lambda^b = D^b(u), \quad (A-7)$$

$$\Lambda^{d-1} D^{d*}(u) \Lambda^d = D^d(u). \quad (A-8)$$

First we note that any two unitary matrices, say Λ_1 and Λ_2 , both of which satisfy either (A-6) or (A-7), can only differ by a phase. Since the relation $D^j(u) = \Lambda_1^{-1} D^{j*}(u) \Lambda_1 = \Lambda_2^{-1} D^{j*}(u) \Lambda_2$ implies $[\Lambda_2^{-1} \Lambda_1, D^j(u)] = 0$ for all u , and by Schur's lemma¹⁴ we get $\Lambda_2^{-1} \Lambda_1 = a I$; taking the determinant of this equation on both sides yields $|a| = 1$. It is well known¹ that $d^j(\pi) D^{j*}(u) d^{j-1}(\pi) = D^j(u)$ where $d_{mm',j}(\pi) = (-1)^{j+m'} \delta_{m,-m'}$, so in general $\Lambda_{mm',j} = \eta_j (-1)^{m'} \delta_{m,-m'}$, with η_j an arbitrary phase independent of m' . From (A-4) we immediately get $\eta(b,\beta) = \eta_b (-1)^\beta$ and $\eta(d,\delta) = \eta_d (-1)^\delta$. It is clear that (A-3) implies

$$\eta_{\bar{a}} = (-1)^{2a} \eta_a, \quad (A-9)$$

so we see it is an intrinsic characteristic that η_a and $\eta_{\bar{a}}$ have opposite signs when $2a$ is an odd integer.

FOOTNOTES AND REFERENCES

- * This work was done under the auspices of the United States Atomic Energy Commission.
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